

WEAKLY PROPER GROUP ACTIONS, MANSFIELD'S IMPRIMITIVITY AND TWISTED LANDSTAD DUALITY

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ABSTRACT. Using the theory of weakly proper actions of locally compact groups recently developed by the authors, we give a unified proof of both reduced and maximal versions of Mansfield's Imprimitivity Theorem and obtain a general version of Landstad's Duality Theorem for twisted group coactions. As one application, we obtain the stabilization trick for arbitrary twisted coactions, showing that every twisted coaction is Morita equivalent to an inflated coaction.

1. INTRODUCTION

The main goal of this paper is to show that the theory of weakly proper actions of locally compact groups developed by the authors in [1, 2] can be used to give unified proofs and/or generalizations of some of the central results about (twisted) coactions of groups. More specifically, we want to explore Mansfield's Imprimitivity Theorem (and its generalizations) as well as Landstad Duality for twisted coactions of groups from the point of view of the theory of weakly proper actions and their generalized fixed-point algebras.

In [22] Mansfield proved his main result, today called Mansfield's Imprimitivity Theorem, which says that for a (reduced) coaction $\delta: B \rightarrow \mathcal{M}(B \otimes C_r^*(G))$ of a locally compact group G on a C^* -algebra B and an amenable normal closed subgroup $N \subseteq G$, the crossed product $B \rtimes_{\delta|} \widehat{G/N}$ by the restricted coaction $\delta|: B \rightarrow \mathcal{M}(B \otimes C_r^*(G/N))$ of G/N is Morita equivalent to $B \rtimes_{\delta} \widehat{G} \rtimes_{\widehat{\delta}} N$, the crossed product by the dual N -action $\widehat{\delta}$. The bimodule implementing this equivalence is obtained as a certain completion of a special dense $*$ -subalgebra $\mathcal{D} \subseteq B \rtimes_{\delta} \widehat{G}$, often called the *Mansfield subalgebra*. Over time, several authors – see [12, 17, 18] – generalized Mansfield's theorem in different directions by allowing non-amenable and even non-normal closed subgroups of G in combination with different classes of coactions including *full* normal or maximal coactions of G (the word “full” means that we consider coactions of the full group C^* -algebra $C^*(G)$). We should emphasize that the theory of “full normal” coactions is equivalent to the theory of coactions by the reduced group algebra $C_r^*(G)$ (see [25]).

The version of Mansfield's theorem for normal coactions can be obtained from the theory of Rieffel proper actions ([27, 28]) by proving that the dual action of N

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on $B \rtimes_{\delta} \widehat{G}$ is proper (in Rieffel's sense) with respect to Mansfield's subalgebra $\mathcal{D} \subseteq B \rtimes_{\delta} \widehat{G}$. Indeed, this fact has been first observed in Mansfield's original paper (see [22, §7]) and it has been used to obtain generalizations of Mansfield's Imprimitivity Theorem to non-normal and/or non-amenable subgroups in [12, 14, 17]. On the other hand, the maximal version of Mansfield's theorem (obtained in [18]) is proved in an indirect way by analysing relations between several imprimitivity bimodules (such as Green's imprimitivity bimodule and Katayama's bimodule).

One of our goals in this paper is to show that both the maximal and normal versions of Mansfield's Imprimitivity Theorem can be obtained by considering full or reduced generalized fixed-point algebras for appropriate weakly proper actions. While the reduced generalized fixed-point algebras have been introduced by Rieffel in the 1980's ([27]), the theory of full fixed-point algebras has been introduced only recently in the quite general situation of weakly proper G -algebras by the authors in [1]. Recall that a G -action α on a C^* -algebra A is called weakly proper if there is a proper G -space X and a G -equivariant non-degenerate $*$ -homomorphism $C_0(X) \rightarrow \mathcal{M}(A)$. We then call A a weakly proper $X \rtimes G$ -algebra (or just a weak $X \rtimes G$ -algebra). For such algebras we constructed in [1] a Hilbert module $\mathcal{F}_{\mu}(A)$ over the μ -crossed product $A \rtimes_{\alpha, \mu} G$ for any given crossed-product norm $\|\cdot\|_{\mu}$ on $C_c(G, A)$ which lies between the reduced crossed-product norm $\|\cdot\|_r$ and the maximal crossed-product norm $\|\cdot\|_u$. The algebra of compact operators $A_{\mu}^G = \mathcal{K}(\mathcal{F}_{\mu}(A))$ is a completion of the *generalized fixed-point algebra with compact supports*:

$$(1.1) \quad A_c^G = C_c(G \backslash X) \cdot \{m \in \mathcal{M}(A)^G : m \cdot C_c(X), C_c(X) \cdot m \subseteq A_c\} \cdot C_c(G \backslash X),$$

where $A_c = C_c(X) \cdot A \cdot C_c(X)$ and $\mathcal{M}(A)^G$ denotes the algebra of G -fixed points in the multiplier algebra $\mathcal{M}(A)$. If the action of G on X is free and proper, $\mathcal{F}_{\mu}(A)$ implements a Morita equivalence $A_{\mu}^G \sim A \rtimes_{\alpha, \mu} G$.

Given a G -coaction (B, δ) , the crossed product $B \rtimes_{\delta} \widehat{G}$ may be viewed as a weak $G \rtimes G$ -algebra in a canonical way by taking the dual G -action $\widehat{\delta}$ and the canonical homomorphism $j_B : C_0(G) \rightarrow \mathcal{M}(B \rtimes_{\delta} \widehat{G})$, where $X = G$ is endowed with the right translation action of G . In particular, if H is a closed subgroup of G , we may restrict the G -action to H and view $B \rtimes_{\delta} \widehat{G}$ as a weak $G \rtimes H$ -algebra. Therefore, by the general theory of weakly proper actions explained above, we get a Hilbert bimodule $\mathcal{F}_{\mu}^H(B \rtimes_{\delta} \widehat{G})$ implementing a Morita equivalence $(B \rtimes_{\delta} \widehat{G})_{\mu}^H \sim B \rtimes_{\delta} \widehat{G} \rtimes_{\widehat{\delta}, \mu}^H H$ for any crossed-product norm $\|\cdot\|_{\mu}$ on $C_c(H, B \rtimes_{\delta} \widehat{G})$. The only remaining point to get Mansfield's theorem is to suitably identify the fixed-point algebra $(B \rtimes_{\delta} \widehat{G})_{\mu}^H$ with a sort of "crossed product" $B \rtimes_{\delta|_{\mu}} \widehat{G/H}$ by the homogeneous space G/H whenever this is defined. In fact, we prove that if N is a normal closed subgroup of G and $\mu = u$ or $\mu = r$ denotes either the maximal or reduced crossed-product norm (for both groups G and N), then $(B \rtimes_{\delta} \widehat{G})_{\mu}^N$ is indeed isomorphic to the crossed product $B_{\mu} \rtimes_{\delta_{\mu}|} \widehat{G/N}$ by the restricted coaction, where (B_{μ}, δ_{μ}) denotes either the maximalization (for $\mu = u$) or the normalization (for $\mu = r$) of (B, δ) . For non-normal subgroups $H \subseteq G$, it follows almost by definition that $(B \rtimes_{\delta} \widehat{G})_r^H$ identifies with the crossed product $B_r \rtimes_{\delta, r} \widehat{G/H}$ as defined in [6]. This has been observed before in [12, Theorem 3.1] and [13, Proposition 5.2]. Our results indicate that it would be useful to *define* the *full crossed product* $B \rtimes_{\delta, u} \widehat{G/H}$ of B by the

restriction of a G -coaction δ to the homogeneous space G/H as the maximal fixed-point algebra $(B \rtimes_{\delta} \widehat{G})_u^H$. Our Morita equivalence $(B \rtimes_{\delta} \widehat{G})_u^H \sim (B \rtimes_{\delta} \widehat{G}) \rtimes_{\widehat{\delta}, u}^{\widehat{}} H$ then automatically provides a full version of Mansfield's Imprimitivity Theorem for crossed products by G/H .

We should stress that our results are completely independent from Mansfield's original idea of constructing certain dense subalgebras \mathcal{D}_H and \mathcal{D} of $B \rtimes_{\delta, r} \widehat{G/H}$ and $B \rtimes_{\delta} \widehat{G}$, respectively. Nevertheless, we show that our results are compatible with Mansfield's constructions by showing that the algebra \mathcal{D}_H of Mansfield lies inductive limit dense in the fixed-point algebra $(B \rtimes_{\delta} \widehat{G})_c^H$, and hence also dense in any fixed-point algebra $(B \rtimes_{\delta} \widehat{G})_{\mu}^H$. In particular, this implies that one also obtains the full fixed-point algebra $(B \rtimes_{\delta} \widehat{G})_u^H$ as a certain completion of \mathcal{D}_H .

Mansfield's theorem motivated Phillips and Raeburn to introduce twisted coactions of groups in [23]. If $N \subseteq G$ is a closed normal subgroup of G , a twist over G/N for a (full) coaction (B, δ) of G is a unitary corepresentation $\omega \in \mathcal{M}(B \otimes C^*(G/N))$ of G/N such that the restriction $\delta|$ of δ to G/N is implemented by conjugation with ω and such that δ coacts trivially on the first leg of ω . For each such twisted coaction, one can form the twisted crossed product $B \rtimes_{\delta, \omega} \widehat{G}$ which is the quotient of the untwisted crossed product $B \rtimes_{\delta} \widehat{G}$ by a certain (twisting) ideal. Another main result of this paper will be a Landstad Duality Theorem for twisted coactions of groups which will, in particular, provide us with the notion of *maximalizations* of twisted coactions: we prove that for a weak $G \rtimes N$ -algebra A , there is a twisted coaction $(\delta_{\mu}^N, \omega_{\mu}^N)$ on A_{μ}^N – for suitable crossed-product norms μ on $C_c(N, A)$ – and a natural isomorphism $A_{\mu}^N \rtimes_{\delta_{\mu}, \omega_{\mu}} \widehat{G} \cong A$ of weak $G \rtimes N$ -algebras. Conversely, if we start with any given twisted coaction (δ, ω) , the twisted crossed product $A := B \rtimes_{\delta, \omega} \widehat{G}$ carries a canonical structure as a weak $G \rtimes N$ -algebra, and for suitable crossed-product norms $\|\cdot\|_{\mu}$ on $C_c(N, B \rtimes_{\delta, \omega} \widehat{G})$ we obtain a twisted $(G, G/N)$ -coaction $(\delta_{\mu}^N, \omega_{\mu}^N)$ on $B_{\mu}^N := (B \rtimes_{\delta, \omega} \widehat{G})_{\mu}^N$ such that $\mathcal{F}_{\mu}^N(B \rtimes_{\delta, \omega} \widehat{G})$ implements a Morita equivalence between B_{μ}^N and $B \rtimes_{\delta, \omega} \widehat{G} \rtimes_{\widehat{\delta}, \mu}^{\widehat{}} N$ which provides a version of Katayama duality for such twisted cosystems. We then show:

- (1) Given an arbitrary twisted coaction (B, δ, ω) there exists a unique norm $\|\cdot\|_{\mu}$ on $C_c(N, B \rtimes_{\delta, \omega} \widehat{G})$ such that $(B, \delta, \omega) \cong (B_{\mu}^N, \delta_{\mu}^N, \omega_{\mu}^N)$. In particular, (B, δ, ω) satisfies the above version of Katayama duality for $\|\cdot\|_{\mu}$. Moreover, we show that (B, δ, ω) is Morita equivalent to the trivially twisted inflated bidual coaction $(\text{Inf } \widehat{\delta}_{\mu}, 1)$ on $B \rtimes_{\delta, \omega} \widehat{G} \rtimes_{\widehat{\delta}, \mu}^{\widehat{}} N$. This gives the *stabilization trick for arbitrary twisted coactions* extending the main result of [9] where the stabilization trick was shown for amenable N .
- (2) There are canonical epimorphisms $B_u^N \twoheadrightarrow B \twoheadrightarrow B_r^N$ which are equivariant for the twisted coactions (δ_u^N, ω_u^N) , (δ, ω) , and (δ_r^N, ω_r^N) , respectively, such that the resulting homomorphisms

$$B_u^N \rtimes_{\delta_u^N, \omega_u^N} \widehat{G} \twoheadrightarrow B \rtimes_{\delta, \omega} \widehat{G} \twoheadrightarrow B_r^N \rtimes_{\delta_r^N, \omega_r^N} \widehat{G}$$

are isomorphisms of weakly proper $G \rtimes N$ -algebras.

By the previous discussion we already know that the twisted cosystem $(B_u^N, \delta_u^N, \omega_u^N)$ satisfies full Katayama duality $B_u^N \sim_M (B_u^N \rtimes_{\delta_u^N, \omega_u^N} \widehat{G}) \rtimes_u^{\widehat{}} N$ and the twisted cosystem $(B_r^N, \delta_r^N, \omega_r^N)$ satisfies reduced Katayama duality $B_r^N \sim_M$

$(B_r^N \rtimes_{\delta_r^N, \omega_r} \widehat{G}) \rtimes_r N$. They therefore give twisted analogues of a *maximalization* and a *normalization* of the coaction (δ, ω) , thus extending similar concepts for ordinary coactions as introduced in [7] and [25] and we obtain complete twisted analogues of the results on exotic coactions obtained in [1].

The outline of the paper is as follows: after a short preliminary section (§2) we prove in §3 some useful general results on fixed-point algebras including a theorem on iterated fixed-point algebras for normal subgroups: if A is a weakly proper $X \rtimes G$ -algebra and N is a closed normal subgroup of G , then $A_u^G \cong (A_u^N)_u^{G/N}$ and similarly for the reduced fixed-point algebra. This result will give the main tool for proving our versions of Mansfield's theorem in §4. The results on twisted Landstad Duality and maximalizations and normalizations for twisted coactions are given in the final section, §5. As one application, we give a new proof of the decomposition theorem $B \rtimes_{\delta} \widehat{G} \cong (B \rtimes_{\delta} \widehat{G/N}) \rtimes_{\delta, \omega} \widehat{G}$ of Phillips and Raeburn in [23].

2. PRELIMINARIES

To fix notation and for the reader's convenience we recall in this section some basic definitions and constructions from [1, 2] about weakly proper actions and their generalized fixed-point algebras as well as some notation on coactions that will be needed in this paper.

Let G be a locally compact group and let A be a G -algebra, that is, a C^* -algebra endowed with a (strongly continuous) G -action $\alpha: G \rightarrow \text{Aut}(A)$. We endow $C_c(G, A)$ with the usual $*$ -algebra structure by

$$f * g(t) := \int_G f(s) \alpha_s(g(s^{-1}t)) \, ds, \quad f^*(t) := \Delta(t)^{-1} \alpha_t(f(t^{-1})^*),$$

where Δ denotes the modular function of G . The full and reduced crossed product, denoted $A \rtimes_{\alpha} G$ and $A \rtimes_{\alpha, r} G$, respectively, are also defined in the usual way as completions of $C_c(G, A)$ with respect to the universal and reduced C^* -norms $\|\cdot\|_u$ and $\|\cdot\|_r$, respectively – the latter is defined in terms of the regular representation $\Lambda: C_c(G, A) \rightarrow \mathcal{L}(L^2(G, A))$. More generally, we call a *crossed-product norm* any C^* -norm $\|\cdot\|_{\mu}$ between the full and reduced norm and write $A \rtimes_{\alpha, \mu} G$ for the corresponding C^* -algebra completion, sometimes called an *exotic crossed product*. Certain special exotic crossed products of this type have been constructed in [15]: they are associated to crossed-product norms coming from G -invariant ideals in the Fourier-Stieltjes algebra $B(G)$ of G .

Now let X be a locally compact Hausdorff G -space with left G -action $G \times X \rightarrow X$, $(t, x) \mapsto t \cdot x$. We usually denote by $\tau: G \rightarrow \text{Aut}(C_0(X))$ the corresponding action given by $(\tau_t(f))(x) = f(t^{-1} \cdot x)$. An important special situation will be the case where $X = G$ is endowed with (right) translation G -action $t \cdot g := gt^{-1}$.

By a *weak $X \rtimes G$ -algebra* we mean a C^* -algebra A endowed with a G -action α and a G -equivariant non-degenerate $*$ -homomorphism $\phi: C_0(X) \rightarrow \mathcal{M}(A)$. We often write $f \cdot a := \phi(f)a$ and $a \cdot f := a\phi(f)$ for $f \in C_0(X)$, $a \in A$. We are mainly interested in the situation where X is a proper G -space, in which case we say that A is a weakly proper $X \rtimes G$ -algebra (notice that these actions are also proper in Rieffel's sense [27] by [28, Theorem 5.7]). Recall that the G -action on X is proper if and only if for all compact subsets $K, L \subseteq X$, the set $\{t \in G : t \cdot K \cap L \neq \emptyset\}$ is compact in G . In this situation, the space $\mathcal{F}_c(A) := C_c(X) \cdot A$ can be endowed with a canonical structure of a pre-Hilbert module over $C_c(G, A) \subseteq A \rtimes_{\alpha, \mu} G$ (for any

crossed-product norm $\|\cdot\|_\mu$) with inner product and right $C_c(G, A)$ -action given by

$$\langle\langle \xi | \eta \rangle\rangle_{C_c(G, A)}|_t := \Delta(t)^{-1/2} \xi^* \alpha_t(\eta), \quad \xi * f = \int_G \Delta(t)^{-1/2} \alpha_t(\xi \cdot f(t^{-1})) dt$$

for all $\xi, \eta \in \mathcal{F}_c(A)$, $f \in C_c(G, A)$ and $t \in G$. The Hilbert $A \rtimes_{\alpha, \mu} G$ -module completion of $\mathcal{F}_c(A)$ is denoted by $\mathcal{F}_\mu(A)$ (sometimes also $\mathcal{F}_\mu^G(A)$ if it is important to keep track of the group G). The C^* -algebra of compact operators on $\mathcal{F}_\mu(A)$ can be canonically identified with a completion A_μ^G of the generalized fixed-point algebra with compact supports A_c^G (see (1.1)) via the (left) A_c^G -valued inner product and left action on $\mathcal{F}_c(A)$ given by

$$A_c^G \langle\langle \xi | \eta \rangle\rangle := \mathbb{E}(\xi \eta^*) = \int_G^{\text{st}} \alpha_t(\xi \eta^*) dt, \quad a \cdot \xi := a\xi \text{ (multiplication in } \mathcal{M}(A)),$$

where $\mathbb{E}(a) := \int_G^{\text{st}} \alpha_t(a) dt$ denotes the strict (unconditional) integral ([10]) whenever this makes sense (which is the case for elements $a = \xi \eta^* \in A_c := C_c(X) \cdot A \cdot C_c(X)$). We shall write \mathbb{E}^G if it is important to keep track of the group in the notation. Note that in this construction $\mathcal{F}_c(A)$ becomes a (partial) A_c^G - $C_c(G, A)$ -pre-imprimitivity bimodule in which all possible pairings are jointly continuous with respect to the respective inductive limit topologies (see [1, Definition 2.11] for the definition of these topologies). This is shown in [1, Lemma 2.12] and the proof of that lemma also shows that the $\mathbb{E} : A_c \rightarrow A_c^G$ is inductive limit continuous as well.

If the above construction is applied to the special case where $A = C_0(X)$, we obtain a $C_0(X) \rtimes_\tau G$ -Hilbert module $\mathcal{F}(X) := \mathcal{F}(C_0(X))$ with algebra of compact operators isomorphic to $C_0(G \backslash X)$. Recall that $C_0(X) \rtimes_\tau G \cong C_0(X) \rtimes_{\tau, r} G$, that is, there is only one crossed-product norm $\mu = u = r$ on $C_c(G, C_0(X))$ because the action is proper. This is definitely not the case in general: every exotic crossed product appears as an (exotic) μ -generalized fixed point algebra A_μ^G (see [2, Corollary 3.25]). The relation between $\mathcal{F}(X)$ and $\mathcal{F}_\mu(A)$ is given by the (balanced) tensor product decomposition (obtained in [1, Proposition 2.9]):

$$\mathcal{F}_\mu(A) \cong \mathcal{F}(X) \otimes_{C_0(X) \rtimes_\tau G} (A \rtimes_{\alpha, \mu} G).$$

In particular, if the action on X is free, this decomposition implies that $\mathcal{F}_\mu(A)$ is full as a right Hilbert $A \rtimes_{\alpha, \mu} G$ -module and hence may be viewed as an imprimitivity bimodule between A_μ^G and $A \rtimes_{\alpha, \mu} G$. One special class of weakly proper actions where the above theory of generalized fixed point algebras can be successfully applied comes from crossed products by group coactions. Recall that a (full) coaction of G on a C^* -algebra B is a non-degenerate $*$ -homomorphism $\delta : B \rightarrow \mathcal{M}(B \otimes C^*(G))$ satisfying $(\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta_G) \circ \delta$ and such that $\delta(B)(1 \otimes C^*(G)) = B \otimes C^*(G)$ (“non-degeneracy” of the coaction), where $\delta_G : C^*(G) \rightarrow \mathcal{M}(C^*(G) \otimes C^*(G))$, $\delta_G(u_t) = u_t \otimes u_t$ for all $t \in G$, denotes the comultiplication of $C^*(G)$ and $G \ni t \mapsto u_t \in \mathcal{M}(C^*(G))$ denotes the universal representation. Although we have mentioned reduced coactions in the introduction, meaning (injective) coactions modelled on $C_r^*(G)$ in place of $C^*(G)$, we only work with full coactions in the main body of this paper.

Given a coaction (B, δ) , one can assign the crossed product $B \rtimes_\delta \widehat{G}$ which is endowed with a universal covariant representation pair $j_B : B \rightarrow \mathcal{M}(B \rtimes_\delta \widehat{G})$ and $j_G : C_0(G) \rightarrow \mathcal{M}(B \rtimes_\delta \widehat{G})$ in such a way that elements of the form $j_B(b)j_G(f)$ linearly span a dense subspace of $B \rtimes_\delta \widehat{G}$. By a covariant representation we mean

a pair of (non-degenerate) $*$ -homomorphisms $\pi, \sigma: B, C_0(G) \rightarrow \mathcal{M}(D)$, for some C^* -algebra D , satisfying

$$(\pi \otimes \text{id})(\delta(b)) = (\sigma \otimes \text{id})(\omega_G)(\pi(a) \otimes 1)(\sigma \otimes \text{id})(\omega_G)^* \quad \text{for all } b \in B,$$

where $\omega_G \in \mathcal{M}(C_0(G) \otimes C^*(G))$ is the unitary represented by the function $t \mapsto u_t$. The universality of (j_B, j_G) means that any such pair (π, σ) gives rise to a (unique) non-degenerate $*$ -homomorphism $\pi \rtimes \sigma: B \rtimes_\delta \widehat{G} \rightarrow \mathcal{M}(D)$ with $(\pi \rtimes \sigma) \circ j_B = \pi$ and $(\pi \rtimes \sigma) \circ j_G = \sigma$. The theory of crossed products by coactions turns out to be “amenable”, in the sense that the regular representation of $B \rtimes_\delta \widehat{G}$ into $\mathcal{M}(B \otimes \mathcal{K}(L^2G))$ given by the covariant pair $(\pi, \sigma) = ((\text{id} \otimes \lambda) \circ \delta, 1 \otimes M)$, where λ denotes the left regular representation of G and M the representation of $C_0(G)$ by multiplication operators, is faithful for every coaction. In other words, $B \rtimes_\delta \widehat{G}$ may be identified with the image of the regular representation in $\mathcal{M}(B \otimes \mathcal{K}(L^2G))$. On the other hand, the representation j_B need not be faithful in general (as happens for some dual coactions on full crossed products by actions of non-amenable G). A coaction is said to be *normal* if j_B is injective.

The crossed product $B \rtimes_\delta \widehat{G}$ carries a *dual action* $\widehat{\delta}$ of G given on generators by the formula:

$$\widehat{\delta}_t(j_B(b)j_G(f)) = j_B(b)j_G(\tau_t(f)),$$

where τ denotes the right translation action of G on itself: $\tau_t(f)|_s = f(st)$. This action turns j_G into a G -equivariant homomorphism and therefore enriches $A = B \rtimes_\delta \widehat{G}$ with the structure of a weakly proper $G \rtimes G$ -algebra. The double (full) crossed product $B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} G$ is, in general, not isomorphic to $B \otimes \mathcal{K}(L^2G)$, but there is a canonical surjection

$$\Phi: B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} G \twoheadrightarrow B \otimes \mathcal{K}(L^2G)$$

which is defined as the integrated form $\Phi := (\pi \rtimes \sigma) \rtimes (1 \otimes \rho)$, where $(\pi, \sigma) = ((\text{id} \otimes \lambda) \circ \delta, 1 \otimes M)$ is the regular covariant representation of (B, δ) and ρ denotes the right regular representation of G on L^2G . The coaction (B, δ) is called *maximal* if Φ is an isomorphism. Maximal coactions are exactly those which are Morita equivalent to dual coactions on full crossed products by actions. In general, there is a (unique, up to isomorphism) *maximalization* (B_u, δ_u) of (B, δ) which is a maximal coaction together with an equivariant surjection $B_u \rightarrow B$ inducing an isomorphism $B_u \rtimes_{\delta_u} \widehat{G} \xrightarrow{\sim} B \rtimes_\delta \widehat{G}$ (of weak $G \rtimes G$ -algebras). Similarly, there is a *normalization* (B_r, δ_r) of (B, δ) , that is, a normal coaction with an equivariant surjection $B \rightarrow B_r$ inducing an isomorphism $B \rtimes_\delta \widehat{G} \cong B_r \rtimes_{\delta_r} \widehat{G}$ (of weak $G \rtimes G$ -algebras) in such a way that the canonical surjection Φ factors through an isomorphism

$$\Phi_r: B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}_r} G \xrightarrow{\sim} B_r \otimes \mathcal{K}(L^2G).$$

More generally, Φ determines a crossed-product norm $\|\cdot\|_\mu$ on $C_c(G, A)$ for $A = B \rtimes_\delta \widehat{G}$ in such a way that Φ factors through an isomorphism

$$\Phi_\mu: B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}_\mu} G \twoheadrightarrow B \otimes \mathcal{K}(L^2G).$$

It follows from [1, Theorem 4.6] that there is a coaction δ_μ^G on the μ -generalized fixed-point algebra A_μ^G and Corollary 4.7 in [1] says that $(B, \delta) \cong (A_\mu^G, \delta_\mu^G)$. In this situation we say that (B, δ) is a μ -coaction, or that it satisfies μ -duality (which is implemented by Φ_μ). In particular, a coaction is maximal (resp. normal) if and only if $(B, \delta) \cong (A_u^G, \delta_u^G)$ (resp. $(B, \delta) \cong (A_r^G, \delta_r^G)$).

Summarizing, we may recover every coaction of G as a coaction of the form (A_μ^G, δ_μ^G) for some weak $G \rtimes G$ -algebra A . Moreover, for crossed-product norms associated to ideals in $B(G)$ as in [15], the assignment $A \mapsto (A_\mu^G, \delta_\mu^G)$ is an equivalence between the categories of weak $G \rtimes G$ -algebras and G -coactions satisfying μ -duality (see [1, Theorem 7.2]).

In this paper we extend these results and describe the category of weak $G \rtimes N$ -algebras, for N a closed normal subgroup of G , in terms of coactions of G twisted over G/N . This will be done in §5, where we review the definition of twisted coactions and derive the relevant results.

3. GREEN TWISTED ACTIONS AND ITERATED FIXED-POINT ALGEBRAS

Let G be a locally compact group and $N \subseteq G$ a normal closed subgroup. In most of this section $\|\cdot\|_\mu$ will denote either the maximal or reduced crossed-product norm.

Let (B, β) be a G -algebra. Recall that a (Green) *twist* for β is a strictly continuous group homomorphism $v: N \rightarrow \mathcal{UM}(A)$ satisfying

$$\alpha_n(a) = v_n a v_{n^{-1}} \quad \text{and} \quad \alpha_t(v_n) = v_{tnt^{-1}} \quad \forall t \in G, n \in N.$$

In this case we also say that (β, v) is a (Green) *twisted action* of (G, N) on a C^* -algebra B , or that (B, β, v) is a (G, N) -algebra. If v is the trivial twist, that is, $v_n = 1$ for all n , then β is trivial on N and hence factors through a G/N -action $\dot{\beta}$. Conversely, If $(B, \dot{\beta})$ is a G/N -algebra, then we may *inflate* $\dot{\beta}$ to a G -action $\text{Inf } \dot{\beta}$ on B and this is a (G, N) -twisted action with respect to the trivial twist. Hence, we may view (G, N) -algebras as generalizations of G/N -algebras. The maximal twisted crossed product $B \rtimes_{\beta, v} (G, N)$ can be constructed as the universal completion of the convolution algebra $C_c(G, B, v)$ consisting of all continuous functions $f: G \rightarrow B$ with compact supports modulo N which satisfy the relation $f(ns) = f(s)v_{n^{-1}}$ for all $s \in G, n \in N$. Convolution and involution on $C_c(G, B, v)$ are defined by

$$f * g(s) = \int_{G/N} f(t) \beta_t(g(t^{-1}s)) \, d_{G/N} t N \quad \text{and} \quad f^*(s) = \Delta_G(s^{-1}) \beta_s(f(s^{-1}))^*.$$

We always choose Haar measures on G, N , and G/N in such a way that the formula

$$(3.1) \quad \int_G \varphi(s) \, d_G s = \int_{G/N} \left(\int_N \varphi(sn) \, d_N n \right) d_{G/N} s N$$

holds for all $\varphi \in C_c(G)$. The non-degenerate $*$ -representations of $B \rtimes_{\beta, v} G$ are in one-to-one correspondence with the covariant representations (π, U) of (B, β) which preserve the twist v in the sense that $\pi(v_n) = U_n$ for all $n \in N$. Any such covariant representation integrates to a $*$ -representation $\pi \rtimes U$ of $C_c(G, B, v)$ by putting

$$\pi \rtimes U(f) = \int_{G/N} \pi(f(s)) U_s \, d_{G/N} s N.$$

The universal norm $\|\cdot\|_u$ on $C_c(G, B, v)$ is then given as

$$\|f\|_u = \sup_{(\pi, U)} \|\pi \rtimes U(f)\|$$

where (π, U) runs through all twisted covariant representations of (B, β, v) . Alternatively, $B \rtimes_{\beta, v} (G, N)$ can be obtained as the quotient of the untwisted crossed product by the ideal

$$I_v := \bigcap \{ \ker \pi \rtimes U : (\pi, U) \text{ is a twisted covariant representation of } (B, \beta, v) \}.$$

Note that we have a canonical isomorphism $C_c(G, B, 1_N) \cong C_c(G/N, B)$ which induces an isomorphism $B \rtimes_{\text{Inf } \dot{\beta}, 1_N} (G, N) \cong B \rtimes_{\dot{\beta}} G/N$, if $\dot{\beta}$ is an action of G/N on B . Twisted actions of this kind have been introduced by Phil Green in [11] and we refer to his paper for more details.

If (A, α, v^α) and (B, β, v^β) are two twisted (G, N) -algebras, then a (G, N) -equivariant A - B -correspondence (\mathcal{E}, γ) is a G -correspondence (\mathcal{E}, γ) between (A, α) and (B, β) which preserves the twists in the sense that

$$\gamma_n(\xi) = v_n^\alpha \cdot \xi \cdot v_{n^{-1}}^\beta \quad \forall \xi \in \mathcal{E}, n \in N.$$

If, in addition, \mathcal{E} is an imprimitivity A - B -bimodule, we say that the twisted actions (A, α, v^α) and (B, β, v^β) are Morita equivalent. By the version of the Packer-Raeburn stabilization trick given in [5], we know that every (G, N) -twisted action is Morita equivalent to a twisted (G, N) -action with trivial twist, *i.e.*, to a G/N -action.

Given a twisted (G, N) -action (B, β, v) and a Hilbert B, G -module (\mathcal{E}, γ) , the *crossed-product module* (or *descent*) $\mathcal{E} \rtimes_{\gamma, v} (G, N)$ is the Hilbert $B \rtimes_{\beta, v} (G, N)$ -module defined as the completion of the space $C_c(G, \mathcal{E}, v)$ of all continuous functions $x: G \rightarrow \mathcal{E}$ with compact support mod N satisfying $x(ns) = x(s)v_{n^{-1}}$ for all $s \in G$ and $n \in N$, endowed with the structure of a pre-Hilbert module over $C_c(G, B, v)$ given by:

$$x \cdot f|_t := \int_{G/N} x(s) \gamma_s(f(s^{-1}t)) \, d_{G/N}(tN),$$

$$\langle\langle x|y \rangle\rangle_t := \int_{G/N} \beta_{s^{-1}}(\langle x(s)|y(ts) \rangle) \, d_{G/N}(tN).$$

Given a G -algebra (A, α) , the crossed product $A \rtimes_{\alpha|} N$ (where $\alpha|$ denotes the restriction of α to N) carries a twisted (G, N) -action $(\tilde{\alpha}, \iota_N)$ given by $\tilde{\alpha}_t(f)|_n := \delta(t)\alpha_t(f(t^{-1}nt))$, where $\delta(t) = \frac{\Delta_G(t)}{\Delta_{G/N}(tN)}$ for all $t \in G$, and ι_N is the canonical homomorphism $N \rightarrow \mathcal{M}(A \rtimes N)$, $\iota_N(n)(f)|_s = \alpha_n(f(n^{-1}s))$. Sometimes $(\tilde{\alpha}, \iota_N)$ is called the *decomposition twisted action* of (G, N) on $A \rtimes_{\alpha|} N$. There is a canonical isomorphism $A \rtimes_{\mu} N \rtimes_{\mu} (G, N) \cong A \rtimes_{\mu} G$ for $\mu = u$ or $\mu = r$ (see [11, Proposition 1] and [21, Proposition 5.2]). Note that $(\tilde{\alpha}, \iota_N)$ factors through a twisted action (which we also denote $(\tilde{\alpha}, \iota_N)$) on a given exotic crossed product $A \rtimes_{\alpha|, \mu} N$ if and only if the ideal

$$I_\mu := \ker(A \rtimes_{\alpha|} N \rightarrow A \rtimes_{\alpha|, \mu} N)$$

is $\tilde{\alpha}$ -invariant. However, it is not clear how crossed-product norms for actions of N, G , and G/N should be related to each other in general to obtain the description of the G -crossed products as iterated crossed products.

We shall need twisted actions for the proof that for any weakly proper $X \rtimes G$ -algebra (A, α, ϕ) we have canonical isomorphisms

$$(A_u^N)^{G/N} \cong A_u^G \quad \text{and similarly} \quad (A_r^N)^{G/N} \cong A_r^G,$$

where $\|\cdot\|_u$ and $\|\cdot\|_r$ denote, as usual, the universal or reduced norms on crossed products by G, N , and G/N , respectively. Note that in the case where $G = N \times H$ is a direct product of groups, this result has been shown in [2, Lemma 5.17]. Observe

that by restricting the action α to N provides us with the weakly proper $X \rtimes N$ -algebra $(A, \alpha|_N, \phi)$. We then denote by

$$\mathbb{E}^N : A_c \rightarrow A_c^N; \mathbb{E}^N(a) = \int_N^{\text{st}} \alpha_s(a) d_N(s)$$

the corresponding surjective “conditional expectation”.

Proposition 3.2. *Suppose that (A, α, ϕ) is a weakly proper $X \rtimes G$ -algebra and let $\|\cdot\|_\mu$ be any crossed-product norm on $C_c(N, A)$ such that the corresponding ideal $I_\mu \subseteq A \rtimes_{\alpha|_N} N$ is invariant under the decomposition action $\tilde{\alpha}$. Then the formula*

$$\gamma_t^N(\xi) := \delta(t)^{1/2} \alpha_t(\xi)$$

for $\xi \in \mathcal{F}_c^N(A)$ extends to a G -action $\gamma^N : G \rightarrow \text{Aut}(\mathcal{F}_\mu^N(A))$ which is compatible with the decomposition G -action $\tilde{\alpha}$ of G on $A \rtimes_{\alpha|_N, \mu} N$. The corresponding G -action $\alpha^N := \text{Ad} \gamma^N$ on $A_\mu^N \cong \mathcal{K}(\mathcal{F}_\mu^N(A))$ is given on the dense subalgebra A_c^N by the restriction of α to $A_c^N \subseteq \mathcal{M}(A)$ and satisfies the equation

$$(3.3) \quad \alpha_t^N(\mathbb{E}^N(a)) = \alpha_t(\mathbb{E}^N(a)) = \delta(t) \mathbb{E}^N(\alpha_t(a)) \quad \forall a \in A_c, t \in G.$$

This action is trivial on N (hence is inflated from an action of G/N) and $(\mathcal{F}_\mu^N(A), \gamma)$ is a correspondence between the twisted actions $(A_\mu^N, \alpha^N, 1_N)$ and $(A \rtimes_{\alpha|_N, \mu} N, \tilde{\alpha}, \iota_N)$. If N acts freely on X , this correspondence will be a (G, N) -equivariant Morita equivalence.

Proof. We know that $\mathcal{F}_\mu^N(A) \cong \mathcal{F}_X^N \otimes_{C_0(X) \rtimes_{\tau} N} A \rtimes_\mu N$ as Hilbert $A \rtimes_\mu N$ -modules, via the map that sends $f \otimes g \in C_c(X) \odot C_c(N, A)$ to

$$f * g = \int_N \Delta_N(s)^{-1/2} \alpha_s(f \cdot g(s^{-1})) d_N(s) \in C_c(X) \cdot A = \mathcal{F}_c(A).$$

It was observed in [3, Remark 5.8] that $\mathcal{F}_X^N = \mathcal{F}^N(C_0(X))$ carries a G -action $\tilde{\tau}^N$, given by $\tilde{\tau}_t^N(f) = \delta(t)^{1/2} \tau_t(f)$ for all $f \in C_c(X)$, which is compatible with the twisted decomposition (G, N) -action on $C_0(X) \rtimes N$ and the G/N -action on $X_N := N \backslash X$. Hence, there is a G -action on $\mathcal{F}_X^N \otimes_{C_0(X) \rtimes_{\tau} N} A \rtimes_\mu N$ given on $C_c(X) \odot C_c(N, A)$ by the formula $\gamma_t^N(f \odot g) = \tilde{\tau}_t(f) \odot \tilde{\alpha}_t(g)$. Using the isomorphism $\mathcal{F}_\mu^N(A) \cong \mathcal{F}_X^N \otimes_{C_0(X) \rtimes_{\tau} N} A \rtimes_\mu N$ we may view γ^N as an action on $\mathcal{F}_\mu^N(A)$ and a straightforward computation shows that $\gamma_t(\xi) = \delta(t)^{1/2} \alpha_t(\xi)$ for all $\xi \in \mathcal{F}_c^N(A)$. The corresponding action $\alpha^N := \text{Ad} \gamma^N$ on $A_\mu^N \cong \mathcal{K}(\mathcal{F}_\mu^N(A))$ is given, for all $\xi, \eta \in \mathcal{F}_c^N(A)$, by:

$$\alpha_t^N(\mathbb{E}^N(\xi \eta^*)) =_{A_\mu^N} \langle \gamma_t^N \xi | \gamma_t^N \eta \rangle = \delta(t) \mathbb{E}^N(\alpha_t(\xi \eta^*)) = \alpha_t(\mathbb{E}^N(\xi \eta^*)),$$

where the last equation follows from a straightforward computation using the fact that $\int_N \varphi(tnt^{-1}) dn = \delta(t) \int_N \varphi(n) dn$ for every integrable function φ on N . This proves (3.3). Since the elements in A_c^N are fixed by α_n for all $n \in N$, it follows that α^N is trivial on N .

It is straightforward to check that $\gamma_n^N(\xi) * g = \xi * (\iota_N(n^{-1}) \cdot g)$ for all $n \in N$, $\xi \in \mathcal{F}_c^N(A)$ and $g \in C_c(N, A)$ and hence $\gamma_n^N(\xi) = \xi \cdot \iota_N(n^{-1})$, which implies the compatibility of γ^N with the twists. The last assertion follows from the fact that $\mathcal{F}_\mu^N(A)$ is an A_μ^N - $A \rtimes_\mu N$ imprimitivity bimodule if N acts freely on X . \square

If (A, α, ϕ) is a weakly proper $X \rtimes G$ -algebra as in the proposition and if N is a closed normal subgroup of G , then G/N acts properly on $N \backslash X$ in a canonical way and we observe that the homomorphism

$$(3.4) \quad \phi^N : C_0(N \backslash X) \rightarrow \mathcal{M}(A_\mu^N) \cong \mathcal{L}_{A \rtimes N}(\mathcal{F}_\mu^N(A))$$

induced by ϕ corresponds to the canonical left $C_0(N \backslash X)$ -action on $\mathcal{F}_\mu^N(A) \cong \mathcal{F}^N(X) \otimes_{C_0(X) \rtimes N} A \rtimes_\mu N$ given by

$$(3.5) \quad \phi^N(f)m = \phi(f)m \quad \forall m \in A_c^G$$

(where ϕ has been tacitly extended to $C_b(X)$ and we view $C_0(N \backslash X)$ as a subalgebra of $C_b(X)$ in the usual way). From this it is easy to see that ϕ^N is G/N -equivariant and hence (A_μ^N, α^N) is a weakly proper $N \backslash X \rtimes G/N$ -algebra. Thus we may study the iterated fixed-point algebra $(A_\mu^N)_\mu^{G/N}$, if μ stands either for the universal or for the reduced crossed-product norms. For the corresponding conditional expectation $\mathbb{E}^{G/N} : (A_\mu^N)_c \rightarrow (A_\mu^N)_c^{G/N}$, we get the following:

Lemma 3.6. *Let (A, α, ϕ) and $N \subseteq G$ be as above. Then $\mathbb{E}^N(A_c)$ is inductive limit dense in $(A_\mu^N)_c$ and hence $\mathbb{E}^{G/N}(\mathbb{E}^N(A_c))$ is inductive limit dense in $(A_\mu^N)_c^{G/N}$. Moreover, we have*

$$\mathbb{E}^{G/N}(\mathbb{E}^N(a)) = \mathbb{E}^G(a) \quad \forall a \in A_c.$$

Hence $A_c^G = \mathbb{E}^G(A_c) = \mathbb{E}^{G/N}(\mathbb{E}^N(A_c))$ is inductive limit dense in $(A_\mu^N)_c^{G/N}$.

Proof. The first assertion follows from the fact that \mathbb{E}^N and $\mathbb{E}^{G/N}$ are continuous with respect to the inductive limit topologies and that $A_c^N = \mathbb{E}^N(A_c)$ is inductive limit dense in $(A_\mu^N)_c$. The second assertion follows from (3.1). \square

In what follows we abuse slightly the notation and write $C_c(G, \mathcal{F}_c^N(A), \iota_N)$ for the space of all functions $g \in C_c(G, \mathcal{F}_\mu^N(A), \iota_N)$ such that there exists an $f \in C_c(X)$ with $g(s) = f \cdot g(s) \in \mathcal{F}_c^N(A)$ for all $s \in G$. We leave it as an exercise for the reader to check that $C_c(G, \mathcal{F}_c^N(A), \iota_N)$ is inductive limit dense (with respect to compact supports mod N) in $C_c(G, \mathcal{F}_\mu^N(A), \iota_N)$.

Proposition 3.7. *Let (A, α, ϕ) be a weakly proper $X \rtimes G$ -algebra and consider the corresponding weakly proper $N \backslash X \rtimes G/N$ -algebra $(A_\mu^N, \alpha^N, \phi^N)$ as in Proposition 3.2, where μ stands either for the maximal or for the reduced crossed-product norms. Then there is a canonical isomorphism*

$$\mathcal{F}_\mu^{G/N}(A_\mu^N) \otimes_{A_\mu^N \rtimes_{G/N}} (\mathcal{F}_\mu^N(A) \rtimes (G, N)) \xrightarrow{\sim} \mathcal{F}_\mu^G(A)$$

as Hilbert $A \rtimes_\mu N \rtimes_\mu (G, N) \cong A \rtimes_\mu G$ -modules via the map sending $a \otimes g \in A_c^N \odot C_c(G, \mathcal{F}_c^N(A), \iota_N)$ to $a * g := \int_{G/N} \Delta_G(s)^{-1/2} \alpha_s(a \cdot g(s^{-1})) d_{G/N}(sN)$.

Proof. Let us first observe that $A_c^N = C_c(N \backslash X) \cdot A_c^N$ is indeed a dense subspace of $\mathcal{F}_\mu^{G/N}(A_\mu^N)$, and that the function $s \mapsto h(s) := \Delta_G(s)^{-1/2} \alpha_s(a \cdot g(s^{-1}))$ is constant on N -orbits (and has compact support mod N) so that the integral over G/N defining $a * g$ makes sense and gives an element of A . In fact, since $g \in C_c(G, \mathcal{F}_c^N(A), \iota_N)$, we have $g(n^{-1}s^{-1}) = g(s^{-1}) \cdot \iota_N(n) = \Delta_G(n)^{1/2} \alpha_{n^{-1}}(g(s))$ and since a is N -invariant, this implies

$$h(sn) = \Delta_G(sn)^{-1/2} \alpha_{sn}(a \cdot g(n^{-1}s^{-1})) = \Delta_G(s)^{-1/2} \alpha_s(a \cdot g(s^{-1})) = h(s)$$

for all $s \in G$ and $n \in N$. Now observe that $a * g \in \mathcal{F}_\mu^G(A) = C_c(X) \cdot A$. In fact, take $K \subseteq G$ compact such that $\text{supp}(h) \subseteq KN$ and let $\varphi \in C_c(G)$ with $\int_N \varphi(sn) d_N(n) = 1$ for all $s \in K$. Then

$$\int_{G/N} h(s) d_{G/N}(sN) = \int_{G/N} \int_N h(sn) \varphi(sn) d_N(n) d_{G/N}(sN) = \int_G h(s) \varphi(s) ds.$$

Now, take $f \in C_c(X)$ with $g(s) = f \cdot g(s)$ for all $s \in G$. Since $a \in A_c^N$, we have $a \cdot f \in A_c = C_c(X) \cdot A \cdot C_c(X)$, so there is $\psi \in C_c(X)$ with $a \cdot f = \psi \cdot b$ for some $b \in A$. But then

$$a * g = \int_G \varphi(s) \Delta_G(s)^{-1/2} \tau_s(\psi) \cdot \alpha_s(b \cdot g(s^{-1})) ds$$

which is easily seen to be an element of $\mathcal{F}_c(A)$ (compare this to the formula (2.10) in [1]). We therefore have a well-defined linear map from the dense subspace $A_c^N \odot C_c(G, \mathcal{F}_c^N(A), \iota_N)$ of $\mathcal{F}_\mu^{G/N}(A_\mu^N) \otimes_{A_\mu^N \rtimes G/N} \mathcal{F}_\mu^N(A) \rtimes (G, N)$ into the dense subspace $\mathcal{F}_c(A)$ of $\mathcal{F}(A)$. Now, a computation as in the proof of [1, Proposition 2.9] shows that this map preserves inner products and has dense range and hence extends to an isomorphism $\mathcal{F}_\mu^{G/N}(A_\mu^N) \otimes_{A_\mu^N \rtimes G/N} \mathcal{F}_\mu^N(A) \rtimes (G, N) \xrightarrow{\sim} \mathcal{F}_\mu^G(A)$. \square

Using the above proposition, we are now able to show the desired isomorphism $(A_\mu^N)_\mu^{G/N} \cong A_\mu^G$ for $\mu = u$ and $\mu = r$. For reduced norms and *free* proper actions, this result has been obtained in [14, Theorem 4.5]. Our method of proof is, however, quite different from [14] and works for reduced and universal norms as well as for non-free proper actions.

Theorem 3.8. *For a weakly proper $X \rtimes G$ -algebra A , there is an isomorphism $(A_\mu^N)_\mu^{G/N} \cong A_\mu^G$ extending the inclusion map $A_c^G = \mathbb{E}^{G/N}(A_c^N) \subseteq (A_\mu^N)_\mu^{G/N}$ into $A_c^G \subseteq A_\mu^G$ (where $\|\cdot\|_\mu$ denotes either the universal or the reduced crossed-product norm).*

Proof. Let $\psi: \mathcal{E} \xrightarrow{\sim} \mathcal{F}_\mu^G(A)$, $\psi(a \otimes g) = a * g$, be the isomorphism of Proposition 3.7, where $\mathcal{E} := \mathcal{F}_\mu^{G/N}(A_\mu^N) \otimes_{A_\mu^N \rtimes G/N} (\mathcal{F}_\mu^N(A) \rtimes_\mu (G, N))$. This isomorphism induces an isomorphism $\tilde{\psi}: \mathcal{K}(\mathcal{E}) \xrightarrow{\sim} \mathcal{K}(\mathcal{F}_\mu^G(A)) = A_\mu^G$ determined by the equation

$$\tilde{\psi}(T)(\psi(a \otimes g)) = \psi(T(a \otimes g)).$$

On the other hand, since $A_\mu^N \rtimes G/N \cong \mathcal{K}(\mathcal{F}_\mu^N(A) \rtimes_\mu (G, N))$, we have a canonical isomorphism $(A_\mu^N)_\mu^{G/N} = \mathcal{K}(\mathcal{F}_\mu^{G/N}(A_\mu^N)) \xrightarrow{\sim} \mathcal{K}(\mathcal{E})$ sending an operator $S \in \mathcal{K}(\mathcal{F}_\mu^{G/N}(A_\mu^N))$ to the operator $S \otimes 1 \in \mathcal{K}(\mathcal{E})$ given by $(S \otimes 1)(a \otimes g) = S(a) \otimes g$.

We therefore get an isomorphism $(A_\mu^N)_\mu^{G/N} \xrightarrow{\sim} A_\mu^G$ sending $S \in (A_\mu^N)_\mu^{G/N}$ to $\tilde{\psi}(S \otimes 1) \in A_\mu^G$. Applying this to $S = \mathbb{E}^{G/N}(b) = \mathbb{E}^G(c)$ for $b = \mathbb{E}^N(c)$, $c \in A_c$, we

see that S is G -invariant. Hence we get:

$$\begin{aligned}
 \tilde{\psi}(S)(\psi(a \otimes g)) &= \psi(Sa \otimes g) = Sa * g \\
 &= \int_{G/N} \Delta_G(s)^{-1/2} \alpha_s(Sa \cdot g(s^{-1})) \, d_{G/N} s N \\
 &= S \left(\int_{G/N} \Delta_G(s)^{-1/2} \alpha_s(a \cdot g(s^{-1})) \, d_{G/N} s N \right) \\
 &= S \cdot (a * g) = S \cdot \psi(a \otimes g)
 \end{aligned}$$

so that $\tilde{\psi}: (A_\mu^N)^{G/N} \rightarrow A_\mu^G$ is the extension of the identity map on $\mathbb{E}^{G/N}(A_c^N)$. \square

Suppose that H is a closed subgroup of a group G and that (A, α) is an H -algebra. Then $C_0(G) \otimes A$ becomes a weak $G \rtimes (G \times H)$ -algebra with respect to the structure map $\psi: f \mapsto f \otimes 1$ from $C_0(G)$ into $\mathcal{ZM}(C_0(G) \otimes A)$. We let G act on $C_0(G) \otimes A$ via $\tau \otimes \text{id}$, where τ denotes the left translation action of G on itself, and we let H act on $C_0(G) \otimes A$ via $\sigma \otimes \alpha$, with $\sigma_h(f)(s) = f(sh)$ for all $f \in C_0(G)$, $s \in G$, $h \in H$. The actions of G and H on $C_0(G) \otimes A$ clearly commute and make the structure map ψ equivariant with respect to both G and H actions. Since H acts properly on G , the restriction of the action to H gives $C_0(G) \otimes A$ the structure of a proper $G \rtimes H$ -algebra and we can form the H -fixed-point algebras $(C_0(G) \otimes A)_\mu^H$ for this structure. Since the structure map ψ takes its values in the center $\mathcal{ZM}(C_0(G) \otimes A)$, it follows from [2, Theorem 3.28] that they do not depend on the given crossed-product norm $\|\cdot\|_\mu$ on $C_c(H, C_0(G) \otimes A)$ (indeed, for centrally proper actions all such norms coincide with the universal norm $\|\cdot\|_u$). Note that the G -action on $C_0(G) \otimes A$ factors to a G -action on the H -fixed-point algebra $(C_0(G) \otimes A)^H$ (we may now omit the norm μ in the notation).

The algebra $(C_0(G) \otimes A)^H$ is actually well known under the name of *induced algebra* $\text{Ind}_H^G(A, \alpha)$ and can be described as follows:

$$\text{Ind}_H^G(A, \alpha) = \left\{ F \in C_b(G, A) : \begin{array}{l} F(sh) = \alpha_{h^{-1}}(F(s)) \, \forall s \in G, h \in H \\ \text{and } (sH \mapsto \|F(s)\|) \in C_0(G/H) \end{array} \right\}.$$

Indeed, identifying $\mathcal{M}(C_0(G) \otimes A)$ with the set of bounded strictly continuous functions from G to A , it is an easy exercise to check that $(C_0(G) \otimes A)_c^H$ is just the set of functions in $\text{Ind}_H^G(A, \alpha)$ which have compact supports mod H . In this picture, the G -action is given as the *induced action*

$$\text{Ind } \alpha: G \rightarrow \text{Aut}(\text{Ind}_H^G(A, \alpha)); \text{Ind } \alpha_s(F(t)) = F(s^{-1}t) \quad \forall s, t \in G.$$

If $A = C_0(Y)$ for an H -space Y , we get $\text{Ind}_H^G C_0(Y) \cong C_0(G \times_H Y)$, where the *induced G -space* $G \times_H Y$ is defined as the quotient $H \backslash (G \times Y)$ under the action of H on $G \times Y$ given by $h(s, y) = (sh^{-1}, hy)$. Moreover, if we start with a weakly proper $Y \rtimes H$ -algebra (A, α, ϕ) , the algebra $C_0(G) \otimes A$ actually becomes a weakly proper $(G \times Y) \rtimes (G \times H)$ -algebra via the obvious structure map $\psi: C_0(G \times Y) \rightarrow \mathcal{M}(C_0(G) \otimes A)$. It follows then from [2, Proposition 3.12] that the H -fixed-point algebras $(C_0(G) \otimes A)_\mu^H$ and the corresponding modules $\mathcal{F}_\mu(C_0(G) \otimes A)$ coincide, no matter whether we regard $C_0(G) \otimes A$ as a weakly proper $(G \rtimes Y) \rtimes H$ -algebra or a weakly proper $G \rtimes H$ -algebra. But if we view it as a $(G \times Y) \rtimes (G \times H)$ -algebra, we see that $\text{Ind}_H^G(A, \alpha) = (C_0(G) \otimes A)^H$ carries the structure of a weakly proper

$(G \times_H Y) \rtimes G$ -algebra, and by Theorem 3.8 we obtain isomorphisms

$$(3.9) \quad \begin{aligned} (\mathrm{Ind}_H^G(A, \alpha))_\mu^G &\cong ((C_0(G) \otimes A)_\mu^H)^G \cong (C_0(G) \otimes A)_\mu^{G \times H} \\ &\cong ((C_0(G) \otimes A)_\mu^G)^H \cong A_\mu^H, \end{aligned}$$

where $\|\cdot\|_\mu$ denotes either the universal or the reduced crossed-product norm (everywhere). The last isomorphism in (3.9) is induced by the H -equivariant isomorphism $A \cong \mathrm{Ind}_G^G(A, \mathrm{id})$; $a \mapsto 1_G \otimes a$. We summarize our discussion as follows:

Proposition 3.10. *Let H be a closed subgroup of G and let (A, α, ϕ) be a weakly proper $Y \rtimes H$ -algebra. Let $\|\cdot\|_\mu$ denote either the universal or reduced crossed-product norm for both G and H . Then there is an isomorphism*

$$(3.11) \quad A_\mu^{H, \alpha} \xrightarrow{\sim} (\mathrm{Ind}_H^G(A, \alpha))_\mu^{G, \mathrm{Ind} \alpha},$$

sending $m \in A_c^{H, \beta} \subseteq \mathcal{M}(A)^H$ to the constant function $t \mapsto m$ from G to $\mathcal{M}(A)$ viewed as an element of $(\mathrm{Ind}_H^G(A, \alpha))_c^{G, \mathrm{Ind} \beta} \subseteq \mathcal{M}(C_0(G) \otimes A)^G$.

Proof. The only statement which is not instantly clear from the above discussion is the special description of the isomorphism $A_\mu^H \cong (\mathrm{Ind}_H^G(A, \alpha))_\mu^G$. But this follows easily from the description of the isomorphism $((C_0(G) \otimes A)_\mu^G)^H \cong A_\mu^H$ in (3.9) and the fact that, according to Theorem 3.8, all other isomorphisms in (3.9) are induced by the identity map on $(C_0(G) \otimes A)_c^{G \times H}$. \square

Remark 3.12. Later we shall apply the above proposition to the special situation in which $Y = G$ equipped with the right translation action of H . Let (A, α, ϕ) be a $G \rtimes H$ -algebra. Notice that the induced space $G \times_H G$ is G -homeomorphic to $G/H \times G$ via $[(s, t)] \mapsto (sH, st)$. If we forget the factor G/H , we see that $\mathrm{Ind}_H^G(A, \alpha)$ carries a structure of a weakly proper $G \rtimes G$ -algebra with structure map $\psi : C_0(G) \rightarrow \mathcal{M}(\mathrm{Ind}_H^G(A, \alpha))$ given by the formula

$$(\psi(f)F)(s) = \phi(\tau_{s^{-1}}(f))F(s) \quad \forall f \in C_0(G), F \in \mathrm{Ind}_H^G(A, \alpha).$$

It follows from [2, Proposition 3.12] that the G -fixed-point algebra for this $G \rtimes G$ -structure on $\mathrm{Ind}_H^G(A, \alpha)$ coincides with the G -fixed-point algebra for the $(G \times_H G) \rtimes G$ -structure, hence Proposition 3.10 will still apply if we just consider the $G \rtimes G$ -structure.

Suppose now that (A, α, ϕ) is a weakly proper $X \rtimes G$ -algebra and that H is a closed subgroup of G . Then $(A, \alpha|, \phi)$, where $\alpha|$ denotes the restriction of α to H , is a weakly proper $X \rtimes H$ -algebra, and we close this section by proving an isomorphism

$$\mathcal{F}_\mu^G(A) \otimes_{A \rtimes_\mu G} \mathcal{X}_{H, \mu}^G(A) \cong \mathcal{F}_\mu^H(A),$$

where $\|\cdot\|_\mu$ denotes either the universal or the reduced crossed-product norm. Here $\mathcal{X}_{H, \mu}^G(A)$ denotes Green's $C_0(G/H, A) \rtimes_\mu G$ - $A \rtimes_\mu H$ imprimitivity bimodule of [11, §2]. Recall that it is the completion of $C_c(G, A)$ viewed as a $C_c(H, A)$ -pre-Hilbert module with respect to the module action and inner product given by the formulas

$$\begin{aligned} \xi \cdot \varphi(t) &= \int_H \gamma_H(h) \xi(th) \alpha_{th}(\varphi(h^{-1})) \, dh, \\ \langle\langle \xi | \eta \rangle\rangle_{C_c(H, A)}(h) &= \gamma_H(h) \int_G \alpha_{s^{-1}}(\xi(s)^* \eta(sh)) \, ds \end{aligned}$$

where $\gamma_H(h) := \sqrt{\Delta_G(h)\Delta_H(h^{-1})}$ for $h \in H$. The above formulas are taken from [29, Theorem 4.15]. The left action of $C_c(G, C_0(G/H, A)) \subseteq C_0(G/H, A) \rtimes_\mu G$ on $C_c(G, A)$ is given by the formula

$$f * \xi(s) = \int_G f(t, sH) \alpha_t(\xi(t^{-1}s)) \, dt.$$

The G -equivariant imbedding of A into $\mathcal{M}(C_0(G/H, A))$ as constant functions induces the left action of $A \rtimes_\mu G$ on $\mathcal{X}_{H,\mu}^G(A)$ given on the level of functions on $f, \xi \in C_c(G, A)$ by the usual convolution $(f \cdot \xi)(s) = f * \xi(s) = \int_G f(t) \alpha_t(\xi(t^{-1}s)) \, dt$. Combining these formulas with the formulas for the pre-imprimitivity modules $\mathcal{F}_c^G(A)$ and $\mathcal{F}_c^H(A)$ as given in §2 we get the following:

Proposition 3.13. *Let A be a weakly proper $X \rtimes G$ -algebra and let H be a closed subgroup of G . Let $\mu = u$ or $\mu = r$. Then there is an isomorphism*

$$\mathcal{F}_\mu^G(A) \otimes_{A \rtimes_{\alpha,\mu} G} \mathcal{X}_{H,\mu}^G(A) \xrightarrow{\sim} \mathcal{F}_\mu^H(A)$$

*of Hilbert $A \rtimes_{\beta|\mu} H$ -modules which sends an elementary tensor $a \otimes \xi$ in $\mathcal{F}_c^G(A) \odot C_c(G, A)$ to $a * \xi := \int_G \Delta_G(t)^{-1/2} \alpha_t(a \cdot \xi(t^{-1})) \, dt$.*

The induced $$ -homomorphism on compact operators $A_\mu^G \rightarrow \mathcal{M}(A_\mu^H)$ is given by the identity map on A_c^G by viewing $m \in A_c^G$ as a multiplier of A_μ^H via multiplication in $\mathcal{M}(A)$: $a \cdot m = am$ and $m \cdot a = ma$ for all $a \in A_c^H$.*

Proof. It is enough to check that the map $a \otimes \xi \mapsto a * \xi$ preserves inner products and has dense range. For the inner products we let $a, b \in \mathcal{F}_c(A)$ and $\xi, \eta \in C_c(G, A)$ and we compute

$$\begin{aligned} \langle a \otimes \xi | b \otimes \eta \rangle_{C_c(H,A)}(h) &= \langle \xi | \langle a | b \rangle_{C_c(G,A)} \cdot \eta \rangle_{C_c(H,A)}(h) \\ &= \gamma_H(h) \int_G \int_G \Delta_G(t)^{-1/2} \alpha_{s^{-1}}(\xi(s)^* \alpha_h(a^* \alpha_t(b \eta(t^{-1}sh)))) \, dt \, ds. \end{aligned}$$

On the other hand, we compute

$$\begin{aligned} \langle a * \xi | b * \eta \rangle_{C_c(H,A)}(h) &= \Delta_H(h)^{-1/2} (a * \xi)^* \alpha_h(b * \eta) \\ &= \Delta_H(h)^{-1/2} \int_G \int_G \Delta_G(st)^{-1/2} \alpha_s(\xi(s^{-1})^* a^*) \alpha_{ht}(b \xi(t^{-1})) \, dt \, ds. \end{aligned}$$

If we apply the transformation $s \mapsto s^{-1}$ followed by the transformation $t \mapsto h^{-1}s^{-1}t$ to the above integral, we see that $\langle a \otimes \xi | b \otimes \eta \rangle_{C_c(H,A)}(h) = \langle a * \xi | b * \eta \rangle_{C_c(H,A)}(h)$ for all $h \in H$. A similar but easier computation shows that $a * (\xi \cdot \varphi) = (a * \xi) \cdot \varphi$ for all $a \in \mathcal{F}_c^G(A)$, $\xi \in C_c(G, A)$ and $\varphi \in C_c(H, A)$, which then implies that the map $a \otimes \xi \mapsto a * \xi$ extends to an isometric $A \rtimes_\mu H$ -Hilbert-module map from $\mathcal{F}_\mu^G(A) \otimes_{A \rtimes_{\alpha,\mu} G} \mathcal{X}_{H,\mu}^G(A)$ into $\mathcal{F}_\mu^H(A)$. Surjectivity of this map follows from standard approximative unit arguments as done, for example, in the proof of [2, Proposition 3.32]. For the final assertion, it is enough to check that given $m \in A_c^G$, we have $(m \cdot a) * \xi = m \cdot (a * \xi)$ for all $a \in \mathcal{F}_c^G(A)$ and $\xi \in C_c(G, B)$. This follows from a simple computation using that m is G -invariant. \square

Observe that the canonical homomorphism $A_\mu^G \rightarrow \mathcal{M}(A_\mu^H)$ can be used to induce representations from A_μ^H to A_μ^G . The above proposition says that this corresponds to the induction process from representations of $A \rtimes_\mu H$ to representations of $A \rtimes_\mu G$ via Green's imprimitivity bimodule. This is especially interesting if the involved actions are saturated, in which case $\mathcal{F}_\mu^G(A)$ and $\mathcal{F}_\mu^H(A)$ are imprimitivity bimodules

implementing Morita equivalences $A_\mu^G \sim A \rtimes_\mu G$ and $A_\mu^H \sim A \rtimes_\mu H$, so that we get bijections between the spaces of representations $\text{Rep}(A_\mu^G) \cong \text{Rep}(A \rtimes_\mu G)$ and $\text{Rep}(A_\mu^H) \cong \text{Rep}(A \rtimes_\mu H)$. In this situation the induction process $\text{Rep}(A \rtimes_\mu H) \rightarrow \text{Rep}(A \rtimes_\mu G)$ is therefore essentially equivalent to $\text{Rep}(A_\mu^H) \rightarrow \text{Rep}(A_\mu^G)$, but the latter might be easier to describe in some situations.

4. MANSFIELD'S IMPRIMITIVITY THEOREM

As a consequence of our previous results, we deduce Mansfield's Imprimitivity Theorem for both universal and reduced crossed-product norms. So in what follows next we let $\delta : B \rightarrow M(B \otimes C^*(G))$ be a coaction of G on the C^* -algebra B . Recall from §2 that $B \rtimes_\delta \widehat{G}$ carries a canonical weakly proper $G \rtimes G$ -structure $(B \rtimes_\delta \widehat{G}, j_G, \widehat{\delta})$ which restricts to a $G \rtimes H$ -structure $(B \rtimes_\delta \widehat{G}, j_G, \widehat{\delta}|_H)$ for every closed subgroup H of G . Since right translation of H on G is free, we see that $\mathcal{F}_\mu^H(B \rtimes_\delta \widehat{G})$ implements a $(B \rtimes_\delta \widehat{G})_\mu^H - (B \rtimes_\delta \widehat{G}) \rtimes_\mu H$ imprimitivity bimodule for every crossed-product norm $\|\cdot\|_\mu$ on $C_c(H, B \rtimes_\delta \widehat{G})$.

In what follows, we want to compare this result with the various versions of Mansfield's Imprimitivity Theorem for coactions which give rise to Morita equivalences between $B_\mu \rtimes_{\delta_\mu} \widehat{G/N}$ and $(B \rtimes_\delta \widehat{G}) \rtimes_\mu N$, where the notation (B_μ, δ_μ) indicates that we have to be careful about the type of coactions we may consider here. Indeed, we shall restrict below to the two cases where $\|\cdot\|_\mu$ is either the universal norm $\|\cdot\|_u$ or the reduced norm $\|\cdot\|_r$. Then, as explained in §2, (B_u, δ_u) is the maximalization and (B_r, δ_r) is the normalization of (B, δ) . Recall that for any coaction $\delta : B \rightarrow M(B \otimes C^*(G))$, the restriction $\delta|$ of δ to the quotient group G/N is given by the composition

$$\delta| : B \xrightarrow{\delta} M(B \otimes C^*(G)) \xrightarrow{\text{id}_B \otimes q_N} M(B \otimes C^*(G/N)),$$

where $q_N : C^*(G) \rightarrow C^*(G/N)$ denotes the canonical quotient map.

Theorem 4.1 (Mansfield's Imprimitivity Theorem). *Let (B, δ) be a G -coaction and equip the crossed product $B \rtimes_\delta \widehat{G}$ with the canonical weak $G \rtimes G$ -algebra structure. Then there are canonical isomorphisms*

$$B_r \rtimes_{\delta_r|} \widehat{G/N} \cong (B \rtimes_\delta \widehat{G})_r^N \quad \text{and} \quad B_u \rtimes_{\delta_u|} \widehat{G/N} \cong (B \rtimes_\delta \widehat{G})_u^N.$$

In particular, if (B, δ) is normal, $\mathcal{F}_r^N(B \rtimes_\delta \widehat{G})$ becomes a $B \rtimes_{\delta|} \widehat{G/N} - (B \rtimes_\delta \widehat{G}) \rtimes_r N$ imprimitivity bimodule and if (B, δ) is maximal, then $\mathcal{F}_u^N(B \rtimes_\delta \widehat{G})$ becomes a $B \rtimes_{\delta|} \widehat{G/N} - (B \rtimes_\delta \widehat{G}) \rtimes_u N$ imprimitivity bimodule.

Remark 4.2. We should remark that the isomorphism for normal coactions has been established before by Quigg and Raeburn in [26, Proposition 4.1] in the case where N is amenable, and shortly after that by Kaliszewski and Quigg in [18] for arbitrary closed normal subgroups N . Both proofs rely heavily on Mansfield's original proof of his imprimitivity theorem and they use the Mansfield algebra $\mathcal{D} \subseteq B \rtimes_\delta \widehat{G}$ as a dense $*$ -algebra which implements properness in Rieffel's sense ([27]). So a priori, the fixed-point algebras and the bimodules considered in those papers could be different from ours, but we shall see below that they are not.

The isomorphism for maximal coactions is new but a version of Mansfield's Imprimitivity Theorem for maximal coactions has been shown by Kaliszewski and

Quigg in [18] using quite different techniques. The above gives a unified treatment to all of these different versions and does not rely on Mansfield's techniques.

Proof of Theorem 4.1. In what follows let $A := B \rtimes_{\delta} \widehat{G}$. It follows then from [1, Theorem 4.6] that $(B_u, \delta_u) \cong (A_u^G, \delta_u^G)$ and $(B_r, \delta_r) \cong (A_r^G, \delta_r^G)$ where the coactions δ_{μ}^G , with $\mu = u$ or $\mu = r$, are given by the formulas

$$(4.3) \quad \delta_{\mu}^G(m) = (j_G \otimes \text{id})(\omega_G)(m \otimes 1)(j_G \otimes \text{id})(\omega_G)^* \quad \text{for all } m \in A_c^G$$

(see also [1, Remark 4.14] for the correct interpretation of this formula). On the other hand, A_{μ}^N is a weak $G/N \rtimes G/N$ -algebra and by Proposition 3.7 we have a canonical isomorphism $\Psi : A_{\mu}^G \xrightarrow{\cong} (A_{\mu}^N)_{\mu}^{G/N}$ given via the canonical inclusion of A_c^G into $(A_{\mu}^N)_c^{G/N}$. Applying [1, Theorem 4.6] again, we see that $(A_{\mu}^N)_{\mu}^{G/N}$ carries a G/N -coaction $\delta_{\mu}^{G/N}$ given by

$$\delta_{\mu}^{G/N}(n) = (j_G^N \otimes \text{id})(\omega_{G/N})(n \otimes 1)(j_G^N \otimes \text{id})(\omega_{G/N})^* \quad \text{for all } n \in (A_{\mu}^N)_c^{G/N},$$

where $j_G^N : C_0(G/N) \rightarrow \mathcal{M}(A_{\mu}^N)$ is the structural homomorphism induced from $j_G : C_0(G) \rightarrow \mathcal{M}(A)$. It is given by the equation $j_G^N(\mathbb{E}^{\tau, N}(f)) = \mathbb{E}^{\alpha, N}(j_G(f))$ for all $f \in C_c(G)$. We claim that $\delta_{\mu}^{G/N}$ corresponds to the restriction $\delta_{\mu}^G|$ of δ_{μ}^G to the quotient G/N via the isomorphism $A_{\mu}^G \cong (A_{\mu}^N)_{\mu}^{G/N}$. Notice that j_G^N , once composed with the canonical homomorphism $\kappa : A_{\mu}^N \rightarrow \mathcal{M}(A)$, coincides with the restriction of j_G to $C_0(G/N) \subseteq C_b(G)$, that is, $\kappa \circ j_G^N = j_G|_{C_0(G/N)}$. Then, if $q_N : C^*(G) \rightarrow C^*(G/N)$ is the quotient map, we get

$$(j_G \otimes q_N)(\omega_G) = (j_G \otimes \text{id})(\omega_{G/N}) = (\kappa \circ j_G^N \otimes \text{id})(\omega_{G/N}),$$

so that

$$\begin{aligned} \delta_{\mu}^G|(m) &= (\text{id} \otimes q) \circ \delta_{\mu}^G(m) = (\kappa \circ j_G^N \otimes \text{id})(\omega_{G/N})(m \otimes 1)(\kappa \circ j_G^N \otimes \text{id})(\omega_{G/N})^* \\ &= (\kappa \otimes \text{id})(\delta_{\mu}^{G/N}(m)), \end{aligned}$$

for $m \in A_c^G \subseteq (A_{\mu}^N)_c^{G/N}$, which proves the claim.

Now [1, Theorem 4.6] applied to the weak $G/N \rtimes G/N$ -algebra A_{μ}^N gives an isomorphism $A_{\mu}^N \cong (A_{\mu}^N)_{\mu}^{G/N} \rtimes_{\delta_{\mu}^{G/N}} \widehat{G/N}$ and if we combine this with the $\widehat{G/N}$ -isomorphisms $((A_{\mu}^N)_{\mu}^{G/N}, \delta_{\mu}^{G/N}) \cong (A_{\mu}^G, \delta_{\mu}^G) \cong (B_{\mu}, \delta_{\mu})$ and the fact that $B_{\mu} \rtimes_{\delta_{\mu}} \widehat{G} \cong B \rtimes_{\delta} \widehat{G}$ for $\mu = u, r$, we finally obtain a chain of isomorphisms

$$(B \rtimes_{\delta} \widehat{G})_{\mu}^N = A_{\mu}^N \cong (A_{\mu}^N)_{\mu}^{G/N} \rtimes_{\delta_{\mu}^{G/N}} \widehat{G/N} \cong A_{\mu}^G \rtimes_{\delta_{\mu}|} \widehat{G/N} \cong B_{\mu} \rtimes_{\delta_{\mu}|} \widehat{G/N}.$$

This finishes the proof. \square

In what follows next, we want to compare our module $\mathcal{F}_{\mu}^H(B \rtimes_{\delta} \widehat{G})$ with Mansfield's original construction in [22] which provides us with an explicit description of a dense submodule of the $(B \rtimes_{\delta} \widehat{G})_{\mu}^H - (B \rtimes_{\delta} \widehat{G}) \rtimes_{\mu} H$ bimodule $\mathcal{F}_{\mu}^H(B \rtimes_{\delta} \widehat{G})$ and a subalgebra \mathcal{D}^H sitting densely inside the fixed-point algebra with compact supports $(B \rtimes_{\delta} \widehat{G})_c^H$ with respect to any chosen norm μ on $C_c(H, B \rtimes_{\delta} \widehat{G})$ as above.

Notation 4.4 (cf. [22]). For a locally compact group G we let $B(G) \cong C^*(G)^*$ denote the Fourier-Stieltjes algebra and we denote by $A(G) \subseteq B(G)$ the Fourier algebra of G , i.e., the set of matrix coefficients of the regular representation λ_G of G . For $w \in B(G)$ we let $\delta_w : B \rightarrow B$ denote the composition $\delta_w(b) = (\text{id}_B \otimes w) \circ \delta(b)$.

Let $A_c(G) := A(G) \cap C_c(G) \subseteq B(G)$. For a compact subset $E \subseteq G$ we denote by $C_E(G)$ the set of functions $f \in C_c(G)$ with support in E . Recall that $\mathbb{E}^H : C_c(G) \rightarrow C_c(G/H)$ denotes the surjective linear map given by

$$\mathbb{E}^H(f)(gH) = \int_H f(gh) \, dh.$$

For $w \in A_c(G)$ and $E \subseteq G$ compact, let

$$\mathcal{D}_{w,E,H} := \overline{j_B(\delta_w(B))j_G(\mathbb{E}^H(C_E(G)))} \subseteq \mathcal{M}(B \rtimes_\delta \widehat{G})$$

and

$$\mathcal{D}_H := \bigcup \{ \mathcal{D}_{(w,E,H)} : w \in A_c(G), E \subseteq G \text{ compact} \}.$$

We call \mathcal{D}_H the H -Mansfield subalgebra of $\mathcal{M}(B \rtimes_\delta \widehat{G})$. We simply write \mathcal{D} in the case where $H = \{e\}$.

We should note for later use that the general assumption on our coactions being non-degenerate – in the sense that $\delta(B)(1_B \otimes C^*(G)) = B \otimes C^*(G)$ – implies that

$$(4.5) \quad B_c := \delta_{A_c(G)}(B) = \{ \delta_w(B) : w \in A_c(G) \}$$

is norm dense in B . This follows from [20, Theorem 5] together with the fact that $A_c(G)$ is weak- $*$ dense in $A(G)$. Mansfield shows the following:

Lemma 4.6. *Let (B, δ) be a coaction of G and let H be a closed subgroup of G . Then*

- (1) \mathcal{D}_H is a dense $*$ -subalgebra of $C^*(j_B(B)j_G(C_0(G/H))) \subseteq \mathcal{M}(B \rtimes_\delta \widehat{G})$ and we have $\mathcal{D}_H = j_G(C_c(G/H)) \cdot \mathcal{D}_H = \mathcal{D}_H \cdot j_G(C_c(G/H))$.
- (2) \mathcal{D} is a dense $*$ -subalgebra of $B \rtimes_\delta \widehat{G}$.

Proof. The first assertion in (1) follows from [22, Lemma 11] and the equation $\mathcal{D}_H = j_G(C_c(G/H)) \cdot \mathcal{D}_H = \mathcal{D}_H \cdot j_G(C_c(G/H))$ is a consequence of [22, Lemma 9] (see also [12, Lemma 3.2]). Note that Mansfield proved both lemmas in the full generality of arbitrary closed subgroups H of G . Item (2) follows from [22, Theorem 12] in the special case $H = \{e\}$. \square

In the special case where $H = G$ we get the $*$ -algebra $\mathcal{D}_G = \bigcup_{w \in A_c(G)} \overline{j_B(\delta_w(B))}$. This $*$ -algebra has an easier description:

Lemma 4.7. $B_c = \delta_{A_c(G)}(B)$ is a dense $*$ -subalgebra of B and $j_B : B_c \rightarrow \mathcal{D}_G$ is an isomorphism of $*$ -algebras. In particular, $\mathcal{D}_G = j_B(\delta_{A_c(G)}(B))$.

Proof. We already observed above that $B_c = \delta_{A_c(G)}(B)$ is dense in B and the fact that it is a $*$ -subalgebra of B follows from items (ii)-(iv) of [22, Lemma 1]. To see, for instance, that B_c is a vector subspace, observe that, by [22, Lemma 1(ii)], if $b_1, b_2 \in B$, $v_1, v_2 \in A_c(G)$ and $w \in A_c(G)$ is such that $w = 1$ on $\text{supp}(v_1) \cup \text{supp}(v_2)$, then $\delta_{v_1}(a_1) + \delta_{v_2}(a_2) = \delta_{wv_1}(a_1) + \delta_{wv_2}(a_2) = \delta_w(\delta_{v_1}(a_1) + \delta_{v_2}(a_2))$. Items (iii) and (iv) in [22, Lemma 1] imply that B_c is a $*$ -subalgebra of B .

To show that j_B gives a $*$ -isomorphism $B_c \xrightarrow{\sim} \mathcal{D}_G$, we first show that it is surjective, that is, $j_B(\delta_{A_c(G)}(B)) = \mathcal{D}_G$. For this assume that $w \in A_c(G)$ is fixed and that $(b_n)_n$ is a sequence in B and $b \in B$ such that $j_B(\delta_w(b_n)) \rightarrow j_B(b)$. It suffices to show that $j_B(b) = j_B(\delta_v(b))$ for some $v \in A_c(G)$.

We first note that $I = \ker j_B$ is annihilated by $\delta_w : B \rightarrow B$ for any $w \in A(G)$. This follows from the fact that the kernel $\ker \lambda_G \subseteq C^*(G)$ is annihilated by the elements

in $A(G)$ viewed as linear functionals of $C^*(G)$. Indeed, if we realize $B \rtimes_{\delta} \widehat{G}$ as a subalgebra of $\mathcal{M}(B \otimes \mathcal{K}(L^2 G))$ via the covariant representation $((\text{id} \otimes \lambda_G) \circ \delta, 1 \otimes M)$, we see that $\ker j_B = \ker(\text{id}_B \otimes \lambda_G) \circ \delta$. Moreover, since for $w \in A(G)$ the linear functional on $C^*(G)$ associated to w factors through a functional w_r of $C_r^*(G)$, we see that δ_w is given by the composition

$$\delta_w = (\text{id}_B \otimes w_r) \circ (\text{id}_B \otimes \lambda_G) \circ \delta.$$

Hence $\ker j_B = \ker(\text{id}_B \otimes \lambda_G) \circ \delta \subseteq \ker \delta_w$. Suppose now that $j_B(\delta_w(b_n)) \rightarrow j_B(b)$. By passing to a subsequence, if necessary, we may choose elements $c_n \in \ker j_B$ with $\delta_w(b_n) + c_n \rightarrow b$ in B . Now let $v \in A_c(G)$ such that $v \equiv 1$ on $\text{supp } w$. Then

$$\delta_w(b_n) = \delta_{vw}(b_n) = \delta_v(\delta_w(b)) = \delta_v(\delta_w(b_n) + c_n) \rightarrow \delta_v(b),$$

and hence $j_B(\delta_w(b_n)) \rightarrow j_B(\delta_v(b))$, which proves that $j_B(b) = j_B(\delta_v(b)) \in j_B(B_c)$, hence $j_B(B_c) = \mathcal{D}_G$.

We now use item (ii) of [22, Lemma 1], that is, the fact that $\delta_{wv}(b) = \delta_w(\delta_v(b))$ for all $w, v \in A_c(G)$, to show that $j_B : B_c \rightarrow \mathcal{D}_G$ is injective and hence an isomorphism of $*$ -algebras. For this assume that $j_B(\delta_w(b)) = 0$ for some $w \in A_c(G)$ and $b \in B$. Let $v \in A_c(G)$ such that $vw = w$. Since $\ker j_B \subseteq \ker \delta_v$, we get $0 = \delta_v(\delta_w(b)) = \delta_{vw}(b) = \delta_w(b)$, and the result follows. \square

For later use, we should also note that \mathcal{D}_H is a bimodule over $B_c \cong \mathcal{D}_G$ with bimodule operations given by the usual multiplication inside $\mathcal{M}(B \rtimes_{\delta} \widehat{G})$, that is,

$$(4.8) \quad \delta_w(b) \cdot d = j_B(\delta_w(b))d \quad \text{and} \quad d \cdot \delta_w(b) = dj_B(\delta_w(b)),$$

for $d \in \mathcal{D}_H$. This gives a canonical imbedding of $B_c = \delta_{A_c(G)}(B)$ into the (algebraic) multiplier algebra $\mathcal{M}(\mathcal{D}_H)$.

Recall from [1] that for any weakly proper $G \rtimes H$ -algebra A and for any given crossed-product norm $\|\cdot\|_{\mu}$ on $C_c(H, A)$, the A_{μ}^H - $A \rtimes_{\mu} H$ -imprimitivity bimodule $\mathcal{F}_{\mu}(A)$ is given as the completion of the pre-imprimitivity A_c^H - $C_c(H, A)$ bimodule $\mathcal{F}_c(A) := C_c(G) \cdot A$. Moreover, we showed in [1, Lemma 2.12] that all bimodule operations are continuous with respect to the inductive limit topologies on A_c^H , $C_c(H, A)$ and $\mathcal{F}_c(A)$, respectively, and that in all three spaces, inductive limit convergence implies norm-convergence in their respective completions for any chosen norm $\|\cdot\|_{\mu}$. Similarly, the canonical “conditional expectation”

$$\mathbb{E}^H : A_c \rightarrow A_c^H, \quad \mathbb{E}^H(x) = \int_H^{st} \alpha_t(x) \, dt,$$

is inductive limit continuous on $A_c = C_c(G) \cdot A \cdot C_c(G)$ and we have $A_c^H = \mathbb{E}^H(A_c)$. Recall also that the inductive limit topology on $C_c(H, A)$ is the usual one, and that a net $(a_i)_{i \in I}$ in $\mathcal{F}_c(A)$ (resp. in A_c) converges in the inductive limit topology to some a if it converges to a in norm and there exists an $f \in C_c(G)$ such that $a_i = f \cdot a_i$ (resp. $a_i = f \cdot a_i \cdot f$) for all $i \in I$. Similarly, a net $(b_i)_{i \in I}$ in A_c^H converges to $b \in A_c^H$ in the inductive limit topology, if it converges in $\mathcal{M}(A)$ in norm and if the following are satisfied:

- (1) there exists a $\psi \in C_c(G/H)$ such that $\psi \cdot b_i \cdot \psi = b_i$ for all $i \in I$, and
- (2) for all $f \in C_c(G)$ the net $b_i \cdot f$ converges to $b \cdot f$ in the inductive limit topology of A_c

(recall that $b \cdot f \in A_c$ for all $b \in A_c^H$ and $f \in C_c(G)$, where the multiplication is performed inside $\mathcal{M}(A)$).

Recall that we always use the notation $f \cdot a$ for $\phi(f)a$ if $\phi : C_0(G) \rightarrow \mathcal{M}(A)$ is the given structure map. In the case where $A = B \rtimes_{\delta} \widehat{G}$, this structure map is given by the canonical map $j_G : C_0(G) \rightarrow \mathcal{M}(B \rtimes_{\delta} \widehat{G})$. Thus we get the following:

Lemma 4.9. *Let (B, δ) be a coaction of G and let H be a closed subgroup of G . Then*

- (1) \mathcal{D} is inductive limit dense in both $(B \rtimes_{\delta} \widehat{G})_c$ and $\mathcal{F}_c(B \rtimes_{\delta} \widehat{G})$.
- (2) \mathcal{D}_H is inductive limit dense in $(B \rtimes_{\delta} \widehat{G})_c^H$.

In particular, every generalized fixed-point algebra $(B \rtimes_{\delta} \widehat{G})_{\mu}^H$ is a norm completion of \mathcal{D}_H for some suitable norm.

Proof. Since \mathcal{D} is norm dense in $B \rtimes_{\delta} \widehat{G}$ it follows that $\mathcal{D} = j_G(C_c(G)) \cdot \mathcal{D} \cdot j_G(C_c(G))$ is inductive limit dense in $j_G(C_c(G)) \cdot (B \rtimes_{\delta} \widehat{G}) \cdot j_G(C_c(G)) = (B \rtimes_{\delta} \widehat{G})_c$ and a similar argument shows density in $\mathcal{F}_c(B \rtimes_{\delta} \widehat{G})$. To check that \mathcal{D}_H is a subalgebra of $(B \rtimes_{\delta} \widehat{G})_c^H$, we first observe that \mathcal{D}_H lies in the (classical) fixed-point algebra $\mathcal{M}(B \rtimes_{\delta} \widehat{G})^H$. Moreover, multiplying $\mathcal{D}_H = j_G(C_c(G/H)) \cdot \mathcal{D}_H \cdot j_G(C_c(G/H))$ with $j_G(f)$ for some $f \in C_c(G)$ from either side gives an element in $\mathcal{D} \subseteq (B \rtimes_{\delta} \widehat{G})_c$. Thus it follows easily from the definition of the fixed-point algebra with compact supports as given in (1.1) that $\mathcal{D}_H \subseteq (B \rtimes_{\delta} \widehat{G})_c^H$. Since

$${}_{(B \rtimes_{\delta} \widehat{G})_c^H} \langle\langle \mathcal{D} | \mathcal{D} \rangle\rangle \subseteq \mathbb{E}(\mathcal{D}) \subseteq \mathcal{D}_H$$

and since ${}_{(B \rtimes_{\delta} \widehat{G})_c^H} \langle\langle \mathcal{D} | \mathcal{D} \rangle\rangle$ is inductive limit dense in $(B \rtimes_{\delta} \widehat{G})_c^H$ due to the facts that \mathcal{D} is inductive limit dense in $(B \rtimes_{\delta} \widehat{G})_c$ and that every element in $(B \rtimes_{\delta} \widehat{G})_c^H$ can be written as an inner product of two elements in $(B \rtimes_{\delta} \widehat{G})_c$, it follows that \mathcal{D}_H is inductive limit dense in $(B \rtimes_{\delta} \widehat{G})_c^H$. The final assertion now follows from the fact that inductive limit convergence implies norm convergence in $(B \rtimes_{\delta} \widehat{G})_{\mu}^H$ with respect to any given crossed-product norm $\|\cdot\|_{\mu}$ on $C_c(H, B \rtimes_{\delta} \widehat{G})$. \square

Remark 4.10. In the case $H = G$, the above result shows that for any coaction (B, δ) of G , the dense *-subalgebra $B_c = \delta_{A_c(G)}(B)$ of B maps faithfully onto the inductive limit dense *-subalgebra $j_B(B_c) = \mathcal{D}_G$ of $(B \rtimes_{\delta} \widehat{G})_c^G$. It follows then from the above lemma that for a given norm $\|\cdot\|_{\mu}$ on $C_c(G, B \rtimes_{\delta} \widehat{G})$ the μ -fixed-point algebra $B_{\mu} := (B \rtimes_{\delta} \widehat{G})_{\mu}^G$ can be obtained as a completion of $B_c \cong \mathcal{D}_G$ with respect to a suitable norm induced from $\|\cdot\|_{\mu}$ via the bimodule $\mathcal{F}_{\mu}(B \rtimes_{\delta} \widehat{G})$. In this picture, the canonical epimorphisms

$$B_u \twoheadrightarrow B \twoheadrightarrow B_r,$$

with $B_u := (B \rtimes_{\delta} \widehat{G})_u^G$ and $B_r := (B \rtimes_{\delta} \widehat{G})_r^G$, respectively, are given by the identity map on B_c and it is easy to check that these maps are \widehat{G} -equivariant with respect to the coactions δ_u, δ , and δ_r , respectively (use $(B, \delta) \cong (B_{\mu}, \delta_{\mu})$ for a suitable crossed-product norm $\|\cdot\|_{\mu}$ and formula (4.3)). This gives a very concrete picture for the maximalization (B_u, δ_u) and the normalization (B_r, δ_r) of (B, δ) and their connections to the given coaction (B, δ) .

Suppose now that $H = N$ is a closed *normal* subgroup of G , and δ is an arbitrary coaction of G on B . Consider the representation $j_B \rtimes j_G : B \rtimes_{\delta} \widehat{G/N} \rightarrow \mathcal{M}(B \rtimes_{\delta} \widehat{G})$. It is clear that it maps the dense subset $i_B(B_c) i_{G/N}(C_c(G/N))$ of $B \rtimes_{\delta} \widehat{G/N}$ onto the dense subspace $j_B(B_c) j_G(C_c(G/N))$ of \mathcal{D}_N (where $(i_B, i_{G/N})$ denotes the

canonical covariant representation of $(B, \delta|)$ into $\mathcal{M}(B \rtimes_{\delta|} \widehat{G/N})$, which implies that $j_B \rtimes j_G|$ maps $B \rtimes_{\delta|} \widehat{G/N}$ onto the closure $(B \rtimes_{\delta} \widehat{G})_r^N$ of D_N in $\mathcal{M}(B \rtimes_{\delta} \widehat{G})$, that is,

$$\text{Im}(j_B \rtimes j_G|) \cong (B \rtimes_{\delta} \widehat{G})_r^N.$$

Moreover, by Lemma 3.1 in [17], $j_B \rtimes j_G|$ is faithful if and only if $\ker(j_B) \subseteq \ker(i_B)$ (this also follows from our Lemma 5.11). In particular, if δ is a normal coaction, i.e., if $j_B : B \rightarrow \mathcal{M}(B \rtimes_{\delta} \widehat{G})$ is injective, then $j_B \rtimes j_G|$ can be viewed as an isomorphism:

$$B \rtimes_{\delta|} \widehat{G/N} \xrightarrow{\sim} (B \rtimes_{\delta} \widehat{G})_r^N.$$

This gives an alternative (and probably more concrete) description of the isomorphism $B \rtimes_{\delta|} \widehat{G/N} \cong (B \rtimes_{\delta} \widehat{G})_r^N$ of Theorem 4.1 for normal coactions.

In the case of maximal coactions $\delta = \delta_u$, there are canonical *-homomorphisms

$$l_B : B \rightarrow \mathcal{M}((B \rtimes_{\delta} \widehat{G})_u^N) \quad \text{and} \quad l_{G/N} : C_0(G/N) \rightarrow \mathcal{M}((B \rtimes_{\delta} \widehat{G})_u^N)$$

in which for $b \in B_c$, the element $l_B(b)$ acts on the dense subalgebra \mathcal{D}_N via multiplication inside $\mathcal{M}(B \rtimes_{\delta} \widehat{G})$ (see equation (4.8)). Similarly, if $f \in C_c(G/N)$, then $l_{G/N}(f)$ is determined via the obvious left and right actions of $j_G(f)$ on \mathcal{D}_N . The following corollary is then a straightforward consequence of Theorem 4.1.

Corollary 4.11. *Let (B, δ) be a maximal coaction of G and let N be a normal subgroup of G . Then there is a unique covariant homomorphism $(l_B, l_{G/N})$ of $(B, \delta|)$ into $\mathcal{M}((B \rtimes_{\delta} \widehat{G})_u^N)$ given on B_c and $C_c(G/N)$ as above such that the integrated form $l_B \rtimes l_{G/N}$ implements the isomorphism*

$$l_B \rtimes l_{G/N} : B \rtimes_{\delta|} \widehat{G/N} \rightarrow (B \rtimes_{\delta} \widehat{G})_u^N$$

of Theorem 4.1.

Proof. In the notation of Theorem 4.1, we have $B = B_u$ and $A = B \rtimes_{\delta} \widehat{G}$. If we follow the arguments given in the proof of that theorem we see that the isomorphism $B \rtimes_{\delta|} \widehat{G/N} \cong (B \rtimes_{\delta} \widehat{G})_u^N$ is given on the level of $(B \rtimes_{\delta} \widehat{G})_c^N \subseteq \mathcal{M}(B \rtimes_{\delta} \widehat{G})$ by sending $B_c \subseteq (B \rtimes_{\delta} \widehat{G})_c^G$ to $j_B(B_c) \subseteq \mathcal{M}(B \rtimes_{\delta} \widehat{G})$ and $C_c(G/N)$ to $j_G|(C_c(G/N)) \subseteq \mathcal{M}(B \rtimes_{\delta} \widehat{G})$. But this means that it coincides on these dense subspaces of B and $C_0(G/N)$ with l_B and $l_{G/N}$, which implies the result. \square

Of course, the results presented in this section also provide versions of Mansfield's imprimitivity theorems for “crossed products by homogeneous spaces” as considered in [6, 12]: if δ is a coaction of G and H is a closed subgroup of G , then the *reduced crossed product* $B \rtimes_{\delta, r} \widehat{G/H}$ of B by the homogeneous space G/H is defined in [6] as

$$B \rtimes_{\delta, r} \widehat{G/H} = \overline{j_B(B)j_G(C_0(G/H))} \subseteq \mathcal{M}(B \rtimes_{\delta} \widehat{G}).$$

Since the reduced fixed-point algebra is the closure of $(B \rtimes_{\delta} \widehat{G})_c^H$ inside $\mathcal{M}(B \rtimes_{\delta} \widehat{G})$, which coincides with the closure of \mathcal{D}_H by Lemma 4.9, we see that

$$B \rtimes_{\delta, r} \widehat{G/H} \cong (B \rtimes_{\delta} \widehat{G})_r^H$$

and our results provide us with an imprimitivity bimodule $\mathcal{F}_r^H(B \rtimes_{\delta} \widehat{G})$ between $B \rtimes_{\delta, r} \widehat{G/H}$ and $(B \rtimes_{\delta} \widehat{G}) \rtimes_r H$. This coincides with the one obtained in [12]. On the other extreme, it makes perfect sense to *define* the *universal crossed product*

$B \rtimes_{\delta,u} \widehat{G/H}$ of B by the homogeneous space G/H as the universal fixed-point algebra

$$(4.12) \quad B \rtimes_{\delta,u} \widehat{G/H} := (B \rtimes_{\delta} \widehat{G})_u^H,$$

which then provides us with the Morita equivalence $\mathcal{F}_u^H(B \rtimes_{\delta} \widehat{G})$ between $B \rtimes_{\delta,u} \widehat{G/H}$ and $(B \rtimes_{\delta} \widehat{G}) \rtimes_u H$. Note that for normal N and arbitrary coactions δ we get isomorphisms

$$B \rtimes_{\delta,r} \widehat{G/N} \cong B_r \rtimes_{\delta_r|} \widehat{G/N} \quad \text{and} \quad B \rtimes_{\delta,u} \widehat{G/N} \cong B_u \rtimes_{\delta_u|} \widehat{G/N}$$

but it is important here to use the normalization (B_r, δ_r) in the first isomorphism and the maximalization (B_u, δ_u) in the second, since in general we do not have isomorphisms between $B_u \rtimes_{\delta_u|} \widehat{G/N}$ and $B_r \rtimes_{\delta_r|} \widehat{G/N}$ (e.g., take $N = G$, in which case we obtain the algebras B_u and B_r , which are often different if G is not amenable).

As far as we know, there was no general definition of the universal crossed product by homogeneous spaces as in (4.12) before. However, such crossed products have been defined in the special case of dual coactions, *i.e.*, in the case $B = A \rtimes_{\alpha} G$ with dual coaction $\delta = \widehat{\alpha}$ for some G -algebra (A, α) . In this situation the crossed product $B \rtimes_{\delta,u} \widehat{G/H}$ has been defined in [6] as the crossed product $C_0(G/H, A) \rtimes_{\tau \otimes \alpha} G$, where here τ denotes left translation of G on G/H (see the discussion before Lemma 2.4 in [6]). Let us now check that both definitions agree in this case. By the Imai-Takai Duality Theorem ([8, Theorem A.67]), we have a canonical isomorphism of weak $G \rtimes G$ -algebras:

$$B \rtimes_{\delta} \widehat{G} \cong A \otimes \mathcal{K}(L^2 G)$$

where $A \otimes \mathcal{K}(L^2 G)$ is endowed with the G -action $\alpha \otimes \text{Ad}_\rho$ and the structure map $1 \otimes M: C_0(G) \rightarrow \mathcal{M}(A \otimes \mathcal{K}(L^2 G))$. But then Proposition 5.7 in [2] shows that

$$(B \rtimes_{\delta} \widehat{G})_u^H \cong (A \otimes \mathcal{K}(L^2 G))_u^H \cong \text{Ind}_H^G(A) \rtimes_{\text{Ind}_\alpha} G \cong C_0(G/H, A) \rtimes_{\tau \otimes \alpha} G.$$

5. TWISTED LANDSTAD DUALITY

Let G be a locally compact group and N a closed normal subgroup of G . In this section we are going to study weak $G \rtimes N$ -algebras, that is, C^* -algebras A endowed with an N -action α of N and an N -equivariant non-degenerate $*$ -homomorphism $\phi: C_0(G) \rightarrow \mathcal{M}(A)$, where $C_0(G)$ is endowed with right translation action of N . Since this action is free and proper, the corresponding Hilbert $A \rtimes_{\alpha,\mu} N$ -module $\mathcal{F}_\mu(A)$ implements a Morita equivalence between $A_\mu^N \cong \mathcal{K}(\mathcal{F}_\mu(A))$ and $A \rtimes_{\alpha,\mu} N$ for every crossed-product norm $\|\cdot\|_\mu$ on $C_c(N, A)$. Assume now that $\|\cdot\|_\mu$ is a norm for which the dual N -coaction $\widehat{\alpha}$ on $A \rtimes_{\alpha,u} N$ factors through a coaction $\widehat{\alpha}_\mu$ on $A \rtimes_{\alpha,\mu} N$. By Lemma 4.12 in [1], we know that $\mathcal{F}_\mu^N(A)$ carries a G -coaction $\delta_{\mathcal{F}}$ given by:

$$(5.1) \quad \delta_{\mathcal{F}}(\xi) = (\phi \otimes \text{id})(w_G)(\xi \otimes 1) \quad \text{for all } \xi \in \mathcal{F}_c(A) = \phi(C_c(G))A$$

which implements a Morita equivalence between the inflation $\text{Inf } \widehat{\alpha}_\mu$ of the dual coaction $\widehat{\alpha}_\mu$ on $A \rtimes_{\alpha,\mu} N$, and the G -coaction δ_μ^N on A_μ^N induced by $\delta_{\mathcal{F}}$ which is given by:

$$(5.2) \quad \delta_\mu^N(m) = (\phi \otimes \text{id})(\omega_G)(m \otimes 1)(\phi \otimes \text{id})(\omega_G)^* \quad \text{for all } m \in A_c^N$$

(see [1, Remark 4.14]). Our first goal is to show that δ_μ is a twisted coaction in the following sense.

Definition 5.3. A twisted coaction of $(G, G/N)$ on a C^* -algebra B is a pair (δ, ω) consisting of a (non-degenerate) G -coaction $\delta: B \rightarrow \mathcal{M}(B \otimes C^*(G))$ of G on B and a *twisting unitary* over G/N , meaning a unitary multiplier $\omega \in \mathcal{UM}(B \otimes C^*(G/N))$ satisfying:

- (i) $(\omega \otimes 1)(\text{id} \otimes \sigma_{G/N, G/N})(\omega \otimes 1) = (\text{id} \otimes \delta_{G/N})(\omega)$;
- (ii) $(\delta \otimes \text{id}_{G/N})(\omega) = (\text{id}_A \otimes \sigma_{G/N, G})(\omega \otimes 1)$; and
- (iii) $\delta|(b) = \omega(b \otimes 1)\omega^*$ for all $b \in B$,

where $\sigma_{G/N, G/N}$ and $\sigma_{G/N, G}$ denote the flip isomorphisms on $C^*(G/N) \otimes C^*(G/N)$ and $C^*(G/N) \otimes C^*(G)$, respectively, and $\delta| := (\text{id} \otimes q_N) \circ \delta$ denotes the restriction of δ to G/N , where $q_N: C^*(G) \rightarrow C^*(G/N)$ is the quotient map.

Equivalently (see [26]), a twisted coaction can be defined as a pair (δ, ς) consisting of a coaction $\delta: B \rightarrow \mathcal{M}(B \otimes C^*(G))$ and a non-degenerate $*$ -homomorphism $\varsigma: C_0(G/N) \rightarrow \mathcal{M}(B)$ satisfying:

- (1) (ι, ς) is a covariant representation of $(B, \delta|)$ into $\mathcal{M}(B)$, where $\iota: B \rightarrow \mathcal{M}(B)$ denotes the inclusion map, that is,

$$\delta|(b) = (\varsigma \otimes \text{id})(\omega_{G/N})(b \otimes 1)(\varsigma \otimes \text{id})(\omega_{G/N}^*) \quad \text{for all } b \in B; \text{ and}$$

- (2) $\delta(\varsigma(f)) = \varsigma(f) \otimes 1$ for all $f \in C_0(G/N)$.

In this case, ς is called the *twisting homomorphism* for (B, δ) .

If the twisting homomorphism ς is given, the unitary twist ω can be recovered from ς by $\omega = (\varsigma \otimes \text{id})(\omega_{G/N})$. Conversely, every unitary twist is of this form by [26, Lemma A.1], and in this case we say that ς is the twisting homomorphism associated to ω or that ω is the twisting unitary associated to ς . We refer to [9, 23, 26] for further information on twisted coactions.

To simplify the writing, we shall use standard leg numbering notation like $\omega_{12} := \omega \otimes 1$, $\omega_{23} := 1 \otimes \omega$ and $\omega_{13} = (\text{id} \otimes \sigma)(\omega)$, where σ is some suitable flip automorphism (like $\sigma_{G/N, G/N}$ or $\sigma_{G/N, G}$ as above). With this notation, the two first conditions in the above definition can be reformulated as:

$$(i) \ \omega_{12}\omega_{13} = (\text{id} \otimes \delta_{G/N})(\omega) \quad \text{and} \quad (ii) \ (\delta \otimes \text{id}_{G/N})(\omega) = \omega_{13}.$$

The first condition can be interpreted by saying that $\omega \in \mathcal{UM}(B \otimes C^*(G/N))$ is a corepresentation of G/N on B . Observe that, in this case, if $\psi: B \rightarrow \mathcal{M}(C)$ is a $*$ -homomorphism, then $(\psi \otimes \text{id})(\omega)$ is a corepresentation of G/N on C .

Definition 5.4. Let (B, δ, ω) be a twisted coaction of $(G, G/N)$. We say that a covariant representation (π, σ) of (B, δ) *preserves the twist* if

$$(5.5) \quad (\sigma \otimes \text{id}_{G/N})(\omega_{G/N}) = (\pi \otimes \text{id}_{G/N})(\omega).$$

In this case we also say that (π, σ) is a covariant representation of (B, δ, ω) .

A *twisted crossed product* for (B, δ, ω) is a C^* -algebra C endowed with a covariant representation (k_B, k_G) of (B, G, ω) into $\mathcal{M}(C)$ such that $k_B(B)k_G(C_0(G))$ is linearly dense in C and such that for every other twisted covariant representation (π, σ) into $\mathcal{M}(D)$ there exists a unique non-degenerate representation $\pi \rtimes_\omega \sigma: C \rightarrow \mathcal{M}(D)$ such that $\pi = (\pi \rtimes_\omega \sigma) \circ k_B$ and $\sigma = (\pi \rtimes_\omega \sigma) \circ k_G$.

Remark 5.6. If $\varsigma: C_0(G/N) \rightarrow \mathcal{M}(B)$ is the twisting homomorphism associated to ω , then a covariant representation (π, σ) preserves the twist if and only if $\sigma|_{C_0(G/N)} = \pi \circ \varsigma$ (see [23, Remark 2.6]).

Every twisted coaction admits a twisted crossed product which is uniquely determined up to isomorphism and denoted by $B \rtimes_{\delta, \omega} \widehat{G}$. If $B \rtimes_{\delta} \widehat{G}$ denotes the (untwisted) crossed product for (B, δ) and (j_B, j_G) is its universal covariant representation, then $B \rtimes_{\delta, \omega} \widehat{G}$ can be realized as the quotient $B \rtimes_{\delta, \omega} \widehat{G} = (B \rtimes_{\delta} \widehat{G})/I_{\omega}$, where

$$I_{\omega} = \bigcap \{ \ker(\pi \rtimes \sigma) : (\pi, \sigma) \text{ is a covariant representation preserving the twist} \}$$

is the *twisting ideal*. In this picture, the universal covariant representation (k_B, k_G) is just the composition $(q_{\omega} \circ j_B, q_{\omega} \circ j_G)$, where $q_{\omega}: B \rtimes_{\delta} \widehat{G} \rightarrow B \rtimes_{\delta} \widehat{G}/I_{\omega}$ is the quotient map and $\pi \rtimes_{\omega} \sigma$ is the unique factorization of $\pi \rtimes \sigma$ through $(B \rtimes_{\delta} \widehat{G})/I_{\omega}$.

Example 5.7. (1) The inflation $\text{Inf } \delta$ of an N -coaction $\delta: B \rightarrow \mathcal{M}(B \otimes C^*(N))$ is a *trivially twisted coaction* over G/N , that is, a twisted coaction of $(G, G/N)$ with respect to the trivial unitary twist $\omega = 1$ (this corresponds to the *trivial twisting homomorphism* $\varsigma: C_0(G/N) \rightarrow \mathcal{M}(B)$ defined by $\varsigma(f) = f(eN)1_B$ for all $f \in C_0(G/N)$). The twisted crossed product $B \rtimes_{\text{Inf } \delta, 1} \widehat{G}$ is canonically isomorphic to the original (untwisted) crossed product $B \rtimes_{\delta} \widehat{N}$ (see [9, 23]).

(2) Given an arbitrary coaction $\delta: B \rightarrow \mathcal{M}(B \otimes C^*(G))$ of G on B , the restricted crossed product $B \rtimes_{\delta|} \widehat{G/N}$ carries a canonical twisted $(G, G/N)$ -coaction $(\tilde{\delta}, \tilde{\omega})$: the coaction $\tilde{\delta}$ is the integrated form $\pi \rtimes \sigma$ of the covariant representation

$$(\pi, \sigma) = ((j_B \otimes \text{id}) \circ \delta, j_{G/N} \otimes 1),$$

where $(j_B, j_{G/N})$ denotes the universal covariant representation of $(B, \delta|)$ and the twisting homomorphism for $\tilde{\delta}$ is $j_{G/N}$, hence $\tilde{\omega} = (j_{G/N} \otimes \text{id})(\omega_{G/N})$. Moreover, there is a canonical isomorphism (see [23] and also Remark 7.12 in [26]):

$$(B \rtimes_{\delta|} \widehat{G/N}) \rtimes_{\tilde{\delta}, \tilde{\omega}} \widehat{G} \cong B \rtimes_{\delta} \widehat{G}.$$

In Corollary 5.25 below we derive this decomposition isomorphism also as a consequence of our results.

If ω is a twisting unitary over G/N for (B, δ) , then the twisting ideal I_{ω} is N -invariant with respect to the dual action $\widehat{\delta}$, so that $\widehat{\delta}$ induces an N -action on the twisted crossed product $B \rtimes_{\delta, \omega} \widehat{G}$, which we denote by $\widehat{\delta}^{\omega}$. If (k_B, k_G) denotes the universal twisted covariant representation, the homomorphism $k_G: C_0(G) \rightarrow \mathcal{M}(B \rtimes_{\delta, \omega} \widehat{G})$ is N -equivariant with respect to the right translation action of N on G , and hence $B \rtimes_{\delta, \omega} \widehat{G}$ carries a canonical structure as a weakly proper $G \rtimes N$ -algebra.

The following result is well known. In [26, Theorem 4.4] it is shown for normal *amenable* subgroups and *reduced coactions*, that is, injective non-degenerate coactions of $C_r^*(G)$. But it is pointed out in the proof of [17, Theorem 4.3] that the proof of [26, Theorem 4.4] extends to arbitrary (*i.e.*, also non-amenable) normal subgroups if one replaces reduced coactions by *full* coactions of $C^*(G)$ as we are using here.

Proposition 5.8 (Quigg-Raeburn). *For a twisted $(G, G/N)$ -coaction (B, δ, ω) , there is a G -equivariant isomorphism $\chi: B \rtimes_{\delta} \widehat{G} \xrightarrow{\sim} \text{Ind}_N^G(B \rtimes_{\delta, \omega} \widehat{G})$ sending $x \in B \rtimes_{\delta} \widehat{G}$ to the function $t \mapsto (k_B \rtimes k_G)(\widehat{\delta}_{t^{-1}}(x)) \in \text{Ind}_N^G(B \rtimes_{\delta, \omega} \widehat{G})$.*

Remark 5.9. It is clear that the extension of χ to the multiplier algebra $\mathcal{M}(B \rtimes_{\delta} \widehat{G})$ sends $j_B(b)$ to the constant function $\tilde{k}_B(b) = (t \mapsto k_B(b)) \in \mathcal{M}(\text{Ind}_N^G(B \rtimes_{\delta, \omega} \widehat{G}))$ and

$j_G(f)$ to the element $\tilde{k}_G(f) \in \mathcal{M}(\text{Ind}_N^G(B \rtimes_{\delta, \omega} \widehat{G}))$ determined by the function $t \mapsto k_G(\tau_{t^{-1}}(f)) \in \mathcal{M}(B \rtimes_{\delta, \omega} \widehat{G})$ with $\tau_t(f)|_s = f(st)$. Hence, χ becomes an isomorphism between the weak $G \rtimes G$ -algebras $(B \rtimes_{\delta} \widehat{G}, j_G, \widehat{\delta})$ and $(\text{Ind}_N^G(B \rtimes_{\delta, \omega} \widehat{G}), \tilde{k}_G, \text{Ind } \widehat{\delta}^\omega)$.

Observe that the above proposition implies, in particular, that there is a central homomorphism $\psi: C_0(G/N) \rightarrow \mathcal{ZM}(B \rtimes_{\delta} \widehat{G})$ corresponding to the canonical homomorphism $f \mapsto f \otimes 1$ from $C_0(G/N)$ into $\mathcal{ZM}(\text{Ind}_N^G(B \rtimes_{\delta, \omega} \widehat{G}))$. A formula for ψ is given in [26, Theorem 4.4]: it is a certain convolution of $j_G|_{C_0(G/N)}$ and $j_B \circ \varsigma$, where ς is the twisting homomorphism $C_0(G/N) \rightarrow \mathcal{M}(B)$ associated to ω .

An easy consequence of the above proposition is the following:

Corollary 5.10. *For a twisted coaction (B, δ, ω) , the (untwisted) coaction (B, δ) is normal if and only if k_B is injective. Moreover, we have $\ker(j_B) = \ker(k_B)$.*

Proof. Recall that δ is normal if and only if $j_B: B \rightarrow \mathcal{M}(B \rtimes_{\delta} \widehat{G})$ is injective. Thus it is enough to prove the final assertion. Let $q := k_B \rtimes k_G: B \rtimes_{\delta} \widehat{G} \rightarrow B \rtimes_{\delta, \omega} \widehat{G}$ denote the canonical surjection and let $\chi: B \rtimes_{\delta} \widehat{G} \xrightarrow{\sim} \text{Ind}_N^G(B \rtimes_{\delta, \omega} \widehat{G})$ be the isomorphism of Proposition 5.8. Then, for all $b \in B$, since $j_B(b)$ is $\widehat{\delta}$ -invariant, we have

$$\chi(j_B(b))|_t = q(\widehat{\delta}_{t^{-1}}(j_B(b))) = q(j_B(b)) = k_B(b).$$

This implies $\ker(j_B) = \ker(k_B)$ because χ is injective. \square

The following result appears as Corollary 4.10 in [26] where, again, amenability of N is required due to the use of reduced coactions. However, using Proposition 5.8 above, the same proof as given in [26] applies to full normal coactions and non-amenable N .

Lemma 5.11 (Quigg-Raeburn). *Let (π, σ) be a covariant representation of a twisted $(G, G/N)$ -coaction (B, δ, ω) into $\mathcal{M}(D)$. Assume that (B, δ) is normal. Then $\pi \rtimes_{\omega} \sigma$ is faithful if and only if π is faithful and there is an action of N on the image of $\pi \rtimes_{\omega} \sigma$ making σ into an N -equivariant homomorphism. (The last condition is equivalent to saying that $\ker(\pi \rtimes_{\omega} \sigma)$ is an N -invariant ideal in $B \rtimes_{\delta, \omega} \widehat{G}$.)*

Suppose that (B, δ, ω) is a twisted $(G, G/N)$ -coaction and let (B_r, δ_r) be the normalization of (B, δ) . Then it is well known that $B \rtimes_{\delta} \widehat{G} \cong B_r \rtimes_{\delta_r} \widehat{G}$. In particular, covariant representations of (B, δ) correspond bijectively to covariant representations of (B_r, δ_r) . This correspondence can be described as follows: recall that (B_r, δ_r) can be realized as $B_r = j_B(B) \cong B / \ker j_B$ and δ_r is given on $j_B(B)$ by conjugation with the unitary $(j_G \otimes \text{id})(\omega_G)$. If (π, σ) is a covariant representation of (B, δ) , the equation $(\pi \rtimes \sigma) \circ j_B = \pi$ implies that $\ker(j_B) \subseteq \ker(\pi)$, so that π factors through a homomorphism π_r of B_r and the pair (π_r, σ) is a covariant representation of (B_r, δ_r) . The assignment $(\pi, \sigma) \mapsto (\pi_r, \sigma)$ is then a bijective correspondence between covariant representations of (B, δ) and (B_r, δ_r) .

Moreover, if $\varrho: B \twoheadrightarrow B_r$ denotes the quotient map, the unitary twist ω for δ induces a twist $\omega_r := (\varrho \otimes \text{id})(\omega)$ for δ_r , and a covariant representation (π, σ) of (B, δ) preserves the twist ω if and only if (π_r, σ) preserves the twist ω_r . It follows that the canonical surjection $\varrho: B \rightarrow B_r$ induces an isomorphism of weak $G \rtimes N$ -algebras:

$$(5.12) \quad \varrho \rtimes_{\omega} \widehat{G}: B \rtimes_{\delta, \omega} \widehat{G} \xrightarrow{\sim} B_r \rtimes_{\delta_r, \omega_r} \widehat{G}.$$

Using this observation, we obtain the following generalization of Lemma 5.11 to arbitrary twisted coactions:

Lemma 5.13. *Let (π, σ) be a covariant representation of the $(G, G/N)$ -twisted coaction (B, δ, ω) such that there exists an action of N on the image of $\pi \rtimes_{\omega} \sigma$ making σ into an N -equivariant homomorphism. Then $\pi \rtimes_{\omega} \sigma$ is faithful if and only if $\ker \pi = \ker k_B (= \ker (\varrho: B \rightarrow B_r))$.*

Recall that two twisted coactions (B, δ_B, ω_B) and (C, δ_C, ω_C) are *Morita equivalent* if there is an imprimitivity A – B -bimodule \mathcal{E} carrying a G -coaction $\delta_{\mathcal{E}}$ compatible with δ_A and δ_B and satisfying $\delta_{\mathcal{E}}|(x) := (\text{id} \otimes q_N) \circ \delta_{\mathcal{E}}(x) = \omega_A(x \otimes 1)\omega_B^*$ for all $x \in \mathcal{E}$. The following result is a twisted version of the Landstad Duality Theorem for coactions we proved in [1] (which is, in turn, a generalization of the main result in [24]).

Theorem 5.14. *Let (A, α, ϕ) be a weak $G \rtimes N$ -algebra and let $\|\cdot\|_{\mu}$ be a crossed-product norm on $C_c(N, A)$ for which the dual N -coaction $\hat{\alpha}$ on $A \rtimes_{\alpha, u} N$ factors through a coaction $\hat{\alpha}_{\mu}$ on $A \rtimes_{\alpha, \mu} N$. Then*

- (i) *the G -coaction δ_{μ}^N on A_{μ}^N given by equation (5.2) is twisted over G/N with twisting homomorphism $\phi^N: C_0(G/N) \rightarrow \mathcal{M}(A_{\mu}^N)$ induced from the structural homomorphism $\phi: C_0(G) \rightarrow \mathcal{M}(A)$ as in (3.5), i.e., $\omega_{\mu} := (\phi^N \otimes \text{id})(\omega_{G/N})$ is the twisting unitary for δ_{μ}^N .*
- (ii) *The coaction $\delta_{\mathcal{F}}$ on $\mathcal{F} = \mathcal{F}_{\mu}^N(A)$ implements a Morita equivalence between $(A_{\mu}^N, \delta_{\mu}^N, \omega_{\mu})$ and the trivially twisted coaction $(A \rtimes_{\alpha, \mu} N, \text{Inf } \hat{\alpha}_{\mu}, 1)$. Moreover, (A_u^N, δ_u^N) is a maximal G -coaction and (A_r^N, δ_r^N) is a normal G -coaction.*
- (iii) *If $\kappa: A_{\mu}^N \rightarrow \mathcal{M}(A)$ is the canonical representation given by the extension of the inclusion map $A_c^N \hookrightarrow \mathcal{M}(A)$ (see [1, Proposition 3.5]), then the pair (κ, ϕ) is a covariant representation of $(A_{\mu}^N, \delta_{\mu}^N, \omega_{\mu})$ into $\mathcal{M}(A)$ and the corresponding integrated form $\kappa \rtimes_{\omega} \phi$ is an isomorphism $A_{\mu}^N \rtimes_{\delta_{\mu}^N, \omega_{\mu}} \hat{G} \xrightarrow{\sim} A$ of weak $G \rtimes N$ -algebras.*

Proof. To prove (i), we have to verify the conditions in (1) and (2) in Definition 5.3 for the homomorphism ϕ^N and the coaction δ_{μ}^N . The condition $\delta_{\mu}^N \circ \phi^N = \phi^N \otimes 1$ follows from equation (5.2) and the relation $(f \otimes 1)\omega_G = \omega_G(f \otimes 1)$ for all $f \in C_0(G)$ (remember that $\omega_G \in \mathcal{M}(C_0(G) \otimes C^*(G))$).

In order to prove the condition $\delta_{\mu}^N|(m) = \omega_{\mu}(m \otimes 1)\omega_{\mu}^*$ for all $m \in A_c^N$ we choose $f \in C_c(G/N)$ such that $m = \phi(f)m$. We then compute, for $z \in C^*(G)$,

$$(\phi \otimes q_N)(\omega_G)(m \otimes q_N(z)) = (\phi \otimes \text{id})(\text{id} \otimes q_N)(\omega_G(f \otimes z))(m \otimes 1)$$

and also

$$(\phi^N \otimes \text{id})(\omega_{G/N})(m \otimes q_N(z)) = (\phi^N \otimes \text{id})(\omega_{G/N}(f \otimes q_N(z)))(m \otimes 1).$$

Now observe that $(\text{id} \otimes q_N)(\omega_G(f \otimes z))$ is the function in $C_b(G, C^*(G/N))$ given by $s \mapsto f(sN)u_{sN}q_N(z)$, which is constant on N -cosets and factors to the function $\omega_{G/N}(f \otimes q_N(z))$ in $C_c(G/N, C^*(G/N))$. Since $\phi \otimes \text{id}$ restricts to $\phi^N \otimes \text{id}$ on this space, we conclude that $(\phi \otimes q_N)(\omega_G)(m \otimes q_N(z)) = (\phi^N \otimes \text{id})(\omega_{G/N})(m \otimes q_N(z))$ for all $m \in A_c^N$ and $z \in C^*(G)$, which then implies that $(\phi \otimes q_N)(\omega_G) = (\phi^N \otimes \text{id})(\omega_{G/N})$

in $\mathcal{M}(A_\mu^N \otimes C^*(G/N))$. Therefore

$$\begin{aligned}\delta_\mu^N|(m) &= (\phi \otimes q_N)(\omega_G)(m \otimes 1)(\phi \otimes q_N)(\omega^*) \\ &= (\phi^N \otimes \text{id})(\omega_{G/N})(m \otimes 1)(\phi^N \otimes \text{id})(\omega_{G/N}^*) = \omega_\mu(m \otimes 1)\omega_\mu^*.\end{aligned}$$

Therefore $(\delta_\mu^N, \omega_\mu)$ is a twisted action of $(G, G/N)$ on A_μ^N . Moreover, the same argument just used to verify axiom (1) in Definition 5.3 yields, for all $\xi \in \mathcal{F}_c(A)$,

$$\delta_{\mathcal{F}}|(\xi) = (\text{id} \otimes q_N)\delta_{\mathcal{F}}(\xi) = (\phi \otimes q_N)(\omega_G)(\xi \otimes 1) = (\phi^N \otimes \text{id})(\omega_{G/N})(\xi \otimes 1) = \omega_\mu(\xi \otimes 1)$$

which is saying that $(\mathcal{F}, \delta_{\mathcal{F}})$ implements the desired Morita equivalence between $(A_\mu^N, \delta_\mu^N, \omega_\mu)$ and $(A \rtimes_{\alpha, \mu} N, \text{Inf } \widehat{\alpha}, 1)$. This proves the first assertion in (ii) and the second assertion follows from the fact that maximality and normality of coactions are preserved by Morita equivalence and by inflation of coactions (see [7, Proposition 3.5], [18, Proposition 7.3] and [8, Lemma 3.19]).

Finally, to prove (iii) we first observe that (κ, ϕ) is a covariant representation of (A_μ^N, δ_μ^N) , that is, that

$$(\kappa \otimes \text{id})(\delta_\mu^N(a)) = (\phi \otimes \text{id})(\omega_G)(\kappa(a) \otimes 1)(\phi \otimes \text{id})(\omega_G^*)$$

for all $a \in A_\mu^G$. Of course, it suffices to verify this equation for $a \in A_c^G$ and then it follows directly from formula (5.2). Therefore (κ, ϕ) is a covariant representation of (A_μ^N, δ_μ^N) into $\mathcal{M}(A)$ and an argument similar to that given in the proof of [24, Lemma 3.10(2)] (replacing G by N where appropriate) shows that the image of $\kappa \rtimes \phi$ is A , so that we may view $\kappa \rtimes \phi$ as a surjective $*$ -homomorphism from $A_\mu^N \rtimes_{\delta_\mu^N} \widehat{G}$ onto A . Moreover, it is easy to see that $\kappa \rtimes \phi$ commutes with the N - and $C_0(G)$ -actions. Since $\phi|_{C_0(G/N)} = \kappa \circ \phi^N$, it follows from Remark 5.6 that the covariant representation (κ, ϕ) preserves the twist $\omega_\mu = \phi^N \otimes \text{id}(\omega_{G/N})$. We need to show that the $G \rtimes N$ -equivariant $*$ -homomorphism

$$\kappa \rtimes_{\omega_\mu} \phi : A_\mu^N \rtimes_{\delta_\mu^N, \omega_\mu} \widehat{G} \twoheadrightarrow A$$

is injective. But this follows from Lemma 5.13 and the fact that $\kappa : A_\mu^N \rightarrow \mathcal{M}(A)$ factors through a faithful map $\kappa_r : A_r^N \rightarrow \mathcal{M}(A)$, hence $\ker \kappa$ coincides with the kernel of the normalization morphism $A_\mu^N \twoheadrightarrow A_r^N$. \square

In what follows next we want to show that, conversely, every twisted coaction (δ, ω) is of the kind as in Theorem 5.14 for the weak $G \rtimes N$ -algebra (A, α, ϕ) with

$$A = B \rtimes_{\delta, \omega} \widehat{G}, \quad \alpha = \widehat{\delta}^\omega, \quad \text{and} \quad \phi = k_{C_0(G/N)}.$$

In order to prepare the result, we show

Lemma 5.15. *Suppose that (δ, ω) is a twisted coaction of $(G, G/N)$ on the C^* -algebra B . Let $\|\cdot\|_\mu$ denote either the full crossed-product norm $\|\cdot\|_u$ or the reduced crossed-product norm $\|\cdot\|_r$ for crossed products by N and G . Then the cosystems*

$$((B \rtimes_\delta \widehat{G})_\mu^G, \delta_\mu^G) \quad \text{and} \quad ((B \rtimes_{\delta, \omega} \widehat{G})_\mu^N, \delta_\mu^N)$$

are isomorphic. The isomorphism maps an element b of the inductive limit dense subalgebra $B_c \cong j_B(B_c)$ of $(B \rtimes_\delta \widehat{G})_c^G$ to the element $k_B(b)$ in the inductive limit dense subalgebra $k_B(B_c) \subseteq (B \rtimes_{\delta, \omega} \widehat{G})_c^N$, where $B_c = \delta_{A_c(G)}(B) \subseteq B$ (compare with Remark 4.10). In particular, we get $(B \rtimes_\delta \widehat{G})_r^G = k_B(B) \subseteq \mathcal{M}(B \rtimes_{\delta, \omega} \widehat{G})$.

Proof. Write $A := B \rtimes_{\delta, \omega} \widehat{G}$. By Proposition 5.8, we have $\text{Ind}_N^G(A) \cong B \rtimes_{\delta} \widehat{G}$ as weak $G \rtimes G$ -algebras. Then [1, Theorem 4.6] implies that $((\text{Ind}_N^G(A))_u^G, \delta_u^G)$ is the maximalization and $((\text{Ind}_N^G(A))_r^G, \delta_r^G)$ is the normalization of (B, δ) . The coactions δ_μ^G for $\mu = u, r$ are given by the formula (as in equation (5.2))

$$(5.16) \quad \delta_\mu^G(x) = (\tilde{\phi} \otimes \text{id})(\omega_G)(x \otimes 1)(\tilde{\phi} \otimes \text{id})(\omega_G^*) \quad \text{for all } x \in (\text{Ind}_N^G(A))_c^G,$$

where $\tilde{\phi}$ is the structural homomorphism $C_0(G) \rightarrow \mathcal{M}(\text{Ind}_N^G(A))$, which is given by $\tilde{\phi}(f)|_t = \phi(\tau_{t^{-1}}(f))$ (with $\tau_t(f)|_s = f(st)$) acting on an element $F \in \text{Ind}_N^G A$ by pointwise multiplication. Now, Proposition 3.10 yields a canonical isomorphism $\psi: A_\mu^N \xrightarrow{\sim} (\text{Ind}_N^G(A))_\mu^G$, which sends $m \in A_c^N$ to the constant function $G \rightarrow \mathcal{M}(A)$, $t \mapsto m$, which defines an element of $(\text{Ind}_N^G(A))_c^G \subseteq \mathcal{M}(\text{Ind}_N^G(A))_c^G$.

We now show that ψ is equivariant with respect to the G -coactions δ_μ^N on A_μ^N and δ_μ^G on $(\text{Ind}_N^G(A))_\mu^G$, that is, $\delta_\mu^G \circ \psi = (\psi \otimes \text{id}) \circ \delta_\mu^N$, so that ψ becomes an isomorphism of \widehat{G} -algebras. To prove this, recall that δ_μ^N is given as in equation (5.2) by the formula

$$\delta_\mu^N(m) = (\phi \otimes \text{id})(\omega_G)(m \otimes 1)(\phi \otimes \text{id})(\omega_G^*) \quad \text{for all } m \in A_c^N.$$

As explained in [1, Remark 4.14], the right hand side of this equation is, a priori, an element of $\mathcal{M}(A \otimes C^*(G))$, but can be interpreted as an element of $(A \otimes C^*(G))_c^N \hookrightarrow A_\mu^N \otimes C^*(G)$. A similar interpretation is used for δ_μ^G in (5.16). Now we observe that the isomorphism $\psi \otimes \text{id}: A_\mu^N \otimes C^*(G) \xrightarrow{\sim} (\text{Ind}_N^G(A))_\mu^G \otimes C^*(G)$ sends an element $x \in (A \otimes C^*(G))_c^N$ to the constant function $t \mapsto x$ from G to $\mathcal{M}(A \otimes C^*(G))$ viewed as an element of $\mathcal{M}((\text{Ind}_N^G(A))_\mu^G \otimes C^*(G))$. We will apply this to $x = \delta_\mu^N(m)$. We need to show that the element $\delta_\mu^G(\psi(m)) \in \mathcal{M}(\text{Ind}_N^G(A) \otimes C^*(G))$ is sent via the canonical inclusion $\mathcal{M}(\text{Ind}_N^G(A) \otimes C^*(G)) \hookrightarrow \mathcal{M}(C_0(G, A \otimes C^*(G)))$ to the constant function $t \mapsto \delta_\mu^N(m)$ from G to $\mathcal{M}(A \otimes C^*(G))$. But given $t \in G$, observe that $(\tilde{\phi} \otimes \text{id})(\omega)|_t = (\phi \otimes \text{id})(\omega_G)(1 \otimes u_{t^{-1}})$, where $t \mapsto u_t$ denotes the universal representation of G into $\mathcal{M}(C^*(G))$ (remember that $\omega_G(s) = u_s$). Therefore,

$$\begin{aligned} \delta_\mu^G(\psi(m))|_t &= (\tilde{\phi} \otimes \text{id})(\omega_G)(\psi(m) \otimes 1)(\tilde{\phi} \otimes \text{id})(\omega_G^*)|_t \\ &= (\phi \otimes \text{id})(\omega_G)(1 \otimes u_{t^{-1}})(\psi(m) \otimes 1)(1 \otimes u_t)(\phi \otimes \text{id})(\omega_G^*) \\ &= (\phi \otimes \text{id})(\omega_G)(\psi(m) \otimes 1)(\phi \otimes \text{id})(\omega_G^*) = \delta_\mu^N(m). \end{aligned}$$

This proves that ψ is \widehat{G} -equivariant. Finally, it follows from Remark 5.9 and the description of the inclusion of A_c^N into $(\text{Ind}_N^G(A))_c^G$ via constant functions that the isomorphism $A_\mu^N \cong (\text{Ind}_N^G A)_\mu^G \cong (B \rtimes_{\delta} \widehat{G})_\mu^G$ maps $k_B(B_c)$ bijectively onto $j_B(B_c) \subseteq (B \rtimes_{\delta} \widehat{G})_c^G$. This implies the last assertion of the lemma. \square

Remark 5.17. The above lemma together with Lemma 4.7 imply in particular that $k_B: B_c \rightarrow k_B(B_c)$ is an isomorphism of $*$ -algebras. Hence we may regard B_c as an inductive limit dense subalgebra of $(B \rtimes_{\delta, \omega} \widehat{G})_c^N$. In particular, we see that for any crossed-product norm $\|\cdot\|_\mu$ on $C_c(N, B \rtimes_{\delta} \widehat{G})$, the corresponding fixed-point algebra $B_\mu^N := (B \rtimes_{\delta, \omega} \widehat{G})_\mu^N$ can be regarded as a completion of B_c with respect to a suitable norm. Moreover, if the chosen norm $\|\cdot\|_\mu$ admits a dual coaction on $(B \rtimes_{\delta, \omega} \widehat{G}) \rtimes_\mu N$, we obtain a twisted coaction $(\delta_\mu^N, \omega_\mu)$ as in Theorem 5.14.

Theorem 5.18. *Let (B, δ, ω) be a twisted coaction of $(G, G/N)$. Then there exist $(G, G/N)$ -equivariant epimorphisms*

$$B_u^N \xrightarrow{q_u} B \xrightarrow{q_r} B_r^N$$

given by the identity map on B_c viewed as a dense $$ -subalgebra of all three algebras such that the induced morphisms*

$$B_u^N \rtimes_{\delta_u^N, \omega_u} \widehat{G} \xrightarrow{q_u \rtimes \widehat{G}} B \rtimes_{\delta, \omega} \widehat{G} \xrightarrow{q_r \rtimes \widehat{G}} B_r^N \rtimes_{\delta_r^N, \omega_r} \widehat{G}$$

are isomorphisms of weakly proper $G \rtimes N$ -algebras. Moreover there exists a unique crossed-product norm $\|\cdot\|_\mu$ on $C_c(N, B \rtimes_{\delta, \omega} \widehat{G})$ which admits a $(\delta_\mu^N, \omega_\mu)$ - (δ, ω) equivariant isomorphism $B_\mu^N \cong B$ extending the identity on B_c , and then $\mathcal{F}_\mu^N(B \rtimes_{\delta, \omega} \widehat{G})$ induces a Morita equivalence between the twisted cosystems

$$(B, \delta, \omega) \quad \text{and} \quad ((B \rtimes_{\delta, \omega} \widehat{G}) \rtimes_\mu N, \text{Inf}(\widehat{\delta\omega})_\mu, 1_{G/N}).$$

Proof. It follows from item (ii) of Theorem 5.14 together with Lemma 5.15 that (B_u^N, δ_u^N) is a maximalization of (B, δ) and (B_r^N, δ_r^N) is a normalization of (B, δ) . Thus it follows from Remark 4.10 that the identity map on B_c induces $\delta_u^N, \delta, \delta_r^N$ equivariant epimorphisms $B_u^N \xrightarrow{q_u} B \xrightarrow{q_r} B_r^N$. By continuity, the composition $q_r \circ q_u$ extends the identity map on $(B \rtimes_{\delta, \omega} \widehat{G})_c^G$, hence we see that B can be obtained as a completion of $B_c^N := (B \rtimes_{\delta, \omega} \widehat{G})_c^G$ with respect to a suitable norm $\|\cdot\|_\nu$. It follows then from the Rieffel-correspondence applied to the $(G, G/N)$ -equivariant B_u^N – $(B \rtimes_{\delta, \omega} \widehat{G}) \rtimes_\mu N$ equivalence bimodule $\mathcal{F}_u^N(B \rtimes_{\delta, \omega} \widehat{G})$ that there exists a unique crossed-product norm $\|\cdot\|_\mu$ on $C_c(N, B \rtimes_{\delta, \omega} \widehat{G})$ which admits a dual coaction $(\widehat{\delta\omega})_\mu$ of N such that $\mathcal{F}_u^N(B \rtimes_{\delta, \omega} \widehat{G})$ factors through a δ – $\text{Inf}(\widehat{\delta\omega})_\mu$ equivariant B – $(B \rtimes_{\delta, \omega} \widehat{G}) \rtimes_\mu N$ equivalence bimodule. Since all bimodule operations extend the operations on the dense submodule $\mathcal{F}_c^N(B \rtimes_{\delta, \omega} \widehat{G})$, it follows that $B \cong B_\mu^N$ with isomorphism given via the identity on B_c (or even on $(B \rtimes_{\delta, \omega} \widehat{G})_c^N$). Thus, the theorem will follow from item (iii) of Theorem 5.14 if we can show that this isomorphism intertwines the twists ω_μ and ω . The latter will follow if we can show that the corresponding homomorphisms $\zeta_\omega, \zeta_{\omega_\mu} : C_0(G/N) \rightarrow \mathcal{M}(B)$ coincide. Recall that we regard B_c as a subset of $B_c^N = (B \rtimes_{\delta, \omega} \widehat{G})_c^N$ via the identification $B_c \cong k_B(B_c) \subseteq B_c^N$. By item (i) of Theorem 5.14 we have $\zeta_{\omega_\mu}(f)k_B(b) = k_G(f)k_B(b)$ for all $f \in C_c(G/N)$, $b \in B_c$. On the other hand, since (k_B, k_G) preserves the twist ω , Remark 5.6 implies that $k_B(\zeta(f)b) = k_B(\zeta(f))k_B(b) = k_G(f)k_B(b)$, which shows the desired identity. \square

As a direct corollary of the last assertion of the theorem we get:

Corollary 5.19 (Stabilization trick for arbitrary coactions). *Every twisted coaction (δ, ω) of $(G, G/N)$ is Morita equivalent to an inflated twisted coaction $(\inf \epsilon, 1)$ for some coaction ϵ of N .*

Remark 5.20. In [9] the stabilization trick has been proved for twisted (reduced) coactions of $(G, G/N)$ with N amenable. As remarked in [17] (see comments before Theorem 5.5 in [17]), the same ideas carry over to prove a stabilization trick for twisted (full) coactions for arbitrary (non-amenable) N under the assumption that the underlying G -coaction is normal. Our result works for all twisted coactions (δ, ω) .

Using the above results, we may now generalize the notion of “maximal coactions”, “normal coactions”, and “ μ -coactions” for a given crossed-product norm $\|\cdot\|_\mu$ on $C_c(G, B \rtimes_\delta \widehat{G})$ as discussed in [1, 7, 16] to the category of twisted coactions:

Definition 5.21. Let (δ, ω) be a twisted coaction of $(G, G/N)$ on a C^* -algebra B . Let $\|\cdot\|_\mu$ be the unique norm on $C_c(N, B \rtimes_{\delta, \omega} \widehat{G})$ (given by Theorem 5.18) such that $\mathcal{F}_u^N(B \rtimes_{\delta, \omega} \widehat{G})$ factors through a $B-(B \rtimes_{\delta, \omega} \widehat{G}) \rtimes_\mu N$ Morita equivalence. We then say that (δ, ω) is a μ -twisted coaction on B . If $\mu = u$, we say that (δ, ω) is a maximal twisted coaction and if $\mu = r$, we say that (δ, ω) is a normal twisted coaction.

Remark 5.22. Let (B, δ, ω) be a twisted coaction of $(G, G/N)$, and consider the weak $G \rtimes N$ -algebra $A = B \rtimes_{\delta, \omega} \widehat{G}$. Then the twisted coaction (δ_u^N, ω_u) on A_u^N serves as a *maximalization* of (δ, ω) and (δ_r^N, ω_r) on A_r^N serves as a *normalization* of (δ, ω) , while, for an arbitrary crossed-product norm $\|\cdot\|_\mu$ which admits a dual coaction $(\widehat{\delta}^\omega)_\mu$, $(\delta_\mu^N, \omega_\mu)$ may be regarded as a μ -ization of (δ, ω) .

In this language, (δ, ω) is a μ -coaction if and only if $(\delta, \omega) \cong (\delta_\mu^N, \omega_\mu)$. Thus we see that we get complete twisted analogues of the results obtained in [1]. Although we do not develop this here, we remark that it is also possible to obtain an analogue of the categorical Landstad Duality Theorem [1, Theorem 7.2] for twisted coactions by using essentially the same ideas as used there.

The following result follows immediately from item (ii) of Theorem 5.14.

Corollary 5.23. *A twisted coaction (B, δ, ω) of $(G, G/N)$ is maximal (resp. normal) if and only if the (untwisted) coaction (B, δ) is a maximal (resp. normal) coaction of G . In particular, if N is an amenable closed subgroup of G , then every G -coaction (B, δ) which is twisted over G/N is both maximal and normal.*

Recall that a *unitary coaction* is a G -coaction (B, δ) which is twisted over G (that is, $N = \{e\}$ is the trivial group in the above notation). Equivalently, this is the same as a weak $G \rtimes \{e\}$ -algebra, that is, a C^* -algebra B with a non-degenerate representation $\phi: C_0(G) \rightarrow \mathcal{M}(B)$. The G -coaction δ is then recovered by the formula $\delta(b) = (\phi \otimes \text{id})(\omega_G)(b \otimes 1)(\phi \otimes \text{id})(\omega_G)^*$. The above corollary immediately implies the following result (see also [4, Proposition A1]).

Corollary 5.24. *Every unitary coaction is maximal and normal.*

As already mentioned previously, the following decomposition theorem is well known. (It has been proved by Phillips and Raeburn in [23] for amenable N and reduced coactions. But in [26, Remark 7.12] Quigg and Raeburn stated that the amenability of N is actually not necessary if one works with full coactions.) As an application of our methods, we now derive an alternative proof for this theorem:

Corollary 5.25 (Phillips-Raeburn). *For an arbitrary G -coaction (B, δ) and a normal closed subgroup $N \subseteq G$, there is a canonical isomorphism of weak $G \rtimes N$ -algebras:*

$$B \rtimes_\delta \widehat{G} \cong (B \rtimes_{\delta|_N} \widehat{G/N}) \rtimes_{\tilde{\delta}, \tilde{\omega}} \widehat{G},$$

where $(\tilde{\delta}, \tilde{\omega})$ denotes the twisted $(G, G/N)$ -coaction on $B \rtimes_{\delta|_N} \widehat{G/N}$ as described in Example 5.7 above.

Proof. Let A be the weak $G \rtimes N$ -algebra $B \rtimes_\delta \widehat{G}$. For the crossed-product norms $\mu = u$ or $\mu = r$, it follows from Theorem 4.1 that $A_\mu^N \cong B \rtimes_{\delta_\mu} \widehat{G/N}$ and we leave it as an exercise for the reader to check that the isomorphism is equivariant for the twisted coaction $(\delta_\mu^N, \omega_\mu)$ and the decomposition coaction $(\tilde{\delta}_\mu, \tilde{\omega}_\mu)$. It follows from Theorem 5.14(iii) that we have a natural isomorphism

$$(5.26) \quad (B_\mu \rtimes_{\delta_\mu} \widehat{G/N}) \rtimes_{\tilde{\delta}_\mu, \tilde{\omega}_\mu} \widehat{G} \cong A_\mu^N \rtimes_{\delta_\mu^N, \omega_\mu^N} \widehat{G} \cong A$$

of weak $G \rtimes N$ -algebras for $\mu = u$ and $\mu = r$. On the other hand, since (B_u, δ_u) is the maximalization and (B_r, δ_r) is the normalization of (B, δ) there are equivariant surjections $B_u \rightarrow B \rightarrow B_r$ which therefore induce surjections

$$B_u \rtimes_{\delta_u} \widehat{G/N} \rightarrow B \rtimes_\delta \widehat{G/N} \rightarrow B_r \rtimes_{\delta_r} \widehat{G/N}$$

which are morphisms of $(G, G/N)$ -coactions and hence also induce surjections

$$(B_u \rtimes_{\delta_u} \widehat{G/N}) \rtimes_{\tilde{\delta}_u, \tilde{\omega}_u} \widehat{G} \twoheadrightarrow (B \rtimes_\delta \widehat{G/N}) \rtimes_{\tilde{\delta}, \tilde{\omega}} \widehat{G} \twoheadrightarrow (B_r \rtimes_{\delta_r} \widehat{G/N}) \rtimes_{\tilde{\delta}_r, \tilde{\omega}_r} \widehat{G}.$$

Moreover, by equation (5.26), the composition of the two epimorphisms above is an isomorphism and the first and the third algebras are isomorphic to $B \rtimes_\delta \widehat{G}$, so $(B \rtimes_\delta \widehat{G/N}) \rtimes_{\tilde{\delta}, \tilde{\omega}} \widehat{G}$ must also be isomorphic to $B \rtimes_\delta \widehat{G}$, as desired. \square

We finish with the following consequence of the Landstad Duality Theorem 5.14, which shows that Mansfield's Imprimitivity Theorem 4.1 can be enriched to an equivalence of twisted coactions. This therefore yields a natural connection between the two main topics of this paper.

Corollary 5.27. *Let (B, δ) be a maximal coaction of G . Then there is a coaction $\delta_{\mathcal{F}_u^N}$ on Mansfield's $B \rtimes_\delta \widehat{G/N} - B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} N$ imprimitivity bimodule $\mathcal{F}_u^N(B \rtimes_\delta \widehat{G})$ which is compatible with the canonical twisted coactions on both algebras, namely, the decomposition twisted coaction $(\tilde{\delta}, \tilde{\omega})$ on $B \rtimes_\delta \widehat{G/N}$ and the trivially twisted coaction $(\text{Inf } \widehat{\delta}|, 1)$ on $B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} N$. In other words, $(\mathcal{F}_u^N(B \rtimes_\delta \widehat{G}), \delta_{\mathcal{F}_u^N})$ is a Morita equivalence of twisted coactions*

$$(B \rtimes_\delta \widehat{G/N}, \tilde{\delta}, \tilde{\omega}) \sim (B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} N, \text{Inf } \widehat{\delta}|, 1).$$

A similar result holds for normal coactions (B, δ) if we replace the universal crossed products by the reduced crossed products everywhere.

Proof. This follows directly from Theorem 5.14(ii) (applied to $A = B \rtimes_\delta \widehat{G}$) and the fact (already observed in the proof of Corollary 5.25) that the decomposition twisted coaction $(\tilde{\delta}, \tilde{\omega})$ corresponds to the twisted coaction $(\delta_\mu^N, \omega_\mu)$ under the canonical isomorphism $B \rtimes_\delta \widehat{G/N} \cong (B \rtimes_\delta \widehat{G})_\mu^N$. \square

Remark 5.28. We should remark that for maximal coactions, the equivalence

$$(B \rtimes_\delta \widehat{G/N}, \tilde{\delta}) \sim (B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} N, \text{Inf } \widehat{\delta}|)$$

is one of the main results of [18], where it has also been proved that $(B, \delta) \mapsto (\mathcal{F}_u^N(B \rtimes_\delta \widehat{G}), \delta_{\mathcal{F}_u^N})$ may be interpreted as an equivalence between the crossed-product functors $(B, \delta) \mapsto (B \rtimes_\delta \widehat{G/N}, \tilde{\delta})$ and $(B, \delta) \mapsto (B \rtimes_\delta \widehat{G} \rtimes_{\widehat{\delta}} N, \text{Inf } \widehat{\delta}|)$ if we

restrict to maximal coactions (B, δ) . Our result shows that the natural twists involved match up, so that $\mathcal{F}_u^N(B \rtimes_{\delta} \widehat{G})$ may be viewed also as an equivalence between the functors $(B, \delta) \mapsto (B \rtimes_{\delta|} \widehat{G/N}, \widehat{\delta}, \widehat{\omega})$ and $(B, \delta) \mapsto (B \rtimes_{\delta} \widehat{G} \rtimes_{\widehat{\delta}|} N, \text{Inf } \widehat{\delta|}, 1)$. Moreover, it follows from our Proposition 3.2 that $\mathcal{F}_u^N(B \rtimes_{\delta} \widehat{G})$ carries a G -action which is compatible with the natural twisted actions of (G, N) on the left and right coefficient algebras, namely, the inflation $\text{Inf } \widehat{\delta|}$ of the dual G/N -action on $B \rtimes_{\delta|} \widehat{G/N}$ (viewed as a trivially twisted action of (G, N)), and the decomposition twisted action $(\widehat{\delta|}, \iota_N)$ on $B \rtimes_{\delta} \widehat{G} \rtimes_{\widehat{\delta}|} N$. Therefore $\mathcal{F}_u^N(B \rtimes_{\delta} \widehat{G})$ also provides an equivalence between the functors $(B, \delta) \mapsto (B \rtimes_{\delta|} \widehat{G/N}, \text{Inf } \widehat{\delta|}, 1)$ and $(B, \delta) \mapsto (B \rtimes_{\delta} \widehat{G} \rtimes_{\widehat{\delta}|} N, \widehat{\delta|}, \iota_N)$.

An analogue of these equivalences follows also for normal coactions (B, δ) via the bimodule $\mathcal{F}_r^N(B \rtimes_{\delta} \widehat{G})$. This case has been shown before in [8, Theorem 4.21]; see also [19].

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