TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 367, Number 12, December 2015, Pages 8843–8860
S 0002-9947(2015)06438-1
Article electronically published on March 13, 2015

WEAKLY HOLOMORPHIC MODULAR FORMS AND RANK TWO HYPERBOLIC KAC-MOODY ALGEBRAS

HENRY H. KIM, KYU-HWAN LEE, AND YICHAO ZHANG

Abstract. In this paper, we compute basis elements of certain spaces of weight 0 weakly holomorphic modular forms and consider the integrality of Fourier coefficients of the modular forms. We use the results to construct automorphic correction of the rank 2 hyperbolic Kac-Moody algebras $\mathcal{H}(a)$, $a = 4, 5, 6$, through Hilbert modular forms explicitly given by Borcherds lifts of the weakly holomorphic modular forms. We also compute asymptotics of the Fourier coefficients as they are related to root multiplicities of the rank 2 hyperbolic Kac-Moody algebras. This work is a continuation of an earlier work of the first and second authors, where automorphic correction was constructed for $\mathcal{H}(a)$, $a = 3, 11, 66$.

Introduction

The relationship between affine Kac-Moody algebras and Jacobi modular forms is well understood through the works of Macdonald [20], Kac, Peterson, Wakimoto [12, 13] and others. A more mysterious relationship between hyperbolic Kac-Moody algebras and automorphic forms was perceived by Lepowsky and Moody [19] and Feingold and Frenkel [5], and further investigated by Borcherds [1], and Gritsenko and Nikulin [7]. Surprisingly, it turned out that hyperbolic Kac-Moody algebras should be "corrected" to be precisely related to automorphic forms. Namely, hyperbolic Kac-Moody algebras need to be extended to generalized Kac-Moody superalgebras so that the denominator functions may become automorphic forms. The resulting generalized Kac-Moody superalgebras and automorphic forms are called *automorphic correction* of the hyperbolic Kac-Moody algebras.

In a series of papers [6–9], Gritsenko and Nikulin constructed automorphic correction of many rank 3 hyperbolic Kac-Moody algebras. In a recent paper of Kim and Lee [17], it was shown that rank 2 symmetric hyperbolic Kac-Moody algebras $\mathcal{H}(a)$, $a \geq 3$, form infinite families through chains of embeddings. (See [14, 17] for the definition of $\mathcal{H}(a)$.) Moreover, they considered three specific families and constructed automorphic correction for the first algebra in each family, i.e. $\mathcal{H}(3), \mathcal{H}(11)$ and $\mathcal{H}(66)$. Their construction of automorphic correction utilized Hilbert modular forms given by Borcherds products associated to weakly holomorphic modular forms of prime levels, which were written explicitly by Bruinier and Bundschuh [4].

More precisely, Borcherds products are associated to vector-valued modular forms ([2]), and Bruinier and Bundschuh established an isomorphism in [4] between vector-valued modular forms and scalar-valued modular forms in the case of *prime*

Received by the editors September 19, 2013 and, in revised form, January 21, 2014 and April 2, 2014.

2010 *Mathematics Subject Classification.* Primary 11F22; Secondary 17B67, 11F41.

The first author was partially supported by an NSERC grant.

levels, and could write Borcherds products associated to scalar-valued forms. The automorphic corrections in [17] are associated with Borcherds products associated to scalar-valued weakly holomorphic modular forms of prime levels $p = 5, 13, 17$.

Only these three families could be considered in [17] because the available information on weakly holomorphic forms was limited at that time. In order to obtain automorphic correction of other rank 2 hyperbolic Kac-Moody algebras, we first need to extend the isomorphism between vector-valued modular forms and scalar-valued modular forms to *non-prime* levels, and to explicitly write Borcherds products associated to the scalar-valued modular forms. Furthermore, it is necessary to compute basis elements of the spaces of scalar-valued modular forms and to show integrality of their Fourier coefficients. Since such Fourier coefficients are root multiplicities of generalized Kac-Moody superalgebras, the integrality is crucial. In the previous paper [17], the integrality was shown only partially and was assumed to be true.

In this paper, we overcome the obstacles mentioned above and consider more general families of rank 2 hyperbolic Kac-Moody algebras $\mathcal{H}(a)$ attached to the generalized Cartan matrix $\left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$. Let $a^2 - 4 = Ns^2$, $s \in \mathbb{N}$, and suppose that N is the fundamental discriminant of $\mathbb{Q}(\sqrt{a^2-4})$. Then we obtain automorphic correction of $\mathcal{H}(a)$ in the cases $N = 12, a = 4$; $N = 8$, $a = 6$; $N = 21$, $a = 5$. Actually, a generalization of the isomorphism between vector-valued forms and scalar-valued forms has already been established by Y. Zhang in his recent preprint [26], and we use the results throughout this paper.

After introducing notation in Section 1, we compute some basis elements f_m $(m \geq 1)$ with the principal part $\frac{1}{s(m)} q^{-m}$ in the spaces of weight 0 scalar-valued modular forms for the discriminants $N = 12, 8, 21$, where $s(m)$ is a normalizing factor. We start with η-quotients and use SAGE to compute the Fourier coefficients of f_m up to q^{18}. An outline of the computation and a presentation of the results are the contents of Section 2. The integrality of Fourier coefficients is considered in Section 3. We use Sturm's theorem [24] and show how one can check the integrality of Fourier coefficients. In particular, we check the integrality for f_1 in each of the cases $N = 12, 8, 21$.

In Section 4, we explicitly write the Borcherds products associated to the scalar-valued modular forms following [2–4, 26]. They are Hilbert modular forms on the quadratic extension $\mathbb{Q}[\sqrt{N}]$. After that, we obtain automorphic correction of the hyperbolic Kac-Moody algebras $\mathcal{H}(a)$, $a = 4, 5, 6$, in Section 5, using the construction in parallel with that of the cases considered in [17]. Namely, the Borcherds product associated to f_1 provides the automorphic correction (Theorem 5.7).

The Fourier coefficients of f_1 are root multiplicities of the generalized Kac-Moody superalgebras in automorphic correction and give natural bounds for root multiplicities of the original rank 2 hyperbolic Kac-Moody algebras. There are no known results on the asymptotic behavior of the root multiplicities of rank 2 hyperbolic Kac-Moody algebras, and it is valuable to know the asymptotic behavior of the Fourier coefficients as it gives information on upper bounds for root multiplicities of rank 2 hyperbolic Kac-Moody algebras. In Section 6, we investigate asymptotics of the Fourier coefficients of f_1's using the method of Hardy-Ramanujan-Rademacher.

For other rank 2 symmetric hyperbolic Kac-Moody algebras, the situation is somewhat different since the obstruction spaces of weight 2 cusp forms are nontrivial (see Lemma 2.1). In fact we prove that f_1 does not exist if $N > 21$

(Lemma 2.3). Hence for $N > 21$, we need new ideas in order to construct automorphic correction. We hope that we can come back to these issues in the future.

1. Special subspace of scalar-valued modular forms

In this section, we recall some definitions and notation from [26]. Let $N_1 > 1$ be a square-free integer. Let $F = \mathbb{Q}(\sqrt{N_1})$ and let $\mathcal{O}_F$ be its ring of integers. Let N be the discriminant of $F/\mathbb{Q}$. Thus if $N_1 \equiv 2, 3 \bmod 4$, then $N = 4N_1$ and $\mathcal{O}_F = \mathbb{Z}[\sqrt{N_1}]$, and if $N_1 \equiv 1 \bmod 4$, then $N = N_1$ and $\mathcal{O}_F = \mathbb{Z}\left[\frac{\sqrt{N_1}+1}{2}\right]$. Let $\mathrm{N}(x)$ and $\mathrm{tr}(x)$ denote the norm and trace of $x \in F/\mathbb{Q}$, respectively. If $\mathfrak{d}$ is the different of $F/\mathbb{Q}$, we know that

$$\mathfrak{d}^{-1} = \{x \in F : \mathrm{tr}(x\mathcal{O}_F) \subset \mathbb{Z}\} = \begin{cases} \frac{1}{2}\mathbb{Z} + \frac{1}{2\sqrt{N_1}}\mathbb{Z}, & N_1 \equiv 2, 3 \bmod 4; \\ \frac{1}{\sqrt{N_1}}\mathcal{O}_F, & N_1 \equiv 1 \bmod 4. \end{cases}$$

Define the following lattice $L = \mathbb{Z}^2 \oplus \mathcal{O}_F$ with the quadratic form

$$q(a, b, \gamma) = \mathrm{N}(\gamma) - ab, \quad a, b \in \mathbb{Z}, \gamma \in \mathcal{O}_F.$$

The corresponding bilinear form is given by

$$((a_1, b_1, \gamma_1), (a_2, b_2, \gamma_2)) = \mathrm{tr}(\gamma_1\gamma_2') - a_1b_2 - a_2b_1.$$

We see that L is an even lattice of signature $(2, 2)$. Its dual lattice is $L' = \mathbb{Z}^2 \oplus \mathfrak{d}^{-1}$, hence the discriminant form is given by $D = L'/L \cong \mathfrak{d}^{-1}/\mathcal{O}_F$. The level of D is N. Denote $q \bmod 1$ on D also by q.

Let k be an even integer. Let ρ_D be the Weil representation of $SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$; that is, if $\{e_\gamma : \gamma \in D\}$ is the standard basis for the group algebra $\mathbb{C}[D]$, then the action

$$\rho_D(T)e_\gamma = e(q(\gamma))e_\gamma,$$
$$\rho_D(S)e_\gamma = \frac{1}{\sqrt{N}} \sum_{\delta \in D} e(-(\gamma, \delta))e_\delta$$

defines the unitary representation ρ_D of $SL_2(\mathbb{Z})$ on $\mathbb{C}[D]$. Here $e(x) = e^{2\pi i x}$, and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ are the standard generators of $SL_2(\mathbb{Z})$.

Let $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$ be the congruence subgroup of matrices whose left lower entry is divisible by N. The weight k slash operator on a function f on the upper half plane is defined as

$$(f|_k M)(\tau) = (\det M)^{\frac{k}{2}}(c\tau + d)^{-k} f(M\tau), \text{ for } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R}).$$

We extend the definition of slash operator to a vector-valued form: for $F = \sum_\gamma F_\gamma e_\gamma$, we define $F|_k M = \sum_\gamma (F_\gamma|_k M)e_\gamma$. Now we let $\mathcal{A}(k, \rho_D)$ be the space of modular forms of weight k and type ρ_D. That is, $F = \sum_\gamma F_\gamma e_\gamma \in \mathcal{A}(k, \rho_D)$ if $F|_k M = \rho_D(M)F$ for any $M \in SL_2(\mathbb{Z})$ and $F_\gamma = \sum_{n \in q(\gamma)+\mathbb{Z}} a(\gamma, n)q^n$ with at most finitely many negative power terms. Let $\mathcal{M}(k, \rho_D)$ and $\mathcal{S}(k, \rho_D)$ denote the space of holomorphic forms and the space of cusp forms, respectively. We denote by $\mathcal{A}^{\mathrm{inv}}(k, \rho_D)$ the subspace of modular forms that are invariant under $\mathrm{Aut}(D)$. Similarly, we have $\mathcal{M}^{\mathrm{inv}}(k, \rho_D)$. As we noted in the introduction, Borcherds products are associated to vector-valued modular forms in $\mathcal{A}(k, \rho_D)$. However, vector-valued modular forms are not easy to handle, and it is useful if we have an isomorphism between vector-valued modular forms and scalar-valued modular forms. It was

done in the case of prime levels in [4]. It turns out that we need to consider a special subspace whose Fourier expansion is supported on either the squares or the non-squares modulo the prime. Non-prime level cases are more delicate, and it was resolved only recently by the third author [26].

Given a Dirichlet character χ modulo N, we denote by $A(N,k,\chi)$ ($M(N,k,\chi)$, $S(N,k,\chi)$, respectively), the space of holomorphic functions f on the upper half plane that satisfy

$$(f|_k M)(\tau) = \chi(d) f(\tau), \quad \text{for all } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

and that are meromorphic (or are holomorphic or vanish, respectively) at all cusps. The functions in the space $A(N,k,\chi)$ are called *weakly holomorphic*.

We define the character $\chi_D := \left(\frac{N}{\cdot}\right)$. For any positive integer m, we denote by $\omega(m)$ the number of distinct prime divisors of m. Define a subspace $A^\delta(N,k,\chi_D)$ of $A(N,k,\chi_D)$ for each $\delta = (\delta_p)_{p|N} \in \{\pm 1\}^{\omega(N)}$ as follows:

$$A^\delta(N,k,\chi_D) = \left\{ f = \sum_n a(n) q^n \in A(N,k,\chi_D) \,\middle|\, a(n) = 0 \right.$$
$$\left. \text{if } \chi_p(n) = -\delta_p \text{ for some } p \mid N \right\}.$$

The condition we impose on the Fourier coefficients of the functions in $A^\delta(N,k,\chi_D)$ will be called the *δ-condition*. Then by [26], Proposition 3.10, $A(N,k,\chi_D) = \bigoplus_\delta A^\delta(N,k,\chi_D)$, where δ runs over $\{\pm 1\}^{\omega(N)}$. Then we have

Theorem 1.1 ([26, Theorem 3.16])**.** *There is an isomorphism between $\mathcal{A}^{\mathrm{inv}}(k,\rho_D)$ and $A^\epsilon(N,k,\chi_D)$ for a certain ϵ defined below.*

If p is odd, we see that the p-component of χ is $\chi_p = \left(\frac{\cdot}{p}\right)$. If $2 \mid N$, the 2-component of χ is given by

$$\chi_2 = \begin{cases} \left(\frac{-4}{\cdot}\right), & \text{if } N_1 \equiv 3 \bmod 4, \\ \left(\frac{2}{\cdot}\right), & \text{if } N_1 \equiv 2 \bmod 8, \\ \left(\frac{-2}{\cdot}\right), & \text{if } N_1 \equiv 6 \bmod 8. \end{cases}$$

We set χ_1 to be the trivial character. For each positive integer m, write $\chi_m = \prod_{p|m} \chi_p$. For each prime $p \mid N$, we define

$$\epsilon_p = \chi_p(-1), \quad \text{if } p \text{ is odd}, \quad \epsilon_2 = \begin{cases} -1, & \text{if } N_1 \equiv 3 \bmod 4, \\ \chi_{N_1/2}(-1), & \text{if } N_1 \equiv 2 \bmod 4. \end{cases}$$

Then we set

$$\epsilon = (\epsilon_p)_{p|N} \in \{\pm 1\}^{\omega(N)}.$$

We also define $\epsilon^* = (\epsilon_p^*)_{p|N}$ to be $\epsilon_p^* = \chi_p(-1)\epsilon_p$. Then we can see easily that $\epsilon_p^* = 1$ for each prime $p \mid N$. Note that if $p \mid N$, $p \equiv 3 \bmod 4$, then the Fourier expansion of $f \in A^\epsilon(N,k,\chi_D)$ is of the form $f = \sum_{n \geq n_0} a_n q^n$, $a_1 = 0$. We also note that if $f = q + O(q^2) \in A^\delta(N,k,\chi_D)$ for some δ, then $\delta = \epsilon^*$.

We shall employ the notation $p^l || N$ for a prime number p and non-negative integers l, N, to mean that $p^l \mid N$ but $p^{l+1} \nmid N$.

2. Computation of basis elements

Recall the following lemma on the obstruction of the existence of weakly holomorphic modular forms. Let $s(m) = 2^{\omega((m,N))}$ for each $m \in \mathbb{Z}$.

Lemma 2.1 ([26, Theorem 4.5]). *Let $P(q) = \sum_{n<0} a(n)q^n$ be a polynomial in q^{-1} that satisfies the ϵ-condition; namely $a(n) = 0$ if $\chi_p(n) = -\epsilon_p$ for some $p \mid N$. There exists $f \in A^\epsilon(N, k, \chi_D)$ with prescribed principal part $P(q)$ if and only if*

$$\sum_{n<0} s(n)a(n)b(-n) = 0,$$

for each $g = \sum_{n>0} b(n)q^n \in S^{\epsilon^*}(N, 2-k, \chi_D)$.

We consider the cases of $N = 12$, $N = 8$ and $N = 21$. With SAGE, we can easily see that in all of these cases we have $S(N, 2, \chi_D) = \{0\}$, hence there are no obstructions for the existence of elements in $A^\epsilon(N, 0, \chi_D)$. We explicitly compute the basis elements of the space $A^\epsilon(N, 0, \chi_D)$. We denote by f_m, $m \in \mathbb{Z}_{>0}$, the function in the space $A^\epsilon(N, 0, \chi_D)$ with the principal part $\frac{1}{s(m)}q^{-m}$, i.e. $f_m = \frac{1}{s(m)}q^{-m} + O(1)$. We will compute the Fourier coefficients of f_m up to q^{18}. We briefly explain how the computations are performed. Firstly, we compute the weight, character, and behavior at all cusps of the η-quotients of corresponding level ([21]). Secondly, we compute the order of zeros of f_m at all cusps. Then we multiply f_m by a suitable quotient of those η-quotients to bring f_m to a holomorphic modular form of certain weight and character. Finally, with SAGE, we can obtain a basis of modular forms of the same weight and character, and by solving a linear system we find our f_m.

Before we list our f_m explicitly in each case, we first prove the uniqueness of f_m if it exists. Note that in the case of $2 \nmid N_1$ this is known by Corollary 3.13 in [26]. For the general case, we pass to the vector-valued forms using the isomorphism in [26].

Lemma 2.2. *If f_m exists, then it is unique.*

Proof. It is enough to prove that if $f \in A^\epsilon(N, 0, \chi_D)$ is holomorphic at ∞, then $f = 0$. The isomorphism from $A^\epsilon(N, 0, \chi_D)$ to $\mathcal{A}^{\text{inv}}(0, \rho_D)$, constructed in [26], is denoted by ψ and its inverse by ϕ. Let $F = \psi(f) \in \mathcal{A}^{\text{inv}}(0, \rho_D)$. By Theorem 3.16 in [26], we see that F_γ is holomorphic at ∞ for each $\gamma \in D$. Let $W = \text{span}_{\mathbb{C}}\{F_\gamma\}$ and $W' = \text{span}_{\mathbb{C}}\{F_0|M \colon M \in SL_2(\mathbb{Z})\}$. Therefore all functions in W, hence in W', are holomorphic at ∞, since we know that $W = W'$ ([26, Section 3]). It follows that $F_0|M$ is holomorphic at ∞ for each $M \in SL_2(\mathbb{Z})$ and $F_0 \in M(N, 0, \chi_D)$. So $F_0 = 0$, $F = 0$ and $f = \phi(F) = 0$. □

2.1. $N = 12$. In this case, $\epsilon_2 = \epsilon_3 = -1$. By Theorem 4.5 in [26], we know that f_m exists if and only if $m \equiv 0, 1, 4, 6, 9, 10 \bmod 12$. We list the first few Fourier coefficients of f_m as follows:

$$\begin{aligned} f_1 &= q^{-1} + 1 + 2q^2 + q^3 - 2q^6 - 2q^8 + 4q^{12} + 4q^{14} - q^{15} + O(q^{18}), \\ f_4 &= \frac{1}{2}q^{-4} + \frac{5}{2} - 2q^2 + 16q^3 + 22q^6 - 35q^8 - 160q^{11} + \frac{209}{2}q^{12} \\ &\quad - 172q^{14} + 416q^{15} + O(q^{18}), \end{aligned}$$

$$
\begin{aligned}
f_6 &= \frac{1}{4}q^{-6} + 3 + \frac{27}{2}q^2 - 16q^3 + 36q^6 + 162q^8 - 864q^{11} + 292q^{12} \\
&\quad + 1080q^{14} - 1440q^{15} + O(q^{18}), \\
f_9 &= \frac{1}{2}q^{-9} + 5 - 54q^2 + 6q^3 - 330q^6 + 1782q^8 + 54q^{11} + 4884q^{12} \\
&\quad - 20844q^{14} - \frac{87}{2}q^{15} + O(q^{18}), \\
f_{10} &= \frac{1}{2}q^{-10} + 2 - 40q^2 - 160q^3 + \frac{1045}{2}q^6 - 1460q^8 + 11840q^{11} + 9080q^{12} \\
&\quad - 20235q^{14} - 59456q^{15} + 88440q^{18} + O(q^{20}), \\
f_{12} &= \frac{1}{4}q^{-12} + \frac{3}{2} + 54q^2 + 144q^3 + 606q^6 + 3807q^8 + 35424q^{11} + 14184q^{12} \\
&\quad + 69444q^{14} + 106144q^{15} + 177246q^{18} + O(q^{20}).
\end{aligned}
$$

For $m > 12$, f_m can be obtained by multiplying one of the above by $j(12\tau)$ and then eliminating other negative power terms.

2.2. $N = 8$**.** Note that $\epsilon_2 = 1$. We know that f_m exists if and only if $m \equiv 0, 1, 2, 4, 6, 7 \bmod 8$. We can explicitly compute the following list:

$$
\begin{aligned}
f_1 &= q^{-1} + 2 + 2q + 4q^2 - 4q^4 - 8q^6 + q^7 + 12q^8 - 2q^9 + 16q^{10} - 24q^{12} \\
&\quad - 32q^{14} - q^{15} + 44q^{16} + 4q^{17} + O(q^{18}), \\
f_2 &= \frac{1}{2}q^{-2} + 3 + 8q - 3q^2 + 14q^4 - 24q^6 - 64q^7 + 42q^8 + 120q^9 - 80q^{10} \\
&\quad + 132q^{12} - \frac{447}{2}q^{14} - 576q^{15} + 370q^{16} + 912q^{17} + O(q^{18}), \\
f_4 &= \frac{1}{2}q^{-4} + 5 - 16q + 28q^2 + 89q^4 + 280q^6 - 896q^7 + 730q^8 - 2288q^9 + 1744q^{10} \\
&\quad + 3984q^{12} + 8480q^{14} - 24448q^{15} + 17366q^{16} - 48928q^{17} + O(q^{18}), \\
f_6 &= \frac{1}{2}q^{-6} + 2 - 48q - 72q^2 + 420q^4 - 1708q^6 + 6528q^7 + 6012q^8 \\
&\quad - 21200q^9 - \frac{36669}{2}q^{10} + 51128q^{12} - 133056q^{14} + 419200q^{15} \\
&\quad + 325644q^{16} - 1000800q^{17} + O(q^{18}), \\
f_7 &= q^{-7} + 16 + 7q - 224q^2 - 1568q^4 + 7616q^6 + 128q^7 + 29792q^8 + 14q^9 \\
&\quad - 101248q^{10} - 310464q^{12} + 878336q^{14} - 896q^{15} + 2328928q^{16} - 7q^{17} + O(q^{18}), \\
f_8 &= \frac{1}{2}q^{-8} + 9 + 96q + 168q^2 + 1460q^4 + 8016q^6 + 34048q^7 + 34737q^8 \\
&\quad + 136608q^9 + 130144q^{10} + 434472q^{12} + 1330368q^{14} + 4533504q^{15} \\
&\quad + 3799986q^{16} + 12556992q^{17} + O(q^{18}).
\end{aligned}
$$

As in the case of $N = 12$, we can compute f_m when $m > 8$ using $j(8\tau)$ and the above list.

2.3. $N = 21$**.** This is a case when N_1 is composite. Note that in this case $\epsilon_3 = \epsilon_7 = -1$ and $\epsilon_3^* = \epsilon_7^* = 1$. Therefore f_m exists if and only if

$$m \equiv 0, 1, 4, 7, 9, 15, 16, 18 \bmod 21.$$

By similar computations in the previous cases, we obtain

$$\begin{aligned} f_1 &= q^{-1} + \frac{1}{2} + q^3 + q^5 - q^6 - q^{14} - q^{17} + 2q^{20} + q^{21} + q^{24} - 2q^{27} \\ &\quad -q^{33} - q^{35} - 2q^{38} + 3q^{41} + 2q^{42} + 3q^{45} + q^{47} - 4q^{48} + O(q^{49}). \end{aligned}$$

Here we show more terms as the level is large. Similar computations can give all f_m with $m \leq 21$ and then using $j(21\tau)$ we can have all f_m.

We prove a lemma for later use in Section 5, which is interesting in its own right.

Lemma 2.3. *The modular form f_1 exists if and only if $1 < N \leq 21$.*

Proof. The fundamental discriminants for $1 < N \leq 21$ are $N = 5, 8, 12, 13, 17, 21$. Now it can be checked that $S(N, 2, \chi_D) = \{0\}$ for $1 < N \leq 21$. It follows from Lemma 2.1 that f_1 exists in $A^\epsilon(N, 0, \chi_D)$. Now assume that $N > 21$. Using dimension formulas [23, Section 6.3, page 98], one can see that $S(N, 2, \chi_D) \neq \{0\}$. Since χ_D is primitive, the space $S(N, 2, \chi_D)$ consists of newforms $f = \sum_{n\geq 1} b(n)q^n$. In particular, if f is a Hecke eigenform, we have $b(1) \neq 0$. It follows that there exists $g = q + O(q^2) \in S^\delta(N, 2, \chi_D)$ for some δ. By the definition of ϵ^*, $\delta = \epsilon^*$. Hence by Lemma 2.1, f_1 does not exist. □

3. Integrality of Fourier coefficients

In this section, we prove the integrality of Fourier coefficients of f_1 in the above cases and the case when $N = 17$, using Sturm's theorem. More precisely, if $f_1 = \sum_n a(n)q^n$, we prove that $s(n)a(n) \in \mathbb{Z}$. The case $N = 17$ was considered in [17], where the integrality was assumed to be true.

For any congruence subgroup Γ, we denote by $M(\Gamma, k)$ the space of holomorphic modular forms of weight k for Γ. For a commutative ring R, let $R[\![q]\!]$ be the ring of power series in q over R. We begin with Sturm's theorem.

Theorem 3.1 ([24])**.** *Let Γ be any congruence subgroup of $SL_2(\mathbb{Z})$, let $\mathcal{O}_F$ be the ring of integers in a number field F, and let $\mathfrak{p}$ be any prime ideal. Assume $f = \sum_n a_n q^n \in M(\Gamma, k) \cap \mathcal{O}_F[\![q]\!]$. If $a_n \in \mathfrak{p}$ for $n \leq \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma]$, then $a_n \in \mathfrak{p}$ for all n.*

Corollary 3.2. *Let Γ be any congruence subgroup of $SL_2(\mathbb{Z})$. Assume $f = \sum_n a_n q^n \in M(\Gamma, k) \cap \mathbb{Q}[\![q]\!]$ with bounded denominator. If $a_n \in \mathbb{Z}$ for $n \leq \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma]$, then $a_n \in \mathbb{Z}$ for all n.*

Proof. Let M be the smallest positive integer such that $Mf \in \mathbb{Z}[\![q]\!]$, and we need to prove that $M = 1$. Suppose $M > 1$ and let p be any prime divisor of M. Now $Mf \in M(\Gamma, k) \cap \mathbb{Z}[\![q]\!]$ and $p \mid Ma_n$ for all n up to $\frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma]$. By Theorem 3.1, we have $p \mid Ma_n$ for all n. Therefore, $p^{-1}Mf \in \mathbb{Z}[\![q]\!]$, contradicting the minimality of M. □

Example 3.3. In the case when $N = p = 17$, we have $[SL_2(\mathbb{Z}) : \Gamma_1(17)] = 288$, so the Sturm's bound is 96. By Proposition 8 in [4] or Proposition 4.7 in [26], f_m has bounded denominator for each m. In this case we see that $\eta(\tau)^7\eta(17\tau)f_1 \in M(\Gamma_1(17), 4)$ (Mayer [22] used a different η-product). Because the constant term of f_1 is $1/2$, we consider the form $g = \eta(\tau)^7\eta(17\tau)(f_1 - \frac{1}{2})$ instead. With SAGE, we can explicitly see that all of the first 96 Fourier coefficients of g are integral, hence all Fourier coefficients of g are integral by Corollary 3.2. Therefore, all Fourier

coefficients of $f_1 - \frac{1}{2}$ are integral. We can similarly do the cases $N = 12, 8$ and 21. Here we only briefly mention some details:

- $N = 12$: we have $f_1(\tau)\eta(\tau)^2\eta(3\tau)^{-2}\eta(4\tau)\eta(6\tau)^2\eta(12\tau) \in M(\Gamma_1(12), 2)$, so the Sturm's bound is 16 and we readily see the integrality.
- $N = 8$: we have $f_1(\tau)\eta(\tau)^{-2}\eta(2\tau)^3\eta(4\tau)\eta(8\tau)^2 \in M(\Gamma_1(8), 2)$, so the Sturm's bound is 8 and the integrality follows easily.
- $N = 21$: Because the constant term is $1/2$, we consider

$$\left(f_1(\tau) - \frac{1}{2}\right)\eta(\tau)^{12}\eta(3\tau)^{-3}\eta(7\tau)^3 \in M(\Gamma_1(21), 6).$$

  Then the Sturm's bound is 192 and the integrality also follows via SAGE.

Remark 3.4. The integrality of $s(n)a(n)$ is expected to hold generally for a *reduced modular form* like f_m. This type of integrality is more precise than the naive integrality $a(n) \in \mathbb{Z}$. This question is raised in [25]; see Section 6 therein for details.

4. Borcherds products

In this section, we explicitly write Borcherds products corresponding to modular forms in $A^\epsilon(N, 0, \chi_D)$. We will use the Hilbert modular forms given by these Borcherds products to establish automorphic correction of some rank 2 hyperbolic Kac-Moody algebras in the next section.

Let $F = \mathbb{Q}(\sqrt{N_1})$ for $N_1 > 1$, a square-free integer, and let N be the fundamental discriminant as before. We keep the notation in Section 1. We write x' for the conjugate of an element $x \in F$. Then $\mathrm{tr}(x) = x + x'$ and $N(x) = xx'$. Denote by ε_0 the fundamental unit in F; in particular $\varepsilon_0 > 1$. Recall that we have the lattice $L = \mathbb{Z}^2 \oplus \mathcal{O}_F$ with the quadratic form $q(a, b, \lambda) = N(\lambda) - ab$ for $a, b \in \mathbb{Z}$ and $\lambda \in \mathcal{O}_F$. We also have the dual lattice $L' = \mathbb{Z}^2 \oplus \mathfrak{d}^{-1}$ of L, the discriminant form $D = L'/L$, and $\chi_D = \left(\frac{N}{\cdot}\right)$.

Denote by $\mathbf{\Gamma}_F = SL_2(\mathcal{O}_F)$ the Hilbert modular group. Let $\mathbb{H}$ be the upper half plane; we use (z_1, z_2) as a standard variable on $\mathbb{H} \times \mathbb{H}$ and write (y_1, y_2) for its imaginary part. For every positive integer m, we denote by $T(m)$ the $\mathbf{\Gamma}_F$-invariant algebraic divisor on $\mathbb{H} \times \mathbb{H}$ defined by:

$$T(m) = \sum_{\substack{(a,b,\lambda)\in L'/\{\pm 1\} \\ ab - N(\lambda) = m/N}} \{(z_1, z_2) \in \mathbb{H} \times \mathbb{H} : az_1z_2 + \lambda z_1 + \lambda' z_2 + b = 0\}.$$

Moreover, define for $m > 0$ the subset in $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$:

$$S(m) = \bigcup_{\substack{\lambda \in \mathfrak{d}^{-1} \\ -N(\lambda) = m/N}} \{(y_1, y_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \colon \lambda y_1 + \lambda' y_2 = 0\}.$$

Fix a weakly holomorphic form $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A^\epsilon(N, 0, \chi_D)$. Each connected component $\mathcal{W}$ of the space

$$\mathbb{R}_{>0} \times \mathbb{R}_{>0} - \bigcup_{\substack{n<0 \\ a(n)\neq 0}} S(-n)$$

is called a *Weyl chamber* associated to f, following the terminology in [3]. We say $\lambda \in F$ is positive with respect to a Weyl chamber $\mathcal{W}$, if $\lambda y_1 + \lambda' y_2 > 0$ for all

vectors $(y_1, y_2) \in \mathcal{W}$, in which case we write $(\lambda, \mathcal{W}) > 0$. For each positive integer m and each Weyl chamber $\mathcal{W}$, define

$$R(m, \mathcal{W}) = \{\lambda \in \mathfrak{d}^{-1} : \ N(\lambda) = -m/N, \ (\varepsilon_0^{-2}\lambda, \mathcal{W}) < 0, \ (\lambda, \mathcal{W}) > 0\}.$$

When N is a prime $p \equiv 1 \bmod 4$, Bruinier and Bundschuh explicitly described Borcherds products corresponding to weight 0 weakly holomorphic modular forms in [4]. With the above settings and computations, we extend their description to cover non-prime cases in Theorem 4.1 below. The general construction of Borcherds can be found in his seminal paper [2].

Theorem 4.1 (cf. [4, Theorem 9])**.** *Let $f = \sum_{n\in\mathbb{Z}} a(n)q^n \in A^\epsilon(N, 0, \chi_D)$ be such that $s(n)a(n) \in \mathbb{Z}$ for all $n < 0$. Then there is a meromorphic function $\Psi(z_1, z_2)$ on $\mathbb{H} \times \mathbb{H}$ with the following properties:*

(1) Ψ is a meromorphic Hilbert modular form for $\mathbf{\Gamma}_F$ with some unitary character of finite order. The weight of Ψ is equal to $s(0)a(0)/2$.

(2) The divisor of Ψ is determined by the principal part (at ∞) of f and equals

$$\sum_{n<0} s(n)a(n)T(-n).$$

(3) Let $\mathcal{W}$ be a Weyl chamber attached to f. Then the function Ψ has the Borcherds product expansion

$$\Psi(z_1, z_2) = e(\rho z_1 + \rho' z_2) \prod_{\substack{\nu\in\mathfrak{d}^{-1} \\ (\nu,\mathcal{W})>0}} (1 - e(\nu z_1 + \nu' z_2))^{s(N\nu\nu')a(N\nu\nu')}, \tag{4.2}$$

where $e(z) = e^{2\pi i z}$. Furthermore, the Weyl vector ρ associated to f and $\mathcal{W}$ is given by

$$\rho = \rho_{f,\mathcal{W}} = \frac{1}{\varepsilon_0^2 - 1} \sum_{m>0} s(-m)a(-m) \sum_{\lambda\in R(m,\mathcal{W})} \lambda.$$

The product converges normally for all (z_1, z_2) with $y_1 y_2 \gg 0$.

(4) There exists a positive integer c such that Ψ^c has integral rational Fourier coefficients with greatest common divisor 1.

Proof. Most of the statements follow from Borcherds' Theorem 13.3 in [2] and the isomorphism between vector-valued and scalar-valued modular form spaces (Theorem 1.1). For the computation of the Weyl vector, one can see Section 3.2 in [3]. The last statement follows from Proposition 4.7 in [26], except the case when $N_1 \equiv 2 \bmod 4$. Actually, by a similar argument utilized in Lemma 2.2, we can see that Proposition 4.7 in [26] is also true for the case $N_1 \equiv 2 \bmod 4$ when δ is specified to be ϵ (or ϵ^*). □

Remark 4.3. When $f = q^{-1} + O(1)$, we can compute Weyl vectors more explicitly. For example, if we choose $\mathcal{W}$ to be the one that contains $(1, \varepsilon_0)$, then

$$\rho_{f,\mathcal{W}} = \begin{cases} \frac{\varepsilon_0}{\sqrt{N}} \frac{1}{\mathrm{tr}(\varepsilon_0)} & \text{if } N(\varepsilon_0) = -1, \\ \frac{1+\varepsilon_0}{\mathrm{tr}(\sqrt{N}\varepsilon_0)} & \text{if } N(\varepsilon_0) = 1. \end{cases} \tag{4.4}$$

This formula is given in Example 3.11 in [3].

4.1. **Computing the weights.** In this subsection, we explain how to compute the weights of the Hilbert modular forms in Theorem 4.1 and will consider the case $N = 12$ as an example. Let f_m be defined as above, and the corresponding vector-valued modular form will be denoted by $F_m = \sum_\gamma F_{\gamma,m} e_\gamma$. By Theorem 13.3 in [2], the weight of Ψ_f is given by $\frac{1}{2}a_0(0)$ where $a_0(0)$ is the constant coefficient of $F_{0,m}$. By Theorem 3.16 in [26], this is in turn given by

$$\frac{1}{2}a_0(0) = \frac{1}{2}s(0)a(0),$$

where $a(0)$ is the 0th Fourier coefficient of f_m. According to this, the weights in the case of $N = 12$, where $s(0) = 4$, are given by

m	1	4	6	9	10	12
$a(0)$	1	5/2	3	5	2	3/2
weight	2	5	6	10	4	3

Alternatively, one can also compute the weight by the theorem on obstructions ([26, Theorem 4.5]) as noted in [4]. More precisely, $a(0)$ is given by the Eisenstein series, $E^{\epsilon^*} = 1 + \sum_{n>0} B(n)q^n$, in the dual modular form space, as follows:

$$a(0) = -\frac{1}{s(0)} \sum_{n<0} s(n)a(n)B(-n),$$

and the weight is then $-\frac{1}{2}\sum_{n<0} s(n)a(n)B(-n)$. Therefore, the principal part determines the weight explicitly. It follows that if $s(n)a(n) \in \mathbb{Z}$ for all negative n, the weight is half integral (or integral). In the case $N = 12$, the constant term $a(0)$ of f_m is given by $-\frac{B(m)}{4}$. Since we have

$$E^{\epsilon^*} = 1 - 4q - 10q^4 - 12q^6 - 20q^9 - 8q^{10} - 6q^{12} - 56q^{13} - 34q^{16} + O(q^{18}),$$

we see that $a(0)$ matches the data given above for each m.

Remark 4.5. (1) The computation of $a(0)$ by $B(m)$ is a special case of a more general phenomenon, *Zagier duality*. See Theorem 5.7 in [25]; note that we normalize E^{ϵ^*} differently therein.

(2) Since $-\frac{B(m)}{2}$, if it is non-zero and $m > 0$, represents the weight of some holomorphic Hilbert modular form, it agrees with the fact that the coefficients $B(m)$ $(m > 0)$ are all non-positive and integral.

5. Automorphic correction

In this section, we construct automorphic correction of some rank 2 hyperbolic Kac-Moody algebras. We begin with the definition of automorphic correction. More details on automorphic correction can be found in [7, 10, 15, 17].

5.1. **Definition.** A Kac-Moody algebra $\mathfrak{g}$ is called *Lorentzian* if its generalized Cartan matrix is given by a set of simple roots of a Lorentzian lattice M, namely, a lattice with a non-degenerate integral symmetric bilinear form $(\cdot,\cdot)$ of signature $(n, 1)$ for some integer $n \geq 1$. A vector $\alpha \in M$ is a root if $(\alpha, \alpha) > 0$ and (α, α) divides $2(\alpha, \beta)$ for all $\beta \in M$. Let Π be a set of (real) simple roots. Then the generalized Cartan matrix A is given by

$$A = \left(\frac{2(\alpha,\alpha')}{(\alpha,\alpha)}\right)_{\alpha,\alpha'\in\Pi}.$$

The Weyl group W is a subgroup of $O(M)$. Consider the cone

$$V(M) = \{\beta \in M \otimes \mathbb{R} \,|\, (\beta, \beta) < 0\},$$

which is a union of two half cones. One of these half cones is denoted by $V^+(M)$. The reflection hyperplanes of W partition $V^+(M)$ into fundamental domains, and we choose one fundamental domain $\mathfrak{D} \subset V^+(M)$ so that the set Π of (real) simple roots is orthogonal to the fundamental domain $\mathfrak{D}$. Then

$$\mathfrak{D} = \{\beta \in \overline{V^+(M)} \,|\, (\beta, \alpha) \leq 0 \text{ for all } \alpha \in \Pi\}.$$

We have a Weyl vector $\rho \in M \otimes \mathbb{Q}$ satisfying $(\rho, \alpha) = -(\alpha, \alpha)/2$ for each $\alpha \in \Pi$.

Define the complexified cone $\Omega(V^+(M)) = M \otimes \mathbb{R} + iV^+(M)$. Let $L = \left(\begin{smallmatrix} 0 & -m \\ -m & 0 \end{smallmatrix}\right) \oplus M$ be an extended lattice for some $m \in \mathbb{N}$. We consider the quadratic space $V = L \otimes \mathbb{Q}$ with the quadratic form induced from the bilinear form on L. Let $V(\mathbb{C})$ be the complexification of V and let $P(V(\mathbb{C})) = (V(\mathbb{C}) - \{0\})/\mathbb{C}^*$ be the corresponding projective space. Let $\mathcal{K}^+$ be a connected component of

$$\mathcal{K} = \{[Z] \in P(V(\mathbb{C})) : (Z, Z) = 0,\ (Z, \bar{Z}) < 0\}, \tag{5.1}$$

and let $O_V^+(\mathbb{R})$ be the subgroup of elements in $O_V(\mathbb{R})$ which preserve the components of $\mathcal{K}$.

For $Z \in V(\mathbb{C})$, write $Z = X + iY$ with $X, Y \in V(\mathbb{R})$. Let $\Gamma \subseteq O_L^+ := O_L \cap O_V^+(\mathbb{R})$ be a subgroup of finite index. Then Γ acts on $\mathcal{K}$ discontinuously. Set

$$\widetilde{\mathcal{K}}^+ = \{Z \in V(\mathbb{C}) - \{0\} : [Z] \in \mathcal{K}^+\}.$$

Let $k \in \frac{1}{2}\mathbb{Z}$ and let χ be a multiplier system of Γ. Then a meromorphic function $\Phi : \tilde{\mathcal{K}}^+ \longrightarrow \mathbb{C}$ is called a *meromorphic modular form* of weight k and multiplier system χ for the group Γ, if

(1) Φ is homogeneous of degree $-k$, i.e. $\Phi(cZ) = c^{-k}\Phi(Z)$ for all $c \in \mathbb{C} - \{0\}$,
(2) Φ is invariant under Γ, i.e. $\Phi(\gamma Z) = \chi(\gamma)\Phi(Z)$ for all $\gamma \in \Gamma$.

Define a map $\Omega(V^+(M)) \to \mathcal{K}$ by $z \mapsto \left[\frac{(z,z)}{2m}\, e_1 + e_2 + z\right]$, where $\{e_1, e_2\}$ is the basis for $\left(\begin{smallmatrix} 0 & -m \\ -m & 0 \end{smallmatrix}\right)$. Then the space $\mathcal{K}^+$ is canonically identified with $\Omega(V^+(M))$.

Consider a meromorphic automorphic form $\Phi(z)$ on $\Omega(V^+(M))$ with respect to a subgroup $\Gamma \subseteq O_L^+$ of finite index. The function $\Phi(z)$ is called an *automorphic correction* of the Lorentzian Kac-Moody algebra $\mathfrak{g}$ if it has a Fourier expansion of the form

$$\Phi(z) = \sum_{w \in W} \det(w) \left(e\left(-(w(\rho), z)\right) - \sum_{a \in M \cap \mathfrak{D},\, a \neq 0} m(a)\, e(-(w(\rho + a), z)) \right), \tag{5.2}$$

where $e(x) = e^{2\pi i x}$ and $m(a) \in \mathbb{Z}$ for all $a \in M \cap \mathfrak{D}$.

We note that $O_V^+(\mathbb{R})$ is the orthogonal group $O(n+1, 2)$, and when $n = 2$, the automorphic forms on $O(3, 2)$ are Siegel modular forms since $SO(3, 2)$ is isogeneous to Sp_4. When $n = 1$, which is our case in this paper, the automorphic forms on $O(2, 2)$ are Hilbert modular forms since $SO(2, 2)$ is isogeneous to $SL_2 \times SL_2$. We also note that the denominator of $\mathfrak{g}$ is $\sum_{w \in W} \det(w) e\left(-(w(\rho), z)\right)$, which is not an automorphic form on $\Omega(V^+(M))$ in general, and one can see from (5.2) that the denominator of $\mathfrak{g}$ is corrected to be an automorphic form $\Phi(z)$.

An automorphic correction $\Phi(z)$ defines a generalized Kac-Moody superalgebra $\mathcal{G}$ as in [10] so that the denominator of $\mathcal{G}$ is $\Phi(z)$. In particular, the function $\Phi(z)$ determines the set of imaginary simple roots of $\mathcal{G}$ in the following way: First,

assume that $a \in M \cap \mathfrak{D}$ and $(a,a) < 0$. If $m(a) > 0$, then a is an even imaginary simple root with multiplicity $m(a)$, and if $m(a) < 0$, then a is an odd imaginary simple root with multiplicity $-m(a)$. Next, assume that $a_0 \in M \cap \mathfrak{D}$ is primitive and $(a_0, a_0) = 0$. Then we define $\mu(na_0) \in \mathbb{Z}$, $n \in \mathbb{N}$, by

$$1 - \sum_{k=1}^{\infty} m(ka_0)t^k = \prod_{n=1}^{\infty} (1-t^n)^{\mu(na_0)},$$

where t is a formal variable. If $\mu(na_0) > 0$, then na_0 is an even imaginary simple root with multiplicity $\mu(na_0)$; if $\mu(na_0) < 0$, then na_0 is an odd imaginary simple root with multiplicity $-\mu(na_0)$.

The generalized Kac-Moody superalgebra $\mathcal{G}$ will also be called an *automorphic correction* of $\mathfrak{g}$. Using the denominator identity for $\mathcal{G}$, the automorphic form $\Phi(z)$ can be written as the infinite product

$$\Phi(z) = e(-(\rho, z)) \prod_{\alpha \in \Delta(\mathcal{G})^+} (1 - e(-(\alpha, z)))^{\mathrm{mult}(\mathcal{G},\alpha)},$$

where $\Delta(\mathcal{G})^+$ is the set of positive roots of $\mathcal{G}$ and $\mathrm{mult}(\mathcal{G}, \alpha)$ is the root multiplicity of α in $\mathcal{G}$.

5.2. **Rank 2 hyperbolic Kac-Moody algebras.** Let $A = \left(\begin{smallmatrix} 2 & -a \\ -a & 2 \end{smallmatrix}\right)$ be a generalized Cartan matrix with $a \geq 3$, and let $\mathcal{H}(a)$ be the hyperbolic Kac-Moody algebra associated with the matrix A. We write $\mathfrak{g} = \mathcal{H}(a)$ if there is no need to specify a. Let $\{h_1, h_2\}$ be the set of simple coroots in the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h_1 + \mathbb{C}h_2 \subset \mathfrak{g}$. Let $\{\alpha_1, \alpha_2\} \subset \mathfrak{h}^*$ be the set of simple roots, let $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ be the root lattice, and define $\mathfrak{h}^*_{\mathbb{Q}} = \mathbb{Q}\alpha_1 + \mathbb{Q}\alpha_2$ and $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}\alpha_1 + \mathbb{R}\alpha_2$. The set of roots of $\mathfrak{g}$ will be denoted by Δ, the set of positive (resp. negative) roots by Δ^+ (resp. by Δ^-), and the set of real (resp. imaginary) roots by Δ_{re} (resp. by Δ_{im}). We will use the notation Δ^+_{re} to denote the set of positive real roots. Similarly, we use Δ^+_{im}, Δ^-_{re} and Δ^-_{im}.

Let $F = \mathbb{Q}(\sqrt{a^2-4})$ and let N be the discriminant of F. We define $s \in \mathbb{N}$ by $a^2 - 4 = Ns^2$. We keep the notation in Section 1 for the quadratic field F. We set

$$\eta = \frac{a + \sqrt{a^2-4}}{2} = \frac{a + s\sqrt{N}}{2}.$$

Then we have $\eta' = \eta^{-1}$ and $1 + \eta^2 = a\eta$. The simple reflection corresponding to α_i in the root system of $\mathfrak{g}$ is denoted by r_i $(i = 1, 2)$ and the Weyl group by W. The eigenvalues of $r_1 r_2$ as a linear transformation on $\mathfrak{h}^*$ are η^2 and η^{-2}. Let γ^+ be an eigenvector for η^2 and we set $\gamma^- = r_2\gamma^+$. Then γ^- is an eigenvector for η'^2. Specifically, we choose

$$\gamma^+ = \frac{\alpha_1 + \eta'\alpha_2}{s} \quad \text{and} \quad \gamma^- = \frac{\alpha_1 + \eta\alpha_2}{s}.$$

We define a symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ to be given by the Cartan matrix A with respect to $\{\alpha_1, \alpha_2\}$. Then we have $(\gamma^+, \gamma^+) = (\gamma^-, \gamma^-) = 0$ and $(\gamma^+, \gamma^-) = -N$.

We will use the column vector notation for the elements in $\mathfrak{h}^*$ with respect to the basis $\{\gamma^+, \gamma^-\}$, i.e. we write $\binom{x}{y}$ for $x\gamma^+ + y\gamma^-$. Then we have

$$\alpha_1 = \frac{1}{\sqrt{N}} \begin{pmatrix} \eta \\ -\eta' \end{pmatrix} \quad \text{and} \quad \alpha_2 = \frac{1}{\sqrt{N}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

It follows that $\mathfrak{h}^*_{\mathbb{Q}} = \{\binom{x}{x'} \mid x \in F\}$. A symmetric bilinear form $\langle \cdot, \cdot \rangle$ on F is defined by $\langle x, y \rangle = -N \operatorname{tr}(xy')$. We define a map $\psi : \mathfrak{h}^*_{\mathbb{Q}} \to F$ by $\binom{x}{x'} \mapsto x$. Then the map ψ is an isometry from $(\mathfrak{h}^*_{\mathbb{Q}}, (\cdot, \cdot))$ to $(F, \langle \cdot, \cdot \rangle)$. In particular, the root lattice $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is mapped onto a sublattice of $\mathcal{O}/\sqrt{N}$.

Let ω_i $(i = 1, 2)$ be the fundamental weights of $\mathfrak{g}$. Then we have

$$\omega_1 = \frac{1}{4 - a^2}(2\alpha_1 + a\alpha_2) \qquad \text{and} \qquad \omega_2 = \frac{1}{4 - a^2}(a\alpha_1 + 2\alpha_2).$$

In the column vector notation,

$$\omega_1 = \frac{-1}{sN}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \omega_2 = \frac{-1}{sN}\begin{pmatrix} \eta \\ \eta' \end{pmatrix}.$$

We define $\rho := -(\omega_1 + \omega_2) = \frac{1}{sN}\left(\begin{smallmatrix} 1+\eta \\ 1+\eta' \end{smallmatrix}\right)$. The simple reflections have the matrix representations

$$r_1 = \begin{pmatrix} 0 & \eta^2 \\ \eta'^2 & 0 \end{pmatrix} \quad \text{and} \quad r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Weyl group W also acts on F by

$$r_1 x = \eta^2 x' \quad \text{and} \quad r_2 x = x' \qquad \text{for } x \in F,$$

so that the isometry ψ is W-equivariant. Since $W = \{(r_1 r_2)^i, r_2(r_1 r_2)^i \mid i \in \mathbb{Z}\}$, we calculate the set of positive real roots and obtain

$$\Delta^+_{\mathrm{re}} = \left\{ \frac{1}{\sqrt{N}}\begin{pmatrix} \eta^j \\ -\eta'^j \end{pmatrix} (j > 0), \qquad \frac{1}{\sqrt{N}}\begin{pmatrix} -\eta'^j \\ \eta^j \end{pmatrix} (j \geq 0) \right\}.$$

We can also obtain a description of the set of positive imaginary roots. See [14] for details.

5.3. **Hilbert modular forms.** We put $M = \psi^{-1}(\mathfrak{d}^{-1}) \subset \mathfrak{h}^*_{\mathbb{Q}}$. Then M is of signature $(1, 1)$ and the Kac-Moody algebra $\mathfrak{g} = \mathcal{H}(a)$ is Lorentzian. We take the Weyl group W for the reflection group of M, and choose the cone

$$V^+(M) = \{x\gamma^+ + y\gamma^- \in \mathfrak{h}^*_{\mathbb{R}} \mid x > 0,\ y > 0\}. \tag{5.3}$$

We set $\Pi = \{\alpha_1, \alpha_2\}$ and obtain the Weyl chamber

$$\mathfrak{D} = \{\beta \in V^+(M) \mid (\beta, \alpha_i) \leq 0,\ i = 1, 2\} = \mathbb{R}_{\leq 0}\,\omega_1 + \mathbb{R}_{\leq 0}\,\omega_2.$$

The Weyl vector is given by $\rho = -(\omega_1 + \omega_2)$.

From our choice of $V^+(M)$ in (5.3), we have the complexified cone

$$\Omega(V^+(M)) = M \otimes \mathbb{R} + iV^+(M) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : Im(z_1) > 0,\ Im(z_2) > 0 \right\} \subset \mathfrak{h}^*$$

with respect to the basis $\{\gamma^+, \gamma^-\}$. Then $\Omega(V^+(M))$ is naturally identified with $\mathbb{H}^2$. We choose the extended lattice $L = \left(\begin{smallmatrix} 0 & -N \\ -N & 0 \end{smallmatrix}\right) \oplus M$. Now it can be shown that an automorphic form on $\Omega(V^+(M))$ is a Hilbert modular form through the identification $\Omega(V^+(M)) \cong \mathbb{H}^2$. See [3, 17] for details. As we identify $\mathbb{H}^2$ with $\Omega(V^+(M)) \subset \mathfrak{h}^*$, the Weyl group W acts on $\mathbb{H}^2$; in particular, we have

$$r_1(z_1, z_2) = (\eta^2 z_2, \eta'^2 z_1) \quad \text{and} \quad r_2(z_1, z_2) = (z_2, z_1).$$

We also define a pairing on $F \times \mathbb{H}^2$ by

$$(\nu, z) = -N\,(\nu z_2 + \nu' z_1), \tag{5.4}$$

for $\nu \in F$ and $z = (z_1, z_2) \in \mathbb{H}^2$.

Our automorphic correction will be a Hilbert modular form with respect to the congruence subgroup $\mathbf{\Gamma}_0(N)$ defined by

$$\mathbf{\Gamma}_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathcal{O}_F) : a, b, d \in \mathcal{O}_F,\ c \in (N) \right\} \subset O_L^+,$$

where $(N) \subset \mathcal{O}_F$ is the principal ideal generated by N. We will need the following lemma, whose proof is essentially the same as that of Lemma 5.13 of [17].

Lemma 5.5. *Let $g(z)$ be a Hilbert modular form with respect to $SL_2(\mathcal{O}_F)$. Define $f(z) = \overline{g}(Nz)$, where $\overline{g}(z_1, z_2) = g(z_2, z_1)$. Then the function $f(z)$ is a Hilbert modular form with respect to the congruence subgroup $\mathbf{\Gamma}_0(N)$.*

5.4. **Construction of automorphic correction.** Assume that $1 < N \leq 21$. Then the modular form f_1 exists by Lemma 2.3. In particular, we consider the following three cases: (1) $N = 12$, $a = 4$; (2) $N = 8$, $a = 6$; (3) $N = 21$, $a = 5$. The fundamental units ε_0 are: $2 + \sqrt{3}$, $1 + \sqrt{2}$, $\frac{5+\sqrt{21}}{2}$, respectively, and $\eta = \varepsilon_0$ for $N = 12, 21$, and $\eta = \varepsilon_0^2$ for $N = 8$.

We choose the Weyl chamber $\mathcal{W}$ attached to f_1 which contains the point $(1, \varepsilon_0)$. Then the Weyl vector is given by the formula (4.4). In the case $N(\varepsilon_0) = -1$, the point $(\varepsilon_0^{-1}, \varepsilon_0)$ lies in the same Weyl chamber $\mathcal{W}$. (See Example 3.11 in [3].)

Recall that we have the isometry $\psi : \mathfrak{h}_{\mathbb{Q}}^* \to F$ by $\binom{\nu}{\nu'} \mapsto \nu$. One can check that

$$\psi(\rho) = \frac{1}{sN}(1 + \eta) = \rho_{f_1, \mathcal{W}}, \tag{5.6}$$

and we write $\rho = \rho_{f_1, \mathcal{W}}$ if there is no peril of confusion.

First, assume that $\nu \gg 0$ and $(\nu, \mathcal{W}) > 0$ for $\nu \in \mathfrak{d}^{-1}$. Then $N(\nu) > 0$ and $\langle \nu, \nu \rangle = -N\mathrm{tr}(\nu\nu') < 0$. Thus ν corresponds to an imaginary root of $\mathcal{H}(a)$ by Proposition 5.10 of [11]. We can also check $\langle \rho, \nu \rangle < 0$ using $\nu + \varepsilon_0 \nu' > 0$ or $\varepsilon_0^{-1}\nu + \varepsilon_0 \nu' > 0$ if $N(\varepsilon_0) = -1$. Thus ν is positive, i.e. $\nu \in \psi(\Delta_{\mathrm{im}}^+)$.

Next, assume that $\nu \not\gg 0$ and $(\nu, \mathcal{W}) > 0$ for $\nu \in \mathfrak{d}^{-1}$. Then $N(\nu) < 0$ and $a(N\nu\nu') \neq 0$ only for ν with $N(\nu) = \nu\nu' = -1/N$, in which case $s(N\nu\nu')a(N\nu\nu') = 1$. Further, $\langle \nu, \nu \rangle = 2$, and ν corresponds to a positive real root of $\mathcal{H}(a)$, i.e. $\nu \in \psi(\Delta_{\mathrm{re}}^+)$.

Now, from the above observations, the Borcherds product (4.2) can be written:

$$\begin{aligned}
\Psi(z_1, z_2) &= e(\rho z_1 + \rho' z_2) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ (\nu, \mathcal{W}) > 0}} (1 - e(\nu z_1 + \nu' z_2))^{s(N\nu\nu')a(N\nu\nu')} \\
&= e(\rho z_1 + \rho' z_2) \prod_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} (1 - e(\nu z_1 + \nu' z_2))^{s(N\nu\nu')a(N\nu\nu')} \\
&\quad \times \prod_{\substack{\nu \in \mathfrak{d}^{-1}, \nu + \varepsilon_0 \nu' > 0 \\ N(\nu) = -1/N}} (1 - e(\nu z_1 + \nu' z_2))
\end{aligned}$$

$$= e(\rho z_1 + \rho' z_2) \prod_{\nu \in \psi(\Delta_{\mathrm{im}}^+)} (1 - e(\nu z_1 + \nu' z_2))^{s(N\nu\nu')a(N\nu\nu')}$$
$$\times \prod_{\nu \in \psi(\Delta_{\mathrm{re}}^+)} (1 - e(\nu z_1 + \nu' z_2)).$$

Define $\Phi(z) = \Phi(z_1, z_2) = \Psi(Nz_2, Nz_1)$. Then by Lemma 5.5, Φ is a Hilbert modular form with respect to $\mathbf{\Gamma}_0(N)$. By (5.4), Φ can be written as

$$\Phi(z) = e(-(\rho, z)) \prod_{\nu \in \psi(\Delta_{\mathrm{im}}^+)} (1 - e(-(\nu, z)))^{s(N\nu\nu')a(N\nu\nu')} \prod_{\nu \in \psi(\Delta_{\mathrm{re}}^+)} (1 - e(-(\nu, z))).$$

As in [17], we can prove

$$\Phi(wz) = \det(w)\Phi(z) \quad \text{for } w \in W.$$

This in turn implies that $\Phi(z)$ can be written as

$$\Phi(z) = \sum_{w \in W} \det(w) \left(e(-(w(\rho), z)) - \sum_{\nu \in M \cap \mathfrak{D}, \nu \neq 0} m(\nu) e(-(w(\rho + \nu), z)) \right).$$

This is exactly the form for the automorphic correction in (5.2), and hence it provides an automorphic correction for $\mathcal{H}(a)$. So we have obtained:

Theorem 5.7. *Let $\mathcal{H}(a)$ be the rank 2 symmetric hyperbolic Kac-Moody algebra, and let N be the discriminant of the quadratic field $\mathbb{Q}(\sqrt{a^2 - 4})$. Assume that $1 < N \leq 21$. Then the Hilbert modular form Φ provides an automorphic correction for the hyperbolic Kac-Moody algebra $\mathcal{H}(a)$. In particular, there exists a generalized Kac-Moody superalgebra $\widetilde{\mathcal{H}}$ whose denominator function is the Hilbert modular form Φ.*

Remark 5.8. The prime level cases $N = 5, 13, 17$ were established in Theorem 5.16 of [17]. The above Theorem 5.7 includes three non-prime level cases $N = 12, 8, 21$. Altogether, we have constructed automorphic correction for the hyperbolic Kac-Moody algebra $\mathcal{H}(a)$ corresponding to each of the fundamental discriminants $1 < N \leq 21$. We summarize it in a table:

N	5	8	12	13	17	21
a	3	6	4	11	66	5

If $N > 21$, the modular form f_1 does not exist by Lemma 2.3, and we need a new idea to construct automorphic correction.

6. Asymptotics of Fourier coefficients

In this section, we obtain asymptotics of Fourier coefficients of the modular forms f_1 defined in Section 2. Note that the Fourier coefficients of f_1 are root multiplicities of the generalized Kac-Moody superalgebra $\widetilde{\mathcal{H}}$ with some modifications and give natural bounds for root multiplicities of $\mathcal{H}(a)$. Hence, asymptotics of the Fourier coefficients give information on upper bounds for root multiplicities of $\mathcal{H}(a)$.

We apply the result of J. Lehner [18] on Fourier coefficients of modular forms using the method of Hardy-Ramanujan-Rademacher to our special case. We refer

to [18] for unexplained notation. (See [16, 17] for the details.) Let

$$f_1 = q^{-1} + \sum_{n=0}^{\infty} a(n) q^n.$$

Since f_1 is holomorphic at all other cusps except at ∞ (Corollary 3.6 of [25] and Corollary 3.14 of [26]), we can show that

$$a(n) = \begin{cases} \frac{2\pi}{3\sqrt{n}} \sin \frac{\pi(n-1)}{2} \sin \frac{\pi(n-1)}{3} I_1\left(\frac{\pi\sqrt{n}}{3}\right) \\ \qquad + O\left(n^{\frac{1}{4}} \left(\log \frac{\pi\sqrt{n}}{3}\right) I_1\left(\frac{\pi\sqrt{n}}{6}\right)\right), & \text{if } N = 12, \\ \frac{\pi}{\sqrt{n}} \sin \frac{\pi(n-1)}{2} \sin \frac{\pi(n-1)}{4} I_1\left(\frac{\pi\sqrt{n}}{2}\right) \\ \qquad + O\left(n^{\frac{1}{4}} \left(\log \frac{\pi\sqrt{n}}{2}\right) I_1\left(\frac{\pi\sqrt{n}}{4}\right)\right), & \text{if } N = 8, \\ \frac{2\pi}{\sqrt{21n}} \left(\frac{n}{21}\right) I_1\left(\frac{4\pi\sqrt{n}}{\sqrt{21}}\right) \sum_{v^2 \equiv -n \bmod 21} e\left(\frac{2v}{21}\right) \\ \qquad + O\left(n^{\frac{1}{4}} \left(\log \frac{4\pi\sqrt{n}}{21}\right) I_1\left(\frac{2\pi\sqrt{n}}{21}\right)\right), & \text{if } N = 21, \end{cases}$$

where I_1 is the Bessel I-function, and it has the asymptotic expansion $I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 + O(\frac{1}{x})\right)$.

Remark 6.1. Following [16, 17], when $N = 12$, we can show that f_{12} has non-negative Fourier coefficients: Note the following Fourier expansions along various cusps: f_{12} is holomorphic at $\frac{1}{2}$, $\frac{1}{6}$, and by the ϵ-condition,

$$f_{12}\left(\tfrac{1}{3} - \tfrac{1}{\tau}\right) = -\tfrac{i}{2} q^{-\frac{1}{12}} + O(1); \quad f_{12}\left(\tfrac{1}{4} - \tfrac{1}{\tau}\right) = -\tfrac{i\sqrt{3}}{4} q^{-\frac{1}{12}} + O(1);$$

$$f_{12}\left(-\tfrac{1}{\tau}\right) = \tfrac{\sqrt{3}}{2} q^{-\frac{1}{12}} + O(1).$$

Then if $f_{12} = \frac{1}{4} q^{-12} + \sum_{n=0}^{\infty} a(n) q^n$, then $a(n) \geq 0$ for all n, and

$$\begin{aligned} a(n) &= \tfrac{\pi}{\sqrt{n}} \left(\tfrac{1}{\sqrt{3}} \left(\sin \tfrac{\pi n}{2} \sin \tfrac{\pi n}{3} - \sin \tfrac{2\pi n}{3}\right) + \tfrac{1}{2}\left(1 - \sin \tfrac{\pi n}{2}\right)\right) I_1\left(\tfrac{4\pi\sqrt{n}}{\sqrt{12}}\right) \\ &+ O\left(n^{\frac{1}{4}} \left(\log \tfrac{4\pi\sqrt{n}}{\sqrt{12}}\right) I_1\left(\tfrac{2\pi\sqrt{n}}{\sqrt{12}}\right)\right). \end{aligned}$$

Similarly, when $N = 8$, f_8 has non-negative Fourier coefficients: f_8 is holomorphic at $\frac{1}{2}$ and $\frac{1}{4}$, and $f_8\left(-\frac{1}{\tau}\right) = \sqrt{2} q^{-\frac{1}{8}} + O(1)$. So if $f_8 = \frac{1}{2} q^{-8} + \sum_{n=0}^{\infty} a(n) q^n$, then $a(n) \geq 0$ for all n, and

$$a(n) = \tfrac{\pi}{\sqrt{n}} \left(1 + \sqrt{2} \sin \tfrac{\pi n}{2} \sin \tfrac{\pi n}{4}\right) I_1(\pi\sqrt{2n}) + O\left(n^{\frac{1}{4}} \left(\log \pi\sqrt{2n}\right) I_1\left(\tfrac{\pi\sqrt{n}}{\sqrt{2}}\right)\right).$$

Acknowledgments

We thank J. Bruinier who explained how to extend the result in his paper with Bundschuh [4] to general discriminant forms. We thank the referee and the editor for helpful remarks.

References

[1] Richard E. Borcherds, *Monstrous moonshine and monstrous Lie superalgebras*, Invent. Math. **109** (1992), no. 2, 405–444, DOI 10.1007/BF01232032. MR1172696 (94f:11030)

[2] Richard E. Borcherds, *Automorphic forms with singularities on Grassmannians*, Invent. Math. **132** (1998), no. 3, 491–562, DOI 10.1007/s002220050232. MR1625724 (99c:11049)

[3] Jan Hendrik Bruinier, *Hilbert modular forms and their applications*, The 1-2-3 of modular forms, Universitext, Springer, Berlin, 2008, pp. 105–179, DOI 10.1007/978-3-540-74119-0_2. MR2447162 (2009g:11057)

[4] Jan Hendrik Bruinier and Michael Bundschuh, *On Borcherds products associated with lattices of prime discriminant*, Ramanujan J. **7** (2003), no. 1-3, 49–61, DOI 10.1023/A:1026222507219. Rankin memorial issues. MR2035791 (2005a:11057)

[5] Alex J. Feingold and Igor B. Frenkel, *A hyperbolic Kac-Moody algebra and the theory of Siegel modular forms of genus* 2, Math. Ann. **263** (1983), no. 1, 87–144, DOI 10.1007/BF01457086. MR697333 (86a:17006)

[6] Valery A. Gritsenko and Viacheslav V. Nikulin, *Igusa modular forms and "the simplest" Lorentzian Kac-Moody algebras*, Mat. Sb. **187** (1996), no. 11, 27–66.

[7] Valeri A. Gritsenko and Viacheslav V. Nikulin, *Siegel automorphic form corrections of some Lorentzian Kac-Moody Lie algebras*, Amer. J. Math. **119** (1997), no. 1, 181–224. MR1428063 (98g:11056)

[8] Valeri A. Gritsenko and Viacheslav V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. I*, Internat. J. Math. **9** (1998), no. 2, 153–199, DOI 10.1142/S0129167X98000105. MR1616925 (99b:11040)

[9] Valeri A. Gritsenko and Viacheslav V. Nikulin, *Automorphic forms and Lorentzian Kac-Moody algebras. II*, Internat. J. Math. **9** (1998), no. 2, 201–275, DOI 10.1142/S0129167X98000117. MR1616929 (99b:11041)

[10] V. A. Gritsenko and V. V. Nikulin, *On the classification of Lorentzian Kac-Moody algebras* (Russian, with Russian summary), Uspekhi Mat. Nauk **57** (2002), no. 5(347), 79–138, DOI 10.1070/RM2002v057n05ABEH000553; English transl., Russian Math. Surveys **57** (2002), no. 5, 921–979. MR1992083 (2004f:17034)

[11] Victor G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, Cambridge, 1990. MR1104219 (92k:17038)

[12] Victor G. Kac and Dale H. Peterson, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), no. 2, 125–264, DOI 10.1016/0001-8708(84)90032-X. MR750341 (86a:17007)

[13] Victor G. Kac and Minoru Wakimoto, *Modular and conformal invariance constraints in representation theory of affine algebras*, Adv. in Math. **70** (1988), no. 2, 156–236, DOI 10.1016/0001-8708(88)90055-2. MR954660 (89h:17036)

[14] Seok-Jin Kang and Duncan J. Melville, *Rank* 2 *symmetric hyperbolic Kac-Moody algebras*, Nagoya Math. J. **140** (1995), 41–75. MR1369479 (97c:17039)

[15] Henry H. Kim and Kyu-Hwan Lee, *Automorphic correction of the hyperbolic Kac-Moody algebra* E_{10}, J. Math. Phys. **54** (2013), no. 9, 091701, 13, DOI 10.1063/1.4820562. MR3135585

[16] Henry H. Kim and Kyu-Hwan Lee, *Root multiplicities of hyperbolic Kac-Moody algebras and Fourier coefficients of modular forms*, Ramanujan J. **32** (2013), no. 3, 329–352, DOI 10.1007/s11139-013-9474-2. MR3130654

[17] Henry H. Kim and Kyu-Hwan Lee, *Rank 2 symmetric hyperbolic Kac-Moody algebras and Hilbert modular forms*, J. Algebra **407** (2014), 81–104, DOI 10.1016/j.jalgebra.2014.03.003. MR3197153

[18] Joseph Lehner, *Discontinuous groups and automorphic functions*, Mathematical Surveys, No. VIII, American Mathematical Society, Providence, R.I., 1964. MR0164033 (29 #1332)

[19] James Lepowsky and Robert V. Moody, *Hyperbolic Lie algebras and quasiregular cusps on Hilbert modular surfaces*, Math. Ann. **245** (1979), no. 1, 63–88, DOI 10.1007/BF01420431. MR552580 (81c:10030)

[20] I. G. Macdonald, *Affine root systems and Dedekind's* η*-function*, Invent. Math. **15** (1972), 91–143. MR0357528 (50 #9996)

[21] Yves Martin, *Multiplicative* η*-quotients*, Trans. Amer. Math. Soc. **348** (1996), no. 12, 4825–4856, DOI 10.1090/S0002-9947-96-01743-6. MR1376550 (97d:11070)

[22] Sebastian Mayer, *Hilbert modular forms for the fields* $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{13})$, $\mathbb{Q}(\sqrt{17})$, Dissertation (2007).

[23] William Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007. With an appendix by Paul E. Gunnells. MR2289048 (2008d:11037)

[24] Jacob Sturm, *On the congruence of modular forms*, Number theory (New York, 1984), Lecture Notes in Math., vol. 1240, Springer, Berlin, 1987, pp. 275–280, DOI 10.1007/BFb0072985. MR894516 (88h:11031)

[25] Yichao Zhang, *Zagier duality and integrality of Fourier coefficients for weakly holomorphic modular forms*, to appear in J. Number Theory, arXiv: 1308.1037 (2013).

[26] Yichao Zhang, *An isomorphism between scalar-valued modular forms and modular forms for Weil representations*, to appear in Ramanujan J., arXiv:1307.4390 (2014).

Department of Mathematics, University of Toronto, Toronto, Ontario M5S 2E4, Canada – and – Korea Institute for Advanced Study, Seoul, Korea

E-mail address: henrykim@math.toronto.edu

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269

E-mail address: khlee@math.uconn.edu

Department of Mathematics, University of Connecticut, Storrs, Connecticut 06269

E-mail address: yichao.zhang@uconn.edu

Current address: Department of Mathematics, The University of Arizona, Tucson, Arizona 85721

E-mail address: yichaozhang@math.arizona.edu