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CORNERS OF CUNTZ-KRIEGER ALGEBRAS

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ABSTRACT. We show that if A is a unital C^*-algebra and B is a Cuntz-Krieger algebra for which $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$, then A is a Cuntz-Krieger algebra. Consequently, corners of Cuntz-Krieger algebras are Cuntz-Krieger algebras.

1. Introduction

The Cuntz-Krieger algebras were introduced by J. Cuntz and W. Krieger in 1980, [9], as C^*-algebras arising from dynamical systems. This class of C^*-algebras has since shown up in several contexts, including the classification program since the Cuntz-Krieger algebras with finitely many ideals are examples of non-simple purely infinite C^*-algebras. It has been known since M. Enomoto and Y. Watatani introduced graph algebras in 1980 in [12] that Cuntz-Krieger algebras are the graph algebras arising from finite graphs with no sinks and no sources (see also [16]), but no characterization in terms of outer properties has been established for the Cuntz-Krieger algebras.

We show in Theorem 3.12 that the Cuntz-Krieger algebras are the graph algebras arising from finite graphs with no sinks, and conclude that a graph algebra is a Cuntz-Krieger algebra if and only if it is unital and the rank of its K_0-group equals the rank of its K_1-group. Using this, we show in Theorem 4.8 that if a unital C^*-algebra is stably isomorphic to a Cuntz-Krieger algebra, then it is isomorphic to a Cuntz-Krieger algebra.

As a corollary to Theorem 4.8, we see that corners of Cuntz-Krieger algebras are Cuntz-Krieger algebras; see Corollary 4.10. It is quite surprising that the class of Cuntz-Krieger algebras has this permanence property since the larger class of graph algebras do not (the graph algebra $\mathsf{M}_{2^\infty} \otimes \mathbb{K}$ provides a counterexample). Moreover, this shows that corners of Cuntz-Krieger algebras are semiprojective (Corollary 4.11), as Cuntz-Krieger algebras are semiprojective. Our results also show that a unital corner of a stabilized Cuntz-Krieger algebra is semiprojective since a stabilized Cuntz-Krieger algebra is semiprojective. It was conjectured by B. Blackadar in [6, Conjecture 4.4] that a full corner of a semiprojective C^*-algebra is semiprojective. He showed in [6, Proposition 2.7] that a full unital corner of a semiprojective C^*-algebra is semiprojective. Recently, S. Eilers and T. Katsura showed in [11] that a corner of a unital graph C^*-algebra that is semiprojective is also semiprojective. Corollary 4.11 is a special case of their results since every Cuntz-Krieger algebra is isomorphic to a unital semiprojective graph algebra. Semiprojectivity is easy in our case since the graphs are finite. Thus we do not need any results from [11].

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2. Definitions and preliminaries

Definition 2.1. Let $E = (E^0, E^1, s_E, r_E)$ be a countable directed graph. A Cuntz-Krieger E-family is a set of mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and a set $\{s_e \mid e \in E^1\}$ of partial isometries satisfying the following conditions:

(CK0) $s_e^* s_f = 0$ if $e, f \in E^1$ and $e \neq f$,
(CK1) $s_e^* s_e = p_{r_E(e)}$ for all $e \in E^1$,
(CK2) $s_e s_e^* \leq p_{s_E(e)}$ for all $e \in E^1$, and
(CK3) $p_v = \sum_{e \in s_E^{-1}(v)} s_e s_e^*$ for all $v \in E^0$ with $0 < |s_E^{-1}(v)| < \infty$.

The *graph algebra* $C^*(E)$ is defined as the universal C^*-algebra given by these generators and relations.

Definition 2.2. Let E be a directed graph, and let $v \in E^0$ be a vertex in E. The vertex v is called *regular* if $s_E^{-1}(v)$ is finite and nonempty. If $s_E^{-1}(v)$ is empty, v is called a *sink*, and if $r_E^{-1}(v)$ is empty, v is called a *source*. If $s_E^{-1}(v)$ is infinite, v is called an *infinite emitter*.

Definition 2.3. A graph algebra of a finite graph with no sinks and no sources is called a *Cuntz-Krieger algebra*.

J. Cuntz and W. Krieger originally defined a Cuntz-Krieger algebra as the universal C^*-algebra determined by a collection of partial isometries satisfying relations determined by a finite matrix with entries in $\{0, 1\}$. It follows from [15, Section 4] that the class of Cuntz-Krieger algebras coincides with the class of graph C^*-algebras of finite graphs with no sinks or sources, and moreover, if E is a finite graph with no sinks or sources, $C^*(E)$ coincides with the Cuntz-Krieger algebra associated with the edge matrix of E.

In their study of Cuntz-Krieger algebras, Cuntz and Krieger often imposed Condition (I) on their matrices, which is equivalent to imposing Condition (L) on the graph. In work after Cuntz and Krieger, particularly in [2], it was shown that Condition (I) was not necessary and that loops without exits would produce ideals in the associated C^*-algebra that are Morita equivalent to $C(\mathbb{T})$. In this paper, we do not assume our matrices satisfy Condition (I), thus obtaining results for C^*-algebras without real rank zero and results where the C^*-algebras have uncountably many ideals or are commutative. We use the language of graph algebras in order to provide us with nice combinatorial models of Cuntz-Krieger algebras, thus motivating us to define a Cuntz-Krieger algebra as in Definition 2.3.

Definition 2.4. Let E be a directed graph. A path $\alpha = e_1 e_2 \cdots e_n$ in E with $r_E(\alpha) := r(e_n) = s_E(e_1) =: s_E(\alpha)$ is called a *cycle*. A cycle $\alpha = e_1 e_2 \cdots e_n$ is called *vertex-simple* if $s_E(e_i) \neq s_E(e_j)$ for all $i \neq j$.

We refer to $s_E(\alpha)$ as the *base point* of the cycle α. In particular, an edge e in E with $s_E(e) = r_E(e)$ is called a *cycle of length one with base point* $s_E(e)$.

Definition 2.5. Let E be a directed graph. For vertices v, w in E, we write $v \geq w$ if there is a path in E from v to w, i.e., a path α in E with $s_E(\alpha) = v$ and $r_E(\alpha) = w$. Let S be a subset of E^0. We write $v \geq S$ if there exists $u \in S$ such that $v \geq u$.

Let H be a subset of E^0. The subset H is called *hereditary* if for all $v \in H$ and $w \in E^0$, $v \geq w$ implies $w \in H$ and H is called *saturated* if $r_E(s_E^{-1}(v)) \subseteq H$ implies $v \in H$ for all regular vertices v in E.

For a hereditary subset H in E^0, we let I_H denote the ideal in $C^*(E)$ generated by $\{p_v \mid v \in H\}$, where $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family generating $C^*(E)$.

Definition 2.6. Let E be a countable directed graph. Let γ denote the gauge action on $C^*(E)$, i.e., the action γ of the circle group $\mathbb{T}$ on $C^*(E)$ for which $\gamma_z(s_e) = zs_e$ and $\gamma_z(p_v) = p_v$ for all $z \in \mathbb{T}$, $e \in E^1$, and $v \in E^0$. An ideal I in $C^*(E)$ is called *gauge invariant* if $\gamma_z(I) \subseteq I$ for all $z \in \mathbb{T}$.

When E is a row-finite graph, the map $H \mapsto I_H$ defines a lattice isomorphism between the saturated hereditary subsets in E^0 and the gauge invariant ideals in $C^*(E)$; see [5, Theorem 4.1].

Definition 2.7. Let E and F be directed graphs. A *graph homomorphism* $f\colon E \to F$ consists of two maps $f^0\colon E^0 \to F^0$ and $f^1\colon E^1 \to F^1$ satisfying $r_F \circ f^1 = f^0 \circ r_E$ and $s_F \circ f^1 = f^0 \circ s_E$. A graph homomorphim $f\colon E \to F$ is called a *CK-morphism* if f^0 and f^1 are injective and f^1 restricts to a bijection from $s_E^{-1}(v)$ onto $s_F^{-1}(f^0(v))$ for all regular vertices v in E.

If E is a subgraph of F, we call it a *CK-subgraph* if the inclusion $E \to F$ is a CK-morphism.

The definition of a CK-morphism between arbitrary graphs was introduced by K. R. Goodearl in [13]. Let **CKGr** be the category whose objects are arbitrary directed graphs and whose morphisms are CK-morphisms. Goodearl showed that there is a functor L_K from the category **CKGr** to the category of algebras over a field K. The functor L_K assigns to an object E the Leavitt path algebra $L_K(E)$. Goodearl also proved in [13, Corollary 3.3] that for every CK-morphism ϕ, the K-algebra homomorphism $L_K(\phi)$ is injective. We now prove the analog of [13, Corollary 3.3] where the category of K-algebras is replaced by the category of C^*-algebras and the functor assigns to an object E the graph algebra $C^*(E)$.

Lemma 2.8. *Let E and F be countable directed graphs, let $f\colon E \to F$ be a CK-morphism. Let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ be a universal Cuntz-Krieger E-family generating $C^*(E)$, and let $\{q_v, t_e \mid v \in F^0, e \in F^1\}$ be a universal Cuntz-Krieger F-family generating $C^*(F)$.*

Then the assignments $p_v \mapsto q_{f^0(v)}$ and $s_e \mapsto t_{f^1(e)}$ induce an injective $*$-homomorphism $\phi\colon C^*(E) \to C^*(F)$ with image equal to the subalgebra of $C^*(F)$ generated by $\{q_v, t_e \mid v, s_F(e) \in f^0(E^0)\}$.

Proof. Using the fact that f is a CK-morphism, one can verify that $\{q_{f^0(v)}, t_{f^1(e)} \mid v \in E^0, e \in E^1\}$ is a Cuntz-Krieger E-family in $C^*(F)$. The universal property of $C^*(E)$ now implies that the $*$-homomorphism ϕ exists. Since ϕ intertwines the canonical gauge actions on $C^*(E)$ and $C^*(F)$ and since $\phi(p_v) = q_{f^0(v)} \neq 0$ for all $v \in E^0$, the gauge invariant uniqueness theorem implies that ϕ is injective. Since f is a CK-morphism, the sets $f^1(E^1)$ and $\{e \in F^1 \mid s_F(e) \in f^0(E^0)\}$ coincide. It now follows that $\phi(C^*(E))$ is equal to the subalgebra of $C^*(F)$ generated by $\{q_v, t_e \mid v, s_F(e) \in f^0(E^0)\}$. ☐

3. Graph algebras over finite graphs with no sinks

Assumption 3.1. *Throughout the rest of the paper, unless stated otherwise, all graphs will be countable and directed.*

Definition 3.2. Let E be a graph, let $v_0 \in E^0$ be a vertex, and let n be a positive integer. Define a graph $E(v_0, n)$ as follows:

$$E(v_0, n)^0 = E^0 \cup \{v_1, v_2, \ldots, v_n\},$$
$$E(v_0, n)^1 = E^1 \cup \{e_1, e_2, \ldots, e_n\}$$

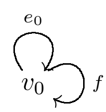
where $r_{E(v_0,n)}$ and $s_{E(v_0,n)}$ extend r_E and s_E respectively and $r_{E(v_0,n)}(e_i) = v_{i-1}$ and $s_{E(v_0,n)}(e_i) = v_i$.

Definition 3.3. Let E be a graph, let $e_0 \in E^1$ be an edge, and let n be a positive integer. Define a graph $E(e_0, n)$ as follows:

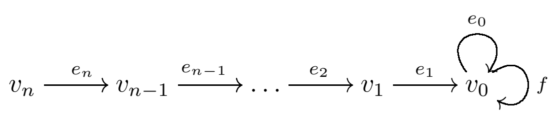
$$E(e_0, n)^0 = E^0 \cup \{v_1, v_2, \ldots, v_n\},$$
$$E(e_0, n)^1 = \big(E^1 \setminus \{e_0\}\big) \cup \{e_1, e_2, \ldots, e_{n+1}\}$$

where $r_{E(e_0,n)}$ and $s_{E(e_0,n)}$ extend r_E and s_E respectively, $r_{E(e_0,n)}(e_i) = v_{i-1}$ for $i = 2, \ldots, n+1$ and $s_{E(e_0,n)}(e_i) = v_i$ for $i = 1, \ldots, n$, and $r_{E(e_0,n)}(e_1) = r_E(e_0)$ and $s_{E(e_0,n)}(e_{n+1}) = s_E(e_0)$.

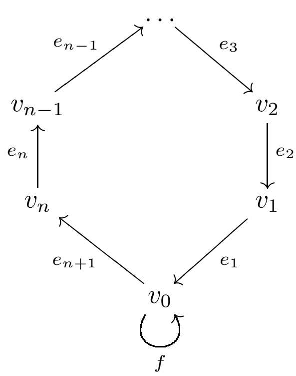
Example 3.4. Let E be the graph



Then $E(v_0, n)$ is the graph



and $E(e_0, n)$ is the graph



Proposition 3.5. *Let E be a graph, let $e_0 \in E^1$ be an edge, and let n be a positive integer. Define $v_0 = r_E(e_0)$. Then $C^*(E(v_0, n)) \cong C^*(E(e_0, n))$.*

Proof. Let $\{s_e, p_v \mid e \in E(e_0, n)^1, v \in E(e_0, n)^0\}$ be a universal Cuntz-Krieger $E(e_0, n)$-family generating $C^*(E(e_0, n))$. For each $v \in E(v_0, n)^0$ and $e \in E(v_0, n)^1$

set

$$Q_v = p_v,$$
$$T_e = \begin{cases} s_e, & \text{if } e \neq e_0, \\ s_{e_{n+1}} s_{e_n} \cdots s_{e_1}, & \text{if } e = e_0. \end{cases}$$

We will show that $\{T_e, Q_v \mid e \in E(v_0,n)^1, v \in E(v_0,n)^0\}$ is a Cuntz-Krieger $E(v_0,n)$-family that generates $C^*(E(e_0,n))$. It is clear that $Q_v Q_w = 0$ for all $v \neq w$. Let $e, f \in E(v_0,n)^1$ with $e \neq f$. Then

$$\begin{aligned} T_e^* T_f &= \begin{cases} s_e^* s_f, & \text{if } e \neq e_0 \text{ and } f \neq e_0, \\ s_{e_1}^* s_{e_2}^* \dots s_{e_{n+1}}^* s_f, & \text{if } e = e_0, \\ s_e^* s_{e_{n+1}} s_{e_n} \cdots s_{e_1}, & \text{if } f = e_0, \end{cases} \\ &= 0. \end{aligned}$$

The last two cases hold true because $g \neq e_{n+1}$ for all $g \in E(v_0,n)^1$.

Now let $e \in E(v_0,n)^1$. Then

$$\begin{aligned} T_e^* T_e &= \begin{cases} s_e^* s_e, & \text{if } e \neq e_0, \\ s_{e_1}^* s_{e_2}^* \dots s_{e_{n+1}}^* s_{e_{n+1}} s_{e_n} \cdots s_{e_1}, & \text{if } e = e_0, \end{cases} \\ &= \begin{cases} p_{r_{E(e_0,n)}(e)}, & \text{if } e \neq e_0, \\ p_{r_{E(e_0,n)}(e_1)}, & \text{if } e = e_0, \end{cases} \\ &= p_{r_{E(v_0,n)}(e)} \\ &= Q_{r_{E(v_0,n)}(e)} \end{aligned}$$

and

$$\begin{aligned} T_e T_e^* &= \begin{cases} s_e s_e^*, & \text{if } e \neq e_0, \\ s_{e_{n+1}} s_{e_n} \cdots s_{e_1} s_{e_1}^* s_{e_2}^* \dots s_{e_{n+1}}^*, & \text{if } e = e_0, \end{cases} \\ &\leq \begin{cases} p_{s_{E(e_0,n)}(e)}, & \text{if } e \neq e_0, \\ p_{s_{E(e_0,n)}(e_{n+1})}, & \text{if } e = e_0, \end{cases} \\ &= \begin{cases} p_{s_{E(v_0,n)}(e)}, & \text{if } e \neq e_0, \\ p_{s_E(e_0)}, & \text{if } e = e_0, \end{cases} \\ &= Q_{s_{E(v_0,n)}(e)}. \end{aligned}$$

Let $v \in E(v_0,n)^0$ be a regular vertex. Note that v is a regular vertex in $E(e_0,n)$. Suppose $v = v_i$ for some $i = 1, \dots, n$. Then $s_{E(e_0,n)}^{-1}(v_i) = \{e_i\} = s_{E(v_0,n)}^{-1}(v_i)$. Hence,

$$Q_v = Q_{v_i} = p_{v_i} = s_{e_i} s_{e_i}^* = T_{e_i} T_{e_i}^*.$$

Suppose $v \neq v_i$ for $i = 1, \dots, n$. We break this into two cases. Suppose $e_{n+1} \notin s_{E(e_0,n)}^{-1}(v)$. Then $v \neq s_E(e_0)$. Since $v \neq v_i$ for $i = 1, \dots, n$ and $v \neq s_E(e_0)$, we have that $s_{E(v_0,n)}^{-1}(v) \cap \{e_0, e_1, \dots, e_n\} = \emptyset$ and $s_{E(e_0,n)}^{-1}(v) \cap \{e_1, \dots, e_n, e_{n+1}\} = \emptyset$. Thus,

$$s_{E(e_0,n)}^{-1}(v) = s_E^{-1}(v) = s_{E(v_0,n)}^{-1}(v).$$

Hence,

$$Q_v = p_v = \sum_{e \in s^{-1}_{E(e_0,n)}(v)} s_e s_e^* = \sum_{e \in s^{-1}_{E(v_0,n)}(v)} s_e s_e^* = \sum_{e \in s^{-1}_{E(v_0,n)}(v)} T_e T_e^*.$$

Suppose $e_{n+1} \in s^{-1}_{E(e_0,n)}(v)$. Then $v = s_{E(e_0,n)}(e_{n+1}) = s_E(e_0)$, which implies that $e_0 \in s^{-1}_{E(v_0,n)}(v)$. Note that $s_{e_i} s_{e_i}^* = p_{v_i}$ for all $i = 1, 2, \dots, n$. Thus,

$$\begin{aligned}
Q_v = p_v &= \sum_{e \in s^{-1}_{E(e_0,n)}(v)} s_e s_e^* \\
&= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} s_{e_{n+1}}^* \\
&= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} p_{v_n} s_{e_{n+1}}^* \\
&= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} s_{e_n} s_{e_n}^* s_{e_{n+1}}^* \\
&\vdots \\
&= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} s_e s_e^* + s_{e_{n+1}} s_{e_n} \cdots s_{e_1} s_{e_1}^* \dots s_{e_n}^* s_{e_{n+1}}^* \\
&= \sum_{e \in s_E^{-1}(v) \setminus \{e_0\}} T_e T_e^* + T_{e_0} T_{e_0}^* \\
&= \sum_{e \in s^{-1}_{E(v_0,n)}(v)} T_e T_e^*.
\end{aligned}$$

We have just shown that $\{T_e, Q_v \mid e \in E(v_0,n)^1, v \in E(v_0,n)^0\}$ is a Cuntz-Krieger $E(v_0,n)$-family. Suppose $\{t_e, q_v \mid e \in E(v_0,n)^1, v \in E(v_0,n)^0\}$ is a universal Cuntz-Krieger $E(v_0,n)$-family generating $C^*(E(v_0,n))$. Then there exists a $*$-homomorphism $\psi\colon C^*(E(v_0,n)) \to C^*(E(e_0,n))$ such that

$$\psi(q_v) = Q_v,$$
$$\psi(t_e) = T_e$$

for all $e \in E(v_0,n)^1$ and $v \in E(v_0,n)^0$.

Note that the only generator of $C^*(E(e_0,n))$ that is not included in

$$\{T_e, Q_v \mid e \in E(v_0,n)^1, v \in E(v_0,n)^0\}$$

is $s_{e_{n+1}}$. In this case, recall again that

$$p_{v_i} = s_{e_i} s_{e_i}^*$$

for all $i = 1, 2, \dots, n$. Therefore,

$$\begin{aligned} T_{e_0}T_{e_1}^* \dots T_{e_n}^* &= s_{e_{n+1}}s_{e_n} \dots s_{e_1}s_{e_1}^* \dots s_{e_n}^* \\ &= s_{e_{n+1}}s_{e_n} \dots s_{e_2}p_{v_1}s_{e_2}^* \dots s_{e_n}^* \\ &= s_{e_{n+1}}s_{e_n} \dots s_{e_2}p_{r_{E(e_0,n)(e_2)}}s_{e_2}^* \dots s_{e_n}^* \\ &= s_{e_{n+1}}s_{e_n} \dots s_{e_2}s_{e_2}^* \dots s_{e_n}^* \\ &\vdots \\ &= s_{e_{n+1}}s_{e_n}s_{e_n}^* \\ &= s_{e_{n+1}}p_{v_n} \\ &= s_{e_{n+1}}p_{r_{E(e_0,n)}(e_{n+1})} \\ &= s_{e_{n+1}}. \end{aligned}$$

Hence, $s_{e_{n+1}} \in \psi(C^*(E(v_0, n)))$, which implies that ψ is surjective.

Note that the cycle structure of $E(v_0, n)$ is determined by the cycle structure of E and vice versa. Moreover, the cycles of $E(v_0, n)$ with no exits are in one-to-one correspondence to the cycles of $E(e_0, n)$ with no exits. Let $\alpha = f_1 f_2 \cdots f_m$ be a vertex-simple cycle in $E(v_0, n)$ with no exits. Suppose $s_{E(v_0,n)}(f_i) \neq s_{E(v_0,n)}(e_0)$. Then α is a vertex-simple cycle in $E(e_0, n)$ with no exits. Thus, s_α is a unitary in $C^*(E(e_0, n))$ with spectrum $\mathbb{T}$. Hence,

$$\psi(t_\alpha) = s_\alpha,$$

which implies that $\psi(t_\alpha)$ is a unitary in $C^*(E(e_0, n))$ with spectrum $\mathbb{T}$. Suppose $s_{E(v_0,n)}(f_i) = s_{E(v_0,n)}(e_0)$. Then $\alpha = e_0 f_2 \cdots f_n$ since α is a vertex-simple cycle in $E(v_0, n)$ with no exits. Note that

$$\psi(t_\alpha) = s_{e_{n+1}}s_{e_n} \cdots s_{e_1}s_{f_2} \cdots s_{f_n} = s_\beta$$

and $\beta = e_{n+1}e_n \cdots e_1 f_2 \cdots f_n$ is a vertex-simple cycle in $E(e_0, n)$ with no exits. Hence, $\psi(t_\alpha) = s_\beta$ is a unitary in $C^*(E(e_0, n))$ with spectrum $\mathbb{T}$.

From the above paragraph and the fact that $\psi(q_v) = p_v \neq 0$ for all $v \in E(v_0, n)^0$, by Theorem 1.2 of [19], ψ is injective. Therefore, ψ is an isomorphism. □

Definition 3.6. Let E be a graph and let H be a hereditary subset of E^0. Consider the set

$$F(H) = \{\alpha \in E^* \mid \alpha = e_1 e_2 \dots e_n, s_E(e_n) \notin H, r_E(e_n) \in H\}.$$

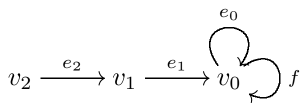
Let $\overline{F}(H)$ be another copy of $F(H)$ and write $\overline{\alpha}$ for the copy of α in $\overline{F}(H)$. Define a graph $E(H)$ as follows:

$$\begin{aligned} E(H)^0 &= H \cup F(H), \\ E(H)^1 &= s_E^{-1}(H) \cup \overline{F}(H) \end{aligned}$$

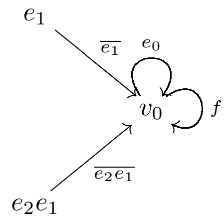
and extend s_E and r_E to $E(H)$ by defining $s_{E(H)}(\overline{\alpha}) = \alpha$ and $r_{E(H)}(\overline{\alpha}) = r_E(\alpha)$.

Note that $E(H)$ is just the graph $(H, r_E^{-1}(H), r_E, s_E)$ together with a source for each $\alpha \in F(H)$ with exactly one edge from α to $r_E(\alpha)$.

Example 3.7. Suppose E is the graph



and $H = \{v_0\}$. Then $F(\{v_0\}) = \{e_1, e_2e_1\}$. Therefore, the graph



represents the graph $E(\{v_0\})$.

Theorem 3.8. *Let E be a graph and let H be a hereditary subset of E^0. Suppose*

$$(E^0 \setminus H, r_E^{-1}(E^0 \setminus H), r_E, s_E)$$

is a finite acyclic graph and $v \geq H$ for all $v \in E^0 \setminus H$. Assume furthermore that the set $s^{-1}(E^0 \setminus H) \cap r^{-1}(H)$ is finite. Then $C^*(E) \cong C^*(E(H))$.

Proof. Let $\{s_e, p_v \mid e \in E^1, v \in E^0\}$ be a universal Cuntz-Krieger E-family generating $C^*(E)$. For $v \in E(H)^0$ define

$$Q_v := \begin{cases} p_v, & \text{if } v \in H, \\ s_\alpha s_\alpha^*, & \text{if } v = \alpha \in F(H) \end{cases}$$

and for $e \in E(H)^1$ define

$$T_e := \begin{cases} s_e, & \text{if } e \in s_E^{-1}(H), \\ s_\alpha, & \text{if } e = \overline{\alpha} \in \overline{F}(H). \end{cases}$$

We shall show that $\{T_e, Q_v \mid e \in E(H)^1, v \in E(H)^0\}$ is a Cuntz-Krieger $E(H)$-family in $C^*(E)$. To begin, we see that the Q_v are mutually orthogonal projections and the T_e are partial isometries with mutually orthogonal ranges. (The orthogonality follows from the fact that an element in $F(H)$ cannot extend an element in $F(H)$ and the fact that $s_E(\alpha) \notin H$ for all $\alpha \in F(H)$.)

To see the Cuntz-Krieger relations hold, we consider cases for $e \in E(H)^1$. If $e \in s_E^{-1}(H)$, then $r_H(e) \in H$ and

$$T_e^* T_e = s_e^* s_e = p_{r_E(e)} = Q_{r_{E(H)}(e)}.$$

If $e = \overline{\alpha} \in F(H)$, then $r_E(\alpha) \in H$ and

$$T_e^* T_e = T_{\overline{\alpha}}^* T_{\overline{\alpha}} = s_\alpha^* s_\alpha = p_{r_E(\alpha)} = Q_{r_E(\alpha)} = Q_{r_{E(H)}(\overline{\alpha})} = Q_{r_{E(H)}(e)}.$$

For the second Cuntz-Krieger relation, we again let $e \in E(H)^1$ and consider cases. If $e \in s_E^{-1}(H)$, then

$$Q_{s_{E(H)}(e)} T_e = p_{s_E(e)} s_e = s_e = T_e.$$

If $e = \overline{\alpha} \in \overline{F}(H)$, then

$$Q_{s_{E(H)}(e)} T_e = Q_\alpha T_{\overline{\alpha}} = s_\alpha s_\alpha^* s_\alpha = s_\alpha = T_{\overline{\alpha}} = T_e.$$

Thus $Q_{s_{E(H)}(e)}T_e = T_e$ for all $e \in E(H)^1$, so that $T_eT_e^* \leq Q_{s_{E(H)}(e)}$ for all $e \in E(H)^1$, and the second Cuntz-Krieger relation holds.

For the third Cuntz-Krieger relation, suppose that $v \in E(H)^0$ and that v is regular. If $v \in H$, then the set of edges that v emits in $E(H)$ is equal to the set of edges that v emits in E, and hence

$$Q_v = p_v = \sum_{\{e\in E^1 | s_E(e)=v\}} s_e s_e^* = \sum_{\{e\in E(H)^1 | s_{E(H)}(e)=v\}} T_eT_e^*.$$

If $v \in F(H)$, then $v = \alpha$ with $r_{E(H)}(\alpha) \in H$, and the element $\overline{\alpha}$ is the unique edge in $E(H)^0$ with source v so that

$$Q_v = s_\alpha s_\alpha^* = T_eT_e^*.$$

Thus the third Cuntz-Krieger relation holds, and

$$\{T_e, Q_v \mid e \in E(H)^1, v \in E(H)^0\}$$

is a Cuntz-Krieger $E(H)$-family in $C^*(E)$.

If $\{q_v, t_e \mid v \in E(H)^0, e \in E(H)^1\}$ is a universal Cuntz-Krieger $E(H)$-family generating $C^*(E(H))$, then by the universal property of $C^*(E(H))$ there exists a $*$-homomorphism $\phi : C^*(E(H)) \to C^*(E)$ with $\phi(q_v) = Q_v$ for all $v \in E(H)^0$ and $\phi(t_e) = T_e$ for all $e \in E(H)^1$.

We shall show injectivity of ϕ by applying the generalized Cuntz-Krieger uniqueness theorem of [19]. To verify the hypotheses, we first see that if $v \in E(H)^0$, then $\phi(q_v) = Q_v \neq 0$. Second, if $e_1 \dots e_n$ is a vertex-simple cycle in $E(H)$ with no exits, then since the cycles in $E(H)$ come from cycles in E all lying in the subgraph given by

$$(H, s_E^{-1}(H), s_E, r_E),$$

we must have that $e_i \in E^1$ for all $1 \leq i \leq n$, and $e_1 \dots e_n$ is a cycle in E with no exits. Thus $\phi(t_{e_1\dots e_n}) = \phi(t_{e_1}) \dots \phi(t_{e_n}) = s_{e_1} \dots s_{e_n} = s_{e_1\dots e_n}$ is a unitary whose spectrum is the entire circle. It follows from the generalized Cuntz-Krieger uniqueness theorem, stated in Theorem 1.2 of [19], that ϕ is injective.

We now show that ϕ is surjective. Let $e \in E^1$ such that $r_E(e) \in H$. If $s_E(e) \in H$, then

$$s_e = T_e = \phi(t_e) \in \operatorname{im}(\phi).$$

Suppose $s_E(e) \notin H$. So, $e \in F(H)$. Hence,

$$s_e = T_{\overline{e}} = \phi(t_{\overline{e}}) \in \operatorname{im}(\phi).$$

We now show that $s_e \in \operatorname{im}(\phi)$ for all $e \in r_E^{-1}(E^0 \setminus H)$. By assumption, $v \geq H$ for all $v \in E^0 \setminus H$. Define for each k the subset D_k of $E^0 \setminus H$ as the set of vertices v for which k is the maximal length of a path from v to H. Put $D_0 = H$, and note that for $k \geq 1$, all vertices in D_k are regular. By induction on $k \geq 1$ we will show for every path α in E with $r_E(\alpha) \in D_k$ that $s_\alpha \in \operatorname{im}(\phi)$.

For $k = 1$ and α a path in E with $r_E(\alpha) \in D_1$, we note that $r_E(e) \in H$ for all $e \in s_E^{-1}(r_E(\alpha))$. Hence

$$\begin{aligned} s_\alpha = s_\alpha p_{r_E(\alpha)} &= \sum_{e \in s_E^{-1}(r_E(\alpha))} s_\alpha s_e s_e^* \\ &= \sum_{e \in s_E^{-1}(r_E(\alpha))} T_{\overline{\alpha e}} T_{\overline{e}}^* \\ &= \sum_{e \in s_E^{-1}(r_E(\alpha))} \phi(t_{\overline{\alpha e}} t_{\overline{e}}^*) \in \operatorname{im}(\phi) \end{aligned}$$

since $\alpha e, e \in F(H)$.

For $k > 1$ and α a path in E with $r_E(\alpha) \in D_k$, we note that for all $e \in s_E^{-1}(r_E(\alpha))$ there is a $j < k$ for which $r_E(\alpha e) = r_E(e) \in D_j$. Hence

$$s_\alpha = \sum_{e \in s_E^{-1}(r_E(\alpha))} s_{\alpha e} s_e^* \in \operatorname{im}(\phi).$$

We have just shown that $s_e \in \operatorname{im}(\phi)$ for all $e \in E^1$. We now show that $p_v \in \operatorname{im}(\phi)$ for all $v \in E^0$. Note that if $v \in E^0$ and v is not a regular vertex, then $v \in H$. Hence, $p_v = Q_v = \phi(q_v)$. Let $v \in E^0$ be a regular vertex. Then for each $e \in s_E^{-1}(v)$, we have that $s_e, s_e^* \in \operatorname{im}(\phi)$. Therefore,

$$p_v = \sum_{e \in s_E^{-1}(v)} s_e s_e^* \in \operatorname{im}(\phi).$$

Since $\{p_v, s_e \mid v \in E^0, e \in E^1\} \subseteq \operatorname{im}(\phi)$ and $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ generates $C^*(E)$ we have that ϕ is surjective. Therefore, ϕ is an isomorphism. □

Definition 3.9. Let E be a graph, let $v_0 \in E^0$ be a vertex in E, and let n be a positive integer. Define a graph $E'(v_0, n)$ as follows:

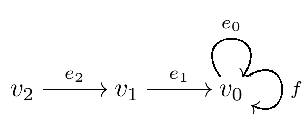
$$\begin{aligned} E'(v_0, n)^0 &= E^0 \cup \{v_1, v_2, \ldots, v_n\}, \\ E'(v_0, n)^1 &= E^1 \cup \{e_1, e_2, \ldots, e_n\} \end{aligned}$$

where $r_{E'(v_0,n)}$ and $s_{E'(v_0,n)}$ extend r_E and s_E respectively, and $r_{E'(v_0,n)}(e_i) = v_0$ and $s_{E'(v_0,n)}(e_i) = v_i$ for all $i = 1, \ldots n$.

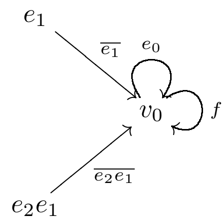
Corollary 3.10. *Let E be a graph, let $v_0 \in E^0$ be a vertex, and let n be a positive integer. Then $C^*(E(v_0, n)) \cong C^*(E'(v_0, n))$.*

Proof. Note that E^0 is a hereditary subset of $E(v_0, n)^0$. Moreover $(E(v_0, n)^0 \setminus E^0, r_{E(v_0,n)}^{-1}(E(v_0, n)^0 \setminus E^0), r_{E(v_0,n)}, s_{E(v_0,n)})$ is a finite acyclic graph and for each $v \in E(v_0, n)^0 \setminus E^0$, there exists a path in $E(v_0, n)$ from v to E^0. Finally, $s_{E(v_0,n)}(E(v_0, n)^0 \setminus E^0) \cap r_{E(v_0,n)}^{-1}(E^0)$ is finite. Thus, by Theorem 3.8, $C^*(E(v_0, n)) \cong C^*(E(v_0, n)(E^0))$. One can verify that the graph $E(v_0, n)(E^0)$ is isomorphic to the graph $E'(v_0, n)$. Thus, $C^*(E(v_0, n)(E^0)) \cong C^*(E'(v_0, n))$. □

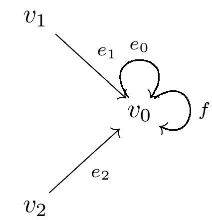
Example 3.11. We give an example to illustrate the proof of Corollary 3.10. Consider the graph E of Example 3.4. Then $E(v_0, 2)$ is the graph



and thereby $E(v_0, 2)(\{v_0\})$ is the graph



which is isomorphic to the graph $E'(v_0, 2)$



Theorem 3.12. *Let E be a graph. Then the following are equivalent:*

(1) *E is a finite graph with no sinks.*
(2) *$C^*(E)$ is isomorphic to a Cuntz-Krieger algebra.*
(3) *$C^*(E)$ is unital and*

$$\operatorname{rank}(K_0(C^*(E))) = \operatorname{rank}(K_1(C^*(E))).$$

Proof. We first show that (1) implies (2). Suppose E is a finite graph with no sinks. Remove the sources from E, and remove the vertices that then become sources; repeat this procedure finitely many times to get a subgraph F of E that has no sinks and no sources. Notice that F^0 is a hereditary subset of E^0, that

$$(E^0 \setminus F^0, r_E^{-1}(E^0 \setminus F^0), r_E, s_E)$$

is a finite acyclic graph, and that for each $v \in E^0 \setminus F^0$ there exists a path in E from v to F^0. Therefore, by Theorem 3.8, $C^*(E) \cong C^*(E(F^0))$. We can apply Corollary 3.10 and Proposition 3.5 as many times as needed (but finitely many times) to get a finite graph E_1 with no sinks and no sources such that $C^*(E(F^0)) \cong C^*(E_1)$. Note that $C^*(E_1)$ is a Cuntz-Krieger algebra and $C^*(E) \cong C^*(E_1)$.

We next show that (2) implies (3). Suppose $C^*(E)$ is isomorphic to a Cuntz-Krieger algebra. Then $C^*(E)$ is unital. Moreover, by the K-theory computation (Theorem 3.1 of [10]),

$$\operatorname{rank}(K_0(C^*(E))) = \operatorname{rank}(K_1(C^*(E))).$$

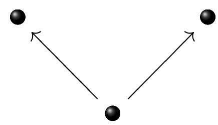
We now show that (3) implies (1). Suppose $C^*(E)$ is unital. Then E^0 is a finite set. Since

$$\operatorname{rank}(K_0(C^*(E))) = \operatorname{rank}(K_1(C^*(E))),$$

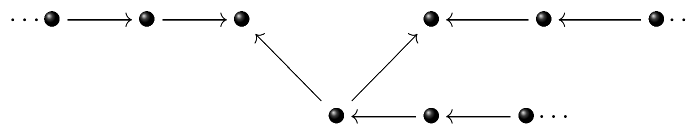
by the K-theory computation (Theorem 3.1 of [10]), E has no singular vertices. Hence, E is a finite graph with no sinks. □

Definition 3.13. Let E be a graph and let SE be the graph obtained by adding an infinite head to every vertex of E:

E :



SE :



We call SE the *stabilization* of E.

Theorem 3.14. *Let E be a graph with finitely many vertices and let T be a finite hereditary subset of $(SE)^0$ such that $E^0 \subseteq T$. Set*

$$p_T = \sum_{v \in T} p_v$$

where $\{s_e, p_v \mid e \in (SE)^1, v \in (SE)^0\}$ is a universal Cuntz-Krieger SE-family generating $C^*(SE)$. Then p_T is a full projection in $C^*(SE)$ and there exists a subgraph F of SE such that $C^*(F) \cong p_T C^*(SE) p_T$.

If in addition $C^*(E)$ is a Cuntz-Krieger algebra, then $p_T C^*(SE) p_T$ is a Cuntz-Krieger algebra.

Proof. The smallest saturated subset of $(SE)^0$ containing T is $(SE)^0$. Hence, p_T is a full projection.

Let $F = (T, s_{SE}^{-1}(T), r_{SE}, s_{SE})$. We claim that F is a CK-subgraph of SE. It is clear that F is a subgraph of SE. We will show that $s_E^{-1}(v) = s_F^{-1}(v)$ for all $v \in F^0$. Let $v \in F^0$. Suppose $v \in E^0$. Then

$$s_{SE}^{-1}(v) = s_E^{-1}(v) = s_F^{-1}(v).$$

Suppose $v \in T \setminus E^0$. Then $s_{SE}^{-1}(v) = \{e\} = s_F^{-1}(v)$ for some e. Since F is a CK-subgraph of SE, we have by Lemma 2.8 that $C^*(F)$ is isomorphic to the subalgebra of $C^*(SE)$ generated by

$$\{p_v, s_e \mid s_{SE}(e), v \in T\},$$

which we denote by B. We claim that $p_T C^*(SE) p_T = B$.

Note that B is unital with unit p_T. Note that if $e \in s_{SE}^{-1}(T)$, then $s_{SE}(e)$ and $r_{SE}(e)$ are elements of T. Therefore, for all $v \in T$ and all $e \in s_{SE}^{-1}(T)$,

$$p_v = p_T p_v p_T \in p_T C^*(SE) p_T$$

and

$$s_e = p_{s_{SE}(e)} s_e p_{r_{SE}(e)} = p_T p_{s_{SE}(e)} s_e p_{r_{SE}(e)} p_T \in p_T C^*(SE) p_T.$$

Hence, B is a subalgebra of $p_T C^*(SE) p_T$.

Let α be a finite path in SE. Suppose $s_{SE}(\alpha)$ is not an element of T. Then

$$p_T p_{s_{SE}(\alpha)} = 0.$$

If $s_{SE}(\alpha) \in T$, then

$$p_T p_{s_{SE}(\alpha)} = p_{s_{SE}(\alpha)}.$$

From these observations, we get that

$$p_T s_\alpha s_\beta^* p_T = \begin{cases} s_\alpha s_\beta^*, & \text{if } s_{SE}(\alpha), s_{SE}(\beta) \in T, \\ 0, & \text{otherwise.} \end{cases}$$

Since $e \in s_{SE}^{-1}(T)$ implies that $s_{SE}(e)$ and $r_{SE}(e)$ are elements of T, we have that α is a path in F if $s_{SE}(\alpha) \in T$. Therefore, if $s_{SE}(\alpha), s_{SE}(\beta) \in T$, then $s_\alpha, s_\beta^* \in B$. Hence,

$$p_T s_\alpha s_\beta^* p_T$$

is an element of B for all paths α and β in SE. We have just shown that $B = p_T C^*(SE) p_T$, which implies that $C^*(F) \cong B = p_T C^*(SE) p_T$.

Assume that $C^*(E)$ is isomorphic to a Cuntz-Krieger algebra. Then by Theorem 3.12, the graph E is finite and has no sinks. Since F is a graph obtained from the graph E by adding a finite head to some vertices of E, the graph F is finite with no sinks. By Theorem 3.12, $C^*(F)$ is a Cuntz-Krieger algebra. □

4. Unital C^*-algebras that are stably isomporphic to Cuntz-Krieger algebras

Definition 4.1. For a C^*-algebra A and projections $p \in \mathsf{M}_n(A)$ and $q \in \mathsf{M}_m(A)$, we say p is *Murray-von Neumann equivalent* to q, denoted $p \sim q$, if there exists $v \in \mathsf{M}_{m,n}(A)$ with $p = v^*v$ and $q = vv^*$.

For a projection p in a C^*-algebra A and $n \in \mathbb{N}$, np will denote the projection

$$\underbrace{p \oplus p \cdots \oplus p}_{n\text{-times}} \in \mathsf{M}_n(A).$$

Lemma 4.2. *Let E be a graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E-family generating $C^*(E)$. Let $v \in E^0$ and assume that v is a regular vertex. Then*

$$p_v \sim \sum_{e \in s^{-1}(v)} p_{r_E(e)}.$$

Proof. The result follows directly from the Cuntz-Krieger relations; see Definition 2.1. □

Lemma 4.3. *Let E be a row-finite graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E-family generating $C^*(E)$. Let $v, w \in E^0$ with $v \neq w$. If there is a path from v to w in E, then there exists a family $(m_u(v,w))_{u \in E^0}$ of non-negative integers satisfying*

$$p_v \sim p_w + \sum_{u \in E^0} m_u(v,w) p_u$$

with all but finitely many $m_u(v,w)$ equal to zero. Moreover, $m_v(v,w)$ can be chosen such that

$$m_v(v,w) \geq |\{e \in E^1 \mid s_E(e) = r_E(e) = v\}|.$$

Proof. Let $e_1 \cdots e_n$ denote a path in E from v to w, so that $e_1, \dots, e_n \in E^1$ with $r_E(e_i) = s_E(e_{i+1})$ for all $i \in \{1, \dots, n-1\}$, $s_E(e_1) = v$, and $r_E(e_n) = w$. Define $v_i = r_E(e_i)$ for $i \in \{1, \dots, n\}$ and $v_0 = v$. Then by Lemma 4.2,

$$\begin{aligned} p_v &\sim p_{v_1} + \sum_{e \in s^{-1}(v) \setminus \{e_1\}} p_{r_E(e)} \\ &\sim p_{v_2} + \sum_{e \in s^{-1}(v_1) \setminus \{e_2\}} p_{r_E(e)} + \sum_{e \in s^{-1}(v_0) \setminus \{e_1\}} p_{r_E(e)} \\ &\vdots \\ &\sim p_w + \sum_{i=1}^{n-1} \left(\sum_{e \in s^{-1}(v_{i-1}) \setminus \{e_i\}} p_{r_E(e)} \right). \end{aligned}$$

Define $(m_u(v,w))_{u \in E^0}$ as the non-negative integer scalars in the above linear combination of $(p_u)_{u \in E^0}$, i.e., such that

$$\sum_{i=1}^{n-1} \left(\sum_{e \in s^{-1}(v_{i-1}) \setminus \{e_i\}} p_{r_E(e)} \right) = \sum_{u \in E^0} m_u(v,w) p_u.$$

This defines $(m_u(v,w))_{u \in E^0}$ for any pair $v, w \in E^0$ for which there is a path from v to w.

The last statement is clear from the construction of $m_u(v,w)$. $\square$

Theorem 4.4 (Theorem 3.5 of [3])**.** *Let E be a row-finite graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E-family generating $C^*(E)$. Any projection in $C^*(E) \otimes \mathbb{K}$ is Murray-von Neumann equivalent to a projection of the form $\sum_{u \in E^0} m_u p_u$ with all but finitely many m_u equal to zero.*

Lemma 4.5. *Suppose E is a row-finite graph in which every vertex is the base point of at least one cycle of length one. Then every subset in E^0 is saturated.*

Proof. Let H be a subset in E^0. Since every vertex in E is the base point of at least one cycle of length one, E is a graph with no sinks. This fact and the fact that E is row-finite imply that every vertex in E is a regular vertex. To show that H is saturated we must show that $r_E(s_E^{-1}(v)) \subseteq H$ implies $v \in H$ for all $v \in E^0$. Let v be a vertex in E such that $r_E(s_E^{-1}(v)) \subseteq H$. By assumption there exists $e \in s_E^{-1}(v)$ such that $v = r_E(e) = s_E(e)$. Hence, $v \in r_E(s_E^{-1}(v))$, which implies that $v \in H$. $\square$

Lemma 4.6. *Let E be a finite graph and let $\{p_v, s_e \mid v \in E^0, e \in E^1\}$ denote a universal Cuntz-Krieger E-family generating $C^*(E)$. Assume that E has no sinks and no sources and every vertex of E is a base point of at least one cycle of length one.*

Let p be a norm-full projection in $C^*(E) \otimes \mathbb{K}$. Then there exists a family $(m_u)_{u \in E^0}$ of integers satisfying

$$p \sim \sum_{u \in E^0} m_u p_u$$

and $m_u \geq 1$ for all $u \in E^0$.

Proof. By Theorem 4.4, there exists a family $(n_u)_{u\in E^0}$ of non-negative integers satisfying

$$p \sim \sum_{u\in E^0} n_u p_u.$$

Set $S_0 = \{u \in E^0 \mid n_u \neq 0\}$ and let H_0 be the smallest hereditary subset of E^0 that contains S_0. By Lemma 4.5, H_0 is saturated. Set $q = \sum_{v\in S_0} p_v \in I_{H_0}$. Note that the ideal generated by $q\otimes e_{11}$ is equal to the ideal generated by $\sum_{u\in E^0} n_u p_u$, where $\{e_{ij}\}_{i,j}$ is a system of matrix units for $\mathbb{K}$. Since $p \sim \sum_{u\in E^0} n_u p_u$, we have that the ideal generated by $q \otimes e_{11}$ is equal to the ideal generated by p. Thus, $q \otimes e_{11}$ is a norm-full projection in $C^*(E) \otimes \mathbb{K}$, which implies that q is a norm-full projection in $C^*(E)$. Hence, $I_{H_0} = C^*(E)$, which implies that $H_0 = E^0$. Therefore, for every $w \in E^0$, there exists $v \in S_0$ such that $v \geq w$.

Set $E^0 \setminus S_0 = \{w_0, w_1, \ldots, w_m\}$. Let $v \in S_0$ such that $v \geq w_0$. By Lemma 4.3,

$$p_v \sim p_{w_0} + \sum_{u\in E^0} m_u(v, w_0) p_u$$

where $m_u(v, w_0) \geq 0$ and

$$m_v(v, w_0) \geq |\{e \in E^1 \mid s_E(e) = r_E(e) = v\}| \geq 1.$$

By making such replacements for all $w_0, \ldots, w_m$ we achieve

$$p \sim \sum_{u\in E^0} n'_u p_u$$

where $n'_u \geq 1$ for all $u \in E^0$. □

Proposition 4.7. *Let E be a finite graph with no sinks and no sources, and assume that every vertex of E is a base point of at least one cycle of length one. Let p be a norm-full projection in $C^*(E) \otimes \mathbb{K}$. Then there exists a finite graph F that has no sinks and no sources such that $C^*(F) \cong p(C^*(E) \otimes \mathbb{K})p$.*

Proof. Let SE be the stabilization of E, as defined in Definition 3.13. Let $\{e_{ij}\}_{i,j}$ be a system of matrix units for $\mathbb{K}$. By Proposition 9.8 of [1] and its proof, there exists an isomorphism $\phi\colon C^*(E) \otimes \mathbb{K} \to C^*(SE)$ such that

$$K_0(\phi)([p_v \otimes e_{11}]) = [p_v]$$

for all $v \in E^0$. Let p be a norm-full projection in $C^*(E) \otimes \mathbb{K}$. By Lemma 4.6, p is Murray-von Neumann equivalent to $\sum_{u\in E^0} m_u p_u$ with $m_u \geq 1$ for all $u \in E^0$. Therefore, since $C^*(SE)$ has weak cancellation by Corollary 7.2 of [3], $\phi(p)$ is Murray-von Neumann equivalent to $p_T \in C^*(SE)$ such that T is a finite, hereditary subset of $(SE)^0$ with $E^0 \subseteq T$. By Theorem 3.14, $p_T C^*(SE) p_T \cong C^*(F)$ for some finite graph F with no sinks and no sources. Note that $p(C^*(E) \otimes \mathbb{K})p \cong \phi(p)C^*(SE)\phi(p) \cong p_T C^*(SE) p_T$. Therefore, $p(C^*(E) \otimes \mathbb{K})p \cong C^*(F)$. □

The following theorem answers a question asked by George A. Elliott at the NordForsk Closing Conference at the Faroe Islands, May 2012.

Theorem 4.8. *Let A be a unital C^*-algebra.*

(1) *If A is stably isomorphic to a Cuntz-Krieger algebra, then A is isomorphic to a Cuntz-Krieger algebra.*

(2) *Let A be a unital, nuclear, separable C^*-algebra with finitely many ideals and let $X = \operatorname{Prim}(A)$. If $A \otimes \mathcal{O}_\infty$ is KK_X-equivalent to a Cuntz-Krieger algebra with real rank zero and primitive ideal space X, then $A \otimes \mathcal{O}_\infty$ is isomorphic to a Cuntz-Krieger algebra of real rank zero.*

Proof. We first prove (1). Let B be a Cuntz-Krieger algebra such that $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Note that $B = C^*(F)$ such that F is a finite graph with no sinks and no sources. By Theorem 5.2 of [18], collapsing a regular vertex that is not a base point of a cycle of length one preserves stable isomorphism classes. Therefore, since F is a finite graph with no sinks and no sources, we can apply Theorem 5.2 of [18] a finite number of times to get a finite graph E with no sinks and no sources, and every vertex of E is a base point of at least one cycle of length one, such that $C^*(F) \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. Hence, $A \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. Let $\phi\colon A \otimes \mathbb{K} \to C^*(E) \otimes \mathbb{K}$ be an isomorphism.

Let $\{e_{ij}\}_{i,j}$ be a system of matrix units for $\mathbb{K}$. Since $1_A \otimes e_{11}$ is a norm-full projection in $A \otimes \mathbb{K}$, $p = \phi(1_A \otimes e_{11})$ is a norm-full projection in $C^*(E) \otimes \mathbb{K}$. By Proposition 4.7, $p(C^*(E) \otimes \mathbb{K})p$ is isomorphic to a Cuntz-Krieger algebra. Note that $(1_A \otimes e_{11})(A \otimes \mathbb{K})(1_A \otimes e_{11}) \cong p(C^*(E) \otimes \mathbb{K})p$ and $A \cong (1_A \otimes e_{11})(A \otimes \mathbb{K})(1_A \otimes e_{11})$. Therefore, A is isomorphic to a Cuntz-Krieger algebra.

We will now use (1) to prove (2). Let B be a Cuntz-Krieger algebra with real rank zero such that $A \otimes \mathcal{O}_\infty$ is KK_X-equivalent to B and $\operatorname{Prim}(B) \cong X$. By Folgerung 4.3 of [14], $A \otimes \mathcal{O}_\infty \otimes \mathbb{K} \cong B \otimes \mathbb{K}$. Therefore, $A \otimes \mathcal{O}_\infty$ is a unital C^*-algebra stably isomorphic to a Cuntz-Krieger algebra with real rank zero. By (1), we have that $A \otimes \mathcal{O}_\infty$ is isomorphic to a Cuntz-Krieger algebra. Since $A \otimes \mathcal{O}_\infty$ is stably isomorphic to a C^*-algebra with real rank zero, $A \otimes \mathcal{O}_\infty$ has real rank zero. Therefore, $A \otimes \mathcal{O}_\infty$ is isomorphic to a Cuntz-Krieger algebra with real rank zero. □

Corollary 4.9. *Let A be a C^*-algebra. Then the following are equivalent:*

(1) *A is a Cuntz-Krieger algebra.*
(2) *$\mathsf{M}_n(A)$ is a Cuntz-Krieger algebra for all $n \in \mathbb{N}$.*
(3) *$\mathsf{M}_n(A)$ is a Cuntz-Krieger algebra for some $n \in \mathbb{N}$.*

Proof. (1) implies (2) follows from Theorem 4.8. (2) implies (3) is obvious. Suppose $\mathsf{M}_n(A)$ is a Cuntz-Krieger algebra for some $n \in \mathbb{N}$. In particular, $\mathsf{M}_n(A)$ is a unital C^*-algebra with $1_{\mathsf{M}_n(A)} = [x_{ij}]$. A computation shows that x_{11} is a multiplicative identity for A. Therefore, A is a unital C^*-algebra. Since $A \otimes \mathbb{K} \cong \mathsf{M}_n(A) \otimes \mathbb{K}$ and since $\mathsf{M}_n(A)$ is a Cuntz-Krieger algebra, by Theorem 4.8, A is a Cuntz-Krieger algebra. □

Corollary 4.10. *Let A be a Cuntz-Krieger algebra.*

(1) *If p is a non-zero projection in A, then pAp is isomorphic to a Cuntz-Krieger algebra.*
(2) *If p is a non-zero projection in $A \otimes \mathbb{K}$, then $p(A \otimes \mathbb{K})p$ is isomorphic to a Cuntz-Krieger algebra.*

Proof. We first prove (1) in the case when p is a norm-full projection. Suppose p is a norm-full projection. By Corollary 2.6 of [8], $pAp \otimes \mathbb{K} \cong A \otimes \mathbb{K}$. Therefore, pAp is a unital C^*-algebra that is stably isomorphic to a Cuntz-Krieger algebra. By Theorem 4.8, pAp is isomorphic to a Cuntz-Krieger algebra.

We now prove the general case in (1). Let $A = C^*(E)$ where E is a finite graph with no sinks and no sources. Let p be a non-zero projection of A. Set

$$I = \text{ the ideal in } C^*(E) \text{ generated by } p.$$

Note that $pAp \subseteq I$, which implies that $pAp \subseteq pIp$. Since $pIp \subseteq pAp$, we have that $pAp = pIp$. Thus, pIp is a norm-full hereditary subalgebra of I. By Corollary 2.6 of [8], $pIp \otimes \mathbb{K} \cong I \otimes \mathbb{K}$.

Since I is generated by a projection p, by Theorem 7.1 and the proof of Theorem 5.3 of [3], I is a gauge-invariant ideal of $C^*(E)$. Thus, by Theorem 3.7 of [4], there exists a hereditary saturated subset H of E^0 such that I_H is I; see Definition 2.5.

Let $E_H = (H, s_E^{-1}(H), r_E, s_E)$. By Proposition 3.4 of [4], $I_H \otimes \mathbb{K} \cong C^*(E_H) \otimes \mathbb{K}$. Note that E_H is a finite graph with no sinks. By Proposition 3.1 of [18], we may continue to remove the sources to obtain a finite graph F with no sinks and no sources such that $C^*(E_H) \otimes \mathbb{K} \cong C^*(F) \otimes \mathbb{K}$. Hence, $C^*(F)$ is a Cuntz-Krieger algebra and $pAp = pIp$ is a unital C^*-algebra that is stably isomorphic to $C^*(F)$. By Theorem 4.8, pAp is isomorphic to a Cuntz-Krieger algebra.

We now prove (2). Let p be a non-zero projection in $A \otimes \mathbb{K}$. Recall that $A = C^*(E)$, where E is a finite graph with no sinks and no sources. By Theorem 4.4, there exists a non-empty subset S of E^0 and a collection of positive integers $\{m_v\}_{v \in S}$ such that p is Murray-von Neumann equivalent to $\sum_{v \in S} m_v p_v$. Set $q = \sum_{v \in S} p_v$. Then q is a non-zero projection in A, and by (1), we have that $qAq \cong C^*(F)$ for some finite graph F with no sinks and no sources. By Theorem 5.3 of [3], p and $q \otimes e_{11}$ generate the same ideal of $A \otimes \mathbb{K}$. Hence, $qAq \otimes \mathbb{K} \cong (q \otimes e_{11})(A \otimes \mathbb{K})(q \otimes e_{11}) \otimes \mathbb{K} \cong p(A \otimes \mathbb{K})p \otimes \mathbb{K}$. Therefore, $p(A \otimes \mathbb{K})p$ is stably isomorphic to a Cuntz-Krieger algebra. By Theorem 4.8, $p(A \otimes \mathbb{K})p$ is isomorphic to a Cuntz-Krieger algebra. □

Corollary 4.11. *Let A be a Cuntz-Krieger algebra. If p is a projection in $A \otimes \mathbb{K}$, then $p(A \otimes \mathbb{K})p$ is semiprojective. If p is a projection in A, then pAp is semiprojective.*

Proof. This follows from Corollary 4.10 since by Corollary 2.24 of [7] all Cuntz-Krieger algebras are semiprojective. □

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References

[1] Gene Abrams and Mark Tomforde, *Isomorphism and Morita equivalence of graph algebras*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3733–3767, DOI 10.1090/S0002-9947-2011-05264-5. MR2775826 (2012c:46127)

[2] Astrid an Huef and Iain Raeburn, *The ideal structure of Cuntz-Krieger algebras*, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 611–624, DOI 10.1017/S0143385797079200. MR1452183 (98k:46098)

[3] P. Ara, M. A. Moreno, and E. Pardo, *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory **10** (2007), no. 2, 157–178, DOI 10.1007/s10468-006-9044-z. MR2310414 (2008b:46094)

[4] Teresa Bates, Jeong Hee Hong, Iain Raeburn, and Wojciech Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math. **46** (2002), no. 4, 1159–1176. MR1988256 (2004i:46105)

[5] Teresa Bates, David Pask, Iain Raeburn, and Wojciech Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324 (electronic). MR1777234 (2001k:46084)

[6] Bruce Blackadar, *Semiprojectivity in simple C^*-algebras*, Operator algebras and applications, Adv. Stud. Pure Math., vol. 38, Math. Soc. Japan, Tokyo, 2004, pp. 1–17. MR2059799 (2005g:46101)

[7] Bruce Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), no. 2, 249–275. MR813640 (87b:46074)

[8] Lawrence G. Brown, *Stable isomorphism of hereditary subalgebras of C^*-algebras*, Pacific J. Math. **71** (1977), no. 2, 335–348. MR0454645 (56 #12894)

[9] Joachim Cuntz and Wolfgang Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), no. 3, 251–268, DOI 10.1007/BF01390048. MR561974 (82f:46073a)

[10] D. Drinen and M. Tomforde, *Computing K-theory and* Ext *for graph C^*-algebras*, Illinois J. Math. **46** (2002), no. 1, 81–91. MR1936076 (2003k:46103)

[11] S. Eilers and T. Katsura, *Semiprojectivity and properly infinite projections in graph C^*-algebras*, preprint.

[12] Masatoshi Enomoto and Yasuo Watatani, *A graph theory for C^*-algebras*, Math. Japon. **25** (1980), no. 4, 435–442. MR594544 (83d:46069a)

[13] K. R. Goodearl, *Leavitt path algebras and direct limits*, Rings, modules and representations, Contemp. Math., vol. 480, Amer. Math. Soc., Providence, RI, 2009, pp. 165–187, DOI 10.1090/conm/480/09374. MR2508151 (2010h:16073)

[14] Eberhard Kirchberg, *Das nicht-kommutative Michael-Auswahlprinzip und die Klassifikation nicht-einfacher Algebren* (German, with English summary), C^*-algebras (Münster, 1999), Springer, Berlin, 2000, pp. 92–141. MR1796912 (2001m:46161)

[15] Alex Kumjian, David Pask, Iain Raeburn, and Jean Renault, *Graphs, groupoids, and Cuntz-Krieger algebras*, J. Funct. Anal. **144** (1997), no. 2, 505–541, DOI 10.1006/jfan.1996.3001. MR1432596 (98g:46083)

[16] M. H. Mann, Iain Raeburn, and C. E. Sutherland, *Representations of finite groups and Cuntz-Krieger algebras*, Bull. Austral. Math. Soc. **46** (1992), no. 2, 225–243, DOI 10.1017/S0004972700011862. MR1183780 (93k:46046)

[17] Jonathan Rosenberg and Claude Schochet, *The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor*, Duke Math. J. **55** (1987), no. 2, 431–474, DOI 10.1215/S0012-7094-87-05524-4. MR894590 (88i:46091)

[18] Adam P. W. Sørensen, *Geometric classification of simple graph algebras*, Ergodic Theory Dynam. Systems **33** (2013), no. 4, 1199–1220, DOI 10.1017/S0143385712000260. MR3082546

[19] Wojciech Szymański, *General Cuntz-Krieger uniqueness theorem*, Internat. J. Math. **13** (2002), no. 5, 549–555, DOI 10.1142/S0129167X0200137X. MR1914564 (2003h:46083)

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