ABSTRACT BERGMAN KERNEL EXPANSION AND ITS APPLICATIONS

CHIUNG-JU LIU AND ZHIQIN LU

Abstract. We give a purely complex geometric proof of the existence of the Bergman kernel expansion. Our method actually provides a sharper estimate, and in the case that the metrics are real analytic, we prove that the remainder decays faster than any polynomial.

Contents

1. Introduction 1467
2. Preliminaries 1469
3. An abstract version of Bergman kernel expansion 1471
4. Peak sections 1480
5. On the generalization of a lemma of Ruan 1484
Appendix A. K-coordinates and K-frames 1488
Acknowledgements 1494
References 1494

1. INTRODUCTION

Let $M$ be an $n$-dimensional algebraic manifold in a certain projective space $\mathbb{CP}^N$. The hyperplane line bundle of $\mathbb{CP}^N$, restricting to an ample line bundle $L$ of $M$, is a polarization of $M$. A Kähler metric $g$ is called a polarized metric if the corresponding Kähler form represents the first Chern class $c_1(L)$ of $L$. Given any polarized Kähler metric $g$, there is a Hermitian metric $h_L$ on $L$ whose Ricci form is equal to the Kähler form $\omega_g$.

The Bergman kernel, defined as a sequence of smooth (bundle-valued) functions, plays one of the central roles in Kähler-Einstein geometry. In [19], Tian proved the $\mathcal{C}^2$-convergence theorem of the Bergman metric (whose Kähler potential is the Bergman kernel). A far-reaching generalization of Tian’s theorem was obtained by Catlin [4] and Zelditch [21], using the existence of the parametrix of the Bergman or Szegő kernel (cf. [3]):

Theorem 1.1 (Catlin, Zelditch). Let $M$ be a compact complex manifold of dimension $n$ (over $\mathbb{C}$) and let $(L, h_L) \to M$ be a positive Hermitian holomorphic line bundle and $(E, h_E)$ a Hermitian vector bundle of rank $r$. Let $g$ be the Kähler metric
on $M$ corresponding to the Kähler form $\omega_g = \text{Ric}(h_L)$. Then there is an asymptotic expansion of the Bergman kernel $\mathfrak{B}_m(x)$:

$\mathfrak{B}_m(x) \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots$

for certain smooth coefficients $a_j(x) \in \text{Hom}(E, E)$ with $a_0 = I$. More precisely, for any $s$, the following inequalities hold:

$\|\mathfrak{B}_m(x) - \sum_{k=0}^{s} a_k(x)m^{n-k}\|_{C^\mu} \leq C_{s,\mu}m^{-s-1}$,

where $C_{s,\mu}$ depends on $s, \mu$, the manifold $M$, and the bundles $L, E$.

In [13] the first three coefficients were computed together with a general algorithm of computing any coefficients.

There are several different proofs of the above expansion theorem after Catlin and Zelditch. In [1], Berman, Berndtsson, and Sjöstrand gave a direct, constructive approach, avoiding using the parametrix of the Bergman kernel. In [5], Dai, Liu, and Ma gave a heat kernel approach of the expansion. In Engliš [9], the first four coefficients were computed. More recently, Xu [20] gave a graph-theoretic interpretation of the coefficients of the expansion.

Because of the importance of the Bergman kernel expansion in complex geometry, we would like to give a sharper estimate of the Bergman expansion which is particularly useful in the collapsing case [12].

In this paper, we first prove an abstract version of the Bergman kernel expansion which is of its own interest. One of the innovations we obtain is that by systematically using the $K$-coordinate systems and $K$-frames, we are able to prove, in the abstract setting, that $C^0$ expansion implies $C^\mu$ expansion. Consequently, we generalize Theorem 1.1.

**Theorem 1.2.** We use the same notation as in Theorem 1.1. Let $\varepsilon > 0$ be an absolute number (for example, we can take $\varepsilon = 1/16$). Let $S_1, \cdots, S_k$ be peak sections, where $k = \lceil \varepsilon (\log m)^2 \rceil$. Let $\mathfrak{B}_m^{k,\text{peak}}$ be the Bergman kernel with respect to the peak sections $S_1, \cdots, S_k$ (see Definition 3.5 and Definition 4.1). Then

1. $\mathfrak{B}_m^{k,\text{peak}}$ has a $C^\infty$ asymptotic expansion;
2. the expansion stables to the Bergman kernel expansion, that is,

$$\|\mathfrak{B}_m(x) - \mathfrak{B}_m^{k,\text{peak}}(x)\|_{C^\mu} \leq \frac{C}{m^{s-n+1}}$$

for any $s \leq [\varepsilon (\log m)^2]$.

In particular, for any $\mu \geq 0$ and any increasing sequence $\beta(m, \mu) \to \infty$, there exists an increasing integer sequence $\alpha(m, \mu) \to \infty$ such that

$$(1.3) \quad \left\|\mathfrak{B}_m(x) - \sum_{k=0}^{\alpha(m, \mu)} a_k m^{n-k}\right\|_{C^\mu} \leq \frac{\beta(m, \mu)}{m^{\alpha(m, \mu)-n+1}}.$$

Since $\alpha(m, \mu) \to \infty$, for any fixed $s$, if $m$ is large enough, we must have $\alpha(m, \mu) > s$. Therefore, our theorem is a generalization of Theorem 1.1.

The stretch from $(s + 1)$ terms in Theorem 1.1 to $(\alpha(m, \mu) + 1)$ terms in Theorem 1.2 provides a neat estimate in the real analytic case. When the metrics of $M, L, E$ are real analytic, we are able to improve (1.3). More precisely, we prove the following result.
**Theorem 1.3.** With the notation as in the above theorem, suppose that the Hermitian metrics $h_L$ and $h_E$ are real analytic at a fixed point $x$. Then the expansion

$$\sum_{j=0}^{\infty} a_j(x)m^{-j}$$

is convergent in $C^\mu$ ($\mu \geq 0$) for $m$ large. Moreover, we have

$$(1.4) \quad \|\mathcal{B}_m(x) - \sum_{k=0}^{\infty} a_k(x)m^{n-k}\|_{C^\mu} \leq m^ne^{-\varepsilon(\log m)^3}$$

for some absolute constant $\varepsilon > 0$.

**Remark 1.1.** Both of the above two theorems are sharper than Theorem 1.1, but it is not clear to the authors that estimate (1.4) is optimal. In [11], the first author proved the best possible estimate $e^{-\varepsilon m}$ for the compact Riemann surface (of genus $g \geq 2$) of constant Gauss curvature. But this is only a very special case.

The following corollary of the above theorem provides bounds of the Bergman kernel coefficients, which is of independent interest.

**Corollary 1.1.** Assume that the metrics of $M, L, E$ are real analytic. Then there exists a constant $C > 1$ such that the coefficients $a_j$ of the Bergman kernel expansion satisfy

$$\|a_j(x)\|_{C^\mu} \leq C^j$$

for all $j \geq 0$.

We remark that using the method in Lu-Shiffman (cf. [15, Lemma 2.5]), similar results of this paper also hold for the off-diagonal Bergman kernel.

In [1], a local reproducing kernel and then the local Bergman kernel were constructed explicitly. A relation between the local Bergman kernel and the Bergman kernel with respect to the peak sections (see Definition 3.5) and its relation to the heat kernel parametrix would provide insights into the family version of the Catlin-Zelditch expansion. In particular, in both [1] and our paper, we only require the existence of a “good” holomorphic coordinate system, but we don’t explicitly require that the injectivity radius have a lower bound. This observation may prove to be useful when we study the Bergman kernels on a family of Kähler manifolds.

The method used in proving the main results of this paper is completely elementary. It is similar to that in Shiffman [18], where we have to deal with matrices with increasing size. We believe that our methods, including the discussion of the family version of $K$-coordinates and $K$-frames in the Appendix, are useful in many other places in complex geometry.

The paper is organized as follows. In Section 3, we prove Theorem 3.1, an abstract version of the Bergman kernel expansion. The main results of this paper follow naturally from this abstract version and the hard analysis in Sections 4-5. In Section 4, we define the peak sections and verify assumption (1) of Theorem 3.1 in Section 5 we verify assumptions (2), (3), (4) of Theorem 3.1 and in the Appendix, we include the discussion of the family version of $K$-coordinates and $K$-frames.

2. Preliminaries

Throughout this paper, we use $C$ as a constant, which may differ from line to line. Let $(M, g)$ be a Kähler manifold, and let $(E, h_E)$ be a Hermitian vector bundle
of rank $r$. For $U, V \in \Gamma(M, E)$, the pointwise and the $L^2$ inner products are defined as

$$\langle U(x), V(x) \rangle_{h_E}$$

and

$$\langle U, V \rangle = \int_M \langle U(x), V(x) \rangle_{h_E} dV_g,$$

respectively, where $dV_g = \omega^n_{g}/n!$ is the volume form of $g$.

Assume that $T_1, \cdots, T_d$ is an orthonormal basis of $H^0_{L^2}(M, E)$. Let

$$e_1, \cdots, e_r$$

be holomorphic local frames of $E$. Let

$$T_j = \sum_{\alpha=1}^r a_{j\alpha} e_\alpha$$

for $1 \leq j \leq d$. Then $a_{j\alpha}$ are local holomorphic functions. Let $A(x)$ denote the $d \times r$ matrix $(a_{j\alpha}(x))$. The Bergman kernel, which is an element of Hom $(E_y, E_x)$ for any $x, y \in M$, is defined by

$$\mathcal{B}(x, y) = H(y) A^*(y) A(x),$$

where $H = (h_{\alpha\beta}) = (\langle e_\alpha, e_\beta \rangle)$ is the metric matrix and $A^*$ is the complex conjugate of $A$.

The name Bergman kernel is justified by the following property. For any $f \in \Gamma_{L^2}(M, E)$, if we write $f$ as a row vector under the frame (2.2), then

$$\int_M f(y) \mathcal{B}(x, y) dV_g$$

is the orthogonal projection from $\Gamma_{L^2}(M, E)$ to $H^0(M, E)$.

In this paper, we study $\mathcal{B}(x, y) = H A^* A$, which we also call the Bergman kernel.

The Bergman kernel $\mathcal{B}(x)$ is independent of the choice of orthonormal basis $T_1, \cdots, T_d$. Let $S_1, \cdots, S_d$ be any basis of $H^0(M, E)$. Let

$$F = (F_{ij}) = (S_i, S_j).$$

Let $P$ be a matrix such that $PFP^* = I$. If we write

$$S_j = \sum_{\alpha=1}^r b_{j\alpha} e_\alpha,$$

then the Bergman kernel can be represented by

$$\mathcal{B}(x) = H(PB)^* PB = HB^* F^{-1} B,$$

where $B = (b_{j\alpha})$ is a local $d \times r$ matrix-valued function.

Now we assume that the Kähler manifold $M$ is polarized, that is, there is an ample Hermitian line bundle $(L, h_L)$ over $M$ such that the Kähler metric $g$ is the curvature $\text{Ric}(h_L)$. Let $m$ be a large positive integer. Let $\mathcal{B}_m(x)$ be the Bergman kernel of the bundle $L^m \otimes E$, which is a sequence of $\text{Hom}(L^m \otimes E, L^m \otimes E) (= \text{Hom}(E, E))$-valued smooth functions. The main purpose of this paper is to study the asymptotic expansion of these functions.
3. An abstract version of Bergman kernel expansion

We begin with some abstract discussions of the bundle-valued function expansions.

Definition 3.1. We say a sequence of matrix-valued functions \( f_m(x) \) has a strongly \( C^\mu \) asymptotic expansion if there exists a sequence \( \sigma(m) \to \infty \) for \( m \to \infty \), such that for any \( s < \sigma(m) \) and \( \mu \geq 0 \), we have

\[
\left\| f_m(x) - m^n \left( a_0(x) + \frac{a_1(x)}{m} + \cdots + \frac{a_s(x)}{m^s} \right) \right\|_{C^\mu} \leq \frac{C(s, \mu)}{m^{s-n+1}}
\]

(3.1)

for matrix-valued functions \( a_0(x), \ldots, a_s(x), \ldots \), and a constant \( C(s, \mu) \) independent of \( m \). If the metrics of \( M, L, E \) are real analytic, then we further assume that there exists a constant \( C_1 > 1 \), independent of \( s \) and \( x \), such that \( C(s, \mu) \leq C_1^{n+1} \).

It is natural to believe that a sequence of functions \( f_m(x) \) has a strongly \( C^\mu \) asymptotic expansion if and only if at any point of \( M \) it has a strongly \( C^\mu \) asymptotic expansion. However, we can’t prove such a statement. Since the \( C^\mu \)-norm depends on the choices of local coordinates and frames, in order to obtain \( \Box \), we need to prove the existence of a smooth family of “good” coordinates and frames at any point of the manifold. We first make the following definition.

Definition 3.2. Let \( \mu \) be a nonnegative integer. We say a sequence of matrix-valued functions \( f_m(x) \) has a strongly \( C^\mu \) asymptotic expansion at any point if there exists a constant \( C \) independent of \( m \) such that for any \( x_0 \in M \), there exist a local \( K \)-coordinate system and \( K \)-frames at \( x_0 \) of order \( p > \mu + \sigma(m) \) and matrix-valued functions \( f^{\mu}_0(x), \ldots, f^{\mu}_n(x), \ldots \) in a neighborhood of \( x_0 \) with the following property: for any \( s < \sigma(m) \), \( \mu' \leq \mu \), there exist constants \( C(s, \mu') \) which are independent of \( m \) and \( s \) such that

\[
\left| D^{\mu'}_{x_0} f_m(x) - m^n \left( f^{\mu'}_0(x) + \frac{f^{\mu'}_1(x)}{m} + \cdots + \frac{f^{\mu'}_s(x)}{m^s} \right) \right|(x_0) \leq \frac{C(s, \mu')}{m^{s-n+1}},
\]

(3.2)

where \( D^{\mu'}_{x_0} \) is the \( \mu' \)-th derivative with respect to the above-mentioned \( K \)-coordinate system and frames centered at \( x_0 \).

In particular, we say \( f_m(x) \) has a strongly \( C^0 \) asymptotic expansion\(^2\) at any point if for any \( s < \sigma(m) \)

\[
\left| f_m(x_0) - m^n \left( a_0(x_0) + \frac{a_1(x_0)}{m} + \cdots + \frac{a_s(x_0)}{m^s} \right) \right| \leq \frac{C(s)}{m^{s-n+1}}
\]

(3.3)

for any \( x_0 \in M \), where \( C(s) \) are constants independent of \( m \) and \( x_0 \).

Similar to the previous definition, if the metrics of \( M, L, E \) are real analytic, then we further assume that there exists a constant \( C_1 > 1 \), independent of \( s \), such that \( C \leq C_1^{n+1} \).

Lemma 3.1. For any positive integer \( \mu \), a sequence of matrix-valued functions \( f_m \) has a strongly \( C^\mu \) asymptotic expansion if \( f_m \) has a strongly \( C^\mu \) asymptotic expansion at any point.

---

\(^1\)Strictly speaking, these functions are vector-valued functions whose components are matrix-valued functions. This is because a \( \mu \)-th derivative has many components, but for the sake of simplicity, we don’t emphasize this fact.

\(^2\)By this notation, we have \( a_j^y = a_j \) for any \( j \geq 0 \).
Proof. We claim that for any nonnegative integers \( \mu, s \), we have
\[
a_\mu^s(x) = D_x^\mu a_s(x),
\]
where \( D_x^\mu \) is the \( \mu \)'-th derivative with respect to the above-mentioned \( K \)-coordinate system and frames centered at \( x \).

For \( \mu = 0 \), the claim follows by definition. Inductively, assume that the claim is proved to be true for any \( \mu' \leq \mu - 1 \) and for any \( s' \leq s \) when \( \mu' = \mu \). We shall prove that
\[
a_\mu^{s+1}(x) = D_x^\mu a_{s+1}(x).
\]
Without loss of generality, we only need to prove the above statement on an open neighborhood \( U \) of a fixed point \( x_0 \). By Corollary \[A.1\] and by shrinking \( U \) if necessary, there exists a holomorphic family of \( K \)-coordinate systems on \( U \). In what follows, we shall fix such a family of coordinate systems.

We define a sequence of functions
\[
\xi_{m,s}(x) = m^{s+1} \left( \frac{f_m(x)}{m^n} - \left( a_0(x) + \cdots + \frac{a_s(x)}{m^s} \right) \right).
\]
By \( (3.2) \) and the inductive assumption, \( D_x^\mu(\xi_{m,s}(x)) \) are uniformly convergent to \( a_\mu^{s+1}(x) \) on \( U \). Using the chain rule, for any \( \mu' \leq \mu \), we have
\[
D_{x_0}^{\mu'}(\xi_{m,s}(x)) = F_{\mu'}(D_x^{\mu'}(\xi_{m,s}(x)), D_x^{\mu'-1}(\xi_{m,s}(x)), \ldots, \xi_{m,s}(x))
\]
for functions \( F_{\mu'} \) depending only on the family of \( K \)-coordinate systems. Therefore, the functions \( \{\xi_{m,s}(x)\} \) are convergent on \( U \) in the \( C^\mu \)-norm to some function \( \tilde{a}_{s+1}(x) \). By taking the limit in \( (3.5) \) and the inductive assumption, we have
\[
D_{x_0}^{\mu}(\tilde{a}_{s+1}(x)) = F_\mu(a_\mu^{s+1}(x), D_x^{\mu-1}(a_{s+1}(x)), \ldots, a_{s+1}(x)).
\]
On the other hand, by the chain rule again, we have
\[
D_{x_0}^{\mu}(\tilde{a}_{s+1}(x)) = F_\mu(D_x^{\mu}(a_{s+1}(x)), D_x^{\mu-1}(a_{s+1}(x)), \ldots, a_{s+1}(x)).
\]
By further shrinking \( U \) if necessary, we can represent both \( a_\mu^{s+1}(x) \) and \( D_x^{\mu}(a_{s+1}(x)) \) as the exact same polynomial of
\[
D_{x_0}^{\mu}(\tilde{a}_{s+1}(x)), D_x^{\mu-1}(a_{s+1}(x)), \ldots, a_{s+1}(x).
\]
Therefore they must be equal:
\[
a_\mu^{s+1} = D_x^{\mu}(a_{s+1}(x)),
\]
and the claim is proved.

Lemma \ref{lem:3.1} then follows from \( (3.2) \) and \( (3.5) \).

Let \( P = (p_1, \cdots, p_n) \) be a multiple index and let \( 1 \leq \alpha \leq r \). Define the lexicographical order on the set of \( \alpha \)'s, that is, \( (P, \alpha) < (Q, \beta) \) if
\begin{enumerate}
\item \( \sum p_i < \sum q_i \), or
\item \( p_1 = q_1, \cdots, p_i = q_i \) but \( p_{i+1} < q_{i+1} \) for some \( 0 \leq \ell \leq n - 1 \), or
\item \( P = Q \), but \( \alpha < \beta \).
\end{enumerate}
Such an order gives rise to the function $j = \sigma(P, \alpha)$. For example, $1 = \sigma((0, \cdots, 0), 1)$, $2r + 2 = \sigma((0, 1, \cdots, 0), 2)$, etc. Its inverse function is defined by $(P, \alpha) = (R(j), \zeta(j))$. For an example, $(R(1), \zeta(1)) = ((0, \cdots, 0), 1)$ and $(R(2r + 2), \zeta(2r + 2)) = ((0, 1, \cdots, 0), 2)$, etc.

**Definition 3.3.** Let $\sigma(m)$ be sequences of positive integers such that $\sigma(m) \to \infty$ as $m \to \infty$. Let $x_0 \in M$ be a fixed point. At $x_0$, let $(z_1, \cdots, z_n)$ be $K$-coordinates and let $e_L, (e_1, \cdots, e_r)$ be $K$-frames for $L, E$, respectively (of order $\mu + \sigma(m)$ if the metrics are smooth and of order $+\infty$ if the metrics are analytic). Let $S_1, \cdots, S_d$ be a basis of $H^0(M, L^m \otimes E)$. We say that it is a strongly regular basis at $x_0$ if for the smallest $k$ such that $|R(k)| \geq \sigma(m)$,

1. for $1 \leq j \leq k$, $S_j(z) = z^{R(j)}e_L^m \otimes e_{\zeta(j)} + o(|z|^\sigma(m))$;
2. for $j > k$, $S_j(z) = o(|z|^\sigma(m))$.

**Remark 3.1.** The motivation of the above definition is the existence of peak sections (Definition 4.1).

For the rest of the paper, we shall let

$$\sigma(m) = s = \lfloor \varepsilon (\log m)^2 \rfloor,$$

where $\varepsilon > 0$ is a small absolute positive number. Let $\delta_{-1} = 0$ and

$$\delta_j = r \left( \frac{n - 1 + j}{n - 1} \right)$$

for $j = 0, \cdots, s$. Then $\delta_j$ is the number of indices $(P, \alpha)$ such that $|P| = j$ and $1 \leq \alpha \leq r$. Let

$$\delta_{s+1} = r \sum_{j=1}^{s} \delta_j.$$

Then we have

$$k = r \sum_{j=1}^{s} \delta_j \leq 2rs^2.$$

In particular,

$$|R(k)| = s.$$

**Definition 3.4.** A strongly regular basis $\{S_j\}_{j=1, \cdots, d}$ is called almost orthonormal if

$$\frac{|(S_i, S_j)|}{\|S_i\| \cdot \|S_j\|} \leq \frac{C}{m^{1 + (|R(j)| - |R(i)|)/2}}$$

for any $i < j \leq 2k$, and

$$\begin{align*}
(S_i, S_j) &= 0, \quad i \leq k, j > 2k; \\
(S_i, S_j) &= \delta_{ij}, \quad i, j > 2k,
\end{align*}$$

\[\text{By math induction, } \delta_{s+1} = r \binom{n+s}{n}.\]
where the constant $C$ is independent of $m$ and $x_0$. Moreover, if the metrics of $M, L, E$ are real analytic, then we assume that there exists a constant $C_1 > 1$, independent of $m, x_0$ and $i, j$, such that

$$C \leq C_1^{||R(j)||-||R(i)||}.$$  

Let $F = ((S_i, S_j))_{i,j=1,...,d}$ be the $d \times d$ metric matrix. For an almost orthonormal basis, we can write $F$ as

$$F = \begin{pmatrix} F_1 & I_{d-2k} \end{pmatrix},$$

where $F_1$ is a $(2k) \times (2k)$ matrix. In order to study the inverse matrix of $F$, we only need to study the inverse matrix of $F_1$.

In what follows, we shall prove our key observation: for an almost orthonormal basis, up to order $[\varepsilon (\log m)^2]$, the expansions of the $k \times k$ minor of $F^{-1}$ only depend on $(S_i, S_j)$ for $i, j \leq k$ and are independent of the rest of the matrix! Since we shall pick $(S_1, \cdots, S_k)$ as peak sections later, we reduce the expansion of Catlin-Zelditch ($\mathcal{B}_m$) into the expansions $\mathcal{B}^k_{m,peak}$. That is, we reduce the problem to finding the expansion of the inverse of the inner products of the peak sections.

Let

$$F_2 = \Lambda^{-1} F_1 \Lambda^{-1},$$

where

$$\Lambda = \begin{pmatrix} \|S_1\| & & \\ & \ddots & \\ & & \|S_k\| \end{pmatrix}.$$  

We represent the matrix $F_2$ as a block matrix. For any $0 \leq \xi, \eta \leq s + 1$, define the matrix $A_{\xi\eta}$ to be the $\delta_\xi \times \delta_\eta$ matrix whose entries are

$$\delta_{ij} = \frac{(S_i, S_j)}{||S_i|| \cdot ||S_j||}$$

for

$$\sum_{\gamma=-1}^{\xi-1} \delta_{\gamma} < i \leq \sum_{\gamma=-1}^{\xi} \delta_{\gamma}; \quad \sum_{\gamma=-1}^{\eta-1} \delta_{\gamma} < j \leq \sum_{\gamma=-1}^{\eta} \delta_{\gamma}.$$  

Let $A$ be the block matrix defined by

$$A = (A_{\xi\eta})_{(0 \leq \xi, \eta \leq s + 1)}.$$  

Then we have

$$F_2 = I - A.$$

Define $||A||$ to be the maximum entries of the matrix $A$. Let column($A$) be the number of the columns of the matrix $A$. Then we have

$$||A + B|| \leq ||A|| + ||B||;$$

$$(3.8) \quad ||AB|| \leq \text{column}(A) \cdot ||A|| \cdot ||B||.$$

\footnote{By this definition, $A_{s+1,s+2} = 0$ and $A_{s+2,s+1} = 0$.}
By (3.7), we have
\[
\|A_\xi\eta\| \leq \frac{C}{m^{1 + \frac{1}{2}|\xi - \eta|}}.
\]
Using the expansion
\[
F_2^{-1} = I + A + A^2 + \cdots
\]
for any fixed \((\xi_0, \eta_0)\), we have
\[
(F_2^{-1} - I)_{\xi_0\eta_0} = \sum_{k=1}^{\infty} \sum_{i_1, \ldots, i_{k-1}} A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0}.
\]

By (3.6), \(\text{column}(A) \leq 2rs^n\). Therefore by (3.6) and (3.8), for fixed \(\xi_0, i_1, \ldots, i_{k-1}, \eta_0\), we have
\[
\|A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0}\| \leq \left(\frac{C \cdot 4rs^n}{m}\right)^k.
\]
Therefore
\[
\left\| \sum_{i_1, \ldots, i_{k-1}} A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0} \right\| \leq (s+1) \left(\frac{C \cdot 4rs^n}{m}\right)^k.
\]
For fixed \(s\), if \(m\) is large enough, then we have
\[
\left\| \sum_{k=s+1}^{\infty} \sum_{i_1, \ldots, i_{k-1}} A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0} \right\| \leq \frac{C}{m^{s+1}}.
\]
Similarly, we consider the terms
\[
\sum_{k=1}^{s} \sum_{\text{some } i_j = s+1} A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0}.
\]
If some \(i_j = s + 1\), we must have
\[
|\xi_0 - i_1| + |i_1 - i_2| + \cdots + |i_{k-1} - \eta_0| \geq 2s + 2 - \xi_0 - \eta_0.
\]
Thus by (3.9) we have
\[
\left\| \sum_{k=1}^{s} \sum_{\text{some } i_j = s+1} A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0} \right\| \leq \frac{C}{m^{s+1} - \frac{1}{2}(\xi_0 + \eta_0)}.
\]
Consequently, we have
\[
\left\| (F_2^{-1})_{\xi_0 \eta_0} - \left( I_{\xi_0 \eta_0} + \sum_{k=1}^{s} \sum_{i_1, \ldots, i_{k-1} = 0} A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0} \right) \right\| \leq \frac{C}{m^{s+1} - \frac{1}{2}(\xi_0 + \eta_0)}.
\]
If the metrics of $M, L, E$ are real analytic, then using the same estimate as above, we will get

\[
\left\| (F_2^{-1})_{\xi_0\eta_0} \right\| \leq \left( \frac{C_1 (\log m)^{4n}}{m} \right)^{s+1 - \frac{1}{2}(\xi_0 + \eta_0)}
\]

(3.11)

for $s \geq \lceil \varepsilon (\log m)^2 \rceil$.

Let

\[
\delta(\xi_0, \eta_0, i_1, \cdots, i_{k-1}) = |\xi_0 - i_1| + |i_1 - i_2| + \cdots + |i_{k-1} - \eta_0|.
\]

We use the notation $f(m) \ll g(m)$ to denote that

1. both $f(m), g(m)$ are Taylor series of $m^{-1}$;
2. the coefficients of $g(m)$ are nonnegative;
3. the coefficients of $g(m) - f(m)$ are nonnegative.

By the assumption of $A_{ij}$, we have

\[
\| A_{ij} \| \ll \left( \frac{C}{m} \right)^{1 + \frac{|i-j|}{2}} \cdot \frac{1}{1 - \frac{C}{m}}
\]

for some constant $C > 1$. It follows that

\[
\sum_{k=1}^{s} \sum_{i_1, \cdots, i_{k-1}=0}^{s} A_{\xi_0 i_1} A_{i_1 i_2} \cdots A_{i_{k-1} \eta_0} \ll \sum_{k=1}^{s} \sum_{i_1, \cdots, i_{k-1}=0}^{s} \left( \frac{C}{m} \right)^{k + \delta(\xi_0, \eta_0, i_1, \cdots, i_{k-1})} \left( 1 - \frac{C}{m} \right)^{-k}.
\]

(3.12)

Let $\#\delta$ be the number of $i_1, \cdots, i_{k-1}$ such that $\delta(\xi_0, \eta_0, i_1, \cdots, i_{k-1}) = \delta$. In the expansion of the left hand side of (3.12), the coefficient of $m^{-b}$ is not more than

\[
C^b \sum_{\delta+k \leq b} \frac{(b-\delta)!}{k!(b-\delta-k)!} \cdot \#\delta.
\]

By Lemma 3.2 below, the coefficient of $m^{-b}$ is not more than

\[
C^b \sum_{\delta+k \leq b} \frac{(b-\delta)!}{k!(b-\delta-k)!} \cdot \#\delta \leq C^b
\]

for a possibly larger constant $C$.

Replacing $s$ in (3.11) by $s + 1 + \frac{1}{2}(\xi_0 + \eta_0)$, we get

\[
\left\| (F_2^{-1})_{\xi_0\eta_0} \right\| \leq \left( \frac{C_1 (\log m)^{4n}}{m} \right)^{s+2}.
\]

(3.13)

In the analytic case, $s \to \infty$ as $m \to \infty$. 

---

*In the analytic case, $s \to \infty$ as $m \to \infty$.)*
By the estimate on the coefficients of $m^{-b}$, we obtain
\[
\| (F_2^{-1})_{\xi_0\eta_0} - \text{the expansion up to order } s \| \\
\leq \left( \frac{C}{m} \right)^{s+1} + \left( \frac{C_1 (\log m)^{4n}}{m} \right)^{s+2} \leq \left( \frac{C}{m} \right)^{s+1} .
\] (3.14)

By using the same method, when the metrics are smooth, from (3.10), we obtain
\[
\| (F_2^{-1})_{\xi_0\eta_0} - \text{the expansion up to order } s \| \leq \frac{C}{m^{s+1}} ,
\] (3.15)
where $C$ is a constant independent of $m$ (but may depend on $s$).

We need the following combinatorial lemma to complete the above estimate.

**Lemma 3.2.** There exists a constant $C > 1$ such that
\[
\# \delta \leq C^{\delta+k} .
\]

**Proof.** It is well known that the number of nonnegative solutions of the equation
\[
a_1 + \cdots + a_k = \delta
\]
is equal to
\[
\frac{(\delta + k - 1)!}{(k-1)! \delta!} \leq C^{\delta+k} .
\]
On the other hand, $\# \delta$ is not more than the number of solutions of the equations $i_{j-1} - i_j = \pm a_j$ for $1 \leq j \leq k$ (where we define $i_0 = \xi_0, i_k = \eta_0$). Thus we have the estimate
\[
\# \delta \leq 2^k C^{\delta+k} \leq C^{\delta+k}
\]
for some larger constant $C > 1$. \hfill \Box

Now we state the following abstract version of the Bergman kernel expansion:

**Theorem 3.1.** Let $\sigma(m) = [\varepsilon (\log m)^2]$ for a sufficiently small absolute number $\varepsilon > 0$. Let $k$ be the smallest positive integer such that $|R(k)| \geq \sigma(m)$. Let $s = |R(k)|$. Assume that for any $x_0 \in M$, there exists a basis $\{S_j\}_{j=1,\ldots,d}$ of $H^0(M, E \otimes L^m)$ such that

1. it is a strongly regular basis in the sense of Definition 3.3;
2. it is almost orthonormal in the sense of Definition 3.4;
3. the expressions
\[
\frac{(S_i, S_j)}{\|S_i\| \cdot \|S_j\|}
\]
have strongly $C^0$ asymptotic expansions (in the sense of (3.3))
\[
\frac{(S_i, S_j)}{\|S_i\| \cdot \|S_j\|} \sim \frac{r_l}{m^l} + \frac{r_{l+1}}{m^{l+1}} + \cdots
\]
for $|R(i)| + |R(j)| \leq \mu$ with $l \geq 1 + (|R(j)| - |R(i)|)/2$. Moreover, we have $|r_t| \leq C$, where $C$ is independent of $i, j$ for all $l \leq t \leq s$. If the metrics of $M, L, E$ are real analytic, we further assume that there exists a constant $C_1 > 1$, independent of $m, x_0$ and $i, j$, such that $C \leq C_1'$.
(4) the expressions $\|S_i\|$ have strongly $C^0$ asymptotic expansions (in the sense of (3.3))

$$\|S_i\| \sim r_i + \frac{r_{i+1}}{m^{l+1}} + \cdots$$

for $|R(i)| \leq \mu$, where $l = (n + |R(i)|)/2$, and $\|r_i\|_{C^\alpha} \geq c > 0$. Moreover, we have $|r_i| \leq C$, where $C$ is independent of $i,j$ for all $l \leq t \leq s$. If the metrics of $M,L,E$ are real analytic, we further assume that there exists a constant $C_1 > 1$, independent of $m,x_0$ and $i,j$, such that $C \leq C_1^s$.

Then we have the following Catlin-Zelditch type expansion: For any increasing sequence $\beta(m,\mu) \to \infty$, there exists a sequence $\alpha(m,\mu) \to \infty$ such that

$$\|B_m - \sum_{k=0}^{\alpha(m,\mu)} a_k m^{n-k}\|_{C^\mu} \leq \frac{\beta(m,\mu)}{m^{\alpha(m,\mu) - n + 1}}.$$  

Moreover, if the metrics of $M,L,E$ are real analytic, then we have the sharper estimate

$$\|B_m - \sum_{k=0}^{\infty} a_k m^{n-k}\|_{C^\mu} \leq m^n e^{-\varepsilon(\log m)^3}$$

for a small absolute constant $\varepsilon$.

**Proof.** Let $a$ and $H$ be the Hermitian metrics of $L$ and $E$, respectively. Then by (2.4), near a fixed point $x_0$,

$$B_m = a^m H B^* F^{-1} B.$$  

It is apparent that any derivatives of the functions $a^m$ and $H$ at $x_0$ are polynomials of $m$. For the rest of the paper, we shall repeatedly use the following elementary fact: if functions $f_m,g_m$ have asymptotic expansions, so does their product.

Thus in order to prove the result, we only need to prove the existence of the strongly $C^\mu$ asymptotic expansion of $B^* F^{-1} B$.

Using (2.4) and the definition of $S_j$, we have

$$b_{ij} = \delta_{\alpha,\zeta(j)} z^{R(j)} + o(|z|^{\sigma(m)}),$$

where $\delta_{ij}$ is the Kronecker symbol. It follows that

$$\frac{\partial^P b_{ij}}{\partial z^P} \bigg|_{z=0} = \delta_{\alpha,\zeta(j)} \delta_{P,R(j)} R(j)!$$

for any $|P| \leq \mu < \sigma(m)$.

By a straightforward computation, the $(i,j)$-th entry of the derivatives of the matrix $B^* F^{-1} B$ is

$$\left( \frac{\partial |P| + |Q|}{\partial z^P} (B^* F^{-1} B) \bigg|_{z=0} \right)_{\alpha\beta} = P! Q!(F^{-1})_{ij}$$

for any $1 \leq \alpha, \beta \leq r; |P| + |Q| \leq \mu$, where $i = \sigma(P,\alpha), j = \sigma(Q,\beta)$.

By (3.15) and assumption (4) in Theorem 3.1, the above expression has a strongly $C^0$ asymptotic expansion. This implies that the Bergman kernel has a $C^\mu$ expansion at a point in the sense of Definition 3.2. By Lemma 3.1 for any $s < \sigma(m) - \mu$, (2.2) is valid. Without loss of generality, we assume that the constants $C_{s,\mu}$ in (2.2)
are increasing with respect to $s$ and is divergent to $\infty$ as $s \to \infty$. Let $\beta(m, \mu) < \sigma(m) - \mu$ be an increasing positive sequence that is divergent to $\infty$ as $m \to \infty$. Let $\alpha(m, \mu) \to \infty$ be an increasing sequence of integers such that

$$C_{\alpha(m, \mu), \mu} < \beta(m, \mu).$$

Then (3.16) is valid.

If the metrics are real analytic, and we use (3.14) for $s$ and $s - 1$, we have

$$\| (F^{-1}_{2})_{\xi_{0} \eta_{0}} \| = \| \text{the expansion up to order } s \| \leq \left( \frac{C}{m} \right)^{s+1};$$

$$\| (F^{-1}_{2})_{\xi_{0} \eta_{0}} \| = \| \text{the expansion up to order } s - 1 \| \leq \left( \frac{C}{m} \right)^{s}.$$

Let $b_{j}$ be the coefficient of the $m^{-j}$ in the expansion of $F^{-1}_{2}$. From the above inequality, we have $\| b_{j} \|_{C'} \leq C^{j}$. Therefore, the coefficients $a_{j}$ of the Bergman kernel expansion satisfy

$$\| a_{j} \|_{C'} \leq C^{j}$$

for $j \geq 0$. Consequently, we have

$$\| \sum_{k=s+1}^{\infty} a_{k} m^{n-k} \|_{C'} \leq m^{n} \left( \frac{C}{m} \right)^{s+1}.$$

On the other hand, since we already obtained the estimate

$$\| B_{m} - \sum_{k=0}^{s} a_{j} m^{n-k} \|_{C'} \leq m^{n} \left( \frac{C}{m} \right)^{s+1},$$

by combining the above two inequalities and taking $s = \lceil \varepsilon (\log m) \rceil$ we obtain (3.17).

We end this section by making the following relative version of the Bergman kernel.

**Definition 3.5.** Let $\{ S_{1}, \cdots, S_{d} \}$ be a basis of $H^{0}(M, L^{m} \otimes E)$. Then the Bergman kernel $B_{m, \text{peak}}^{k}$ with respect to part of the basis $\{ S_{1}, \cdots, S_{k} \}$ is defined as

$$B_{m, \text{peak}}^{k} = a^{m} H(B_{k})^{*} F_{k}^{-1} B_{k},$$

where $B_{k}, F_{k}$ are defined similarly to (2.3) and (2.4). That is, $F_{k}$ is the $k \times k$ matrix defined by

$$(F_{k})_{ij} = (S_{i}, S_{j}),$$

and $B_{k}$ is the $k \times r$ matrix whose entries $(b_{k})_{j\alpha}$ are defined by

$$S_{j} = \sum_{\alpha=1}^{r} (b_{k})_{j\alpha} e_{\alpha}$$

for some basis $e_{1}, \cdots, e_{r}$. 

4. Peak sections

The peak sections were introduced in [19] (see also [13]). The results of this section are mostly known, except we make efforts to extend our estimate up to order $[\varepsilon (\log m)^2]$, which grows to infinity as $m \gg 0$.

We begin with citing the following well-known result (cf. [6,19]).

**Proposition 4.1.** Suppose that $(M, g)$ is an $n$-dimensional compact Kähler manifold, $(L, h_L)$ is a Hermitian line bundle over $M$, and $(E, h_E)$ is a Hermitian vector bundle over $M$. We assume that $\text{Ric}(h_L) = \omega_g$ defines the Kähler metric of $M$. Let $\Theta_E$ be the curvature of $E$ with respect to $h_E$, and let $I_E$ be the identity map of $E$. Let $\Psi$ be a function on $M$ which can be approximated by a decreasing sequence of smooth functions $\{\Psi_\ell\}_{\ell=1}^\infty$. Assume that the endomorphisms

$$\Theta_\ell = \Theta_E + I_E \otimes \partial \bar{\partial} \Psi_\ell + \frac{2\pi}{\sqrt{-1}} I_E \otimes (\text{Ric}(h) + \text{Ric}(g))$$

on $\bigwedge^{0,1}(M, L \otimes E)$ satisfy

$$\langle \Theta_\ell(\varphi), \varphi \rangle_{g^* \otimes h_L \otimes h_E} \geq C \|\varphi\|^2_{g^* \otimes h_L \otimes h_E}$$

for $\varphi \in \bigwedge^{0,1}(M, L \otimes E)$, where $C > 0$ is a constant independent of $\ell$. Then for any $w \in \bigwedge^{0,1}(M, L \otimes E)$ with $\bar{\partial} w = 0$ and

$$\int_M \|w\|^2_{g^* \otimes h_L \otimes h_E} e^{-\Psi} dV_g < \infty,$$

there exists $u \in C^\infty(M, L \otimes E)$ such that $\bar{\partial} u = w$ and

$$\int_M \|u\|^2_{h_L \otimes h_E} e^{-\Psi} dV_g \leq \frac{1}{C} \int_M \|w\|^2_{g^* \otimes h_L \otimes h_E} e^{-\Psi} dV_g.$$

\qed

**Remark 4.1.** The choice of $u$ is, of course, not unique. In order to construct the peak sections, we need to fix the solution of the equation $\bar{\partial} u = w$. Let $\Delta$ be the $(0,1)$-Laplacian on bundle $L \otimes E$ with respect to the corresponding metrics and the weight function $\Psi$. By [4,11] and the Weitzenböck formula, the smallest eigenvalue $\lambda_1$ of $\Delta$ satisfies $\lambda_1 \geq C$. It follows that the Green’s operator $G$ of $\Delta$ exists and is a bounded operator on $L^2$ spaces.

Let

$$u = \bar{\partial}^* G w.$$

Then we have

$$\bar{\partial} u = \bar{\partial} \bar{\partial}^* G w = (\bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}) G w = w,$$

because $w$ is $\bar{\partial}$-closed and because $\bar{\partial} \bar{\partial}^*$ commutes with $G$. Moreover, we have

$$\int_M \|u\|^2_{h_L \otimes h_E} e^{-\Psi} dV_g = \int_M \langle u, \bar{\partial}^* G w \rangle_{h_L \otimes h_E} e^{-\Psi} dV_g$$

$$= \int_M \langle w, G w \rangle_{g^* \otimes h_L \otimes h_E} e^{-\Psi} dV_g \leq \frac{1}{C} \int_M \|w\|^2_{g^* \otimes h_L \otimes h_E} e^{-\Psi} dV_g.$$

Thus section $u$ defined above satisfies (4.2). For the rest of the paper, when we use Proposition 4.1, we shall fix this unique solution.
Let $\varepsilon = 1/16$, and let $p' = \lfloor \varepsilon (\log m)^2 \rfloor$. Let $K_1$ be the upper bound of the curvature operator of $g$, and let $K_2$ be the upper bound of the curvature operator of $h_E$. Throughout the rest of the paper, we use $C$ to denote a positive constant, which may be different line by line, depending only on $n$, $r$, $p'$, $k_1$, $k_2$, and the injectivity radius $\delta$ of $M$.

Let $x_0 \in M$ be a fixed point. Let $(z_1, \cdots, z_n)$ be holomorphic coordinates on an open neighborhood $U$ of $x_0$ in $M$. Define the function $|z|$ by

$$|z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$$

for $z \in U$. We further assume that $(z_1, \cdots, z_n)$ are the $K$-coordinates centered at $x_0$ and that $e_L, \{e_\alpha\}$ are the $K$-frames of order $p'$. We assume that $g_{ij}, a$, and $h_{\alpha \overline{\beta}}$ are the local representatives of the metrics of $M, L, \text{and } E$, respectively.

Let $Z_{n+}^n$ be the set of $n$-tuples of integers $(p_1, \cdots, p_n)$ such that $p_i \geq 0$ for $i = 1, \cdots, n$. Let $P = (p_1, \cdots, p_n)$. Define

$$z^P = z_1^{p_1} \cdots z_n^{p_n}$$

and

$$p = p_1 + \cdots + p_n.$$

Let $\eta$ be a smooth cut-off function

$$\eta(t) = \begin{cases} 1 & \text{for } t < \frac{1}{2}, \\ 0 & \text{for } t \geq 1, \end{cases}$$

satisfying $0 \leq -\eta'(t) \leq 4$ and $|\eta''(t)| \leq 8$.

The following lemma is a revised version of Tian’s Lemma [19, Lemma 1.2].

**Lemma 4.1.** For any $1 \leq \alpha \leq r$ and $P = (p_1, \cdots, p_n) \in Z_{n+}^n$ such that $p' \geq p = p_1 + \cdots + p_n$, there is a holomorphic global section $S_{P,\alpha, m}^{p'}$ of $H^0(M, L^m \otimes E)$ satisfying

$$\left| \int_M \|S_{P,\alpha, m}^{p'}\|^2_{h_L^m \otimes h_E} dV_g - 1 \right| \leq Ce^{-\frac{1}{4}(\log m)^2}. \tag{4.4}$$

$S_{P,\alpha, m}^{p'}$ can be decomposed as

$$S_{P,\alpha, m}^{p'} = \tilde{S}_{P,\alpha, m} - u_{P,\alpha, m} \tag{4.5}$$

for $\tilde{S}_{P,\alpha, m}, u_{P,\alpha, m} \in C^\infty(M, L^m \otimes E)$. The support of $\tilde{S}_{P,\alpha, m}$ is within $U$ and we can write

$$\tilde{S}_{P,\alpha, m}(z) = \lambda_{P,\alpha} \eta \left( \frac{m|z|^2}{(\log m)^2} \right) z^P e_L^m \otimes e_\alpha, \tag{4.6}$$

where

$$\lambda_{P,\alpha}^{-2} = \int_{|z| \leq \log m / \sqrt{m}} |z^P|^2 a^m h_{\alpha \overline{\beta}} dV_g. \tag{4.7}$$

The smooth section $u_{P,\alpha, m}$, which solves the equation

$$\overline{\partial} u_{P,\alpha, m} = \overline{\partial} \tilde{S}_{P,\alpha, m}, \tag{4.8}$$

satisfies

$$u_{P,\alpha, m}(z) = O(|z|^p') \quad \text{if } z \in U, \tag{4.9}$$
and
\begin{equation}
\int_M \|u_{P,\alpha,m}\|_{L^2(m^\alpha E)}^2 e^{-\Psi} dV_g \leq C e^{-\frac{1}{4}(\log m)^2}.
\end{equation}

Let $C(\mathbb{R})$ be the space of real functions on $\mathbb{R}$ and let the norm of a function $f$ in $C(\mathbb{R})$ be
\[ \|f\| = \sup_{x \in \mathbb{R}} |f(x)e^{-x}| \]
if the right hand side of the above is finite.

For any positive integer $p$, the norm of the function $x^p$ is $p^p e^{-p}$. The function $x^p e^{-x}$ peaks at $x = p$. As $x \to \infty$, the function decays very fast. With this picture in mind, we justify the following definition.

**Definition 4.1.** The sections $\{S_{P,\alpha,m}^0\}$, which look like $z^P$ near $x_0$, are called peak sections.

**Proof of Lemma 4.1.** Define the weight function
\[ \Psi(z) = (2n + 2p')\eta \left( \frac{m|z|^2}{(\log m)^2} \right) \log \left( \frac{m|z|^2}{(\log m)^2} \right). \]
A straightforward computation gives
\begin{equation}
\sqrt{-1} \partial \bar{\partial} \Psi \geq -\frac{8m(2n + 2p')}{(\log m)^2} \omega_g \geq -\frac{9}{10} m \omega_g
\end{equation}
for $m$ large enough. Let
\begin{equation}
\Theta = \Theta_E + I_E \otimes \partial \bar{\partial} \Psi + \frac{2\pi}{\sqrt{-1}} I_E \otimes (\text{Ric}(h_L^m) + \text{Ric}(g)).
\end{equation}

Using (4.11), we verify that
\begin{equation}
\langle \Theta(\varphi), \varphi \rangle \geq \left( \frac{1}{10} m - K_1 - K_2 \right) \|\varphi\|_{g^* \otimes h_L^m \otimes h_E}^2
\end{equation}
for $\varphi \in \Lambda^{0,1}(M, L^m \otimes E)$. For $P \in \mathbb{Z}_+^n$, let
\begin{equation}
w_{P,\alpha,m} = \partial \bar{S}_{P,\alpha,m}.
\end{equation}
Since $\partial \bar{\partial} w_{P,\alpha,m} \equiv 0$ and
\[ \int_M \|w_{P,\alpha,m}\|_{g^* \otimes h_L^m \otimes h_E}^2 e^{-\psi} dV_g < +\infty, \]
by Proposition 4.1 and Remark 4.1, there exists a smooth section $u_{P,\alpha,m}$ that solves (4.8) such that
\begin{equation}
\int_M \|u_{P,\alpha,m}\|_{L^2(m^\alpha E)}^2 e^{-\psi} dV_g \leq \frac{20}{m} \int_M \|w_{P,\alpha,m}\|_{g^* \otimes h_L^m \otimes h_E}^2 e^{-\psi} dV_g < \infty.
\end{equation}
Since
\[ \int_M \|u_{P,\alpha,m}\|_{E}^2 e^{-\psi} dV_g < +\infty, \]
we must have \( u_{P,\alpha,m}(x) = O(|x|^{\eta}) \). This proves (4.9). In order to verify (4.10), we need to estimate the right hand side of (4.15). By the definition of function \( \eta \), we have

\[
\frac{20}{m} \int_{\mathcal{M}} \|w_{P,\alpha,m}\|_{\mathcal{H}_L^{\pi\otimes h_E}}^2 e^{-\psi} dV_g \\
\leq \frac{320}{m} \lambda_{P,\alpha}^2 \int_{\log \frac{m}{\sqrt{2m}} \leq |z| \leq \log \frac{m}{\sqrt{2m}}} m^2 \left( \log m \right)^4 |z_k\overline{z}_l g^{\alpha\beta} |z|^{2} a_{m} h_{\alpha\pi} e^{-\psi} dV_g \\
\leq C \lambda_{P,\alpha}^2 \left( \log m \right)^{2(|P|-1)} \int_{\log \frac{m}{\sqrt{2m}} \leq |z| \leq \log \frac{m}{\sqrt{2m}}} a_{m}^2 dV_0,
\]

where \( dV_0 \) is the volume form of \( \mathbb{C}^n \):

\[
dV_0 = \left( \frac{-1}{2\pi} \right)^n d|z_1| \wedge |z_1| \wedge \cdots \wedge |z_n| \wedge |d\bar{z}_n|.
\]

Since \( e_L \) is a \( K \)-frame, we have

\[
a = e^{\text{log } a} = e^{-|z|^2 + O(|z|^4)}
\]
on \( U \). Thus we have

\[
\int_{\mathcal{M}} \|u_{P,\alpha,m}\|_{\mathcal{H}_L^{\pi\otimes h_E}}^2 e^{-\psi} dV_g \leq C \lambda_{P,\alpha}^2 \frac{\left( \log m \right)^{2(|P|-1)}}{m^{|P|}} \int_{|z| \geq \log \frac{m}{\sqrt{2m}}} e^{-m|z|^2} dV_0 \\
= C \lambda_{P,\alpha}^2 \frac{\left( \log m \right)^{2(|P|-1+n)}}{m^{|P|+n}} e^{-\frac{1}{2} \left( \log m \right)^2}.
\]

On the other hand, by Lemma 4.2 below, for \( m \) large enough

\[
\lambda_{P,\alpha}^2 \geq C \frac{P!}{m^{n+|P|}}
\]

and therefore

\[
\lambda_{P,\alpha} \leq C m^{\frac{n+|P|}{2}} / \sqrt{P!}.
\]

Putting (4.17) and (4.16) together, we obtain (4.10) for \( m \) large enough. Finally, we have

\[
\left| \int_{\mathcal{M}} \|\tilde{S}_{P,\alpha,m}\|_{\mathcal{H}_L^{\pi\otimes h_E}}^2 dV_g - 1 \right| \leq C \lambda_{P,\alpha}^2 \frac{\left( \log m \right)^{2(|P|-1+n)}}{m^{|P|+n}} \int_{\log \frac{m}{\sqrt{2m}} \leq |z| \leq \log \frac{m}{\sqrt{2m}}} a_{m}^2 dV_g.
\]

Using the same estimate as above, we obtain

\[
\left| \int_{\mathcal{M}} \|\tilde{S}_{P,\alpha,m}\|_{\mathcal{H}_L^{\pi\otimes h_E}}^2 dV_g - 1 \right| \leq C e^{-\frac{1}{3} \left( \log m \right)^2}
\]

for \( m \) large enough. (4.4) then follows from (4.5) and all the above estimates. \( \square \)

**Lemma 4.2.** For any multi-index \( P \) such that \(|P| \leq \frac{1}{3} \left( \log m \right)^2 \), we have

\[
\frac{P!}{m^{n+|P|}} \left( 1 - 2^n e^{-\frac{1}{4} \left( \log m \right)^2} \right) \leq \int_{|z| \leq \log \frac{m}{\sqrt{2m}}} |z|^P |z|^{-2q} e^{-m|z|^2} dV_0 \leq \frac{P!}{m^{n+|P|}}.
\]

**Proof.** We quote the following elementary identity:

\[
\int_{\mathbb{C}^n} |z|^P |z|^{-2q} e^{-m|z|^2} dV_0 = \frac{(|P|+n-1+q)! P!}{(|P|+n-1)! m^{n+|P|+q}},
\]

where \( q \geq 0 \) and \( P \) is a multi-index.
Using the above identity, the second inequality of (4.18) follows. To prove the first inequality of (4.18), we need to show that

\[
\int_{|z| \geq \log m} |z^P|^2 e^{-m|z|^2} \, dV_0 \leq 2^n \frac{P_!}{m^{n+|P|}} e^{-\frac{1}{2} (\log m)^2}.
\]

Rescaling \( z \) to \( \sqrt{mz} \), the above inequality (4.18) becomes

(4.20) \[
\int_{|z| \geq \log m} |z^P|^2 e^{-|z|^2} \, dV_0 \leq 2^n P_! e^{-\frac{1}{2} (\log m)^2}.
\]

We have

\[
\int_{|z| \geq \log m} |z^P|^2 e^{-|z|^2} \, dV_0 \leq e^{-\frac{1}{2} (\log m)^2} \int_{|z| \geq \log m} |z^P|^2 e^{-\frac{1}{2} |z|^2} \, dV_0,
\]

and by rescaling \( z \) to \( \sqrt{2}z \), we obtain

\[
\int_{|z| \geq \log m} |z^P|^2 e^{-|z|^2} \, dV_0 \leq e^{-\frac{1}{2} (\log m)^2} 2^{P_! + n} \int_{|z| \geq \log m} |z^P|^2 e^{-\frac{1}{2} |z|^2} \, dV_0 = 2^{|P| + n} P_! \frac{1}{m^{n+|P|}} e^{-\frac{1}{2} (\log m)^2}.
\]

The lemma then follows from the assumption \( |P| \leq \frac{1}{4} (\log m)^2 \).

**Definition 4.2.** For fixed \( m, p' \), let \( k \) be the smallest integer such that \( |R(k)| \geq p' \). Let

\[ S_j = \lambda_{R(j), \zeta(j)}^{m-1} S_{R(j), \zeta(j)}^{p'} \]

for \( 0 \leq j \leq k \). It is apparent that \( S_1, \cdots, S_k \) are linearly independent. We extend \( \{S_j\}_{j=1, \cdots, k} \) to \( \{S_j\}_{j=1, \cdots, d} \) so that \( S_j = o(|z|^{p'}) \) for \( j > k \).

**Corollary 4.1.** For any \( x_0 \in M \), the above basis \( \{S_j\} \) is a strongly regular basis. Such a basis satisfies assumption (1) of Theorem 3.1. \( \square \)

5. ON THE GENERALIZATION OF A LEMMA OF RUAN

W. D. Ruan discovered the almost orthogonality between the peak sections and the sections of high orders. For trivial vector bundle \( E \), we quote Ruan’s Lemma [17] Lemma 3.2] as follows.

**Lemma 5.1 (Ruan).** Let \( S_P = S_{P, m}^{p'} \) be a section constructed in Lemma 4.1. Define

\[
\|S\| = \sqrt{\int_M |S|^2 \, h_L \otimes g \, dV_g}
\]

for \( S \in H^0(M, L^m) \). Let \( T \) be another section of \( L^m \). Near \( x_0 \), \( T = f e_L^m \) for a holomorphic function \( f \). When we say \( T \)’s Taylor expansion at \( x_0 \), we mean the Taylor expansion of \( f \) at \( x_0 \) under the coordinate system \( (z_1, \cdots, z_n) \).

1. If \( z^P \) is not in \( T \)’s Taylor expansion at 0, then

\[
(S_P, T) = O \left( \frac{1}{m} \right) \|T\|.
\]

2. If \( T \) contains terms \( z^Q \) for \( |Q| \geq |P| + \sigma \) in the Taylor expansion, then

\[
(S_P, T) = O \left( \frac{1}{m^{1+\frac{\sigma}{2}}} \right) \|T\|.
\]
In order to use Ruan’s result, we need to give the precise bound in the above lemma.

We say that a smooth function \( f \) on \( U \subset \mathbb{C}^n \) is regular if, in its Taylor expansion, there are no \( z^P, z^P z_j, z^P z_j \) terms. We begin with the following

**Lemma 5.2.** Let \( f \) be a holomorphic function on \( U \subset \mathbb{C}^n \), let \( a > 0 \) be a positive smooth function and \( \xi \) a complex smooth function on \( U \) such that \( \xi(0) = 1 \). Assume that both \( a \) and \( \xi \) are regular functions. Define

\[
\|f\| = \sqrt{\int_{|z| \leq \frac{\log m}{\sqrt{m}}} |f|^2 a^m dV_0}.
\]

Assume that \( |P| \leq \varepsilon (\log m)^2 \) for some \( \varepsilon > 0 \).

1. If \( z^P \) is not in \( f \)'s Taylor expansion at 0, then

\[
\left| \int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^P a^m \xi dV_0 \right| \leq \frac{C \sqrt{|P|}}{m^{1 + |P|/2}} \|f\|.
\]

2. If \( f \) only contains terms \( z^Q \) for \( |Q| \geq |P| + \sigma \) and \( \sigma > 0 \) in the Taylor expansion, then

\[
(5.1) \quad \left| \int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^P a^m \xi dV_0 \right| \leq \left( \frac{(\log m)^{10}}{m} \right)^{1 + \frac{n + |P| + \sigma}{2}} \|f\|
\]

for \( m > m(\sigma) \), where \( m(\sigma) \) is a constant depending on \( \sigma \) and the functions \( a, \xi \).

3. Moreover, if the metrics are analytic, then we have

\[
(5.2) \quad \left| \int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^P a^m \xi dV_0 \right| \leq C^\sigma \left( \frac{(\log m)^{10}}{m} \right)^{1 + \frac{n + |P| + \sigma}{2}} \|f\|
\]

for \( m > m(\sigma) \).

**Proof.** We shall omit the proof of (1), since it is similar to that of (2).

Let \( \log a + |z|^2 = \zeta \). Since \( a \) is a regular function, \( \zeta = O(|z|^4) \). It follows that

\[
(5.3) \quad |m\zeta| \leq \frac{1}{m^{3/4}}
\]

for \( |z| \leq \frac{\log m}{\sqrt{m}} \) and \( m \) is sufficiently large. As a result, the functions \( a^m \) and \( e^{-m|z|^2} \) are mutually bounded by a constant independent of \( m \). In particular,

\[
\|f\| \leq C \sqrt{\int_{|z| \leq \frac{\log m}{\sqrt{m}}} |f|^2 e^{-m|z|^2} dV_0}
\]

for some constant \( C > 0 \). We shall use this fact below repeatedly without further notice.

Let \( \zeta = \zeta_1 + \zeta_2 \) such that \( \zeta_1 \) is the Taylor’s polynomial of \( \zeta \) of order \( 2\sigma + 1 \), and let \( \xi = \xi_1 + \xi_2 \) such that \( \xi_1 \) is the Taylor’s polynomial of \( \xi \) of order \( 2\sigma + 1 \). Then for \( m \) large enough, we have

\[
(5.4) \quad |e^{m\zeta} \xi - \sum_{k=0}^{2\sigma+1} \frac{m^k}{k!} \zeta^k \xi_1| \leq \frac{1}{m^{1 + \frac{\sigma}{2}}}
\]
for $|z| \leq \frac{\log m}{\sqrt{m}}$. Therefore using the Cauchy inequality, we get

$$
\left| \int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^P \overline{f} e^{-m|z|^2} (e^{m\zeta} \xi - \sum_{k=0}^{2\sigma+1} \frac{m^k}{k!} \zeta_1^k \xi_1) dV_0 \right|
$$

(5.5)

$$
\leq \frac{C \sqrt{P}}{m^{1+\frac{n+|P|+\sigma}{2}}} \|f\| \leq \left( \frac{\log m}{m} \right)^{1+\frac{n+|P|+\sigma}{2}} \cdot \|f\|
$$

by (4.19) and the fact that $|P| \leq \varepsilon (\log m)^2$.

Note that $\zeta_1^k \xi_1$ is a polynomial in $z$ and $\overline{z}$. Let

$$
(5.6) \quad \zeta_1^k \xi_1 = \sum_{IJ} \zeta_{IJ} z^I \overline{z}^J.
$$

The coefficients

(5.7) \quad |\zeta_{IJ}| = \frac{1}{I!J!} \left| \frac{\partial^{|I|+|J|}(\zeta_1^k \xi_1)}{\partial z^I \partial \overline{z}^J}(0) \right| = \frac{1}{I!J!} \left| \frac{\partial^{|I|+|J|}(\zeta_1^k \xi_1)}{\partial z^I \partial \overline{z}^J}(0) \right| \leq C(\sigma)

for $C(\sigma) > 0$ and for $m$ large enough.

If $|I| - |J| < \sigma$, then by the assumption on $f$ and by symmetry,

$$
(5.8) \quad \int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^P \overline{f} e^{-m|z|^2} m^k \zeta_{IJ} z^I \overline{z}^J dV_0 = 0.
$$

On the other hand, under the $K$-coordinates and $K$-frames, in the expansion of $\zeta$, there is no $z^P$ or $\overline{z}^P$ terms. Thus in (5.6), we must have $|J| \geq 2$. If $|I| - |J| \geq \sigma$ and $|J| \geq 2$, then

$$
|I| + |J| - 2k \geq 2 + \sigma.
$$

Hence,

$$
(5.9) \quad \left| \sum_{|I|+|J|-2k\geq 2+\sigma} \zeta_{IJ} z^I \overline{z}^J \right| \leq C(\sigma) |z|^{2k+2+\sigma}.
$$

Using (5.7), (5.8), (4.19), and the above estimate, we have

$$
\left| \int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^P \overline{f} e^{-m|z|^2} m^k \zeta_1^k \xi_1 dV_0 \right|
$$

$$
\leq m^k \|f\| \cdot \left( \int_{|z| \leq \frac{\log m}{\sqrt{m}}} |z^P|^2 \left| \sum_{|I|+|J|-2k\geq 2+\sigma} \zeta_{IJ} z^I \overline{z}^J \right|^2 e^{-m|z|^2} dV_0 \right)^{1/2}
$$

$$
\leq C(\sigma) (\log m)^{-1} \left( \frac{\log m}{m} \right)^{1+\frac{n+|P|+\sigma}{2}} \cdot \|f\|.
$$

Combining the above inequality and (5.4), (5.5), we proved that

$$
(5.10) \quad \left| \int_{|z| \leq \frac{\log m}{\sqrt{m}}} z^P \overline{f} e^{-m|z|^2} \xi_0 dV_0 \right| \leq C(\sigma) (\log m)^{-1} \left( \frac{\log m}{m} \right)^{1+\frac{n+|P|+\sigma}{2}} \cdot \|f\|.
$$

Since $m$ is sufficiently large, the above inequality implies (5.1).
The proof of (3) is similar. We notice that in the analytic case, (5.4) becomes
\[ |e^{m\zeta} - \sum_{k=0}^{2\sigma+1} \frac{m^k}{k!} \zeta^k \xi_1| \leq \frac{C}{m^{1+\frac{\sigma}{2}}} \]
for some constant \( C > 0 \) and (5.5) becomes
\[ \left| \int_{|z| \leq \frac{\log m}{\sqrt{m}} e^{-m|z|^2}} (e^{m\zeta} - \sum_{k=0}^{2\sigma+1} \frac{m^k}{k!} \zeta^k \xi_1) dV_0 \right| \leq C \sigma \left( \frac{(\log m)^2}{m} \right)^{1 + \frac{\sigma}{2}} \|f\| \]
for a possibly larger constant \( C \).

Therefore, we have
\[ \sum_{|I|+|J|-2k\geq 2+\sigma} |\zeta_{IJ} z^I z^J| \leq C \sigma |z|^{2k+2+\sigma} \]
for some constant \( C \). (3) follows from the above analytic version of (5.4), (5.5), and (5.9). This completes the proof of the lemma.

By taking \( \xi = (\det g_{ij}) h_{\alpha\beta} \), we proved the following results, which extend the estimates up to \( \varepsilon (\log m)^2 \).

**Theorem 5.1.** Let \( S_{P,\alpha} = S'_{P,\alpha,m} \) for \( |P| \leq p' \), \( 1 \leq \alpha \leq r \) be the peak sections defined in Lemma 4.1 and let \( T \) be a section with vanishing order at \( x_0 \) at least \( p' \). Then for \( m > m(\sigma) \), we have

(1) \[ \left| (S_{P,\alpha}, T) \right| \leq \frac{C}{m^{3/2}} \|T\|. \]

(2) Moreover,
\[ \left| (S_{P,\alpha}, T) \right| \leq \frac{C}{m^{1+\frac{1}{2}(p' - |P|)}} \|T\|. \]

(3) If the metrics are analytic, then
\[ \left| (S_{P,\alpha}, T) \right| \leq \frac{C\sigma}{m^{1+\frac{1}{2}(p' - |P|)}} \|T\|. \]

Taking a special case, we can prove that

**Corollary 5.1.** The basis \( \{S_j\} \) is an almost orthonormal basis and assumption (2) of Theorem 3.1 is satisfied.

**Theorem 5.2.** For fixed \( i, j \) such that \( |R(i)| + |R(j)| \leq \mu \) and \( \alpha = \zeta(i), \beta = \zeta(j) \), there is an asymptotic expansion
\[ (S_i, S_j) \sim \frac{1}{m^{n+\frac{1}{2}(|R(i)|+|R(j)|)}} \left( A_{0ij} + \frac{A_{1ij}}{m} + \ldots \right) \]
for \( r \times r \) matrices \( A_i \) in the sense that for any \( \mu > 0 \),

1. If the metrics are smooth, then for any \( s \leq \varepsilon (\log m)^2 \), we have

\[
\left| \left( S_i, S_j \right) - \frac{1}{m^{n+\frac{1}{2}(|R(i)|+|R(j)|)}} \left( \frac{A_{0ij}}{m} + \cdots + \frac{A_{sij}}{m^s} \right) \right| \leq \frac{C}{m^{s+1}}.
\]

2. If the metrics are real analytic, then for \( s = [\varepsilon (\log m)^2] \), we have

\[
\left| \left( S_i, S_j \right) - \frac{1}{m^{n+\frac{1}{2}(|R(i)|+|R(j)|)}} \left( \frac{A_{0ij}}{m} + \cdots + \frac{A_{sij}}{m^s} \right) \right| \leq \frac{Cs^{s+1}}{m^{s+1}},
\]

where

\[
\| A_{\xi ij} \|_{C^\mu} \leq C_{\xi} \quad \text{for } \xi \geq 0.
\]

Proof. In [13, Proposition 2.1], (1) was proved for the trivial line bundle \( E \). The proof of the general case is the same so we omit the proof. If the metrics are real analytic, the Taylor expansions of all the metric matrices are convergent. Therefore using the same method as in (1), we prove the second part of the theorem. \( \square \)

Corollary 5.2. The assumptions (3), (4) of Theorem 3.1 are valid.

Proofs of Theorem 1.2 and Theorem 1.3: These two theorems follow from Theorem 3.1 Corollary 4.1 Corollary 5.1 and Corollary 5.2. \( \square \)

Proof of Corollary 1.1: This is the same as the proof of 3.19. \( \square \)

Appendix A. \( K \)-coordinates and \( K \)-frames

Let \( \mu \) be a nonnegative integer. It is well known that other than \( \mu = 0 \), the \( C^\mu \)-norm on the space of smooth functions depends on the choice of local coordinate systems. For two different atlases of the manifold, the two different \( C^\mu \)-norms are equivalent (mutually bounded). Therefore the underlying topology is intrinsically defined by these norms.

In the proof of the main results of this paper, we need to treat uncountably many local coordinate systems. Therefore, it is necessary to look into the details of the definition of \( C^\mu \)-norms.

Let \( P \) be a multiple index: \( P = (p_1, \cdots, p_n) \), where \( p_1, \cdots, p_n \) are nonnegative numbers. Let \( |P| = p_1 + \cdots + p_n \), and let \( P! = p_1! \cdots p_n! \). Define

\[
z^P = z_1^{p_1} \cdots z_n^{p_n}.
\]

Let \( f \) be a smooth function on an open set \( U \) of \( M \) with holomorphic local coordinates \( z = (z_1, \cdots, z_n) \). Define

\[
|D^k f| = \sum_{|P|+|Q|=k} \frac{k!}{P!Q!} |D^P, Q f|
\]

for \( k \geq 0 \), where

\[
P = (p_1, \cdots, p_n), \quad Q = (q_1, \cdots, q_n),
\]

and \( D \) is the differential operator

\[
D^P, Q f = \frac{\partial^{P|+|Q|} f}{\partial z_1^{p_1} \cdots \partial z_n^{p_n} \partial z_1^{q_1} \cdots \partial z_n^{q_n}}.
\]
Lemma A.1. For any $|D^k f|$ defines a nonnegative function on $U$. The $C^\mu$-norm of $f$ on $U$ is defined by
\[ \|f\|_{C^\mu} = \max \sup_{k \leq \mu} |D^k f|(x). \]
The definition of $|D^k f|$ depends on the local coordinates. We use the notation $|D^k f|_z$ to denote such a dependence. Let $w = (w_1, \cdots, w_n)$ be other local coordinates on $U$. Then there is a constant $C > 1$, depending on $z, w$, such that
\[ C^{-1} |D^k f|_z \leq |D^k f|_w \leq C |D^k f|_z. \]

Let $M$ be an $n$-dimensional algebraic manifold with a positive Hermitian line bundle $(L, h_L) \to M$. Let $(E, h_E)$ be a Hermitian vector bundle of rank $r$ over $M$. Suppose that the Kähler form $\omega$ of the Kähler metric $g$ is defined by the curvature $\text{Ric}(h_L)$ of $h_L$. That is, under local coordinates $(z_1, \cdots, z_n)$ at a fixed point $x_0$, we have
\[ \omega_g = -\frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta = 1}^n \partial \bar{\partial} \log a \, d z_\alpha \wedge d \bar{z}_\beta = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta = 1}^n g_{\alpha\bar{\beta}} \partial_\alpha \partial_{\bar{\beta}}, \]
where $a$ is the local representation of the Hermitian metric $h_L$.

Definition A.1. Let $p > 0$ be any positive integer. Let $x_0 \in M$ be a point. Let $(z_1, \cdots, z_n)$ be a holomorphic coordinate system centered at $x_0$. Let $(g_{\alpha\bar{\beta}})$ be the Kähler metric matrix. If it satisfies
\[ g_{\alpha\bar{\beta}}(x_0) = \delta_{\alpha\beta}; \]
\[ \frac{\partial^{p_1 + \cdots + p_n} g_{\alpha\bar{\beta}}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(x_0) = 0 \]
for $\alpha, \beta = 1, \cdots, n$ and any nonnegative integers $(p_1, \cdots, p_n)$ with $p > p_1 + \cdots + p_n \neq 0$, then we call the coordinate system a $K$-coordinate system of order $p$.

The existence of a $K$-coordinate system was known to the string theorists in the 1980s. However, the result had been known to Bochner a long time ago. We refer to [2] for the proof of the existence of $K$-coordinates.

If the metric is analytic, then we can take $p$ to be $+\infty$. In this case, the $K$-coordinate system is unique up to an affine transformation.

Similar to the above, we make the following definition

Definition A.2. Let $e_L$ be a local holomorphic frame of $L$ at $x_0$. If for $p > 0$, the local representation function $a$ of the Hermitian metric $h_L$ satisfies
\[ a(x_0) = 1, \quad \frac{\partial^{p_1 + \cdots + p_n} a}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(x_0) = 0 \]
for any nonnegative integers $(p_1, \cdots, p_n)$ with $p > p_1 + \cdots + p_n \neq 0$, then we call $e_L$ a $K$-frame of order $p$. If $a$ is analytic, then again we can take $p = +\infty$.

The following lemma is similar to the results above. We sketch the proof here.

Lemma A.1. For any $p > 0$, it is possible to choose a local holomorphic frame $\{e_1, \cdots, e_r\}$ of order $p$ on $U$ such that
\[ h_{ij}(x_0) = \delta_{ij}; \]
\[ \frac{\partial^{p_1 + \cdots + p_n} h_{ij}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(x_0) = 0 \]
for any nonnegative integers \((p_1, \cdots, p_n)\) with \(p > p_1 + \cdots + p_n \neq 0\), where \(h_{ij} = \langle e_i, e_j \rangle\). If the metric is analytic, then we can take \(p = +\infty\). Moreover, any derivative of \(h_{ij}\) at \(x_0\) can be represented as a polynomial of curvatures of both \(E\) and \(M\) and their derivatives.

**Proof.** We first choose a holomorphic frame \(\{e_1, \cdots, e_r\}\) such that \(h_{ij}(x_0) = \delta_{ij}\). Let the matrix \(H = (h_{ij})\). Let the Taylor expansion of \(H\) be

\[
H \sim I + A + \overline{A} + B,
\]

where \(I\) is the identity matrix, \(A\) is the holomorphic part of the Taylor expansion, \(\overline{A}\) is the complex conjugate of \(A\), and \(B\) is the mixed part, that is, the entries of \(B\) are composed of \(z\)'s and \(\overline{z}\)'s.

Since \(P^T = H\), we must have \(A = AT\). Let \(p > 2\) be an integer. Let \(A_p\) be the first \(p\) terms in the formal series \(A\). Define a new frame \(\{f_1, \cdots, f_r\}\) such that

\[
e_i = f_i + (A_p)_{ij} f_j.
\]

It is not hard to see that under the new frame, the metric matrix is \(\tilde{H} = (I + A_p)^{-1}H(I + A_p)^{-1}\). A straightforward computation shows that in the Taylor expansion of \(\tilde{H}\), there are no holomorphic or anti-holomorphic parts up to order \(p\).

Finally, we have the formula

\[
\frac{\partial^2 h_{ij}}{\partial z_\alpha \partial \overline{z}_\beta} = (\Theta_E)_{ij\alpha\beta} + h^{kl} \frac{\partial h_{ij}}{\partial z_\alpha} \cdot \frac{\partial h_{kl}}{\partial \overline{z}_\beta}.
\]

By the above equation and by induction, all derivatives of \((h_{ij})\) at \(x_0\) can be expressed as polynomials of \(\Theta_E\), the curvature of \(M\), and their covariant derivatives.

Because of the above lemma, we will call the frames \(e_L\) and \(\{e_j\}\) \(K\)-frames, that is, the local holomorphic coordinates, the local frames of \(L\), and the local frame of \(E\) will satisfy \((A.1)\), \((A.2)\), and \((A.3)\).

As we have discussed before, \(K\)-coordinates and \(K\)-frames are not unique. However, in what follows, we will write out explicitly a smooth family of \(K\)-frames and \(K\)-coordinates near any given point.

Let \(a(z) = \langle e_L, e_L \rangle\), \(H(z) = (h_{ij})\), \(G(z) = (g_{ij})\). Let \(P\) be the multiple index: \(P = (p_1, \cdots, p_n)\). We define \(|P| = \sum p_i\); \(P! = p_1! \cdots p_n!\); \(z^P = z_1^{p_1} \cdots z_n^{p_n}\); and

\[
f^{(P)}(z) = \frac{\partial^P f}{\partial z^P}(z)
\]

for a smooth function \(f\). Using this notation, \((A.1)\), \((A.2)\), and \((A.3)\) can be written as

\[
g_{\alpha\beta}(x_0) = \delta_{\alpha\beta}, a(x_0) = 1, h_{ij}(x_0) = \delta_{ij},
\]

and

\[
(g_{\alpha\beta})^{(P)}(x_0) = 0, a^{(P)}(x_0) = 0, (h_{ij})^{(P)}(x_0) = 0
\]

for \(0 < |P| < p\).

---

\(7\)If \(H\) is analytic, the Taylor expansion must be convergent. If \(H\) is smooth, the expansion is understood as *formal*: it doesn’t have to converge, and even if it does, it doesn’t have to converge to the matrix-valued function \(H\).
For any $p > 2$ and any $t \in U$ with $|t|$ very small, we define
\[ a_{ij}(t) = \frac{\partial^2 \log a(z)}{\partial z_i \partial \bar{z}_j} \bigg|_{z=t}, \]
\[ b^P_1(z, t) = \sum_{1 \leq |P| \neq 0} \frac{(\log a)^{(P)}(t)}{P!} (z - t)^P, \]
\[ b^P_2(z, t) = \sum_{j=1}^n \sum_{1 < |P| \leq p} \frac{1}{P!} \frac{\partial (\log a)^{(P)}(t)}{\partial \bar{t}_j} (z - t)^P (\bar{z}_j - \bar{t}_j). \]
We write
\[ \log a(z) = \log a(t) + a_{ij}(t)(z_i - t_i)(\bar{z}_j - \bar{t}_j) + b^P_1(z, t) + b^P_2(z, t) + c^p(z, t). \]
By the definition of $c^p$, we have
\[ \frac{\partial^P c^p}{\partial z^P} \bigg|_{z=t} = 0, \quad \frac{\partial^P c^p}{\partial \bar{z}_j \partial z^P} \bigg|_{z=t} = 0 \]
for $|P| \leq p$ and $1 \leq j \leq n$. Let $G^p(z, t), R^p(z, t)$ be the matrix-valued functions such that
\[ G^p(z, t)_{ij} = -\frac{\partial^2 b^P_1(z, t)}{\partial z_i \partial \bar{z}_j}, \quad R^p(z, t)_{ij} = -\frac{\partial^2 c^p(z, t)}{\partial z_i \partial \bar{z}_j}. \]
Then we have
\[ G(z) = G(t) + G^p(z, t) + \overline{G^p(z, t)^T} + R^p(z, t) \]
with
\[ \frac{\partial^P R^p}{\partial z^P} \bigg|_{z=t} = 0 \]
for $0 \leq |P| \leq p - 1$.

Let $D(t)$ be a smooth Hermitian matrix-valued function such that $D(0) = I$ and $D(t)^2 = G(t)$. Such a function exists. For example, if $\sum a_j x^j$ is the Taylor expansion of the function $\sqrt{1+x}$, then we can define $D(t) = \sum a_j (G(t) - I)^j$.

We use the following matrix notation: $z' = (z'_1, \cdots, z'_n)$, $z = (z_1, \cdots, z_n)$, $t = (t_1, \cdots, t_n)$, and
\[ \frac{\partial b^P_2}{\partial \bar{z}} = (\frac{\partial b^P_2}{\partial \bar{z}_1}, \cdots, \frac{\partial b^P_2}{\partial \bar{z}_n}). \]
We define
\[ z' = zD(t) - tD(t) - \frac{\partial b^P_2(z, t)}{\partial \bar{z}} D(t)^{-1}. \]
Then $z'$ is a holomorphic coordinate system centered at $t$. Differentiating the above, we have
\[ dz' = dz(D(t) + G^p(z, t)D(t)^{-1}). \]
Under the new coordinates, the metric-matrix is
\[ P(z, t)^{-1}G(z)(P(z, t)^{-1})^*, \]
where $P(z, t) = D(t) + G^p(z, t)D(t)^{-1}$. By (A.6), the above matrix is equal to
\[ I + P(z, t)^{-1}(R^p(z, t) - G^p(z, t)G^{-1}(t)G^p(z, t)^T)(P(z, t)^{-1})^T. \]
If \( z = t \), then the above matrix is the identity matrix. Moreover, since \( P(z, t) \) is holomorphic with respect to \( z \) and \( G^p(t, t) = 0 \), from (A.7), we have
\[
\frac{\partial P}{\partial z} \bigg|_{z=t} P(z, t)^{-1}(R^p(z, t) - G^p(z, t)G^{-1}(t)G^p(z, t)) (P(z, t)^{-1})^T = 0
\]
for any \( |P| < p - 1 \). By the chain rule, we conclude that \( z' \) is a \( K \)-coordinate system for any \( t \).

Using a similar way, we can construct \( K \)-frames as well.

We let
\[
\xi^p(z, t) = \sum_{|P| \leq p} \frac{a(P)(t)}{P!} (z - t)^P;
\]
(A.9)
\[
B^p(z, t) = \sum_{|P| \neq 0} \frac{H(P)(t)}{P!} (z - t)^P,
\]
and we write
(A.10)
\[
a(z) = a(t) + \xi^p(z, t) + \overline{\xi^p(z, t)} + \eta^p(z, t); \\
H(z) = H(t) + B^p(z, t) + B^p(z, t)^T + C^p(z, t).
\]

As before, we let \( K(t) \) be a smooth Hermitian matrix-valued function such that \( K(t)^2 = H(t) \). We can rewrite
(A.11) \[
a(z) = \left( \sqrt{a(t)} + \frac{\xi^p(z, t)}{\sqrt{a(t)}} \right) \left( \sqrt{a(t)} + \frac{\overline{\xi^p(z, t)}}{\sqrt{a(t)}} \right) + \tilde{\eta}^p(z, t);
\]
(A.12) \[
H(z) = (K(t) + B^p(z, t)K(t)^{-1})(K(t) + B^p(z, t)K(t)^{-1})^* + \tilde{C}^p(z, t),
\]
where
\[
\tilde{\eta}^p(z, t) = \eta^p(z, t) - \frac{|\xi^p(z, t)|^2}{a(t)}; \\
\tilde{C}^p(z, t) = C^p(z, t) - B^p(z, t)H(t)B^p(z, t)^T.
\]

As above, we know that
(A.13) \[
\frac{\partial P}{\partial z} \bigg|_{z=t} \tilde{\eta}^p(z, t) = 0;
\]
(A.14) \[
\frac{\partial P}{\partial z} \bigg|_{z=t} \tilde{C}^p(z, t) = 0
\]
for \(|P| < p\).

Define
(A.15) \[
e_L(t) = \left( \sqrt{a(t)} + \frac{\xi^p(z, t)}{\sqrt{a(t)}} \right)^{-1} e_L;
\]
(A.16) \[
e(t) = e((K(t) + B^p(z, t)K(t)^{-1})^{-1})^T,
\]
where \( e = (e_1, \ldots, e_r) \) and \( e(t) = (e_1(t), \ldots, e_r(t)) \).
Then by (A.11), we have
\[
\langle e_L(t), e_L(t) \rangle = 1 + \tilde{\eta}^p(z, t) \left( \sqrt{a(t)} + \frac{\xi^p(z, t)}{\sqrt{a(t)}} \right)^{-2}.
\]

By (A.13), we know that in the Taylor expansion of \(\langle e_L(t), e_L(t) \rangle\) at \(t\), there are no holomorphic parts up to the order \(p\). This proves that \(e_L(t)\) is a \(K\)-frame for any \(t\). Similarly, let
\[
Q(z, t) = K(t) + B^p(z, t)K(t)^{-1}.
\]

Then by (A.12), we have
\[
e(t)^T e(t) = I + Q(z, t)^{-1} \tilde{C}^p(z, t)(Q(z, t)^{-1})^T.
\]

By (A.14), \(e(t)\) is a \(K\)-frame for any \(t\).

In summary, we prove

**Lemma A.2** (The first stability lemma). For \(|t|\) small, (A.8), (A.13), and (A.16) define a smooth family of \(K\)-frames and \(K\)-coordinates. Moreover, for any \(t\), the \(K\)-frames and \(K\)-coordinates are defined on the set
\[|z'| < \delta/3,
\]
where \(\delta\) is the injectivity radius of \(M\) and \((z'_1, \ldots, z'_n)\) are the \(K\)-coordinate system at \(t\).

\(\square\)

**Corollary A.1.** Let \(f\) be a smooth function on \(M\). Assume that at any \(x_0 \in M\) and for any \(K\)-coordinates \((z_1, \ldots, z_n)\) at \(x_0\),
\[
\sup_{x_0, k \leq \mu} |D^k f|(x_0) \leq 1.
\]
Then there is a constant \(C(\mu)\), depending on \(\mu\), such that
\[
\|f\|_{C^\mu} \leq C(\mu).
\]

\(\square\)

By the above lemma and the continuity, the following must be true: for any \(+\infty > p > 0\) and \(x \in M\), there exists a \(\rho = \rho_{x,p} > 0\) such that:

1. For each \(|t| < \rho\), the smooth \(K\)-frames and \(K\)-coordinates exist and are at least of size \(\rho\), that is, \(||z'|| < \rho\) is contained in the \(K\)-coordinate system.
2. Define
\[
B_{1,k}(t) = \sum_{\ell \leq k} \frac{1}{\ell!} \|D^\ell \log a(t)\|((\delta/3)^\ell);
\]
\[
B_{2,k}(t) = \max_{i,j} \sum_{\ell \leq k} \frac{1}{\ell!} \|D^\ell h_{Ei\overline{j}}(t)\|((\delta/3)^\ell);
\]
\[
A_k(t) = \sum_{k_1 + k_2 = k} B_{1,k_1}(t) \cdot B_{2,k_2}(t),
\]
where \(a(t)\) and \(h_{Ei\overline{j}}(t)\) are the metric representations under the \(K\)-frames and \(K\)-coordinates. Then \(A_k(t) \leq 2A_k(0)\) for any \(p > k \geq 0\).
3. Let \(U_t = \{|z'| < \rho\}\) be the \(K\)-coordinate neighborhood. Then on \(U_t\), we have
\[
\log a(t) > \frac{1}{2}, \quad g_{ij}(t) > \frac{1}{2} \delta_{ij}, \quad h_{ij} \geq \frac{1}{2} \delta_{ij}.
\]
Since $M$ is compact, finitely many of the above neighborhoods cover $M$. Therefore, we are able to define the norms $A_k$ as in the last section.

We have the following

**Lemma A.3** (The second stability lemma). For any $+\infty \geq p > 0$ and for any point $x \in M$, there exist $K$-frames and $K$-coordinates of order $p$ such that:

1. $A_k(t) \leq 2A_k(0)$, where $t \in U_t$.
2. On $U_t$, (A.17) is valid.

**Proof.** The lemma essentially follows from continuity and compactness. □

**Acknowledgements**

The first author thanks Chin-Lung Wang for his encouragement during the preparation of this paper. She also thanks Chang-Shou Lin for his support during her stay in TIMS. The second author thanks Kengo Hirachi for his interest in our work and the discussions on the topic for many years.

**References**


Department of Mathematics, National Taiwan University, Taipei, Taiwan 106
E-mail address: cjliu4@ntu.edu.tw

Department of Mathematics, University of California, Irvine, California 92697-3875
E-mail address: zlu@uci.edu