

THE TOPOLOGICAL COMPLEXITY OF CANTOR ATTRACTORS FOR UNIMODAL INTERVAL MAPS

SIMIN LI AND WEIXIAO SHEN

ABSTRACT. For a non-flat C^3 unimodal map with a Cantor attractor, we show that for any open cover \mathcal{U} of this attractor, the complexity function $p(\mathcal{U}, n)$ is of order $n \log n$. In the appendix, we construct a non-renormalizable map with a Cantor attractor for which $p(\mathcal{U}, n)$ is bounded from above for any open cover \mathcal{U} .

1. INTRODUCTION

In this paper, we will consider the topological complexity of a unimodal interval map $f : [0, 1] \rightarrow [0, 1]$ restricted to an invariant Cantor set X . For an open cover \mathcal{U} of X , let $N(\mathcal{U})$ be the minimal cardinality of a subcover of \mathcal{U} . For open covers \mathcal{U}, \mathcal{V} of X , let $\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. The *topological complexity function* of an open cover \mathcal{U} is the non-decreasing function

$$p(\mathcal{U}, n) = N\left(\bigvee_{i=0}^{n-1} f^{-i}\mathcal{U}\right), \quad n = 1, 2, \dots$$

Recall that the topological entropy of $f : X \rightarrow X$ is by definition

$$h_{\text{top}}(f|_X) = \sup_{\mathcal{U}} \lim_{n \rightarrow \infty} \frac{1}{n} \log p(\mathcal{U}, n),$$

where the supremum is taken over all open covers of X . The complexity functions can be used to characterize the dynamical behavior of some systems with topological entropy zero. For example, it is proved in [1, Proposition 2.2] that a system is equicontinuous if and only if $\sup_{n \geq 1} p(\mathcal{U}, n) < \infty$ holds for each open cover \mathcal{U} ,

A continuous map $f : [0, 1] \rightarrow [0, 1]$ is called *unimodal* if there exists a unique $c \in (0, 1)$ (called the *turning point*) such that f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$. In order to apply a convenient version of the Koebe principle, we shall always assume that f is C^3 outside c , $f'(x) \neq 0$ for $x \neq c$, and c is non-flat, i.e., there exist C^3 local diffeomorphisms ϕ, ψ defined on a neighborhood of 0 with $\phi(0) = c$, $\psi(0) = f(c)$, and a real number $\ell = \ell_c > 1$ (called the *order of c*), such that $|\psi^{-1} \circ f \circ \phi(x)| = |x|^\ell$ holds when $|x|$ is small. The turning point c is also called a *critical point*.

Let \mathcal{A} denote the collection of unimodal maps with the above properties and let \mathcal{A}_* denote the collection of $f \in \mathcal{A}$ which have all periodic points hyperbolic repelling.

Received by the editors December 16, 2012 and, in revised form, December 2, 2013.
 2010 *Mathematics Subject Classification*. Primary 37E05; Secondary 37C70, 37B40.

We are interested in the case where X is a Cantor attractor. Following [22], a (minimal) *metric attractor* is a compact invariant subset $X \subset [0, 1]$ such that $\text{Rel}(X) := \{x \in [0, 1] : \omega(x) \subset X\}$ has positive Lebesgue measure, but no invariant compact proper subset of X has this property. Metric attractors were studied in [3] under an additional assumption that f has negative Schwarzian derivative, although most of their work extends to maps in the class \mathcal{A} after [16]; see also [14]. In particular, it was shown that a metric attractor of $f \in \mathcal{A}$ can be one of the following forms: an attracting periodic orbit, or the union of a cycle of periodic intervals, or a Cantor set. In the last case, the Cantor attractor X must coincide with $\omega(c)$ and $h_{\text{top}}(f|_X) = 0$; see [3, Section 11].

Our main result is the following theorem.

Main Theorem. *Let $f \in \mathcal{A}_*$ be a unimodal map with critical point c . Suppose that $\omega(c)$ is a Cantor attractor. Then for each open cover \mathcal{U} of $\omega(c)$, there is a constant $C > 0$ such that the complexity function of $f|_{\omega(c)}$ satisfies $p(\mathcal{U}, n) \leq Cn \log n$ for $n > 1$.*

One may wonder whether $p(\mathcal{U}, n)$ has a lower bound for some (small) open cover \mathcal{U} . It is well known that when f is infinitely renormalizable, $\omega(c)$ is a Cantor attractor, $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to an adding machine and hence $p(\mathcal{U}, n)$ is bounded for each open cover \mathcal{U} of $\omega(c)$. Even in the non-renormalizable case, there exists a unimodal map with a Cantor attractor for which $f : \omega(c) \rightarrow \omega(c)$ is again topologically conjugate to an adding machine, as we show in Theorem 6.1 in §6. On the other hand, in Corollary 5.6, we prove that for interval maps with *special combinatorics* (including the well-studied Fibonacci case), $p(\mathcal{U}, n)/n$ is bounded away from zero for small open covers of $\omega(c)$.

By considering open covers formed by nice intervals and their entry domains, we reduce the Main Theorem to an estimate of the number of children of symmetric nice intervals. See the Reduced Main Theorem in §2.2.

A Cantor attractor of non-infinitely renormalizable map $f \in \mathcal{A}$ is often called a *wild attractor* because its basin of attraction is of the first Baire category. The existence of wild attractors for unimodal maps with the Fibonacci combinatorics was obtained in [5]. This result was generalized in [4] to unimodal maps with “Fibonacci-like” combinatorics. While a sufficient and necessary combinatorial condition compatible with the existence of a wild attractor seems far from being reached, the dynamics of unimodal maps restricted to wild attractors was studied in [6, 8, 17], among others. Our construction in Theorem 6.1 was motivated by [2], and the proof uses a result of [4].

Entropy zero systems. There have appeared quite a few notions to measure the complexity of topological dynamical systems of zero topological entropy. In the following, we shall mention two of them. In [13], a notion called *topological entropy dimension* was introduced. (The metric entropy dimension was introduced earlier in [11].) A topological dynamical system $f : X \rightarrow X$ has zero (upper) topological entropy dimension if for every open cover \mathcal{U} and $\alpha > 0$, $n^{-\alpha} \log p_n(\mathcal{U}) \rightarrow 0$ as $n \rightarrow \infty$. So our main theorem clearly implies that the topological entropy dimension of $f|_{\omega(c)}$ is zero.

Another notion we would like to mention is the *topological sequence entropy* introduced in [12]. For an increasing sequence $(n_k)_{k=1}^{\infty}$ of positive integers, the

sequence entropy of $f : X \rightarrow X$ is

$$h(T, (n_k)_{k=1}^\infty) = \sup_{\mathcal{U}} \limsup_{k \rightarrow \infty} \frac{1}{n_k} \log N \left(\bigvee_{i=1}^k T^{-n_i}(\mathcal{U}) \right),$$

where the supremum is taken over all open covers of X . There are systems which have zero topological entropy but positive topological sequence entropy. A system is called a *null system* if the topological sequence entropy is zero for every sequence $(n_k)_{k=1}^\infty$. We shall see that the dynamics in a Cantor attractor are not necessarily null. In fact, in [6, Theorem 3 (2)], Bruin, Keller and Pierre constructed a unimodal map $f \in \mathcal{A}_*$ together with a symbolic dynamical system (Ω, T) such that

- (Ω, T) is minimal, uniquely ergodic and weakly mixing with respect to its unique invariant probability measure μ ;
- f has a wild attractor $\omega(c)$;
- $f|_{\omega(c)}$ is a factor of (Ω, T) .

Theorem 1.1. *For the above example of Bruin-Keller-Pierre, $f : \omega(c) \rightarrow \omega(c)$ has zero entropy dimension but is not null.*

Proof. By the Main Theorem, $f : \omega(c) \rightarrow \omega(c)$ has entropy dimension zero. Let us show that $f : \omega(c) \rightarrow \omega(c)$ is not null. Indeed, since T is minimal, for each non-empty open set $U \subset \Omega$, $\Omega = \bigcup_{n=0}^{+\infty} T^{-n}U$, and therefore $\mu(U) > 0$. Since T is weakly mixing with respect to μ , $T \times T$ is ergodic with respect to $\mu \times \mu$. As $\mu \times \mu$ has positive measure on each non-empty open set, it follows that $T \times T$ is topologically transitive; see [25, Theorem 5.16]. Thus

$$\widehat{f} := f \times f : \omega(c) \times \omega(c) \rightarrow \omega(c) \times \omega(c)$$

is topologically transitive. Arguing by contradiction, assume that $f|_{\omega(c)}$ is null. Then by [15, Theorem 4.3], it is an almost one to one extension of an equicontinuous system $g : X \rightarrow X$. As $\omega(c)$ is a Cantor set, X is not a singleton, hence $g \times g$ is not topologically transitive. On the other hand, $g \times g$ is a factor of \widehat{f} , hence topologically transitive. A contradiction! □

2. NICE INTERVALS AND CHILDREN

In this section we shall prove the Main Theorem in the infinitely renormalizable case and deduce it from a Reduced Main Theorem in the non-infinitely renormalizable case.

Consider a unimodal map $f : [0, 1] \rightarrow [0, 1]$ in \mathcal{A}_* . Let c denote the critical point of f and let ℓ be the order of c . Without loss of generality, we may assume $f(0) = f(1) = 0$. We will also assume that f is geometrically symmetric near c .

2.1. Notation and terminologies. Given a subset V of $[0, 1]$ and an integer $k \geq 0$, we say that a component J of $f^{-k}(V)$ is a *pull back of V by f^k* . We say that such a pull back is

- *critical* if it contains the critical point c ;
- *diffeomorphic* if f^k maps J diffeomorphically onto a component of V ;
- *unimodal* if $J \ni c$ and f^{k-1} maps a neighborhood of $f(J)$ diffeomorphically onto a component of V .

For $T \subset [0, 1]$, let

$$D(T) = \{x \in [0, 1] : f^k(x) \in T \text{ for some } k \geq 1\}.$$

The *first entry map* $R_T : D(T) \rightarrow T$ is defined as $x \mapsto f^{k(x)}(x)$, where $k(x)$ is the *entry time* of x into T , i.e., the minimal positive integer such that $f^{k(x)}(x) \in T$. The map $R_T|_{(D(T) \cap T)}$ is called the *first return map* of T . A component of $D(T)$ (resp. $D(T) \cap T$) is called an *entry domain* (resp. *return domain*) of T . Let $\mathcal{L}_x(T)$ denote the entry domain containing x .

Let us call an open set $T \subset [0, 1]$ *nice* if $f^n(\partial T) \cap T = \emptyset$ for all $n \geq 0$ and T does not contain a fixed point of f . It is well known that for such an open set T ,

- pull backs of a nice set are again nice;
- if J_j is a pull back of T by f^{k_j} , $j = 1, 2$, and $k_1 \geq k_2$, then $J_1 \cap J_2 = \emptyset$ or $J_1 \subset J_2$;
- the entry time is constant in any component of $D(T)$, so the first entry map $R_T : D(T) \rightarrow T$ is continuous.

Moreover, if $f \in \mathcal{A}_*$, then there exists an arbitrarily small symmetric nice interval $T \ni c$. See for example [20].

A nice interval $T \ni c$ is called *symmetric* if $f(\partial T)$ consists of a single point. A unimodal pull back of a nice interval $T \ni c$ is also called a *child* of T .

We say that f is *persistently recurrent* if for each symmetric nice interval $T \ni c$, the number of children of T is finite. The following is well known.

Proposition 2.1 (Blokh-Lyubich [3]). *Suppose that $f \in \mathcal{A}_*$ has a Cantor attractor A . Then $A = \omega(c) \ni c$, A is a minimal set and f is persistently recurrent.*

Given a bounded interval I and a constant $\tau > 0$, let τI denote the open interval which is concentric with I and has length $\tau|I|$. We say that a bounded interval J is τ -well inside an interval I if $I \supset (1 + 2\tau)J$, i.e., both components of $I \setminus J$ have length at least $\tau|J|$.

A nice interval I is called τ -*nice*, if each return domain of I is τ -well inside I .

A closed interval I is called a *restrictive interval* if I contains c in its interior and there exists an integer $s \geq 2$ such that $I, f(I), \dots, f^{s-1}(I)$ have pairwise disjoint interior and such that $f^s(I) \subset I, f^s(\partial I) \subset \partial I$. The integer s is called the *period* of I and $f^s : I \rightarrow I$ is called a *renormalization* of f . The map f is called *infinitely renormalizable* if there exists a restrictive interval with an arbitrarily large period.

2.2. Nice covers. Assume that $f \in \mathcal{A}_*$ has a non-periodic recurrent critical point c such that $\omega(c)$ is minimal. We say that an open cover \mathcal{Y} of $\omega(c)$ is *nice* if there is a symmetric nice interval Y such that \mathcal{Y} is the collection of components of $Y \cup D(Y)$ which intersect $\omega(c)$. For such an open cover, let

$$(1) \quad q(\mathcal{Y}, n) = \#\{\text{components of } f^{-n}(Y \cup D(Y)) \text{ intersecting } \omega(c)\}.$$

For each nice interval $Y \ni c$, let $\nu(Y)$ denote the number of children of Y , and for each $n \geq 0$, let Y_{-n} denote the component of $f^{-n}(D(Y) \cup Y)$ which contains c . We shall use the following lemma.

Lemma 2.2. *For any symmetric nice interval Y and the corresponding nice cover \mathcal{Y} , we have*

$$p(\mathcal{Y}, n + 1) \leq q(\mathcal{Y}, n)$$

for each $n \geq 0$. Moreover, assuming that f is persistently recurrent, then for n large enough, we have

$$q(\mathcal{Y}, n) \leq \sum_{i=0}^{n-1} \nu(Y_{-i}).$$

Proof. For $x \in \omega(c)$ and $n \geq 0$, if $Z_n(x)$ is the component of $f^{-n}(D(Y) \cup Y)$ which contains x , then $f^j(Z_n(x))$, $0 \leq j \leq n$, is contained in a component of $D(Y) \cup Y$. It follows that $Z_n(x)$ is contained in an element of $\bigvee_{j=0}^n f^{-j}\mathcal{Y}$. The first inequality follows.

Let us prove the second inequality; assume that n is so large that Y has no child with transition time greater than n . For each component J of $f^{-n}(D(Y) \cup Y)$, there exists a minimal integer $n' = n'(J) \in \{0, 1, \dots, n\}$ such that $f^{n'}(J)$ contains the critical point c . Let $\mathcal{J}_{n'}$ denote the collection of all components J of $f^{-n}(D(Y) \cup Y)$ with $n'(J) = n'$ and $J \cap \omega(c) \neq \emptyset$. Clearly, \mathcal{J}_0 has at most one element. Let us show that $\mathcal{J}_n = \emptyset$. Indeed, any element J is a diffeomorphic pull back of Y by f^m for some $m = m(J) \geq n$, and if $t(J) > 1$ is the entry time of c into J , then the pull back of J by $f^{t(J)}$ containing 0 is a child of Y with transition time $\geq m + t(J) > n$, which is ruled out by our assumption on n . A similar argument shows that for each $n > n' > 0$,

$$(2) \quad \#\mathcal{J}_{n'} \leq \nu(Y_{-n+n'}).$$

Indeed, each $J \in \mathcal{J}_{n'}$ is a diffeomorphic pull back of $Y_{-n+n'}$ by $f^{n'}$, so if $t(J)$ is the first entry time of c to J , then the component of $f^{-t(J)}(J)$ which contains c is a child of $Y_{-n+n'}$ with transition time $t(J) + n'$. As different J 's correspond to different $t(J)$'s, (2) follows. Thus

$$q(\mathcal{Y}, n) \leq \sum_{i=1}^{n-1} \nu(Y_{-i}) + 1 \leq \sum_{i=0}^{n-1} \nu(Y_{-i}).$$

□

Remark 2.3. In Theorem 5.5, we shall show that $q(\mathcal{Y}, n)/n$ is bounded away from zero. However, this does not imply a lower bound for $p(\mathcal{Y}, n)$, because an element of $\bigvee_{j=0}^n f^{-j}\mathcal{Y}$ may contain a large number of components of $f^{-n}(D(Y) \cup Y)$ intersecting $\omega(c)$.

The following Reduced Main Theorem is the main step of our proof of the Main Theorem.

Reduced Main Theorem. *Suppose that $f \in \mathcal{A}_*$ is non-renormalizable and that $\omega(c)$ is a Cantor attractor. For each symmetric nice interval $Y \ni c$, there exists $n_0 = n_0(Y) \geq 2$ such that if T is a critical pull back of Y by f^n for some $n \geq n_0$, then the number of children of T is bounded from above by $C \log n$, where $C > 0$ is a constant depending only the critical order.*

Proof of the Main Theorem. We may assume that f is non-renormalizable, as in the infinitely renormalizable case the Main Theorem is well known (see for example

[21, Proposition III.4.5]), and the finitely renormalizable case can be reduced to the non-renormalizable case.

Given a nice interval $Y \ni c$, let \mathcal{Y} denote the corresponding nice cover of $\omega(c)$. Since f has no wandering interval ([21, Chapter IV]), the maximal length of elements of \mathcal{Y} tends to zero as $|Y| \rightarrow 0$. Thus for any open cover \mathcal{U} of $\omega(c)$, there exists a small symmetric nice interval $Y \ni c$ such that \mathcal{Y} is a refinement of \mathcal{U} ; hence $p(\mathcal{U}, n) \leq p(\mathcal{Y}, n)$. By Lemma 2.2, it follows that

$$p(\mathcal{U}, n + 1) \leq \sum_{i=0}^{n-1} \nu(Y_{-i}),$$

provided that n is large enough. By Proposition 2.1, $\nu(Y_{-i})$ is finite for each i . By the Reduced Main Theorem, there exists n_0 such that for $i \geq n_0$, $\nu(Y_{-i}) \leq C \log i$. Thus $p(\mathcal{U}, n + 1) = O(n \log n)$. □

Remark 2.4. In [9], it is proved that a Fibonacci-like unimodal map has sublinear complexity, i.e., $p(\mathcal{U}, n) \leq Cn$ for some constant $C > 0$ and each open cover \mathcal{U} . For a Fibonacci-like unimodal map, the numbers of children of nice intervals are bounded by a constant. Therefore their result is compatible with ours.

It is not clear to us whether the upper bounds appearing in the Reduced Main Theorem are optimal. Indeed, the following simpler problem is open:

Problem. Give a positive integer $N \geq 2$, does there exist a real number ℓ_0 such that if $f \in \mathcal{A}_*$ has critical order $\ell > \ell_0$ and satisfies the property that each nice interval has at most N children, then does f have a wild attractor?

In [4, Section 6] Bruin gave a sufficient condition in terms of a different combinatorial language (the kneading map) for the existence of wild attractors. Note that Bruin’s condition prohibits the existence of saddle-node-like returns which however does not seem to be an obstruction for the existence of wild attractors.

2.3. Idea of proof of the Reduced Main Theorem. We introduce the notion of “empty space”, for each small symmetric nice interval, at the beginning of §4. Roughly speaking, we fix a suitable neighborhood Λ of $\omega(c)$, and consider the subset $\Lambda(T)$ of T consisting of points which return to T before escaping the neighborhood Λ . The “empty space” $\xi(T)$ measures the proportion of $T \setminus \Lambda(T)$ in T : The smaller $\xi(T)$ is, the smaller the proportion of $T \setminus \Lambda(T)$ in T . The assumption that $\omega(c)$ is a Cantor attractor implies that $\xi(T) \rightarrow 0$ as $|T| \rightarrow 0$.

Most of our effort is to estimate the distortion $\xi(T)$ under unimodal pull back. There are two important principles lying in the proof:

- If a symmetric nice interval T has many children, then all young children J are τ -nice with a large τ , i.e., all the return domains lie deep inside J .
- If a symmetric nice interval T is τ -nice and J is a child of T , then $\xi(J)/\xi(T)$ is bounded away from zero. Moreover, if τ is large and $\xi(T)$ is close to zero, then $\xi(J)$ becomes much bigger than $\xi(T)$.

The proof of the Reduced Main Theorem occupies the next three sections. In §3, we study the size of children of a given nice interval and the geometry of their return domains. In §4, we study the distortion of “empty space” under pull backs. In both cases, the presence of a central cascade is an unpleasant situation and responsible for most complications of the arguments. The proof of the Reduced Main Theorem is completed in §5.

3. REAL BOUNDS

Consider a map $f \in \mathcal{A}_*$ with a recurrent critical point c . We say a constant is *universal* if it depends only on ℓ . In this section, we shall obtain upper bounds of the length of children of given nice intervals and the geometry of their return domains. The main result is Proposition 3.7.

3.1. Preliminaries. The Koebe principle is the main tool to control distortion in one-dimensional dynamics. The following version was taken from [7, Proposition 1], whose proof is based on previous results in the literature, in particular [24, Theorem C].

Theorem 3.1. *There exists $\eta(f) > 0$ such that the following holds. Let $s \geq 1$ be an integer and let T be an interval. Assume that $f^s|_T$ is a diffeomorphism onto its image and that $|f^s(T)| < \eta(f)$. If J is a subinterval of T such that $f^s(J)$ is τ -well inside $f^s(T)$, then*

1. for any $x, y \in J$,

$$0.9 \left(\frac{\tau}{1 + \tau} \right)^2 \leq \frac{|Df^s(x)|}{|Df^s(y)|} \leq \frac{1}{0.9} \left(\frac{1 + \tau}{\tau} \right)^2;$$

2. J is τ' -well inside T , where $\tau' = \frac{0.9\tau^2}{1 + 2\tau}$.

Given a symmetric nice interval $I \ni c$, we shall use the following notation: $I^0 = I$ and I^{k+1} is the return domain of I^k that contains c . The sequence

$$I^0 \supset I^1 \supset I^2 \supset \dots$$

is often called the *principal nest* starting from I . The first return map $R_{I^n} : I^{n+1} \rightarrow I^n$ is called *central* if $R_{I^n}(c) \in I^{n+1}$ and *non-central* otherwise. We say that $R_{I^n} : I^{n+1} \rightarrow I^n$ is *high* if $R_{I^n}(I^{n+1}) \ni c$ and *low* otherwise.

The following Real Bounds Theorem was first proved by Martens [20] in the case that f has negative Schwarzian derivative, and extended to general smooth unimodal maps in [16].

Theorem 3.2. *There exists a universal constant $\rho > 0$ such that for any small symmetric nice interval $I^0 \ni c$, the following hold:*

- (i) *If $R_{I^0} : I^1 \rightarrow I^0$ is non-central and low, then I^1 is ρ -well inside I^0 .*
- (ii) *If $R_{I^0} : I^1 \rightarrow I^0$ is non-central and high, then I^2 is ρ -well inside I^1 .*
- (iii) *If I^1 is not ρ -well inside I^0 , then f^{s-1} maps a neighborhood of $f(I^1)$ diffeomorphically onto a ρ -scaled neighborhood of I^0 , where s is the return time of c into I^0 . In particular, the map $f^{s-1}|_{f(I^1)}$ has uniformly bounded distortion: for any $x, y \in I^1$,*

$$|Df^{s-1}(f(x))| \leq K(\rho)|Df^{s-1}(f(y))|,$$

where $K(\rho) > 1$ is a constant.

A sequence of open intervals $\{T_j\}_{j=0}^s$ is called a *chain* if for each $j = 0, 1, \dots, s-1$, T_j is a component of $f^{-1}(T_{j+1})$. The *order* of the chain is the number of j 's with $0 \leq j < s$ such that T_j contains the critical point c .

The following theorem is an improvement of [24, Theorem C(1)] for unimodal maps, which gives the relationship between the constants τ and τ' .

Theorem 3.3. *Assume that f is not infinitely renormalizable. For any $\tau > 0$ there exists $\tau' > 0$, such that the following holds. Let $c \in J \subset I$ be small symmetric nice intervals such that J is τ -well inside I . Then for any $x \in D(J)$, $\mathcal{L}_x(J)$ is τ' -well inside $\mathcal{L}_x(I)$. Moreover, for each constant $\tau_* > 0$ there exist constants $C = C(\tau_*) > 0, \alpha = \alpha(\tau_*) > 0$ such that if $\tau > \tau_*$, then we can choose τ' such that*

$$(3) \quad \tau' \geq C\tau^\alpha.$$

Proof. By Theorem 3.1 and non-flatness of the critical point, it suffices to prove the statement for $x \in D(J) \setminus I$. Let $I^0 = I$. Let $m(0) = 0$ and $1 \leq m(1) < m(2) < \dots$ be all the non-central return moments, i.e., the return map $R_{I^{m(k)-1}}$ is non-central. Since f is not infinitely renormalizable, $|I^n| \rightarrow 0$ as $n \rightarrow \infty$, provided that I is small enough. So there exists $k \geq 0$ such that

$$I^0 \supset I^{m(1)} \supset \dots \supset I^{m(k)} \not\supseteq J \subset I^{m(k+1)}.$$

Define $\tau_i, 1 \leq i \leq k + 1$, such that

$$\frac{|I^{m(i-1)}|}{|I^{m(i)}|} := 1 + 2\tau_i, \text{ for } 1 \leq i \leq k,$$

and

$$\frac{|I^{m(k)}|}{|J|} := 1 + 2\tau_{k+1}.$$

Then

$$(4) \quad 1 + 2\tau = \frac{|I|}{|J|} = \prod_{i=1}^k \frac{|I^{m(i-1)}|}{|I^{m(i)}|} \cdot \frac{|I^{m(k)}|}{|J|} = \prod_{i=1}^{k+1} (1 + 2\tau_i).$$

For each $1 \leq i \leq k$, the first entry map $R_{I^{m(i)}} : \mathcal{L}_x(I^{m(i)}) \rightarrow I^{m(i)}$ can be extended diffeomorphically onto $I^{m(i-1)}$ (see Lemma 3.8). By Theorem 3.1, $\mathcal{L}_x(I^{m(i)})$ is τ'_i -well inside $\mathcal{L}_x(I^{m(i-1)})$, where $\tau'_i = 0.9 \frac{\tau_i^2}{1+2\tau_i}$. Similarly, since $I^{m(k)} \supset J \supset I^{m(k+1)}$, the first entry map $R_J : \mathcal{L}_x(J) \rightarrow J$ can be extended diffeomorphically onto $I^{m(k)}$, and $\mathcal{L}_x(J)$ is τ'_{k+1} -well inside $\mathcal{L}_x(I^{m(k)})$, where $\tau'_{k+1} = 0.9 \frac{\tau_{k+1}^2}{1+2\tau_{k+1}}$. In conclusion, $\mathcal{L}_x(J)$ is τ' -well inside $\mathcal{L}_x(I)$, where

$$(5) \quad 1 + 2\tau' = \prod_{i=1}^{k+1} (1 + 2\tau'_i).$$

Let us prove that τ' is bounded away from zero. By Theorem 3.2, for each $2 \leq i \leq k$, we have $\tau_i \geq \rho$. So we are done if $k \geq 2$. If $k \leq 1$, then by (4), $(1 + 2\tau_i)^2 \geq 1 + 2\tau$ holds for $i = 1$ or 2 , thus τ' is bounded from below by a positive constant depending on τ .

Now assume τ is bounded from below by a constant $\tau_* > 0$ and let us prove (3). Let $\rho_* = \min(\tau_*, \rho)$, and let

$$\mathcal{I} = \{1 \leq i \leq k + 1 : (1 + 2\tau_i)^4 > 1 + 2\rho_*\}.$$

Then $\{2, \dots, k\} \subset \mathcal{I}$. So by (4),

$$(6) \quad \prod_{i \in \mathcal{I}} (1 + 2\tau_i) \geq \frac{1 + 2\tau}{\sqrt{1 + 2\rho_*}} \geq \sqrt{1 + 2\tau}.$$

For each $i \in \mathcal{I}$, τ'_i is bounded away from zero, so there exists a constant $\mu \in (0, 1)$ such that $\tau'_i \geq \mu\tau_i$. Thus by (5),

$$1 + 2\tau' \geq \prod_{i \in \mathcal{I}} (1 + 2\mu\tau_i) \geq \prod_{i \in \mathcal{I}} (1 + 2\tau_i)^\mu.$$

Together with (6), this implies

$$1 + 2\tau' \geq (1 + 2\tau)^{\mu/2}.$$

The inequality (3) follows. □

Recall that a child $J \ni c$ of a symmetric nice $I \ni c$ is a unimodal pull back of I by f^s for some $s \geq 1$. The integer s is called a *transition time* from J to I .

Lemma 3.4. *Let J be a child of I with transition time s . Then for each $x \in J$, the return time of x to J is not less than s .*

Proof. Let $\{J_i\}_{i=0}^s$ be the chain with $J_0 = J$ and $J_s = I$. Since $f^{s-1} : J_1 \rightarrow J_n$ is diffeomorphic, $c \notin J_i$ for each $1 \leq i \leq s - 1$. Therefore $J_i \cap J = \emptyset$ ($1 \leq i \leq s - 1$), since otherwise $J_i \supset J \ni c$, which is impossible. For each $x \in J$, $f^i(x) \in J_i$ ($1 \leq i \leq s - 1$). If $f^k(x) \in J$, then $k \geq s$. □

Lemma 3.5. *Let $I \ni c$ be a small nice interval and let J be a child of I . Assume that J is τ -well inside I . Then J is a τ' -nice interval, where $\tau' > 0$ depends only on τ . Moreover, when τ is sufficiently large, we have $\tau' \geq C\tau^\alpha$ for some constants $C > 0, \alpha > 0$.*

Proof. Let s be the transition time of J into I . Take an arbitrary $x \in D(J) \cap J$ and let r be the first return time of x into J . By Lemma 3.4, $r \geq s$. Let $U := f^s(\mathcal{L}_x(J))$. By Theorem 3.3, $\mathcal{L}_{f^s(x)}(J)$ is τ_1 -well inside I , where $\tau_1 > 0$ is a constant depending only on τ , and when $\tau > 1$, there exist constants $C_1 > 0$ and $\alpha_1 > 1$ such that $\tau_1 \geq C_1\tau^{\alpha_1}$. Since f^{s-1} maps a neighborhood of $f(J)$ diffeomorphically onto I , by the Koebe principle and non-flatness of the critical point, $\mathcal{L}_x(J)$ is τ' -well inside J , for some constant $\tau' > 0$ depending only on τ_1 . Moreover when τ is sufficiently large, $\tau_1 > 1$, and we can choose $\tau' = C_0\tau_1^{1/\ell}$, where C_0 is a constant. Thus the lemma holds with $C = C_0C_1^{1/\ell}$ and $\alpha = \alpha_1/\ell$. □

Lemma 3.6. *There exists a universal constant $\rho_0 > 0$ such that if $I \ni c$ is a small nice interval and $J \neq I^1$ is a child of I , then J is ρ_0 -well inside I .*

Proof. Let s be the return time of c to I and let $m \geq 1$ be minimal such that $f^s(c) \in I^{m-1} \setminus I^m$. Note that $J \subset I^m$ and $f^s(J) \subset I^{m-1} \setminus I^m$.

Let $\rho > 0$ be the constant appearing in Theorem 3.2. If I^m is ρ -well inside I^{m-1} , then J is ρ -well inside I , and we are done. So assume that I^m is not ρ -well inside I^{m-1} . Then $R_{I^{m-1}} : I^m \rightarrow I^{m-1}$ is a high return, and $f^{s-1}|f(I^m)$ has uniformly bounded distortion. Since $f^s(I^m)$ is definitely larger than $f^s(J)$, it follows that $|f(J)|/|f(I^m)|$ is bounded away from one, hence J is uniformly well inside $I^m \subset I$. □

3.2. Central cascade. By a *central cascade*, we mean a sequence of symmetric nice intervals

$$T \supset T^1 \supset \dots \supset T^m \quad (m \geq 1)$$

which contain c such that

- T^{i+1} is the central return domain of T^i , for each $0 \leq i < m$;
- the first return times of c to T, T^1, \dots, T^{m-1} are all the same.

So R_{T^i} are central for all $0 \leq i \leq m - 2$. A central cascade is called *maximal* if $R_T(c) \notin T^m$.

Proposition 3.7. *Let $T = T^0 \ni c$ be a small symmetric nice interval and let $T^0 \supset T^1 \supset T^2 \supset \dots \supset T^m$ be a maximal central cascade. Assume that T^1 is τ -well inside T^0 . Let $i \in \{1, 2, \dots, m\}$ and let $J_1 \supseteq J_2 \supseteq \dots$ be all the children of T^i . Then there exist constants $C > 0$ and $0 < \lambda < \lambda_0 < 1$, depending only on τ , such that*

1. for each $k = 1, 2, \dots$, we have $|J_k| \leq \lambda^{k-1}|T^i|$;
2. for each $k \geq 2$, J_k is $C\lambda_0^{-k}$ -nice.

To prove this proposition, let us first introduce some notation. For $y \in D(T^0)$, let $r(y)$ denote the first entry time of y into T^0 , and let $s = r(c)$, so $R_{T^0}|T^1 = f^s|T^1$. Let

$$E(T) = \bigcup_{i=0}^{m-1} \{x \in T^i \setminus T^{i+1} : R_T^i(x) \in D(T)\},$$

and for each $x \in E(T) \cap (T^i \setminus T^{i+1})$, let

$$t(x) = is + r(f^{is}(x)).$$

Moreover, let $F = F_T : E(T) \rightarrow T$ be defined as

$$F(x) = f^{t(x)}(x).$$

Clearly, $t(x)$ is constant on each component J of $E(T)$.

We shall also need the following notation:

- $Q = f^{-s}(T^m) \cap T^m$;
- V is the component of $f^{-s}(E(T))$ which contains c ;
- $X = f^{-s}(E(T)) \cap (T^m \setminus (Q \cup V))$.

Lemma 3.8. (i) *The map F maps each component J of $E(T)$ diffeomorphically onto T .*

(ii) *For each $x \in D(T^m) \setminus (Q \cup V)$, if k is the entry time of x to T^m , then f^k maps a neighborhood $W(x)$ of $\mathcal{L}_x(T^m)$ diffeomorphically onto T . Moreover, if, in addition, $x \in X$, then $W(x) \subset X$.*

Proof. We first prove the statement (i). If J is a component of $E(T)$ in $T^0 \setminus T^1$, then J is a non-central return domain and $F = R_T$, so F maps J diffeomorphically onto T . Now let J be a component of $E(T)$ in $T^i \setminus T^{i+1}$ for some $1 \leq i < m$. Since f^{is} maps a component of $T^i \setminus T^{i+1}$ diffeomorphically onto a component of $T^0 \setminus T^1$, $f^{is} : J \rightarrow J' := f^{is}(J)$ is a diffeomorphism and J' is a component of $E(T)$ in $T^0 \setminus T^1$. Since $t|J = t|J' + is$, $F|J = R_T|J' \circ f^{is}|J$ maps J diffeomorphically onto T .

Let us prove statement (ii). Let us distinguish a few cases.

Case 1. $x \in T \setminus T^m$. In this case, $f^k|_{\mathcal{L}_x(T^m)}$ can be written as an iterate of F , so the statement follows from (i). Note that $W(x) \subset E(T)$.

Case 2. $x \notin T$. Let $k' \leq k$ be the first entry time of x to T . Then $f^{k'} : \mathcal{L}_x(T) \rightarrow T$ is a diffeomorphism. So the statement holds if $k' = k$. If $k' < k$, then $f^{k'}(\mathcal{L}_x(T^m)) = \mathcal{L}_{f^{k'}(x)}(T^m)$ and we are reduced to Case 1.

Case 3. $x \in D(T^m) \cap X$. Then $k > s$ and $x' = f^s(x) \in D(T^m) \cap (T^{m-1} \setminus T^m)$. Let $W_0(x)$ and $W_0(x')$ denote the component of $X \cup E(T)$ which contains x and x' respectively. By the definition of X , $f^s : W_0(x) \rightarrow W_0(x')$ is a diffeomorphism. So we are reduced to Case 1 again. □

A nice interval I is called τ -non-central nice if all its return domains, except possibly the one containing c , are τ -well inside I . The following is an immediate consequence of Lemma 3.8.

Lemma 3.9. *Assume that T^1 is τ -well inside T^0 . Then for each $1 \leq i < m$, T^i is a τ' -non-central-nice interval, where τ' depends only on τ .*

Proof. Note that for each return domain U of T^i , $U \neq T^{i+1}$, the first return map $R_{T^i}|_U$ can be written in the form $F^n|_U$ for some $n \geq 1$. By Lemma 3.8, it follows that $f^k : U \rightarrow T^i$ extends to a diffeomorphism $f^k : \widehat{U} \rightarrow T^0$ and $\widehat{U} \subset T^i \setminus T^{i+1}$, where $k \geq 1$ is the first return time of J into T^i . Since $T^i \subset T^1$ is τ -well inside T , by the Koebe principle, U is well inside \widehat{U} , hence well inside T^i . □

Lemma 3.10. *For any $\tau > 0$, there exists $\tau' > 0$ such that if $T \ni c$ is a small τ -non-central-nice interval and $J_2 \subset J_1$ are children of T , then J_2 is τ' -well inside J_1 .*

Proof. Let $s_1 < s_2$ be the transition time of J_1, J_2 to T respectively. Let s be the maximal integer such that $s_1 \leq s < s_2$ and $x := f^s(c) \in T$. Then $s_2 - s$ is the return time of x into T and $\mathcal{L}_x(T)$ does not contain c . By assumption, $\mathcal{L}_x(T)$ is τ -well inside T , so by Theorem 3.3, the component of $f^{-(s-s_1)}(\mathcal{L}_x(T))$ containing $f^{s_1}(c)$ is τ' -well inside T , where $\tau' > 0$ is a constant. By Theorem 3.1 and non-flatness of the critical point, J_2 is well inside J_1 . □

These lemmas imply Proposition 3.7 immediately unless

$$(7) \quad (1 + 2\rho)T^m \supset T^{m-1}.$$

To deal with the case when (7) holds, we need the following three lemmas.

Assume (7). Then by Theorem 3.2, $R_{T^{m-1}} : T^m \rightarrow T^{m-1}$ is high, so Q consists of two intervals, each of which is mapped diffeomorphically onto T^m by f^s . Let Q_+, Q_- denote the components of Q such that $f^s|_{Q_+}$ is monotone increasing. Let b be the unique fixed point of $f^s|_{Q_-}$, let $\hat{b} = (f^s|_{Q_+})^{-1}(b)$ and let $B = (b, \hat{b})$.

Lemma 3.11. *There exist universal constants $K > 1$ and $\sigma > 0$ such that the following hold:*

- (i) *For any $x, y \in T^m$, $|Df^{s-1}(f(x))| \leq K|Df^{s-1}(f(y))|$;*
- (ii) *$|(f^s)'(x)| \leq K$ holds for all $x \in T^m$;*
- (iii) *for any measurable $A \subset T^m$, $\frac{|A|}{|T^m|} \leq K \left(\frac{|f^s(A)|}{|T^m|} \right)^{1/\ell}$;*

- (iv) f^s maps a neighborhood Z of Q_+ diffeomorphically onto its image and $\widehat{Z} := f^s(Z) \supset Z \cup T^m \cup (1 + 2\sigma)Q_+$;
- (v) $\sigma|T^m| \leq |B| \leq (1 - \sigma)|T^m|$.

Proof. By Theorem 3.2, f^{s-1} maps a neighborhood G_1 of $f(T^m)$ diffeomorphically onto $G := (1 + 2\rho)T^{m-1}$. By the real Koebe principle, there exists $K > 1$ such that (i) holds. For $x \in T^m$, we have

$$|(f^s)'(x)| = |f'(x)| |(f^{s-1})'(f(x))| \leq K \frac{|f^s(T^m)|}{|f(T^m)|} |f'(x)|.$$

Since $|f^s(T^m)| \leq |T^{m-1}| \leq (1 + 2\rho)|T^m|$, by the non-flatness, it follows that statement (ii) holds by redefining the constant K . Statement (iii) follows from (i) in a similar way. For (iv) and (v), assume for definiteness that Q_+ lies to the left of c . Let $G_0 = f^{-1}(G_1)$ and let $Z = (u, c)$ be the left component of $G_0 \setminus \{c\}$. Then f^s maps Z diffeomorphically onto its image and $\widehat{Z} := f^s(Z) \supset T^m$. Since $f^s(u)$ is the left endpoint of G , we have $f^s(Z) \supset (1 + 2\sigma)Q_+$, where $\sigma = \min(1, \rho)$. If $Z \not\subset f^s(Z)$, then f^s would map $Z \setminus T^m$ into itself and hence f^s would have an attracting fixed point, which is not possible. Thus $f^s(Z) \supset Z$. Statement (iv) is proved. Statement (iv) follows from (ii), since $f^s(Q_- \cap \overline{B}) \supset Q_- \setminus B$ and $f^s(Q_- \setminus B) \supset Q_- \cap B$. \square

Lemma 3.12. *Assume that $(1 + 2\rho)T^m \not\subset T^{m-1}$. Then there exists a universal constant $\theta \in (0, 1)$ such that if P is an interval such that $f^{js}(P) \subset Q$ for $j = 0, 1, \dots, N - 1$, then*

$$|P| \leq \theta^N |T^m|.$$

Proof. Let

$$\mathcal{P}_n = \{x \in T^m : f^{is}(x) \in Q \text{ for } 0 \leq i < n\}$$

and

$$\mathcal{P}_n^* = \{x \in \mathcal{P}_n : f^{ns}(x) \in T^m \setminus \{b, \hat{b}\}\}.$$

Note that each component of \mathcal{P}_n is the union of three intervals of \mathcal{P}_n^* , up to two points (corresponding to preimages of b and \hat{b}). As each component of \mathcal{P}_n , $n \geq 1$, is at most of length $|T^m|/2$, it suffices to show there exist universal constants $C_* > 0$ and $\theta_* \in (0, 1)$ such that for each component P_n^* of \mathcal{P}_n^* we have

$$(8) \quad |P_n^*| \leq C_* \theta_*^n |T^m|.$$

Let $Q_{+,1} = Q_+$ and $Q_{-,1} = Q_-$, and for each $j > 1$, let $Q_{+,j} = (f^s|_{Q_+})^{-1}(Q_{+,j-1})$ and $Q_{-,j} = (f^s|_{Q_-})^{-1}(Q_{-,j-1})$. Then $Q_{+,j}$ are symmetric to $Q_{-,j}$ with respect to c .

Claim 1. There exists a universal constant $\theta_1 \in (0, 1)$ such that

$$|Q_{+,j}| = |Q_{-,j}| \leq \theta_1^j |T^m|.$$

Indeed, by (iv) of Lemma 3.11, for each $j \geq 1$, f^{js} maps a neighborhood of $Q_{+,j}$ diffeomorphically onto \widehat{Z} . Since $f^{js}(Q_{+,j+1}) = Q_+$, and $f^{js}(Q_{+,j}) = T^m$, it follows by the Koebe principle that $|Q_{+,j+1}|/|Q_{+,j}|$ is uniformly bounded away from 1. The claim follows.

Let

$$\mathcal{B}_n = \{x \in T^m : f^{js}(x) \in Q \text{ for } 0 \leq j < n, f^{ns}(x) \in B\} \subset \mathcal{P}_n^*.$$

For each component B_n of \mathcal{B}_n , f^{ns} maps a neighborhood of B_n diffeomorphically onto T^m . By (v) of Lemma 3.11, B is uniformly well inside T^m . By the Koebe principle, there exists a universal constant $K_1 > 1$ such that

$$(9) \quad \sup_{x,y \in B_n} \frac{|(f^{ns})'(x)|}{|(f^{ns})'(y)|} \leq K_1.$$

Claim 2. There exists a universal constant $\theta_2 \in (0, 1)$ such that for each component B_n of \mathcal{B}_n , $n = 0, 1, \dots$, we have

$$(10) \quad |B_n| \leq \theta_2^n |T^m|.$$

To prove this claim, let $\mathcal{B} = \bigcup_{n=1}^\infty \mathcal{B}_n \subset D(B)$. For each $x \in \mathcal{B}$, the first entry time of x into B is of the form $k(x)s$, where $k(x) \geq 1$ is an integer. For $x \in \mathcal{B} \setminus B$, we have $f^{js}(x) \in Q_+$ for $1 \leq j < k(x)$, so $\mathcal{L}_x(B) \subset Q_{+,k(x)} \cup Q_{-,k(x)}$. Thus by Claim 1, we have

$$(11) \quad |\mathcal{L}_x(B)| \leq \theta_1^{k(x)} |T^m| \text{ holds for all } x \in \mathcal{B} \setminus B.$$

Let us now show that there exists a universal constant $\theta_3 \in (0, 1)$ such that

$$(12) \quad |\mathcal{L}_x(B)| \leq \theta_3^{k(x)} |B| \text{ holds for all } x \in \mathcal{B} \cap B.$$

Indeed, $\mathcal{L}_x(B)$ lies in a component of $B \setminus \{c\}$, so

$$(13) \quad |\mathcal{L}_x(B)| \leq |B|/2.$$

In particular, if $k(x) = 1$, then (12) holds with $\theta_3 = 1/2$. If $k(x) > 1$, then $f^s(x) \in \mathcal{B} \setminus B$ and $f^s(\mathcal{L}_x(B)) = \mathcal{L}_{f^s(x)}(B)$. So by (11) and part (iii) and (v) of Lemma 3.11, we have

$$\frac{|\mathcal{L}_x(B)|}{|B|} \leq \sigma_2^{-1} \frac{|\mathcal{L}_x(B)|}{|T^m|} \leq \frac{K_2}{\sigma_2} \theta_1^{(k(x)-1)/\ell}.$$

Together with (13), it follows that (12) holds for a suitable choice of θ_3 .

Now let us prove (10). Take a component B_n of \mathcal{B}_n , $n \geq 1$. Let $1 \leq n_1 < n_2 < \dots < n_k = n$ be all the positive integers such that $f^{n_i s}(B_n) \subset B$ and let B_{n_i} be the component of \mathcal{B}_{n_i} which contains B_n . Then $B_{n_1} \supset B_{n_2} \supset \dots \supset B_{n_k} = B_n$. For each $1 \leq i < k$, $Y_i := f^{n_i s}(B_{n_{i+1}})$ is a component of \mathcal{B} with entry time $(n_{i+1} - n_i)s$. By (9),

$$\frac{|B_{n_{i+1}}|}{|B_{n_i}|} \leq \frac{K_1 |Y_i|}{|B| + (K_1 - 1) |Y_i|}.$$

Thus by (12), there exists $\theta_4 \in (0, 1)$ such that $|B_{n_{i+1}}| \leq \theta_4^{n_{i+1} - n_i} |B_{n_i}|$. So $|B_n| \leq \theta_4^{n - n_1} |B_{n_1}|$. Let $\theta_2 = \max(\theta_1, \theta_3, \theta_4)$. By (11) and (12), $|B_{n_1}| \leq \theta_2^{n_1} |T^m|$. Thus (10) holds.

Now let us complete the proof. Let P_n^* be a component of \mathcal{P}_n^* . We may assume $P_n^* \notin \mathcal{B}_n$, for otherwise Claim 2 applies. Write $P_i^* = f^{(n-i)s}(P_n^*)$. Let n_0 be maximal in $\{0, 1, \dots, n\}$ such that $P_i^* \cap B = \emptyset$ for all $0 \leq i \leq n_0$. Since $f^s(Q \setminus B) \cap (Q_- \setminus B) = \emptyset$, we have $f^{is}(P_{n_0}^*) \subset Q_+$ for all $1 \leq i < n_0$. So $P_{n_0}^* \subset Q_{+,n_0}$ or $P_{n_0}^* \subset Q_{-,n_0}$. By Claim 1, we have $|P_{n_0}^*| \leq \theta_1^{n_0} |T^m|$. If $n_0 = n$, then we are done again. Assume $n_0 < n$. Then $P_{n_0+1}^* \subset B$. By parts (iii) and (v) of Lemma 3.11, $|P_{n_0+1}^*| \leq C'_1 \theta_1^{n_0+1} |B|$ holds for some universal constants $C'_1 > 0$

and $\theta'_1 \in (0, 1)$. Let B_{n-n_0-1} be the component of \mathcal{B}_{n-n_0-1} which contains P_n^* . By (9), we have

$$|P_n^*| \leq K_1 |B_{n-n_0-1}| \frac{|P_{n_0+1}^*|}{|B|}.$$

By Claim 2, the inequality (8) follows. □

Lemma 3.13. *Assume $(1 + 2\rho)T^m \not\subset T^{m-1}$. Let $y \in V$ and let t be the first return time of y to T^m . Assume that $f^{js}(f^t(y)) \in Q$ for all $j = 0, 1, \dots, n - 1$ and let H be the component of $f^{-ns-t}(T^m)$ which contains y . Then*

$$|H| \leq \theta_0^n |T^m|,$$

where $\theta_0 \in (0, 1)$ is a constant depending on τ .

Proof. Let $\delta \in (0, \rho)$ be such that $|T^{m-1}| = (1 + 2\delta)|T^m|$. Since $f^s(V) \subset T^{m-1} \setminus T^m$ we have $|f^s(V)| \leq \delta|T^m|$. By part (i) of Lemma 3.11, $|fV| \leq K\delta|f(T^m \setminus V)|$. By non-flatness, there exist universal constants $K_1 > 1$ and $\eta_1 \in (0, 1)$ such that

$$\frac{|V|}{|T^m|} \leq \min(\eta_1, K_1\delta^{1/\ell}) =: \eta.$$

Since $H \subset V$, we obtain

$$(14) \quad |H| \leq \eta|T^m|.$$

Take $\gamma \in (0, 1)$ such that $K_1^\gamma \eta_1^{1-\gamma} = 1$.

Case 1. $\delta < \theta^{n/2}$. Then

$$\eta \leq (K_1\delta^{1/\ell})^\gamma \eta_1^{1-\gamma} \leq \theta^{n\gamma/(2\ell)},$$

so we are done in this case.

Case 2. $\delta \geq \theta^{n/2}$. By Lemma 3.12, $|f^t(H)| \leq \theta^n |T^m|$. So $f^t(H)$ is $\theta^{-n/2}$ -well inside T^{m-1} . By Lemma 3.8 (ii), f^{t-1} maps an interval $W \ni f(y)$ diffeomorphically onto T^{m-1} . Let W_0 be the component of $f^{-1}(W)$ which contains y . Then $W_0 \subset T^m$. By the Koebe principle and non-flatness, we obtain that

$$|H| \leq C\theta^{n/2\ell} |W_0| \leq C\theta^{n/2\ell} |T^m|,$$

where $C = C(\theta)$ is a constant. Together with (14), this implies the statement. □

Proof of Proposition 3.7. The second statement follows from the first by Lemma 3.5. In the following we shall prove the first statement.

By Lemmas 3.9 and 3.10, the first statement holds in the case $1 \leq i < m$. In the following, we shall estimate the size of children of T^m .

If $(1 + 2\rho)T^m \subset T^{m-1}$, then by Lemma 3.5, T^m is ρ' -nice for some $\rho' > 0$, and so we are done again by Lemma 3.10. We assume from now on that $(1 + 2\rho)T^m \not\subset T^{m-1}$, so that Lemmas 3.11, 3.12 and 3.13 apply.

For each $i = 1, 2, \dots$, let S_i denote a transition time from J_i to T^m . By definition, f^{S_i-1} maps an interval \hat{J}_i which contains $f(J_i)$ diffeomorphically onto T^m . Let $i(1) = \inf\{i \geq 1 : f^{S_i}(c) \notin Q\}$, and define inductively

$$i(j + 1) = \inf\{i > i(j) : f^{S_i}(c) \notin Q\}.$$

For $i \in \{1, 2, \dots, i(1)\}$, applying Lemma 3.13 to $y = c$ and $n = i$, we obtain that $|J_i| \leq \theta_0^i |T^m|$.

It remains to show that for each $i(j) < i \leq i(j+1)$, $|J_i| \leq \theta_2^{i-i(j)} |J_{i(j)}|$ holds for some constant $\theta_2 = \theta_2(\tau) \in (0, 1)$. To this end, let $y := f^{S_{i(j)}(c)}$ and we distinguish two cases.

Case 1. $y \in X$. Then $k := S_{i(j+1)} - S_{i(j)}$ is the first return time of y into T^m . By Lemma 3.8, f^k maps an interval $W(y)$ with $y \in W(y) \subset X$ diffeomorphically onto $T \supset (1 + 2\tau)T^m$. By Lemma 3.12, $|f^{S_{i(j)+1}(J_i)}| \leq \theta^{i-i(j)-1} |T^m|$, so $f^{S_{i(j)+1}(J_i)}$ is $\tau\theta^{i(j)-i+1}$ -well inside T . By the Koebe principle, $f^{S_{i(j)}(J_i)}$ is $\tau'\theta^{i(j)-i}$ -well inside $W(y)$ for some constants $\tau' > 0$ and $\theta' \in (0, 1)$. Applying the Koebe principle again to the diffeomorphism $f^{S_{i(j)}-1} : \widehat{J}_{i(j)} \rightarrow T^m$ and using the non-flatness of the critical point, we obtain the desired estimate.

Case 2. $y \in V$. In this case, applying Lemma 3.13 to y and $n = i - i(j)$, we obtain that $|f^{S_{i(j)}(J_i)}| \leq \theta_0^{i-i(j)} |T^m|$. So $f^{S_{i(j)}(J_i)}$ is $C\tilde{\theta}_0^{i(j)-i}$ -well inside T^m for some constants $C > 0$ and $\theta_0 \in (0, 1)$. Applying the Koebe principle again to the diffeomorphism $f^{S_{i(j)}-1} : \widehat{J}_{i(j)} \rightarrow T^m$ and the non-flatness of the critical point, we obtain the desired estimate. \square

4. PULL BACK OF EMPTY SPACE

In this section, we will assume that f is non-renormalizable and that $\omega(c)$ is a wild attractor. We fix a suitable neighborhood Λ of $\omega(c)$. For each small nice interval $T \ni c$, let $\Lambda(T)$ denote the set of points in T which return to T before escaping Λ . Then we define a parameter $\xi(T)$ which measures the relative size of the complement of $\Lambda(T)$ (“empty space”) in T , and study the distortion of this parameter under pull back by f . The main results are Propositions 4.5 and 4.6.

Fix a small symmetric nice interval $I \ni c$ such that the union Λ of components of $D(I) \cup I$ which intersects $\omega(c)$ satisfies the property that $[f^2(c), f(c)] \setminus \Lambda$ has non-empty interior. For $T \subset I$, let

$$\Lambda(T) = \{x \in [0, 1] : \exists k \geq 1 \text{ such that } x, f(x), \dots, f^{k-1}(x) \in \Lambda \text{ and } f^k(x) \in T\}.$$

For an interval $Y \subset [0, 1]$ and a set $Y' \subset Y$, we will define a number $\lambda(Y'|Y)$ to measure how much the subset Y' occupies in Y . Let \mathcal{F}_Y be the set of diffeomorphisms of the form $f^s : J \rightarrow Y$. Define

$$\lambda(Y'|Y) = \sup_{\phi \in \mathcal{F}_Y} \frac{|\phi^{-1}(Y')|}{|\phi^{-1}(Y)|},$$

and

$$\xi(Y'|Y) = 1 - \lambda(Y'|Y).$$

For a nice interval $T \subset I$ with $T \ni c$, define

$$\xi(T) = \xi(\Lambda(T) \cap T|T).$$

Remark. For each small nice interval T , $\xi(T) > 0$. Indeed, f is topologically transitive on $[f(c), f^2(c)] \supset T$, so $T \setminus \Lambda(T)$ has non-empty interior. Moreover, since $\omega(c)$ is minimal, $\partial T \cap \omega(c) = \emptyset$, so there exists $\delta = \delta(T) > 0$ such that each diffeomorphism $f^s : J \rightarrow T$ extends to a diffeomorphism onto the δ -neighborhood of T . By the Koebe principle, there exists a constant $C = C(T) > 0$ such that

$$\frac{|(f^s|J)^{-1}(T \setminus \Lambda(T))|}{|J|} \geq C \frac{|T \setminus \Lambda(T)|}{|T|} > 0.$$

Lemma 4.1. *Suppose that f has a wild attractor. Let $T_n \ni c$ be a sequence of nice intervals such that $|T_n| \rightarrow 0$ as $n \rightarrow \infty$. Then $\xi(T_n) \rightarrow 0$.*

Proof. Assume by contradiction that there exists a sequence of nice intervals $T_n \ni c$ and a constant $\lambda > 0$ such that $|T_n| \rightarrow 0$ and $\xi(T_n) \geq \lambda$, $n = 1, 2, \dots$. Since f has a wild attractor, the non-escaping set

$$\mathcal{J} = \{x \in [0, 1] : f^k(x) \in \Lambda \text{ for all } k = 0, 1, \dots\}$$

has positive Lebesgue measure. Let $X = \{x \in \mathcal{J} : \omega(x) \ni c\}$. Then by Mäné’s theorem [19], $|X| = |\mathcal{J}| > 0$. Let n_0 be large such that $|X \setminus T_{n_0}| > 0$ and let $x \in X \setminus T_{n_0}$ be a Lebesgue density point of X . For each $n \geq n_0$, let s_n be the first entry time of x under f to T_n and let $J_n = \mathcal{L}_x(T_n)$. Then $f^{s_n} : J_n \rightarrow T_n$ is a diffeomorphism. Since $f^{s_n}(X) \subset \Lambda(T_n)$, we have

$$\frac{|J_n \cap X|}{|J_n|} \leq \frac{|J_n \cap f^{-s_n}(\Lambda(T_n))|}{|J_n|} \leq \lambda(T_n \cap \Lambda(T_n)|T_n) \leq 1 - \lambda.$$

Since f has no wandering interval [21], $|J_n| \rightarrow 0$. This contradicts the assumption that x is a Lebesgue density point of X . □

The following lemma is an improvement of [17, Lemma 4.11].

Lemma 4.2. *Let T be small interval. Let U_1, U_2, \dots and W_1, W_2, \dots, W_k be pairwise disjoint subintervals of T and $Y \subset (\bigcup_i U_i) \cup (\bigcup_{i=1}^k W_i)$. Assume that*

- for each i , $\lambda(Y \cap U_i|U_i) \leq \lambda$;
- for each $1 \leq j \leq k$, W_j is τ -well inside T .

Then there exists $\varepsilon = \varepsilon(k, \tau) \in (0, 1)$ such that

$$(15) \quad 1 - \lambda(Y|T) \geq (1 - \varepsilon)(1 - \lambda).$$

Moreover, for a fixed k , $\varepsilon(k, \tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$.

Proof. Let $\mu_k(\tau) = (1 + \tau^{-1})^{-k}$. We first prove that

$$(16) \quad \sum_{j=1}^k |W_j| \leq (1 - \mu_k(\tau))|T|.$$

Without loss of generality, we may assume that W_1, W_2, \dots, W_k lie from left to right in T . Let $\delta_j = |W_j|/|T|$ and let $\rho = 1 - \sum_{j=1}^k \delta_j$. Since the left component of $T \setminus W_j$ has length at least $\tau|W_j|$, we obtain

$$\rho \geq \tau\delta_1,$$

and for each $j = 2, 3, \dots, k$,

$$\rho + \sum_{j'=1}^{j-1} \delta_{j'} \geq \tau\delta_j.$$

By induction, it follows that for each $j = 1, 2, \dots, k$,

$$\delta_j \leq \frac{\rho}{\tau} (1 + \tau^{-1})^{j-1}.$$

Since $\rho + \delta_1 + \dots + \delta_k = 1$, this implies that

$$\rho \geq (1 + \tau^{-1})^{-k} = \mu_k(\tau);$$

hence

$$\delta_1 + \delta_2 + \dots + \delta_k \leq 1 - \mu_k(\tau).$$

The inequality (16) is proved.

Now let $\tau' = 0.9 \frac{\tau^2}{1+2\tau}$ and $\varepsilon(k, \tau) = 1 - \mu_k(\tau')$. Clearly, for a fixed k , $\varepsilon(k, \tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$. It remains to show that (15) holds with $\varepsilon = \varepsilon(k, \tau)$. To this end, take an arbitrary diffeomorphism $\phi : T' \rightarrow T$ from the class \mathcal{F}_T . Let U'_i, W'_j, Y' be the preimages of U_i, W_j, Y under ϕ respectively. By the Koebe principle, W'_j is τ' -well inside T' . Therefore as above, we obtain

$$\sum_{i=1}^k |W'_j| \leq (1 - \mu_k(\tau'))|T'|.$$

For each $i \geq 1$,

$$|Y' \cap U'_i| \leq \lambda(Y \cap U_i|U_i)|U'_i| \leq \lambda|U'_i|.$$

Putting $U' = \bigcup_i U'_i$ and $W' = \bigcup_j W'_j$, we have $Y' \subset U' \cup W'$. Thus

$$\begin{aligned} |T' \setminus Y'| &\geq \sum_i |U'_i \setminus Y'| + |T' \setminus (U' \cup W')| \\ &\geq (1 - \lambda)(|U'| + |T' \setminus (U' \cup W')|) \\ &= (1 - \lambda)|T' \setminus W'| \\ &\geq (1 - \lambda)\mu_k(\tau')|T'|. \end{aligned}$$

The inequality (15) follows. □

Lemma 4.3. *Let T be a small interval, let T' be a unimodal pull back of T by f^s , and let $Y' \subset T'$. Assume that $Y := f^s(Y')$ is covered by subintervals U_i of T , $i = 0, 1, 2, \dots$, such that*

- for each $i \geq 0$, U_i is τ -well inside T ;
- for each $i \geq 1$, $\lambda(Y|U_i) \leq \lambda$.

Then

$$(17) \quad 1 - \lambda(Y'|T') \geq (1 - \varepsilon(\tau))(1 - \lambda).$$

Moreover, $\varepsilon(\tau) = O(\tau^{-1/\ell})$ as $\tau \rightarrow \infty$.

Proof. For each $i \geq 1$, $f^{-s}(U_i) \cap T'$ has at most two components, and if $f^s(c) \in U_i$, then $f^{-s}(U_i) \cap T'$ is an interval. Let U'_j be the components of $f^{-s}(\bigcup_{i=0}^\infty U_i)$ such that for each $j \geq 4$, $f^s|U'_j$ is a diffeomorphism onto U_i for some $i \geq 1$; hence

$$\lambda(Y'|U'_j) \leq \lambda(Y|U_i) \leq \lambda.$$

By the Koebe principle, each U'_j is τ' -well inside T' , where τ' is a constant depending only on τ and $\tau' = O(\tau^{1/\ell})$ as $\tau \rightarrow \infty$. Thus by Lemma 4.2, the statement follows. □

Lemma 4.4. *Let $T \subset I$ be a nice interval that contains c such that T^1 is τ -well inside T and let $K \subset T \setminus T^1$ be a component of $\Lambda(T)$. Then there exists $\varepsilon = \varepsilon(\tau) > 0$, such that*

$$(18) \quad \lambda(K \cap \Lambda(T^1)|K) \leq \frac{\varepsilon}{\xi(T) + \varepsilon}.$$

Moreover, $\varepsilon(\tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$.

Proof. Let $U_0 = T \setminus \Lambda(T)$, $V_0 = (\Lambda(T) \cap T) \setminus T^1$ and $W_0 = T^1$. Moreover, for each $k \geq 1$, inductively define

$$\begin{aligned} U_k &= \{x \in V_{k-1} : R_T^k(x) \in U_0\}, \\ V_k &= \{x \in V_{k-1} : R_T^k(x) \in V_0\}, \\ W_k &= \{x \in V_{k-1} : R_T^k(x) \in W_0\}. \end{aligned}$$

Since T^1 is τ -well inside T , by the Koebe principle, there exists $\varepsilon = \varepsilon(\tau) > 0$ such that for each $\psi \in \mathcal{F}_T$, we have

$$|\psi^{-1}(T^1)| \leq \varepsilon |\psi^{-1}(T)|,$$

where $\varepsilon = \varepsilon(\tau) = (1 + 2\theta)^{-1}$ and $\theta = 0.9\tau^2/(1 + 2\tau)$. So $\varepsilon(\tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$.

For each component J of V_{k-1} , $R_T^k|_J$ is a diffeomorphism onto T , so for each $\phi \in \mathcal{F}_J$, we have $R_T^k|_J \circ \phi \in \mathcal{F}_T$. Therefore,

$$|\phi^{-1}(U_k \cap J)| \geq \xi(T) |\phi^{-1}(J)|,$$

and

$$|\phi^{-1}(W_k \cap J)| \leq \varepsilon |\phi^{-1}(J)|.$$

So

$$\frac{|\phi^{-1}(U_k \cap J)|}{|\phi^{-1}(W_k \cap J)|} \geq \frac{\xi(T)}{\varepsilon}.$$

By Mañé's theorem [19], $\bigcap_k V_k$ has measure zero. It follows that

$$\frac{|\phi^{-1}(T \cap W)|}{|\phi^{-1}(T)|} \leq \frac{\varepsilon}{\xi(T) + \varepsilon},$$

where

$$W := \bigcup_{k=0}^{\infty} W_k.$$

Thus,

$$\lambda(T \cap W|T) \leq \frac{\varepsilon}{\xi(T) + \varepsilon}.$$

For each component K of V_0 , since the first return map R_T maps $K \cap \Lambda(T^1)$ onto W , we have $\lambda(K \cap W|K) \leq \lambda(T \cap W|T)$. The lemma follows. \square

Proposition 4.5. *For any $\tau > 0$, there exists $\varepsilon = \varepsilon(\tau) \in (0, 1)$, such that for any τ -nice interval $T \subset I$ with $T \ni c$ and any child J of T ,*

$$\frac{\xi(J)}{\xi(T)} \geq \frac{1 - \varepsilon}{\xi(T) + \varepsilon}.$$

Moreover, $\varepsilon(\tau) = O(\tau^{-1/\ell})$ as $\tau \rightarrow \infty$.

Proof. Let s be a transition time of J to T . By Lemma 3.4 $Y := f^s(\Lambda(J) \cap J) \subset \Lambda(T^1) \cup T^1$. Let U_0, U_1, \dots be the components of $\Lambda(T) \cap T$ such that $U_0 \ni c$. Then for all $i \geq 0$, U_i is τ -well inside T . By Lemma 4.4, for each $i \geq 1$, $\xi(Y|U_i) \geq \xi(T)/(\xi(T) + \varepsilon_1)$, where $\varepsilon_1 = \varepsilon_1(\tau) = O(\tau^{-1})$ as $\tau \rightarrow \infty$. By Lemma 4.3, the statement follows. \square

The previous proposition says that the empty space of a unimodal pull back does not decrease too much. Now we will show that the central cascade does not influence the empty space too much as well.

Definition 4.1. Given a maximal central cascade $T \supset T^1 \supset \dots \supset T^m$, an *inheritor* of T is, by definition, a child J of $T^{m'}$ for some $0 \leq m' \leq m$ such that $J \subsetneq T^m$.

Proposition 4.6. *Let $T \supset T^1 \supset \dots \supset T^m$ be a maximal central cascade, where $T \ni c$ is a small symmetric τ -nice interval. Then there is a constant $C = C(\tau) > 0$ such that for each inheritor J of T , we have*

$$\xi(J) \geq C\xi(T).$$

Proof. Let us first prove the proposition under the following assumption:

(*) each component of $D(T) \cap (T \setminus T^1)$ is τ -well inside $T \setminus T^1$.

Let E_T, F_T, V, Q , and X be as defined in §3.2. Let V_0 be the component of $T^m \setminus \overline{Q}$ which contains c . Let $Q' = Q \cap D(X \cup V_0)$. Note that $X \cup V_0$ is a nice set, and for each component K of Q' , the first entry time of K into $X \cup V_0$ is of the form ns and f^{ns} maps a neighborhood of K in Q diffeomorphically onto T^m .

Claim 1. There exists a constant $\tau' > 0$ such that

- (1a) for each $0 \leq i < m$, each component of $E(T) \cap (T^i \setminus T^{i+1})$ is τ' -well inside T^i ;
- (1b) V is τ' -well inside V_0 ;
- (1c) each component of X is τ' -well inside T^m ;
- (1d) each component of Q' is τ' -well inside T^m .

Proof of Claim 1. (1a). By assumption, if K is a component of $E(T) \cap (T \setminus T^1)$, then K is τ -well inside $T \setminus T^1 \subset T$. For each $1 \leq i < m$, f^{is} maps each component of $T^i \setminus T^{i+1}$ diffeomorphically onto $T \setminus T^1$, and for each component K of $E(T) \cap (T^i \setminus T^{i+1})$, $K' = f^{is}(K)$ is a component of $E(T) \cap (T \setminus T^1)$. The statement follows by the Koebe principle.

(1b). Note that V_0 is a unimodal pull back of the component of $T^{m-1} \setminus T^m$ which contains $f^s(c)$. Since $f^s(V) \subset E(T)$, by (1a), $f^s(V)$ is well inside $T^{m-1} \setminus T^m$. Thus the statement follows by the Koebe principle and non-flatness of the critical point. (We need to redefine the constant τ' .)

(1c). It also follows from (1a) by the Koebe principle and non-flatness of the critical point.

(1d) follows from (1b) and (1c) and the observation on the components Q' by the Koebe principle. □

Claim 2. There exists a constant $C_0 > 0$ such that

- (2a) for each component K of $E(T) \cap (T \setminus T^1)$, we have

$$\xi(\Lambda(T^1) \cap K|K) \geq C_0\xi(T);$$

- (2b) for each $1 \leq i < m$ and each component K of $E(T) \cap (T^i \setminus T^{i+1})$, we have

$$\xi(\Lambda(T^i) \cap K|K) \geq C_0\xi(T);$$

- (2c) for each component K of X , we have

$$\xi(\Lambda(V) \cap K|K) \geq C_0\xi(T);$$

- (2d) for the interval V_0 , we have

$$\xi((\Lambda(V) \cup V) \cap V_0|V_0) \geq C_0\xi(T);$$

(2e) for each component K of Q' , we have

$$\xi(\Lambda(V) \cap K|K) \geq C_0 \xi(T).$$

Proof of Claim 2. (2a) follows from Lemma 4.4.

(2b) follows from (2a) and the observation that $K' = f^{is}(K)$ is a component of $E(T) \cap (T \setminus T^1)$ and $f^{is}(K \cap \Lambda(T^i)) \subset K' \cap \Lambda(T^i) \subset K' \cap \Lambda(T^1)$.

(2c) follows similarly.

(2d). The set $(\Lambda(V) \cup V) \cap V_0$ is covered by V and the components of $X \cap V_0$. The statement follows from (1b) and (2c) by Lemma 4.2.

(2e) follows from (2c) and (2d) by the observation on Q' . □

Now suppose $J \subsetneq T^m$ is a child of $T^{m'}$ for some $0 \leq m' \leq m$. Let t be a transition time from J to $T^{m'}$. Note that $J \subset V$, so

$$Y := f^t(\Lambda(J) \cap J) \subset (\Lambda(J) \cup J) \cap T^{m'} \subset (\Lambda(V) \cup V) \cap T^{m'}.$$

Let U_0, U_1, U_2, \dots be the components of $E(T) \cap T^{m'}$, $X \setminus \overline{Q \cup V_0}$, V_0 and Q' . These sets cover Y . Each of these intervals are uniformly well inside $T^{m'}$ and $\xi(Y|U_i) \geq C_0 \xi(T)$. By Lemma 4.3, it follows that $\xi(J) \geq C \xi(T)$, where $C > 0$ is a constant.

We have completed the proof of the proposition under the assumption (*). For the general case, by Proposition 4.5, we may assume $m \geq m' \geq 2$, so $T^1 \supset T^2 \supset \dots \supset T^m$ is also a maximal central cascade. We claim that each component K of $D(T^1) \cap (T^1 \setminus T^2)$ is τ_1 -well inside $T^1 \setminus T^2$ for some $\tau_1 > 0$. Indeed, the first return time of K into T^1 is greater than s . Since T^1 is well inside T , $f^s(K)$ is well inside a component K' of $D(T)$, by Theorem 3.3. Thus $f^s(K)$ is well inside $T \setminus T^1$, which implies that K is well inside $T^1 \setminus T^2$. Applying the above argument to the maximal central cascade $T^1 \supset T^2 \supset \dots \supset T^m$ proves the statement. □

5. PROOF OF THE REDUCED MAIN THEOREM

We continue to assume that f is non-renormalizable and has a Cantor attractor $\omega(c)$. Fix a neighborhood Λ of $\omega(c)$ as in the previous section. Let $Y \ni c$ be a symmetric nice interval which is necessarily contained in $(f(c), f^2(c))$. Let $\mathcal{N}_Y = \{n \geq 1 : f^n(c) \in Y\}$, and for each $n \in \mathcal{N}_Y$, let Y_{-n} denote the pull back of Y by f^n which contains c . To obtain an upper bound for the number of children of Y_{-n} , we first apply Propositions 3.7, 4.5 and 4.6 to obtain lower bounds on $\xi(Y_{-n})$, together with niceness control on young children. Then we apply Proposition 4.5 again to obtain the desired upper bound: if Y_{-n} has too many children, then some grandchild of Y_{-n} has a large “empty space” which is ruled out by Lemma 4.1.

Since f is non-renormalizable, we have

$$(19) \quad \lim_{\substack{n \in \mathcal{N}_Y \\ n \rightarrow \infty}} |Y_{-n}| = 0.$$

For $n \in \mathcal{N}_Y$ with $n \geq 1$, we shall define a positive integer $M_n(Y)$, called the *essential order* of Y_{-n} . Let $\{Y_i\}_{i=-n}^0$ be the chain with $Y_0 = Y$. Let $0 = i_0 > i_1 > \dots > i_p = -n$ be all the integers such that $Y_{i_j} \ni c$. So Y_{i_j} is a child of $Y_{i_{j-1}}$ with the transition time $s_j = i_{j-1} - i_j$, for each $1 \leq j \leq p$. By Lemma 3.4, $s_1 \leq s_2 \leq \dots \leq s_p$. Define

$$M_n(Y) = \#\{s_j : 1 \leq j \leq p\}.$$

Let

$$\mathcal{N}_M(Y) = \{n \in \mathcal{N}_Y : M_n(Y) \leq M\}.$$

Proposition 5.1. *There exists a universal constant $C_0 > 0$ such that for any symmetric nice interval $Y \ni c$ and $n \in \mathcal{N}_Y$, we have $M_n(Y) \leq C_0 \log n$. Moreover, there exists a universal constant κ such that the transition time from the second child of Y_{-n} to Y_{-n} is greater than κn .*

Proof. Let $n' > n$ be such that $n' \in \mathcal{N}_Y$ and such that $Y_{-n'}$ is the second child of Y_{-n} . Let $\{Y_i\}_{i=-n'}^0, \{i_j\}_{j=0}^{p'}$ and $\{s_j\}_{j=1}^{p'}$ be defined as above, and let $M' = M_{n'}(Y)$. Note that $M := M_n(Y) = M' - 1$. Define $m(0) = 0, m(1) = 1$, and define inductively integers $m(1) < m(2) < \dots < m(M') \leq p'$ by

$$m(j) = \inf\{m > m(j - 1) : s_m > s_{m(j-1)}\}, \quad j = 2, 3, \dots$$

For $1 \leq j \leq M$, let r_j denote the minimal return time of points in $Y_{i_{m(j)-1}} \cap \omega(c)$ to $Y_{i_{m(j)-1}}$. Let us show that for each $2 \leq j \leq M'$,

$$(20) \quad s_{m(j)} \geq s_{m(j-1)}(m(j) - m(j - 1)) + r_{j-1}.$$

Indeed, in the case $m(j) = m(j - 1) + 1$, this is clear as $s_{m(j)} - s_{m(j-1)}$ is a return time of $f^{s_{m(j-1)}}(c)$ to $Y_{i_{m(j-1)-1}}$. When $m(j) - m(j - 1) > 1$, observe that $f^{s_{m(j-1)}}(c) \in Y_{i_{m(j)-2}} \setminus Y_{i_{m(j)-1}}$, and hence

$$f^{ks_{m(j-1)}}(c) \in Y_{i_{m(j)-k-1}} \setminus Y_{i_{m(j)-k}}$$

for each $1 \leq k \leq m(j) - m(j - 1)$. Since $Y_{i_{m(j)}}$ is a child of $Y_{i_{m(j)-1}}$, we have $s_{m(j)} > (m(j) - m(j - 1))s_{m(j-1)}$ and that $s_{m(j)} - (m(j) - m(j - 1))s_{m(j-1)}$ is a return time of $f^{(m(j)-m(j-1))s_{m(j-1)}}(c)$ to $Y_{i_{m(j)-1}}$. The inequality (20) follows.

By Lemma 3.4, for $2 \leq j \leq M'$, $r_j \geq s_{m(j-1)}$. Thus for $3 \leq j \leq M'$,

$$(21) \quad s_{m(j)} \geq s_{m(j-1)}(m(j) - m(j - 1)) + s_{m(j-2)} > s_{m(j-1)} + s_{m(j-2)}.$$

Thus $s_{m(j)}$ grows at least as fast as the Fibonacci sequence. Since $n \geq s_{m(M)}$, it follows that $M \leq C_0 \log n$ for some universal constant $C_0 > 0$.

To prove the last statement, note that $s_{m(M')}$ is the transition time from $Y_{-n'}$ to Y_{-n} . Write $S_j = s_{m(j)}(m(j + 1) - m(j))$. Then (21) implies that S_j is strictly increasing in j and $S_{j+1} \geq S_j + S_{j-2}$. As $n = S_1 + S_2 + \dots + S_M$, it follows that S_M/n is bounded away from zero. By (21) again, $n' - n = s_{m(M+1)}$ is at least comparable to n . \square

Remark 5.2. In the proof, we only used the fact that f is non-renormalizable. Thus this proposition holds whenever f is non-renormalizable.

This proposition allows us to obtain a lower bound on $q(\mathcal{Y}, n)$ for a nice cover \mathcal{Y} , which implies a lower bound for the topological complexity function for maps with special combinatorics. See Theorem 5.5 and Corollary 5.6 at the end of this section.

Lemma 5.3. *Given a symmetric nice interval Y , for each $M \geq 1$, we have $\#\mathcal{N}_M(Y) < \infty$.*

Proof. It suffices to prove that $\mathcal{N}_1(Y)$ is finite, since for each $n \in \mathcal{N}_M(Y)$ with $M \geq 2$, there exists $n' \in \mathcal{N}_{M-1}(Y)$ such that $n \in \mathcal{N}_1(Y_{-n'})$.

By Proposition 2.1, the set

$$\mathcal{N}_1^o(Y) = \{s : Y_{-s} \text{ is a child of } Y\}$$

is finite. For each $n \in \mathcal{N}_1(Y) \setminus \mathcal{N}_1^o(Y)$, there exists $s \in \mathcal{N}_1^o(Y)$ such that $n - s \in \mathcal{N}_1(Y)$ and such that Y_{-n} is a child of Y_{-n+s} . Since $|Y_{-n+s}| \geq |f^s(c) - c|$ is

bounded away from zero, by (19), $n - s$ is bounded from above. It follows that $\mathcal{N}_1(Y) \setminus \mathcal{N}_1^o(Y)$, hence $\mathcal{N}_1(Y)$, is finite. \square

In particular, for each symmetric nice interval $Y \subset I$,

$$\widehat{\xi}_M(Y) = \inf\{\xi(Y_{-n}) : n \in \mathcal{N}_M(Y)\} > 0.$$

Lemma 5.4. *Let $Y \ni c$ be a small symmetric nice interval, let $n \in \mathcal{N}_Y$ be such that $M_n(Y) \geq 3$, and let Y_{-n} be the critical pull back of Y by f^n . Let $K_1 \supset K_2 \supset \dots$ be all the children of Y_{-n} . Then for each $k \geq 2$, K_k is $C\lambda_0^{-k}$ -well inside Y_{-n} and*

$$(22) \quad \xi(K_k) \geq n^{-\beta} \widehat{\xi}_2(Y),$$

where $C > 0$, $\lambda_0 \in (0, 1)$ and $\beta > 0$ are universal constants.

Proof. Let $\{Y_i\}_{i=-n}^0$ be the chain with $Y_0 = Y$ and define $m(0), m(1), \dots$ as above. Define $T_j = Y_{i_j}$ for $0 \leq j \leq p$. Note that $T_{m(2)}$ is of the form $Y_{-n'}$ for some $n' \in \mathcal{N}_2(Y)$, so

$$\xi(T_{m(2)}) \geq \widehat{\xi}_2(Y).$$

Let us first prove that there exists a universal constant $\tau > 0$ such that for each $2 \leq j \leq M$, $T_{m(j)}$ is a τ -nice interval. If either $T_{m(j)}$ is well inside $T_{m(j)-1}$ or $T_{m(j)-1}$ is well inside $T_{m(j)-2}$, then by Lemma 3.5 we are done. Thus, by Lemma 3.6, we may assume that $T_{m(j)}$ is the first child of $T_{m(j)-1}$ and that $T_{m(j)-1}$ is the first child of $T_{m(j)-2}$, i.e. $s_{m(j)}$ is the first return time of c into $T_{m(j)-1}$ and $s_{m(j)-1}$ is the first return time of c into $T_{m(j)-2}$. Since $s_{m(j)} > s_{m(j)-1}$, it follows that $R_{T_{m(j)-2}} : T_{m(j)-1} \rightarrow T_{m(j)-2}$ is non-central. By Theorem 3.2, it follows that $T_{m(j)}$ is uniformly well inside $T_{m(j)-1}$, and thus we are done.

Now let us show that there exists $\kappa \in (0, 1)$ such that $\xi(T_{m(j)}) \geq \kappa \xi(T_{m(j-1)})$ for each $3 \leq j \leq M$. Indeed, by Proposition 4.5, such an estimate holds if $m(j) = m(j-1) + 1$. So assume $m(j) > m(j-1) + 1$. By Lemma 3.4, it follows that $T_{m(j-1)} \supset T_{m(j-1)+1} \supset \dots \supset T_{m(j)-1}$ is a central cascade, i.e., $s_k = s_{m(j-1)}$ is the first return time to T_{k-1} for each $m(j-1) < k \leq m(j) - 1$. Since $s_{m(j)} > s_{m(j-1)}$, $T_{m(j)}$ is an inheritor of $T_{m(j-1)}$. So by Proposition 4.6, the statement follows.

Similarly for each $k \geq 2$, K_k is either a child or an inheritor of $T_{m(M)}$, so $\xi(K_k) \geq \kappa \xi(T_{m(M)})$. Thus

$$\xi(K_k) \geq \kappa^{M-1} \xi(T_{m(2)}) \geq \kappa^{M-1} \widehat{\xi}_2(Y).$$

By Proposition 5.1, the statement follows.

If $Y_{-n} = T_{m(M)}$, then by Lemma 3.10, the children of Y_{-n} are well nested, so the niceness of K_k follows from Lemma 3.5. If $Y_{-n} \subsetneq T_{m(M)}$, then for each $k \geq 2$, the same conclusion follows from Proposition 3.7. \square

Proof of the Reduced Main Theorem. By (19) and Lemma 5.3, we may assume that Y is small so that Lemma 5.4 applies.

Let $n \in \mathcal{N}_Y$ be so large that $M_n(Y) \geq 3$ and let Y_0 be the critical pull back of Y under f^n . Assume the number N of children of Y_0 is at least 2 and let K_N denote the N -th child of Y_0 . Let L be the first child of K_N . Then by Lemma 5.4, $\xi(K_N) \geq n^{-\beta} \widehat{\xi}_2(Y)$ and K_N is a $C\lambda_0^{-N}$ -nice interval, where $C > 0$ and $\lambda_0 \in (0, 1)$ are universal constants. So by Proposition 4.5, we have

$$\xi(L) \geq \frac{1 - C_0 \lambda_0^{N/\ell}}{\xi(K_N) + C_0 \lambda_0^{N/\ell}} \xi(K_N) \geq \frac{1 - C_0 \lambda_0^{N/\ell}}{\widehat{\xi}_2(Y) + C_0 n^\beta \lambda_0^{N/\ell}} \widehat{\xi}_2(Y),$$

where $C_0 > 0$ is a constant. On the other hand, by Lemma 4.1, when n is large enough, we have $\xi(L) \leq \widehat{\xi}_2(Y)/2$. Since $\widehat{\xi}_2(Y) \leq 1$, it follows that $N = O(\log n)$. \square

We end this section with the following theorem.

Theorem 5.5. *Let $f \in \mathcal{A}_*$ be a non-renormalizable unimodal map with a non-periodic recurrent critical point c and such that $\omega(c)$ is minimal. Let Y be a symmetric nice interval, let \mathcal{Y} denote the collection of components of $D(Y) \cup Y$ which intersects $\omega(c)$ and let $q(\mathcal{Y}, n)$ be defined as in (1). Then*

$$q(\mathcal{Y}, n) \geq \kappa_0 n$$

holds for all n large enough, where $\kappa_0 > 0$ is a universal constant.

Proof. Let N denote the maximal entry time of a point in $Y \cap \omega(c)$ into Y , and for each $n \geq 0$, let T_n denote the connected component of $f^{-n}(D(Y) \cup Y)$ which contains c . Clearly, $T_0 \supset T_1 \supset \dots$.

Fix a large positive integer n , and let m_0 be a minimal positive integer such that $n - m_0 < \kappa m_0$, where $\kappa > 0$ is as in Proposition 5.1. Then $n - (m_0 - 1) \geq \kappa(m_0 - 1)$, and hence

$$n - m_0 \geq \kappa m_0 - (1 + \kappa) \geq \frac{\kappa n}{1 + \kappa} - 1 - \kappa.$$

Let m be the minimal integer such that $m \geq m_0$ and $f^m(c) \in Y$. Then $m < m_0 + N$, so

$$n - m > \kappa_0 n,$$

provided that n is large enough, where $0 < \kappa_0 < \kappa/(1 + \kappa)$. By Proposition 5.1, the second child of T_m has transition time greater than $\kappa m > n - m$.

Note that for any symmetric nice interval $I \supset I'$, if J is a child of I with transition time r , then I' has a child J' with transition time at least r such that $J' \subset J$. Indeed, if $s \geq 0$ is the minimal integer such that $f^{s+r}(c) \in I'$, then I' has a child with transition time $s + r$. Thus for each $m \leq i < n$, there exists r_i such that

- T_{m+r_m} is the second child of T_m ;
- T_{i+r_i} is a child of T_i , $m \leq i < n$;
- $r_m \leq r_{m+1} \leq \dots \leq r_{n-1}$.

Therefore

$$n - 1 + r_{n-1} > n - 2 + r_{n-2} > \dots > m + r_m > n.$$

For each $m \leq i < n$, let J_i denote the pull back of T_i by f^{n-i} which contains $f^{i+r_i-n}(c)$. Then each J_i is a component of $f^{-n}(Y \cup D(Y))$ intersecting $\omega(c)$. Note that f^{n-i} maps J_i diffeomorphically onto $T_i \ni c$. It follows that for $m \leq i < i' < n$, $J_i \cap J_{i'} = \emptyset$, for otherwise, $J_i = J_{i'}$ is mapped onto $T_{i'}$ diffeomorphically by $f^{n-i'}$ and mapped onto T_i diffeomorphically by f^{n-i} , which is absurd. Therefore, $q_n(\mathcal{Y}) \geq n - m \geq \kappa_0 n$. \square

Let us say that a map $f \in \mathcal{A}_*$ has *special combinatorics* if $\omega(c) \ni c$ is minimal and there exists a symmetric nice interval Y such that for each $n = 0, 1, \dots$, $Y_n \setminus Y_{n+1}$ has exactly one component intersecting $\omega(c)$, where $Y_0 = Y$ and Y_{n+1} is the return domain of Y_n which contains c . Such a map is necessarily non-renormalizable.

Corollary 5.6. *Suppose that $f \in \mathcal{A}_*$ has special combinatorics. Then for any small open cover \mathcal{U} of the Cantor set $\omega(c)$, the topological complexity function satisfies*

$$p(\mathcal{U}, n + 1) \geq \kappa_0 n \text{ for all } n \text{ large enough.}$$

Proof. Let Y be a symmetric nice interval as in the definition above. We first show that if J is a pull back of Y which intersects $\omega(c)$ and $J \not\supseteq c$, then $\widehat{J} \cap \omega(c) = \emptyset$, where \widehat{J} is the interval lying on a different side of c as J and with $f(\widehat{J}) = f(J)$. Indeed, let $n \geq 0$ be maximal such that $J \subset Y_n$. So $J \not\subset Y_{n+1}$. As both Y_{n+1} and J are pull backs of Y_0 and $J \not\supseteq c$, we have $J \cap Y_{n+1} = \emptyset$. Since only one component of $Y_n \setminus Y_{n+1}$ intersects $\omega(c)$, it follows that $\widehat{J} \cap \omega(c) = \emptyset$.

Let \mathcal{Y} denote the nice cover associated with Y . Let us show that $q(\mathcal{Y}, n) = p(\mathcal{Y}, n + 1)$ for each $n \geq 0$. To this end, it suffices to show that each element of $\bigvee_{j=0}^n f^{-j}\mathcal{Y}$ has at most one component intersecting $\omega(c)$. Arguing by contradiction, assume that there exists $n \geq 0$ such that some element K of $\bigvee_{j=0}^n f^{-j}\mathcal{Y}$ has at least two components intersecting $\omega(c)$. If n is minimal with the last property, then K has exactly two components intersecting $\omega(c)$ which are symmetric around c , which is impossible by what we proved in the previous paragraph.

By Theorem 5.5, it follows that $p(\mathcal{Y}, n + 1) \geq \kappa_0 n$ for all n large enough. For each open cover \mathcal{U} which is a refinement of \mathcal{Y} , $p(\mathcal{U}, n + 1) \geq p(\mathcal{Y}, n + 1) \geq \kappa_0 n$. \square

Remark 5.7. The well-studied Fibonacci unimodal maps have special combinatorics. See for example [18, Section 6].

6. APPENDIX: A WILD ADDING MACHINE

Theorem 6.1. *There exists a unimodal map $f \in \mathcal{A}_*$ which has a wild attractor $\omega(c)$ such that $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to an adding machine and hence equicontinuous.*

Following [10], we define an adding machine as follows. Let $\alpha = (p_1, p_2, \dots)$ be a sequence of integers where each $p_i \geq 2$. Let Δ_α denote the set of all sequences (x_1, x_2, \dots) , where $x_i \in 0, 1, \dots, p_i - 1$ for each i . We use the product topology on Δ_α . For each α , an adding machine map $f_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$ is defined as:

$$f_\alpha(x_1, x_2, \dots) = \begin{cases} \left(\overbrace{(0, 0, \dots, 0)}^{l-1}, x_l + 1, x_{l+1}, \dots \right) & \text{if } x_i = p_i - 1 \text{ for } i < l \\ & \text{and } x_l < p_l - 1, \\ (0, 0, 0, \dots) & \text{if } x_i = p_i - 1 \text{ for all } i. \end{cases}$$

It is clear that $f_\alpha : \Delta_\alpha \rightarrow \Delta_\alpha$ is minimal and equicontinuous. It is well known that for an infinitely renormalizable map $f \in \mathcal{A}_*$, $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to an adding machine; see [21, Proposition III.4.5]. (The definition of an adding machine there is slightly different, but equivalent to the one above.)

In [2] the authors constructed uncountably many non-renormalizable unimodal maps such that $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to a (generalized) adding machine, hence equicontinuous. It seems that their construction only gives non-persistent recurrent maps. To obtain an equicontinuous wild attractor, we shall modify their construction to obtain a unimodal map with a wild attractor.

We start with the following lemma which gives a sufficient condition for a non-renormalizable unimodal map for which $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to an adding machine.

Lemma 6.2. *Let $f \in \mathcal{A}_*$ be a unimodal map with a recurrent critical point c . Assume that for each $n = 1, 2, \dots$, there exists a nice interval $T_n \ni c$ together with three distinct return domains T'_n, Q_n, \widehat{Q}_n such that*

- (i) $|T_n| \rightarrow 0$;
- (ii) $T'_n \ni c$ and $f(Q_n) = f(\widehat{Q}_n)$;
- (iii) for each $x \in T'_n \cap \omega(c)$, $R_{T'_n}(x) \in Q_n \cup \widehat{Q}_n$;
- (iv) for each $x \in (Q_n \cup \widehat{Q}_n) \cap \omega(c)$, $R_{T'_n}(x) \in T'_n$.

Then $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to an adding machine.

Proof. By passing to a subsequence if necessary, we may assume $T_n \supset T_{n+1}$ for each $n \geq 1$. For each interval n , we shall construct a cover \mathcal{U}_n of $\omega(c)$, such that

- \mathcal{U}_n consists of finitely many pairwise disjoint close subsets of $\omega(c)$ which are cyclically permuted by f ;
- \mathcal{U}_{n+1} is a refinement of \mathcal{U}_n ;
- the maximum diameter of elements of \mathcal{U}_n converges to 0 as $n \rightarrow \infty$.

It is well known that the existence of such covers \mathcal{U}_n implies that $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to some adding machine map. See for example [2, Theorem 1.1] and the references therein.

To this end, let k_n and l_n denote the return time of T'_n and Q_n into T_n respectively. Define

$$U_j^n = \begin{cases} f^j(T'_n \cap \omega(c)) & \text{if } 0 \leq j < k_n; \\ f^{j-k_n}((Q \cup \widehat{Q}) \cap \omega(c)) & \text{if } k_n \leq j \leq k_n + l_n, \end{cases}$$

and let $\mathcal{U}_n = \{U_j^n : 0 \leq j < k_n + l_n\}$. Then the assumptions imply that $f(U_j^n) \subset U_{j+1}^n$ for each $0 \leq j < k_n + l_n$ and $U_{k_n+l_n}^n \subset U_0^n$. Moreover, condition (iv) implies that $\omega(c) \cap \partial T_n = \emptyset$, and hence $\omega(c) \cap \partial P = \emptyset$ for each entry domain P of T_n . Thus each U_j^n is a closed subset of $\omega(c)$, and $\bigcup_{j=0}^{k_n+l_n-1} U_j^n = \omega(c)$. Let us show that $\{U_j^n\}_{j=0}^{k_n+l_n-1}$ are pairwise disjoint. If $0 \leq j_1 < j_2 < k_n$ or $k_n \leq j_1 < j_2 < k_n + l_n$, then for any $a_1 \in U_{j_1}$ and $a_2 \in U_{j_2}$, a_1 and a_2 have different return times to T_n ; thus $U_{j_1}^n \cap U_{j_2}^n = \emptyset$. If $0 \leq j_1 < k_n \leq j_2 < k_n + l_n$, then for $a_1 \in U_{j_1}$ and $a_2 \in U_{j_2}$, $R_{T'_n}(a_1) \in Q_n \cap \widehat{Q}_n$ is different from $R_{T'_n}(a_2) \in T'_n$; thus we also have $U_{j_1}^n \cap U_{j_2}^n = \emptyset$.

Since $T_{n+1} \subset T_n$, each component of $D(T_{n+1}) \cup T_{n+1}$ is contained in a component of $D(T_n) \cup T_n$. It follows that \mathcal{U}_{n+1} is a refinement of \mathcal{U}_n . Since f has no wandering interval, the supremum of length of components of $D(T_n) \cup T_n$ tends to 0; thus the maximal diameter of elements of \mathcal{U}_{T_n} tends to 0. □

We shall now describe the combinatorial property of our example in terms of the first return map to symmetric nice intervals. For a unimodal map f with $f(0) = f(1) = 0$ and $f(c) > c$, there is an orientation-reversing fixed point q . For $x \in [0, 1] \setminus \{c\}$, let \hat{x} denote the preimage of $f(x)$ other than x , and let $\hat{c} = c$. Let $I_0 = (\hat{q}, q)$, and whenever I_k is defined and c returns to I_k , define I_{k+1} to be the return domain of I_k containing c . (So let $I_0 \supset I_1 \supset \dots$ be the principal nest starting from I_0 .) Let R_{I_k} denote the first return map to I_k . If $R_{I_k}(c)$ returns to I_k , let J_{k+1} denote the return domain of I_k containing $R_{I_k}(c)$, and let $\widehat{J}_{k+1} = \{\hat{x} : x \in J_{k+1}\}$. These objects depend of course on f , and when we want to emphasize the map f , we write $I_k^f, J_k^f, R_{I_k}^f$, etc.

Proposition 6.3. *There is a unimodal map $f : [0, 1] \rightarrow [0, 1]$ with the following properties:*

- (1) $f(c) > c$, $f^2(c) < \hat{q}$ and $f^3(c), f^5(c) \in I_0$;
- (2) I_{k+1} and J_{k+1} are defined and disjoint for all $k = 0, 1, \dots$;
- (3) $R_{I_k}(I_{k+1}) \supset I_{k+1} \ni c$;
- (4) $R_{I_k}|_{I_{k+1}} = R_{I_{k-1}}^2|_{I_{k+1}}$ for each $k \geq 1$;
- (5) when k is odd, $R_{I_{k-1}}(J_{k+1}) \subset \hat{J}_k$ and $R_{I_k}|_{J_{k+1}} = R_{I_{k-1}}^2|_{J_{k+1}}$;
- (6) when $k \geq 2$ is even, $R_{I_k}|_{J_{k+1}} = R_{I_{k-1}}|_{J_{k+1}}$.

Moreover, for each $\ell > 1$, there is $a \in (0, 1)$ such that the unimodal

$$(23) \quad f(x) = a(1 - |2x - 1|^\ell)$$

has the above properties.

Let $r_0 = 3, t_0 = 2$, and for each $k \geq 0$, define inductively

$$(24) \quad r_{k+1} = r_k + t_k, \text{ for all } k = 0, 1, \dots,$$

and

$$(25) \quad t_{k+1} = \begin{cases} r_k & \text{if } k \text{ is odd,} \\ r_{k+1} & \text{if } k \text{ is even.} \end{cases}$$

In fact, r_k and t_k are the return time of c and $R_{I_k}(c)$ to I_k for a map satisfying properties (1)-(6).

Proof. It suffices to construct a continuous unimodal map with the properties (1)-(6). In fact, each $\ell > 1, x \mapsto a(1 - |2x - 1|^\ell), 0 \leq a \leq 1$, is a full family, and thus the existence of a map of the form (23) follows. See [21, Section II.4].

To construct such a continuous unimodal map, we argue as in [23, Section 2.1]. We shall first construct inductively a sequence of unimodal maps $f_n : [0, 1] \rightarrow [0, 1], n = 0, 1, \dots$, with the same turning pint c , such that the intervals $I_0^{f_n}, I_1^{f_n}, \dots, I_{n+1}^{f_n}, J_1^{f_n}, J_2^{f_n}, \dots, J_{n+1}^{f_n}, \hat{J}_1^{f_n}, \hat{J}_2^{f_n}, \dots, \hat{J}_{n+1}^{f_n}$ are well defined and such that properties (1)-(6) hold for $f = f_n$ and $k \leq n$. Moreover, our construction satisfies

- (a) $f_{n+1} = f_n$ for $x \in [0, 1] \setminus I_{n+1}$ for all $n \geq 0$;
- (b) $I_k^{f_n} = I_k^{f_{n-1}}$ and $J_k^{f_n} = J_k^{f_{n-1}}$ for all $n \geq 1$ and $k \leq n$;
- (c) $2|I_{n+2}^{f_{n+1}}| \leq |I_{n+1}^{f_n}|$;
- (d) $2|f_{n+1}(I_{n+2}^{f_{n+1}})| \leq |f_n(I_{n+1}^{f_n})|$.

For the starting step, we take f_0 to be an arbitrary unimodal map for which the critical point c satisfies $f_0(c) > f_0^4(c) > c = f_0^5(c) > f_0^3(c) > f_0^2(c)$. It is straightforward to check that for this map f_0 , the return time of c to $I_0^{f_0} = (\hat{q}^{f_0}, q^{f_0})$ is equal to 3 and the return time of $f_0^3(c)$ to $I_0^{f_0}$ is 2, so $I_1^{f_0}$ and $J_1^{f_0}$ are well defined and disjoint. Moreover, the map f_0^3 is monotone increasing on the right component of $I_0^{f_0} \setminus \{c\}$, and thus $R_{I_0^{f_0}}^{f_0}(I_1^{f_0}) \supset I_1^{f_0} \ni c$. Properties (4)-(6) are null in this case.

For the induction step, assuming that f_n is defined such that properties (1)-(6) hold for $f = f_n$ and $k \leq n$, we shall modify the map f_n on the interval $I_{n+1}^{f_n}$. To be definite, let us assume that n is even. Note that r_n is the return time of c into $I_n^{f_n}$ and t_n is the return time of $f_n^{r_n}(c)$ into $I_n^{f_n}$. Let $U_{n+1} = (f_n^{r_n}|_{I_{n+1}^{f_n}})^{-1}(J_{n+1}^{f_n})$ and let $V_{n+1} = (f_n^{r_n}|_{I_{n+1}^{f_n}})^{-1}(\hat{J}_{n+1}^{f_n})$. Then $U_{n+1} \ni c$ is an interval, V_{n+1} has two components and $U_{n+1} \cap V_{n+1} = \emptyset$. Moreover, each component $V_{n+1,i}, i = 1, 2$, of V_{n+1} is mapped homeomorphically onto $I_n^{f_n}$ by $f_n^{r_n+t_n} = f_n^{r_{n+1}}$, so it contains an

interval $W_{n+1,i}$ such that $f_n^{r_{n+1}}$ maps $W_{n+1,i}$ homeomorphically onto $I_{n+1}^{f_n}$. Let K_{n+1} be the component of $f_n^{-(r_{n+1}-1)}(I_n^{f_n})$ which contains $f_n(c)$. Then $f_n^{r_{n+1}-1}$ maps K_{n+1} homeomorphically onto $I_n^{f_n}$ and $f_n(U_{n+1}) \subset K_{n+1}$, $f_n(\partial U_{n+1}) \subset \partial K_{n+1}$. Let K'_{n+1} denote the subinterval of K_{n+1} which is mapped onto $I_{n+1}^{f_n}$. Let $\varphi_n : K_{n+1} \rightarrow K_{n+1}$ be an orientation-preserving homeomorphism such that $f_n^{r_{n+1}-1}(\varphi_n(f_n(c)))$ is contained in a component W_{n+1,i_1} for some $i_1 \in \{1, 2\}$ and such that $f_n^{r_{n+1}-1}(\varphi_n(f_n(U_{n+1})))$ contains W_{n+1,i_2} for $i_2 \in \{1, 2\} \setminus \{i_1\}$. Note that we can choose φ_n such that $\varphi_n^{-1}(K'_{n+1})$ is deep inside K_{n+1} . Define $f_{n+1} = f_n$ outside U_{n+1} and $f_{n+1} = \varphi_n \circ f_n$ on U_{n+1} . Then f_{n+1} is the map which we looked for. Indeed, $J_{n+2}^{f_{n+1}} = W_{n+1,i_1}$ and $I_{n+2}^{f_{n+1}} = (f_n|_{U_{n+1}})^{-1}(\varphi_n^{-1}(K'_{n+1}))$ can be made much smaller than $I_{n+1}^{f_n}$.

So these unimodal maps f_n have been constructed. Properties (a) and (d) imply that $\{f_n\}$ is equicontinuous, while (a) and (c) imply that $f_n(x)$ is eventually constant for each $x \in [0, 1] \setminus \{c\}$. Thus f_n converges to a continuous unimodal map f as $n \rightarrow \infty$. By continuity, $f^{r_n}(c) = \lim_{m \rightarrow \infty} f_m^{r_n}(c)$ enters the closure of $I_n := I_n^{f_n}$, so c is a recurrent critical point, which then implies $f^i(c)$ is disjoint from the boundary of I_n for any $i, n \geq 0$. It is then easily verified that conditions (1)-(6) hold for f by continuity. \square

Now let us fix a unimodal map (23) with properties (1)-(6). We first show that

Proposition 6.4. *The map $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to an adding machine.*

Proof. Let $V_k = I_k \cup J_k \cup \widehat{J}_k$ when k is odd, and $V_k = I_k \cup J_k$ when k is even. By induction, it is easy to see that for each $0 \leq l \leq k$,

- $f^j(I_{k+1}) \cap I_l \subset V_{l+1}$ holds for $0 \leq j < r_k$;
- $f^j(J_{k+1}) \cap I_l = f^j(\widehat{J}_{k+1}) \cap I_l \subset V_{l+1}$ holds for $0 \leq j < t_k$.

It follows that

$$\omega(c) \cap I_k \subset \overline{V_{k+1}}.$$

Applying Lemma 6.2 to $T_n = I_{2n}$, $T'_n = I_{2n+1}$, $Q_n = J_{2n+1}$ and $\widehat{Q}_n = \widehat{J}_{2n+1}$, we conclude that $f : \omega(c) \rightarrow \omega(c)$ is topologically conjugate to an adding machine. \square

To show that when $\ell > 1$ is large enough, the map has a wild attractor, we shall apply [4, Theorem 6.1]. To this end, we shall recall a combinatorial language, called a *kneading map*, which was used there. Assume $f \in \mathcal{A}_*$. The *closest precritical points* z_k and *cutting times* S_k are defined as follows: $S_0 := 1$, $z_0 := f^{-1}(c) \cap (0, c)$. Inductively,

$$S_{k+1} := \min\{n > S_k : f^{-n}(c) \cap (z_k, c) \neq \emptyset\},$$

and

$$z_{k+1} := f^{-S_{k+1}}(c) \cap (z_k, c).$$

For each $k = 1, 2, \dots$, $f^{S_{k+1}}$ is monotone on the interval $[z_k, c]$. So $f^{S_{k+1}-S_k}$ maps $[c, c_{S_k}]$ monotonically onto an interval containing c . Thus $S_{k+1} - S_k$ is also a cutting time. This implies that there is an integer $Q(k+1)$ such that

$$S_{k+1} = S_k + S_{Q(k+1)}.$$

Define also $Q(0) = 0$. The function $k \mapsto Q(k)$, $k = 0, 1, \dots$, is called the *kneading map* of f .

Lemma 6.5. *The cutting times of f not smaller than r_1 are the following:*

$$r_1, r_1 + r_0, r_2, r_3, r_2 + r_3, r_4, r_5, \dots$$

Proof. For each $k = 1, 2, \dots$, f^{r_k} maps each component of $I_{k+1} \setminus \{c\}$ monotonically onto an interval containing c ; thus r_k is a cutting time. If $k \geq 2$ is even, then $f^{t_k} = f^{r_{k-1}}$ is monotone on $[c_{r_k}, c] \subset I_k$; hence there is no cutting time between r_k and $r_k + t_k = r_{k+1}$. Now assume that k is odd. We need to show that $r_k + r_{k-1}$ is the only cutting time between r_k and r_{k+1} . To this end, let s be the cutting time next to r_k . So s is the smallest integer with $s > r_k$ and such that f^{s-r_k} maps $[c_{r_k}, c]$ monotonically onto an interval containing c in the interior. Since $c_{r_k} \in J_{k+1}$, and $f^{r_{k-1}}$ is monotone on $[c_{r_k}, c]$, thus $s \geq r_k + r_{k-1}$. Note that there is a component K_{k+1} of $f^{-r_{k-1}}(I_k)$ between I_{k+1} and J_{k+1} , and $f^{r_{k-1}}$ maps K_{k+1} monotonically onto I_k . Thus $s = r_k + r_{k-1}$. Finally since $c_{r_k+r_{k-1}} \in I_k$, $f^{t_{k-1}}$ is monotone on $[c_{r_k+r_{k-1}}, c]$, there is no cutting time between $r_k + r_{k-1}$ and $r_k + r_{k-1} + t_{k-1} = r_{k+1}$. \square

Lemma 6.6. *There exist positive integers k_1 and N such that when $k \geq k_1$ we have*

$$Q(k + 1) \geq Q^2(k) + 1,$$

and

$$k - Q(k) \leq N.$$

Proof. The previous lemma implies that there exists $k_0 \geq 1$ such that for each $m = 0, 1, \dots$,

$$S_{k_0+3m+1} = r_{2m+1}, S_{k_0+3m+2} = r_{2m+1} + r_{2m}, S_{k_0+3m+3} = r_{2m+2}.$$

Using (24) and (25), we easily find

$$\begin{aligned} S_{k_0+3m+2} - S_{k_0+3m+1} &= r_{2m} = S_{k_0+3m}, \\ S_{k_0+3m+3} - S_{k_0+3m+2} &= t_{2m+1} - r_{2m} = r_{2m-1} = S_{k_0+3m-2}, \\ S_{k_0+3m+4} - S_{k_0+3m+3} &= r_{2m+3} - r_{2m+2} = r_{2m+1} = S_{k_0+3m+1}, \end{aligned}$$

which implies that when k is large enough,

$$Q(k) = \begin{cases} k - 5 & \text{if } k - k_0 \equiv 0 \pmod 3, \\ k - 3 & \text{if } k - k_0 \equiv 1 \pmod 3, \\ k - 2 & \text{if } k - k_0 \equiv 2 \pmod 3, \end{cases}$$

and hence

$$Q^2(k) = \begin{cases} k - 8 & \text{if } k - k_0 \equiv 0 \pmod 3, \\ k - 6 & \text{if } k - k_0 \equiv 1 \pmod 3, \\ k - 7 & \text{if } k - k_0 \equiv 2 \pmod 3. \end{cases}$$

Thus the lemma holds with $N = 5$ and k_1 large enough. \square

Corollary 6.7. *When $\ell > 1$ is sufficiently large, f has a wild attractor.*

Proof. This follows directly from Lemma 6.6 by [4, Theorem 6.1]. \square

Proof of Theorem 6.1. Let f be a unimodal map of the form (23) which satisfies properties (1)-(6) as in Proposition 6.3. By (6.4), $f : \omega(c) \rightarrow \omega(c)$ is equicontinuous. By Corollary 6.7, $\omega(c)$ is a wild attractor provided that ℓ is sufficiently large. \square

ACKNOWLEDGMENT

The first named author was supported by NSFC grant 11271344, and the second named author was supported by MOE AcRF Tier 1 grant R146-000-128-133. The authors would like to thank the anonymous referee for valuable comments which led to a revision of this paper.

REFERENCES

- [1] F. Blanchard, B. Host, and A. Maass, *Topological complexity*, Ergodic Theory Dynam. Systems **20** (2000), no. 3, 641–662, DOI 10.1017/S0143385700000341. MR1764920 (2002b:37019)
- [2] Louis Block, James Keesling, and Michal Misiurewicz, *Strange adding machines*, Ergodic Theory Dynam. Systems **26** (2006), no. 3, 673–682, DOI 10.1017/S0143385705000635. MR2237463 (2007c:37037)
- [3] A. M. Blokh and M. Yu. Lyubich, *Measurable dynamics of S -unimodal maps of the interval*, Ann. Sci. École Norm. Sup. (4) **24** (1991), no. 5, 545–573. MR1132757 (93f:58132)
- [4] Henk Bruin, *Topological conditions for the existence of absorbing Cantor sets*, Trans. Amer. Math. Soc. **350** (1998), no. 6, 2229–2263, DOI 10.1090/S0002-9947-98-02109-6. MR1458316 (99f:58064)
- [5] H. Bruin, G. Keller, T. Nowicki, and S. van Strien, *Wild Cantor attractors exist*, Ann. of Math. (2) **143** (1996), no. 1, 97–130, DOI 10.2307/2118654. MR1370759 (96m:58145)
- [6] Henk Bruin, Gerhard Keller, and Matthias St. Pierre, *Adding machines and wild attractors*, Ergodic Theory Dynam. Systems **17** (1997), no. 6, 1267–1287, DOI 10.1017/S0143385797086392. MR1488317 (98m:58083)
- [7] H. Bruin, J. Rivera-Letelier, W. Shen, and S. van Strien, *Large derivatives, backward contraction and invariant densities for interval maps*, Invent. Math. **172** (2008), no. 3, 509–533, DOI 10.1007/s00222-007-0108-4. MR2393079 (2009d:37065)
- [8] Henk Bruin, Weixiao Shen, and Sebastian van Strien, *Existence of unique SRB-measures is typical for real unicritical polynomial families* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 3, 381–414, DOI 10.1016/j.ansens.2006.02.001. MR2265674 (2007j:37062)
- [9] H. Bruin and O. Volkova, *The complexity of Fibonacci-like kneading sequences*, Theoret. Comput. Sci. **337** (2005), no. 1-3, 379–389, DOI 10.1016/j.tcs.2005.02.001. MR2141232 (2006a:37007)
- [10] Jorge Buescu and Ian Stewart, *Liapunov stability and adding machines*, Ergodic Theory Dynam. Systems **15** (1995), no. 2, 271–290, DOI 10.1017/S0143385700008373. MR1332404 (96i:58083)
- [11] Sébastien Ferenczi and Kyewon Koh Park, *Entropy dimensions and a class of constructive examples*, Discrete Contin. Dyn. Syst. **17** (2007), no. 1, 133–141. MR2257422 (2007i:37014)
- [12] T. N. T. Goodman, *Topological sequence entropy*, Proc. London Math. Soc. (3) **29** (1974), 331–350. MR0356009 (50 #8482)
- [13] Dou Dou, Wen Huang, and Kyewon Koh Park, *Entropy dimension of topological dynamical systems*, Trans. Amer. Math. Soc. **363** (2011), no. 2, 659–680, DOI 10.1090/S0002-9947-2010-04906-2. MR2728582 (2012a:37030)
- [14] Jacek Graczyk, Duncan Sands, and Grzegorz Świątek, *Metric attractors for smooth unimodal maps*, Ann. of Math. (2) **159** (2004), no. 2, 725–740, DOI 10.4007/annals.2004.159.725. MR2081438 (2005e:37080)
- [15] W. Huang, S. M. Li, S. Shao, and X. D. Ye, *Null systems and sequence entropy pairs*, Ergodic Theory Dynam. Systems **23** (2003), no. 5, 1505–1523, DOI 10.1017/S0143385702001724. MR2018610 (2004i:37012)
- [16] O. S. Kozlovski, *Getting rid of the negative Schwarzian derivative condition*, Ann. of Math. (2) **152** (2000), no. 3, 743–762, DOI 10.2307/2661353. MR1815700 (2002e:37050)
- [17] Simin Li and Weixiao Shen, *Hausdorff dimension of Cantor attractors in one-dimensional dynamics*, Invent. Math. **171** (2008), no. 2, 345–387, DOI 10.1007/s00222-007-0083-9. MR2367023 (2009h:37077)
- [18] Mikhail Lyubich and John Milnor, *The Fibonacci unimodal map*, J. Amer. Math. Soc. **6** (1993), no. 2, 425–457, DOI 10.2307/2152804. MR1182670 (93h:58080)

- [19] Ricardo Mañé, *Hyperbolicity, sinks and measure in one-dimensional dynamics*, Comm. Math. Phys. **100** (1985), no. 4, 495–524. MR806250 (87f:58131)
- [20] Marco Martens, *Distortion results and invariant Cantor sets of unimodal maps*, Ergodic Theory Dynam. Systems **14** (1994), no. 2, 331–349, DOI 10.1017/S0143385700007902. MR1279474 (96c:58108)
- [21] Wellington de Melo and Sebastian van Strien, *One-dimensional dynamics*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 25, Springer-Verlag, Berlin, 1993. MR1239171 (95a:58035)
- [22] John Milnor, *On the concept of attractor*, Comm. Math. Phys. **99** (1985), no. 2, 177–195. MR790735 (87i:58109a)
- [23] T. Nowicki, S. van Strien, *Polynomial maps with a Julia set of positive measure*, arXiv:math/9402215.
- [24] Sebastian van Strien and Edson Vargas, *Real bounds, ergodicity and negative Schwarzian for multimodal maps*, J. Amer. Math. Soc. **17** (2004), no. 4, 749–782 (electronic), DOI 10.1090/S0894-0347-04-00463-1. MR2083467 (2005i:37043)
- [25] Peter Walters, *An introduction to ergodic theory*, Graduate Texts in Mathematics, vol. 79, Springer-Verlag, New York-Berlin, 1982. MR648108 (84e:28017)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, AND
WU WEN-TSUN KEY LABORATORY OF MATHEMATICS, USTC, HEFEI, 230026, PEOPLE'S REPUBLIC
OF CHINA

E-mail address: `lsm@ustc.edu.cn`

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, BLOCK S17, 10 LOWER
KENT RIDGE ROAD, SINGAPORE 119076

E-mail address: `matsw@nus.edu.sg`