CONDITIONS FOR THE VANISHING
OF THE GENUS-2 G-FUNCTION

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ABSTRACT. In this paper we give some sufficient conditions for the vanishing
of the genus-2 G-function, which was introduced by B. Dubrovin, S. Liu and
Y. Zhang in 2012. As a corollary we prove their conjecture for the vanishing
of the genus-2 G-function for ADE singularities.

In this paper, we study properties of the genus-2 potential function $F_2$ for
semisimple Frobenius manifolds. In [DZ], Dubrovin and Zhang proved that higher
genus potential functions can be uniquely constructed from the Virasoro constraints
and the genus-0 potential function for semisimple Frobenius manifolds. They also
gave an explicit formula for $F_2$ for semisimple Frobenius manifolds under the as-
sumption of the genus-2 Virasoro conjecture. In [L07], by studying genus-2 topo-
logical recursion relations for Gromov-Witten invariants, the first author of this
paper obtained a much simpler formula of $F_2$ in terms of genus-0 Gromov-Witten
invariants for compact symplectic manifolds with semisimple quantum cohomology,
and proved the genus-2 Virasoro conjecture using this formula. Since the proofs of
[L07] only used universal equations for Gromov-Witten invariants which were ob-
tained from the splitting principle and relations in the tautological rings of moduli
spaces of stable curves, the formula found in [L07] holds for the genus-2 generating
functions of descendant invariants of any semisimple cohomological field theory.
Both formulas in [DZ] and [L07] used canonical coordinates for semisimple Frobe-
nius manifolds (or a generalization of canonical coordinates for the big phase space
of the Frobenius manifolds). However geometric invariants for cohomological field
theories usually appear as coefficients of the formal power series representations
of potential functions in terms of flat coordinates. In general, explicit formulas for the
transition between flat coordinates and canonical coordinates are not easy to find.
Therefore from a geometric point of view, it is desirable to have a formula for $F_2$
purely in terms of flat coordinates.

Recently in [DLZ], B. Dubrovin, S. Liu and Y. Zhang decomposed $F_2$ as a sum-
mation of two functions:

$$F_2 = F_2' + G^{(2)},$$

where the first part $F_2'$ can be expressed purely in terms of flat coordinates, and
the second part $G^{(2)}$, called the genus-2 G-function, still depends on the canoni-
cal coordinates. An explicit formula for the genus-2 G-function can be found in
Appendix A. We will not define $F'_2$ since most of this paper does not need this function. Interested readers are referred to [DLZ] for the exact form of $F'_2$. It is an interesting problem to find conditions under which the genus-2 G-function vanishes, i.e. $G^{(2)} = 0$. In such cases, we will have a formula for $F_2$ completely expressed in terms of flat coordinates.

The main purpose of this paper is to give a set of geometric conditions which force the vanishing of the genus-2 G-function. Let $(\mathcal{H}, \eta, \Lambda)$ be a cohomological field theory of rank $N$, where $\mathcal{H}$ is an $N$-dimensional vector space with a non-degenerate pairing $\eta$ and $\Lambda := \{\Lambda_{g,n}\}$ are $n$-linear $H^\bullet(\mathcal{M}_{g,n})$-valued forms on $\mathcal{H}$, i.e. $\Lambda_{g,n} \in Hom(\mathcal{H}^{\otimes n}, H^\bullet(\mathcal{M}_{g,n}))$, satisfying the axioms of cohomological field theory (cf. [KM]). Let $\{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ be a fixed basis of $\mathcal{H}$, and $\eta^{\alpha \beta}$ entries for the inverse of the matrix of $\eta$ in this basis. We will use $\eta^{\alpha \beta}$ to raise indices, for example, $\gamma_\alpha = \sum_\beta \eta^{\alpha \beta} \gamma_\beta$ for any $\alpha$.

Let $\langle \tau_{n_1}(\gamma_{\alpha_1}) \cdots \tau_{n_k}(\gamma_{\alpha_k}) \rangle_g$ be genus-$g$ descendant invariants of the cohomological field theory determined by $\Lambda$. We will identify $\tau_0(\gamma_\alpha)$ with $\gamma_\alpha$ and call $\langle \gamma_{\alpha_1} \cdots \gamma_{\alpha_k} \rangle_g$ primary invariants. The genus-0 primary invariants of a cohomological field theory define a Frobenius manifold structure on $\mathcal{H}$. Consider the following three conditions:

\begin{align*}
(C1) & \quad \sum_{\alpha, \beta} \langle \gamma_\alpha \gamma_\alpha \gamma_\beta \gamma_\alpha \cdots \gamma_{\alpha_k} \rangle_0 = 0, \\
(C2) & \quad \langle \gamma_{\alpha_1} \cdots \gamma_{\alpha_k} \rangle_1 = 0, \\
(C3) & \quad \langle \gamma_{\alpha_1} \cdots \gamma_{\alpha_k} \rangle_2 = 0 \text{ and } \langle \tau_{1}(\gamma_{\alpha_1}) \gamma_{\alpha_2} \cdots \gamma_{\alpha_k} \rangle_2 = 0,
\end{align*}

for all $\alpha_1, \ldots, \alpha_k$, $k \geq 1$. Condition (C1) is inspired by Lemma 2.5 in [DLZ]. The relation between this condition and Lemma 2.5 in [DLZ] will be explained in Section 3. The proof of Lemma 2.5 in [DLZ] for ADE singularities was quite complicated and was done via a case by case study (cf. Section 3.1 to Section 3.4 in [DLZ]). We will give a unified and much simpler proof of this lemma in Section 3 (cf. Remark 3.3). By the genus-1 constitutive relation of Dijkgraaf and Witten, the genus-1 generating function $F_1$ can be written as a sum of a genus-0 function together with a function which still depends on genus-1 primary invariants (cf. [DW]). The latter part of this decomposition was called the genus-1 G-function by Dubrovin and Zhang. Condition (C2) is equivalent to the genus-1 G-function being equal to a constant. The first part of condition (C3) is a necessary condition for the vanishing of the genus-2 G-function since the restriction of $F'_2$ to the small phase space (i.e. $\mathcal{H}$) is zero. Under the assumption of condition (C2), it can be shown that the second part of condition (C3) is also a necessary condition for the vanishing of the genus-2 G-function due to the string equation. The main result of this paper is the following.

**Theorem 0.1.** For any semisimple cohomological field theory satisfying conditions (C1), (C2) and (C3), the genus-2 G-function vanishes.

It can be shown that for the FJRW theory of ADE singularities (cf. [FJR]), the three conditions (C1)-(C3) are all satisfied. Therefore an immediate consequence of Theorem 0.1 is the following.

**Corollary 0.2.** The genus-2 G-function vanishes for the cohomological field theories associated to ADE singularities.

It was conjectured in [DLZ] that the genus-2 G-function vanishes for the cohomological field theories associated to ADE singularities and Gromov-Witten theory.
for \( \mathbb{P}^1 \)-orbifolds of ADE type. Corollary 0.2 gives an affirmative answer to this conjecture for ADE singularities. Recently Y. Fu, S. Liu, Y. Zhang, and C. Zhou proved this conjecture for \( A_n \) singularities using a completely different method (cf. [FLZZ]). Their proof depends on specific structures of Frobenius manifolds associated to \( A_n \) singularities, which do not apply to singularities of DE type. In comparison, our method gives a more geometric explanation and applies to all ADE singularities, and possibly also applies to other cohomological field theories as well. We will discuss the case for \( \mathbb{P}^1 \)-orbifolds of ADE type in a forthcoming paper.

This paper is organized as follows. In section 1 we introduce notation and review basic theories needed in this paper. We prove Theorem 0.1 in section 2 and prove Corollary 0.2 in section 3. In section 4 we make some remarks on how to simplify \( F_2 \) if the genus-2 G-function is zero and condition (C2) is satisfied. We give the precise definition for the genus-2 G-function in the appendix.

1. Preliminaries

Properties of the quantum product and idempotents on the big phase space for Gromov-Witten theory were studied in [L02], [L06] and [L07]. These properties can be easily extended to any cohomological field theory. In this section we will review the concepts and properties needed for the proof of Theorem 0.1.

Let \((\mathcal{H}, \eta, \Lambda)\) be a cohomological field theory of rank \( N \). Fix a basis \( \{\gamma_{\alpha} | \alpha = 1, \cdots, N\} \) of \( \mathcal{H} \) with \( \gamma_1 \) equal to the identity of the Frobenius manifold structure on \( \mathcal{H} \). As a convention, repeated Greek letter indices are summed up over their entire ranges from 1 to \( N \). The vector space \( \mathcal{H} \) is also called the small phase space.

The big phase space is defined to be \( \mathcal{P} := \prod_{n=0}^{\infty} \mathcal{H} \), and the corresponding basis for the \( n \)-th copy of \( \mathcal{H} \) in this product is denoted by \( \{\tau_n(\gamma_{\alpha}) | \alpha = 1, \cdots, N\} \) for \( n \geq 0 \). Let \( t_{n}^{a} \) be the coordinates on \( \mathcal{P} \) with respect to the standard basis \( \{\tau_n(\gamma_{\alpha}) | \alpha = 1, \cdots, N, \ n \geq 0\} \). These coordinates are called the flat coordinates on \( \mathcal{P} \). We will also identify the small phase space \( \mathcal{H} \) with the subspace of \( \mathcal{P} \) defined by \( t_{n}^{a} = 0 \) for \( n > 0 \).

The genus-\( g \) potential function \( F_{g} \) is a formal power series of \( t = (t_{n}^{a}) \) with coefficients given by \( \langle \tau_{n_1}(\gamma_{a_1}) \cdots \tau_{n_k}(\gamma_{a_k}) \rangle \). Derivatives of \( F_{g} \) with respect to \( t_{n_1}^{a_1}, \cdots, t_{n_k}^{a_k} \) are denoted by \( \langle \tau_{n_1}(\alpha_1) \cdots \tau_{n_k}(\alpha_k) \rangle \). We will identify \( \tau_n(\gamma_{\alpha}) \) with \( \partial/\partial \tau_n^{\alpha} \) as vector fields on the big phase space. We will also write \( \tau_0(\gamma_{\alpha}) \) simply as \( \gamma_{\alpha} \). Any vector field of the form \( \sum_{\alpha} f_{\alpha} \gamma_{\alpha} \), where \( f_{\alpha} \) are functions on the big phase space, is called a primary vector field. We use \( \tau_+ \) and \( \tau_- \) to denote the operators which shift the level of descendants, i.e.

\[
\tau_{\pm} \left( \sum_{n, \alpha} f_{n, \alpha} \tau_n(\gamma_{\alpha}) \right) = \sum_{n, \alpha} f_{n, \alpha} \tau_{n \pm 1}(\gamma_{\alpha})
\]

where \( f_{n, \alpha} \) are functions on the big phase space.

Define a k-tensor \( \langle \cdots \rangle \) by

\[
\langle W_1 W_2 \cdots W_k \rangle := \sum_{m_1, \alpha_1, \cdots, m_k, \alpha_k} f_{m_1, \alpha_1}^{1} \cdots f_{m_k, \alpha_k}^{k} \frac{\partial^k}{\partial t_{m_1}^{\alpha_1} \cdots \partial t_{m_k}^{\alpha_k}} F_{g}
\]

for vector fields \( W_i = \sum_{n, \alpha} f_{n, \alpha}^{i} \) where \( f_{n, \alpha}^{i} \) are functions on the big phase space. This tensor is called the k-point (correlation) function. For any vector fields
\( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) on the big phase space, the *quantum product* of \( \mathcal{W}_1 \) and \( \mathcal{W}_2 \) is defined by
\[
\mathcal{W}_1 \circ \mathcal{W}_2 := \langle \mathcal{W}_1 \mathcal{W}_2 \gamma^\alpha \rangle_0 \gamma_\alpha.
\]
This is a commutative and associative product, but it does not have an identity.

Let
\[
\mathcal{X} := - \sum_{m, \alpha} (m + d_\alpha) \tilde{t}_m^\alpha \tau_m (\gamma_\alpha) - \sum_{m, \alpha, \beta} C_\alpha^\beta \tilde{t}_m^\alpha \tau_{m-1} (\gamma_\beta)
\]
be the *Euler vector field* on the big phase space, where \( \tilde{t}_m^\alpha = t_m^\alpha - \delta_{m,1} \delta_{\alpha,1} \) and the constants \( d_\alpha \) and \( C_\alpha^\beta \) are uniquely determined by the fact that the restriction of \( \mathcal{X} \) to the small phase space is the *Euler vector field* of the Frobenius manifold structure on \( \mathcal{H} \). For example, for quantum cohomology of any compact symplectic manifold \( M, \mathcal{H} = H^*(M, \mathbb{C}) \), \( d_\alpha = \frac{1}{2} (\text{dimension of } \gamma_\alpha) - \frac{1}{2} (\text{dimension of } \gamma_1) - 1 \), and the matrix \( \mathcal{C} = (C_\alpha^\beta) \) is defined by \( c_1(M) \cup \gamma_\alpha = C_\beta^\alpha \gamma_\beta \) where \( c_1(M) \) is the first Chern class of \( M \).

The quantum multiplication by \( \mathcal{X} \) is an endomorphism of the space of primary vector fields on the big phase space. If this endomorphism has distinct eigenvalues at generic points, the big phase space is called *semisimple*. In this case, let \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) be the eigenvectors with corresponding eigenvalues \( u_1, \ldots, u_N \), i.e.
\[
\mathcal{X} \circ \mathcal{E}_i = u_i \mathcal{E}_i
\]
for each \( i = 1, \ldots, N \). \( \mathcal{E}_i \) is considered as a vector field on the big phase space, and \( u_i \) is considered as a function on the big phase space. They satisfy the following properties:
\[
\mathcal{E}_i \circ \mathcal{E}_j = \delta_{ij} \mathcal{E}_i, \quad [\mathcal{E}_i, \mathcal{E}_j] = 0, \quad \mathcal{E}_i u_j = \delta_{ij}
\]
for any \( i \) and \( j \). Vector fields \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) are called *idempotents* on the big phase space. When restricted to the small phase space, \( \{u_1, \ldots, u_N\} \) gives the canonical coordinate system on semisimple Frobenius manifold \( \mathcal{H} \) and \( \mathcal{E}_1, \ldots, \mathcal{E}_N \) coincide with coordinate vector fields of this system (cf. [D]).

Let
\[
\mathcal{S} := - \sum_{m, \alpha} \tilde{t}_m^\alpha \tau_{m-1} (\gamma_\alpha)
\]
be the *string vector field* on the big phase space. For any vector fields \( \mathcal{W} \) and \( \mathcal{V} \) on the big phase space, define
\[
\langle \mathcal{W}, \mathcal{V} \rangle := \langle \mathcal{S} \mathcal{W} \mathcal{V} \rangle_0.
\]
This bilinear form generalizes the pairing \( \eta \) on the small phase space. It is non-degenerate only when restricted to the space of primary vector fields. Define
\[
g_i := \langle \mathcal{E}_i, \mathcal{E}_i \rangle, \quad h_i = \sqrt{g_i}.
\]
Since \( \langle \mathcal{E}_i, \mathcal{E}_j \rangle = 0 \) if \( i \neq j \), functions \( g_1, \ldots, g_N \) completely determine the pairing \( \langle \cdot, \cdot \rangle \) in the semisimple case. When representing expressions involving flat primary fields \( \gamma_\alpha \) in terms of idempotents, the following simple fact is very useful (cf. [L06]): For any tensor \( Q \),
\[
Q(\gamma_\alpha, \gamma^\alpha, \cdots) = \sum_{i=1}^N \frac{1}{g_i} Q(\mathcal{E}_i, \mathcal{E}_i, \cdots).
\]
Define the rotation coefficients on the big phase space by
\[ r_{ij} := \frac{E_j}{\sqrt{g_j}} \sqrt{g_i}. \]
(4)
These functions were introduced in [L06] to study relations among \( k \)-point functions in the semisimple case. For \( i \neq j \), the restriction of \( r_{ij} \) to the small phase space coincides with the definition of rotation coefficients for semisimple Frobenius manifolds defined in [D]. This is no longer true when \( i = j \). We will use \( \gamma_{ij} \) to denote the rotation coefficients defined in [D]. In [D], \( \gamma_{ii} \) is defined to be 0. But \( r_{ii} \) may not be 0 according to our definition. Using the definition of \( r_{ij} \) in equation (4), the so-called string equation on the small phase space \( \mathcal{H} \) has the form
\[ \sum_j r_{ij} h_j = 0 \]
(5)
for all \( i \).

Now we briefly review basic properties of rotation coefficients which will be used later. Readers are referred to [L06] for more details. First, \( r_{ij} = r_{ji} \), for all \( i, j \).

Define
\[ v_{ij} := (u_j - u_i) r_{ij}. \]
When taking derivatives of rotation coefficients, the following functions naturally appear:
\[ \theta_{ij} := \frac{1}{u_j - u_i} \left( r_{ij} + \sum_k r_{ik} v_{jk} \right) \]
for \( i \neq j \). Obviously,
\[ \theta_{ij} + \theta_{ji} = -\sum_k r_{ik} r_{jk} \]
(6)
for any \( i \neq j \). First derivatives of rotation coefficients are given by the formula
\[ E_k r_{ij} = r_{ik} r_{jk} + \begin{cases} 0 & \text{if } i \neq j \neq k, \\ \theta_{ij} & \text{if } k = i \neq j, \\ \sqrt{\frac{\theta_{ik}}{g_{ik}}} \theta_{ik} & \text{if } i = j \neq k, \\ -2 \sum_l r_{il}^2 + \sum_{p \neq i} \sqrt{\frac{\theta_{ip}}{g_{ip}}} \theta_{ip} + \frac{1}{g_{ii}} \langle \tau^2 (S), \mathcal{E}_i \rangle & \text{if } i = j = k. \end{cases} \]
(7)
Notice that on the small phase space \( \langle \tau^2 (S), \mathcal{E}_i \rangle = 0 \).

Define
\[ \Omega_{ij} := \frac{1}{u_j - u_i} \left( \theta_{ij} - \theta_{ji} + \sum_{k,l} r_{il} r_{jk} v_{kl} \right) \]
for \( i \neq j \). Then
\[ \Omega_{ij} = \Omega_{ji} \]
for all \( i \neq j \). These functions arise naturally in the second order derivatives of rotation coefficients because
\[ E_j \theta_{ij} = \left( r_{jj} - \sqrt{\frac{g_j}{g_i}} r_{ij} \right) \theta_{ij} - \Omega_{ij} \]
for \( i \neq j \). We might consider \( \theta_{ij} \) and \( \Omega_{ij} \) as functions having poles of order 1 and 2 respectively in terms of \( u_1, \ldots, u_N \).
As in [L07], we will use the following notation for genus-0 and genus-1 $k$-point functions:

\[ z_{i_1, \ldots, i_k} := \langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle_0 \]

and

\[ \phi_{i_1, \ldots, i_k} := \langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle_1. \]

In [L06], it was proved that genus-0 4-point functions satisfy the following properties: For $i \neq j$,

\begin{enumerate}
  \item $z_{iiii} = -g_ir_{ii}$,
  \item $z_{jjii} = -z_{jjii} = -\sqrt{g_ig_j}r_{ij}$,
  \item $z_{ijkl} = 0$, otherwise.
\end{enumerate}

(8)

It was also proved that the genus-1 1-point functions are given by

\[ 24\phi_i = -12 \sum_j r_{ij}v_{ij} - \sum_j h_i^{-1}r_{ij} \]

for all $i$. Higher point genus-0 and genus-1 functions can be computed recursively using the following formula:

\begin{align*}
  \langle \mathcal{E}_{i_1} \mathcal{E}_{i_2} \cdots \mathcal{E}_{i_{k+1}} \rangle_g \\
  = \mathcal{E}_{i_{k+1}} \langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle_g - \left( \sum_{j=1}^k r_{ij} h_j^{-1} \sqrt{\frac{g_{i_{k+1}}}{g_{i_j}}} \right) \langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_k} \rangle_g \\
  - \sum_{j=1}^k r_{ij} h_j^{-1} \sqrt{\frac{g_{i_{k+1}}}{g_{i_j}}} \langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_j} \mathcal{E}_{i_{k+1}} \rangle_g \\
  + \sum_{j=1}^k \delta_{i_{k+1}, i_j} \sum_p r_{p,i_{k+1}} \sqrt{\frac{g_{i_{k+1}}}{g_p}} \langle \mathcal{E}_{i_1} \cdots \mathcal{E}_{i_{j-1}} \mathcal{E}_{i_{j+1}} \mathcal{E}_{i_p} \rangle_g \\
  \end{align*}

(10)

for all $g \geq 0$. For example, the genus-1 2-point functions are given by

\[ 24\phi_{ij} = 12r_{ij}^2 + \sum_l \{ r_{il}r_{jl} h_l h_j^{-2} h_i h_j - r_{ij} r_{il} h_j h_l^{-1} - r_{ij} r_{jl} h_l h_i^{-1} \} - \{ \theta_{ij} h_j h_i^{-1} + \theta_{ji} h_i h_j^{-1} \} \]

(11)

\[ -24r_{ij} \left( \frac{h_j}{h_i} \phi_i + \frac{h_i}{h_j} \phi_j \right) \]

for $i \neq j$, and

\[ 24\phi_{ii} = 12r_{ii}^2 + \sum_j \{ r_{ij}^2 (-10 + \frac{h_i^2}{h_j^2}) - 2r_{ii}r_{ij} h_i \} - \sum_{j \neq i} \{ \theta_{ij} h_j h_i^{-1} + \theta_{ji} h_i h_j^{-1} \} \]

\[ -48r_{ii} \phi_i + 24 \sum_j r_{ij} h_j h_i \phi_j - \frac{1}{h_i^2} \langle \tau_+^2(S), \mathcal{E}_i \rangle \]

(12)

for all $i$.

For any vector field $\mathcal{W}$, define

\[ T(\mathcal{W}) := \tau_+(\mathcal{W}) - \langle \mathcal{W} \gamma^\alpha \rangle_0 \gamma^\alpha. \]
The operator $T$ was introduced in [L02] in order to simplify topological recursion relations for Gromov-Witten invariants. The following properties of operator $T$ were proved in [L06]:

(i) $T(\mathcal{E}_j) u_i = 0$,
(ii) $T(\mathcal{E}_j) g_i = -2\delta_{ij} g_i$,
(iii) $T(\mathcal{E}_k) r_{ij} = \delta_{ij} r_{ik} \sqrt{\frac{g_k}{g_i}}$,
(iv) $T(\mathcal{E}_k) \theta_{ij} = -r_{ij} r_{ik} \sqrt{\frac{g_k}{g_i}}$ for $i \neq j$,
(v) $T(\mathcal{E}_k) \Omega_{ij} = \theta_{ji} r_{ik} \sqrt{\frac{g_k}{g_i}} + \theta_{ij} r_{jk} \sqrt{\frac{g_k}{g_j}}$ for $i \neq j$,
(vi) $T(\mathcal{E}_k)(\tau^2(S), \mathcal{E}_i) = -\delta_{ik}(\tau^2(S), \mathcal{E}_i)$,
(vii) $T(\mathcal{E}_k)(\tau^3(S), \mathcal{E}_i) = -\delta_{ik}(\tau^3(S), \mathcal{E}_i)$.

(13)

2. Proof of the main theorem

Recall that for a semisimple Frobenius manifold $\mathcal{H}$, the genus-2 G-function $G^{(2)}(u, u_x, u_{xx})$ defined in [DLZ] has the following form:

$$G^{(2)}(u, u_x, u_{xx}) = \sum_i G^{(2)}_{i}(u, u_x) u_{xx}^i + \sum_{i \neq j} G^{(2)}_{ij}(u) \left(\frac{u_j}{u_x}\right)^3$$

$$+ \frac{1}{2} \sum_{i,j} P^{(2)}_{ij}(u) u_{x}^i u_{x}^j + \sum_i Q^{(2)}_{i}(u) (u_{x}^i)^2$$

(14)

where $G^{(2)}_{i}$, $G^{(2)}_{ij}$, $P^{(2)}_{ij}$, and $Q^{(2)}_{i}$ are functions on the jet space of $\mathcal{H}$. Precise definitions for these functions can be found in Appendix A. In this formula, $u_i = u^i$ for $i = 1, \ldots, N$ are the canonical coordinates on the small phase space $\mathcal{H}$, i.e. the restriction of functions $u_i$ defined in Section 1 to $\mathcal{H}$. Let $w^\alpha$ for $\alpha = 1, \ldots, N$ be the flat coordinates on $\mathcal{H}$ with respect to the basis $\{\gamma_1, \ldots, \gamma_N\}$; i.e. $w^\alpha$ is the restriction of $t_0^\alpha$ to the small phase space $\mathcal{H}$. Under the transformation

$$w^\alpha = \langle \gamma_1 \gamma^\alpha \rangle_0,$$

(15)

every function on the small phase space can also be viewed as a function on the big phase space. In particular, $u_{i,x} = u_{x}^i$ means the first order derivative of $u_i$ with respect to $x := t_0^i$ after transformation (15). The functions $u_i, u_{i,x}, u_{i,xx}, \ldots$ for $i = 1, \ldots, N$ form a coordinate system for the jet space of the semisimple Frobenius manifold $\mathcal{H}$ (cf. [DZ]).

To prove that the genus-2 G-function is 0, it suffices to show that functions $G^{(2)}_{i}$ and $Q^{(2)}_{i} + \frac{1}{2} P^{(2)}_{ii}$ for all $i$, $G^{(2)}_{ij}$ and $P^{(2)}_{ij} + P^{(2)}_{ji}$ for $i \neq j$ are equal to 0. In this section, we prove that all these functions vanish under conditions (C1), (C2) and (C3). The main idea of the proof is to first represent all these functions in terms of rotation coefficients $r_{ij}$ and functions $h_i$, $u_i$, $\theta_{ij}$, and $\Omega_{ij}$ defined in Section 1. We then use conditions (C1), (C2) and (C3) to get rid of $u_i$, $\theta_{ij}$, and $\Omega_{ij}$ and obtain expressions for the above four types of functions only involving $r_{ij}$ and $h_i$. Then straightforward calculations show that all these functions vanish.

The following lemma will be very useful in the proof.
Lemma 2.1. Under the condition (C2), on the small phase space the following identities hold:

\[
\sum_j r_{ij} u_{ij} = -\frac{1}{12} \sum_j r_{ij} \frac{h_i}{h_j}
\]

for all \(i\), and

\[
\theta_{ij} \frac{h_j}{h_i} + \theta_{ji} \frac{h_i}{h_j} = 12 r_{ij}^2 + \sum_l \left( r_{il} r_{jl} \frac{h_i h_j}{h_l^2} - r_{ij} r_{il} \frac{h_j}{h_l} - r_{ij} r_{jl} \frac{h_i}{h_l} \right)
\]

for all \(i \neq j\).

Proof. First note that condition (C2) is equivalent to the vanishing of the restriction of genus-1 correlation functions to the small phase space, i.e.

\[
\phi_{i_1 \ldots i_k} | \mathcal{H} = 0
\]

for any \(\{i_1, \ldots, i_k\}\), with \(k \geq 1\).

Equation (16) follows from equation (9) and \(\phi_i | \mathcal{H} = 0\). Equation (17) follows from equation (11), \(\phi_i | \mathcal{H} = 0\), and \(\phi_{ij} | \mathcal{H} = 0\) for \(i \neq j\). □

In the definition of the genus-2 G-function (see Appendix A), the following functions were introduced to simplify expressions:

\[
H_i := \frac{1}{2} \sum_{j \neq i} u_{ij} \gamma_{ij}^2
\]

where \(u_{ij} := u_i - u_j\) and \(\gamma_{ij}\) are the rotation coefficients defined in [D]. An immediate application of equation (16) is to get rid of functions \(u_{ij}\) in \(H_i\), i.e.

\[
H_i = \frac{1}{24} \sum_k r_{ik} \frac{h_i}{h_k}
\]

when condition (C2) is satisfied.

2.1. Vanishing of \(G_{ij}^{(2)}\). In this subsection, we prove \(G_{ij}^{(2)} = 0\) for all \(i \neq j\) under condition (C2). In the definition of \(G_{ij}^{(2)}\) given in Appendix A, \(\partial_i := \frac{\partial}{\partial u_i}\) is equal to taking a derivative along direction \(E_i\) on the small phase space. Therefore we can compute \(\partial_i h_j\) and \(\partial_i \gamma_{jk}\) using formulas (4) and (7). We can also replace \(H_i\) by the right hand side of equation (19). We then obtain the following expression for \(G_{ij}^{(2)}\):

\[
G_{ij}^{(2)} = -\frac{r_{ij}}{5760 h_i h_j} \left\{ \theta_{ij} \frac{h_j}{h_i} + \theta_{ji} \frac{h_i}{h_j} - 12 r_{ij}^2 - \sum_l \left( r_{il} r_{jl} \frac{h_i h_j}{h_l^2} - r_{ij} r_{il} \frac{h_j}{h_l} - r_{ij} r_{jl} \frac{h_i}{h_l} \right) \right\}
\]

The right hand side of this equation is understood as a function obtained by applying transformation (13) to a function on the small phase space. Therefore by equation (17), we have \(G_{ij}^{(2)} = 0\). Most formulas in the proof of Theorem 0.1 will be understood in a similar way. To make the paper concise, we will not repeat this argument later.

This simple proof of \(G_{ij}^{(2)} = 0\) is rather typical of the rest of the proof for Theorem 0.1. To prove the vanishing of other parts of the genus-2 G-function, we will use similar ideas, although the computations will be much more involved.
2.2. Vanishing of $G^{(2)}_i$. In this subsection, we show $G^{(2)}_i = 0$ for all $i$ under conditions (C1) and (C2). We write

$$G^{(2)}_i = G^{(2)}_{i,1} + G^{(2)}_{i,2}$$

where $G^{(2)}_{i,1}$ explicitly involves jet coordinates $u_{k,x}$, but $G^{(2)}_{i,2}$ does not. Precise definitions for $G^{(2)}_{i,1}$ and $G^{(2)}_{i,2}$ are given in Appendix A. We will compute these two functions separately.

First note that for any function $f$, we have

$$\frac{\partial_x f}{\partial x} = \sum_k u_{k,x} \frac{\partial f}{\partial x}$$

by the chain rule. Using formulas (4) and (7), we can then compute all partial derivatives involved in the definitions of $G^{(2)}_{i,1}$ and $G^{(2)}_{i,2}$. We can also compute $H_i$ using equation (19). After straightforward calculations, we obtain the following formulas:

$$G^{(2)}_{i,1} = \sum_{k \neq i} \frac{1}{1920h_i^2} \left\{ \theta_{ik} \frac{h_k}{h_i} + \theta_{ki} \frac{h_i}{h_k} - 12r^2_{ik} - \sum_l \left( r_{il} r_{kl} \frac{h_i h_k}{h_l^2} - r_{ik} r_{il} \frac{h_k}{h_l} - r_{ik} r_{kl} \frac{h_l}{h_i} \right) \right\} \frac{\partial_x u_k}{\partial x} u_i$$

$$+ \sum_{k \neq i} \frac{\theta_{ik}}{5760h_i h_k} - \frac{r^2_{ik}}{1440h_i^2} - \sum_k r^2_{ik} 1920h_i^2 + \sum_k \frac{r_{ik} r_{ik}}{1440h_i h_k}$$

and

$$G^{(2)}_{i,2} = \sum_{k \neq i} \frac{\theta_{ik}}{2880h_i h_k} - \sum_{k \neq i} \frac{\theta_{ki} h_k}{384h_i^3} + \sum_{k \neq i} \frac{7\theta_{ki}}{2880h_i h_k} - \sum_j \frac{17r_{ij} r_{ij}}{2880h_i h_j} + \sum_k \frac{r_{ik} r_{kl} h_i}{1440h_k^3}$$

$$- \sum_k \frac{19r^2_{ik}}{720h_i^2} + \sum_k \frac{r^2_{ik}}{1440h_k^2} + \sum_{j,k} \frac{r_{ik} r_{ij}}{2880h_j h_k} + \sum_{j,k} \frac{7r_{ik} r_{jk}}{2880h_i h_j} - \sum_k \frac{h_i r_{il} r_{kl}}{2880h_k h_l^2}$$

(20) + \frac{23r^2_{ik}}{720h_i^2}.

By equation (17), the coefficient for $\frac{\partial_x u_k}{\partial x} u_i$ in $G^{(2)}_{i,1}$ becomes 0 under condition (C2). So we have

$$G^{(2)}_{i,1} = \sum_{k \neq i} \frac{\theta_{ik}}{5760h_i h_k} - \frac{r^2_{ik}}{1440h_i^2} - \sum_k \frac{r^2_{ik}}{1920h_k^2} + \sum_k \frac{r_{ik} r_{ik}}{1440h_i h_k}$$

(21)

Next we will get rid of first order poles $\theta_{ik}$ and $\theta_{ki}$ in equations (21) and (20) using functions depending only on $\{r_{ij} \mid i,j = 1, \ldots, N\}$ and $\{h_i \mid i = 1, \ldots, N\}$. Here we also need condition (C1). A key ingredient in this process is the following lemma:

**Lemma 2.2.** Under conditions (C1) and (C2), on the small phase space we have

$$2 \sum_{i \neq k} \frac{\theta_{ki}}{h_i} = \sum_i \left( 7 \frac{h_k}{h_i} r^2_{ik} - 6 \frac{r^2_{ik}}{h_k^2} \frac{h_i h_k}{h_i} - 2 \frac{h_k^2}{h_i^2} r_{ik} r_{ik} \right) + \sum_{i,j} \left( \frac{h_k^2}{h_i h_j} r_{ij} r_{jk} - \frac{h_k}{h_i h_j} r_{ik} r_{jk} \right)$$

for all $k$. 


Proof. Condition (C1) is equivalent to saying that the restriction of $\langle \gamma_0 \gamma^\alpha \gamma_0 \gamma^\beta \rangle_0$ to the small phase space is constant. By equation (3), we have

$$\langle \gamma_0 \gamma^\alpha \gamma_0 \gamma^\beta \rangle_0 = \sum_{i,j=1}^{N} \frac{1}{g_i g_j} \langle E_i E_i E_j E_j \rangle_0 = \sum_{i,j=1}^{N} \frac{1}{g_i g_j} z_{iiij}.$$ 

So equation (8) implies

$$\langle \gamma_0 \gamma^\alpha \gamma_0 \gamma^\beta \rangle_0 = \sum_{i,j} \frac{r_{ij}}{h_i h_j} - 2 \sum_{i} \frac{r_{ii}}{h_i^2}.$$  

(23)

Therefore if condition (C1) is satisfied, on the small phase space, we have

$$\sum_{i,j} \frac{r_{ij}}{h_i h_j} - 2 \sum_{i} \frac{r_{ii}}{h_i^2} = \text{constant.}$$

Taking derivatives of this equation along the direction of $E_k$, we have

$$\sum_{i \neq k} \left( \frac{h_i}{h_k^3} + \frac{h_i}{h_k^3} \right) \theta_{ik} - 2 \sum_{i \neq k} \frac{r_{ik} r_{jk}}{h_i h_j} = \sum_{i,j} \frac{r_{ik} r_{jk}}{h_i h_j} - 2 \sum_{i} \frac{r_{ik}^2}{h_i^2} - \sum_{i,j} \frac{r_{ij} h_k (r_{ik} h_i + r_{jk} h_j)}{h_i^2 h_j^2}$$

(24)

$$+ 4 \sum_{i} \frac{h_k r_{ik} r_{ii}}{h_i^3} + 2 \sum_{i} \frac{r_{ik}^2}{h_k^2}.$$ 

We then convert the first summation on the left hand side to an expression similar to the second summation on the left hand side together with some function depending only on $r_{ij}$ and $h_i$. To see how this can be done, we write

$$\sum_{i \neq k} \frac{h_k}{h_i^3} \theta_{ik} = \sum_{i \neq k} \left\{ \frac{h_k}{h_i} \theta_{ik} \right\} \frac{1}{h_i^2}$$

and use equation (17) to replace $\frac{h_k}{h_i} \theta_{ik}$ by $\frac{h_k}{h_i} \theta_{ki}$ plus a function depending only on $r_{ij}$ and $h_i$. In this way, we can see that

$$\sum_{i \neq k} \frac{h_k}{h_i^3} \theta_{ik} = -\sum_{i \neq k} \left\{ \frac{1}{h_i h_k} \theta_{ki} \right\} + \{ \text{some function of } r \text{ and } h \}. $$

(25)

Here “some function of $r$ and $h$” means a function which depends only on $h_i$ for all $i$ and $r_{ij}$ for all $i$ and $j$. For simplicity, we omit the precise form of such functions. Note that we can use equation (6) to switch indices in $\theta_{ik}$. So by a similar calculation using equation (17), we also obtain

$$\sum_{i \neq k} \frac{h_i}{h_k^3} \theta_{ik} = -\sum_{i \neq k} \left\{ \frac{1}{h_i h_k} \theta_{ki} \right\} + \{ \text{some function of } r \text{ and } h \}.$$ 

After plugging these terms into equation (24) and multiplying the resulting expression by $-\frac{1}{2} h_k$, we obtain equation (22). \qed

Remark 2.3. Combining equations (6) and (22), we can express $\sum_{i \neq k} \frac{\theta_{ik}}{h_i}$ in terms of functions $\{ r_{ij} \}$ and $\{ h_i \}$. By equations (6), (17) and (22), we can also express $\sum_{i} h_i \theta_{ki}$ and $\sum_{i} h_i \theta_{ik}$ in terms of functions $\{ r_{ij} \}$ and $\{ h_i \}$. More generally, we can
express both $\sum_{i,k} h_i^p \theta_{ik}$ and $\sum_{i,k} h_i^p \theta_{ki}$ in terms of functions \{rij\} and \{hi\} if p is odd. We also note that repeatedly applying equations (6) and (17) shows that
\[
\theta_{ij} h_i^p h_j^q = \theta_{ij} h_i^{p+2k} h_j^{q-2k} + \{\text{some function of } r \text{ and } h\}
\]
for any integers p, q, and k. This observation is very useful in the process of getting rid of functions \(\theta_{ij}\) in complicated expressions.

Now we come back to the computation of \(G_{ij}^{(2)} = G_{i,1}^{(2)} + G_{i,2}^{(2)}\). After replacing all terms containing \(\theta_{ik}\) and \(\theta_{ki}\) in equations (21) and (20) using the method indicated in Lemma 2.2 and Remark 2.3 we obtain a formula for \(G_{ij}^{(2)}\) which only involves functions \{\(r_{jk} \mid j, k = 1, \ldots, N\}\} and \{\(h_j \mid j = 1, \ldots, N\}\). Straightforward computations then show \(G_{ij}^{(2)} = 0\).

2.3. Vanishing of \(P_{ij}^{(2)} + P_{ji}^{(2)}\). In this subsection and subsection 2.4 we will prove \(P_{ij}^{(2)} + P_{ji}^{(2)} = 0\) and \(\frac{1}{2} P_{ii}^{(2)} + Q_i^{(2)} = 0\). The idea for the proof is to express these functions in terms of functions \(\theta_{kl}, \Omega_{kl}, v_{kl}, r_{kl}\), and \(h_{kl}\) first; then using conditions (C1), (C2), and (C3) we can replace terms containing \(\theta_{kl}, \Omega_{kl}, v_{kl}\) by functions which depend only on \(r_{kl}\) and \(h_{kl}\). After these two steps, the cancellation of terms containing only \(r_{kl}\) and \(h_{kl}\) will be straightforward. The most crucial part in this proof is how to get rid of terms containing \(\theta_{kl}, \Omega_{kl}, v_{kl}\). Therefore we will omit terms containing only \(r_{kl}\) and \(h_{kl}\) in many formulas in these two subsections in order to keep the paper concise. Interested readers may find complete formulas in the version of this paper posted in arXiv:1310.2101. Appendix D of the arXiv version also contains explicit formulas for genus-1 3-point functions \(\phi_{ijk}\) and \(\phi_{iij}\), which are needed in the proof of Lemmas 2.4 and 2.6.

We now prove \(P_{ij}^{(2)} + P_{ji}^{(2)} = 0\) for all \(i \neq j\) under conditions (C1) and (C2). We can use formulas (14) and (17) to compute all partial derivatives involved in the definition of \(P_{ij}^{(2)}\) given in Appendix A. We can also compute functions \(H_i\) using equation (19). We then obtain a formula for \(P_{ij}^{(2)}\) which depends only on \(\{r_{kl} \mid k, l = 1, \ldots, N\}\), \(\{h_k \mid k = 1, \ldots, N\}\), and \(\{\theta_{kl} \mid k, l = 1, \ldots, N\}\). For simplicity, we only give the part of this function which contains first order poles \(\theta_{kl}\):

\[
P_{ij}^{(2)} = \frac{1}{1440} \left( -3 \sum_{k \neq l} \frac{h_i h_j r_{ik \theta_{kj}}}{h_k^4} - \frac{41 r_{ij} \theta_{ij}}{h_i^2} + \frac{6 r_{ij} h_j \theta_{ij}}{h_i^3} - 3 \sum_k r_{ik} h_j h_k \theta_{ij} h_i^2 h_k^2 \
- 4 \sum_{k \neq i} \frac{r_{ij} h_j h_k \theta_{ik}}{h_i^4} - 5 \sum_{k \neq i} \frac{r_{ij} h_j \theta_{ki}}{h_i^2 h_k} \right) + \{\text{some function of } r \text{ and } h\} \tag{26}
\]

Note that we have used equation (8) to get rid of some terms containing \(\theta_{ik}\) when deriving this formula. As in subsection 2.2 the main idea for proving \(P_{ij}^{(2)} + P_{ji}^{(2)} = 0\) is to replace first order poles \(\theta_{kl}\) in \(P_{ij}^{(2)} + P_{ji}^{(2)}\) by functions depending only on \(\{r_{kl}\}\) and \(\{h_k\}\). For simplicity of notation, \(\{r_{kl}\}\) means the set of all functions \(r_{kl}\) where \(k\) and \(l\) run over their entire range. The notation \(\{h_k\}\) is interpreted in a similar way.
Using Remark 2.3, the two terms containing $\theta_{ik}$ and $\theta_{ki}$ in the second line of equation (26) can be expressed as functions of $\{r_{kl}\}$ and $\{h_k\}$. Below we explain how to deal with the first four terms in equation (26). For simplicity, we will omit precise forms of functions depending only on $\{r_{kl}\}$ and $\{h_k\}$.

By equation (17), we have

$$\sum_{k \neq j} \frac{h_i h_j r_{ik} \theta_{kj}}{h_k^4} = -\frac{h_i}{h_j} \sum_{k \neq j} \frac{r_{ik} \theta_{jk}}{h_k^2} + \{\text{some function of } r \text{ and } h\}. \quad (27)$$

Next we will use the following identities:

**Lemma 2.4.** Under condition (C2), on the small phase space we have

$$\theta_{ik} r_{jk} \frac{h_j}{h_i} + \theta_{ki} r_{ij} \frac{h_j}{h_k} + \theta_{kj} r_{ij} \frac{h_i}{h_k} + \theta_{ij} r_{jk} \frac{h_k}{h_i} + \theta_{ji} r_{ik} \frac{h_k}{h_j} = \{\text{some function of } r \text{ and } h\}$$

for $i \neq j \neq k$.

**Proof.** Condition (C2) implies that genus-1 correlation functions on the small phase space $\mathcal{H}$ are 0. For distinct $i$, $j$, $k$, we can compute genus-1 3-point function $\phi_{ijk}$ using formula (10) together with the formulas of genus-1 1-point functions (9) and genus-1 2-point functions (11). The lemma follows from $\phi_{i|\mathcal{H}} = 0$, $\phi_{ij|\mathcal{H}} = 0$, $\phi_{ijk|\mathcal{H}} = 0$ for any $i \neq j \neq k$. \hfill \Box

Multiplying both sides of equation (28) by $\frac{1}{h_k^4}$ and taking the sum over $k$ for $k \notin \{i,j\}$, we obtain

$$\frac{h_i}{h_j} \sum_{k \neq j} \frac{r_{ik} \theta_{jk}}{h_k^2} + \frac{h_j}{h_i} \sum_{k \neq j} \frac{r_{jk} \theta_{ik}}{h_k^2} = -\sum_k \theta_{ij} r_{jk} \frac{1}{h_i h_k} - \sum_k \theta_{ji} r_{ik} \frac{1}{h_j h_k} + 2 \theta_{ij} r_{ij} \frac{1}{h_i h_j} + 2 \theta_{ji} r_{ji} \frac{1}{h_i h_j} + \{\text{some function of } r \text{ and } h\}. \quad (29)$$

In deriving this formula, we have used equation (17) and the fact that both $\sum_{k \neq i} \frac{\theta_{ik} h_k}{h_i}$ and $\sum_{k \neq j} \frac{\theta_{ij} h_k}{h_k}$ can be expressed as functions of $\{r_{kl}\}$ and $\{h_k\}$, which in turn follow from equations (25) and (23). By equations (27) and (29), we get

$$P^{(2)}_{ij} + P^{(2)}_{ji} = \frac{1}{1440} \left( -3 \sum_k r_{jk} \theta_{ij} \frac{1}{h_i h_k} - 3 \sum_k r_{jk} \theta_{ji} \frac{h_i}{h_j^2 h_k} - 3 \sum_k r_{ik} \theta_{ji} \frac{1}{h_j h_k} \right. \right.$$  

$$\left. - 3 \sum_k r_{ik} \theta_{ij} \frac{h_j}{h_i^2 h_k} - \frac{41 r_{ij} \theta_{ij} h_j}{h_i \theta_{ij} h_j + \theta_{ji} h_j^2} + 6 r_{ii} \theta_{ij} h_j^3 \right. \right.$$  

$$\left. + 6 r_{ii} \theta_{ji} \frac{1}{h_i h_j} + 6 r_{ij} \theta_{ji} \frac{h_i^3}{h_j^2} + 6 r_{ji} \theta_{ij} \frac{1}{h_i h_j} \right)$$

$$+ \{\text{some function of } r \text{ and } h\}. \tag{30}$$

Using equation (17), it is easy to see that the right hand side of this equation can be expressed as a function depending only on $\{r_{kl}\}$ and $\{h_k\}$. A straightforward
calculation shows that this function is equal to 0. This proves that \( P_{ij}^{(2)} + P_{ji}^{(2)} = 0 \) for \( i \neq j \).

2.4. **Vanishing of** \( \frac{1}{2} P_{ii}^{(2)} + Q_i^{(2)} \). In this subsection, we prove \( \frac{1}{2} P_{ii}^{(2)} + Q_i^{(2)} = 0 \) for all \( i \) under conditions (C1), (C2), and (C3). In the definition of \( Q_i^{(2)} \) given in Appendix A, there are six terms having \( u_{ik} = u_i - u_k \) in the denominator. After calculating derivatives of rotation coefficients by equation (7) and replacing \( H_k \) by equation (19), these terms become

\[
\frac{1}{1152} \left( \sum_{k,l,i} h_k v_i \theta_{il} - 2 \sum_{k \neq i} \theta_{ik} h_k + \frac{1}{12} \sum_{k,l} r_{ik} r_{il} h_k - \sum_{k \neq i} r_{ii} r_{ik} h_k - 2 \sum_{k \neq i} \frac{\theta_{ki}}{u_{ik} h_i^3} - 2 \sum_{k \neq i} \frac{\theta_{ki}}{u_{ik} h_i^3} \right) - 2 \sum_{k \neq i} \frac{\theta_{ki}}{u_{ik} h_i^3}.\]

The last four terms in this expression do not contain \( u_{ik} \) explicitly. We will see how to deal with other terms in this expression whose dominators contain \( u_{ik} \) or \( u_{il} \). First observe that

\[
\frac{v_{ik}}{u_{ik}} = r_{ik} \left( -1 + \frac{u_i - u_l}{u_i - u_k} \right).
\]

Multiplying both sides of this equation by \( \theta_{il} \) and using the definition of \( \theta_{il} \), we have

\[
\frac{v_{ik} \theta_{il}}{u_{ik}} = -r_{ik} \theta_{il} - \frac{r_{ik}}{u_i - u_k} \left( r_{il} + \sum_{m} r_{im} v_{lm} \right).\]

Plugging this formula into the first term of expression (30), then using the definition of \( \Omega_{ik} \) and equations (5), (6) and (16), we obtain the following formula for the first four terms in expression (30):

\[
-2 \sum_{k \neq i} \theta_{ki} u_{ik} h_i h_k - 2 \sum_{k \neq i} \frac{\theta_{ki}}{u_{ik} h_i^3} - 2 \sum_{k \neq i} \frac{\theta_{ki}}{u_{ik} h_i^3} + \frac{1}{12} \sum_{k,l} r_{ik} r_{il} h_k + \sum_{k \neq i} r_{ii} r_{ik} h_k - \sum_{k \neq i} r_{ii} r_{ik} h_k - \sum_{k \neq i} \frac{\theta_{ki}}{u_{ik} h_i^3} = \sum_{k \neq i} \frac{h_k}{h_i} \Omega_{ik} + \sum_{l \neq i} \frac{1}{h_i} r_{il} \theta_{il} .
\]

Similarly, for the next five terms in expression (30), we have the following formula:

\[
-2 \sum_{k \neq i} \frac{\theta_{ki}}{u_{ik} h_i h_k} + \frac{1}{12} \sum_{k,l} r_{ik} r_{il} u_{ik} h_i h_k - 2 \sum_{k \neq i} \frac{r_{kk} r_{ik}}{u_{ik} h_i h_k} + \sum_{k,l} \frac{r_{ik} r_{il} v_{kl}}{u_{il} h_i h_l} + \sum_{k \neq i} \frac{u_{ik} r_{ik} \theta_{lk}}{u_{il} h_i h_l} = - \sum_{k \neq i} \frac{\Omega_{ki}}{h_i h_k} + \sum_{l \neq i} \frac{r_{il}}{h_i h_l} \theta_{il}.
\]
Consequently, \( \frac{1}{2}P^{(2)}_{ii} + Q^{(2)}_i \) can be expressed as a function of \( \{\Omega_{ij}\}, \{\theta_{jk}\}, \{v_{jk}\}, \{r_{jk}\} \) and \( \{h_j\} \). After getting rid of part of the \( \{\theta_{jk}\} \) terms by using equations (6), (17), Lemma 2.2 and Remark 2.3, we have

\[
5760\left( \frac{1}{2}P^{(2)}_{ii} + Q^{(2)}_i \right) = -5 \sum_{j \neq i} \frac{\Omega_{ij}}{h_i^2} - 100 \sum_{k \neq i} \frac{r_{ik}\theta_{ik}}{h_i^2} + 4 \sum_{k \neq i} \frac{r_{kk}h_i\theta_{ik}}{h_i^3} - 6 \sum_{k \neq i} \frac{h_i^2r_{ik}\theta_{k1}}{h_i^4} + 5 \sum_{k, l \neq i} \frac{r_{kl}\theta_{lk}}{h_i h_l} - 2 \sum_{k, l \neq i} \frac{r_{kl}h_i\theta_{il}}{h_k h_i^2} + 40 \sum_{k, l \neq i} \frac{r_{ik}v_{lk}\theta_{il}}{h_i^2}
\]

(31) \(+ \{\text{some function of } r \text{ and } h\}\)

where \( \tilde{\Omega}_{ij} = \Omega_{ij}\left(\frac{h_j}{h_i}, \frac{h_j}{h_i}\right) \). Obviously, \( \tilde{\Omega}_{ij} = -\tilde{\Omega}_{ji} \). To show \( \frac{1}{2}P^{(2)}_{ii} + Q^{(2)}_i = 0 \), we need to get rid of \( \{\Omega_{ij}\}, \{\theta_{jk}\}, \{v_{jk}\} \) from this function. Here the most difficult term to deal with is

\[
40 \sum_{k, l \neq i} \frac{r_{ik}v_{lk}\theta_{il}}{h_i^2}.
\]

In order to get rid of this term, we need condition (C3). This is the only place in the proof of Theorem 0.1 where condition (C3) is needed.

**Lemma 2.5.** If conditions (C1), (C2), (C3) are satisfied, we have

\[
80 \sum_{j \neq i} \sum_k \theta_{ij}r_{ik}v_{jk} \frac{1}{h_i^2} = -15 \frac{1}{h_i^2} \sum_{j \neq i} \tilde{\Omega}_{ji} - 5 \sum_{j \neq i} \tilde{\Omega}_{ji} \frac{1}{h_j^2} - 24 \sum_{j \neq i} \theta_{ji}r_{jj} \frac{h_i}{h_j^3} - 400 \sum_{j \neq i} \theta_{ji}r_{jj} \frac{1}{h_i^2}
\]

(32) \(+ 22 \sum_{j \neq i} \sum_k \theta_{ji}r_{jk} \frac{1}{h_k h_i} + \{\text{some function of } r \text{ and } h\}\)

for all \( i \).

**Proof.** Condition (C3) is equivalent to

\[
\langle \gamma_\alpha \rangle_2 |_H = 0 \quad \text{and} \quad \langle \tau_1(\gamma_\alpha) \rangle_2 |_H = 0
\]

for all \( \alpha \), where \( H \) is the small phase space. Since

\[
\langle T(\mathcal{E}_i) \rangle_2 = \langle \tau_+(\mathcal{E}_i) \rangle_2 - \langle \mathcal{E}_i \gamma^\alpha \rangle_0 \langle \gamma_\alpha \rangle_2
\]

and \( \mathcal{E}_i \) is primary vector fields, we have

\[
\langle T(\mathcal{E}_i) \rangle_2 |_H = 0.
\]

(33)

On the other hand, a formula expressing \( F_2 \) in terms of genus-0 functions was proved in [L07] (cf. Theorem 3.1 in [L07]). Using this formula and basic properties of the operator \( T \) given in equation (13), we can calculate \( \langle T(\mathcal{E}_i) \rangle_2 = T(\mathcal{E}_i)F_2 \). The formula for \( \langle T(\mathcal{E}_i) \rangle_2 \) obtained this way is rather complicated. But the restriction of this formula to the small phase space has a much simpler form. For example, the restriction of \( \langle \tau^k(S), \mathcal{E}_i \rangle \) to the small phase space is 0 for all \( k \geq 1 \). Moreover,
if conditions (C1) and (C2) are satisfied, we can use equation (17), Lemma 2.2 and Remark 2.3 to get rid of many terms containing \( \theta_{jk} \). We also observe that the product \( \theta_{ij}v_{ij} \) does not contain poles. More precisely

\[
\theta_{ij}v_{ij} = r_{ij}(r_{ij} + \sum_k r_{ik}v_{jk})
\]

by definition of \( \theta_{ij} \) and \( v_{ij} \). After all these simplifications and combinations of like terms, the remaining terms containing \( \theta_{ij} \) and \( \Omega_{ij} \) are exactly those terms appearing in equation (32). In particular, the left hand side of equation (32) comes from the following two terms which appear in \( 5760 \langle \langle T(E_i) \rangle \rangle_2 \)

\[
\sum_{j \neq i} \sum_k \theta_{ij}r_{jk}v_{ik} \frac{1}{h_i^2} + 90 \sum_{j \neq i} \sum_k \theta_{ji}r_{ik}v_{jk} \frac{1}{h_i^2}.
\]

For the first term, we can decompose the function \( v_{ik} \) into two parts,

\[
v_{ik} = r_{ik}u_{kj} + r_{ik}u_{ji},
\]

and observe that \( \theta_{ij}u_{ji} \no \) longer has poles. For the second term in expression (35), we can replace \( \theta_{ji} \) by \( -\theta_{ij} \) using equation (6). After combining these terms together, expression (35) becomes

\[
\sum_{j \neq i} \sum_k \theta_{ij}r_{ik}v_{jk} \frac{1}{h_i^2}
\]

plus a function which depends only on \{\( r_{jk} \), \( h_k \)\}, and \{\( v_{jk} \)\}. The expression (36) is exactly the left hand side of equation (32).

We should also note that there are many terms in \( \langle \langle T(E_i) \rangle \rangle_2 \) which do not contain \{\( \theta_{jk} \)\}, but contain \{\( v_{jk} \)\}. Many such terms can be got rid of using equation (16). Other terms of this type can be canceled with terms produced in the process of obtaining expression (36) and terms produced from applying equation (34) to those terms containing \{\( \theta_{jk}v_{jk} \)\}. Eventually, after straightforward calculations, we obtain equation (32) from \( \langle \langle T(E_i) \rangle \rangle_2 \mid H = 0 \).

By equations (31) and (32), we obtain

\[
5760\{P^{(2)}_{ii} + 2Q^{(2)}_{i}\}
\]

\[
= -5 \frac{1}{h_i^2} \sum_{j \neq i} \tilde{\Omega}_{ji} - 5 \sum_{j \neq i} \tilde{\Omega}_{ji} \frac{1}{h_j^2} + 10 \sum_{k,l} \theta_{kl}r_{ik} \frac{1}{h_j h_i} - 200 \sum_k \theta_{ki}r_{ki} \frac{1}{h_i^2}
\]

\[
- 4 \sum_{k,l} \frac{r_{kl}h_i \theta_{il}}{h_k h_i^2} + 32 \theta_{ik}r_{kk} \frac{1}{h_k^3} + 22 \sum_{j \neq i} \sum_k \theta_{ji} r_{jk} \frac{1}{h_k h_i} - 12 \sum_{k \neq i} \frac{h_i^2 r_{ik} \theta_{ki}}{h_k^4}
\]

(37) + \{some function of \( r \) and \( h \)\}.

Here we omit the terms not containing \{\( \theta_{jk} \)\} or \{\( \Omega_{jk} \)\}. The following lemma will be used to get rid of \( \tilde{\Omega}_{ij} \) in this formula.
Lemma 2.6. Under conditions (C1) and (C2), on the small phase space we have

\[ 0 = \tilde{\Omega}_{ij} + \theta_{ij} \{ 20r_{ij} + 4r_{ii} \frac{h_j}{h_i} - \sum_k (r_{ik} \frac{h_j}{h_k} + r_{jk} \frac{h_i}{h_k}) \} + 2 \sum_{k \neq i} \theta_{ik} r_{jk} \frac{h_j}{h_i} \]

\[ + \{ \text{some function of } r \text{ and } h \} \]

(38)

for all \( i \neq j \).

Proof. By equation (18), \( \phi_{ij} |_{H} = 0 \). The explicit formula for \( \phi_{ij} \) can be obtained by taking derivatives of equation (11) with respect to \( E_i \) and using the recursion relation (10). Using this formula together with equation (17) and Remark 2.3, we obtain equation (38). \( \square \)

Using equation (38) to get rid of \( \Omega_{ij} \) in equation (37) and using equation (17), Lemma 2.2 and Remark 2.3 to simplify the expression, we obtain

\[ 5760 \{ P_{ii}^{(2)} + 2Q_i^{(2)} \} = 32 \sum_{k \neq i} \theta_{ik} r_{ik} \frac{1}{h_i^2} - 16 \sum_{j,k \neq i} \theta_{ik} r_{jk} \frac{1}{h_i h_j} + 32 \sum_{k \neq i} \theta_{ik} r_{kk} \frac{h_i}{h_k^3} \]

\[ + \{ \text{some function of } r \text{ and } h \} \]

(39)

The following lemma is needed in dealing with the first three terms on the right hand side of this equation.

Lemma 2.7. Under conditions (C1) and (C2), on small phase space we have

\[ 4 \sum_{k \neq i} \theta_{ik} r_{ik} \frac{1}{h_i^2} - 2 \sum_{j,k \neq i} \theta_{ik} r_{jk} \frac{1}{h_i h_j} + 4 \sum_{k \neq i} \theta_{ik} r_{kk} \frac{h_i}{h_k^2} \]

\[ = \{ \text{some function of } r \text{ and } h \} \]

for all \( i \).

Proof. We first multiply both sides of equation (28) by \( \frac{1}{h_i^2} \) and take summation over \( j \) and \( k \) for distinct \( i, j, k \). Then by equations (6), (17), Lemma 2.2 and Remark 2.3, we obtain equation (40). \( \square \)

Plugging equation (40) into equation (39), we can express \( \frac{1}{2} P_{ii}^{(2)} + Q_i^{(2)} \) as a function which depends only on \( \{ r_{ik} \} \) and \( \{ h_j \} \). A straightforward but complicated calculation then shows that this function is identically equal to 0. This proves \( \frac{1}{2} P_{ii}^{(2)} + Q_i^{(2)} = 0 \) for all \( i \) and thus finishes the proof of Theorem 0.1.

3. Proof of Corollary 0.2

In this section we will prove \( G^{(2)} = 0 \) for the cohomological field theories associated to \( ADE \) singularities by showing that conditions (C1), (C2), (C3) in Theorem 0.1 are satisfied.

In fact, conditions (C2) and (C3) follow from a simple dimension count. For \( ADE \) singularities, a necessary condition for \( \langle \tau_{l_1} (\gamma_{k_1}) \cdots \tau_{l_s} (\gamma_{k_s}) \rangle_g \neq 0 \) is the dimension condition

\[ 2((\hat{c}_W - 3)(1 - g) + s) = \sum_{i=1}^{s} (2l_i + \deg_W \gamma_i), \]

where \( W \) is a quasi-homogenous polynomial with a singularity of \( ADE \) type, \( \hat{c}_W \) is the central charge of \( W \) and \( \deg_W \gamma_i \) is the \( W \)-degree of \( \gamma_i \) (cf. [FJR] for details).
For $ADE$ singularities, $\hat{c}_W < 1$, $\deg_W \gamma_i < 2$ for all $i$. Therefore the dimension condition implies that
\[ \langle \gamma_{k_1} \cdots \gamma_{k_s} \rangle_1 = 0 \]
and
\[ \langle \tau_m(\gamma_{k_1})\gamma_{k_2} \cdots \gamma_{k_s} \rangle_2 = 0 \]
for all $k_i$ and $m \leq 2$. In particular, conditions (C2) and (C3) are satisfied.

We present two different approaches to condition (C1). The first approach does not involve canonical coordinates for semisimple Frobenius manifolds. It only uses flat coordinates. Recall that the genus-2 Mumford relation has the following form (cf. [Ge] and [L07]): For any vector field $W$,
\[
\langle \tau_2(W) \rangle_2 = \frac{7}{10} \langle \gamma_1 \rangle_1 \langle \{ \gamma^\alpha \circ W \} \rangle_1 + \frac{1}{10} \langle \gamma_1(\gamma^\alpha \circ W) \rangle_1 \\
- \frac{12}{240} \langle \langle W(\gamma_1 \circ \gamma^\alpha) \rangle_1 + \frac{13}{240} \langle \langle W(\gamma_1 \gamma^\alpha \gamma^\beta) \rangle_1 \langle \gamma_1 \rangle_1 \\
+ \frac{1}{960} \langle \langle W(\gamma_1 \gamma^\alpha \gamma^\beta \gamma^\rho) \rangle_1 \rangle.
\]

By definition of operator $T$, we have
\[
\langle \tau_2(W) \rangle_2 = \langle \langle \tau_2(W) \rangle_2 \rangle_2 - \langle \langle W(\gamma_1 \gamma^\alpha \gamma^\beta) \rangle_1 \langle \gamma_1 \rangle_1 \rangle.
\]

By equation (41), we have
\[
\langle \tau_2(W) \rangle_2 \bigg|_{H} = 0
\]
for any primary vector field $W$. Therefore by the Mumford equation and condition (C2), we have
\[
\langle \langle W(\gamma_1 \gamma^\alpha \gamma^\beta \gamma^\rho) \rangle_1 \rangle_0 = 0
\]
for any primary vector field $W$. This proves condition (C1) for $ADE$ singularities.

Our second approach to condition (C1) needs [DLZ, Lemma 2.5]. During this process, we also found a new proof for [DLZ, Lemma 2.5] which is much simpler than the original proof of this lemma in [DLZ]. The following functions were introduced in [DLZ]:
\[
O_1 = \langle \langle \gamma_1 \gamma^\alpha \gamma^\beta \gamma^\rho \rangle_1 \rangle_0 (M^{-1})^{\alpha\alpha'}(M^{-1})^{\beta\beta'}
\]
and
\[
O_2 = \langle \langle \gamma_1 \gamma^\alpha \gamma^\beta \gamma^\rho \rangle_1 \rangle_0 (M^{-1})^{\alpha\alpha'}(M^{-1})^{\beta\beta'}(M^{-1})^{\rho\rho'},
\]
where entries of the matrix $M$ are defined by
\[
M_{\mu\rho} = \langle \langle \gamma_1 \gamma_\mu \gamma_\rho \rangle_1 \rangle_0
\]
for any $\mu$ and $\rho$. The following lemma explains the relation between condition (C1) and the condition
\[
O_1 - O_2 = \text{constant}.
\]
Lemma 3.1. For any cohomological field theory,
\[
\langle\langle \gamma_\alpha \gamma_\beta \gamma_\beta \rangle \rangle_{0, H} |_{s \to b} = O_1 - O_2
\]
where \(\langle\langle \cdots \rangle\rangle_{g, H}\) means the restriction of the function \(\langle\langle \cdots \rangle\rangle_g\) to the small phase space \(H\), and the notation “\(f |_{s \to b}\)” stands for the function on the big phase space obtained from a function \(f\) on the small phase space via the transformation given by equation (15).

Proof. The genus-0 constitutive relation has the following form (cf. [DW]):
\[
\langle\langle \gamma_\alpha \gamma_\beta \rangle \rangle_0 |_{s \to b} = \langle\langle \gamma_\alpha \gamma_\beta \gamma_\beta \rangle \rangle_0 |_{s \to b}.
\]

Taking derivatives with respect to \(\frac{\partial}{\partial t} \rho\), we obtain
\[
\langle\langle \gamma_\alpha \gamma_\beta \gamma_\rho \rangle \rangle_0 |_{s \to b} = \langle\langle \gamma_\alpha \gamma_\beta \gamma_\rho \gamma_\mu \rangle \rangle_0 |_{s \to b} (M^{-1})^{\rho \mu},
\]
Taking derivatives with respect to \(\frac{\partial}{\partial t} \sigma\) again, we obtain
\[
\langle\langle \gamma_\alpha \gamma_\beta \gamma_\rho \gamma_\sigma \rangle \rangle_0 |_{s \to b} = \langle\langle \gamma_\alpha \gamma_\beta \gamma_\rho \gamma_\mu \gamma_\lambda \rangle \rangle_0 |_{s \to b} (M^{-1})^{\rho \mu} (M^{-1})^{\sigma \lambda} + \langle\langle \gamma_1 \gamma_\mu \gamma_\rho \gamma_\sigma \rangle \rangle_0 |_{s \to b} (M^{-1})^{\rho \mu} (M^{-1})^{\sigma \lambda}.
\]

Multiplying both sides of this equation by \((M^{-1})^{\rho \mu} (M^{-1})^{\sigma \lambda}\), we obtain
\[
O_1 = \langle\langle \gamma_\alpha \gamma_\beta \gamma_\rho \gamma_\sigma \rangle \rangle_0 |_{s \to b} + \langle\langle \gamma_1 \gamma_\mu \gamma_\rho \gamma_\sigma \rangle \rangle_0 |_{s \to b} (M^{-1})^{\rho \mu} (M^{-1})^{\sigma \lambda} + O_2.
\]

The lemma is thus proved. \(\square\)

Remark 3.2. For semisimple Frobenius manifolds, it was proved in [DLZ, Lemma 2.4] that on the small phase space
\[
O_1 = \sum_{1 \leq i < j \leq n} r_{ij} \frac{(h_i^2 + h_j^2)^2}{h_i^3 h_j^3}.
\]
After expanding the square in the numerator, it is easy to see that the right hand side of this equation is precisely the restriction of \(\langle\langle \gamma_\alpha \gamma_\beta \gamma_\beta \rangle \rangle_0\) to the small phase space due to equations (5) and (23). This gives an alternative proof for Lemma 3.1 for the special case of semisimple Frobenius manifolds.

By [DLZ, Lemma 2.5], \(O_1 - O_2\) is constant for Frobenius manifolds associated to \(ADE\) singularities and a \(P^1\)-orbifold of \(AD\) type. Therefore Lemma 3.1 implies that condition (C1) also holds for such Frobenius manifolds.

Remark 3.3. Note that the proof of [DLZ, Lemma 2.5] was quite complicated and was done via a case by case study. Lemma 3.1 and our first proof of equation (42) provide a unified and much simpler proof of [DLZ, Lemma 2.5] for \(ADE\) singularities.

Since we have verified all three conditions (C1)–(C3), Corollary 0.2 follows from Theorem 0.1.
4. Further remarks

Due to decomposition (1), if \( G^{(2)} = 0 \), the genus-2 potential function \( F_2 \) can be written as a linear combination of sixteen terms \( Q_1 \) to \( Q_{16} \) as given in [DLZ, Theorem 1.1]. Each term \( Q_i \) can be expressed using flat coordinates, and graphic representations of these terms were also given in [DLZ]. If, in addition, all genus-1 primary invariants are 0, i.e. condition (C2) holds, then the last four terms \( Q_{13} \) to \( Q_{16} \) are redundant. In this case, \( F_2 \) can be expressed as a linear combination of the first twelve terms \( Q_1 \) to \( Q_{12} \). For example, this is true for cohomological field theories satisfying conditions (C1)–(C3), in particular for \( ADE \) singularities.

Recall the following definition from [DLZ]:

\[
Q_{13} = \langle \langle \gamma_{\mu} \gamma_{\sigma} \rangle \rangle_1 \langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\mu}} \gamma_{\gamma_{\sigma}} \rangle \rangle_0 (M^{-1})^{\alpha \beta}(M^{-1})^{\sigma \gamma},
\]

\[
Q_{14} = \langle \langle \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\beta}} \rangle \rangle_1 (M^{-1})^{\alpha \beta},
\]

\[
Q_{15} = \langle \langle \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\beta}} \rangle \rangle_0 \langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\beta}} \rangle \rangle_1 (M^{-1})^{\alpha \beta}(M^{-1})^{\beta \gamma},
\]

\[
Q_{16} = \langle \langle \gamma_{\gamma_{\beta}} \rangle \rangle_1 \langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\beta}} \rangle \rangle_1 (M^{-1})^{\beta \gamma}
\]

where matrix \( M \) is defined by equation (43). The following lemma describes the precise way to get rid of these terms under condition (C2).

**Lemma 4.1.** If all genus-1 primary invariants are zero, then

\[
(i) \quad Q_{16} - \frac{1}{24} Q_{15} = 0,
\]

\[
(ii) \quad Q_{13} = \frac{1}{24} Q_3 - \frac{1}{24} Q_4,
\]

\[
(iii) \quad Q_{14} = \frac{1}{24} Q_1 - \frac{1}{12} Q_2 - \frac{1}{24} Q_3 + \frac{1}{12} Q_4.
\]

**Proof.** If all the genus-1 primary invariants are zero, then the genus-1 constitutive relation has the form (cf. [DW])

\[
F_1 = \frac{1}{24} \log \det \{\eta^{-1} M\} + \text{constant}.
\]

Note that derivatives of \( M^{-1} \) are given by

\[
\frac{\partial}{\partial t_0} (M^{-1})^{\alpha \beta} = -\langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\beta}} \gamma_{\gamma_{\sigma}} \rangle \rangle_0 (M^{-1})^{\alpha \beta}(M^{-1})^{\beta \gamma}.
\]

Taking derivatives of \( F_1 \), we get

\[
\langle \langle \gamma_{\mu} \rangle \rangle_1 = \frac{1}{24} \langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\beta}} \gamma_{\gamma_{\mu}} \rangle \rangle_0 (M^{-1})^{\alpha \beta}
\]

and

\[
\langle \langle \gamma_{\mu} \gamma_{\sigma} \rangle \rangle_1 = -\frac{1}{24} \langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\beta}} \gamma_{\gamma_{\mu}} \rangle \rangle_0 \langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\beta}} \gamma_{\gamma_{\sigma}} \rangle \rangle_0 (M^{-1})^{\alpha \beta}(M^{-1})^{\beta \gamma}
\]

\[
+ \frac{1}{24} \sum_{\alpha, \beta} \langle \langle \gamma_{\gamma_{1}} \gamma_{\gamma_{\alpha}} \gamma_{\gamma_{\beta}} \gamma_{\gamma_{\mu}} \rangle \rangle_0 (M^{-1})^{\alpha \beta}.
\]

Applying equation (46) to the genus-1 1-point function in the definition of \( Q_{16} \), we get \( \frac{1}{24} Q_{15} \). This proves part (i). Part (ii) is obtained by applying equation (47) to the genus-1 2-point function in the definition of \( Q_{13} \). Taking derivatives of equation (47) with respect to \( \gamma_{1} \), we get a formula expressing \( \langle \langle \gamma_{1} \gamma_{\mu} \gamma_{\sigma} \rangle \rangle_1 \) in terms
of genus-0 functions, which implies part (iii) after plugging into the definition of $Q_{14}$. \hfill \Box

Remark 4.2. The involvement of $Q_{15}$ and $Q_{16}$ in $F_2$ is through the expression

$$-\frac{7}{240}Q_{15} + \frac{7}{10}Q_{16}.$$ 

Therefore part (i) of Lemma 4.1 eliminates both $Q_{15}$ and $Q_{16}$.

APPENDIX A. THE GENUS-2 G-FUNCTION

In this appendix, we give the precise definition of the genus-2 G-function $G^{(2)}$ following [DLZ]. Write

$$G^{(2)} = \sum_i G^{(2)}_i(u, u_x)u_{xx}^i + \sum_{i \neq j} G^{(2)}_{ij}(u) \frac{(u_j^i)^3}{u_x^i}$$

(48)

$$+ \frac{1}{2} \sum_{i,j} P^{(2)}_{ij}(u) u_x^i u_x^j + \sum_i Q^{(2)}_i(u) (u_x^i)^2.$$ 

Let $\gamma_{ij}$ be the rotation coefficient on the small phase space as defined in [D]. Note that $\gamma_{ii} = 0$, which is different from our definition of $\gamma_{ii}$ in section 1. For $i \neq j$, $\gamma_{ij}$ is equal to our definition of $r_{ij}$ restricted to the small phase space. Define

$$H_i := \frac{1}{2} \sum_{j \neq i} u_{ij} \gamma_{ij}^2$$

where $u_{ij} := u_i - u_j$. Then the function $G^{(2)}_i$ can be defined as

$$G^{(2)}_i = G^{(2)}_{i,1} + G^{(2)}_{i,2}$$

with

$$G^{(2)}_{i,1} = -\frac{\partial_x h_i H_i}{60 u_{i,x} h_i^3} - \frac{7 \partial_x h_i \partial_x h_i}{5760 u_{i,x} h_i^4} + \sum_k \left( \frac{\gamma_{ik} H_k}{120 h_i h_k} \frac{u_{k,x}}{u_{i,x}} - \frac{\gamma_{ik} \partial_x h_i}{5760 h_i^2 h_k u_{i,x}} - \frac{\gamma_{ik} \partial_k h_k u_{k,x}}{1152 h_i h_k^2 u_{i,x}} \right)$$

$$+ \frac{\partial_x \gamma_{ik} H_k}{1920 u_{i,x} h_i^3} + \frac{\partial_x \gamma_{ik}}{5760 u_{i,x} h_i h_k} + \frac{\partial_k \gamma_{ik} u_{k,x}}{2880 h_i h_k u_{i,x}} - \frac{7 \gamma_{ik}^2 u_{k,x}}{1152 h_i^2 u_{i,x}}$$

$$- \sum_{k,l} \frac{u_{k,x} h_k \gamma_{ik} \gamma_{kl}}{1920 u_{i,x} h_i h_l^2}$$

and

$$G^{(2)}_{i,2} = -\frac{3 \partial_x h_i H_i}{40 h_i^3} + \frac{19(\partial_x h_i)^2}{2880 h_i^4} + \sum_k \left( \frac{\gamma_{ik} H_i}{120 h_i h_k} + \frac{7 \gamma_{ik} H_k}{120 h_i h_k} - \frac{4 \gamma_{ik} \partial_x h_i}{5760 h_i^2 h_k} - \frac{7 \gamma_{ik} \partial_k h_k}{2880 h_i h_k^2} \right)$$

$$+ \frac{\gamma_{ik} \partial_x h_i}{384 h_i^3} + \frac{\partial_x \gamma_{ik} h_k}{384 h_i^3} + \frac{\partial_x \gamma_{ik}}{2880 h_i h_k} + \frac{7 \partial_k \gamma_{ik}}{2880 h_i h_k} + \frac{\gamma_{ik} \partial_k h_k}{2880 h_i h_k^3}$$

$$- \frac{19 \gamma_{ik}^2}{720 h_i^2} + \frac{\gamma_{ik}^2}{1440 h_i^2} - \sum_{k,l} \frac{h_i \gamma_{ik} \gamma_{kl}}{2880 h_k h_l^2}.$$
Other functions in equation (48) are defined in the following way:

\[
G_{ij} = \frac{\gamma_{ij}^2 H_j}{120 h_{ij}^3} + \frac{\gamma_{ij}^3}{480 h_{ij}^3} - \frac{\gamma_{ij}}{5760} \left( \frac{\partial_i \gamma_{ij}}{h_{ij}^2} + \frac{\partial_j \gamma_{ij}}{h_{ij}^2} \right) + \frac{\gamma_{ij}^2}{5760} \left( \frac{\partial_i h_{ij}}{h_{ij}^3} + 3 \frac{\partial_j h_{ij}}{h_{ij}^3} \right)
\]

\[
+ \sum_k \left( \frac{\gamma_{ijk} \gamma_{ijk} h_{ik}}{5760 h_k^3} + \frac{\gamma_{ij}^2}{5760 h_k^3} \left( \gamma_{jk} - \gamma_{ik} h_{ik} \right) \right),
\]

\[
P_{ij}^{(2)} = \frac{2 \gamma_{ij} H_i H_j}{5 h_i h_j} + \frac{\gamma_{ij} \partial_i h_{ij} h_i}{20 h_i h_j^2} + \frac{\gamma_{ij} h_i \partial_j h_{ij} H_j}{20 h_i^3} - \frac{19 \gamma_{ij}^2 H_j}{30 h_j^2} - \frac{\partial_i \gamma_{ij} H_j}{60 h_i h_j}
\]

\[
+ \frac{41 \gamma_{ij}^3}{240 h_i h_j} - \frac{41 \gamma_{ij} \partial_i \gamma_{ij} h_i}{1440 h_i^2 h_j} + \frac{\partial_i \gamma_{ij} \partial_j h_{ij}}{1440 h_i^3 h_j} + \frac{79 \gamma_{ij} \partial_j h_{ij}}{1440 h_j^3}
\]

\[- \frac{\gamma_{ij} \partial_i h_{ij} \partial_j h_{ij}}{720 h_i^2 h_j^2} - \frac{\gamma_{ij} \partial_i h_{ij} (\partial_j h_{ij})^2}{288 h_j^3}
\]

\[
+ \sum_k \left( \frac{\gamma_{ijk} \gamma_{ijk} H_{ik}}{60 h_j h_k} - \frac{\gamma_{ij} \gamma_{ijk} h_{ij} h_k}{60 h_j^2 h_k} - \frac{\gamma_{ij} \gamma_{ijk} h_{ij} H_k}{60 h_j h_k^2} \right) + \frac{\gamma_{ijk} \gamma_{ijk} h_{ij} h_k}{720 h_k^3 h_j}
\]

\[- \frac{\gamma_{ij} \gamma_{ijk} \partial_j h_{jk}}{120 h_j h_k} - \frac{\gamma_{ij} \gamma_{ijk} \partial_k h_{jk}}{720 h_j h_k^2} + \frac{\gamma_{ijk} \gamma_{ijk} h_{ij} \partial_k h_{jk}}{1440 h_i^2 h_j h_k} + \frac{\gamma_{ijk} \partial_j \gamma_{ijk}}{1440 h_j^3 h_k}
\]

\[- \frac{\gamma_{ij}^2 \gamma_{ijk}}{120 h_i h_k} + \frac{7 h_i \gamma_{ij} \gamma_{ijk}^2}{160 h_j^2} + \frac{11 \gamma_{ijk} \gamma_{ijk} h_{jk}}{288 h_k^2} + \frac{h_{ijk}^2 \gamma_{ijk}}{96 h_k^3}
\]

\[
+ \sum_{k,l} \left( \frac{h_{ij} \gamma_{ij} \gamma_{ijkl}}{720 h_k^2 h_l^2} \left( \frac{\gamma_{kl}}{h_i} - \frac{\gamma_{jk}}{2 h_j} \right) - \frac{h_{ij} \gamma_{ijkl} \gamma_{kl}}{720 h_k^2 h_l^2} \right),
\]

and

\[
Q_{i}^{(2)} = \frac{4 H_i^3}{5 h_i^2} - \frac{7 \partial_i h_{ij} H_i}{10 h_i^3} + \frac{7 (\partial_i h_{ij} H_i)^3}{48 h_i^4} + \frac{\gamma_{ij} H_k}{120 h_i h_k} + \frac{(2 H_i + 7 H_k) \partial_i \gamma_{ijk}}{240 h_i h_k}
\]

\[
+ \frac{7 \partial_i \gamma_{ijk} h_{ij} h_k}{240 h_i h_k} + \frac{7 \gamma_{ijk} \partial_i h_{ij} H_k}{80 h_i^2 h_k} + \frac{\gamma_{ij} H_k}{576 u_{ik} h_i h_k} + \frac{(2 H_i + 7 H_k) \partial_i \gamma_{ijk}}{240 h_i h_k}
\]

\[
+ \frac{\gamma_{ijk} H_i}{576 u_{ik} h_i^2} - \frac{31 \gamma_{ij} h_{ij}}{144 h_i^2} + \frac{7 \gamma_{ijk} \partial_i h_{ij} h_k}{720 h_i^3 h_k} + \frac{253 \gamma_{ij} \partial_i h_{ij} h_k}{5760 h_i^3} - \frac{\partial_i \gamma_{ijk} \partial_i h_{ij} h_k}{960 h_i^2 h_k} - \frac{\gamma_{ijk} \partial_i h_{ij} h_k}{2880 h_i^2}
\]

\[
- \frac{7 \partial_i \gamma_{ijk} \partial_i h_{ij} h_k}{1920 h_i^2} + \frac{7 \partial_i \gamma_{ijk} \partial_i h_{ij} h_k}{5760 h_i^4} + \frac{41 \partial_i \gamma_{ijk} \partial_i h_{ij} h_k}{5760 h_i^4} + \frac{\partial_i (h_{ijk} \gamma_{ij}) \partial_i h_{ij} h_k}{2880 h_i^4}
\]

\[
- \frac{113 \gamma_{ijk} \partial_i \gamma_{ijk}}{5760 h_i^2 h_k^2} + \frac{3 (\partial_i \gamma_{ijk} + \partial_k \gamma_{ijk} h_{ik}) \partial_i h_{ij} h_k}{1440 h_i^2 h_k} - \frac{(\partial_i \gamma_{ijk} + \partial_k \gamma_{ijk} h_{ik}) \partial_i h_{ij} h_k}{1440 h_i^2 h_k} - \frac{\partial_k \gamma_{ijk} \partial_i h_{ij} h_k}{2880 h_i^2 h_k} + \frac{\partial_k \gamma_{ijk} \partial_i h_{ij} h_k}{2880 h_i^2 h_k} - \frac{\gamma_{ijk} \partial_i h_{ij} h_k}{2880 h_i^2 h_k} + \frac{\gamma_{ijk} \partial_i h_{ij} h_k}{2880 h_i^2 h_k} + \frac{\gamma_{ijk} \partial_i h_{ij} h_k}{2880 h_i^2 h_k}
\]

\[
+ \sum_{k,l} \left( - \frac{\gamma_{ijkl} \partial_i \gamma_{ijkl}}{2880 h_k h_l^2} + \frac{\gamma_{ijkl} \partial_i \gamma_{ijkl}}{2880 h_k h_l^2} - \frac{\partial_i \gamma_{ijkl} \gamma_{ijkl}}{240 h_i h_k} + \frac{u_{ik} \gamma_{ijkl} \partial_i \gamma_{ijkl}}{1152 u_{ik} h_i h_l} + \frac{\gamma_{ijkl} \partial_i \gamma_{ijkl}}{240 h_i h_k} \right).\]
In these expressions, all summations are taken over the ranges of indices where the denominators do not vanish. The symbol $\partial_x$ means taking derivatives with respect to $x = t_0$ after transformation given by equation (15), and $\partial_i$ means taking derivatives with respect to $u_i$.

**References**


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