TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 11, November 2015, Pages 8071–8106
http://dx.doi.org/10.1090/tran/6330
Article electronically published on April 3, 2015

ENDOMORPHISM ALGEBRAS OF FACTORS OF CERTAIN HYPERGEOMETRIC JACOBIANS

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Abstract. We classify the endomorphism algebras of factors of the Jacobians of certain hypergeometric curves over a field of characteristic zero. Other than a few exceptional cases, the endomorphism algebras turn out to be either a cyclotomic field $E = \mathbb{Q}(\zeta_q)$, or a quadratic extension of E, or $E \oplus E$. This result may be viewed as a generalization of the well known results of the classification of endomorphism algebras of elliptic curves over $\mathbb{C}$.

1. Introduction

Throughout this paper, the word "curve" is reserved for smooth projective curves, and $N \in \mathbb{N}$ denotes an integer strictly greater than 1. If N is a prime power p^r, we write q for it instead. Let k be a field of characteristic zero with algebraic closure $\bar{k}$, and A be an elliptic curve over k. It is a classical result that the absolute endomorphism algebra $\mathrm{End}^0(A) := \mathrm{End}_{\bar{k}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ of A is either $\mathbb{Q}$ or an imaginary quadratic field ([26, Theorem 5.5]). Every elliptic curve over k is defined by a Weierstrass equation of the form

$$C_{f,2}: \quad y^2 = f(x), \tag{1.1}$$

where $f(x) \in k[x]$ is a polynomial of degree 3 without multiple roots. It is very tempting to replace the exponent of y in (1.1) by N and study the curve

$$C_{f,N}: \quad y^N = f(x) \tag{1.2}$$

and its Jacobian variety $J(C_{f,N})$. We are interested in the endomorphism algebra of $J(C_{f,N})$.

There are multiple ways of putting $C_{f,N}$ in a slightly more general context. In one direction (say $k = \mathbb{C}$), we may look at the hypergeometric curves $C_{\lambda,N}$ defined by

$$y^N = x^A(x-1)^B(x-\lambda)^C, \tag{1.3}$$

where $\lambda \in \mathbb{C} - \{0,1\}$. These curves are closely related to Gauss's hypergeometric series $F(a,b,c;z)$ (cf. [29] by J. Wolfart). Assume that $A = B = C = 1$. One may study the exceptional set

$$\mathscr{E}_N = \{\lambda \in \mathbb{C} - \{0,1\} \mid \text{ The Jacobian } J(C_{\lambda,N}) \text{ has complex multiplication}\}.$$

Received by the editors April 23, 2013 and, in revised form, April 25, 2013 and September 4, 2013.

2010 *Mathematics Subject Classification.* Primary 14H40, 11G15.

Key words and phrases. Hypergeometric curves, endomorphism algebras, Jacobians.

When $N = 5$ or 7, De Jong and Noot [7] showed that $\mathscr{E}_N$ is infinite, thus providing counterexamples (in genus $g = 4$ and 6 respectively) to Coleman's conjecture ([6, Conjecture 6]), which predicted that for each fixed $g \geq 4$, there are only a finite number of isomorphic classes of curves of genus g whose Jacobians have complex multiplication. Coleman's conjecture remains open for $g \geq 8$. See [20] for a survey. Nowadays, the question of the finiteness of $\mathscr{E}_N$ is generally seen in the light of the André–Oort Conjecture ([1, p. 215, problem 1], [22], [23, Conjecture 1.5]), which is a conjecture on the special points of Shimura varieties.

Another general setting for $C_{f,N}$ is to allow the degree of $f(x)$ in (1.2) to be an arbitrary number $n \geq 3$, while still requiring that $f(x)$ has no multiple roots. We call such curves *superelliptic curves*. In a series of papers [32], [34], etc., Yu. G. Zarhin determined the endomorphism algebras $\mathrm{End}^0(J(C_{f,N}))$, assuming that $n \geq 5$, $N = q = p^r$ is a prime power coprime to n, and $f(x)$ is irreducible over k with Galois group $\mathrm{Gal}(f)$ equal to either the full symmetric group $\mathbf{S}_n$ or the alternating group $\mathbf{A}_n$ (cf. [32, Theorem 1.1], [34, Theorem 1.1]).

To explain Zarhin's results more clearly, and to state our main theorem, we need to introduce some new concepts. Clearly, $C_{f,N}$ (with an arbitrary $f(x)$) admits a natural periodic automorphism of order N:

$$\delta_N : C_{f,N} \to C_{f,N}, \qquad (x, y) \mapsto (x, \xi_N y),$$

where $\xi_N \in \bar{k}$ is a primitive N-th root of unity. By Albanese functoriality, δ_N induces an automorphism of $J(C_{f,N})$, which will be denoted again by δ_N by an abuse of notation. Thus we obtain an embedding of the cyclic group $G = \mathbb{Z}/N\mathbb{Z}$ into $\mathrm{Aut}_{\bar{k}}(J(C_{f,N}))$, and hence a homomorphism from the group ring $\mathbb{Q}[G]$ to $\mathrm{End}^0(J(C_{f,N}))$. Let $\zeta_N := e^{2\pi i/N} \in \mathbb{C}$, and $\zeta_D := \zeta_N^{N/D}$ for each positive divisor $D \mid N$. The natural isomorphism

$$\mathbb{Q}[G] \cong \mathbb{Q}[T]/(T^N - 1) \cong \prod_{D|N} \mathbb{Q}(\zeta_D)$$

gives rise to an isogeny

$$J(C_{f,N}) \sim \prod_{D|N,\, D\neq 1} J_{f,D}^{\mathrm{new}}. \tag{1.4}$$

Each $J_{f,D}^{\mathrm{new}}$ is isogenous to an abelian subvariety of $J(C_{f,D})$, and $J_{f,N}^{\mathrm{new}}$ is the abelian subvariety of $J(C_{f,N})$ that has not appeared in $J(C_{f,D})$ for any proper divisor D of N before, and thus the name "new part" (see Section 2 for more details). We have a natural embedding

$$\mathfrak{i} : \mathbb{Z}[\zeta_N] \hookrightarrow \mathrm{End}_{\bar{k}}(J_{f,N}^{\mathrm{new}}), \qquad \zeta_N \mapsto \delta_N \mid_{J_{f,N}^{\mathrm{new}}}.$$

Zarhin showed that under aforementioned assumptions, the embedding $\mathfrak{i}$ is in fact an isomorphism, and (recall that $q = p^r$)

$$\mathrm{End}^0(J(C_{f,q})) \cong \prod_{i=1}^{r} \mathbb{Q}(\zeta_{p^i}). \tag{1.5}$$

He also treated the case $n = 3, 4$ assuming some further conditions on the base field and $\mathrm{Gal}(f)$ (cf. [33, Theorem 1.3]). If $q = 2^r$, and $\deg f(x) = 3$, then (1.5) needs to be modified accordingly [33, Theorem 1.4]. Many parts of this paper are based on his results.

We will improve Zarhin's result by removing the extra assumptions and classify $\mathrm{End}^0(J_{f,q}^{\mathrm{new}})$ for all polynomials $f(x)$ of degree 3 with nonzero discriminants. Partial results are also obtained for $\mathrm{End}^0(J_{f,N}^{\mathrm{new}})$ with a general N.

The genus formula for $C_{f,N}$ with a general $f(x)$ and N is given in [15] and [27]. By (2.6) of Section 2, if $f(x)$ is of degree 3 with nonzero discriminant, and $N > 3$, then $\dim J_{f,N}^{\mathrm{new}} = \varphi(N)$. Therefore,

$$\dim_{\mathbb{Q}}(\mathbb{Q}(\zeta_N)) = \varphi(N) = \dim J_{f,N}^{\mathrm{new}}.$$

In a way, $J_{f,N}^{\mathrm{new}}$ generalizes the elliptic curves in the sense that they are abelian varieties naturally equipped with multiplication by cyclotomic fields whose degree coincides with the dimension of the variety.

Theorem 1.1 (Main Theorem)**.** *Let k be a field of characteristic zero, let $q = p^r$ be a prime power, and let $q \geq 9$ if $p = 3$ and $q \geq 4$ if $p = 2$. Let $f(x) \in k[x]$ be a polynomial of degree 3 with no multiple roots, and let $J_{f,q}^{\mathrm{new}}$ be defined as in Definition 2.1. Then one of the following holds for $J_{f,q}^{\mathrm{new}}$:*

(1) $J_{f,q}^{\mathrm{new}}$ is absolutely simple, and $\mathrm{End}^0(J_{f,q}^{\mathrm{new}})$ is one of the following:
- *(1a) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong \mathbb{Q}(\zeta_q)$.*
- *(1b) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong L$, where L is a CM field containing $\mathbb{Q}(\zeta_q)$, and $[L : \mathbb{Q}(\zeta_q)] = 2$.*

(2) $J_{f,q}^{\mathrm{new}}$ is not absolutely simple, and $\mathrm{End}^0(J_{f,q}^{\mathrm{new}})$ is one of the following:
- *(2a) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong \mathbb{Q}(\zeta_q) \oplus \mathbb{Q}(\zeta_q)$, if $p \geq 5$, and $q \neq 5, 7$.*
- *(2b) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong \mathbb{Q}(\zeta_q) \oplus \mathbb{Q}(\zeta_q)$ or $\mathrm{Mat}_2(\mathbb{Q}(\zeta_q))$ if $q = 3^r \geq 27$.*
- *(2c) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong \mathrm{Mat}_2(\mathbb{Q}(\zeta_q))$ if $q = 4, 5$ or 9.*
- *(2d) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong \mathrm{Mat}_3(\mathbb{Q}(\sqrt{-7})) \oplus \mathbb{Q}(\zeta_7)$ if $q = 7$.*
- *(2e) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-1})) \oplus \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-2}))$ if $q = 8$.*
- *(2f) $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) \cong \mathbb{Q}(\zeta_q) \oplus \mathbb{Q}(\zeta_q)$ or $\mathrm{Mat}_2(\mathbb{Q}(\alpha)) \oplus \mathbb{Q}(\zeta_q)$ if $q = 2^r \geq 16$, where $\alpha = 2\sqrt{-1}\sin(2\pi/q))$.*

In particular, $\mathrm{End}^0(J(C_{f,q})) = \prod_{i=1}^{r} \mathrm{End}^0(J_{f,p^i}^{\mathrm{new}})$ is commutative if $p > 7$.

Theorem 1.2. *In addition to the assumptions of Theorem 1.1, we assume that $p \neq 3$, $q \neq 4$, the field k contains a primitive q-th root of unity ξ_q, and $f(x)$ is irreducible over k with Galois group $\mathrm{Gal}(f) \cong \mathbf{S}_3$. Then $J_{f,q}^{\mathrm{new}}$ is absolutely simple. In other words, either (1a) or (1b) in Theorem 1.1 holds, and case (2) does not appear.*

The proofs of Theorem 1.1 and Theorem 1.2 will be given in Section 3.

Remark 1.3. If $q = 3$, then $C_{f,3}$ has genus 1, and $J_{f,3}^{\mathrm{new}} = J(C_{f,3})$ is an elliptic curve with $\mathrm{End}_{\bar{k}}(J(C_{f,3})) \supseteq \mathbb{Z}[\zeta_3]$, the maximal order in the imaginary quadratic field $\mathbb{Q}(\zeta_3)$. Therefore, $\mathrm{End}_{\bar{k}}(J(C_{f,3})) = \mathbb{Z}[\zeta_3]$. Since the class number of $\mathbb{Q}(\zeta_3)$ is one, $J(C_{f,3})$ is isomorphic over $\bar{k}$ to the elliptic curve $y^2 = x^3 + 1$.

Remark 1.4. If $q = 4$, then $\mathrm{End}^0(J_{f,4}^{\mathrm{new}}) = \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-1}))$ for all $f(x) \in k[x]$ of degree 3 with no multiple roots. In other words, $J_{f,4}^{\mathrm{new}}$ is isogenous to the square of the elliptic curve $y^2 = x^3 - x$. This result was first proven by J.W.S. Cassels [4], and an explicit construction of the isogeny is given by J. Guàrdia in [10].

Corollary 1.5. *Let the assumptions be the same as Theorem* 1.1. *We further assume that $p > 7$ and $f(x)$ is a monic polynomial. With a unique change of variable of the form $x \mapsto x - b$ for a suitable $b \in k$, we may assume that $f(x) = x^3 + B_0 x + C_0$.*

(1) If $B_0 = 0$, then $\mathrm{Aut}_{\bar{k}}(C_{f,q}) \cong \mathbb{Z}/3q\mathbb{Z}$.
(2) If $C_0 = 0$, then $\mathrm{Aut}_{\bar{k}}(C_{f,q}) \cong \mathbb{Z}/2q\mathbb{Z}$.
(3) $\mathrm{Aut}_{\bar{k}}(C_{f,q}) \cong \mathbb{Z}/q\mathbb{Z}$ otherwise.

In particular, if $f(x) = x(x-1)(x-\lambda)$, then $\mathrm{Aut}_{\bar{k}}(C_{\lambda,q}) \cong \mathbb{Z}/3q\mathbb{Z}$ if and only if $\lambda = (1 \pm \sqrt{-3})/2$, and $\mathrm{Aut}_{\bar{k}}(C_{\lambda,q}) \cong \mathbb{Z}/2q\mathbb{Z}$ if and only if $\lambda \in \{-1, 2, 1/2\}$. There are only finitely many $\lambda \in \mathbb{C}$ such that the curve $C_{\lambda,q}$ has extra automorphisms.

Remark 1.6. We will also obtain some results on $\mathrm{End}^0(J^{\mathrm{new}}_{f,N})$ for a general N with $\gcd(N, 3) = 1$. For example, if there exists a quadratic field extension $L/\mathbb{Q}(\zeta_N)$ such that

$$\mathrm{End}^0(J^{\mathrm{new}}_{f,N}) \supseteq L \supseteq \mathbb{Q}(\zeta_N),$$

and $N \notin \{4, 10\}$, then it is shown in Corollary 3.7 that $\mathrm{End}^0(J^{\mathrm{new}}_{f,N})$ coincides with L, and $J^{\mathrm{new}}_{f,N}$ is absolutely simple.

Examples 1.7. Here are some examples of $f(x)$ that give rise to the endomorphism algebras in Theorem 1.1. Most of the proofs will be given in Section 4.

(1a) Suppose that $k = \bar{\mathbb{Q}}(t)$, the rational function field of transcendental degree 1 over $\bar{\mathbb{Q}}$, and $f(x) = x^3 - x - t \in k[x]$. By Example 2.3 and Theorem 5.18 of [33], $\mathrm{End}_{\bar{k}}(J^{\mathrm{new}}_{f,q}) \cong \mathbb{Z}[\zeta_q]$ for any prime power $q = p^r \neq 4$.
(1b) If $f(x) = x^3 + 1$ and $3 \nmid N$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f,N}) \cong \mathbb{Q}(\zeta_{3N})$.
(1b') If $f(x) = x^3 - x$ and N is even and coprime to 3, then $\mathrm{End}^0(J^{\mathrm{new}}_{f,N}) \cong \mathbb{Q}(\zeta_{2N})$.
(2a) If $f(x) = x^3 - x$ and $p \geq 5$, and $q \neq 5, 7$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f,q}) \cong \mathbb{Q}(\zeta_q) \oplus \mathbb{Q}(\zeta_q)$.
(2b) If $f(x) = x^3 + 1$ and $q = 3^r \geq 9$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f,q}) \cong \mathrm{Mat}_2(\mathbb{Q}(\zeta_q))$.
(2b') If $f(x) = x^3 - x$ and $q = 3^r \geq 27$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f,q}) \cong \mathbb{Q}(\zeta_q) \oplus \mathbb{Q}(\zeta_q)$.
(2c) If $f(x) = x^3 - x$ and $q = 5, 9$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f,q}) \cong \mathrm{Mat}_2(\mathbb{Q}(\zeta_q))$.
(2d) If $f(x) = x^3 - x$ and $q = 7$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f,7}) \cong \mathrm{Mat}_3(\mathbb{Q}(\sqrt{-7})) \oplus \mathbb{Q}(\zeta_7)$.

Remark 1.8. The examples show that our classification in Theorem 1.1 is complete for those $q = p^r$ with $p \geq 5$, in the sense that there are examples for each case listed in the theorem for those q. However, if $q = 3^r$, we have yet to find examples where $\mathrm{End}^0(J^{\mathrm{new}}_{f,q})$ is a quadratic extension of $\mathbb{Q}(\zeta_{3^r})$ (case (1b)); if $q = 2^r$, we don't have examples for which $J^{\mathrm{new}}_{f,q}$ is not simple (case (2e), (2f)). The remaining cases when q is a power of 2 or 3 are supported by examples.

The paper is organized as follows. In Section 2, we study superelliptic curves $C_{f,N}$ and define the subvariety $J^{\mathrm{new}}_{f,N}$ of $J(C_{f,N})$. In Section 3, we show how the information extracted from the study of $C_{f,N}$ is used in classifying $\mathrm{End}^0(J^{\mathrm{new}}_{f,N})$. Certain arithmetic results needed there are postponed to Section 5. In Section 4, we study the automorphism group of $C_{f,N}$ and construct examples with the given endomorphism algebras in our classification.

2. Superelliptic curve and its Jacobian variety

The goal of this section is to define the abelian subvariety $J^{\mathrm{new}}_{f,N}$ and study its basic properties. Let k be a field with characteristic coprime to N. We assume that

k contains a primitive N-th root of unity ξ_N. Then $\xi_D := (\xi_N)^{N/D}$ is a primitive D-th root of unity for each $D \mid N$. In the latter half of the section, we will restrict to the case that k has characteristic zero.

2.1. We first recall some basic facts about curves and their Jacobian varieties. Let X be a curve over k of genus $g \geq 2$, and $\mathrm{Aut}_k(X)$ its automorphism group over k. It is well known that $\mathrm{Aut}_{\bar{k}}(X)$ (and hence $\mathrm{Aut}_k(X)$) is finite if $g \geq 2$ ([11, Exercise IV.5.2], [13, Chapter 11]). By Albanese functoriality, each $\delta \in \mathrm{Aut}_k(X)$ induces an automorphism of the Jacobian variety $J(X)$, which is still denoted by δ by an abuse of notation. Torelli's theorem ([18, Section 12]) implies that the homomorphism $\mathrm{Aut}_k(X) \to \mathrm{Aut}_k(J(X))$ thus obtained is an embedding. This gives rise to a homomorphism from the group ring $\mathbb{Z}[\mathrm{Aut}_k(X)]$ to the endomorphism ring of $J(X)$:

$$\mathbb{Z}[\mathrm{Aut}_k(X)] \to \mathrm{End}_k(J(X)) \subseteq \mathrm{End}_{\bar{k}}(J(X)). \tag{2.1}$$

Let $\pi : X \to Y$ be a separable map of curves of degree m. It induces two morphisms of the Jacobians:

$$\begin{aligned} \pi^* : J(Y) &\to J(X) && \text{by Picard functoriality;} \\ \pi : J(X) &\to J(Y) && \text{by Albanese functoriality.} \end{aligned}$$

Moreover, $\pi \circ \pi^* = m_{J(Y)}$. So $\ker \pi^* \subseteq J(Y)[m]$, the m-torsions of $J(Y)$. In particular, $J(Y)$ is isogenous to its image $\pi^* J(Y)$:

$$J(Y) \sim \pi^* J(Y) \subseteq J(X). \tag{2.2}$$

Let $G \subseteq \mathrm{Aut}_k(X)$ be a subgroup of order m, and $Y := X/G$ be the quotient curve. The quotient map $\pi : X \to Y$ is separable and finite of degree m. We have

$$\Big(\sum_{g \in G} g\Big) J(X) = \pi^* J(Y). \tag{2.3}$$

In particular, if $Y \cong \mathbb{P}^1$, then $\sum_{g \in G} g = 0 \in \mathrm{End}(J(X))$.

2.2. Suppose that $f(x) \in k[x]$ is a polynomial of degree $n \geq 3$ with factorization $a_0 \prod_{i=1}^{s} (x - \alpha_i)^{m_i}$ in $\bar{k}[x]$, and $\gcd(N, m_1, \cdots, m_s) = 1$. Let $C_{f,N}$ be the curve defined by $y^N = f(x)$ and $J(C_{f,N})$ the Jacobian variety of $C_{f,N}$. The map

$$\pi : C_{f,N} \to \mathbb{P}^1, \qquad (x, y) \mapsto x \tag{2.4}$$

realizes $C_{f,N}$ as a (ramified) cyclic cover of $\mathbb{P}^1$ with covering group $G := \mathrm{Aut}(\pi) \cong \mathbb{Z}/N\mathbb{Z}$. A generator of G is given by

$$\delta_N : C_{f,N} \to C_{f,N}, \qquad (x, y) \mapsto (x, \xi_N y).$$

Let H_D be the subgroup of G of index D. Then the quotient curve $C_{f,N}/H_D$ is isomorphic to $C_{f,D}$ with quotient map

$$\pi_D : C_{f,N} \to C_{f,D}, \qquad (x, y) \mapsto (x, y^{N/D}). \tag{2.5}$$

Definition 2.1. Following [9, Definition 5.1], we call the abelian subvariety

$$J_{f,N}^{\mathrm{old}} := \sum_{D \mid N, D \neq N} \pi_D^* J(C_{f,D})$$

the *old part* of $J(C_{f,N})$, and its orthogonal complement (with respect to the canonical polarization) the *new part* $J_{f,N}^{\mathrm{new}}$. If $N = p$ is a prime, then $J_{f,p}^{\mathrm{new}} := J(C_{f,p})$.

Remark 2.2. If $k = \mathbb{C}$, $J_{f,N}^{\mathrm{new}}$ can also be defined as the complex torus given by a period lattice which is obtained by integration on $C_{f,N}$. For example, in the case where $C_{f,N}$ is hypergeometric, this construction is carried out in [5, Section 3].

2.3. By [9, Corollary 5.4],

$$\dim J_{f,N}^{\mathrm{new}} = \varphi(N)(|R| - 2)/2, \tag{2.6}$$

where

$$R = \{P \in \mathbb{P}^1(\bar{k}) \mid \pi : C_{f,N} \to \mathbb{P}^1 \text{ is ramified over } P\}.$$

Assume that $f(x) = a_0 \prod_{i=1}^{n}(x - \alpha_i)$ has no multiple roots. We have two cases:

- if $N \nmid n$, then $R = \{\alpha_i\}_{i=1}^{n} \cup \{\infty\}$, so $\dim J_{f,N}^{\mathrm{new}} = \varphi(N)(n-1)/2$;
- otherwise, $N \mid n$, then $R = \{\alpha_i\}_{i=1}^{n}$, so $\dim J_{f,N}^{\mathrm{new}} = \varphi(N)(n-2)/2$.

Indeed, in the case $n = Nb$, a simple change of variable of the form $u = 1/(x - \alpha_1)$, $v = y/(x - \alpha_1)^b$ establishes a birational isomorphism between $C_{f,N}$ and $C_{g,N}$ over $k' := k(\alpha_1)$, where $g(x) \in k'[x]$ is of degree $n-1$ without multiple roots ([32, Remark 4.3]).

2.4. We show that there exists a natural embedding $\mathfrak{i} : \mathbb{Z}[\zeta_N] \hookrightarrow \mathrm{End}_k(J_{f,N}^{\mathrm{new}})$. The main idea of the proof is already contained in [9, Lemma 5.2]. It is included here since the construction is needed for Subsection 2.5. Since $G = \mathbb{Z}/N\mathbb{Z}$ is commutative,

$$\mathbb{C}[G] \cong \bigoplus_{\chi \in \widehat{G}(\mathbb{C})} \mathbb{C}_\chi, \tag{2.7}$$

where $\widehat{G}(\mathbb{C}) := \{\chi : G \to \mathbb{C}^\times\}$ is the ($\mathbb{C}$-valued) character group of G, and $\mathbb{C}_\chi := \mathbb{C}$ with the projection map $\mathbb{C}[G] \to \mathbb{C}_\chi$ given by $g \mapsto \chi(g)$ for all $g \in G$. Let χ_N be the generator of $\widehat{G}(\mathbb{C})$ with $\chi_N(\delta_N) = \zeta_N := e^{2\pi i/N}$. For simplicity, we write $\mathbb{C}_a$ for $\mathbb{C}_{\chi_N^a}$ for each $a \in \mathbb{Z}/N\mathbb{Z}$. Let $\epsilon_a \in \mathbb{C}[G]$ be the element associated with the character $(\chi_N)^a$:

$$\epsilon_a = \epsilon_{\chi_N^a} := \frac{1}{N}\sum_{i=0}^{N-1} \chi_N(\delta_N)^{ai}\delta_N^{N-i} = \frac{1}{N}\sum_{i=0}^{N-1} \zeta_N^{ai}\delta_N^{N-i} \in \mathbb{C}[G]. \tag{2.8}$$

The orthogonality of characters implies that ϵ_a is mapped to the primitive idempotent $(0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{C}_a$ on the right hand side of (2.7). Therefore, $\{\epsilon_a\}_{a \in \mathbb{Z}/N\mathbb{Z}}$ is a complete set of primitive pairwise orthogonal idempotents of $\mathbb{C}[G]$. For each $D \in \mathbb{N}$, let $\Phi_D(T) \in \mathbb{Z}[T]$ be the D-th cyclotomic polynomial. We have

$$\mathbb{Q}[G] \cong \mathbb{Q}[T]/(T^N - 1) \cong \bigoplus_{D \mid N} \mathbb{Q}[T]/(\Phi_D(T)), \tag{2.9}$$

and

$$(\mathbb{Q}[T]/(\Phi_D(T))) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \mathbb{C}_{\chi_D^a}, \qquad \text{where } \chi_D = \chi_N^{N/D}. \tag{2.10}$$

Let $\zeta_D := \zeta_N^{N/D} = e^{2\pi i/D}$, and

$$\eta_D := \sum_{a \in (\mathbb{Z}/D\mathbb{Z})^\times} \epsilon_{\chi_D^a} = \frac{1}{N}\sum_{i=0}^{N-1} (\mathrm{Tr}_{\mathbb{Q}(\zeta_D)/\mathbb{Q}}\, \zeta_D^i)\delta_N^{N-i} \in \mathbb{Q}[G]. \tag{2.11}$$

Then η_D is the primitive idempotent in $\mathbb{Q}[G]$ corresponding to the factor $\mathbb{Q}[T]/(\Phi_D(T))$. In particular, $\Phi_D(\delta_N)\eta_D = 0$. Clearly, $N\eta_N \in \mathbb{Z}[G]$ and [9, Corollary 5.3] showed that

$$J_{f,N}^{\text{new}} = (N\eta_N)J(C_{f,N}), \qquad J_{f,N}^{\text{old}} = N(1-\eta_N)J(C_{f,N}), \tag{2.12}$$

and

$$J(C_{f,N}) \sim J_{f,N}^{\text{old}} \times J_{f,N}^{\text{new}}. \tag{2.13}$$

Since δ_N commutes with η_N, the group G acts on $J_{f,N}^{\text{new}}$ as well . So

$$\Phi_N(\delta_N)J_{f,N}^{\text{new}} = N\Phi_N(\delta_N)\eta_N J(C_{f,N}) = 0.$$

Therefore, we have a natural embedding

$$\mathfrak{i} : \mathbb{Z}[\zeta_N] \hookrightarrow \text{End}_k(J_{f,N}^{\text{new}}), \qquad \zeta_N \mapsto \delta_N \mid_{J_{f,N}^{\text{new}}}. \tag{2.14}$$

Similarly, one sees that $(N\eta_D)J(C_{f,N})$ is G-invariant for all $D \mid N$.

2.5. The isogeny in (2.13) can be refined further. Recall that $\{\eta_D\}_{D|N}$ forms a complete set of primitive pairwise orthogonal basis for $\mathbb{Q}[G]$. By the remark below (2.3), $(N\eta_1)J(C_{f,N}) = 0$. So

$$J(C_{f,N}) \sim \prod_{D|N, D\neq 1} (N^2\eta_D)J(C_{f,N}). \tag{2.15}$$

For each $D \mid N$, let

$$\tilde{\epsilon}_D := \frac{1}{|H_D|}\sum_{h\in H_D} h = \frac{D}{N}\sum_{h\in H_D} h \in \mathbb{Q}[G].$$

By (2.3), $(N/D)\tilde{\epsilon}_D J(C_{f,N}) = \pi_D^* J(C_{f,D})$. It was shown in [9, Lemma 5.2] that $\tilde{\epsilon}_D = \sum_{a\in\mathbb{Z}/D\mathbb{Z}} \epsilon_{\chi_D^a}$. In particular, $\eta_D\tilde{\epsilon}_D = \eta_D$.

Clearly, we have the following commutative diagram of morphisms of curves:

$$\begin{array}{ccc} C_{f,N} & \xrightarrow{\pi_D} & C_{f,D} \\ {\scriptstyle\delta_N}\downarrow & & \downarrow{\scriptstyle\delta_D} \\ C_{f,N} & \xrightarrow{\pi_D} & C_{f,D}. \end{array} \tag{2.16}$$

So $\pi_D^*\delta_D^* = \delta_N^*\pi_D^*$. Note that $\delta_N^* = \delta_N^{-1}$, and similarly for δ_D. Hence

$$\delta_N\pi_D^* = \pi_D^*\delta_D. \tag{2.17}$$

It follows that

$$\begin{aligned}(N^2\eta_D)J(C_{f,N}) &= D(N\eta_D)\cdot(N/D)\tilde{\epsilon}_D J(C_{f,N}) = D(N\eta_D)\pi_D^* J(C_{f,D}) \\ &= D\left(\sum_{i=0}^{N-1}(\text{Tr}_{\mathbb{Q}(\zeta_D)/\mathbb{Q}}\,\zeta_D^i)\delta_N^{N-i}\right)\pi_D^* J(C_{f,D}) \qquad \text{(by (2.11))} \\ &= D(N/D)\pi_D^*\left(\sum_{i=0}^{D-1}(\text{Tr}_{\mathbb{Q}(\zeta_D)/\mathbb{Q}}\,\zeta_D^i)\delta_D^{D-i}\right)J(C_{f,D}) \qquad \text{(by (2.17))} \\ &= \pi_D^* J_{f,D}^{\text{new}} \sim J_{f,D}^{\text{new}} \qquad \text{(by (2.2))}.\end{aligned}$$

Therefore,

$$J(C_{f,N}) \sim \prod_{D|N, D\neq 1} J_{f,D}^{\mathrm{new}}. \tag{2.18}$$

This generalizes [32, Corollary 4.12].

Remark 2.3. More generally, let $\pi : C \to C'$ be a cyclic cover of curves with covering group $G = \mathbb{Z}/N\mathbb{Z}$ (cf. [9, Definition 5.1]). The constructions in Subsections 2.4 and 2.5 apply without much change. We see that $\mathbb{Q}(\zeta_N) \hookrightarrow \mathrm{End}(J_C^{\mathrm{new}})$, and

$$J(C) \sim \prod_{D|N, D\neq 1} J_{C/H_D}^{\mathrm{new}} \times J(C')$$

since $(N\eta_1)J(C) = \pi^* J(C')$.

2.6. Let X be a smooth projective curve over k. We write $\lambda_X : J(X) \to J(X)^\vee$ for the canonical polarization of $J(X)$. It is well known that λ_X is an isomorphism. Let $\pi : X \to Y$ be a morphism of curves, and $\pi : J(X) \to J(Y)$ and $\pi^* : J(Y) \to J(X)$ be the induced morphisms of the Jacobians as in Subsection 2.1. We write $\pi^\vee : J(Y)^\vee \to J(X)^\vee$ for the dual homomorphism of π. Then there is a commutative diagram:

$$\begin{array}{ccc} J(Y) & \xrightarrow{\pi^*} & J(X) \\ {\scriptstyle \lambda_Y}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle \lambda_X} \\ J(Y)^\vee & \xrightarrow{\pi^\vee} & J(X)^\vee. \end{array} \tag{2.19}$$

In other words, if we identify each Jacobian with its dual via the canonical polarization, then π and π^* are dual to each other (See [3, Prop. 11.11.6] in the case $k = \mathbb{C}$, and [19, Prop. A.6] in much more generality.)

2.7. Let $i_N : J_{f,N}^{\mathrm{old}} \to J(C_{f,N})$ be the inclusion, and let $\lambda_N := \lambda_{C_{f,N}} : J(C_{f,N}) \to J(C_{f,N})^\vee$ be the canonical principal polarization. By the proof of [21, Theorem 19.1], $J_{f,N}^{\mathrm{new}}$ is the identity component of $\ker(i_N^\vee \circ \lambda_N)$. Similar to Definition 2.1, we define $J_{f,N}^{D-\mathrm{new}}$ to be the orthogonal complement of $\pi_D^* J(C_{f,D})$. Then $J_{f,N}^{D-\mathrm{new}}$ coincides with the identity component of

$$\ker((\pi_D^*)^\vee \circ \lambda_N) = \ker(\lambda_D^{-1} \circ (\pi_D^*)^\vee \circ \lambda_N) = \ker \pi_D, \qquad \text{by (2.19)}.$$

Suppose $D_1 \mid D_2$ and $D_2 \mid N$, then the quotient map $\pi_{D_1} : C_{f,N} \to C_{f,D_1}$ factors as a composition of successive quotient maps $C_{f,N} \to C_{f,D_2} \to C_{f,D_1}$. Therefore,

$$\pi_{D_1}^* J(C_{f,D_1}) \subseteq \pi_{D_2}^* J(C_{f,D_2}) \subseteq J(C_{f,N}).$$

In particular, if $N = q = p^r$ is a prime power, then $J_{f,q}^{\mathrm{old}} = \pi_{q/p}^* J(C_{f,q/p})$, and

$$J_{f,q}^{\mathrm{new}} = J_{f,q}^{(q/p)-\mathrm{new}} = \text{ the identity component of } \ker \pi_{q/p}.$$

2.8. We assume that $f(x) \in k[x]$ satisfies one of the following conditions:

- there exists a root $\alpha \in \bar{k}$ of $f(x)$ whose multiplicity m_α is coprime to N;
- $\deg f(x)$ is coprime to N.

In the first case, there is exactly one point $P \in C_{f,N}(\bar{k})$ corresponding to $(\alpha, 0) \in \mathbb{A}^2(\bar{k})$. (Generally one needs to perform some desingularization to obtain $C_{f,N}$.) Moreover, the covering map $\pi : C_{f,N} \to \mathbb{P}^1$ in (2.4) is totally ramified at P. In the second case, there is exactly one point $P := \infty$ at infinity for $C_{f,N}$, and π is totally ramified at P again. Either way, it follows that $\pi_D : C_{f,N} \to C_{f,D}$ is totally ramified at P for each $D \mid N$.

Let $K_{N,D}$ be the kernel of $\pi_D^* : J(C_{f,D}) \to J(C_{f,N})$. We have seen in Subsection 2.1 that $K_{N,D} \leq J(C_{f,D})[N/D]$. By [18, Section 9], $K_{N,D}^\vee(\bar{k})$ is isomorphic to the covering group of the maximal abelian unramified covering (over $\bar{k}$) of $C_{f,D}$ which is intermediate to $\pi_D : C_{f,N} \to C_{f,D}$. Such a covering must be trivial under our assumption on $f(x)$. Therefore, $K_{N,D}$ is trivial, and π_D^* is an embedding for all $D \mid N$.

Let A be the quotient abelian variety of $J(C_{f,N})$ by $\pi_D^* J(C_{f,D})$. We have an exact sequence

$$0 \to J(C_{f,D}) \xrightarrow{\pi_D^*} J(C_{f,N}) \to A \to 0.$$

Taking the dual exact sequence, we get

$$0 \to A^\vee \to J(C_{f,N})^\vee \xrightarrow{(\pi_D^*)^\vee} J(C_{f,D})^\vee \to 0.$$

By (2.19), we may rewrite the exact sequence as

$$0 \to A^\vee \to J(C_{f,N}) \xrightarrow{\pi_D} J(C_{f,D}) \to 0.$$

Therefore, $\ker \pi_D = A^\vee$ is connected. On the other hand, recall that $J_{f,N}^{D-\mathrm{new}}$ is equal to the identity component of $\ker \pi_D$. It follows that

$$J_{f,N}^{D-\mathrm{new}} = \ker \pi_D = A^\vee. \tag{2.20}$$

Since π_D^* is an embedding,

$$J(C_{f,D})[N/D] = \ker(\pi_D \circ \pi_D^*) \subseteq \ker \pi_D = J_{f,N}^{D-\mathrm{new}}. \tag{2.21}$$

In other words, $\pi_D^* J(C_{f,D}) \cap J_{f,N}^{D-\mathrm{new}} = J(C_{f,D})[N/D]$. Note that both π_D and π_D^* are defined over k, so $J_{f,N}^{D-\mathrm{new}}$ "inherits" from $J(C_{f,D})$ a $\mathrm{Gal}(\bar{k}/k)$-module structure that is isomorphic to $J(C_{f,D})[N/D]$.

In particular, if $N = q = p^r$ is a prime power, and $f(x)$ has no multiple roots, then $J_{f,q}^{\mathrm{new}} = \ker \pi_{q/p}$, and

$$J_{f,q}^{\mathrm{new}}[p] \supseteq \pi_{q/p}^* J(C_{f,q/p})[p] \cong J(C_{f,q/p})[p]. \tag{2.22}$$

We have an exact sequence

$$0 \to J(C_{f,q/p})[p] \to J(C_{f,q/p}) \times J_{f,q}^{\mathrm{new}} \to J(C_{f,q}) \to 0,$$

which makes (2.13) more explicit.

2.9. Let X be a curve over k, and $\mathrm{Lie}_k(J(X))$ be the Lie algebra of $J(X)$, which is canonically isomorphic to the tangent space to $J(X)$ at 0. The Picard functoriality induces a right action of $\mathrm{Aut}_k(X)$ on $\mathrm{Lie}_k(J(X))$. The isomorphism $\mathrm{Lie}_k(J(X)) \cong H^1(X, \mathcal{O}_X)$ given in [18, Proposition 2.1] is $\mathrm{Aut}_k(X)$-equivariant. Combining with the Serre duality [11, Corollary 7.13]), we obtain a perfect and $\mathrm{Aut}_k(X)$-equivariant pairing

$$\Gamma(X, \Omega_X^1) \times \mathrm{Lie}_k(J(X)) \to k, \tag{2.23}$$

where $\mathrm{Aut}_k(X)$ acts on $\Gamma(X, \Omega^1_X)$ from the right via pullbacks. (Over $\mathbb{C}$, this follows directly from the classical definition of the Jacobian [18, Section 2].) Note that $\mathrm{Aut}_k(X)$ also acts on $\mathrm{Lie}_k(J(X))$ from the left via Albanese functoriality, which is just the inverse of the Picard action. Therefore, we will also let $\mathrm{Aut}_k(X)$ act on $\Gamma(X, \Omega^1_X)$ from the left by taking the inverse of the pullback so that (2.23) is again $\mathrm{Aut}_k(X)$-equivariant.

2.10. Since k contains a primitive N-th root of unity ξ_N, any left representation V of $G = \mathbb{Z}/N\mathbb{Z}$ over k splits into a direct sum of subrepresentations, indexed by the k-valued character group $\widehat{G}(k)$ of G:

$$V = \bigoplus_{\chi \in \widehat{G}(k)} V_\chi, \tag{2.24}$$

where $V_\chi := \{v \in V \mid gv = \chi(g)v, \forall\, g \in G\}$.

Recall that $J^{\mathrm{new}}_{f,N} = N\eta_N J(C_{f,N})$. Let $d(N\eta_N) : \mathrm{Lie}_k(J(C_{f,N})) \to \mathrm{Lie}_k(J(C_{f,N}))$ be the induced morphism of Lie algebras of $N\eta_N \in \mathrm{End}(J(C_{f,N}))$. Since the isogeny $J^{\mathrm{old}}_{f,N} \times J^{\mathrm{new}}_{f,N} \to J(C_{f,N})$ in (2.13) is separable with kernel isomorphic to a subgroup of $J(C_{f,N})[N]$, we conclude that $\mathrm{Lie}_k(J^{\mathrm{new}}_{f,N})$ coincides with the image of $d(N\eta_N)$. Clearly, each $\mathrm{Lie}(J(C_{f,N}))_\chi$ is $d(N\eta_N)$ invariant. It follows from (2.11) that $d(N\eta_N)$ acts on $\mathrm{Lie}(J(C_{f,N}))_\chi$ as multiplication by N if $\chi(\delta_N)$ is a primitive N-th root of unity in k, and 0 otherwise. Therefore,

$$\mathrm{Lie}_k(J^{\mathrm{new}}_{f,N}) = \bigoplus_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathrm{Lie}_k(J(C_{f,N}))_{\chi_N^a}, \tag{2.25}$$

where χ_N is the unique character in $\widehat{G}(k)$ such that $\chi_N(\delta_N) = \xi_N$. Let $\mathbb{Z}_+$ denote the set of nonnegative integers. We write $h : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Z}_+$ for the dimension function defined by

$$h(a) = \dim_k \mathrm{Lie}_k(J(C_{f,N}))_{\chi_N^a}. \tag{2.26}$$

2.11. Assume that k has characteristic zero, and $\xi_N = \zeta_N = e^{2\pi i/N}$. We force $G \cong \mathbb{Z}/N\mathbb{Z}$ to act on $\Gamma(C_{f,N}, \Omega^1_{C_{f,N}})$ from the left by taking the inverse of the pullback. It follows from Subsection 2.9 that

$$h(a) = \dim_k \Gamma(C_{f,N}, \Omega^1_{C_{f,N}})_{\chi_N^{-a}}.$$

If $f(x)$ has no multiple roots and $N \nmid n$, then

$$\left\{ \frac{x^{b-1}dx}{y^a} \,\middle|\, 1 \le a < N, 1 \le b \le \left\lfloor \frac{na}{N} \right\rfloor \right\} \tag{2.27}$$

is a basis for $\Gamma(C_{f,N}, \Omega^1_{C_{f,N}})$ by [27, Proposition 2]. Clearly, each $x^{b-1}dx/y^a$ is an eigenvector for $(\delta_N^{-1})^*$ corresponding to eigenvalue ζ_N^a. In particular, if $f(x)$ has no multiple roots, and $N \nmid n$, then

$$h(a) = n - 1 - \left\lfloor \frac{na}{N} \right\rfloor. \tag{2.28}$$

Here $\lfloor t \rfloor$ is the smallest integer less or equal to t (i.e., the floor function). In the floor function of (2.28), we take a to be the unique integer between 0 and $N-1$ for the corresponding residue class. One easily checks that $h(a) + h(-a) = n - 1$ for the function h in (2.28).

Let E be a number field, and k be a field of characteristic zero that contains all conjugates of E. Let $\Sigma_E^k = \{\sigma \mid \sigma : E \hookrightarrow k\}$ be the set of all embeddings of E into k. (We'll drop the superscript k if $k = \mathbb{C}$). Any (E, k)-bimodule V splits into a direct sum of k-vector spaces $V = \bigoplus_{\sigma \in \Sigma_E^k} V_\sigma$, where $V_\sigma := \{v \in V \mid e \cdot v = \sigma(e)v, \forall\, e \in E\}$. Mimicking the definition of CM types, we make the following definition.

Definition 2.4. Let E and k be as above. Suppose that $(X, \mathfrak{i})$ is a pair consisting of an abelian variety X/k together with an embedding $\mathfrak{i} : E \hookrightarrow \mathrm{End}_k^0(X)$. Then $\mathrm{Lie}_k(X)$ is naturally an (E, k)-bimodule. The function $h : \Sigma_E^k \to \mathbb{Z}_+$ defined by

$$h(\sigma) = \dim_k \mathrm{Lie}_k(X)_\sigma$$

is called the *generalized multiplication type* of $(X, \mathfrak{i})$.

2.12. Let the assumptions be the same as in Subsection 2.11. Consider the pair $(J_{f,N}^{\mathrm{new}}, \mathfrak{i})$ with $\mathfrak{i} : \mathbb{Q}(\zeta_N) \hookrightarrow \mathrm{End}^0(J_{f,N}^{\mathrm{new}})$ given in (2.14). Then

$$\Sigma_{\mathbb{Q}(\zeta_N)}^k = \{\sigma_a \mid a \in (\mathbb{Z}/N\,\mathbb{Z})^\times, \text{ and } \sigma_a : E \hookrightarrow k, \quad \zeta_N \mapsto \xi_N^a\},$$

which is naturally identified with the set $(\mathbb{Z}/N\,\mathbb{Z})^\times$. One easily sees that

$$\mathrm{Lie}_k(J_{f,N}^{\mathrm{new}})_{\sigma_a} = \mathrm{Lie}_k(J_{f,N}^{\mathrm{new}})_{\chi_N^a}.$$

Therefore, the generalized multiplication type of $(J_{f,N}^{\mathrm{new}}, \mathfrak{i})$ is given by (2.28) under the aforementioned assumptions on $f(x)$ and N.

2.13. Let $\iota : \mathbb{Q}(\zeta_N) \to \mathbb{Q}(\zeta_N)$ be the complex conjugation, $\bar{\mathfrak{i}} := \mathfrak{i} \circ \iota$, and let $\bar{h}$ be the multiplication type of $(J_{f,N}^{\mathrm{new}}, \bar{\mathfrak{i}})$. Then

$$\bar{h}(a) = h(-a) = \dim_k \Gamma(C_{f,N}, \Omega^1_{C_{f,N}})_{\chi_N^a}. \tag{2.29}$$

This saves us the trouble of going from the left representation $\Gamma(C_{f,N}, \Omega^1_{C_{f,N}})$ of G to its dual representation $\mathrm{Lie}_k(J(C_{f,N}))$ in some calculations. Moreover, (2.28) takes a simpler form

$$\bar{h}(a) = \left\lfloor \frac{na}{N} \right\rfloor. \tag{2.30}$$

Therefore, it is more convenient to replace $(\mathfrak{i}, h)$ with $(\bar{\mathfrak{i}}, \bar{h})$, which we will do in the next section.

3. Complex abelian varieties with given multiplication type

Throughout this section, $E := \mathbb{Q}(\zeta_N)$ is the N-th cyclotomic field, and $(X, \mathfrak{i})$ will denote a pair consisting of a complex abelian variety X together with an embedding $\mathfrak{i} : E \hookrightarrow \mathrm{End}^0(X)$. We will identify E with its image in $\mathrm{End}^0(X)$ via $\mathfrak{i}$ and write $E \subseteq \mathrm{End}^0(X)$. Let $h : (\mathbb{Z}/N\,\mathbb{Z})^\times \to \mathbb{Z}_+$ be the generalized multiplication type of $(X, \mathfrak{i})$. We will classify $\mathrm{End}^0(X)$, using arithmetic properties of h. In the case $X = J_{f,N}^{\mathrm{new}}$, we choose $\mathfrak{i}$ so that h is given by (2.30).

3.1. In general, let $\mathcal{E}$ be a number field, and $g : \Sigma_{\mathcal{E}} \to \mathbb{Z}_+$ be the generalized multiplication type of a pair $(Z, \mathfrak{j})$ consisting of a complex abelian variety Z together with an embedding $\mathfrak{j} : \mathcal{E} \to \mathrm{End}^0(Z)$. The first rational homology group $V_{\mathbb{Q}} := H_1(Z, \mathbb{Q})$ carries naturally a structure of faithful $\mathrm{End}^0(Z)$-module, and hence a structure of $\mathcal{E}$-vector space of dimension $2\dim Z/[\mathcal{E} : \mathbb{Q}]$. In particular, $V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ is a free $\mathcal{E} \otimes_{\mathbb{Q}} \mathbb{C}$ module of rank $2\dim Z/[\mathcal{E} : \mathbb{Q}]$. That is,

$$H_1(Z, \mathbb{C}) = H_1(Z, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma \in \Sigma_{\mathcal{E}}} H_1(Z, \mathbb{C})_\sigma,$$

where each $H_1(Z, \mathbb{C})_\sigma$ is a complex vector space of dimension $2\dim Z/[\mathcal{E} : \mathbb{Q}]$. On the other hand, we have the Hodge decomposition [21, Chapter 1],

$$H_1(Z, \mathbb{C}) = H^{-1,0}(Z) \oplus H^{0,-1}(Z),$$

where $H^{-1,0}(Z)$ and $H^{0,-1}(Z)$ are mutually complex conjugate $\mathbb{C}$-vector spaces of dimension $\dim(Z)$. The splitting is $\mathrm{End}^0(Z)$-invariant and the $\mathrm{End}^0(Z)$-module $H^{-1,0}(Z)$ is canonically isomorphic to $\mathrm{Lie}_{\mathbb{C}}(Z)$. For any $\sigma \in \Sigma_{\mathcal{E}}$, we write $\bar{\sigma}$ for the composition of $\mathcal{E} \xrightarrow{\sigma} \mathbb{C}$ with the complex conjugation map $\mathbb{C} \to \mathbb{C}$. Then

$$H_1(Z, \mathbb{C})_\sigma \cong \mathrm{Lie}_{\mathbb{C}}(Z)_\sigma \oplus \overline{\mathrm{Lie}_{\mathbb{C}}(Z)_{\bar{\sigma}}}.$$

Therefore,

$$g(\sigma) + g(\bar{\sigma}) = 2\dim Z/[\mathcal{E} : \mathbb{Q}]. \tag{3.1}$$

3.2. Let $\mathrm{End}^0(X, \mathfrak{i})$ be the centralizer of $\mathfrak{i}(E)$ in $\mathrm{End}^0(X)$. As $\mathrm{End}^0(X)$ itself is a semisimple $\mathbb{Q}$-algebra, $\mathrm{End}^0(X, \mathfrak{i})$ is a semisimple E-algebra. If $m = 2\dim X/\varphi(N)$, we have

$$E \subseteq \mathrm{End}^0(X, \mathfrak{i}) \subseteq \mathrm{End}_E(H_1(X, \mathbb{Q})) \cong \mathrm{Mat}_m(E).$$

Suppose that $\dim X = \varphi(N) = [E : \mathbb{Q}]$, then we have the following possibilities:

$$\mathrm{End}^0(X, \mathfrak{i}) = \begin{cases} E, \\ L, \\ E \oplus E, \\ \mathrm{Mat}_2(E), \end{cases} \tag{3.2}$$

where L/E is a field extension of degree 2. In the last three cases, X is an abelian variety of CM type, as observed in [33, Theorem 3.1]. We claim that $\mathrm{End}^0(X, \mathfrak{i}) \neq \mathrm{Mat}_2(E)$ if there exists $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ with $h(a) = 1$. Indeed, as in Subsection 2.12,

$$\mathrm{Lie}_{\mathbb{C}}(X) = \bigoplus_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathrm{Lie}_{\mathbb{C}}(X)_a,$$

and each $\mathrm{Lie}(X)_a$ is a $\mathrm{End}^0(X, \mathfrak{i})$-invariant complex vector space of dimension $h(a)$. On the other hand, $\mathrm{Mat}_2(E) \otimes_{\mathbb{Q}} \mathbb{C} \cong \oplus_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \mathrm{Mat}_2(\mathbb{C})$, and a minimal module of $\mathrm{Mat}_2(\mathbb{C})$ is 2-dimensional.

Lemma 3.1. *Suppose that $N \notin \{3, 4, 6, 10\}$ and $f(x) \in \mathbb{C}[x]$ is a polynomial of degree 3 with no multiple roots. Then $\mathrm{End}^0(J_{f,N}^{\mathrm{new}}, \mathfrak{i}) \neq \mathrm{Mat}_2(E)$.*

Proof. The multiplication type function h is given by (2.30). It takes value 1 for some $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ by Proposition 5.1 if $N \notin \{3, 4, 6, 10\}$. □

Remark 3.2. Let $\lambda_X : X \to X^\vee$ be a polarization on X that induces a Rosati involution $\alpha \mapsto \alpha^\dagger$ on $\mathrm{End}^0(X)$. Suppose that E is invariant under the Rosati involution (i.e., $E^\dagger = E$), then its centralizer $\mathrm{End}^0(X, \mathfrak{i})$ is also invariant under the Rosati involution. In particular, if $\mathrm{End}^0(X, \mathfrak{i}) = L$, then L is a CM field. This holds if $X = J_{f,N}^{\mathrm{new}}$ and we take λ_X to be the restriction of the canonical principal polarization of $J(C_{f,N})$ to $J_{f,N}^{\mathrm{new}}$.

3.3. For $s \in (\mathbb{Z}/N\mathbb{Z})^\times$, we write $\theta_s : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$ for the multiplication by s map: $a \mapsto sa$. The generalized multiplication type $h : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Z}_+$ is said to be *primitive* if $h \circ \theta_s \neq h$ for all $s \neq 1$. Suppose that $\dim X = \varphi(N)(n-1)/2$, and h is given by (2.30). Then h is primitive if one of the following conditions holds:

- $\gcd(n, N) = 1$ (by Proposition 5.2);
- $n = 3$, and $N = 3^r \geq 9$ (by [30, Lemma 4.2]).

For the rest of the section, we will carry out a case by case study of the first three cases of (3.2). The case when $\mathrm{End}^0(X, \mathfrak{i}) = E$ and when $N = q = p^r$ is a prime power was treated in [33] for $n = \deg f = 3, 4$, and in [32] and [34] for $n \geq 5$, where it was assumed that $\gcd(q, n) = \gcd(p, n) = 1$. The case when $q = p^r$ and $p \mid n$ was treated in [30]. We will extend these results to a more general N.

First, we state the following theorem of Zarhin [31, Theorem 2.3].

Theorem 3.3. *Let the notation be the same as in Subsection* 3.1. *Suppose that $\mathcal{E}$ (identified with its image via $\mathfrak{j}$) contains the center $\mathfrak{C}_Z$ of* $\mathrm{End}^0(Z)$, *and $\mathcal{E}/\mathfrak{C}_Z$ is Galois, then*

$$g(\sigma \circ \kappa) = g(\sigma), \qquad \forall\, \sigma \in \Sigma_{\mathcal{E}}, \forall\, \kappa \in \mathrm{Gal}(\mathcal{E}/\mathfrak{C}_Z).$$

The statement of [31, Theorem 2.3] is restricted to the case that $\mathcal{E}/\mathbb{Q}$ is Galois. However, its proof shows that the theorem holds as long as $\mathcal{E}$ is Galois over $\mathfrak{C}_Z$.

Lemma 3.4. *Let $\mathfrak{C}_X$ be the center of* $\mathrm{End}^0(X)$. *Suppose that $\mathfrak{C}_X \subseteq E$, and the generalized multiplication type $h : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Z}_+$ of $(X, \mathfrak{i})$ is primitive, then $\mathfrak{C}_X = E$.*

Proof. If $E \neq \mathfrak{C}_X$, by Theorem 3.3, there exists $s \in (\mathbb{Z}/N\mathbb{Z})^\times$, $s \neq 1$ such that $h(a) = h(sa)$ for all $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, which is not the case by our assumption. □

The next proposition generalizes [33, Theorem 4.2] and [32, Corollary 2.2] and follows the main idea of their proofs.

Proposition 3.5. *If* $\mathrm{End}^0(X, \mathfrak{i}) = E$, *and the generalized multiplication type $h : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Z}_+$ of $(X, \mathfrak{i})$ is primitive, then X is absolutely simple, and* $\mathrm{End}^0(X) = E$.

Proof. Since the centralizer of E in $\mathrm{End}^0(X)$ coincides with E, the center $\mathfrak{C}_X$ of $\mathrm{End}^0(X)$ is contained in E. By Lemma 3.4, we have $E = \mathfrak{C}_X$. Hence $\mathrm{End}^0(X) = \mathrm{End}^0(X, \mathfrak{i})$, which equals to E by assumption. □

Proposition 3.6. *Suppose* $\dim X = \varphi(N)$. *If* $\mathrm{End}^0(X, \mathfrak{i}) = L$, *a quadratic extension of E, and the generalized multiplication type h of $(X, \mathfrak{i})$ is primitive, then X is absolutely simple, and* $\mathrm{End}^0(X) = L$.

Proof. As X is an abelian variety of CM type, so by [25, Section II.5], we see that

(i) X is isogenous to a product $Y \times \cdots \times Y$ with a simple abelian variety Y.

(ii) $F := \mathrm{End}^0(Y)$ is a CM subfield of L with $[F : \mathbb{Q}] = 2 \dim Y$.
(iii) $\mathrm{End}^0(X) \cong \mathrm{Mat}_t(F)$, where $t = [L : F]$ is the number of factors of Y in the product $Y \times \cdots \times Y$.
(iv) The center $\mathfrak{C}_X$ of $\mathrm{End}^0(X)$ coincides with F.

Note that we have a tower of fields $E \subseteq EF \subseteq L$. Since $[L : E] = 2$, either $EF = E$ or $EF = L$. We claim that $EF = L$. Suppose otherwise, then $\mathfrak{C}_X = F \subseteq E$. By Lemma 3.4, $E = F$ and hence

$$\mathrm{End}^0(X, \mathfrak{i}) = \mathrm{End}^0(X) \cong \mathrm{Mat}_t(F) \neq L.$$

This contradicts our assumption. Therefore, $EF = L$. If $L = F$, then $t = 1$, and $X = Y$ is simple. Furthermore,

$$\mathrm{End}^0(X) = F = L = \mathrm{End}^0(X, \mathfrak{i}).$$

So for the rest of the proof, we assume that $EF = L$, $F \neq L$ and show that this leads to a contradiction.

Since $E = \mathbb{Q}(\zeta_N)$ is Galois over $\mathbb{Q}$, $L = EF$ is Galois over F with $\mathrm{Gal}(L/F) \subset \mathrm{Gal}(E/\mathbb{Q})$. Let $F_0 := F \cap E$. Then $[F : F_0] = [L : E] = 2$, and F/F_0 is Galois as well. By [17, Theorem VI.1.14] , L/F_0 is Galois, with $\mathrm{Gal}(L/F_0) = \mathrm{Gal}(L/F) \times \mathrm{Gal}(L/E)$. We write ι for the unique generator of $\mathrm{Gal}(L/E) \subset \mathrm{Gal}(L/F_0)$. It commutes with all elements of $\mathrm{Gal}(L/F)$.

Let $g : \Sigma_L \to \mathbb{Z}_+$ be the generalized multiplication type of $(X, L \hookrightarrow \mathrm{End}^0(X))$, and $h_0 : \Sigma_F \to \mathbb{Z}_+$ be the CM type of $(Y, F \hookrightarrow \mathrm{End}^0(Y))$. By (3.1), both g and h_0 take values 0 and 1 only. Since $\mathrm{Lie}_{\mathbb{C}}(X)$ is isomorphic to the direct sum of t copies of $\mathrm{Lie}_{\mathbb{C}}(Y)$, g is induced from h_0 in the following sense:

$$g(\sigma) = 1 \Leftrightarrow h_0(\sigma\mid_F) = 1, \qquad \forall\, \sigma \in \Sigma_L.$$

In particular, $g(\sigma\kappa) = g(\sigma)$ for all $\kappa \in \mathrm{Gal}(L/F)$. On the other hand,

$$h(\sigma\mid_E) = g(\sigma) + g(\sigma\iota), \qquad \forall\, \sigma \in \Sigma_L.$$

It follows that for any $\kappa \in \mathrm{Gal}(L/F)$, $\sigma \in \Sigma_L$,

$$\begin{aligned} h((\sigma\mid_E) \circ (\kappa\mid_E)) &= h(\sigma\kappa\mid_E) = g(\sigma\kappa) + g(\sigma\kappa\iota) = g(\sigma\kappa) + g(\sigma\iota\kappa) \\ &= g(\sigma) + g(\sigma\iota) = h(\sigma\mid_E). \end{aligned}$$

This again contradicts the assumption that h is primitive. □

Corollary 3.7. *Suppose that $f(x) \in \mathbb{C}[x]$ is a polynomial of degree 3 with no multiple roots, $3 \nmid N$ and $N \notin \{4, 10\}$. Suppose further that $\mathrm{End}^0(J_{f,N}^{\mathrm{new}}) \supseteq L \supset E$, where L is a quadratic field extension of E. Then $\mathrm{End}^0(J_{f,N}^{\mathrm{new}}) = L$.*

Proof. Clearly, $\mathrm{End}^0(J_{f,N}^{\mathrm{new}}, \mathfrak{i}) \supseteq L$. By Lemma 3.1, $\mathrm{End}^0(J_{f,N}^{\mathrm{new}}, \mathfrak{i}) \neq \mathrm{Mat}_2(E)$ since $N \neq 4, 10$. It follows from (3.2) that $\mathrm{End}^0(J_{f,N}^{\mathrm{new}}, \mathfrak{i}) = L$. We have mentioned in Subsection 3.3 that the multiplication type of $(J_{f,N}^{\mathrm{new}}, \mathfrak{i})$ is primitive, so the corollary follows from Proposition 3.6. □

We also give the proof of Theorem 1.2.

Proof of Theorem 1.2. By Subsection 3.3, the generalized multiplication type of $(J_{f,q}^{\mathrm{new}}, \mathfrak{i})$ is primitive. It is enough to show that $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}, \mathfrak{i})$ is a simple $\mathbb{Q}$-algebra. Then by (3.2), $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}, \mathfrak{i})$ is either E or a quadratic field extension L of E. We

conclude that $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}) = \mathrm{End}^0(J_{f,q}^{\mathrm{new}}, \mathfrak{i})$ by Proposition 3.5 if $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}, \mathfrak{i}) = E$, or by Proposition 3.6 if $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}, \mathfrak{i}) = L$.

The group $\mathbf{S}_3$ is doubly transitive and k is assumed to contain $\mathbb{Q}(\zeta_q)$. This allows us to first apply [33, Theorem 5.13] since $p \neq 3$ by assumption, and then apply [34, Lemma 3.8]. In the end, we combine the proof of [34, Theorem 3.12] together with Proposition 5.1 using the assumption that $q \neq 4$ to show that $\mathrm{End}^0(J_{f,q}^{\mathrm{new}}, \mathfrak{i})$ is indeed a simple $\mathbb{Q}$-algebra. □

Lemma 3.8. *Let $\mathcal{E}$ be a CM field which is Galois over $\mathbb{Q}$, and let $(Y_i, \mathfrak{j}_i)_{i=1,2}$ be a pair of complex CM abelian varieties of dimension $[\mathcal{E} : \mathbb{Q}]/2$ with embeddings $\mathfrak{j}_i : \mathcal{E} \hookrightarrow \mathrm{End}^0(Y_i)$. Suppose that $g_i : \Sigma_{\mathcal{E}} \to \{0, 1\}$ is the CM type of $(Y_i, \mathfrak{j}_i)$. Then Y_1 is isogenous to Y_2 if and only if there exists $\theta \in \mathrm{Gal}(\mathcal{E}/\mathbb{Q})$ such that $g_1(\sigma\circ\theta) = g_2(\sigma)$ for all $\sigma \in \Sigma_{\mathcal{E}}$.*

Proof. If such a $\theta \in \mathrm{Gal}(\mathcal{E}/\mathbb{Q})$ exists, let $\tilde{\mathfrak{j}}_1 = \mathfrak{j}_1 \circ \theta^{-1}$. Then the CM type of $(Y_1, \tilde{\mathfrak{j}}_1)$ coincides with that of $(Y_2, \mathfrak{j}_2)$. Hence $Y_1 \sim Y_2$ by [25, Section II.6.1]. On the other hand, suppose that $Y_1 \sim Y_2$. By [25, Section 2.5], there exists a simple CM abelian variety Z with $Y_1 \sim Y_2 \sim Z^m$. Let $F := \mathrm{End}^0(Z)$, and $g_Z : \Sigma_F \to \{0, 1\}$ be the CM type of Z. For each $i = 1, 2$, there is an embedding $\tau_i : F \hookrightarrow \mathcal{E} \cong \mathfrak{j}_i(\mathcal{E})$, and $g_i(\sigma) = g_Z(\sigma \circ \tau_i)$ for all $\sigma \in \Sigma_{\mathcal{E}}$. Since $\mathcal{E}/\mathbb{Q}$ is Galois, we may find $\theta \in \mathrm{Gal}(\mathcal{E}/\mathbb{Q})$ such that $\theta \circ \tau_1 = \tau_2$. Then for all $\sigma \in \Sigma_{\mathcal{E}}$, we have $g_1(\sigma \circ \theta) = g_Z(\sigma \circ \theta \circ \tau_1) = g_Z(\sigma \circ \tau_2) = g_2(\sigma)$. □

3.4. Suppose that $\dim X = \varphi(N)$ and $\mathrm{End}^0(X, \mathfrak{i}) = E \oplus E$. Then X is isogenous to $Y_1 \times Y_2$, and each Y_i is an abelian variety of dimension $\varphi(N)/2$ with complex multiplication by E. Let $p_i : E \oplus E \to E$ be the projection onto i-th factor for $i = 1, 2$, and $\mathfrak{j}_i : E \hookrightarrow \mathrm{End}^0(Y_i)$ be the composition of $E \xrightarrow{\mathfrak{i}} \mathrm{End}^0(X, \mathfrak{i}) \xrightarrow{p_i} E \subseteq \mathrm{End}^0(Y_i)$. If we write g_i for the CM type of $(Y_i, \mathfrak{j}_i)$, then $h = g_1 + g_2$. We have

- $Y_1 \sim Y_2$ if and only if $\exists s \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $g_1 \circ \theta_s = g_2$ by Lemma 3.8.
- Y_i is simple if and only if g_i is primitive ([25, Section II.8.2]).

When both Y_1 and Y_2 are simple, $\mathrm{End}^0(Y_1) = \mathrm{End}^0(Y_2) = E$. If $Y_1 \not\sim Y_2$, then $\mathrm{End}^0(X) = \mathrm{End}^0(Y_1) \oplus \mathrm{End}^0(Y_2) = E \oplus E$; otherwise $Y_1 \sim Y_2$, and we have $\mathrm{End}^0(X) = \mathrm{Mat}_2(E)$. On the other hand, say Y_1 is not simple, then the group $\{s \in (\mathbb{Z}/N\mathbb{Z})^\times \mid g_1 \circ \theta_s = g_1\}$ is nontrivial. Let t be the order of this group, and let F be its fixed subfield in $\mathbb{Q}(\zeta_N)$. Then $Y_1 \sim Z^t$, where Z is a simple complex abelian variety with complex multiplication by F. In particular, $\mathrm{End}^0(Y_1) = \mathrm{Mat}_t(F)$.

3.5. Since g_i only takes value in $\{0, 1\}$, $h(a) = 0$ if and only if both $g_1(a)$ and $g_2(a)$ are zero. Note that $h(a) = \lfloor 3a/N \rfloor$ takes value 0 for all $1 \leq a < N/3$ and $\gcd(a, N) = 1$, so the same holds for both $g_1(a)$ and $g_2(a)$. Let $\mathscr{T}_N$ be the set of functions

$$\mathscr{T}_N = \{g : (\mathbb{Z}/N\mathbb{Z})^\times \to \{0, 1\} \mid g(a) + g(-a) = 1, g(a) = 0 \text{ if } 1 \leq a < N/3\}. \tag{3.3}$$

Suppose that $N = q = p^r$ is a prime power. We are interested in the set

$$\mathscr{S}_q := \{s \in (\mathbb{Z}/q\mathbb{Z})^\times \mid s \neq 1, \text{and } \exists g \in \mathscr{T}_q \text{ such that } g \circ \theta_s \in \mathscr{T}_q\}. \tag{3.4}$$

Clearly, $s \in \mathscr{S}_q$ if and only if $s^{-1} \in \mathscr{S}_q$. In Section 5, it will be shown

$$\mathscr{S}_q = \begin{cases} \{2, (q+1)/2\} & \text{if } p \geq 5, \\ \{2, (q+1)/2, q/3-1, 2q/3-1\} & \text{if } p = 3 \text{ and } q \geq 9, \\ \{q/2-1\} & \text{if } p = 2 \text{ and } q \geq 16. \end{cases} \tag{3.5}$$

Moreover, for each $s \in \mathscr{S}_q$, there exists a unique $g \in \mathscr{T}_q$ such that $g \circ \theta_s \in \mathscr{T}_q$. It follows that Y_i fails to be simple only in a few exceptional cases.

Proposition 3.9. *Let $q = p^r$ be a prime power with p odd. Assume that $q \neq 7$, and $q \geq 9$ if $p = 3$. Let $(Y, \mathfrak{j})$ be a pair consisting of a complex abelian variety Y of dimension $\varphi(q)/2$ and an embedding $\mathfrak{j} : \mathbb{Q}(\zeta_q) \hookrightarrow \mathrm{End}^0(Y)$. Suppose that the CM type of $(Y, \mathfrak{j})$ is given by a function $g : (\mathbb{Z}/q\,\mathbb{Z})^\times \to \{0, 1\}$ such that $g(a) = 0$ for all $1 \leq a < q/3$. Then Y is simple with $\mathrm{End}^0(Y) \cong \mathbb{Q}(\zeta_q)$.*

Proof. We need to show that g is primitive. If $p \neq 3$ and $q \neq 7$, this is shown in Corollary 5.5. If $p = 3$ and $q \geq 9$, this is shown in Corollary 5.11. □

Proposition 3.10. *Assume that $\dim X = \varphi(N)$, the multiplication type h of $(X, \mathfrak{i})$ is given by (2.30) with $n = 3$, and $\mathrm{End}^0(X, \mathfrak{i}) = E \oplus E$. Assume further that $N = q = p^r$ is a prime power, $q > 3$ if $p = 3$, and $q > 4$ if $p = 2$. Then $\mathrm{End}^0(X)$ is classified by the following list:*

- *If $p \geq 5$ and $q \neq 5, 7$, then $\mathrm{End}^0(X) = E \oplus E$.*
- *If $q = 5, 9$, then $\mathrm{End}^0(X) = \mathrm{Mat}_2(E)$.*
- *If $q = 7$, then $\mathrm{End}^0(X) = \mathrm{Mat}_3(\mathbb{Q}(\sqrt{-7})) \oplus \mathbb{Q}(\zeta_7)$.*
- *If $q = 3^r \geq 27$, then $\mathrm{End}^0(X)$ is either $E \oplus E$ or $\mathrm{Mat}_2(E)$.*
- *If $q = 8$, then $\mathrm{End}^0(X) = \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-1})) \oplus \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-2}))$.*
- *If $q = 2^r \geq 16$, then $\mathrm{End}^0(X) = E \oplus E$ or $E \oplus \mathrm{Mat}_2(\mathbb{Q}(\alpha))$, where $\alpha = 2\sqrt{-1}\sin(2\pi/q)$.*

Proof. First suppose that $p \geq 5$. By Lemma 5.7, $Y_1 \not\sim Y_2$ if $q \neq 5$. By Proposition 3.9, both Y_1 and Y_2 are simple if $q \neq 7$. It follows that $\mathrm{End}^0(X) = E \oplus E$ when $q \neq 5, 7$. If $q = 5$, then g_1 and g_2 are uniquely determined (up to relabeling) by h, and $g_1 = g_2 \circ \theta_2$ by Remark 5.8. So $Y_1 \sim Y_2$, and $\mathrm{End}^0(X) = \mathrm{Mat}_2(\mathbb{Q}(\zeta_5))$. Similarly, if $q = 7$, by Remark 5.6, g_1 and g_2 are uniquely determined by h up to relabeling. One checks that g_2 is primitive, hence $\mathrm{End}^0(Y_2) = \mathbb{Q}(\zeta_7)$; and

$$g_1 \circ \theta_s = g_1 \Leftrightarrow s \in \langle 2 \rangle \subset (\mathbb{Z}/7\,\mathbb{Z})^\times.$$

So the CM type g_1 is induced from $\mathbb{Q}(\sqrt{-7})$, the fixed subfield of $\mathbb{Q}(\zeta_7)$ by $\langle 2 \rangle \subset \mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$. Therefore, $Y_1 \sim Z^3$, where Z is an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-7})$, and $\mathrm{End}^0(Y_1) = \mathrm{Mat}_3(\mathbb{Q}(\sqrt{-7}))$.

If $p = 3$ and $q = 3^r \geq 9$, then by Proposition 3.9, both Y_1 and Y_2 are simple and $\mathrm{End}^0(Y_i) = E$. If $q = 9$, then by Remark 5.12, there is a unique way (up to labeling) to write $h = g_1 + g_2$, and $g_1 = g_2 \circ \theta_2$. Therefore, $Y_1 \sim Y_2$, and $\mathrm{End}^0(X) = \mathrm{Mat}_2(E)$. If $q \geq 27$, $\mathrm{End}^0(X)$ depends on the specific form of g_i. Suppose that g_1 is of the form given by (5.10). Then $g_2 = g_1 \circ \theta_s$ with $s = q/3 - 1$. So $Y_1 \sim Y_2$ and $\mathrm{End}^0(X) = \mathrm{Mat}_2(E)$. Otherwise, $Y_1 \not\sim Y_2$ by Lemma 5.13, and $\mathrm{End}^0(X) = E \oplus E$.

If $q = 8$, once again, g_1 and g_2 are uniquely determined up to labeling. By Remark 5.14, $g_1 \circ \theta_5 = g_1$, and $g_2 = g_2 \circ \theta_3$. The fixed subfield of $\mathbb{Q}(\zeta_8) = \mathbb{Q}(\sqrt{-1}, \sqrt{-2})$ by $\langle 5 \rangle \subset (\mathbb{Z}/8\,\mathbb{Z})^\times$ is $\mathbb{Q}(\sqrt{-1})$, so $\mathrm{End}^0(Y_1) = \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-1}))$ and Y_1 is isogenous

to the square of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-1})$. Similarly, Y_2 is isogenous to the square of an elliptic curve with complex multiplication by $\mathbb{Q}(\sqrt{-2})$. Therefore, $\mathrm{End}^0(X) = \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-1})) \oplus \mathrm{Mat}_2(\mathbb{Q}(\sqrt{-2}))$.

If $q = 2^r \geq 16$, then $\mathrm{End}^0(X)$ depends on the specific form of g_i's. If g_1 is of the form given in (5.9), by Lemma 5.15,

$$g_1 \circ \theta_s = g_1 \Leftrightarrow s = 1, 2^{r-1} - 1.$$

One easily checks that the fixed subfield of $\mathbb{Q}(\zeta_q)$ by $\langle 2^{r-1} - 1 \rangle$ is $\mathbb{Q}(\alpha)$. By Lemma 5.15 again, $g_2 = h - g_1$ is primitive. So $\mathrm{End}^0(X) = \mathrm{Mat}_2(\mathbb{Q}(\alpha)) \oplus E$. If neither g_1 nor g_2 is of the form in (5.9), then both g_i are primitive, and $Y_1 \not\sim Y_2$ by Lemma 5.16. Therefore, $\mathrm{End}^0(X) = E \oplus E$. □

Proof of Theorem 1.1. By Lemma 3.1, we only need to study the first three cases of (3.2). By Subsection 3.3, the multiplication type h given by (2.30) is primitive. If $\mathrm{End}^0(J^{\mathrm{new}}_{f,q}, \mathfrak{i}) = E$ (or L), then Proposition 3.5 (respectively, Proposition 3.6) shows that $\mathrm{End}^0(J^{\mathrm{new}}_{f,q}) = \mathrm{End}^0(J^{\mathrm{new}}_{f,q}, \mathfrak{i})$. In both cases, $J^{\mathrm{new}}_{f,q}$ is absolutely simple. This takes care of part (1) of the theorem. Part (2) of the theorem corresponds to the case $\mathrm{End}^0(J^{\mathrm{new}}_{f,q}, \mathfrak{i}) = E \oplus E$, which is treated in Proposition 3.10. □

3.6. Recall that $C_{\lambda,q}$ denotes the curve

$$y^q = x(x-1)(x-\lambda),$$

where λ lies on the punctured complex plane with the points 0 and 1 removed. Fix λ such that $J^{\mathrm{new}}_{q,\lambda}$ is not of CM type (such a λ exists if $q \neq 4$; see [33, Example 2.3 and Theorem 5.18], also cf. Examples 1.7 (1a)). We may construct a Shimura datum (G, X) from $J^{\mathrm{new}}_{q,\lambda}$ in the following way. Let V be the $\mathbb{Q}$-vector space $H_1(J^{\mathrm{new}}_{q,\lambda}, \mathbb{Q})$. It carries a natural structure of $\mathbb{Q}(\zeta_q)$-vector spaces of dimension 2. The canonical principal polarization on $J(C_{\lambda,q})$ induces on $V \subseteq H_1(C_{\lambda,q}, \mathbb{Q})$ a nondegenerate alternating $\mathbb{Q}$-bilinear form ψ which satisfies the condition

$$\psi(eu, v) = \psi(u, \bar{e}v), \qquad \forall\, e \in \mathbb{Q}(\zeta_q), \forall\, u, v \in V.$$

Let $\mathrm{CSp}(V, \psi)$ be the group of symplectic similitudes of ψ, and

$$G = \mathrm{GL}_{\mathbb{Q}(\zeta_q)}(V) \cap \mathrm{CSp}(V, \psi) \subset \mathrm{GL}(V).$$

Let $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$ be the Deligne torus, and $h_0 : \mathbb{S} \to G \otimes_{\mathbb{Q}} \mathbb{R}$ the homomorphism of $\mathbb{R}$-algebraic groups that defines the Hodge structure on V. We set $X \subset \mathrm{Hom}_{\mathbb{R}\text{-grps}}(\mathbb{S}, G \otimes_{\mathbb{Q}} \mathbb{R})$ to be the $G(\mathbb{R})$-conjugacy class of h_0. Let $\mathbb{A}_f$ be the ring of finite adeles of $\mathbb{Q}$, and K a compact open subgroup of $G(\mathbb{A}_f)$. By the moduli interpretation of Shimura varieties of PEL-type ([8, Scholie 4.11]), the classifications of the endomorphism algebra in Theorem 1.1 hold for any abelian variety A (with additional structure) corresponding to a complex point on the Shimura variety

$$\mathrm{Sh}_K(G, X) := G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K.$$

Indeed, [8, Scholie 4.11(a)] shows that the generalized multiplication type of $\mathbb{Q}(\zeta_q)$ on $\mathrm{Lie}_{\mathbb{C}}(A)$ coincides with that of $J^{\mathrm{new}}_{\lambda,q}$ for all such A, and Theorem 1.1 was obtained purely by studying the generalized multiplication types.

4. Automorphisms and construction of examples

In this section, we prove Corollary 1.5 and verify the claims in Example 1.7. First we need a simple lemma.

Lemma 4.1. *Let k be an algebraically closed field whose characteristic is coprime to N, and $f(x) = \prod_{i=1}^n (x-\alpha_i)^{e_i} \in k[x]$ with $\gcd(N, e_i) = 1$ for all $1 \le i \le n$. Let $C_{f,N}$ be the smooth projective curve defined by $y^N = f(x)$, and P_i the unique point on $C_{f,N}(k)$ corresponding to $(\alpha_i, 0)$. Then a divisor $D = \sum_{i=1}^n a_i P_i$ of degree 0 is linearly equivalent to 0 if and only if $a_i e_j \equiv a_j e_i \pmod N$ for all $1 \le i, j \le n$.*

Proof. The proof is modeled on that of [32, Lemma 4.7]. Let $\pi : C_{f,N} \to \mathbb{P}^1$ be the map defined in (2.4), and $k(C_{f,N})$ be the rational function field of $C_{f,N}$. We have $\mathrm{Aut}(\pi) = \mathrm{Gal}(k(C_{f,N})/k(\mathbb{P}^1)) = \langle \delta_N \rangle \cong \mathbb{Z}/N\mathbb{Z}$. Suppose that $d := \gcd(\deg f(x), N)$. We write $\deg f(x) = dm$ and $N = dM$ with $\gcd(m, M) = 1$. Choose $s, t \in \mathbb{Z}$ such that $sm + tM = 1$. Let us consider the following birational transformation

$$x = u^{-s} v^{-M}, \qquad y = u^t v^{-m}.$$

The defining equation for $C_{f,N}$ changes into

$$u^d = \prod_{i=1}^n (1 - \alpha_i u^a v^M)^{e_i}.$$

It follows that there are exactly d points in the set $\pi^{-1}(\infty)$, and they are given by $\infty_j := (\xi_N^{Mj}, 0)$ in (u,v)-coordinates for $1 \le j \le d$. By [16, Proposition 11], $\langle \delta_N \rangle$ acts transitively on the set of points $\{\infty_j\}_{j=1}^d$. The inertia group at each ∞_j is necessarily $\langle \delta_N^d \rangle$. If $M > 1$, then $\pi : C_{f,N} \to \mathbb{P}^1$ is ramified at ∞_j with ramification index M. The divisor $D_\infty := \sum_{j=1}^d \infty_j$ is clearly invariant under $\langle \delta_N \rangle$.

Let div denote the divisor of a rational function in $k(C_{f,N})$. Then

$$\mathrm{div}(x - \alpha_i) = NP_i - MD_\infty, \qquad \mathrm{div}(y) = \sum_{i=1}^n e_i P_i - mD_\infty. \tag{4.1}$$

Given any $z \in k(C_{f,N})$, we have

$$\mathrm{div}(\delta_N^* z) = \delta_N^{-1}(\mathrm{div}(z)). \tag{4.2}$$

Clearly, $k(C_{f,N}) = \bigoplus_{c=0}^{N-1} k(x) \cdot y^c$, where $k(x) \cong k(\mathbb{P}^1)$ is the rational function field in one variable. Each $k(x) \cdot y^c$ is a 1-dimensional eigenspace over $k(x)$ for $\delta_N^* : k(C_{f,N}) \to k(C_{f,N})$. Suppose that $D = \sum_{i=1}^n a_i P_i$ is the divisor of a rational function $z \in k(C_{f,N})$. Since D is δ_N-invariant, $\delta_N^* z$ differs from z by a nonzero scalar by (4.2), so $z = w(x) y^c$ for some $w(x) \in k(x)$ and $0 \le c \le N-1$. By (4.1), $\mathrm{div}(w(x))$ must be supported on the set $\{P_1, \cdots, P_n\} \cup \{\infty_1, \cdots, \infty_d\}$. So $w(x) = \varsigma \prod_{i=1}^n (x - \alpha_i)^{\varpi_i}$ for some $\varsigma \in k^\times$ and $\varpi_i \in \mathbb{Z}$. We have

$$\sum_{i=1}^n a_i P_i = \mathrm{div}(w(x) y^c) = \sum_{i=1}^n (c e_i + N \varpi_i) P_i.$$

Indeed $e_j a_i \equiv e_i a_j \pmod N$ for all $1 \le i, j \le n$.

On the other hand, suppose $D = \sum_{i=1}^n a_i P_i$ is a divisor of degree 0 with $e_j a_i \equiv e_i a_j \pmod N$ for all $1 \le i, j \le n$. Let $0 \le c < N$ be the unique integer that represents $e_i^{-1} a_i \in \mathbb{Z}/N\mathbb{Z}$ for all $1 \le i \le n$. Let $\varpi_i = (a_i - c e_i)/N \in \mathbb{Z}$, and $w(x) = \prod_{i=1}^n (x - \alpha_i)^{\varpi_i}$. Then we claim $D = \mathrm{div}(w(x) y^c)$. The coefficients

of each P_i on both sides match by the construction. On the other hand, $\operatorname{div}(w(x)y^c)$ is not supported on ∞_j for any $1 \le j \le d$ since $cm + \sum_{i=1}^n M\varpi_i = cm + \sum_{i=1}^n M(a_i - ce_i)/N = 0$, where we used the fact that $\deg D = \sum_{i=1}^n a_i = 0$. $\square$

For the rest of this section, we assume that k has characteristic zero, and $f(x) = \prod_{i=1}^3 (x - \alpha_i) \in k[x]$ is a monic polynomial of degree 3 with no multiple roots.

4.1. Suppose that $\gcd(N, 3) = 1$, and $f(x) = \prod_{i=1}^3 (x - \alpha_i)$. The set of fixed points of δ_N on $C_{f,N}(\bar{k})$ is

$$\mathfrak{S} := \{P_i := (\alpha_i, 0)\}_{i=1}^3 \cup \{\infty\},$$

where ∞ is the unique point at infinity for $C_{f,N}$. Choose $s, t \in \mathbb{Z}$ such that $3s + Nt = 1$. We have

$$\begin{aligned} \operatorname{div} y^s (x - \alpha_1)^t &= s(P_1 + P_2 + P_3 - 3\infty) + t(NP_1 - N\infty) \\ &= s(P_1 + P_2 + P_3) + tNP_1 - \infty. \end{aligned} \tag{4.3}$$

4.2. Let $\operatorname{Aut}(C_{f,N})$ be the absolute automorphism group of $C_{f,N}$, and H be the normalizer of $\langle \delta_N \rangle$ in $\operatorname{Aut}(C_{f,N})$. Suppose that $H \neq \langle \delta_N \rangle$. We consider an element ϕ in H but not in $\langle \delta_N \rangle$. Then ϕ permutes elements of $\mathfrak{S}$. We claim that $\phi\infty = \infty$ if $N \neq 2, 4$. Otherwise, say $\phi^{-1}\infty = P_1$, then $P_1 \neq \phi^{-1}P_i$ for all $1 \le i \le 3$. Without loss of generality, we assume that $\phi^{-1}P_2 \neq \infty$. Note that

$$\operatorname{div} \phi^* y = \phi^{-1}P_1 + \phi^{-1}P_2 + \phi^{-1}P_3 - 3P_1. \tag{4.4}$$

Using (4.3) to replace ∞ on the right hand side of (4.4), we get a divisor supported on $\{P_1, P_2, P_3\}$ that is linear equivalent to zero. Comparing the coefficient of $\phi^{-1}P_2$ and P_1, we see that $1 \equiv -3 \pmod N$ by Lemma 4.1, which contradicts the assumption that $N \neq 2, 4$.

4.3. Suppose that $\gcd(N, 3) = 1$ and $N \nmid 4$. Let $k(C_{f,N})$ be the field of rational functions of $C_{f,N}$. The fixed subfield of $k(C_{f,N})$ by $\langle \delta_N \rangle$ is $k(x)$, and every element of H sends $k(x)$ to itself, therefore, the action of H on $k(x)$ induces an embedding $H/\langle \delta_q \rangle \subseteq \operatorname{Aut}(k(x)/k)$. It is well known (cf. [24, Corollary 6.65]) that $\operatorname{Aut}(k(x)/k)$ is isomorphic to the group of all linear fractional transformations over k. Since a linear fractional transformation is uniquely determined by its image on any three distinct points, we see that $H/\langle \delta_q \rangle \hookrightarrow \operatorname{Perm}\{P_1, P_2, P_3\} \cong \mathbf{S}_3$.

Let $\phi_x \in H/\langle \delta_q \rangle$ be the automorphism of $k(x)$ induced by $\phi^* : k(C_{f,N}) \to k(C_{f,N})$. Since $\phi \notin \langle \delta_N \rangle$, ϕ_x is nontrivial, so the order of ϕ_x is either 2 or 3. Since $\phi(\infty) = \infty$, $\phi_x(x) = tx + b$, for some $t, b \in k$. Therefore,

$$\phi_x^2(x) = t^2 x + tb + b, \qquad \phi_x^3(x) = t^3 x + (1 + t + t^2)b.$$

If ϕ_x has order 2, then $t = -1$, and if ϕ_x has order 3, $t = \omega$, where ω is a primitive 3rd root of unity. By changing the x coordinate appropriately, we may assume that $\phi_x(x) = -x$ or $\phi_x(x) = \omega x$ respectively. More explicitly, if $\operatorname{ord} \phi_x = 2$, we replace x by $x - b/2$, and if $\operatorname{ord} \phi_x = 3$, we replace x by $x - b(1 - \omega^2)/3$.

Now since ϕ permutes the points P_1, P_2, P_3 and fixes ∞,

$$\begin{aligned} \operatorname{div} \phi^* f(x) &= \phi^{-1}(\operatorname{div} f(x)) = \phi^{-1}(\operatorname{div} y^N) = \phi^{-1} N(P_1 + P_2 + P_3 - 3\infty) \\ &= N(P_1 + P_2 + P_3 - 3\infty) = \operatorname{div} f(x). \end{aligned}$$

In particular, $\phi^* f(x) = cf(x)$ for some $c \in k^*$. On the other hand, $\phi_x(f(x)) = f(\phi_x x)$. Comparing the coefficients, we see that $f(x) = x^3 + B_0 x$ if ϕ_x has order 2,

and $f(x) = x^3 + C_0$ if ϕ_x has order 3. In both cases, the coefficient of x^2 is zero. We note that for $f(x) = x^3 + A_0x^2 + B_0x + C_0$, there exists a unique $b \in k$ such that the coefficient of x^2 in $f(x-b)$ is zero. Indeed, $b = A_0/3$. It follows that $H/\langle\delta_N\rangle$ does not contain both an element of order 2 and an element of order 3. In other words, $H/\langle\delta_N\rangle \neq \mathbf{S}_3$.

If ϕ_x has order 2, $\phi^*(f(x)) = -f(x)$. It follows that $\phi^*(y^N) = -y^N$. If N is odd, then $\phi^*(y) = -\zeta_N^i y$ for some $i \in \mathbb{Z}/N\mathbb{Z}$. If N is even, then $\phi^*(y) = (\zeta_{2N})^{2i+1}y$ for some $i \in \mathbb{Z}/N\mathbb{Z}$. If ϕ_x has order 3, then $\phi^*(f(x)) = f(x)$, so $\phi^*(y^N) = y^N$. Therefore, $\phi^*(y) = \zeta_N^i y$ for some $i \in \mathbb{Z}/N\mathbb{Z}$.

Theorem 4.2. *Let k be an algebraically closed field of characteristic zero, and $f(x) \in k[x]$ a monic polynomial of degree 3 with no multiple roots. After a unique change of variable of the form $x \mapsto x-b$ for some $b \in k$, we may and will assume that $f(x) = x^3 + B_0x + C_0$. Suppose that $(N,3) = 1$ and $N \neq 2,4$. Let $H \subset \mathrm{Aut}(C_{f,N})$ be the normalizer of $\langle\delta_N\rangle$. We have the following cases:*

- *If $B_0C_0 \neq 0$, then $H = \langle\delta_N\rangle \cong \mathbb{Z}/N\mathbb{Z}$.*
- *If $B_0 = 0$ and $C_0 \neq 0$, then $H \cong \mathbb{Z}/3N\mathbb{Z}$, and H is generated by the automorphism $(x,y) \mapsto (\omega x, \zeta_N y)$.*
- *If $B_0 \neq 0$ and $C_0 = 0$, then $H \cong \mathbb{Z}/2N\mathbb{Z}$, and H is generated by the automorphism $(x,y) \mapsto (-x, \zeta_{2N} y)$.*

4.4. Suppose that $k = \mathbb{C}$, and $f(x) = x(x-1)(x-\lambda) = x^3 - (1+\lambda)x^2 + \lambda x$. We take $b = -(1+\lambda)/3$, then

$$f\left(x + \frac{1+\lambda}{3}\right) = x^3 + \left(\frac{\lambda - 1 - \lambda^2}{3}\right)x - \frac{(1+\lambda)(\lambda-2)(2\lambda-1)}{27}.$$

If it is of the form $x^3 + C_0$, then $\lambda = (1 \pm \sqrt{-3})/2$; if it is of the form $x^3 + B_0x$, then $\lambda = -1, 2, 1/2$.

Proof of Corollary 1.5. It was remarked in Subsection 2.1 that

$$\mathrm{Aut}(C_{f,q}) \subseteq \mathrm{Aut}(J(C_{f,q})) \subseteq \mathrm{End}(J(C_{f,q})).$$

By Theorem 1.1, $\mathrm{End}^0(J(C_{f,q}))$ is commutative if $p > 7$, and hence $\mathrm{Aut}(C_{f,q})$ coincides with the normalizer of $\langle\delta_q\rangle$. So Corollary 1.5 follows directly from Theorem 4.2 and Subsection 4.4. □

For the rest of this section, we will try to construct examples of $J^{\mathrm{new}}_{f,N}$ whose endomorphism algebras take the form given in Proposition 3.6 or Proposition 3.10. Our method is to use curves with extra automorphisms.

4.5. Consider the curve $C_{f_1,N}$ with $f_1(x) = x^3 - x$, $3 \nmid N$, $N \in 2\mathbb{Z}$ even, and $N \notin \{4, 10\}$. Let $\gamma_{2N} \in \mathrm{Aut}(C_{f_1,N})$ be the automorphism defined by $(x,y) \mapsto (-x, \zeta_{2N}y)$. Clearly, $\gamma_{2N}^2 = \delta_N$. So $J^{\mathrm{new}}_{f_1,N}$ is γ_{2N}-invariant, and we have an embedding

$$\mathbb{Q}(\zeta_{2N}) \cong \mathbb{Q}(\zeta_N)[T]/(T^2 - \zeta_N) \hookrightarrow \mathrm{End}^0(J^{\mathrm{new}}_{f_1,N}), \qquad T \mapsto \gamma_{2N}\mid_{J^{\mathrm{new}}_{f_1,N}}.$$

It follows from Corollary 3.7 that $J^{\mathrm{new}}_{f_1,N}$ is absolutely simple with $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,N}) = \mathbb{Q}(\zeta_{2N})$.

4.6. Let f_1 be as above, and let $N = q = p^r$ be a prime power with p odd. If $p = 3$, we assume that $q \geq 9$. Let $\gamma_2 \in \mathrm{Aut}(C_{f_1,q})$ be the automorphism defined by $(x, y) \mapsto (-x, -y)$, then γ_2 commutes with δ_N, so $J^{\mathrm{new}}_{f_1,q}$ is γ_2 invariant. With an abuse of notation, we still write γ_2 for the restriction $\gamma_2 \mid_{J^{\mathrm{new}}_{f_1,q}} \in \mathrm{End}(J^{\mathrm{new}}_{f_1,q})$. Clearly, $\gamma_2^2 = \mathrm{Id}$. We claim that $\gamma_2 \neq \pm 1$. Let $b = (q-1)/2$ and $c = (q+1)/2$; then $\gcd(b, q) = \gcd(c, q) = 1$. By (2.27), both dx/y^b and dx/y^c are differentials of the first kind on $C_{f_1,q}$. Clearly they are eigenvectors corresponding to distinct eigenvalues for $\gamma_2^* : \Gamma(C_{f_1,q}, \Omega^1_{C_{f_1,q}}) \to \Gamma(C_{f_1,q}, \Omega^1_{C_{f_1,q}})$. It follows from (2.25) and Subsection 2.9 that both 1 and -1 are eigenvalues of $d\gamma_2 : \mathrm{Lie}_k(J^{\mathrm{new}}_{f_1,q}) \to \mathrm{Lie}_k(J^{\mathrm{new}}_{f_1,q})$. Let $e_1 := (1 + \gamma_2)/2 \in \mathrm{End}^0(J^{\mathrm{new}}_{f_1,q})$ and $e_2 := (1 - \gamma_2)/2 \in \mathrm{End}^0(J^{\mathrm{new}}_{f_1,q})$. Both e_1 and e_2 are nontrivial idempotents of $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,q}, \mathfrak{i})$ with $e_1 + e_2 = 1$. It follows that $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,q}, \mathfrak{i}) = E \oplus E$. By Proposition 3.10,

- If $p \geq 5$ and $q \neq 5, 7$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,q}) = E \oplus E$.
- If $q = 5, 9$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,q}) = \mathrm{Mat}_2(E)$.
- If $q = 7$, then $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,q}) = \mathrm{Mat}_3(\mathbb{Q}(\sqrt{-7})) \oplus \mathbb{Q}(\zeta_7)$.

We claim that $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,q}) = E \oplus E$ if $q = 3^r \geq 27$. Let $Y_i = 2e_i J^{\mathrm{new}}_{f_1,q}$. Then $\mathrm{Lie}(Y_1)$ is the subspace of $\mathrm{Lie}(J^{\mathrm{new}}_{f_1,q})$ on which $d\gamma_2$ acts as identity. On the other hand, if $q/3 < a < 2q/3$, then dx/y^a is γ_2^*-invariant if and only if a is odd. It follows that g_1 (using notation in Subsection 3.4) is given by (5.6). By Lemma 5.13, $g_2 \neq g_1 \circ \theta_s$ for any $s \in (\mathbb{Z}/q\,\mathbb{Z})^\times$, so $\mathrm{End}^0(J^{\mathrm{new}}_{f_1,q}) = E \oplus E$ as claimed.

4.7. Consider the curve $C_{f_0,N}$ with $f_0(x) = x^3 + 1$, $3 \nmid N$, and $N \neq 4, 10$. Let $\gamma_3 : C_{f_0,N} \to C_{f_0,N}$ be the automorphism defined by $(x, y) \mapsto (\omega x, y)$ for a primitive 3rd root of unity $\omega \in k$. As $C_{f_0,N}/\langle\gamma_3\rangle \cong \mathbb{P}^1$, it follows from (2.3) that $\gamma_3 \in \mathrm{End}(J(C_{f_0,N}))$ satisfies the polynomial equation $T^2 + T + 1 = 0$. Since γ_3 commutes with δ_N, $J^{\mathrm{new}}_{f_0,N}$ is γ_3-invariant, and we have an embedding

$$\mathbb{Q}(\zeta_{3N}) \cong \mathbb{Q}(\zeta_N)[T]/(T^2 + T + 1) \hookrightarrow \mathrm{End}^0(J^{\mathrm{new}}_{f_0,N}), \qquad T \mapsto \gamma_3 \mid_{J^{\mathrm{new}}_{f_0,N}}.$$

Now it follows from Corollary 3.7 that $J^{\mathrm{new}}_{f_0,N}$ is absolutely simple with $\mathrm{End}^0(J^{\mathrm{new}}_{f_0,N}) = \mathbb{Q}(\zeta_{3N})$.

This result in itself is not new. A theorem of Kubota-Hazama [12] states that the Jacobian variety of the curve $y^p = x^\ell + 1$ is absolutely simple if p and ℓ are distinct primes. In particular, it follows that if $N = p \neq 3$ is a prime, then $J^{\mathrm{new}}_{f_0,p} = J(C_{f_0,p})$ is simple. More generally, one could realize $C_{f_0,N}$ as a quotient of the Fermat curve $X_{3N} : x^{3N} + y^{3N} = 1$. Therefore, $J^{\mathrm{new}}_{f_0,N}$ is isogenous to a factor of $J(X_{3N})$. One checks that it corresponds to the triple $(N, 3, 2N - 3) \in (\mathbb{Z}/3N\,\mathbb{Z})^3$. There is only one entry (namely, $2N - 3$) that is coprime to $3N$. So it is of Type I in Aoki's classification, and hence simple if $N > 60$ ([2, Theorem 0.2]).

4.8. Let f_0 and γ_3 be as in Subsection 4.7, and let $N = q = 3^r > 3$. We may assume that $\omega = \zeta_q^{3^{r-1}}$. Let $\delta_3 := \delta_q^{3^{r-1}} \in \mathrm{Aut}(C_{f_0,q})$. By (2.27), both dx/y^{q-1} and xdx/y^{q-1} are differentials of the first kind on $C_{f,q}$, and they are eigenvectors corresponding to eigenvalue ω for $\delta_3^* : \Gamma(C_{f_0,q}, \Omega^1_{C_{f_0,q}}) \to \Gamma(C_{f_0,q}, \Omega^1_{C_{f_0,q}})$. On the other hand, $\gamma_3^*(dx/y^{q-1}) = \omega(dx/y^{q-1})$, and $\gamma_3^*(xdx/y^{q-1}) = \omega^2(xdx/y^{q-1})$. In

other words, they correspond to distinct eigenvalues for γ_3^*. Let

$$e_1 = \frac{1}{3}(1 + \delta_3^2\gamma_3 + \delta_3\gamma_3^2), \qquad e_2 = \frac{1}{3}(1 + \delta_3\gamma_3 + \delta_3^2\gamma_3^2)$$

be elements of $\mathrm{End}^0(J_{f_0,q}^{\mathrm{new}})$. Using the fact that both δ_3 and γ_3 satisfy the equation $T^2 + T + 1 = 0$, one sees that e_1 and e_2 are orthogonal idempotents with $e_1 + e_2 = 1$, and it follows from (2.25) and Subsection 2.9 that neither e_1 nor e_2 is zero. Therefore, $\mathrm{End}^0(J_{f_0,q}^{\mathrm{new}}, \mathfrak{i}) = E \oplus E$. Let $Y_1 := 3e_1 J_{f_0,q}^{\mathrm{new}}$, and $Y_2 := 3e_2 J_{f_0,q}^{\mathrm{new}}$. Then $\mathrm{Lie}(Y_1)$ is the subspace of $\mathrm{Lie}(J_{f_0,q}^{\mathrm{new}})$ on which $d(\delta_3^2\gamma_3)$ acts as identity. If $q/3 < a < 2q/3$ and $3 \nmid a$, then dx/y^a is invariant under $(\delta_3^2\gamma_3)^*$ if and only if $a \equiv 2 \pmod 3$. We see that $h = g_1 + g_2$ with g_1 given by (5.10). Since $g_2 = g_1 \circ \theta_s$ with $s = 3^{r-1} - 1$, it follows that Y_2 is isogenous to Y_1, and $\mathrm{End}^0(J_{f_0,q}^{\mathrm{new}}) = \mathrm{Mat}_2(E)$.

Alternatively, one can see that $\mathrm{End}^0(J_{f_0,q}^{\mathrm{new}}) = \mathrm{Mat}_2(E)$ in the following way. Without loss of generality, assume that $k = \mathbb{C}$. Let X_q be the Fermat curve $x^q + y^q = 1$. There exists a cover $X_q \to C_{f_0,q}$ given by $(x, y) \mapsto (-x^{3^{r-1}}, y)$. Therefore, $J_{f_0,q}^{\mathrm{new}}$ is isogenous to a factor of $J(X_q)$. Using the notation of [14], one sees that $J_{f_0,q}^{\mathrm{new}}$ is isogenous to $\mathbb{C}/L_{r,s,t} \times \mathbb{C}/L_{r',s',t'}$ with $(r, s, t) = (2 \cdot 3^{r-1}, 1, 3^r - 2 \cdot 3^{r-1} - 1) \in (\mathbb{Z}/q\mathbb{Z})^3$ and $(r', s', t') = (3^{r-1}, 1, 3^r - 3^{r-1} - 1) \in (\mathbb{Z}/q\mathbb{Z})^3$. Since (ur, us, ut) coincides with (r', s', t') up to permutation with $u = 3^{r-1} + 1 \in (\mathbb{Z}/q\mathbb{Z})^\times$, we have $\mathbb{C}/L_{r,s,t} \sim \mathbb{C}/L_{r',s',t'}$.

5. Arithmetic results

In this section, we prove the arithmetic results mentioned in Section 3. For two real numbers $x \le y$, let $[x, y]_{\mathbb{Z}}$ be the set of integers

$$[x, y]_{\mathbb{Z}} := \{z \in \mathbb{Z} \mid x \le z \le y, \gcd(z, N) = 1\}.$$

Through out this section, $n \ge 3$ is an integer that is not a multiple of N, and h denotes the function

$$h : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{Z}_+, \qquad a \mapsto \left\lfloor \frac{na}{N} \right\rfloor,$$

where a is taken to be in $[1, N-1]_{\mathbb{Z}}$ for the floor function. We have

$$h(a) + h(-a) = n - 1. \tag{5.1}$$

We are particularly interested in the case $n = 3$, where

$$h(a) = \begin{cases} 0 & \text{if} \quad a \in [1, N/3]_{\mathbb{Z}}, \\ 1 & \text{if} \quad a \in [N/3, 2N/3]_{\mathbb{Z}}, \\ 2 & \text{if} \quad a \in [2N/3, N-1]_{\mathbb{Z}}. \end{cases}$$

Proposition 5.1. *Suppose that $n = 3$, and $N \nmid n$. There exists $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $h(a) = 1$ if and only if $N \notin \{4, 6, 10\}$.*

Proof. For one direction, one easily checks by direct calculation that $[N/3, 2N/3]_{\mathbb{Z}} = \emptyset$ if $N \in \{4, 6, 10\}$. For the other direction, we need to show that there exists an $a \in [N/3, 2N/3]_{\mathbb{Z}}$ if $N \notin \{4, 6, 10\}$. The proof will be separated into a few cases based on the factorization type of N.

Suppose $N = p$ is prime. If $N = 2$, we take $a = 1$; if $N = 5$, we take $a = 2$; otherwise, $N \ge 7$, so there exists an integer in $[N/3, 2N/3]_{\mathbb{Z}}$.

If $N = 9$, we take $a = 4$. Suppose that $N = 3N_0$ with $N_0 \ge 4$. If $N_0 \not\equiv 2 \pmod 3$, then let $a = N_0 + 1$; if $N_0 \equiv 2 \pmod 3$, let $a = N_0 + 3$.

Now suppose that N is not a prime and $\gcd(N,3)=1$. Let p_0 be the smallest prime that divides N, $m := N/p_0$ and $a_0 := \lfloor 2p_0/3 \rfloor$. We will choose appropriate $b>0$ such that $a := a_0 m + b$ lies in $[N/3, 2N/3]_{\mathbb{Z}}$. We note that

$$\frac{N}{3} < \frac{2N-2m+3b}{3} \le m\left\lfloor \frac{2p_0}{3} \right\rfloor + b \le \frac{2N-m+3b}{3}. \tag{5.2}$$

By our choice of p_0, every prime factor of m is greater or equal to p_0. In particular, if m is even, then $p_0=2$, and $4 \mid N$. We also note that $m>3$. Indeed, if $m=2$, then $N=4$, contradicting our assumption; moreover $3 \nmid m$ because that N is assumed to be coprime to 3.

If $p_0^2 \mid N$, choose $b=1$, then $m>3b$, so $a<2N/3$ by (5.2). Moreover $a \equiv 1 \pmod p$ for all $p \mid N$. Therefore, $a \in [N/3, 2N/3]_{\mathbb{Z}}$.

If $N=2m$ with m odd, then $a_0 = \lfloor (2\cdot 2)/3 \rfloor = 1$, and $m \ge 7$ since $N \ne 10$. We choose $b=2$ so $a=m+2$ is odd, and for any prime $p \mid m$, $a \equiv 2 \pmod p$. Since $m>3b=6$, $a<2N/3$ by (5.2). If $N=p_0 m$ with $\gcd(p_0,m)=1$ and $p_0 \ge 5$, then $m \ge 7$, and we choose b in the two-element set $\{1,2\}$ such that $p_0 \nmid a$. For any $p \mid m$, $a \equiv b \pmod p$. Hence $\gcd(a,N)=1$. Since $m \ge 7 > 3b$, $a<2N/3$ by (5.2). $\square$

A complex valued function g on $(\mathbb{Z}/N\mathbb{Z})^\times$ is said to be *odd* if $g(-a)=-g(a)$, and *even* if $g(-a)=g(a)$. So $h_{\mathrm{odd}} := h-(n-1)/2$ is an odd function by (5.1). For $s \in (\mathbb{Z}/N\mathbb{Z})^\times$, we write $\theta_s : (\mathbb{Z}/N\mathbb{Z})^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$ for the multiplication by s map: $a \mapsto sa$. Recall that a function $g : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}$ is said to be *primitive* if $g \circ \theta_s = g \Leftrightarrow s=1$. We are going to show that h is primitive if $\gcd(n,N)=1$. Clearly, it is enough to show that h_{odd} is primitive.

5.1. For each $a \in (\mathbb{Z}/N\mathbb{Z})^\times$, we write $[\![a]\!]$ for the unique representative of a in $[1,N-1]_{\mathbb{Z}}$. Then

$$h_{\mathrm{odd}} = \left\lfloor \frac{na}{N} \right\rfloor - \frac{n-1}{2} = \frac{n\,[\![a]\!] - [\![na]\!]}{N} - \frac{n-1}{2}.$$

Let $\mathscr{V}_{\mathrm{odd}}$ denote the space of all complex valued odd functions on $(\mathbb{Z}/N\mathbb{Z})^\times$. A character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ is odd if and only if $\chi(-1)=-1$. The set of odd characters of $(\mathbb{Z}/N\mathbb{Z})^\times$ forms a basis of $\mathscr{V}_{\mathrm{odd}}$. Therefore h_{odd} can be uniquely written as a linear combination $\sum c_\chi \chi$ of the odd characters. Suppose that $\gcd(n,N)=1$. By the orthogonality of the characters,

$$\begin{aligned}
c_\chi &= \frac{1}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} h_{\mathrm{odd}}(a)\overline{\chi(a)} \\
&= \frac{1}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{(n\,[\![a]\!] - [\![na]\!])\overline{\chi(a)}}{N} \qquad \Big(\text{since} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) = 0\Big) \\
&= \frac{n}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{[\![a]\!]\,\overline{\chi(a)}}{N} - \frac{\chi(n)}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{[\![na]\!]\,\overline{\chi(na)}}{N} \\
&= \frac{n-\chi(n)}{\varphi(N)} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} \frac{[\![a]\!]\,\overline{\chi(a)}}{N}.
\end{aligned}$$

Recall that the generalized Bernoulli number $B_{1,\chi}$ is defined (cf. [28, Chapter 4]) to be

$$B_{1,\chi} = \frac{1}{N} \sum_{a \in (\mathbb{Z}/N\mathbb{Z})^\times} [\![a]\!]\, \chi(a).$$

Therefore,

$$c_\chi = (n - \chi(n)) B_{1,\bar{\chi}} / \varphi(N). \tag{5.3}$$

Since $n \geq 2$, $n - \chi(n) \neq 0$. It follows that $c_\chi = 0$ if and only if $B_{1,\bar{\chi}} = 0$.

Proposition 5.2. *If* $\gcd(n, N) = 1$, *then* $h \circ \theta_s = h$ *if and only if* $s = 1$.

Proof. We may and will assume that $N > 2$ since $(\mathbb{Z}/2\mathbb{Z})^\times$ is trivial. Clearly, the proposition is true if and only if it is true for h_{odd}. Moreover, if $h_{\mathrm{odd}} \circ \theta_s = h_{\mathrm{odd}}$, then $h_{\mathrm{odd}} \circ \theta_{s^i} = h_{\mathrm{odd}}$ for all $i \in \mathbb{Z}$. Note that

$$h_{\mathrm{odd}}(1) = \left\lfloor \frac{n}{N} \right\rfloor - \frac{n-1}{2} \leq \frac{n-1}{N} - \frac{n-1}{2} < 0.$$

So $h_{\mathrm{odd}}(-1) = -h_{\mathrm{odd}}(1) > 0$. Therefore -1 is not in the cyclic subgroup $\langle s \rangle$ of $(\mathbb{Z}/N\mathbb{Z})^\times$ generated by s.

Recall that $h_{\mathrm{odd}} = \sum_{\chi(-1)=-1} c_\chi \chi$. So

$$h_{\mathrm{odd}} \circ \theta_s = \sum_{\chi(-1)=-1} c_\chi \chi(s) \chi.$$

By the linear independence of characters, we see that $h_{\mathrm{odd}} \circ \theta_s = h_{\mathrm{odd}}$ if and only if

$$c_\chi = c_\chi \chi(s) \tag{5.4}$$

for all odd characters χ. Following [14, Proposition, p. 1190], we let $S(N)$ be the set of odd characters of $(\mathbb{Z}/N\mathbb{Z})^\times$, and

$$S_0(N) = \{\chi \in S(N) \mid B_{1,\chi} = 0\} \subset S(N).$$

Since $B_{1,\bar{\chi}} = \bar{B}_{1,\chi}$, $S_0(N)$ coincides with the set of odd characters χ with $c_\chi = 0$ by Subsection 5.1. Further, let $T_1(s, N)$ be set

$$\{\chi \in S(N) \mid \chi(s) = 1\} \subset S(N).$$

Assume that $h_{\mathrm{odd}} \circ \theta_s = h_{\mathrm{odd}}$. It follows from (5.4) that

$$S_0(N) \cup T_1(s, N) = S(N). \tag{5.5}$$

Our goal is to show that when $s \neq 1$, the size of both $S_0(N)$ and $T_1(s, N)$ are small compared to that of $S(N)$, which leads to a contradiction.

Let $\mathrm{ord}(s, N)$ denote the order of s in $(\mathbb{Z}/N\mathbb{Z})^\times$. Since -1 is not in the cyclic group generated by s,

$$|T_1(s, N)| = \frac{\varphi(N)}{2\,\mathrm{ord}(s, N)} = \frac{|S(N)|}{\mathrm{ord}(s, N)}.$$

In particular, if $s \neq 1$, then

$$|T_1(s, N)| \leq |S(N)|/2.$$

On the other hand, if $\gcd(N, 6) = 1$, then

$$|S_0(N)| < |S(N)|/6$$

by [14, Proposition, p. 1190]. This clearly contradicts (5.5). For a general N, the proposition follows if we can prove that $|S_0(N)| < |S(N)|/2$. $\square$

Lemma 5.3. *For any integer $N \geq 3$, $|S_0(N)| < |S(N)|/2$.*

Proof. The proof is modeled after that of [14, Proposition, p. 1190]. First, let χ be an odd primitive character with conductor N, then (cf. [28, Chapter 4])

$$L(1,\chi) = \frac{\pi i \tau(\chi)}{N} B_{1,\bar{\chi}},$$

where $\tau(\chi) = \sum_{a=0}^{N-1} \chi(a) e^{2\pi i a/N}$ is the Gauss sum. It is known classically that $|\tau(\chi)| = \sqrt{N}$, and $L(1,\chi) \neq 0$. Therefore, $B_{1,\chi} \neq 0$.

More generally, let χ be a character modulo N with conductor N_0, and χ_0 be the character modulo N_0 that induces χ. Then

$$B_{1,\chi} = B_{1,\chi_0} \prod (1 - \chi_0(p)),$$

where the product is over all prime factors p of N that do not divide N_0. So $B_{1,\chi} = 0$ if and only if there exists a prime factor p of N such that $p \nmid N_0$ and $\chi_0(p) = 1$. In particular, if N is a prime power, then $B_{1,\chi} \neq 0$ for all odd characters χ. Suppose that $N = \prod_{i=1}^m p_i^{e_i}$. For a fixed prime divisor p_i, the number $u(p_i, N)$ of all odd characters χ_0 modulo $N_i := N/p_i^{e_i}$ with $\chi_0(p_i) = 1$ is

$$u(p_i, N) = \begin{cases} 0 & \text{if } p_i^c \equiv -1 \pmod{N_i} \text{ for some } c \in \mathbb{N}, \\ \varphi(N_i)/(2\operatorname{ord}(p_i, N_i)), & \text{otherwise.} \end{cases}$$

Let $v(p_i, N) = 2u(p_i, N)/\varphi(N)$. Then

$$s(N) := \frac{|S_0(N)|}{|S(N)|} \leq \sum_{i=1}^m v(p_i, N) \leq \sum_{i=1}^m \frac{1}{\varphi(p_i^{e_i}) \operatorname{ord}(p_i, N_i)}.$$

We write $w(N)$ for the last sum. Note that it makes sense to talk about u, v, w only if N has at least two distinct prime factors.

Here is a list of some simple facts about $w(N)$.

- By [14, Proposition, p. 1190], $w(N) < 1/6$ if $\gcd(N, 6) = 1$.
- $w(M) \geq w(N)$ if $M \mid N$ and there does not exist a prime p such that $p \mid N$ but $p \nmid M$.
- Suppose that M has at least two factors, and $M \mid N$, then

$$w(N) \leq w(M) + \sum_{p \mid N, p \nmid M} v(p, N).$$

We separate the estimate of $w(N)$ into cases according to the factorization type of N.

Case 1. Assume that N has at least two distinct prime divisors that are greater or equal to 5. Since $\operatorname{ord}(p, N_i) \leq \lfloor \log_p N_i \rfloor + 1$, we have

$$w(N) < \frac{1}{\varphi(2)(\lfloor \log_2(5 \cdot 7) \rfloor + 1)} + \frac{1}{\varphi(3)(\lfloor \log_3(5 \cdot 7) \rfloor + 1)} + \frac{1}{6} = \frac{11}{24} < \frac{1}{2}.$$

Case 2. Assume that $N > 42$, and $N = 2^{e_1} 3^{e_2} p^{e_3}$ with $e_1, e_2, e_3 \geq 0$ and $p \geq 5$.

If $e_1 \geq 1$, $\quad \varphi(2^{e_1}) \operatorname{ord}(2, N/2^{e_1}) \geq \max_{e_1 \geq 1} \{\varphi(2^{e_1})(\lfloor \log_2(N/2^{e_1}) \rfloor + 1)\} \geq 5.$

If $e_2 \geq 1$, $\quad \varphi(3^{e_2}) \operatorname{ord}(3, N/3^{e_2}) \geq \max_{e_2 \geq 1} \{\varphi(3^{e_2})(\lfloor \log_3(N/3^{e_2}) \rfloor + 1)\} \geq 6.$

Moreover, if $e_3 \geq 1$, then $\varphi(p^{e_3})\operatorname{ord}(p, N/p^{e_3}) \geq 8$ since either $\varphi(p^{e_3}) \geq 8$, or $e_3 = 1$ and $p = 5$ or 7, and $\operatorname{ord}(p, N/p^{e_3}) \geq 2$. Overall, we see that

$$w(N) \leq \frac{1}{5} + \frac{1}{6} + \frac{1}{8} = \frac{59}{120} < \frac{1}{2}.$$

Case 3. Assume that $N \leq 42$.

If $N = 6$, then $(\mathbb{Z}/6\mathbb{Z})^\times \cong \mathbb{Z}/2\mathbb{Z}$. There is a unique odd character χ modulo 6, and $B_{1,\chi} = (1-5)/6 \neq 0$.

If $N = 2\ell^{e_2}$ for some odd prime ℓ with $\ell^{e_2} \geq 5$, then $v(\ell, N) = 0$ since $\ell \equiv -1 \pmod 2$. So

$$s(N) \leq v(2, N) \leq \frac{1}{\varphi(2)(\lfloor \log_2 \ell^{e_2} \rfloor + 1)} \leq \frac{1}{3}.$$

If $N = 4 \cdot \ell^{e_2}$ with $e_2 \geq 1$ and $\ell = 3, 7$, then $v(\ell, N) = 0$ since $\ell \equiv -1 \pmod 4$. So

$$s(N) \leq v(2, N) \leq \frac{1}{(\varphi(4)(\lfloor \log_2 \ell^{e_2} \rfloor + 1))} \leq \frac{1}{4}.$$

If $N = 2^{e_1} \cdot 5$, then $v(2, N) = 0$ since $4 \equiv -1 \pmod 5$. So

$$s(N) \leq v(5, N) \leq \frac{1}{\varphi(5)} = \frac{1}{4}.$$

If $N = 24$, then $v(2, 24) = 1/(2\varphi(8)) = 1/8$, and $v(3, 24) = 1/(2\varphi(3)) = 1/4$. So $s(24) \leq w(24) = 1/8 + 1/4 < 1/2$.

If $N = 3p$ with $p \geq 5$, then $\operatorname{ord}(3, p) \geq 3$ since $p \nmid (3^2 - 1)$. So

$$w(N) \leq \frac{1}{3\varphi(3)} + \frac{1}{\varphi(p)} \leq \frac{1}{4} + \frac{1}{6} < \frac{1}{2}.$$

If $N = 30$, then $v(2, 30) = 1/4$, and $v(3, 30) = 0$ since $3^2 \equiv -1 \pmod{10}$, and $v(5, 30) = 0$ since $5 \equiv -1 \pmod 6$. Therefore, $s(30) \leq 1/4$.

If $N = 42$, then $\operatorname{ord}(2, 21) = 6$, $v(3, 42) = 0$ since $3^3 \equiv -1 \pmod{14}$, and $v(7, 42) = 1/6$. So $s(42) \leq 1/6 + 1/6 = 1/3$.

This completes the enumeration of all the positive numbers $N \leq 42$ that are not prime powers. □

For the rest of the section, we focus on the case where $N = q = p^r$ is a prime power and $n = 3$. Then $h(a) = \lfloor 3a/q \rfloor$. Recall that

$$\mathscr{T}_q := \{g : (\mathbb{Z}/q\mathbb{Z})^\times \to \{0, 1\} \mid g(a) + g(-a) = 1, g([1, q/3]_\mathbb{Z}) = 0\},$$
$$\mathscr{S}_q := \{s \in (\mathbb{Z}/q\mathbb{Z})^\times \mid s \neq 1, \text{and } \exists g \in \mathscr{T}_q \text{ such that } g \circ \theta_s \in \mathscr{T}_q\}.$$

Let $g_i : (\mathbb{Z}/q\mathbb{Z})^\times \to \{0, 1\}$ be functions satisfying $g_i(a) + g_i(-a) = 1$ for $i = 1, 2$. If $h = g_1 + g_2$, then $g_i \in \mathscr{T}_q$ for all i.

5.2. We note that if p is odd, then $2 \in \mathscr{S}_q$. Indeed, let $g \in \mathscr{T}_q$ be the function given by

$$g(a) = \begin{cases} 0 & \text{if} \quad a \in [1, q/3]_\mathbb{Z}, \\ 0 & \text{if} \quad a \in [q/3, 2q/3]_\mathbb{Z} \text{ and } a \text{ is even}, \\ 1 & \text{if} \quad a \in [q/3, 2q/3]_\mathbb{Z} \text{ and } a \text{ is odd}, \\ 1 & \text{if} \quad a \in [2q/3, q]_\mathbb{Z}, \end{cases} \tag{5.6}$$

then $g \circ \theta_2 \in \mathscr{T}_q$. On the other hand, if $g' \in \mathscr{T}_q$ is any function such that $g' \circ \theta_2 \in \mathscr{T}_q$, then $g'(a) = 0$ for all even $a \in [q/3, 2q/3]_\mathbb{Z}$. Since q is *odd*, it follows from the

assumption $g'(a) + g'(-a) = 1$ that $g'(a) = 1$ for all odd $a \in [q/3, 2q/3]_{\mathbb{Z}}$. So $g' = g$. In other words, the function with both g and $g \circ \theta_2$ in $\mathscr{T}_q$ is uniquely determined. Since $2^{-1} = (q+1)/2 \in (\mathbb{Z}/q\mathbb{Z})^\times$, $(q+1)/2 \in \mathscr{S}_q$ as well. On the other hand, $-1 \notin \mathscr{S}_q$ since $(g \circ \theta_{-1})(1) = g(q-1) = 1$.

Lemma 5.4. *Suppose that $q = p^r$ with $p \geq 5$. Then $\mathscr{S}_q = \{2, (q+1)/2\}$.*

Proof. Note that the lemma is trivial for $q = 5$ since $(q+1)/2 = 3$ in this case, so the only other nontrivial element in $(\mathbb{Z}/5\mathbb{Z})^\times$ that is not in our list is $s = 4 \equiv -1 \pmod 5$, which is not in $\mathscr{S}_q$ as remarked in subsection 5.2. Henceforth we assume that $q \geq 7$. In particular, $2 \in [0, q/3]_{\mathbb{Z}}$. Suppose that $g \in \mathscr{T}_q$ is a function such that $g \circ \theta_s \in \mathscr{T}_q$. We narrow down the possible s in steps.

Step 1. $s \in [1, 2q/3]_{\mathbb{Z}}$. Otherwise, $(g \circ \theta_s)(1) = g(s) = 1$.

Step 2. We show that the lemma is true for $7 \leq q \leq 23$. This will also give us a glimpse of the idea of the proof for larger q. Note that q is a prime in this case. First suppose that $3 \leq s < q/3$, then $2 < \lfloor q/s \rfloor < q/3$, and

$$q > s \left\lfloor \frac{q}{s} \right\rfloor \geq q - s + 1 > \frac{2q}{3} + 1.$$

So it follows that $g(s \cdot \lfloor q/s \rfloor) = 1$. Contradiction.

Hence we must have $s \in [q/3, 2q/3]_{\mathbb{Z}}$. If $q/3 < s < q/2$, then $2q/3 < 2s < q$, and $(g \circ \theta_s)(2) = g(2s) = 1$. Contradiction again!

Therefore $s \in [q/2, 2q/3]_{\mathbb{Z}}$. Note that the lemma is already proved for $q = 7$ since the only element in the set $[7/2, 14/3]_{\mathbb{Z}}$ is $4 = (7+1)/2$. We further assume that $q \geq 11$. In particular $3 \in [1, q/3]_{\mathbb{Z}}$ and thus $g(3) = 0$. Note that $3q/2 < 3s < 2q$. In order that $g(3s) = 0$ we must have $3s - q < 2q/3$, or equivalently $s < 5q/9$. Recall that $q \leq 23$. So $(q+3)/2 > 5q/9$, and the only element in $[q/2, 5q/9]_{\mathbb{Z}}$ is $(q+1)/2$. The lemma is proved for all $7 \leq q \leq 23$.

We assume that $q \geq 25$ for the rest of the proof. In particular, both 3 and 4 are in $[1, q/3]_{\mathbb{Z}}$.

Step 3. In this step we will show that if $s \geq 3$, then $s \in [q/2, 2q/3]_{\mathbb{Z}}$. One difference from the previous step is that q is not necessarily prime, so we have to avoid using any $1 \leq a < q/3$ that are divisible by p in our proof.

First, we claim that $s \notin [3, q/6]_{\mathbb{Z}}$. Suppose otherwise, then

$$0 < \left\lfloor \frac{q}{s} \right\rfloor - 1 < \left\lfloor \frac{q}{s} \right\rfloor \leq \left\lfloor \frac{q}{3} \right\rfloor,$$

$$q > s \left\lfloor \frac{q}{s} \right\rfloor > s \left(\left\lfloor \frac{q}{s} \right\rfloor - 1 \right) > q - 2s > \frac{2q}{3}.$$

If $p \nmid \lfloor q/s \rfloor$, we set $a = \lfloor q/s \rfloor$, otherwise, we set $a = \lfloor q/s \rfloor - 1$. Then $a \in [2, q/3]_{\mathbb{Z}}$, and $a \cdot s \in [2q/3, q]_{\mathbb{Z}}$, so

$$(g \circ \theta_s)(a) = g(a \cdot s) = 1.$$

This contradicts the assumption of the lemma.

Second, we claim that $s \notin [q/3,\ q/2]_{\mathbb{Z}}$. Suppose this is not true, then $2q/3 < 2s < q$, and $(g \circ \theta_s)(2) = g(2s) = 1$. Contradiction.

Third, if $2q/9 < s < q/3$, then $2q/3 < 3s < q$, and it follows that

$$(g \circ \theta_s)(3) = g(3s) = 1.$$

Again, this leads to a contradiction.

Last, if $q/6 < s < 2q/9$, then $2q/3 < 4s < 8q/9 < q$. Once again the contradiction arises since $(g \circ \theta_s)(4) = g(4s) = 1$. So we must have

$$q/2 < s < 2q/3. \tag{5.7}$$

Step 4. Assume that $s \neq 2$. Then we must show that $s = (q+1)/2$. By (5.7), $3q/2 < 3s < 2q$. Since $g(3s) = g(3s - q) = 0$, we must have $3s - q < 2q/3$, i.e.,

$$q/2 < s < 5q/9. \tag{5.8}$$

Note that $5/9 < 2/3$, so the upper bound for s has been lowered. Now the idea is to repeat this process by taking the products of s with odd numbers $a = 5, 7, 9$, etc., and inductively lower the upper bound until there is no other element left in the interval except $s = (q+1)/2$. But once again extra care must be taken since we need to make sure that the odd numbers that are divisible by p be skipped.

For integers $t \geq 1$, let

$$c_t = \frac{3t-1}{3(2t-1)} = \frac{1}{2} + \frac{1}{6(2t-1)}.$$

Note that $c_1 = 2/3$, $c_2 = 5/9$, and $c_t > c_{t+1}$. Suppose that for a given $t \geq 2$ we have

$$q/2 < s < c_t q.$$

Then

$$tq < (2t+1)s < \frac{(2t+1)(3t-1)q}{3(2t-1)} \leq (t+1)q,$$

$$(t+1)q < (2t+3)s < \frac{(2t+3)(3t-1)q}{3(2t-1)} < (t+2)q.$$

Now assume additionally that $2t+3 < q/3$. If $p \nmid 2t+1$, then $(g \circ \theta_s)(2t+1) = g((2t+1)s) = 0$. Hence $(2t+1)s - tq < 2q/3$. That is,

$$q/2 < s < \frac{(3t+2)q}{3(2t+1)} = c_{t+1}q.$$

Similarly, if $p \nmid 2t+3$, we have $(2t+3)s - (t+1)q < 2q/3$, and hence

$$q/2 < s < \frac{(3t+5)q}{3(2t+3)} = c_{t+2}q.$$

Clearly, either $2t+1$ or $2t+3$ is not divisible by p. Recall that $c_{t+2} < c_{t+1}$. We see that as long as $2t+3 < q/3$,

$$s \in [q/2, c_t q]_{\mathbb{Z}} \quad \Longrightarrow \quad s \in [q/2, c_{t+1}q]_{\mathbb{Z}}.$$

Let $m = [(q-9)/6] > (q-14)/6$. The base case for $t = 2$ is already verified in (5.8). It follows by induction that $s \subseteq [q/2, c_{m+1}q]_{\mathbb{Z}}$. Note that

$$c_{m+1}q - q/2 = \frac{q}{6(2m+1)} < \frac{q}{(2(q-14)+6)} = \frac{q}{2q-22} < 1$$

because $q \geq 25$ by assumption, so the only integer in $[q/2, c_{m+1}q]_{\mathbb{Z}}$ is $(q+1)/2$. Therefore, $s = (q+1)/2$.

Lemma 5.4 follows by combining all the above steps. □

Corollary 5.5. *Suppose that $p \geq 5$, and q is not 7. For any $g \in \mathscr{T}_q$, $g \circ \theta_s = g$ if and only if $s = 1$.*

Proof. Indeed, if $g = g \circ \theta_s$ and $s \neq 1$, then all nontrivial elements of the cyclic group $\langle s \rangle$ lie in the two-element set $\mathscr{S}_q = \{2, (q+1)/2\}$. It follows that $4 = 2^2 \in \mathscr{S}_q$. This is possible only if $q = 7$ whence $(q+1)/2 = 4$ and $2 \in (\mathbb{Z}/7\,\mathbb{Z})^\times$ has order 3. □

Remark 5.6. If $q = 7$, then up to relabeling, h can be uniquely written as $g_1 + g_2$, where h, g_1, g_2 are given by

	1	2	3	4	5	6
h	0	0	1	1	2	2
g_1	0	0	1	0	1	1
g_2	0	0	0	1	1	1

It is clear that $g_1 \circ \theta_2 = g_1$.

Lemma 5.7. *Suppose that p is an odd prime not equal to 3, and $q \neq 5$. There does not exist a function $g : (\mathbb{Z}/q\,\mathbb{Z})^\times \to \{0, 1\}$ that satisfies both of the following conditions:*

(i) $g(a) + g(-a) = 1, \ \forall\, a \in (\mathbb{Z}/q\mathbb{Z})^\times$;

(ii) $h = g + g \circ \theta_s$ *for some* $s \in (\mathbb{Z}/q\,\mathbb{Z})^\times$.

Proof. We prove by contradiction. Suppose such a function g exists. Since $h(a) = 0$ for all $a \in [1, q/3]_\mathbb{Z}$, condition *(ii)* implies that both g and $g \circ \theta_s$ lie in $\mathscr{T}_q$. Clearly, $s \neq 1$ since there exists $a \in (\mathbb{Z}/q\,\mathbb{Z})^\times$ such that $h(a) = 1$. Without loss of generality, we may assume that $s = 2$ so g is given by (5.6). Let us set $a_0 = (q-1)/2$ if $q \equiv 3 \pmod 4$; and $a_0 = (q-3)/2$ if $q \equiv 1 \pmod 4$, then a_0 is odd, $q/3 < a_0 < q/2$, and $(a_0, p) = 1$. But we have $g(a_0) = 1$, and $(g \circ \theta_2)(a_0) = g(2a_0) = 1$. Therefore, $g(a_0) + (g \circ \theta_2)(a_0) = 2$. On the other hand, $f(a_0) = [3a_0/q] = 1$. Contradiction! □

Remark 5.8. Lemma 5.7 fails for $q = 5$ since $h = g_1 + g_2$, where h, g_1 and g_2 are given by the following table:

	1	2	3	4
h	0	1	1	2
g_1	0	1	0	1
g_2	0	0	1	1
$g_2 \circ \theta_2$	0	1	0	1

One sees that $g_2 \circ \theta_2 = g_1$.

Remark 5.9. Let $q = 2^r$ with $r \geq 3$, and $s = 2^{r-1} - 1$. Then for any positive odd number $2t + 1 < 2^{r-1}$, we have

$$s(2t+1) = (2^{r-1} - 1)(2t+1) = 2^r t + 2^{r-1} - (2t+1) \equiv 2^{r-1} - (2t+1) \pmod{2^r}.$$

In particular, $\theta_s([1, 2^{r-1}]_\mathbb{Z}) = [1, 2^{r-1}]_\mathbb{Z}$. Since θ_s is bijective, $\theta_s([2^{r-1}, 2^r]_\mathbb{Z}) = [2^{r-1}, 2^r]_\mathbb{Z}$. Let $g : (\mathbb{Z}/q\mathbb{Z})^\times \to \{0, 1\}$ be the function defined by

$$g(a) = \begin{cases} 0 & \text{if } a \in [1, 2^{r-1}]_\mathbb{Z}, \\ 1 & \text{if } a \in [2^{r-1}, 2^r]_\mathbb{Z}. \end{cases} \tag{5.9}$$

Then $g \in \mathscr{T}_q$ and $g = g \circ \theta_s$ with $s = 2^{r-1} - 1$. In other words, Corollary 5.5 fails for all powers of 2 that are greater or equal to 8.

On the other hand, let $g' \in \mathscr{T}_q$ be any function such that $g' \circ \theta_{(q/2-1)} \in \mathscr{T}_q$ as well. For any $a \in [q/6, q/2]_{\mathbb{Z}}$, we have $g'(a) = (g' \circ \theta_s)(q/2 - a) = 0$ since $q/2 - a \in [0, q/3]_{\mathbb{Z}}$. Combining with the fact that g' also vanishes on $[1, q/3]_{\mathbb{Z}}$, we see that g' coincides with g. In conclusion, if $q = 2^r \geq 8$ and $s = q/2 - 1$, there is a unique $g \in \mathscr{T}_q$ such that $g \circ \theta_s \in \mathscr{T}_q$.

5.3. Suppose that $q = 3^r \geq 9$. We claim that $\mathscr{S}_q \supseteq \{2, (q+1)/2, q/3-1, 2q/3-1\}$. Note that

$$(q/3 - 1)(2q/3 - 1) = 2q^2/9 - q + 1 \equiv 1 \pmod{q},$$

so $(q/3 - 1)^{-1} = 2q/3 - 1 \in (\mathbb{Z}/q\mathbb{Z})^\times$. It is enough to show that $\mathscr{S}_q \ni (q/3 - 1)$. Let $s = q/3 - 1$ and $a = 3t + a_0$ with $a_0 = 1$ or 2. Then

$$sa = qa/3 - a = q(3t + a_0)/3 - a \equiv a_0 q/3 - a \pmod{q}.$$

Suppose $a \in [1, q/3]_{\mathbb{Z}}$. If $a_0 = 1$, then $[\![sa]\!] = q/3 - a \in [1, q/3]_{\mathbb{Z}}$; and if $a_0 = 2$, then $[\![sa]\!] = 2q/3 - a \in [q/3, 2q/3]_{\mathbb{Z}}$. Moreover, if $a \equiv 1 \pmod 3$ and $a \in [q/3, 2q/3]_{\mathbb{Z}}$, then one easily shows that $\exists a' \in [1, q/3]_{\mathbb{Z}}$ such that $[\![sa']\!] = a$. Therefore, if $g \in \mathscr{T}_q$ is a function such that $g \circ \theta_s \in \mathscr{T}_q$, then g must be of the form

$$g(a) = \begin{cases} 0 & \text{if} \quad a \in [1, q/3]_{\mathbb{Z}}, \\ 0 & \text{if} \quad a \in [q/3, 2q/3]_{\mathbb{Z}} \text{ and } a \equiv 1 \pmod{3}, \\ 1 & \text{if} \quad a \in [q/3, 2q/3]_{\mathbb{Z}} \text{ and } a \equiv 2 \pmod{3}, \\ 1 & \text{if} \quad a \in [2q/3, q]_{\mathbb{Z}}. \end{cases} \tag{5.10}$$

Last we note that the order of $s = q/3 - 1 \in (\mathbb{Z}/q\mathbb{Z})^\times$ is 6. Indeed, since $q \geq 9$,

$$s^3 = (q/3 - 1)^3 = q^3/27 - 3(q/3)^2 + 3(q/3) - 1 \equiv -1 \pmod{q}.$$

5.4. Suppose that $s = q/3 - 1$, and let g be as in (5.10) so that $g \circ \theta_s \in \mathscr{T}_q$. Given $a \in [q/3, 2q/3]_{\mathbb{Z}}$, if $a \equiv 1 \pmod 3$, then $(g \circ \theta_s)(a) = g(sa) = 1$; and if $a \equiv 2 \pmod 3$, then $(g \circ \theta_s)(a) = 0$. It follows that

$$h = g + g \circ \theta_{(q/3-1)}$$

for this particular g.

Similar to $[x, y]_{\mathbb{Z}}$, we define

$$(x, y)_{\mathbb{Z}} := \{z \in \mathbb{Z} \mid x < z < y, \quad \gcd(z, q) = 1\}.$$

Lemma 5.10. *Suppose that $q = 3^r \geq 9$. Then $\mathscr{S}_q = \{2, (q+1)/2, q/3-1, 2q/3-1\}$.*

Proof. The proof follows the same ideas as those of Lemma 5.4, except extra care must be taken.

First, if $q = 9$, we only need to prove that $s \neq 4$. Indeed, $2 < 9/3 = 3$, but $(g \circ \theta_4)(2) = g(8) = 1$. We assume that $q \geq 27$ for the rest of the proof. In particular, $q/3 \geq 9$.

Clearly, $s \in [1, 2q/3]_{\mathbb{Z}}$. Similar arguments to Step 3 of Lemma 5.4 show that $s \notin (q/3, q/2)_{\mathbb{Z}}$ and $s \notin (3, q/6)_{\mathbb{Z}}$. Suppose that $s \in (q/6, q/4)_{\mathbb{Z}}$, then $2q/3 \leq 4s \leq q$, and therefore $(g \circ \theta_s)(4) = g(4s) = 1$. Contradiction. Suppose that $s \in (q/4, 2q/7)_{\mathbb{Z}}$, then $5q/3 < 7q/4 < 7s < 2q$, and hence $(g \circ \theta_s)(7) = g(7s) = 1$. Contradiction. We have shown that if $s \in (3, q/2)_{\mathbb{Z}}$, then $s \in (2q/7, q/3)_{\mathbb{Z}}$. In particular, if $q = 27$, then the only integer in $(2q/7, q/3)_{\mathbb{Z}}$ is $q/3 - 1 = 8$. We may further assume that $q \geq 81$.

Let $t \in \mathbb{N}$. Suppose that $s \in (2tq/(6t+1), q/3)_{\mathbb{Z}}$. Notice that this is true for $t = 1$. Moreover,

$$(6t+7)s > \frac{2t(6t+7)q}{6t+1} > (2t+1)q + \frac{2q}{3} = \frac{(6t+5)q}{3}.$$

Indeed,

$$\frac{2t(6t+7)}{6t+1} - \frac{(6t+5)}{3} = \frac{6t(6t+7) - (6t+1)(6t+5)}{3(6t+1)} = \frac{6t-5}{3(6t+1)} > 0.$$

Therefore, if $s \in (2tq/(6t+1), 2(t+1)q/(6t+7))_{\mathbb{Z}}$, then

$$(2t+1)q + \frac{2q}{3} < (6t+7)s < 2(t+1)q.$$

We get a contradiction as long as $6t+7 < q/3$. Since

$$\frac{2t}{6t+1} = \frac{1}{3} - \frac{1}{3(6t+1)},$$

it is an increasing function in t. Our bounds are refined each time we increase t. Take $t_0 = [(q-3)/18] = (q-9)/18$. Then we get

$$\frac{q}{3} - \frac{2t_0 q}{(6t_0+1)} = \frac{q}{3(6t_0+1)} = \frac{q}{q-6} \leq \frac{27}{25}.$$

It follows that if $s \in (2q/7, q/3)_{\mathbb{Z}}$, then $s = q/3 - 1$.

Now assume that $s \in (3q/5, 2q/3)_{\mathbb{Z}}$. If $q = 27$, then the only element in $(3q/5, 2q/3)_{\mathbb{Z}}$ is $s = 17 = 2q/3 - 1$. So we may assume that $q \geq 81$ in this case. Suppose that $s \in (3q/5, 20q/33)_{\mathbb{Z}}$, then

$$7q + \frac{4q}{5} = \frac{39q}{5} < 13s < \frac{260q}{33} = 7q + \frac{29q}{33}.$$

We have $(g \circ \theta_s)(13) = 1$. Contradiction. Now suppose that $s \in (20q/33, 7q/11)_{\mathbb{Z}}$, then

$$6q + \frac{2q}{3} < 11s < 7q.$$

Therefore, $g \circ \theta_s(11) = 1$ but $g(11) = 0$. Contradiction again. So we must have $s \in (7q/11, 2q/3)_{\mathbb{Z}}$.

Now let $t \in \mathbb{N}$. Suppose that $(4t-1)q/(6t-1) < s < 2q/3$. Clearly, this is true for $t = 2$. Now

$$\frac{4t-1}{6t-1} = \frac{2}{3} - \frac{1}{3(6t-1)}.$$

So $\frac{4t-1}{6t-1}$ is an increasing function in t. Moreover,

$$\frac{(4t-1)}{6t-1} \cdot (6t+5) > 4t + 2 + \frac{2}{3}.$$

Indeed,

$$\begin{aligned}\frac{(4t-1)}{6t-1} \cdot (6t+5) - (4t+2+\frac{2}{3}) &= \frac{3(4t-1)(6t+5) - (6t-1)(12t+8)}{3(6t-1)} \\ &= \frac{6t-7}{3(6t-1)} > 0 \qquad \text{if } t \geq 2.\end{aligned}$$

Therefore, if $6t+5 \in (1, q/3)_{\mathbb{Z}}$, then $s \in ((4t-1)q/(6t-1), 2q/3)_{\mathbb{Z}}$ implies that $s \in ((4t+3)q/(6t+5), 2q/3)_{\mathbb{Z}}$.

Let $t_0 = [(q+3)/18] = (q-9)/18$. We then have

$$\frac{2q}{3} - \frac{(4t_0-1)q}{6t_0-1} = \frac{q}{3(6t_0-1)} = \frac{q}{q-12} \leq \frac{27}{23}.$$

This shows the only possible $s \in (3q/5, 2q/3)_{\mathbb{Z}}$ is $s = 2q/3 - 1$.

We need to handle the remaining case $q/2 < s < 3q/5$. Note that $5q/2 < 5s < 3q$, so we must have $5q/2 < 5s < 8q/3$. That is, $q/2 < s < 8q/15$.

Assume that $q/2 < s < (9t-1)q/(3(6t-1))$. Then this holds for $t = 1$.

$$\frac{9t-1}{3(6t-1)} = \frac{1}{2} + \frac{1}{6(6t-1)}.$$

So $\frac{9t-1}{3(6t-1)}$ is a decreasing function in t. On the other hand,

$$\frac{(9t-1)(6t+5)}{3(6t-1)} > (3t+2) + \frac{2}{3}.$$

Indeed,

$$\begin{aligned}\frac{(9t-1)(6t+5)}{3(6t-1)} - ((3t+2) + \frac{2}{3}) &= \frac{(9t-1)(6t+5) - (9t+8)(6t-1)}{3(6t-1)} \\ &= \frac{1}{(6t-1)} > 0.\end{aligned}$$

Therefore, if $6t+5 < q/3$, then $s \in (q/2, (9t-1)q/(3(6t-1)))_{\mathbb{Z}}$ implies that $s \in (q/2, (9t+8)q/(3(6t+5)))_{\mathbb{Z}}$.

Now we take the largest $t_0 = [(q+3)/18] = (q-9)/18$. Then

$$\frac{(9t_0-1)q}{3(6t_0-1)} - \frac{q}{2} = \frac{q}{6(6t_0-1)} = \frac{q}{2q-24} \leq \frac{9}{10} \qquad \text{since } q \geq 27.$$

This takes care of all the cases for $q = 3^r$. □

Corollary 5.11. *Let $q = 3^r \geq 9$. For any $g \in \mathscr{T}_q$, $g \circ \theta_s = g$ if and only if $s = 1$.*

Proof. Suppose that $g \circ \theta_s = g$ and $s \neq 1$. By Lemma 5.10, $s \in \mathscr{S}_q = \{2, (q+1)/2, q/3-1, 2q/3-1\}$, and all nontrivial elements of the cyclic group $\langle s \rangle$ lie in $\mathscr{S}_q$. If $s = 2$ or $(q+1)/2$, then $4 \in \mathscr{S}_q$. On the other hand, if $q = 9$, then $\mathscr{S}_q = \{2, 5\}$; if $q > 9$, then $q/3-1, (q+1)/2$ and $2q/3-1$ are all strictly greater than 4. So $s \notin \{2, (q+1)/2\}$. If $s = q/3-1$ or $2q/3-1$, then the six-element group $\langle s \rangle$ will fix g. However, there are at most 4 elements in $\mathscr{S}_q$. Contradiction. □

Remark 5.12. When $q = 9$, up to labeling, there is a unique way to write $h = g_1 + g_2$, where h, g_1 and g_2 are given by the following table:

	1	2	4	5	7	8
h	0	0	1	1	2	2
g_1	0	0	0	1	1	1
g_2	0	0	1	0	1	1
$g_1 \circ \theta_2$	0	0	1	0	1	1

One sees that $g_2 = g_1 \circ \theta_2$.

Lemma 5.13. *Suppose that $q = 3^r \geq 27$, and $g \in \mathscr{T}_q$ is the function given by* (5.10). *Let $h = g_1 + g_2$ with $g_i \in \mathscr{T}_q$. Then $g_2 = g_1 \circ \theta_s$ for some $s \in (\mathbb{Z}/q\mathbb{Z})^\times$ if and only if the pair (s, g_1) coincides with $(q/3-1, g)$ or $(2q/3-1, g \circ \theta_{(q/3-1)})$. In particular, if $g \notin \{g_1, g_2\}$, then $g_2 \neq g_1 \circ \theta_s$ for any $s \in (\mathbb{Z}/q\mathbb{Z})^\times$.*

Proof. By subsection 5.3, the pairs $(q/3-1, g)$ and $(2q/3-1, g\circ\theta_{(q/3-1)})$ satisfy the required conditions. On the other hand, it follows from Lemma 5.10 that it is enough to show that there does not exist $g' \in \mathscr{T}_q$ such that $h = g' + g'\circ\theta_2$. Suppose that g' is such a function. By subsection 5.2, it must be of the form given by (5.6) in order that $g'\circ\theta_2 \in \mathscr{T}_q$. Let $a_0 = q/3+2$. Then $3 \nmid a_0$, a_0 is odd, and $2q/3 < 2a_0 < q$ since $q \geq 27$. Therefore, $g'(a_0) = 1$, and $(g'\circ\theta_2)(a_0) = g'(2a_0) = 1$. On the other hand, $h(a_0) = 1$, so $h(a_0) \neq g'(a_0) + (g'\circ\theta_2)(a_0)$. □

Remark 5.14. If $q = 8$, then h can be uniquely written as $g_1 + g_2$ (up to relabeling) with h, g_1, g_2 given by

	1	3	5	7
h	0	1	1	2
g_1	0	1	0	1
g_2	0	0	1	1

One easily checks that $g_1 = g_1\circ\theta_5$, and $g_2 = g_2\circ\theta_3$.

Lemma 5.15. *Let $q = 2^r \geq 16$. Then $\mathscr{S}_q = \{q/2-1\}$.*

Proof. Clearly, $s \in [1, 2q/3]_{\mathbb{Z}}$. If $q = 16$, we need to show that $s \notin \{3,5,9\}$. $s \neq 3$ or 5 since $15 = q-1 \in [2q/3, q]_{\mathbb{Z}}$, and both $3, 5 \in [1, q/3]_{\mathbb{Z}}$. On the other hand, if $s = 9$, then $3s = 27 \equiv 11 \pmod{16}$, but $11 \in [2q/3, q]_{\mathbb{Z}}$. Contradiction.

Assume that $q \geq 32$. Suppose that $s \in [3, q/6]_{\mathbb{Z}}$ and we treat it similarly as the previous case. Suppose that $s \in [2q/9, q/3]_{\mathbb{Z}}$, $2q/3 < 3s < q$. Contradiction. Suppose that $s \in [q/6, 2q/9]_{\mathbb{Z}}$, then $3q/2 < 9s < 2q$. Therefore, $s \in [q/6, 5q/27]_{\mathbb{Z}}$. Now $5q/6 < 5s < 25q/27$. Contradiction.

Suppose that $s \in [q/3, q/2]_{\mathbb{Z}}$. Then $5q/3 < 5s < 5q/2$. Therefore, we must have $2q < 5s < 5q/2$, that is, $s \in [2q/5, q/2]$. Suppose that for some $t \in \mathbb{N}$ we have $tq/(2t+1) < s < q/2$. Then

$$\frac{t}{2t+1} = \frac{1}{2} - \frac{1}{2(2t+1)}.$$

So $t/(2t+1)$ is an increasing function in t. If $2t+3 < q/3$, then

$$\frac{(2t+3)t}{2t+1} - (t+\frac{2}{3}) = \frac{3t(2t+3)-(2t+1)(3t+2)}{3(2t+1)} = \frac{2t-2}{3(2t+1)} > 0.$$

Therefore,

$$(t+\frac{2}{3})q < (2t+3)s < (t+3/2)q.$$

Therefore, $(t+1)q < (2t+3)s < (t+3/2)q$, and hence $s \in [(t+1)q/(2t+3), q/2]_{\mathbb{Z}}$.

Take $t_0 \in \mathbb{N}$ such that $2t_0+1$ is the largest odd number smaller than $q/3$, then $t_0 \geq (q-8)/6$.

$$\frac{q}{2(2t_0+1)} = \frac{3q}{2(q-5)} \leq \frac{48}{27} < 2.$$

If $s \in [\frac{t_0 q}{2t_0+1}, q/2]_{\mathbb{Z}}$, then $s = q/2-1$. The case $s \in [q/2, 2q/3]_{\mathbb{Z}}$ is treated similarly as the previous cases, except that s lies in an interval of the form $[q/2, c_t q]$ whose length is less than 1, so there are no integers in it. □

Lemma 5.16. *Suppose that $q = 2^r \geq 16$. There does not exist a function $g \in \mathscr{T}_q$ such that $h = g + g\circ\theta_s$ for any $s \in (\mathbb{Z}/q\mathbb{Z})^\times$.*

Proof. By Lemma 5.15, $s = 1$ or $q/2 - 1$. We have observed in Subsection 5.9 that if $s = q/2 - 1$, then g is given by (5.9). But then it follows $g \circ \theta_s = g$, and $h = 2g$, which is impossible, since there exists $a \in (\mathbb{Z}/q\mathbb{Z})^\times$ such that $h(a) = 1$. □

Lemma 5.17. *Let $q = 2^r \geq 8$, $s = q/2 - 1 \in (\mathbb{Z}/q\mathbb{Z})^\times$, and $\alpha = 2i\sin(2\pi/q)$. Then $\mathbb{Q}(\alpha)$ is the subfield of $\mathbb{Q}(\zeta_q)$ fixed by the subgroup $\langle s \rangle$.*

Proof. Indeed, we assume that $\zeta_q = \exp 2\pi i/q = \cos(2\pi/q) + i\sin(2\pi/q)$. Then $\zeta_q^s = \exp{(q-2)\pi i/q} = -\cos(2\pi/q) + i\sin(2\pi/q)$.

Let K be the subfield of $\mathbb{Q}(\zeta_q)$ fixed by s. It is clear that $2i\sin(2\pi/q) = \zeta_q + \zeta_q^s \in K$. Now ζ_q satisfies the quadratic equation over $\mathbb{Q}(\alpha)$

$$x^2 - 2i\sin(2\pi/q)x - 1 = 0.$$

We have $\mathbb{Q}(\alpha) \subseteq K \subset \mathbb{Q}(\zeta_q)$. We have just shown $[\mathbb{Q}(\zeta_q) : \mathbb{Q}(\alpha)] \leq 2$. However, $[\mathbb{Q}(\zeta_q) : K] \geq 2$. Therefore, $[\mathbb{Q}(\zeta_q) : \mathbb{Q}(\alpha)] = 2$, and $\mathbb{Q}(\alpha) = K$. □

Acknowledgments

The present paper uses several techniques and ideas from a series of papers by Yuri Zarhin. The authors wish to thank him for his pioneering work. Major parts of the work were completed when the first-named author was a post-doc at the National Center for Theoretic Science (NCTS), Hsinchu, Taiwan. He would like to express his gratitude for the generous support of NCTS. He was partially supported by the grant NSC 101-2811-M-001-078. The second-named author was partially supported by the grants NSC 100-2628-M-001-006-MY4 and AS-98-CDA-M01. The authors are grateful to the referee for a careful reading and the helpful comments which eliminated some inaccuracies and improved the exposition of the manuscript.

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