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NILPOTENT AND ABELIAN HALL SUBGROUPS IN FINITE GROUPS

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Abstract. We give a characterization of the finite groups having nilpotent or abelian Hall π-subgroups that can easily be verified using the character table.

1. Introduction

One of the main themes in finite group theory is to study the interaction between global and local structure: if p is a prime and G is a finite group, we seek to analyze the relationship between G and its p-local subgroups. Ideally, a local property of G can be read off from the character table of G.

There are not many theorems analyzing local and global structure from the point of view of two different primes, as we do in the main result of this paper.

Theorem A. *Let G be a finite group, and let p and q be different primes. Then some Sylow p-subgroup of G commutes with some Sylow q-subgroup of G if and only if the class sizes of the q-elements of G are not divisible by p and the class sizes of the p-elements of G are not divisible by q.*

Theorem A, of course, gives us a characterization (detectable in the character table) of when a finite group possesses nilpotent Hall $\{p, q\}$-subgroups. The existence of Hall subgroups (nilpotent or not) is a classical subject in finite group theory, with extensive literature. This characterization in our Theorem A can be seen as a contribution to Richard Brauer's Problem 11. In his celebrated paper [Br], Brauer asks about obtaining information about the existence of (certain) subgroups of a finite group given its character table.

In the course of proving Theorem A we show that for finite simple groups, commuting Sylow subgroups for different primes are actually always abelian; see Theorem 2.1.

Several possible extensions of Theorem A are simply not true. For instance, the fact that $p \nmid |G : \mathbf{C}_G(x)|$ for every q-element $x \in G$ does not guarantee that a Sylow p-subgroup of G commutes with some Sylow q-subgroup of G: the semi-affine groups of order $q^p(q^p - 1)p$ provide solvable examples for every choice of p and q.

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Furthermore, if $p \nmid |G : \mathbf{C}_G(x)|$ for every q-element $x \in G$, it is not even true that P must normalize some Sylow q-subgroup of G. ($G = M_{23}$ for $(q,p) = (3,5)$, or $G = J_4$ with $(q,p) = (3,7)$ are examples.) However, this assertion is true if G is p-solvable or q-solvable, or if $p = 2$, as we will show in Section 4 below.

Recently, the finite groups with nilpotent Hall subgroups have also received attention in [M]. Using Theorem A and the results of [M], we can deduce the following.

Theorem B. *Let G be a finite group, and let π be a set of primes. Then G has nilpotent Hall π-subgroups if and only if for every pair of distinct primes $p, q \in \pi$, the class sizes of the p-elements of G are not divisible by q.*

Theorem B gives an algorithm to determine from the character table if a group has nilpotent Hall π-subgroups. (The somewhat weaker result that the property of having nilpotent Hall subgroups is shared by groups with the same character table was proved in [KS].)

As a consequence of Theorem B and the main results of [NT] and [NST], we now have the following explicit way to detect from the character table of a finite group G whether G possesses an abelian Hall π-subgroup, for any set π of primes.

Theorem C. *Let G be a finite group, and let π be a set of primes. Then G has abelian Hall π-subgroups if and only if the two following conditions hold:*

(i) *For every $p \in \pi$ and every p-element $x \in G$, $|G : \mathbf{C}_G(x)|$ is a π'-integer.*
(ii) *If $\{p\} = \pi \cap \{3,5\}$, then for every irreducible character χ in the principal p-block of G, $\chi(1)$ is not divisible by p.*

The examples of $(G, \pi, p) = (Ru, \{3\}, 3)$ and $(Th, \{5\}, 5)$ show that condition (ii) cannot be removed from Theorem C.

As might be expected, the proofs of Theorems A, B and C use the Classification of Finite Simple Groups.

2. Simple groups and Theorem A

2.1. **Results for simple groups.** The goal of this section is to prove Theorem 2.1, which yields a strong form of Theorem A for simple groups, and another auxiliary result, Theorem 2.2.

Theorem 2.1. *Let X be a finite non-abelian simple group and let p and q be distinct prime divisors of $|X|$. Suppose that $q \nmid |X : \mathbf{C}_X(g)|$ for every p-element $g \in X$ and $p \nmid |C : \mathbf{C}_X(h)|$ for every q-element $h \in X$. Then $p, q > 2$ and X has an abelian Hall $\{p, q\}$-subgroup.*

Indeed, Theorem 2.1 holds for all finite quasi-simple groups, as follows easily from the arguments in this paper. However, since, in contrast to Theorem A, we claim existence of abelian Hall subgroups, Theorem 2.1 obviously does not generalize to arbitrary finite groups.

Theorem 2.2. *Let X be a finite non-abelian simple group of order divisible by an odd prime r. Then X contains a conjugacy class of r-elements of even size.*

We begin with some obvious observations. If $x \in G$, then we denote by x^G the conjugacy class of x in G.

Lemma 2.3. (i) *For each simple group X it suffices to prove the analogue of Theorems* 2.1 *and* 2.2 *for some quasi-simple group L such that $X \cong L/\mathbf{Z}(L)$.*

(ii) *The case* $\min(p,q) = 2$ *of Theorem* 2.1 *follows from Theorem* 2.2.

Proof. (i) Suppose that X satisfies the hypothesis of Theorem 2.1, and that the analogue of Theorem 2.1 holds for some quasi-simple group L with $X = L/\mathbf{Z}(L)$. Let $g \in L$ be any p-element and let $D/Z := \mathbf{C}_X(gZ)$ for $Z := \mathbf{Z}(L)$. Then for any $x \in D$ we have $xgx^{-1} = f(x)g$ for some $f(x) \in Z$. Moreover, $f \in \mathrm{Hom}(D, Z)$, $\mathrm{Ker}(f) = \mathbf{C}_L(g) =: C$, and $f(x)^{|g|} = 1$ for all $x \in G$, i.e. $\mathrm{Im}(f)$ is a p-group. Thus $|D : C|$ is a p-power. It follows that the integers $|(gZ)^X| = |L : D|$ and $|g^L| = |L : C|$ have the same q-part and so $|g^L|$ is coprime to q. Similarly, $|h^L|$ is coprime to p for all q-elements $h \in L$. Since the analogue of Theorem 2.1 holds for L, L contains an abelian Hall $\{p,q\}$-subgroup $P \times Q$, whence $(P \times Q)Z/Z$ is an abelian Hall $\{p,q\}$-subgroup for X.

A similar argument proves the part of the claim concerning Theorem 2.2.

(ii) Suppose $p = 2 < q$. Since $q \mid |X|$, by Theorem 2.2 there is a conjugacy class x^X of q-elements in X of even size, a contradiction. □

Lemma 2.4. *Theorems* 2.1 *and* 2.2 *hold in the case X is an alternating group, a sporadic group, or* ${}^2F_4(2)'$.

Proof. The case of the 26 sporadic groups and ${}^2F_4(2)'$ can be checked directly using [GAP]. (We remark that the only examples among the sporadic groups are J_1 for $\{p,q\} = \{3,5\}$ and J_4 for $\{p,q\} = \{5,7\}$, and in both cases S has cyclic Hall $\{p,q\}$-subgroups.) Suppose that $X = \mathsf{A}_n$ and $n \geq p > q \geq 2$. Then $\mathbf{C}_{\mathsf{S}_n}(g) \cong C_p \times \mathsf{S}_{n-p}$ for a p-cycle $g \in S$. If $p \geq 5$, we have that $|g^{\mathsf{S}_n}| = \binom{n}{p}(p-1)!$ is divisible by $2q$ and so q divides $|g^X|$. If $p = 3$, then $n \geq p+2$ and so $|g^X| = |g^{\mathsf{S}_n}|$ is again divisible by q. □

The rest of the section is devoted to proving Theorems 2.1 and 2.2 for simple groups of Lie type $X \ncong {}^2F_4(2)'$. For the sake of convenience, we rename p, q in Theorem 2.1 to r, s. We will consider the following setup: $X = \mathbf{G}^F/\mathbf{Z}(\mathbf{G}^F)$, where $\mathbf{G}$ is a simple simply connected algebraic group over the algebraic closure of a finite field of characteristic p and $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism. We let q denote the absolute value of all eigenvalues of F on the character group of an F-stable maximal torus of $\mathbf{G}$. Also, we will sometimes use the notation SL^ϵ to denote SL when $\epsilon = +$ and SU when $\epsilon = -$, and similarly for PSL^ϵ, GL^ϵ. Furthermore, $E_6^\epsilon(q)$ denotes $E_6(q)$ when $\epsilon = +$ and ${}^2E_6(q)$ when $\epsilon = -$. In these cases, ϵ will also be treated as $\epsilon 1$ in numerical expressions like $q^m - \epsilon^n$, etc. Finally, for types E_6 and E_7 we will use subscripts sc for groups of simply connected type.

Lemma 2.5. *Theorems* 2.1 *and* 2.2 *hold in the case X is a simple group of Lie type in characteristic $r = p$.*

Proof. By Lemma 2.4 we may assume that $(X, p) \neq ({}^2F_4(2)', 2)$. By Lemma 2.3(i) we may replace X by $G := \mathbf{G}^F$. Set $s := 2$ in the case of Theorem 2.2. By [C, Prop. 5.1.7], G contains a regular unipotent p-element $g \in G$. We claim that s divides $|g^G|$. Indeed, otherwise $s \nmid |g^G|$ and so $\mathbf{C}_G(g)$ contains a Sylow s-subgroup S of G. But every semisimple element in $\mathbf{C}_{\mathbf{G}}(g)$ belongs to $\mathbf{Z}(\mathbf{G})$ by [C, Prop. 5.1.5], whence $S \leq \mathbf{Z}(G)$. It follows that $s \nmid |X|$, a contradiction. □

Proof of Theorem 2.2. Assume the contrary: $|X : \mathbf{C}_X(x)|$ is odd for all r-elements $x \in X$. By Lemmas 2.4 and 2.5 we see that X is a simple group of Lie type in characteristic p for some prime $p \neq r$ and $(X, p) \neq ({}^2F_4(2)', 2)$.

Note that if $1 \neq g \in X$ is a **real** r-element, then $|\mathbf{N}_X(\langle g \rangle) : \mathbf{C}_X(g)|$ is even and so $|g^X|$ is even. Thus X cannot contain any real r-element $g \neq 1$. By [TZ, Prop. 3.1], it follows that

$$X \in \{\mathrm{PSL}_n(q), \mathrm{PSU}_n(q) \mid n \geq 3\} \cup \{P\Omega^{\pm}_{4n+2}(q) \mid n \geq 2\} \cup \{E_6(q), {}^2E_6(q)\}. \tag{1}$$

By Lemma 2.3(i) we may replace X by $G := \mathbf{G}^F$, or by some quotient $L = G/Z$ with $Z \leq \mathbf{Z}(G)$.

Suppose first that $G = \mathrm{SL}_n(q)$ with $n \geq 3$, and set $k := \mathrm{ord}_r(q) \leq n$. If $k \leq n/2$, then $\mathrm{SL}_n(q) \geq \mathrm{Sp}_{2k}(q)$ contains a non-trivial real r-element by [TZ, Prop. 3.1], a contradiction. Similarly, if $2|k$, then again $\mathrm{SL}_n(q) \geq \mathrm{Sp}_k(q)$ contains a non-trivial real r-element. Thus $k > n/2$ and k is odd. Now it is easy to see that $H := \mathrm{SL}_k(q)$ contains an r-element g with $\mathbf{C}_H(g) \cong C_{(q^k-1)/(q-1)}$. Embedding H naturally in $\mathrm{SL}_n(q)$, we get that

$$|g^G| = \frac{|\mathrm{GL}_n(q)|}{(q^k-1) \cdot |\mathrm{GL}_{n-k}(q)|} = q^{\binom{n}{2}-\binom{n-k}{2}} \cdot \frac{\prod_{j=n-k+1}^{n}(q^j-1)}{q^k-1}$$

is even, again a contradiction.

The same argument as above applies to the case $G = \mathrm{SU}_n(q)$ if we replace q by $-q$.

Suppose now that $X = P\Omega^{\epsilon}_{4n+2}(q)$ with $n \geq 2$. Then we replace X by $L = \Omega^{\epsilon}_{4n+2}(q)$. If $r | \prod_{j=1}^{2n}(q^{2j}-1)$, then $\Omega^{\epsilon}_{4n+2}(q) > \Omega_{4n+1}(q)$ contains a non-trivial real r-element by [TZ, Prop. 3.1], a contradiction. So $r \nmid \prod_{j=1}^{2n}(q^{2j}-1)$ but $r|(q^{2n+1}-\epsilon)$. Now it is easy to see that $H := \mathrm{SO}^{\epsilon}_{4n+2}(q)$ contains an r-element $g \in \Omega^{\epsilon}_{4n+2}(q)$ with $\mathbf{C}_H(g) \cong C_{q^{2n+1}-\epsilon}$. It follows that $|g^L|$ is even, a contradiction.

Finally, let $G := E^{\epsilon}_6(q)_{\mathrm{sc}}$. If r divides $|F_4(q)|$, then $G > F_4(q)$ contains a non-trivial real r-element by [TZ, Prop. 3.1], a contradiction. So $r \nmid |F_4(q)|$ but $r|(q^5-\epsilon)(q^9-\epsilon)$. In particular, r is a Zsigmondy prime divisor for $q^5-\epsilon$ or $q^9-\epsilon$. Inspecting the centralizers of semisimple elements in G of order divisible by r, as described in [D], one sees that there exist r-elements h with $|h^G|$ even, again a contradiction.

An alternate way to deal with the simple groups X in (1) is as follows. Let $P \in \mathrm{Syl}_2(G)$. If q is even, then $\mathbf{C}_G(P) \leq \mathbf{Z}(G)P$ by [C, Prop. 5.1.5] and so no non-central r-element $x \in G$ can centralize P, a contradiction. Hence q is odd. In this case, by [GLS, Theorem 4.10.6], there is a commuting product $S(P) \neq 1$ of fundamental $\mathrm{SL}_2(q)$-subgroups in G and a Cartan subgroup H of G normalizing $S(P)$ such that $\mathbf{C}_G(P) \leq S(P)H$. In particular, any odd prime divisor of $|\mathbf{C}_G(P)|$ divides q^2-1. By assumption, some non-trivial r-element $x \in G$ centralizes P, whence $r|(q^2-1)$. But then a direct factor $\mathrm{SL}_2(q)$ of $S(P)$ contains a real r-element $y \neq 1$, again a contradiction. □

We will now prove the following result, which, together with Lemmas 2.4, 2.5 and Theorem 2.2, implies Theorem 2.1.

Theorem 2.6. *Let X be a finite non-abelian simple group of Lie type defined over $\mathbb{F}_q$, q a power of a prime p, and let r and s be distinct odd prime divisors of $|X|$ different from p. Suppose that $s \nmid |X : \mathbf{C}_X(g)|$ for every r-element $g \in X$ and*

$r \nmid |C : \mathbf{C}_X(h)|$ for every s-element $h \in X$. Then X has an abelian Hall $\{r, s\}$-subgroup.

2.2. **Proof of Theorem 2.6 for classical groups.** Throughout this subsection, we assume that X is a simple classical group.

Proposition 2.7. *Theorem 2.6 holds in the case where $X = \mathrm{PSL}_n^\epsilon(q)$, $\epsilon = \pm$, and at least one of the primes r, s divides $q - \epsilon$.*

Proof. For definiteness, assume $r|(q-\epsilon)$. By Lemma 2.3(i), we may replace X by $G := \mathrm{SL}_n^\epsilon(q)$. Let $V = \mathbb{F}_q^n$, respectively $\mathbb{F}_{q^2}^n$, denote the natural G-module for $\epsilon = +$, respectively $\epsilon = -$. In a suitable basis of V, a Sylow r-subgroup R of G contains a subgroup R_T of order $r^{t(n-1)}$, with $r^t := (q-\epsilon)_r$, of the diagonal subgroup $T \cong C_{q-\epsilon}^{n-1}$ of G, and moreover $\mathbf{C}_G(R_T) = T$. By assumption, any s-element $y \in G$ is centralized by a Sylow r-subgroup of G. Thus a conjugate of y is centralized by R_T and so is contained in T. It follows that $|y|$ divides $q - \epsilon$, whence $s|(q-\epsilon)$.

Suppose now that $n \geq r+1$. Then we can find $\alpha \in \overline{\mathbb{F}}_q^\times$ of order $(q^r - \epsilon)_r = r^{t+1}$ and consider the r-element $g \in G$ conjugate (in $\mathbf{G} := \mathrm{SL}_n(\overline{\mathbb{F}}_q)$) to

$$\mathrm{diag}(\alpha, \alpha^{q\epsilon}, \ldots, \alpha^{(q\epsilon)^{r-1}}, \alpha^{-\frac{(q\epsilon)^r-1}{q\epsilon-1}}, \underbrace{1, \ldots, 1}_{n-r-1}).$$

Note that $\alpha^{-\frac{(q\epsilon)^r-1}{q\epsilon-1}}$ has order r. It follows that

$$|g^G| = \frac{|\mathrm{SL}_n^\epsilon(q)|}{(q^r-\epsilon)\cdot|\mathrm{GL}_{n-r-1}^\epsilon(q)|} = \frac{|\mathrm{GL}_n^\epsilon(q)|}{|\mathrm{GL}_{r+1}^\epsilon(q)|\cdot|\mathrm{GL}_{n-r-1}^\epsilon(q)|}\cdot\frac{|\mathrm{GL}_{r+1}^\epsilon(q)|}{(q^r-\epsilon)(q-\epsilon)}$$

which is divisible by

$$\prod_{j=2}^{r-1}(q^j - \epsilon^j)\cdot(q^{r+1}-1),$$

a multiple of s, a contradiction.

We have shown that $n \leq r$, and so $n \leq s$ as well by symmetry. Assume now that $n = r < s$. Then we can find $\beta \in \mathbb{F}_q^\times$ of order s and consider the s-element

$$h := \mathrm{diag}(\beta, \beta^{-1}, \underbrace{1, \ldots, 1}_{r-2}) \in G.$$

Then

$$|h^G| = \frac{|\mathrm{SL}_r^\epsilon(q)|}{(q-\epsilon)\cdot|\mathrm{GL}_{r-2}^\epsilon(q)|} = q^{2r-3}\cdot\frac{q^r-\epsilon}{q-\epsilon}\cdot\frac{q^{r-1}-1}{q-\epsilon}$$

which is divisible by r, again a contradiction.

Consequently, $n < \min(r, s)$. In this case, we have that T contains an abelian Hall $\{r, s\}$-subgroup of G. □

Corollary 2.8. *Suppose that $G = \mathrm{SL}_r^\epsilon(q)$ with $r|(q-\epsilon)$. Then $\mathbf{C}_G(R) = \mathbf{Z}(G) = \mathbf{Z}(R)$ for $R \in \mathrm{Syl}_r(G)$.*

Proof. In the notation of the proof of Proposition 2.7, we can choose $R = \langle R_T, g\rangle$, where g is a permutation matrix of order r in the chosen basis of V. Suppose $x \in \mathbf{C}_G(R)$. Then $x \in \mathbf{C}_G(R_T) = T$ is diagonal. Now the condition $[x, g] = 1$ implies that x acts via scalars on V, and so $x \in \mathbf{Z}(G)$. Also, $\mathbf{Z}(G) = \mathbf{Z}(R)$ as $\dim V = r$. □

In view of Proposition 2.7, in the case X is a simple classical group of Theorem 2.6, we may assume that r, s are both coprime to $q - \epsilon = |\,\mathrm{GL}_n^\epsilon(q) : \mathrm{SL}_n^\epsilon(q)|$ if $X = \mathrm{PSL}_n^\epsilon(q)$. This observation, together with Lemma 2.3(i), implies that in this case we may replace X by $G := \mathrm{GL}_n^\epsilon(q)$. On the other hand, since $r, s > 2$, in the case $X = P\Omega_d^\epsilon(q)$ we may replace X by $G := \mathrm{GO}_d^\epsilon(q)$ (the full orthogonal group on a d-dimensional quadratic space of type ϵ over $\mathbb{F}_q$; in particular, ϵ is vacuous if d is odd). Similarly, $X = \mathrm{PSp}_{2n}(q)$ can be replaced by $G := \mathrm{Sp}_{2n}(q)$. Thus in what follows we will prove Theorem 2.6 for these group G that we just defined:

$$G = \mathrm{GL}_n(q),\ \mathrm{GU}_n(q),\ \mathrm{Sp}_{2n}(q),\ \mathrm{GO}_d^\epsilon(q). \tag{2}$$

Correspondingly, let $\mathbb{F} := \mathbb{F}_q$, $\mathbb{F}_q^2$, $\mathbb{F}_q$, or $\mathbb{F}_q$, $\mathrm{Cl} := \mathrm{GL}$, GU, Sp, or GO, and let $V := \mathbb{F}^n$, $\mathbb{F}^n$, $\mathbb{F}^{2n}$, or $\mathbb{F}^d$, respectively, denote the natural G-module. In the case $\mathrm{Cl} = \mathrm{GL}$, we will assume formally that V is endowed with the zero bilinear form.

As before, let r be an odd prime divisor of $|G|$ coprime to q. For the groups G in (2), let

$$e_r := \mathrm{ord}_r(q),\ \mathrm{ord}_r(-q),\ \mathrm{ord}_r(q),\ \mathrm{ord}_r(q),$$

respectively. If $\mathrm{Cl} = \mathrm{GL}^\epsilon$, let $d_r := e_r$. If $\mathrm{Cl} = \mathrm{Sp}$ or GO, let $d_r := \mathrm{lcm}(2, e_r)$. Furthermore, if $\mathrm{Cl} = \mathrm{GO}$, define $\epsilon_r := +$ if e_r is odd and $\epsilon_r := -$ of e_r is even. It is shown in [GL, Chapter 3, §8] that if Z is a cyclic subgroup of G of order r, then a non-trivial $\mathbb{F}Z$-submodule $V(r)$ of V is orthogonally (with respect to the bilinear or hermitian form on V) indecomposable if and only if

$$\dim_{\mathbb{F}}(V(r)) = d_r$$

and furthermore $V(r)$ has type ϵ_r if $\mathrm{Cl} = \mathrm{GO}$.

Now, as shown in [GL, Chap. 4, §10] and [GLS, Chap. 4, §4.10], a Sylow r-subgroup R of G has the form $R = R_T \rtimes R_W$, with R_T (the "toral part" of R) being homocyclic abelian. Furthermore, there is an orthogonal decomposition of V as $\mathbb{F}R_T$-module:

$$V = V_0 \perp V_1 \perp \ldots \perp V_m,$$

with $V_i \cong V(r)$ for $1 \le i \le m$ (and V_0 may be the zero subspace). Next, $\dim_{\mathbb{F}} V_0 < d_r$ if $\mathrm{Cl} = \mathrm{GL}^\epsilon$ or Sp. If $\mathrm{Cl} = \mathrm{GO}$, then either $\dim_{\mathbb{F}} V_0 < d_r$, or $\dim_{\mathbb{F}} V_0 = d_r$ but V_0 has type $-\epsilon_r$. Moreover,

$$R_T = R_1 \times \ldots \times R_m,$$

with R_i a cyclic subgroup of a cyclic maximal torus T_i of the isometry group $\mathrm{Cl}(V_i)$ of V_i, acting orthogonally indecomposably on V_i and trivially on V_j for all $j \ne i$. Note that $|T_1| = q^{d_r} - \epsilon^{d_r}$ in the case $\mathrm{Cl} = \mathrm{GL}^\epsilon$ and $|T_1| = q^{d_r/2} + (-1)^{e_r}$ in the case $\mathrm{Cl} = \mathrm{Sp}$. If $\mathrm{Cl} = \mathrm{GO}$, then $|T_1| = q^{d_r/2} - \epsilon_r$; in particular,

$$r | (q^{d_r/2} - \epsilon_r). \tag{3}$$

In either case, if $i > 0$ and $1 \ne x \in T_i$, then x fixes no non-zero vector of V_i. Furthermore, for

$$T = T_1 \times \ldots \times T_m$$

one has that

(a) $\mathbf{C}_G(R_T) = \mathrm{Cl}(V_0) \times T$, and
(b) there is a subgroup Σ of $\mathbf{N}_G(T)$ with Σ isomorphic to the symmetric group S_m acting naturally on the sets $\{V_1, \ldots, V_m\}$, $\{T_1, \ldots, T_m\}$, and $\{R_1, \ldots, R_m\}$, and with R_W (the "Weyl part" of R) being a Sylow r-subgroup of Σ.

Again, we can define d_s, $V(s)$, ϵ_s, subgroups $S_1, \ldots, S_{m'}$, cyclic tori $\tilde{T}_1, \ldots, \tilde{T}_{m'}$, and an orthogonal decomposition

$$V = U_0 \perp U_1 \ldots \perp U_{m'}$$

of V as $\mathbb{F}S_T$-module for $S = S_T \rtimes S_W$, where $S \in \mathrm{Syl}_s(G)$; in particular, $S_T = S_1 \times \ldots \times S_{m'}$ and $S_i \leq \tilde{T}_i \leq \mathrm{Cl}(U_i)$. Since the roles of r and s are symmetric, we may assume that

$$d_r \leq d_s, \text{ and } s < r \text{ if } d_r = d_s. \tag{4}$$

Proposition 2.9. *Theorem* 2.6 *holds for simple classical groups X in the case s divides $|T_1|$.*

Proof. (i) By assumption, T_1 contains an element x of order s, where $T_1 \leq \mathrm{Cl}(V_1)$ and V_1 is an orthogonal direct summand of V. Hence $d_r = \dim_{\mathbb{F}} V_1 \geq d_s$, and so $d_r = d_s =: d$ by the choice made in (4). If furthermore $\mathrm{Cl} = \mathrm{GO}$, then $s|(q^{d/2} - \epsilon_r) = |T_1|$ by the assumption and also $s|(q^{d/2} - \epsilon_s) = |\tilde{T}_1|$ by (3). It follows that

$$\epsilon_r = \epsilon_s. \tag{5}$$

In particular, the quadratic spaces V_1 and U_1 are isometric when $\mathrm{Cl} = \mathrm{GO}$. Clearly, V_1 and U_1 are also isometric in the other cases as they have the same dimension.

(ii) Here we show that $m = m'$. Indeed, note that

$$\dim_{\mathbb{F}} U_0 + m'd = \sum_{i=0}^{m'} \dim_{\mathbb{F}} U_i = \dim_{\mathbb{F}} V = \sum_{i=0}^{m} \dim_{\mathbb{F}} V_i = \dim_{\mathbb{F}} V_0 + md. \tag{6}$$

If $\dim_{\mathbb{F}} V_0,\ \dim_{\mathbb{F}} U_0 < d$, then

$$|(m - m')d| = |\dim_{\mathbb{F}} V_0 - \dim_{\mathbb{F}} U_0| < d$$

and so $m = m'$. Assume next that $\dim_{\mathbb{F}} V_0 = d$. It follows that $\mathrm{Cl} = \mathrm{GO}$, and V has type $-\epsilon_r^{m+1}$, as V_0 has type $-\epsilon_r$. Now if $U_0 = 0$, then $m' = m + 1$ by (6), and V has type ϵ_r^{m+1} according to (5), a contradiction. Hence $0 < \dim_{\mathbb{F}} U_0 \leq d$, whence $\dim_{\mathbb{F}} U_0 = d$ as $d | \dim_{\mathbb{F}} V = (m+1)d$, and so $m' = m$ by (6).

It now follows that $\dim_{\mathbb{F}} U_0 = \dim_{\mathbb{F}} V_0$. As mentioned at the end of (i), U_i and V_i are isometric when $i = 1, 2, \ldots, m$. Hence, U_0 and V_0 are isometric by Witt's theorem. So without loss we may now assume that $U_i = V_i$ for all i. In turn, this allows us to write

$$S_T = S_1 \times \ldots \times S_m$$

with $S_i < T_i$. (Indeed, in the case of GL^ϵ, $|T_1| = q^d - \epsilon^d = |\tilde{T}_1|$. If $\mathrm{Cl} = \mathrm{GO}$, then $|T_1| = q^{d/2} - \epsilon_r$ and we have $\epsilon_r = \epsilon_s$ by (5). Finally, when $\mathrm{Cl} = \mathrm{Sp}$, $s|(q^{d/2} + (-1)^{e_r}) = |T_1|$ and $s|(q^{d/2} + (-1)^{e_s}) = |\tilde{T}_1|$ imply $e_r = e_s$. Thus in all cases, the role of the cyclic torus $\tilde{T}_1$ for s can be played by the cyclic torus T_1 for r.)

(iii) Next we show that $s \nmid |\mathrm{Cl}(V_0)|$. This is clear in the case $\dim_{\mathbb{F}} V_0 < d$. In particular, we are done if $\mathrm{Cl} = \mathrm{GL}^\epsilon$ or Sp. Assume now that $\mathrm{Cl} = \mathrm{GO}$ and $\dim_{\mathbb{F}} V_0 = d$. In this case, V_0 has type $-\epsilon_r = -\epsilon_s$, and so $\mathrm{GO}(V_0)$ cannot contain any element of order s as well.

(iv) Now choose an element $y \in G$ of order $|y| = \exp(S) \geq s$. By hypothesis, we may assume that y centralizes R. In particular, $y \in \mathbf{C}_G(R_T) = \mathrm{Cl}(V_0) \times T$. Since

$s \nmid |\operatorname{Cl}(V_0)|$ by (iii), $y \in T$. As $T = T_1 \times \ldots \times T_m$ is homocyclic, $|y| \leq |T_1|_s =: s^a$, and so

$$\exp(S) \leq s^a. \tag{7}$$

Assume in addition that $m \geq s$. Then we may assume that S_W contains an element π that permutes $V_1, \ldots, V_s$ cyclically and fixes each V_j with $j > s$. Also choose $z \in T_1$ of order s^a. It is easy to check that

$$(z\pi)^s = \operatorname{diag}(z, z^\pi, z^{\pi^2}, \ldots, z^{\pi^{s-1}}) \in T_1 \times \ldots \times T_s$$

and so $z\pi \in G$ has order s^{a+1}, contrary to (7).

We have shown that $m < s < r$ (recalling (4)). In this case,

$$R_W = S_W = 1, \ R_T, S_T \leq T,$$

whence T contains $R \times S$, an abelian Hall $\{r, s\}$-subgroup of G, completing the proof. □

Proposition 2.10. *Theorem* 2.6 *holds for simple classical groups in the case* $s \nmid |T_1|$.

Proof. Consider any non-trivial s-element $x \in G$. By assumption, some conjugate gxg^{-1} of x centralizes R, and so $gxg^{-1} \in \mathbf{C}_G(R_T) = \operatorname{Cl}(V_0) \times T$. But $s \nmid |T_1|$, so $gxg^{-1} \in \operatorname{Cl}(V_0)$. It follows that $V_0 \neq 0$ and $\operatorname{Cl}(V_0)$ contains elements of order s. In particular,

$$d_s \leq \dim_{\mathbb{F}} V_0 \leq d_r,$$

whence $\dim_{\mathbb{F}} V_0 = d_r = d_s =: d$ by (4) and $\operatorname{Cl} = \operatorname{GO}$. Since the type of V_0 is $-\epsilon_r$, we must also have that $\epsilon_s = -\epsilon_r$. Furthermore,

$$\dim_{\mathbb{F}} \mathbf{C}_V(x) = \dim_{\mathbb{F}} \mathbf{C}_V(gxg^{-1}) \geq \sum_{i=1}^{m} \dim_{\mathbb{F}} V_i = md. \tag{8}$$

Now we choose

$$x = \operatorname{diag}(z_1, z_2, \ldots z_{m'}) \in S_1 \times \ldots \times S_{m'}$$

with $|z_i| = s$. As noted above, z_i acts fixed-point-freely on U_i, and so

$$\dim_{\mathbb{F}} \mathbf{C}_V(x) = \dim_{\mathbb{F}} U_0 \leq d.$$

Together with (8), this implies that $m = 1$, $\dim_{\mathbb{F}} V = 2d$, $m' = 1$, and $\dim_{\mathbb{F}} U_0 = d$.

We have shown that $V = V_0 \perp V_1 = U_0 \perp U_1$, with $V_1 \cong U_0$ of type $\epsilon_r = -\epsilon_s$ and $U_1 \cong V_0$ of type $\epsilon_s = -\epsilon_r$. Without loss we may assume that $U_0 = V_0$ and $U_1 = V_1$. Now $R = R_T < \operatorname{GO}(V_1)$ and $S < \operatorname{GO}(U_1) = \operatorname{GO}(V_0)$. It follows that

$$G > \operatorname{GO}(V_0) \times \operatorname{GO}(V_1) > R \times S.$$

Therefore $R \times S$ is an abelian Hall $\{r, s\}$-subgroup of G, completing the proof. □

2.3. **Proof of Theorem 2.6 for exceptional groups.** We now turn to the case of exceptional simple groups of Lie type. In what follows, we consider the cyclotomic polynomials, over $\mathbb{Q}(\sqrt{p})$ in the case of Suzuki and Ree groups (see [BM, §F]) and over $\mathbb{Q}$ otherwise, that occur in the generic order of G.

First we record a simple observation:

Lemma 2.11. *Let G be a finite group and r be a prime. Assume that $x \in G$ is an element of order r such that $\mathbf{C}_G(x)$ contains a Sylow r-subgroup R of G. Assume in addition that $\mathbf{C}_G(x)$ contains a normal subgroup D, where $|\mathbf{C}_D(R_1)|$ divides $t|\mathbf{Z}(R_1)|$ for a Sylow r-subgroup R_1 of D and for some integer t. Then $|\mathbf{C}_G(R)|$ divides $t|\mathbf{Z}(R)| \cdot |\mathbf{C}_G(x)/D|$.*

Proof. Note that $x \in \mathbf{Z}(R)$ and we can take $R_1 = D \cap R$. Now,

$$\mathbf{C}_D(R) = \mathbf{C}_D(R_1) \cap \mathbf{C}_D(R)$$

contains a normal subgroup $\mathbf{Z}(R_1) \cap \mathbf{C}_D(R) \leq \mathbf{Z}(R)$ of index dividing t. It follows that $|\mathbf{C}_D(R)|$ divides $t|\mathbf{Z}(R)|$. Since $\mathbf{C}_G(R) \leq \mathbf{C}_G(x)$ and $D \lhd \mathbf{C}_G(x)$, we conclude that $|\mathbf{C}_G(R)|$ divides $t|\mathbf{Z}(R)| \cdot |\mathbf{C}_G(x)/D|$. □

Proposition 2.12. *Let $\mathbf{G}$ be a simple simply connected algebraic group such that $G = \mathbf{G}^F$ is of exceptional type. Let r be an odd prime that either divides the order of the Weyl group of $\mathbf{G}$, or divides two distinct polynomials $\Phi_d(q)$ occurring in the generic order of G. Then one of the following holds for a Sylow r-subgroup R of G:*

(a) $\mathbf{C}_G(R) = \mathbf{Z}(G)\mathbf{Z}(R)$;
(b) $G = E_6(q)_{\mathrm{sc}}$, $r = 5|(q-1)$, $|\mathbf{C}_G(R)|$ *divides* $|\mathbf{Z}(R)|(q-1)^2$;
(c) $G = {}^2E_6(q)_{\mathrm{sc}}$, $r = 5|(q+1)$, $|\mathbf{C}_G(R)|$ *divides* $|\mathbf{Z}(R)|(q+1)^2$;
(d) $G = E_7(q)_{\mathrm{sc}}$, $r = 3$, $|\mathbf{C}_G(R)|$ *divides* $|\mathbf{Z}(R)|(q-\epsilon)$ *where* $q \equiv \epsilon \pmod 3$;
(e) $G = E_7(q)_{\mathrm{sc}}$, $r = 5$, $|\mathbf{C}_G(R)|$ *divides* $|\mathbf{Z}(R)|(q-\epsilon)^3$ *where* $q \equiv \epsilon \pmod 5$;
(f) $G = E_7(q)_{\mathrm{sc}}$, $r = 7$, $|\mathbf{C}_G(R)|$ *divides* $|\mathbf{Z}(R)|(q-\epsilon)$ *where* $q \equiv \epsilon \pmod 7$; *or*
(g) $G = E_8(q)$, $r = 7$, $|\mathbf{C}_G(R)|$ *divides* $|\mathbf{Z}(R)|(q-\epsilon)^2$ *where* $q \equiv \epsilon \pmod 7$.

Proof. If r divides two distinct cyclotomic polynomials occuring in the generic order of G, then r divides the order of the Weyl group of $\mathbf{G}$, by [BM, Cor. 3.13]. Hence the assumptions exclude the Suzuki and Ree groups, except for $G = {}^2F_4(q^2)$ with $r = 3$.

In all cases with $r = 3$, and also when $r = 5$ and $G = E_8(q)$, the normalizer of a Sylow r-subgroup of G is given in [MN, Tab. 1], and conclusion (a) follows, except when $G = E_7(q)$ with $r = 3$. In the latter case, we have $3|(q-\epsilon)$ for some $\epsilon = \pm 1$. Let x be a central element of order 3 in a Levi subgroup $L \leq G$ of type $E_6^\epsilon T_1$, where $|T_1| = q - \epsilon$. Since any reductive overgroup of L has center of order coprime to 3, we must have $L = \mathbf{C}_G(x)$. Note that $\mathbf{C}_G(x)$ contains a Sylow 3-subgroup R of G, and a normal subgroup $D \cong E_6^\epsilon(q)$ of index $q - \epsilon$. Also, $\mathbf{Z}(D) < R_1 := D \cap R \in \mathrm{Syl}_3(D)$ and so by the $E_6^\pm$ case we have that $\mathbf{C}_D(R_1) = \mathbf{Z}(R_1)$. Hence $|\mathbf{C}_G(R)|$ divides $|\mathbf{Z}(R)|(q-\epsilon)$ by Lemma 2.11 (taking $t = 1$).

The only remaining possibilities for primes dividing two cyclotomic polynomials are

(i) $r = 5$, $G = E_6^\epsilon(q)_{\mathrm{sc}}$ and $q \equiv \epsilon \pmod 5$,
(ii) $r = 5, 7$, $G = E_7(q)_{\mathrm{sc}}$ and $r|(q^2-1)$, or
(iii) $r = 7$, $G = E_8(q)$ and $r|(q^2-1)$.

First assume that $G = E_6(q)_{\mathrm{sc}}$. Then, a Sylow 5-subgroup R of G is contained in a Levi subgroup L of type $A_4A_1T_1$. Let $x \in L$ be a central 5-element of L. Since any reductive overgroup of L has center of order coprime to 5, we must have $L = \mathbf{C}_G(x)$. Furthermore, $\mathbf{C}_G(x)$ contains a normal subgroup $D \cong \mathrm{SL}_5(q) \times \mathrm{SL}_2(q)$

of index $q-1$. By Corollary 2.8 the centralizer of a Sylow 5-subgroup of the SL_5-factor is just its center. For the SL_2-factor, the centralizer of a Sylow 5-subgroup has order $q-1$. It follows that $|\mathbf{C}_D(R_1)|$ divides $(q-1)|\mathbf{Z}(R_1)|$ for $R_1 \in \mathrm{Syl}_r(D)$, and so the claim follows by Lemma 2.11 (with $t = q-1$) in this case. Entirely similar arguments apply for ${}^2E_6(q)_{\mathrm{sc}}$.

If $G = E_7(q)_{\mathrm{sc}}$ with $r = 5$, then our assumption gives $5|(q-\epsilon)$ for some $\epsilon = \pm 1$ and again a Sylow 5-subgroup of G lies inside a Levi subgroup L of type $A_4A_2T_1$. As before there exists a 5-element x in the center of L with $L = \mathbf{C}_G(x)$. Furthermore, $\mathbf{C}_G(x)$ contains a normal subgroup $D \cong \mathrm{SL}_5^{\epsilon}(q) \times \mathrm{SL}_3^{\epsilon}(q)$ of index $q-\epsilon$. By Corollary 2.8 the centralizer of a Sylow 5-subgroup of the SL_5-factor is just its center. For the SL_3^{ϵ}-factor, the centralizer of a Sylow 5-subgroup has order $(q-\epsilon)^2$. It follows that $|\mathbf{C}_D(R_1)|$ divides $(q-\epsilon)^2|\mathbf{Z}(R_1)|$ for $R_1 \in \mathrm{Syl}_r(D)$, and so the claim follows by Lemma 2.11 (with $t = (q-\epsilon)^2$) in this case.

If $G = E_7(q)_{\mathrm{sc}}$ with $r = 7$, then our assumption gives $7|(q-\epsilon)$ for some $\epsilon = \pm 1$, and R is contained in a Levi subgroup L of type A_6T_1 and again a 7-element $x \in \mathbf{Z}(L)$ satisfies $L = \mathbf{C}_G(x)$. Furthermore, $\mathbf{C}_G(x)$ contains a normal subgroup $D \cong \mathrm{SL}_7^{\epsilon}(q)$ of index $q-\epsilon$. By Corollary 2.8 the centralizer of a Sylow 7-subgroup of the SL_7-factor is just its center. Hence the claim follows by Lemma 2.11 (taking $t = 1$).

Finally, if $r = 7$ and $G = E_8(q)$, then our assumption gives $7|(q-\epsilon)$ for some $\epsilon = \pm 1$. Here, R is contained in a Levi subgroup of type $A_1A_6T_1$, which contains a central 7-element x with $L = \mathbf{C}_G(x)$. Our previous arguments go through unchanged. □

Next, for Suzuki–Ree groups let Φ_{d_1} be the cyclotomic polynomial occurring in the generic order of G such that $r|\Phi_{d_1}(q)$ as in [BM, App. 2], and similarly Φ_{d_2} for s. For other groups, let d_1 and d_2 be the order of q modulo r, respectively s. Then there exist corresponding Sylow d_i-tori as defined in [BM, 3.14].

Proposition 2.13. *Assume that the Sylow d_1-tori of $\mathbf{G}$ are maximal tori and that s divides a unique cyclotomic factor in the generic order of G. If every s-element of G centralizes a Sylow r-subgroup of G, then $d_1 = d_2$.*

Proof. Let g be an s-element such that $\mathbf{C}_G(g)$ contains a Sylow r-subgroup. Then by [BM, Cor. 3.13], g lies in a maximal torus $\mathbf{T}$ containing a Sylow d_1-torus $\mathbf{T}_1$ of $\mathbf{G}$. Since $\mathbf{T}_1$ is a maximal torus by assumption, $\mathbf{T} = \mathbf{T}_1$. So s divides $\Phi_{d_1}(q) = |\mathbf{T}_1^F|$, which by our assumption implies that $d_2 = d_1$. □

Proposition 2.14. *Assume that $d_1 = d_2$ and neither of r, s divides the order of the Weyl group of $\mathbf{G}$. Then there exists an abelian $\{r, s\}$-Hall subgroup of G.*

Proof. Since neither r nor s divides the order of the Weyl group of $\mathbf{G}$, any Sylow d-torus $\mathbf{T}_d$ of $\mathbf{G}$, where $d = d_1 = d_2$, has the property that $\mathbf{T}_d^F$ contains a Sylow r-subgroup R and a Sylow s-subgroup S of G by [BM, Cor. 3.13]. In particular, $R \times S$ is an abelian Hall $\{r, s\}$-subgroup of G. □

Now we can complete the proof of Theorem 2.6:

Theorem 2.15. *The assertion of Theorem 2.6 holds for X a simple exceptional group of Lie type.*

Proof. As before, there exists a simple, simply connected algebraic group $\mathbf{G}$ with a Steinberg endomorphism F such that $X = G/\mathbf{Z}(G)$, where $G := \mathbf{G}^F$. By Lemma 2.3(i), we may replace X by G.

First assume that r, say, satisfies the assumptions of Proposition 2.12. If both r and s satisfy this condition, then choose r to be the smaller one among r and s. Let R denote a Sylow r-subgroup of G. In case (a) of that result we see that any s-element in $\mathbf{C}_G(R)$ is central in G and so $s \nmid |X|$, a contradiction. In all of the remaining cases (b)–(g) necessarily s divides $q \pm 1$. We will exhibit a Levi subgroup occurring as the centralizer of an s-element of order dividing $q \pm 1$ but not containing a Sylow r-subgroup. For $G = E_6(q)$, take a Levi of type $A_2^2A_1T_1$, cf. [D, p. 120] (note that here $s \geq 7$ by the choice of r), and similarly for ${}^2E_6(q)$. For $G = E_7(q)$ with $r = 3$, we have $s|(q - \epsilon)$, where $q \equiv \epsilon \pmod 3$. But $E_7(q)$ has a Levi subgroup of type D_6T_1, with center a torus of order $q - \epsilon$, and s-elements (which have order at least 5) in that center do not centralize a Sylow 3-subgroup of G. For $E_7(q)$ with $r = 5$ or $r = 7$ we may take a Levi of type $A_3A_2A_1T_1$, and for $E_8(q)$ with $r = 7$ a Levi of type $E_6A_1T_1$ will do.

We may now assume that each of r and s divides a unique cyclotomic polynomial $\Phi_{d_i}(q)$ occurring in the generic order of G, but does not divide the order of the Weyl group of $\mathbf{G}$. If $d_1 = d_2$, we are done by Proposition 2.14. Otherwise, by Proposition 2.13 we have that neither of the Sylow d_i-tori are maximal tori of $\mathbf{G}$. Then d_1 and d_2 are as in Table 1. If, say, the Sylow r-subgroups are cyclic, then G possesses an abelian Hall $\{r, s\}$-subgroup as the Sylow s-subgroups are abelian. So we may assume that the Sylow d_i-tori are non-cyclic. In these cases, the last column of the table gives certain centralizers of r-elements x that do not contain a Sylow s-subgroup. (In all cases, x is chosen in the central torus of the Levi subgroup.) This final contradiction completes the proof of the theorem. □

Table 1. The case of non-maximal non-cyclic Sylow tori

G	d_i	
${}^3D_4(q)$	$1, 2$	$(q+1).A_1(q^3)$
$E_6(q)_{\mathrm{sc}}$	$2, 4;\ 2, 6$	$(q^2-1).{}^2D_4(q)$
$E_6(q)_{\mathrm{sc}}$	$4, 6$	$(q^2+1)(q-1).{}^2A_3(q)$
${}^2E_6(q)_{\mathrm{sc}}$	$1, 3; 1, 4$	$(q^2-1).{}^2D_4(q)$
${}^2E_6(q)_{\mathrm{sc}}$	$3, 4$	$(q^2+1)(q+1).A_3(q)$
$E_7(q)_{\mathrm{sc}}$	$3, 4, 6$	$(q^3 \pm 1).{}^3D_4(q)$

On the last line of Table 1, $\{d_1, d_2\}$ can be any 2-subset of $\{3, 4, 6\}$.

3. Proofs of Theorems A, B and C

We start with a trivial observation.

Lemma 3.1. *Suppose that $N \triangleleft G$, and let p and q be primes. If all the p-elements of G have conjugacy class size not divisible by q, then the same happens in G/N and N.*

Proof. If $x \in N$ is a p-element, then $|N : \mathbf{C}_N(x)|$ divides $|G : \mathbf{C}_G(x)|$, which is not divisible by q, by hypothesis. If $Nx \in G/N$ is a p-element, then $Nx = Ny$ where y is the p-part of x. If $D/N = \mathbf{C}_{G/N}(Nx)$, then $\mathbf{C}_G(y)N/N \leq D/N$ and $|G : D|$ divides $|G : \mathbf{C}_G(y)|$, which is not divisible by q, by hypothesis. □

Lemma 3.2. *Let q be a prime. Suppose that a q-group Q acts coprimely on a finite group N. Let p be a prime and let P be a Q-invariant Sylow p-subgroup of N. Assume that for every $x \in Q$, there exists $n \in N$ such that $[x, P^n] = 1$. Then $[P, Q] = 1$.*

Proof. We argue by induction on $|Q|$. Suppose that R is a maximal subgroup of Q. By induction, we have that $[R, P] = 1$. Suppose that S is another maximal subgroup of Q. Then $[S, P] = 1$, and therefore $[Q, P] = 1$ since $RS = Q$. Hence, we conclude that Q has a unique maximal subgroup. Then $Q/\Phi(Q)$ is cyclic, and therefore $Q = \langle x \rangle$. Now, by hypothesis, there is $n \in N$ such that $[Q, P^n] = 1$. In particular, P^n is Q-invariant. Since P is Q-invariant, by [G, Thm. (6.2.2)] there is $c \in \mathbf{C}_N(Q)$ such that $P = (P^n)^c \leq \mathbf{C}_N(Q)$, as desired. □

We will use the following consequence in several places below.

Corollary 3.3. *Suppose that G is a finite group, and let p, q be different primes. Assume that every q-element of G has conjugacy class of size not divisible by p. Suppose that $N \lhd G$ is a q'-group. If $Q \in \mathrm{Syl}_q(G)$ and $P \in \mathrm{Syl}_p(N)$ is Q-invariant, then $[Q, P] = 1$.*

Proof. Let $x \in Q$. By hypothesis, there is $P_1 \in \mathrm{Syl}_p(G)$ such that $[x, P_1] = 1$. Since $G = \mathbf{N}_G(P)N$ by the Frattini argument, we have that $P^n \leq P_1$ for some $n \in N$, and thus $[x, P^n] = 1$. Now Lemma 3.2 applies. □

Now, we are ready to prove Theorem A of the introduction. Recall that if a finite group G has a nilpotent Hall π-subgroup H, then every π-subgroup of G is contained in some G-conjugate of H by a well-known theorem of Wielandt [W].

Theorem 3.4. *Suppose that G is a finite group, and let p and q be different primes. Then G has nilpotent $\{p, q\}$-Hall subgroups if and only if $q \nmid |G : \mathbf{C}_G(x)|$ for every p-element $x \in G$, and $p \nmid |G : \mathbf{C}_G(y)|$ for every q-element $y \in G$.*

Proof. The "only if" direction is obvious. For the "if" direction, we assume that G satisfies the condition

$$(^*): \quad \begin{array}{l} \text{for every } p\text{-element } x \in G, q \nmid |G : \mathbf{C}_G(x)|, \text{ and} \\ \text{for every } q\text{-element } y \in G, p \nmid |G : \mathbf{C}_G(y)|. \end{array}$$

We prove by induction on $|G|$ that G has a nilpotent Hall $\{p, q\}$-subgroup. Write $\pi := \{p, q\}$.

The condition $(^*)$ is inherited by quotients and normal subgroups, by Lemma 3.1.

Let $1 < N$ be a normal subgroup of G. By induction, we know that G/N has a nilpotent Hall π-subgroup H/N. Suppose that $|N|$ is not divisible by p or q. Then we use the Schur–Zassenhaus theorem in H to get a nilpotent Hall π-subgroup of G.

Suppose now that $|N|$ is not divisible by p. Let $P \in \mathrm{Syl}_p(G)$ and let $Q \in \mathrm{Syl}_q(N)$ be P-invariant (which we know to exist by coprime action). By Corollary 3.3, we have that $[P, Q] = 1$. Now, recall that G/N has a nilpotent Hall π-subgroup H/N. Thus, using the Frattini argument and the Schur–Zassenhaus theorem in the group

$\mathbf{N}_H(Q)/Q$ with respect to the normal subgroup $\mathbf{N}_N(Q)/Q$, we have that $\mathbf{N}_H(Q)/Q$ has a nilpotent Hall π-subgroup U/Q, which we may assume contains P. Now, notice that U is a Hall π-subgroup of G. Write $U/Q = (S/Q) \times (PQ/Q)$, where $S \in \mathrm{Syl}_q(U)$. (In particular, $S \in \mathrm{Syl}_q(G)$.) Then $[P, S] \leq Q$ and $[S, P, P] = 1$. Thus $[S, P] = 1$ by coprime action.

Hence we may assume that the order of every proper normal subgroup is divisible by p and q.

Let N be a minimal normal subgroup of G. Hence $N = S_1 \times \cdots \times S_k$, where S_i is a non-abelian simple group of order divisible by pq, and G transitively permutes the set $\Omega = \{S_1, \ldots, S_k\}$. Let B be the kernel of the action. We claim that G/B is a q'-group. Otherwise, let Bx be an element of order q, where x has q-power order. Now, by hypothesis, we have that $[x, P] = 1$ for some Sylow p-subgroup P of G. Then, $P \cap N \in \mathrm{Syl}_p(N)$, and in fact $P \cap N = (P \cap S_1) \times \ldots \times (P \cap S_k)$. Now, let $y \in P \cap S_i$ be of order p. Then $[y, x] = 1$, and therefore we deduce that x has to normalize S_i. Then $x \in \bigcap_i \mathbf{N}_G(S_i) = B$ and this is a contradiction. This shows that G/B is a q'-group and by symmetry a p'-group. But then B contains both Sylow p-subgroups and q-subgroups of G and therefore, by induction, we may assume that $B = G$. Thus N is a simple group of order divisible by pq.

We show now that G can be assumed to have no proper solvable quotients. Suppose that G/K has prime order, where $K \lhd G$. By induction, we know that K has nilpotent Hall π-subgroups. If G/K is a π'-group, then the nilpotent Hall π-subgroups of G are Hall subgroups of G and we are done. Therefore, we assume (by symmetry) that G/K has order p, so we may write $G = K\langle x\rangle$ for some p-element $x \in G$. By hypothesis, let $Q \in \mathrm{Syl}_q(G) = \mathrm{Syl}_q(K)$ such that $[Q, x] = 1$. Since K has nilpotent Hall π-subgroups, there is $P \in \mathrm{Syl}_p(K)$ such that $[Q, P] = 1$. In particular, $|K : \mathbf{C}_K(Q)|$ is not divisible by p. Since $x \in \mathbf{C}_G(Q)$, we have that $G = K\mathbf{C}_G(Q)$. Hence $|G : \mathbf{C}_G(Q)| = |K : \mathbf{C}_K(Q)|$ is not divisible by p, and there is some Sylow p-subgroup P_1 of G such that $[Q^k, P_1] = 1$. We deduce that G has nilpotent Hall π-subgroups, and in this case the theorem is proved.

Now, since $G/N\mathbf{C}_G(N)$ is isomorphic to a subgroup of $\mathrm{Out}(N)$, then $G/N\mathbf{C}_G(N)$ is solvable and we conclude that $G = N \times \mathbf{C}_G(N)$. Since $\mathbf{C}_G(N)$ has nilpotent Hall π-subgroups by induction, we conclude that N cannot be proper in G, because otherwise N and therefore G would have nilpotent Hall π-subgroups. Thus $G = N$ is simple of order divisible by pq and so Theorem 2.1 applies. □

Theorem B immediately follows from Theorem A, by using the following.

Lemma 3.5. *Let G be a finite group and let π be a set of primes. Assume that π contains at least two prime divisors of $|G|$. If G has nilpotent Hall τ-subgroups for every $\tau \subseteq \pi$ with $|\tau| = 2$, then G has nilpotent Hall π-subgroups.*

Proof. This is [M, Lemma 3.4]. (See the comment that follows the proof.) □

Proposition 2.3 of [KS] provides a different reduction of Theorem B to simple groups. However, it is easier to check Theorem 2.1 than Theorem B for simple groups.

Proof of Theorem C. The "only if" direction is an obvious consequence of the main result of [KM]. For the "if" direction, note that condition (i) implies by Theorem B that G has nilpotent Hall π-subgroups. Assume for instance that Sylow p-subgroups of G are non-abelian for some $p \in \pi$. By [NT, Main Theorem] and condition (i),

$p \in \{3,5\}$. Next, conditions (i), (ii), and the Main Theorem of [NST] imply that $\pi \supseteq \{3,5\}$. Also, using condition (i) and [NST, Theorem B], we see that G admits a non-abelian composition factor S, where $S \in \{Ru, J_4, {}^2F_4(q)'\}$ if $p = 3$ and $S \cong Th$ if $p = 5$. In either case, $|S|$ is divisible by both 3 and 5. Furthermore, S satisfies condition (i) by Lemma 3.1. In particular, if $x \in S$ has order 5, then $\mathbf{C}_S(x)$ contains a Sylow 3-subgroup R of S. This is obviously false for $S \in \{Ru, Th\}$. This does not hold for $S \cong {}^2F_4(q)'$ either, since in this case $\mathbf{C}_S(R) = \mathbf{Z}(R)$ by Proposition 2.12. It follows that Sylow p-subgroups of G are abelian for all $p \in \pi$, and so we are done. □

4. Some solvability conditions

Next, we show that a version of Theorem A is possible under weaker hypotheses if we allow some solvability conditions.

Theorem 4.1. *Let p, q be primes, and let G be a finite group. Assume that all the q-elements have conjugacy class sizes not divisible by p. If G is p-solvable or q-solvable, then a Sylow p-subgroup of G normalizes some Sylow q-subgroup of G.*

Proof. We argue by induction on $|G|$. Assume first that G is p-solvable. Let $K = \mathbf{O}_{p'}(G)$ and let $L/K = \mathbf{O}_p(G/K)$. Let Kx be a q-element of G/K, where x is a q-element of G. Now, $[x, P] = 1$ for some Sylow p-subgroup P of G, by hypothesis. Thus $[Kx, PK/K] = 1$ and therefore Kx centralizes L/K. By Hall-Higman 1.2.3. Lemma, it follows that $Kx \in L/K$, and thus $Kx = K$. We conclude that G/K is a q'-group. In particular, K contains a Sylow q-subgroup of G. Now, P acts coprimely on K, and by coprime action, it follows that P normalizes some Sylow q-subgroup of K, which is a Sylow q-subgroup of G.

Assume now that G is q-solvable. If $1 < N \triangleleft G$, then by induction we know that there exists $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$ such that P normalizes NQ. If $\mathbf{O}_q(G) > 1$, then we set $N = \mathbf{O}_q(G)$, P normalizes $QN = Q$, and we are done. So we may assume that $\mathbf{O}_{q'}(G) = N > 1$. Now Q acts coprimely on N. By coprime action Q normalizes some $P_1 \in \mathrm{Syl}_p(N)$. By Corollary 3.3, we have that $[Q, P_1] = 1$. In particular, $|N : \mathbf{N}_N(Q)|$ is not divisible by p. Now $NQ \triangleleft NQP$ and by the Frattini argument, we have that $NQP = N\mathbf{N}_{NQP}(Q)$. Then $|NQP : \mathbf{N}_{NQP}(Q)| = |N : \mathbf{N}_N(Q)|$ is not divisible by p. Hence some Sylow p-subgroup of NQP (and hence of G) normalizes Q. □

It is an interesting problem to study if the property that a Sylow p-subgroup normalizes some Sylow q-subgroup is detectable by the character table. (More generally, what does the character table of G know about $\nu_q(G)$, the number of Sylow q-subgroups of G?) We can easily solve this question in p-solvable groups.

Theorem 4.2. *Suppose that G is p-solvable. Let $q \neq p$ be another prime. Then some Sylow p-subgroup of G normalizes some Sylow q-subgroup of G if and only if $G/\mathbf{O}_{p'}(G)$ is a q'-group.*

Proof. Suppose that G/N is a q'-group, where $N = \mathbf{O}_{p'}(G)$. Let $P \in \mathrm{Syl}_p(G)$. Then P acts coprimely on N, and by coprime action it stabilizes some Sylow q-subgroup of N, which is a Sylow q-subgroup of G.

Conversely, suppose that $P \in \mathrm{Syl}_p(G)$ normalizes $Q \in \mathrm{Syl}_q(G)$. We show by induction on $|G|$ that $G/\mathbf{O}_{p'}(G)$ is a q'-group. If $N \triangleleft G$, then we have that PN/N normalizes QN/N, so by applying induction in $G/\mathbf{O}_{p'}(G)$, we may assume that

$\mathbf{O}_{p'}(G) = 1$. Now, let $K = \mathbf{O}_p(G)$. By hypothesis, we have that K normalizes Q. Also Q normalizes K. Since $Q \cap K = 1$, then we conclude that $[Q, K] = 1$. Then $Q \leq \mathbf{C}_G(K) \leq K$ by Hall-Higman's Lemma 1.2.3, and $Q = 1$. This concludes the proof. □

Finally, we prove that if the prime 2 is involved in the following form, then we can obtain a certain solvability.

Theorem 4.3. *Let q be an odd prime, and let G be a finite group. If all the q-elements of G have conjugacy class size not divisible by 2, then G is q-solvable. In particular, a Sylow 2-subgroup of G normalizes some Sylow q-subgroup of G.*

Proof. We argue by induction. If $1 < N$ is a proper normal subgroup of G, then G/N and N are q-solvable. So we may assume that G is a non-abelian simple group of order divisible by $2q$ and appeal to Theorem 2.2. The last part follows from Theorem 4.1. □

We cannot reverse the primes in the previous theorem, not even to obtain normalizing conditions between Sylow subgroups. All the 2-elements of $G = J_1$ have conjugacy class not divisible by 5, and no Sylow 5-subgroup of G normalizes any Sylow 2-subgroup of G.

5. Character tables

Recall that

$$|\mathbf{C}_G(x)| = \sum_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2$$

for any $x \in G$. In order to apply our main results Theorems A, B and C to a particular character table, it remains to recognize p-elements in the character table of G for every prime p. This is possible by a well-known theorem of G. Higman (see [I, Thms. 8.20 and 8.21], for instance), which is proved using maximal ideals in rings of algebraic integers. To finish the paper and for the reader's convenience, we outline a very similar method which nevertheless avoids choosing maximal ideals.

Let R be the ring of algebraic integers in $\mathbb{C}$, and let p be a prime. If $\alpha, \beta \in R$, then we write $\alpha \equiv \beta \pmod{p}$ if $\alpha - \beta = p\gamma$ for some $\gamma \in R$. If $\alpha \equiv \beta$, then $\alpha^n \equiv \beta^n$ for every integer n. Also, recall that if $\alpha_1, \ldots, \alpha_k \in R$, then

$$(\alpha_1 + \cdots + \alpha_k)^{p^n} \equiv {\alpha_1}^{p^n} + \cdots + {\alpha_k}^{p^n} \pmod{p}.$$

If $x \in G$, then x_p denotes the p-part of x and $x_{p'}$ denotes the p'-part of x.

Lemma 5.1. *Suppose that χ is a character of a finite group G, and let $x \in G$. Then*

$$\chi(x)^{|x_p|} \equiv \chi(x_{p'})^{|x_p|} \pmod{p}.$$

Thus

$$\chi(x)^{|G|_p} \equiv \chi(x_{p'})^{|G|_p} \pmod{p}.$$

Proof. Let $m := |x_p|$. Write $x = yz$, where $y = x_p$ and $z = x_{p'}$. Let $\mathcal{X}$ be a representation affording χ. We can assume that

$$\mathcal{X}(x) = \mathrm{diag}(\epsilon_1, \ldots, \epsilon_k)$$

is a diagonal matrix. Since y and z are powers of x, we have that

$$\mathcal{X}(y) = \mathrm{diag}(\alpha_1, \ldots, \alpha_k),$$

where $\alpha_i^m = 1$, and

$$\mathcal{X}(z) = \operatorname{diag}(\beta_1, \dots, \beta_k),$$

with $\epsilon_i = \alpha_i \beta_i$. Now

$$\chi(x)^m = (\epsilon_1 + \cdots + \epsilon_k)^m \equiv \beta_1^m + \cdots + \beta_k^m \equiv (\beta_1 + \cdots + \beta_k)^m = \chi(z)^m \pmod{p}.$$

The second assertion easily follows. □

Theorem 5.2. *Let $x, y \in G$, and let p be a prime. Then $x_{p'}$ and $y_{p'}$ are G-conjugate if and only if*

$$\chi(x)^{|G|_p} \equiv \chi(y)^{|G|_p} \pmod{p}$$

for all $\chi \in \operatorname{Irr}(G)$.

Proof. Suppose that $x_{p'}$ and $y_{p'}$ are G-conjugate. By Lemma 5.1, we have that

$$\chi(x)^{|G|_p} - \chi(y)^{|G|_p} \equiv \chi(x_{p'})^{|G|_p} - \chi(y_{p'})^{|G|_p} = 0\,.$$

Conversely, suppose that

$$\chi(x)^{|G|_p} \equiv \chi(y)^{|G|_p} \pmod{p}$$

for all $\chi \in \operatorname{Irr}(G)$. Let I be a maximal ideal of R containing pR, so that $\mathbb{F} := R/I$ is a field. Then we have that

$$\chi(x)^{|G|_p} \equiv \chi(y)^{|G|_p} \pmod{I}$$

by hypothesis. Then

$$(\chi(x) - \chi(y))^{|G|_p} \pmod{I} \equiv \chi(x)^{|G|_p} - \chi(y)^{|G|_p} \pmod{I} = I\,,$$

and therefore $\chi(x) \equiv \chi(y) \pmod{I}$, since $\mathbb{F}$ is a field. But $\chi(x) \equiv \chi(x_{p'}) \pmod{I}$, using the fact that if ϵ is a p-power root of unity, then $\epsilon \equiv 1 \pmod{I}$. Now apply [I, Thm. (8.20)]. □

Corollary 5.3. *We have that $x \in G$ is a p-element if and only if*

$$\chi(x)^{|G|_p} \equiv \chi(1) \pmod{p}$$

for all $\chi \in \operatorname{Irr}(G)$.

Proof. Use Theorem 5.2, and the fact that $n^p \equiv n \pmod{p}$ for every integer n. □

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