PROPERTIES OF SQUEEZING FUNCTIONS AND GLOBAL TRANSFORMATIONS OF BOUNDED DOMAINS

FUSHENG DENG, QI’AN GUAN, AND LIYOU ZHANG

Abstract. The central purpose of the present paper is to study boundary behaviors of squeezing functions on some bounded domains. We prove that the squeezing function of any strongly pseudoconvex domain tends to 1 near the boundary. In fact, such an estimate is proved for the squeezing function on any bounded domain near its globally strongly convex boundary points. We also study the stability properties of squeezing functions on a sequence of bounded domains, and give some comparisons of intrinsic measures and metrics on bounded domains in terms of squeezing functions. As applications, we give new proofs of several well-known results about geometry of strongly pseudoconvex domains, and prove that all Cartan-Hartogs domains are homogeneous regular. Finally, some related problems for further study are proposed.

1. Introduction

In a recent work [5], the authors introduced the notion of squeezing function to study geometric and analytic properties of bounded domains. The squeezing function of a bounded domain is defined as follows:

Definition 1.1. Let $D$ be a bounded domain in $\mathbb{C}^n$. For $z \in D$ and an (open) holomorphic embedding $f : D \to B^n$ with $f(z) = 0$, we set

$$s_D(z, f) = \sup \{ r \mid B^n(0, r) \subset f(D) \}.$$  

The squeezing radius $s_D(z)$ of $D$ at $z$ is defined as

$$s_D(z) = \inf_f \{ s_D(z, f) \},$$

where the supremum is taken over all holomorphic embeddings $f : D \to B^n$ with $f(z) = 0$, $B^n$ is the unit ball in $\mathbb{C}^n$, and $B^n(0, r)$ is the ball in $\mathbb{C}^n$ with center 0 and radius $r$. As $z$ varies, we get a function $s_D$ on $D$, which is called the squeezing function of $D$.

Roughly speaking, $s_D(z)$ describes how much the domain $D$ looks like the unit ball at the point $z$. By definition, it is clear that squeezing functions are invariant under biholomorphic transformations. Namely, if $f : D_1 \to D_2$ is a holomorphic equivalence of two bounded domains, then $s_{D_2} \circ f = s_{D_1}$. Though the definition of squeezing function is simple, it turns out that so many geometric and analytic properties of bounded domains are encoded in their squeezing functions.

Received by the editors April 24, 2013 and, in revised form, January 26, 2014.

2010 Mathematics Subject Classification. Primary 32H02, 32F45.

Key words and phrases. Squeezing function, homogeneous regular domain, globally strongly convex boundary point.

The authors were partially supported by NSFC grants and BNSF(No.1122010).
Some basic properties of squeezing functions were established in [5]. For example, squeezing functions are continuous; and for each \( p \in D \), there exists an extremal map \( f : D \to B^n \) realizing the supremum in Definition 1.1.

In the present paper, we continue to study squeezing functions and their applications to geometry of bounded domains. We first consider the stability properties of the squeezing functions on a monotonic sequence of domains. We prove that, for an increasing sequence of domains convergent to a bounded domain, the squeezing functions of these domains converge to the squeezing function of the limit domain. We also prove a weaker result (an inequality) for the squeezing functions of a decreasing sequence of bounded domains.

A **homogenous regular domain** (introduced in [14]) is a bounded domain whose squeezing function is bounded from below by a positive constant. By the famous Bers embedding ([3]), Teichmüller spaces of compact Riemann surfaces are homogenous regular. In the past decade, the comparisons of various intrinsic metrics on Teichmüller spaces were extensively studied (see e.g. [4], [14], [15], [24]). The equivalence of certain intrinsic measures on Teichmüller spaces was proved in [19]. In [14], it was proved that the Bergman metric, the Kobayashi metric, and the Carathéodory metric on a homogenous regular domain are equivalent. Geometric and analytic properties of homogenous regular domains were systematically studied in [25], where the term of homogenous regular domain was phrased as uniformly squeezing domain. In the present paper, by modifying the methods in [19] and [25], we prove certain comparisons of intrinsic measures and metrics on general bounded domains in terms of their squeezing functions.

The central problem in the theory is to study boundary behaviors of squeezing functions. For a smoothly bounded planar domain \( D \), it was shown in [5] that \( \lim_{z \to \partial D} s_D(z) = 1 \). In this paper, we try to generalize this result to strongly pseudoconvex domains of higher dimensions.

To state the main results, we first introduce the notion of globally strongly convex boundary points. Let \( D \) be a bounded domain in \( \mathbb{C}^n \) and \( p \in \partial D \). We call \( p \) a **globally strongly convex** (g.s.c.) boundary point of \( D \) if \( \partial D \) is \( C^2 \)-smooth and strongly convex at \( p \), and \( D \cap T_p \partial D = \{ p \} \), where \( T_p \partial D \) is the tangent hyperplane of \( \partial D \) at \( p \).

By a result of Fridman and Ma (Theorem 1.3 in [8]), we have \( \limsup_{z \to p} s_D(z) = 1 \) if \( p \) is a g.s.c. boundary point of \( D \). In the present paper, we prove the following.

**Theorem 1.1.** Let \( D \subset \mathbb{C}^n \) be a bounded domain. Assume \( p \in \partial D \) is a g.s.c. boundary point of \( D \). Then \( \lim_{z \to p} s_D(z) = 1 \).

Let \( D \) be a bounded strongly pseudoconvex domain and \( p \in \partial D \). It is well known that there is a local coordinate change near \( p \) such that \( \partial D \) is strongly convex at \( p \) with respect to the new coordinates. It is natural to ask whether the coordinate change can be taken globally on a neighborhood of \( D \). A stronger version of this question is whether there exists a coordinate change on some neighborhood of \( D \) such that, with respect to the new coordinates, \( p \) is a g.s.c. boundary point of \( D \). In a recent work, Diederich and Fornæss and Wold answered this question affirmatively and proved the following deep result:

**Theorem 1.2** ([6]). Let \( D \) be a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary and \( p \in \partial D \). Then there exist a neighborhood \( \tilde{D} \) of \( D \) and a holomorphic (open) embedding \( f : \tilde{D} \to \mathbb{C}^n \) such that \( f(p) \) is a g.s.c. boundary point of \( f(D) \).
Applying Theorem 1.1 and Theorem 1.2, we can prove the following:

**Theorem 1.3.** Let \( D \) be a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary. Then \( \lim_{z \to \partial D} s_D(z) = 1 \). In particular, \( D \) is homogeneous regular.

Based on Theorem 1.3, we can give a different proof of Wong’s result which says that a bounded strongly pseudoconvex domain is holomorphically equivalent to the unit ball if its automorphism group is noncompact, and give new proofs of some other well-known results about geometry of strongly pseudoconvex domains (see \( \S \).

Theorem 1.1 can also be applied to investigate the geometry of Cartan-Hartogs domains, which are certain Hartogs domains over classical bounded symmetric domains. These domains have been extensively studied in the past decade (see e.g. \[1,7,20,21,23,26\]). Motivated by the works in \[14\], Yin proposed a problem of whether all Cartan-Hartogs domains are homogenous regular \[27\]. In this paper, we answer this question affirmatively. Consequently, all Cartan-Hartogs domains are hyperconvex and have bounded geometry; various classical intrinsic metrics, as well as all the measures considered in \( \S \) are equivalent on these domains.

Though squeezing functions are originally defined for bounded domains, a similar idea is possibly applied to algebraic geometry. In fact, by Griffiths’ results on uniformization of algebraic varieties \[12\], we can define squeezing functions on all projective manifolds, which will be studied more carefully elsewhere (see \( \S \)).

The rest of the paper is organized as follows. In \( \S \) we study the stability properties of squeezing functions on monotonic sequences of domains. In \( \S \) we give some comparisons of intrinsic measures and metrics in terms of squeezing functions. We prove Theorem 1.1 and Theorem 1.3 in \( \S \). In \( \S \) we apply the results in the previous sections to study the geometry of Cartan-Hartogs domains and strongly pseudoconvex domains. In the final \( \S \) we propose some related problems for further study.

## 2. Stability of squeezing functions

In this section, we consider the relation between the limit of the squeezing functions of a monotonic sequence of domains and the squeezing function of the limit domain. For an increasing sequence of domains, we have the following.

**Theorem 2.1.** Let \( D \subset \mathbb{C}^n \) be a bounded domain and \( D_k \subset D \) (\( k \in \mathbb{N} \)) be a sequence of domains such that \( \bigcup_k D_k = D \) and \( D_k \subset D_{k+1} \) for all \( k \). Then, for any \( z \in D \), \( \lim_{k \to \infty} s_{D_k}(z) = s_D(z) \).

**Proof.** By the existence of extremal maps w.r.t. squeezing functions (Theorem 2.1 in \[5\]), for each \( k \), there is an injective holomorphic map \( f_k : D_k \to B^n \) such that \( f_k(z) = 0 \) and \( B^n(0, s_{D_k}(z)) \subset f_k(D_k) \). By Montel’s theorem, we may assume that the sequence \( f_k \) converges uniformly on compact subsets of \( D \) to a holomorphic map \( f : D \to \mathbb{C}^n \).

We first prove that \( f \) is injective. Assume \( z \in D_{k_0} \) for some \( k_0 > 0 \); then it is clear that
\[
s_{D_k}(z) \geq \frac{d(z, \partial D_k)}{diam(D_k)} \geq \frac{d(z, \partial D_{k_0})}{diam(D)}
\]
for \( k > k_0 \). So there is a \( \delta > 0 \) such that \( B^n(0, \delta) \subset f_k(D_k) \) for all \( k > k_0 \). Set \( g_k = f_k^{-1} \big|_{B^n(0, \delta)} : B^n(0, \delta) \to D \). By Cauchy’s inequality, \( |\det(dg_k(0))| \) is bounded.
above uniformly for all $k > k_0$ by a positive constant. Hence there exists a constant $c > 0$, such that $|\det(df_k(z))| > c$ for all $k > k_0$. This implies $\det(df(z)) \neq 0$. So the injectivity of $f$ follows from Lemma 2.3 in [8] and the generalized Rouché’s theorem (Theorem 3 in [16]).

Since $f$ is injective, it is an open map (see e.g. Theorem 8.5 in [9]). On the other hand, it is clear that $f(D) \subset B^n$. So we have $f(D) \subset B^n$.

We now prove that $s_D(z) \geq \limsup_k s_{D_k}(z)$. Let $s_{D_k}$ be a subsequence such that $\lim_{k_i \to \infty} s_{D_{k_i}}(z) = \limsup_k s_{D_k}(z) = r$; then, as explained above, we have $r > 0$. Let $\epsilon > 0$ be an arbitrary positive number less than $r$; then $B^n(0, r - \epsilon) \subset f_k(D_{k_i})$ for $k_i$ large enough. Set $h_{k_i} = f_{k_i}^{-1}|B^n(0, r - \epsilon)$; then $\lim_{k_i \to \infty} |\det(dh_{k_i}(z))| = |\det(df^{-1}(0))| \neq 0$. By the argument mentioned above, $h := \lim_{k_i} h_{k_i}$ is injective and hence $h(B^n(0, r - \epsilon)) \subset D$. This implies $f(h(w))$ make sense for all $w \in B^n(0, r - \epsilon)$. It is clear that $f(h(w)) = w$ for all $w \in B^n(0, r - \epsilon)$. So $B^n(0, r - \epsilon) \subset f(D)$ and $s_D(z) \geq r - \epsilon$. Since $\epsilon$ is arbitrary, we get $s_D(z) \geq \limsup_k s_{D_k}(z)$.

Finally, we prove that $s_D(z) \leq \liminf_k s_{D_k}(z)$. Let $s_{D_{k'_i}}$ be a subsequence such that $\lim_{k'_i \to \infty} s_{D_{k'_i}}(z) = \liminf_k s_{D_k}(z)$. By the existence of an extremal map, there exists an injective holomorphic map $\varphi : D \to B^n$ such that $\varphi(z) = 0$ and $B^n(0, s_{D_k}(z)) \subset \varphi(D)$. For arbitrary $0 < \epsilon < s_D(z)$, by assumption, $\varphi^{-1}(B^n(0, s_D(z) - \epsilon)) \subset D_{k_i}$ for $k'_i$ large enough. So, for $k'_i$ large enough, we have $s_{D_{k'_i}}(z) \geq s_D(z) - \epsilon$. This implies $s_D(z) - \epsilon \leq \lim_{k'_i \to \infty} s_{D_{k'_i}}(z)$. Since $\epsilon$ is arbitrary, we get $s_D(z) \leq \liminf_{k'_i \to \infty} s_{D_{k'_i}}(z) = \liminf_k s_{D_k}(z)$.

For a decreasing sequence of bounded domains, we have

**Theorem 2.2.** Let $D \subset \mathbb{C}^n$ be a bounded domain and $D_k \supset D$ ($k \in \mathbb{N}$) be a sequence of domains such that $\bigcap_k D_k = D$ and $D_{k+1} \subset D_k$ for all $k$. Then, for any $z \in D$, $s_D(z) \geq \limsup_k s_{D_k}(z)$.

**Proof.** For each $k$, let $f_k : D_k \to B^n$ be an injective holomorphic map such that $f_k(z) = 0$ and $B^n(0, s_{D_k}(z)) \subset f_k(D_k)$. By Montel’s theorem, we may assume that $\lim_k f_k = f$ exists and gives a holomorphic map from $D$ to $\mathbb{C}^n$. By the same argument as in the proof of Theorem 2.1, we see that $f$ is injective and $f(D) \subset B^n$.

Without loss of generality, we assume $\lim_k s_{D_k}(z) = r$. Then, for any $\epsilon > 0$, $B^n(0, r - \epsilon) \subset f_k(D_k)$ for $k$ large enough. Set $g_k = f_k^{-1}|B^n(0, r - \epsilon) : B^n(0, r - \epsilon) \to D_k$. We can assume $g_k$ converges uniformly on compact subsets of $B^n(0, r - \epsilon)$ to a holomorphic map $g : B^n(0, r - \epsilon) \to \mathbb{C}^n$. Similarly, one can show that $g$ is injective and hence open. It is clear that $g(B^n(0, r - \epsilon)) \subset \bigcap_{k \geq 1} D_k$. For any $k_0 > 0$, by the generalized Rouche’s theorem (Theorem 3 in [16]), $g(B^n(0, r - \epsilon) \cap \partial D_{k_0}) = \emptyset$ since $g_k(B^n(0, r - \epsilon) \cap \partial D_{k_0}) = \emptyset$ for all $k > k_0$. Hence $g(B^n(0, r - \epsilon)) \subset \bigcap_{k \geq 1} D_k = D$. This implies that $f(g(w))$ makes sense for all $w \in B^n(0, r - \epsilon)$. It is clear that $f(g(w)) = w$ for all $w \in B^n(0, r - \epsilon)$. So $B^n(0, r - \epsilon) \subset f(D)$ and $s_D(z) \geq r - \epsilon$. Since $\epsilon$ is arbitrary, we get $s_D(z) \geq r = \lim_k s_{D_k}(z)$. \hfill $\square$

The following example shows that the strict inequality in Theorem 2.2 is possible.

**Example 2.1.** Let $D = \{(z_1, z_2) | 0 < |z_2| < |z_1| < 1\}$ be the Hartogs triangle in $\mathbb{C}^2$. For a positive number $\epsilon$ (small enough), we define a domain $V_\epsilon$ in $\mathbb{C}^2$ as $V_\epsilon = \{(z_1, z_2) | 0 < |z_1| < 1, 0 < |z_2| < \epsilon\}$.
Set $D_\epsilon = D \cup V_\epsilon$. Let $z^j = (z^j_1, z^j_2)$ be a sequence of points in $D$ satisfying the conditions $|z^j_1| \leq (1 + \frac{1}{j})|z^j_2|$ and $|z^j_2| > a$ for all $j$, where $a > 0$ is a fixed constant. Then we have

1) $\lim_{j \to \infty} s_{D_\epsilon}(z^j) = 0$ uniformly with respect to $\epsilon$, and

2) there exists a positive constant $c$, such that $s_D(z^j) \geq c$ for all $j$.

Proof. 1) By Riemann’s removable singularity theorem and Hartogs’s extension theorem, the Carathéodory metric $C_{D_\epsilon}$ on $D_\epsilon$ is given by the restriction on $D_\epsilon$ of the Carathéodory metric on $\Delta \times \Delta$. So the Carathéodory metric on $D_\epsilon$ is not complete and the restriction of $C_{D_\epsilon}$ on $D$ is independent of $\epsilon$. Note that the Carathéodory metric on $\Delta \times \Delta$ is continuous, so for any sufficiently small $\epsilon$, $z^j$ is a Cauchy sequence with respect to $C_{D_\epsilon}$. But the limit of $z^j$ does not lie in $D_\epsilon$. Hence there exists a sequence of positive numbers $r_j$ such that $\lim_j r_j = 0$ and the balls, denoted by $B_\epsilon(z^j, r^j)$, in $D_\epsilon$ with center $z^j$ and radius $r_j$ with respect to $C_{D_\epsilon}$, are not relatively compact in $D_\epsilon$ for all $j$ and all $\epsilon$. Let $f : D_\epsilon \to B^2$ be an injective holomorphic map such that $f(z^j) = 0$ and $B^2(0, s_{D_\epsilon}(z^j)) \subset f(D_\epsilon)$. Note that the distance from $z \in B^2$ to 0 with respect to the Carathéodory metric on $B^2$ is given by

$$\sigma(||z||) = \ln \frac{1 + ||z||}{1 - ||z||},$$

By the decreasing property of the Carathéodory metric, we see that

$$f(B_\epsilon(z^j, \sigma(\frac{s_{D_\epsilon}(z^j)}{2}))) \subset B^2(0, \frac{s_{D_\epsilon}(z^j)}{2})$$

is relatively compact in $f(D_\epsilon)$. This implies that $B_\epsilon(z^j, \sigma(\frac{s_{D_\epsilon}(z^j)}{2}))$ is relatively compact in $D_\epsilon$. So we have $s_{D_\epsilon}(z^j) \leq 2\sigma^{-1}(r^j)$ for all $j$. Hence as $j \to +\infty$, $s_{D_\epsilon}(z^j)$ converge to 0 uniformly with respect to $\epsilon$.

2) The map $\varphi(z_1, z_2) = (z_1, \frac{z_2}{z_1})$ gives a holomorphic isomorphism from $D$ to $\Delta^* \times \Delta^*$. Denote $\varphi(z^j)$ by $(w^j_1, w^j_2)$; then $|w^j_1|, |w^j_2| > a$. Note that the squeezing function on $\Delta^*$ is given by $s_{\Delta^*}(z) = |z|$ (see Corollary 7.2 in [5]), so $s_{\Delta^* \times \Delta^*}(w^j_1, w^j_2) \geq \frac{\sqrt{2}}{2}a$ for all $j$. By the holomorphic invariance of squeezing functions, we get $s_D(z^j) \geq \frac{\sqrt{2}}{2}a$ for all $j$. By the estimate in 1), for $j$ large enough, we have $s_D(z^j) > s_{D_\epsilon}(z^j) + \frac{a}{2}$ for all $\epsilon$. \hfill $\Box$

3. Comparisons of intrinsic measures and metrics

In this section, we give some comparisons of intrinsic measures and metrics on bounded domains in terms of squeezing functions.

3.1. Comparisons of intrinsic measures. Let $D$ be a domain in $\mathbb{C}^n$. The Carathéodory measure on $D$ is defined to be the $(n, n)$-from

$$\mathcal{M}_D^C(z) = M^C_D(z) \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \frac{i}{2} dz_n \wedge d\bar{z}_n,$$

where

$$M^C_D(z) = \sup \{| \det f'(z) |^2; f : D \to B^n \text{ holomorphic with } f(z) = 0 \};$$

and the Eisenman-Kobayashi measure on $D$ is defined to be the $(n, n)$-from

$$\mathcal{M}_D^K(z) = M^K_D(z) \frac{i}{2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge \frac{i}{2} dz_n \wedge d\bar{z}_n,$$
where

\[ M^K_D(z) = \inf \{ 1/|\det f'(0)|^2; f : B^n \to D \text{ holomorphic with } f(0) = z \}. \]

The Carathéodory measure and the Eisenman-Kobayashi measure satisfy the decreasing property. Namely, if \( f : D_1 \to D_2 \) is a holomorphic map between two domains in \( \mathbb{C}^n \), then \( f^* M^{C}_{D_2} \leq M^{C}_{D_1} \) and \( f^* M^K_{D_2} \leq M^K_{D_1} \).

Let \( h \) be a norm on \( \mathbb{C}^n \), and let \( B^n(h) := \{ v \in \mathbb{C}^n | h(v) < 1 \} \) be the unit ball with respect to \( h \). Then the measure of \( h \) is defined as

\[
\frac{\text{vol}(B^n)}{\text{vol}(B^n(h))} i \frac{d\bar{z}_1 \wedge \cdots \wedge i d\bar{z}_n}{2},
\]

where \( \text{vol}(B^n) \) and \( \text{vol}(B^n(h)) \) denote the Euclidean volumes of \( B^n \) and \( B^n(h) \) respectively. Note that the measure of \( h \) is completely determined by \( h \), and independent of the choice of the original inner product on \( \mathbb{C}^n \).

On a bounded domain \( D \), the Kobayashi metric and the Carathéodory metric (see e.g. [13] for an introduction) are nondegenerate; namely, they give norms on the tangent spaces at all points of \( D \). So we can define the measures of the Kobayashi metric and the Carathéodory metric on \( D \) and denote them by \( \tilde{M}^C_D \) and \( \tilde{M}^C_D \) respectively. Since the Kobayashi metric and the Carathéodory metric satisfy the decreasing property (see e.g. [13]), so do their measures. Here one should note that, in general, the Carathéodory (resp. Eisenman-Kobayashi) measure and the measure of the Carathéodory (resp. Kobayashi) metric are different.

On the unit ball \( B^n \), all of the four measures defined above coincide. Let \( M \) and \( M' \) be any two of the four intrinsic measures, i.e., the Carathéodory measure, the Eisenman-Kobayashi measure, the measure of the Carathéodory metric, and the measure of the Kobayashi metric. Then we have the following:

**Theorem 3.1.** Let \( D \) be a bounded domain in \( \mathbb{C}^n \), \( M \) and \( M' \) as above. Then we have

\[
s_D^{2n}(z) M_D'(z) \leq M_D(z) \leq \frac{1}{s_D^{2n}(z)} M'_D(z), \quad z \in D.
\]

In particular, if \( D \) is homogenous regular and \( s_D(z) \geq c > 0 \), then

\[
c^{2n} M_D'(z) \leq M_D(z) \leq \frac{1}{c^{2n}} M'_D(z), \quad z \in D,
\]

and hence \( M_D \) and \( M'_D \) are equivalent.

**Proof.** Let \( f : D \to B^n \) be a holomorphic injective map with \( f(z) = 0 \) and \( B^n(0, s_D(z)) \subset f(D) \). Note that \( M_{B^n} = M_{B^n(0,r)} \). By the decreasing property of \( M \) and \( M' \), we have

\[
s_D^{-2n}(z) M_{B^n}(0) \geq M_{f(D)}(0) \geq M_{B^n}(0),
\]

\[
s_D^{-2n}(z) M_{B^n}(0) \geq M'_{f(D)}(0) \geq M_{B^n}(0).
\]

Note that \( \frac{M_{f(D)}(0)}{M'_{f(D)}(0)} = \frac{M_D(0)}{M'_D(0)} \). We get

\[
s_D^{2n}(z) M'_D(z) \leq M_D(z) \leq \frac{1}{s_D^{2n}(z)} M'_D(z), \quad z \in D. \quad \square
\]
3.2. Comparisons of intrinsic metrics. It is known that the Kobayashi metric and Carathéodory metric on bounded domains are Finsler metrics satisfying the decreasing property. They coincide with each other on the unit ball. It is also well known that the Carathéodory metric on a bounded domain is dominated by its Kobayashi metric. Let $D$ be a bounded domain, and denote by $\mathcal{H}_D^K$ and $\mathcal{H}_D^C$ the Carathéodory metric and the Kobayashi metric on $D$ respectively. With the same argument as in the proof of Theorem 3.1, one can prove the following.

**Theorem 3.2.** Let $D$ be a bounded domain in $\mathbb{C}^n$. Then

$$s_D(z)\mathcal{H}_D^K(z) \leq \mathcal{H}_D^C(z) \leq \mathcal{H}_D^K(z).$$

In particular, if $D$ is homogenous regular and $s_D(z) \geq c > 0$, then, for any $z \in D$,

$$c\mathcal{H}_D^K(z) \leq \mathcal{H}_D^C(z) \leq \mathcal{H}_D^K(z),$$

and hence $\mathcal{H}_D^C$ and $\mathcal{H}_D^K$ are equivalent.

For a bounded domain $D$, we have a comparison between its Carathéodory metric $\mathcal{H}_D^K$ and Kobayashi metric $\mathcal{H}_D^K$ in terms of its squeezing function in Theorem 3.2. The Bergman metric $\mathcal{H}_D^B$ on $D$, which does not satisfy the decreasing property, is invariant under biholomorphic transformations. When $D$ is pseudoconvex, it is well known that there is a unique complete Kähler-Einstein metric on $D$, denoted by $\mathcal{H}_D^{KE}$, with Ricci curvature normalized by $-(n+1)$ [18], which is also invariant under biholomorphic transformations.

**Theorem 3.3.** Let $D$ be a bounded domain in $\mathbb{C}^n$ and $z \in D$, and let $s_D$ be the squeezing function on $D$. Then

$$s_D(z)\mathcal{H}_D^K(z) \leq \mathcal{H}_D^B(z) \leq \frac{2^{n+2} \pi}{s_D(z)} \mathcal{H}_D^K(z).$$

If in addition $D$ is pseudoconvex, then

$$\sqrt[2n]{\frac{1}{n}} s_D(z)\mathcal{H}_D^K(z) \leq \mathcal{H}_D^{KE}(z) \leq \left( \frac{n}{s_D(z)} \right)^{(n-1)/2} \mathcal{H}_D^K(z).$$

**Remark 3.1.** If $D$ is homogenous regular and $s_D(z) \geq c$ for some constant $c > 0$, the above comparisons, with $s_D(z)$ replaced by $c$, was proved in [25]. In this case, the Bergman metric and the Kähler-Einstein metric on $D$ are equivalent to the Kobayashi metric. A slight modification of the method in [25] can be used to give the proof of Theorem 3.3 so we omit it here.

For a metric $h$ on a bounded domain $D$, as explained in the above subsection, we can define the measure $\mathcal{M}^h$ of $h$. If there are two metrics $h$ and $h'$ on $D$ satisfying the condition

$$a(z)h'(z) \leq h(z) \leq b(z)h'(z), z \in D,$$

where $a$ and $b$ are two positive continuous functions on $D$, then the measures $\mathcal{M}^h$ and $\mathcal{M}^{h'}$ satisfy the comparison

$$(a(z))^{2n} \mathcal{M}^{h'}(z) \leq \mathcal{M}^h(z) \leq (b(z))^{2n} \mathcal{M}^{h'}(z), z \in D.$$

In particular, if $h$ and $h'$ are equivalent, then $\mathcal{M}^h$ and $\mathcal{M}^{h'}$ are also equivalent.

We have shown in Theorem 3.1 that the measures of the Kobayashi metric and the Carathéodory metric on a homogenous regular domain are equivalent, and they are equivalent to the Carathéodory measure and the Kobayashi measure. We have
also seen that, on a homogenous regular domain, the Kobayashi metric and the Carathéodory metric are equivalent. By Theorem 3.3, they are equivalent to the Bergman metric and the Kähler-Einstein metric. As a consequence, we have

**Theorem 3.4.** On a homogenous regular domain, the measures of the Kobayashi metric, the Carathéodory metric, the Bergman metric, and the Kähler-Einstein metric are equivalent, and they are equivalent to the Carathéodory measure and the Eisenman-Kobayashi measure.

The equivalence of some of the above measures was established in [19] for Teichmüller spaces, which is one type of homogenous regular domains.

4. Boundary behaviors of squeezing functions

Let \( D \) be a bounded domain and \( p \in \partial D \). Recall that \( p \) is called a **globally strongly convex** (g.s.c.) boundary point of \( D \) if \( \partial D \) is \( C^2 \)-smooth and strongly convex at \( p \), and \( D \cap T_p \partial D = \{ p \} \), where \( T_p \partial D \) is the tangent hyperplane of \( \partial D \) at \( p \). In this section we will prove the following.

**Theorem 4.1.** Let \( D \subset \mathbb{C}^n \) be a bounded domain. If \( p \) is a g.s.c. boundary point of \( D \), then \( \lim_{z \to p} s_D(z) = 1 \).

Note that squeezing functions are invariant under biholomorphic transformations. By the result of Diederich-Fornæss-Wold (Theorem 1.2), we get the following.

**Theorem 4.2.** Let \( D \subset \mathbb{C}^n \) be a bounded strongly pseudoconvex domain with \( C^2 \)-smooth boundary. Then \( \lim_{z \to \partial D} s_D(z) = 1 \).

With Theorem 4.2 we obtain a new proof (see Corollary 4.3) of B. Wong’s result on the characterization of the unit ball by its symmetry from strongly pseudoconvex domains, based on the original idea of localization by Wong.

To prove Theorem 4.1 we need to introduce a new function \( e_D \) on \( \partial D \). Let \( D \) be a bounded domain in \( \mathbb{C}^n \) and \( p \in \partial D \). If \( D \) is \( C^2 \)-smoothly bounded at \( p \) and contained in some ball in \( \mathbb{C}^n \) with boundary point \( p \), then \( e_D(p) \) is defined to be the minimum of the radii of all balls with \( p \) as a boundary point and containing \( D \). If \( D \) is not \( C^2 \)-smoothly bounded at \( p \) or there does not exist such a ball, we set \( e_D(p) = +\infty \). By definition, it is clear that \( p \) is a g.s.c. boundary point of \( D \) if and only if \( e_D(p) < \infty \). The following proposition is an important ingredient in our proof of Theorem 4.1.

**Proposition 4.3.** Let \( D \) be a bounded domain. Then \( e_D \) is an upper semi-continuous function on \( \partial D \).

**Proof.** For \( r > 0 \) and \( q \in \partial D \cap U \), let \( B_{q,r} \) be the ball defined by

\[ |z - (q - r\nabla \rho(q))|^2 < r^2. \]

Let \( r > e_D(p) \) be fixed. We want to prove that, for some neighborhood \( V \subset U \) of \( p \), \( D \subset B_{q,r} \) for all \( q \in \partial D \cap V \). Let

\[ f_r(z,q) = \frac{|z - (q - r\nabla \rho(q))|^2 - r^2}{2r}. \]
By assumption, we can choose a local defining function $\rho$ of $D$ near $p$ such that $||\nabla \rho|| \equiv 1$ and $\text{Hess}(\rho)(p) > c\text{Hess}(f_r(z, p))|_{z=p}$ for some $c > 1$. By continuity, there is a neighborhood $W$ of $p$ such that

$$\text{Hess}(\rho)(q) > c\text{Hess}(f_r(z, q))|_{z=q}$$

for $q \in \partial D \cap W$. We may assume that $W$ is convex and small enough. Then, for any fixed $q \in \partial D \cap W$, we have

$$\rho(z) = \Delta x \cdot \nabla \rho(q) + \sum_{i,j=1}^{2n} h_{i,j}(z, q) \Delta x_i \Delta x_j,$$

where $\Delta x = (\Delta x_1, \cdots, \Delta x_{2n}) = z - q$ is viewed as a vector in $\mathbb{R}^{2n}$. The key point here is that all $h_{i,j}(z, q)$ are continuous on $W \times (W \cap \partial D)$, and $h_{i,j}(q, q) = \frac{\partial^2 \rho}{\partial x_i \partial x_j}(q)$. By (3), replacing $W$ by a sufficiently small relatively open subset of it, we have

$$\rho(z) - f_r(z, q) = \sum_{i,j=1}^{2n} h_{i,j}(z, q) \Delta x_i \Delta x_j - \Delta x \text{Hess}(f_r(z, q))|_{z=q} \Delta x^T > 0$$

for $(z, q) \in W \times (W \cap \partial D)$. This implies that $W \subset B_{q,r}$ for all $q \in \partial D \cap W$.

On the other hand, it is clear that there is an open subset $V$ of $W$ such that $D - W \subset B_{q,r}$ for all $q \in \partial D \cap V$. So, for all $q \in \partial D \cap V$, we have $D \subset B_{q,r}$. This implies $e_D(q) \leq r$. Let $r \searrow e_D(p)$. We see that $e_D$ is upper semi-continuous at $p$. 

We now give the proof of Theorem 4.1.

**Proof.** By scaling if necessary, we can assume $e_D(p) \leq 1$. Let

$$B = \{z \in \mathbb{C}^n : 2\text{Re}z_1 + \sum_{j=1}^n |z_j|^2 < 0\}$$

be the ball with center $(-1, 0, \cdots, 0)$ and radius 1. It was shown in [8] that there exists a series of biholomorphic transformations that map $D$ to a domain, say $D'$ in $B$ and map $p$ to the origin 0, such that $D' \cap \partial B = \{0\}$ and

$$\lim_{r \to 0} s_{D'}(r, 0, \cdots, 0) = 1.$$  

(4)

In other words, $s_{D'}(z)$ tends to 1 as $z$ tends to 0 from the normal direction of $\partial D'$ at the origin. We assume $D \subset B$ and $p$ is the origin. Then the process of transformations given in [8] is as follows.

**Step 1.** After a unitary transformation if necessary, we can assume the defining function $\rho(z)$ of $D$ near $p = 0$ can be written as

$$\rho(z) = 2\text{Re}z_1 + \text{Re} \sum_{i,j=1}^n a_{ij}(p)z_i z_j$$

$$+ \text{Re} \sum_{j=1}^n c_j(p)z_1 \bar{z}_j + \sum_{j=2}^n N_j(p)|z_j|^2 + o(|z|^2).$$

(5)
Let $H_{c,N} : \mathbb{C}^n \to \mathbb{C}^n$ be the biholomorphic map given by

$$H_1(z) = z_1, \quad H_j(z) = z_j + (c_j(p)/(2N_j(p)))z_1, \quad j = 2, \ldots, n.$$ 

Shrinking $H_{c,N}(D)$ in all directions and rescaling it in the $z'$ directions, we obtain a domain $D_1 \subset B$ whose defining function near the origin is of the form

$$\rho'(z) = 2\text{Re}z_1 + 2\text{Re} \sum_{j=1}^{n} b_j z_1 z_j$$

$$+ \text{Re} \sum_{i,j=1}^{n} a_{ij} z_i z_j + d|z_1|^2 + M|z'|^2 + o(|z|^2). \tag{6}$$

**Step 2.** For $\epsilon > 0$, we define a biholomorphic map $f_\epsilon : B \to B$ as:

$$w_1 = \frac{\epsilon z_1}{2 - \epsilon + (1 - \epsilon)z_1}, \quad w' = \frac{\sqrt{\epsilon(2 - \epsilon)}}{2 - \epsilon + (1 - \epsilon)z_1} z'.$$

For $\epsilon$ small enough such that

$$b'_1 = \frac{\epsilon}{2 - \epsilon} b_1, \quad b'_j = \sqrt{\frac{\epsilon}{2 - \epsilon}} b_j$$

satisfy the condition

$$\sum_{j=1}^{n} |b_j| \leq \lambda, \tag{7}$$

where $\lambda$ is a given uniform constant, then the domain $\tilde{D} = G_\nu(f_\epsilon^{-1}(D_1))$ has a defining function near 0 of the form

$$\rho''(z) = 2\text{Re}z_1 + \text{Re} \sum_{i,j=1}^{n} a_{ij} z_i z_j + d|z_1|^2 + M|z'|^2 + o(|z|^2),$$

where the map $G_\nu$ is defined as $G_\nu(z) = (z_1 + \sum_{j=1}^{n} b'_j z_1 z_j, z')$. We choose an integer $N > 8M$ and set $D_2 = \frac{M}{N} \tilde{D}$.

**Step 3.** Let $a_{ij}(p)$ as in Step 1, and set

$$b_{ij} = \frac{1}{4(N - 4 - k/2)} a_{ij}(p), \quad 2 \leq i, j \leq n,$$

where $k < 2(N - 4)$ is a constant. We choose $\epsilon > 0$ small enough and set $D_3 = h \circ F_b \circ f_\epsilon^{-1}(D_2)$, where $F_b : \mathbb{C}^n \to \mathbb{C}^n$ is given by

$$F_b(z_1, z') = \left(z_1 + \sum_{i,j=2}^{n} b_{ij} z_i z_j, z' \right),$$

and $h : \mathbb{C}^n \to \mathbb{C}^n$ is given by $h(z_1, z') = \left(\frac{14}{15} z_1, \sqrt{\frac{14}{15}} z' \right)$.

Repeating Step 3 several times ($\leq 2(N - 4)$) if necessary, we get a domain, say $D'$, whose squeezing function satisfies the property stated in the beginning of the proof.
Now we move forward to prove that \( \lim_{z \to p} s_D(z) = 1 \). Let

\[
l_p = \{ (-r,0') \in \bar{D}' : 0 \leq r < \delta_p \},
\]

where \( \delta_p > 0 \) very small. We have seen that \( s_{D'}(z) \) converges to 1 when \( z \) tends to the origin along \( l_p \).

Let \( l_p^- \) be the inverse image of \( l_p \) in \( D \) under the series of transformations given in the above steps. Then \( l_p^- \) is a smooth curve in \( \bar{D} \) through \( p \). Since all the transformations in the above steps are defined on some neighborhoods of the closure of the domains involved, it is clear that there is a constant \( c_p > 1 \) such that

\[
c_p^{-1} \leq \frac{d(z,p)}{r(z)} \leq c_p
\]

for all \( z \in l_p^- \), where \( d(z,p) \) is the length of the part of \( l_p^- \) between \( z \) and \( p \), and \( (-r(z),0') \in l_p^- \) is the image of \( z \).

The speed of the convergence of \( \{l_p^-\} \) depends on the eigenvalues \( c_j(p) \) and \( N_j(p) \) of the local defining function given in (5). However, a direct computation shows that the convergence is uniform if \( c_j \) and \( N_j \) lie in a bounded set.

We can do the same process for other boundary points near \( p \). Note that \( e_D(p) < 1 \), and by Proposition 1.3 there is a neighborhood \( U \) of \( p \) in \( \partial D \) such that \( e_D(q) < 1 \) for all \( q \in U \). Repeating the above process, we get a smooth curve \( l_q^- \) of length \( \geq \delta_q/c_q \) in \( \bar{D} \) through \( q \) such that \( s_D(z) \) tends to 1 as \( z \) tends to \( q \) along \( l_q^- \).

We need to check how the above steps depend on \( p \). In Step 1, we first meet the map \( H \), which depends on parameters \( c_j(p) \) and \( N_j(p) \) smoothly. By the Gram-Schmidt process, we see that \( c_j(q) \) and \( N_j(q) \) can smoothly vary with respect to \( q \in U \). Moreover this implies that the shrinking and rescaling appearing in Step 1 can be taken uniformly for \( q \in U \). This also implies that the positive numbers \( \epsilon \) appearing in Step 2 and Step 3 can be taken independent of \( q \in U \). So, it is clear that all other transformations appearing in Step 2 and Step 3 also smoothly depend on \( q \). Moreover, for \( q \in U \), \( c_j(q) \) and \( N_j(q) \) vary in a compact set in \( \mathbb{C} \). One can also see that the first and the second derivatives of these transformations are continuous, so \( \delta_q > 0 \) and \( c_q > 0 \) can be taken independent of \( q \).

We choose \( \delta > 0 \) and define a map \( \varphi : [0,\delta) \times U \to \bar{D} \) such that \( \varphi(t,q) \) is the unique point in \( l_q^- \) with \( d(\varphi(t,q),q) = t \). By the above discussion, \( \varphi \) is a smooth map. It is clear that the tangent vector of \( l_p^- \) at \( p \) is not tangential to \( \partial D \). So the differential of \( \varphi \) at \( p \) is a linear isomorphism. By the inverse function theorem, \( \varphi \) is a local diffeomorphism near \( p \). Without loss of generality, we may assume \( \varphi : [0,\delta) \times U \to \varphi((0,\delta) \times U) \) is a diffeomorphism. Hence, for each \( z \in \varphi((0,\delta) \times U) \), there is a unique \( q_z \in U \) such that \( z \in l_{q_z}^- \), and \( d(z,q_z) \) tends to 0 uniformly as \( z \to \partial D \). By the above discussion, we see that \( \lim_{z \to p} s_D(z) = 1 \).

**Corollary 4.4 (22).** Let \( D \) be a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with \( C^2 \)-smooth boundary. If the automorphism group \( \text{Aut}(D) \) of \( D \) is noncompact, then \( D \) is biholomorphic to the unit ball.

**Proof.** If \( \text{Aut}(D) \) is noncompact, then, for any \( z \in D \), there is a sequence \( f_j \in \text{Aut}(D) \) and a point \( p \in \partial D \) such that \( \lim_{j \to \infty} f_j(z) = p \). By the holomorphic invariance of squeezing functions, \( s_D(z) = s_D(f_j(z)) \) for all \( j \). By Theorem 4.3 \( \lim_{j \to \infty} s_D(f_j(z)) = 1 \). Hence \( s_D(z) = 1 \). By Theorem 2.1 in [5], \( D \) is biholomor- phic to the unit ball. \( \square \)
5. Applications

5.1. Geometry of Cartan-Hartogs domains. In this subsection, we apply Theorem 1.1 to investigate squeezing functions and geometry of Cartan-Hartogs domains, which are certain Hartogs domains over classical bounded symmetric domains.

Recall that a classical bounded symmetric domain is a domain of one of the following four types:

\[ D_I(r, s) = \{ Z = (z_{jk}) : I - Z \bar{Z}^t > 0, \text{where } Z \text{ is an } r \times s \text{ matrix} \}, \]

\[ D_{II}(p) = \{ Z = (z_{jk}) : I - Z \bar{Z}^t > 0, \text{where } Z \text{ is a symmetric matrix of order } p \}, \]

\[ D_{III}(q) = \{ Z = (z_{jk}) : I - Z \bar{Z}^t > 0, \text{where } Z \text{ is a skew-symmetric matrix of order } q \}, \]

\[ D_{IV}(n) = \{ Z = (z_1, \cdots, z_n) \in \mathbb{C}^n : 1 + |ZZ^t|^2 - 2ZZ^t > 0, 1 - |ZZ^t| > 0 \}. \]

Let \( \Omega \) be a classical bounded symmetric domain; then the Cartan-Hartogs domain \( \hat{\Omega}_k \) associated to \( \Omega \) is defined to be

\[ \hat{\Omega}_k = \{(Z, W) \in \Omega \times \mathbb{C}^m ; \| W \|^2 < N(Z, Z)^k \}, \]

where \( m \) is a positive integer and \( k \) is a positive real number, \( \| W \| \) is the standard Hermitian norm of \( W \), and the generic norm \( N(Z, Z) \) for \( D_I(r, s) \), \( D_{II}(p) \), \( D_{III}(q) \), \( D_{IV}(n) \) are respectively \( \det(I - Z \bar{Z}^t) \), \( \det(I - Z \bar{Z}^t) \), \( \det(I + Z \bar{Z}^t) \), and \( 1 + |ZZ^t|^2 - 2ZZ^t \).

In [26], Yin computed the automorphism groups and Bergman kernels of Cartan-Hartogs domains explicitly. Motivated by Liu-Sun-Yau’s work [14], Yin proposed the following open problem: are all Cartan-Hartogs domains homogeneous regular [27]? In this section, we give an affirmative answer to this question.

Theorem 5.1. Let \( \hat{\Omega}_k \) be a Cartan-Hartogs domain defined as above:

1. for any \( P_0 = (Z_0, W_0) \in \partial \hat{\Omega}_k \) with \( W_0 \neq 0 \),
   \[ \lim_{P \to P_0} s_{\hat{\Omega}_k}(P) = 1; \]
2. \( \hat{\Omega}_k \) is homogenous regular.

Remark 5.1. The same estimate as in (1) of Theorem 5.1 does not hold for boundary points \( P_0 = (Z_0, W_0) \in \partial \hat{\Omega}_k \) with \( W_0 = 0 \). In fact, such a \( P_0 \) is an accumulation boundary point of \( \hat{\Omega}_k \). If \( \lim_{P \to P_0} s_{\hat{\Omega}_k}(P) = 1 \), by the same argument as in the proof of Corollary 4.4, one can prove that \( s_{\hat{\Omega}_k} \equiv 1 \) and hence \( \hat{\Omega}_k \) is biholomorphic to the unit ball, which is a contradiction since the action of \( \text{Aut}(\hat{\Omega}_k) \) on \( \hat{\Omega}_k \) is not transitive (see [26]).

Consequently, by the work of Yeung in [25] and the results in §8 we have

Corollary 5.2. Let \( D \) be a Cartan-Hartogs domain. Then

1. \( D \) is hyperconvex, i.e., \( D \) admits a bounded exhaustive plurisubharmonic function;
2. the Bergman metric and the Kähler-Einstein metric on \( D \) have bounded geometry, i.e., the curvature is bounded and the injective radius is bounded from below by a positive constant;
(3) the Kobayashi metric, the Carathéodory metric, the Bergman metric, and the Kähler-Einstein metric on \( D \) are equivalent;
(4) all the intrinsic measures considered in \( \mathbb{R}^n \) on \( D \) are equivalent.

Proof of Theorem 5.1 Let \( X : \Omega \times \mathbb{C} \to [0,1) \) be defined as

\[
X(Z,W) = \frac{\|W\|^2}{N(Z,Z)^k} - 1.
\]

Then \( X \) is a defining function of \( \hat{\Omega}_k \) in \( \Omega \times \mathbb{C}^m \).

We give the proof of the theorem in the case that \( \Omega = D_I(r,s) \) is a bounded symmetric domain of the first type in the above list. In this case, \( N(Z,Z) = \det(I - ZZ^t) \). Other cases can be proved with the same argument.

For any point \((Z,W) \in \Omega_k\), there exists an automorphism \( f \) of \( \hat{\Omega}_k \) such that \( f(Z,W) = (0, \cdots, 0, a) \) for some \( a > 0 \) (see [26]). Assume \( \{P_j\} \subset \hat{\Omega}_k \) with \( P_j \to P_0 \) as \( j \to \infty \), and \( \{f_j\} \) are automorphisms of \( \hat{\Omega}_k \) with \( f_j(P_j) = (0, \cdots, 0, a_j) \) \((a_j > 0)\); then \( \lim_{j \to \infty} a_j = 1 \). By the holomorphic invariance and continuity of squeezing functions and Theorem 4.1, it suffices to prove that \((0, \cdots, 0, 1)\) is a g.s.c. boundary point of \( \hat{\Omega}_k \). We now compute the real Hessian \( \text{Hess}(X)(0, \cdots, 0, 1) \) of the defining function \( X \) at \((0, \cdots, 0, 1)\), where \( X(Z,W) = \frac{\|W\|^2}{N(Z,Z)^k} - 1 \) as above.

Let \( z_{jk} = x_{jk} + \sqrt{-1}y_{jk}, 1 \leq j \leq r, 1 \leq k \leq s \). Then \( \frac{\partial}{\partial y_{jk}} = \frac{\partial}{\partial z_{jk}} + \frac{\partial}{\partial \bar{z}_{jk}} \) and

\[
\frac{\partial^2 N}{\partial z_{jk} \partial \bar{z}_{lq}}|_{z=0} = 0 \text{ for all } j, k, l, q. \text{ Note that}
\]

\[
dN(Z,Z) = N(Z,Z) \cdot \text{tr} \left((I - ZZ^t)^{-1} d(I - ZZ^t) \right).
\]

Direct calculations show that

\[
\frac{\partial^2 N(z,z)}{\partial z_{jk} \partial \bar{z}_{lq}}|_{z=0} = -\text{tr} \left(E_{jk} E_{lq}^t \right) = \begin{cases} -1, & j = l, k = q; \\ 0, & \text{otherwise}, \end{cases}
\]

where \( E_{jk} \) denotes an \((r \times s)\)-matrix whose \((j,k)\)-component is 1 and other components are 0. Therefore, we get

\[
\text{Hess}(X)(0, \cdots, 0, 1) = \begin{pmatrix} 2k I_{2rs} & 0 \\ 0 & 2I_{2m} \end{pmatrix}.
\]

Note also that \( \nabla X(0, \cdots, 0, 1) = 2 \frac{\partial}{\partial w_m} \neq 0 \), where \( w_m \) is the real part of \( w_m \), hence \((0, \cdots, 0, 1)\) is a strongly convex boundary point of \( \hat{\Omega}_k \). On the other hand, it is clear that \( \hat{\Omega}_k \cap \{ u_m = 1 \} = \{(0, \cdots, 0, 1)\} \), so \((0, \cdots, 0, 1)\) is a g.s.c. boundary point of \( \hat{\Omega}_k \). This completes the proof of the theorem. \( \square \)

Remark 5.2. For \( k \) tends to 0, the sequence of domains \( \hat{\Omega}_k \) increases to the product domain \( \Omega \times B^m \). By Theorem 2.1 in the present paper and Theorem 7.3 and Theorem 7.4 in [5], we have

\[
\lim_{k \to 0} s_{\hat{\Omega}_k}(Z,W) = s_{\Omega \times B^m}(Z,W) = (1 + c_{\Omega})^{-1/2},
\]

for all \((Z,W) \in \hat{\Omega}_k\), where \( c_{\Omega} = r, p, [q/2] \), 2 for \( \Omega = D_I(r,s), D_{II}(p,q), D_{III}(q), D_{IV}(n) \) respectively.
5.2. Geometry of strongly pseudoconvex domains. In this subsection, we prove some results about the geometry of strongly pseudoconvex domains. These results are well known and play important roles in several complex variables (see e.g. [11, 13]). But we want to show that they are direct consequences of the results in the previous sections. Throughout this subsection, we always assume that $D$ is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $C^2$ boundary.

**Corollary 5.3.** The Carathéodory metric, the Kobayashi metric, the Bergman metric, and the Kähler-Einstein metric on $D$ are equivalent; and the Bergman/Kähler-Einstein metrics on $D$ have bounded geometry.

**Proof.** By Theorem 3.2, $D$ is homogenous regular. So the corollary follows from Theorem 3.2 and Theorem 3.3 in [6] and Theorem 2 in [25].

**Corollary 5.4.** Denote the Carathéodory metric, the Kobayashi metric, the Bergman metric on $D$ by $\mathcal{H}_D^C$, $\mathcal{H}_D^K$, and $\mathcal{H}_D^B$ respectively. Then

1. the metrics admit the following comparisons near the boundary:

$$\lim_{z \to \partial D} \frac{\mathcal{H}_D^K(z)}{\mathcal{H}_D^C(z)} = \sqrt{n+1} \lim_{z \to \partial D} \frac{\mathcal{H}_D^K(z)}{\mathcal{H}_D^B(z)} = 1;$$

2. the sectional curvature of the Bergman metric on $D$ tends to $-\frac{4}{n+1}$ asymptotically near the boundary.

**Proof.** By Theorem 3.2 and Theorem 1.3 we have $\lim_{z \to \partial D} \frac{\mathcal{H}_D^K(z)}{\mathcal{H}_D^B(z)} = 1$. Let $z_n$ be a sequence in $D$ going to $\partial D$. By Theorem 1.3 $\lim_{n \to \infty} s_D(z_n) = 1$. So there exists a sequence of injective holomorphic maps $f_n : D \to B^n$ with $f_n(z_n) = 0$, $f_n(D) \subset f_{n+1}(D)$ and $\bigcup_n f_n(D) = B_n$. Let $K_n(z, \bar{z})$ be the diagonal Bergman kernel of $f_n(D)$ and $K_0$ be the diagonal Bergman kernel of $B^n$. It is well known that $K_n$ converges to $K_0$ uniformly on compact subsets of $B^n$ (see e.g. [13]). Since $K_n$ and $K_0$ can be written as sums of the norms of holomorphic functions, by Cauchy’s inequality, the derivatives of $K_n$ still converge to the derivatives of $K_0$ uniformly on compact subsets of $B^n$. In particular, the Bergman metrics on $f_n(D)$ at 0 converge to the Bergman metric on $B^n$ at 0. By the holomorphic invariance of the Kobayashi metric and the Bergman metric, we see

$$\lim_{n \to \infty} \frac{\mathcal{H}_D^K(z_n)}{\mathcal{H}_D^B(z_n)} = \lim_{n \to \infty} \frac{\mathcal{H}_{f_n(D)}^K(0)}{\mathcal{H}_{f_n(D)}^B(0)} = \frac{s_{B^n}(0)}{s_{B^n}(0)} = (n+1)^{-1/2}.$$

Note that the curvature can be expressed in terms of second order derivatives of the Bergman metric on $D$, and the sectional curvature of the unit ball w.r.t. the Bergman metric is $-\frac{4}{n+1}$. By the holomorphic invariance of the Bergman metric, we see that the sectional curvature of the Bergman metric on $D$ tends to $-\frac{4}{n+1}$ asymptotically near the boundary.

**Corollary 5.5.** Let $\mathcal{M}$ and $\mathcal{M}'$ be any two of the five measures, i.e., the Carathéodory measure, the Eisenman-Kobayashi measure, the measure of the Carathéodory metric, the measure of the Kobayashi metric, and $(n+1)^{-n}$ times of the measure of the Bergman metric on $D$. Then

$$\lim_{z \to \partial D} \frac{\mathcal{M}(z)}{\mathcal{M}'(z)} = 1.$$
6. Further study

In this section, we propose some directions related to the topics in the present paper for further study.

6.1. Squeezing functions on projective manifolds. Squeezing functions are originally defined on bounded domains, which are the classical objects of study in several complex variables. It is natural to consider whether the theory of squeezing functions can be applied to more general complex manifolds. Let $X$ be a complex manifold whose universal covering is $\pi: \tilde{X} \to X$. If $\tilde{X}$ is biholomorphic to a bounded domain, then $s_{\tilde{X}}$ is defined. By the holomorphic invariance of squeezing functions, $s_{\tilde{X}}$ can be pushed down to a continuous function on $X$. If $X$ is compact, $\tilde{X}$ is homogenous regular. This implies a lot of interesting information about the geometry of $X$, as shown in [25]. However, for generic compact complex manifolds, their universal covering cannot be holomorphically equivalent to a bounded domain. So the application of this approach is much more restricted. On the other hand, if restricting ourselves to the context of projective manifolds, we can say more as follows.

By the uniformization theorem, the universal covering of a Riemann surface is either $\mathbb{P}^1$, $\mathbb{C}$ or $\Delta$. In higher dimensions, there is no similar perfect phenomenon. On the other hand, based on Bers’ theory ([2]), Griffiths showed that any point $z$ in a projective manifold $X$ admits a Zariski open neighborhood $U$ such that the universal covering $\tilde{U}$ of $U$ is biholomorphic to a contractible bounded domain in $\mathbb{C}^n$ ([12]). Such a neighborhood $U$ of $z$ is called a Griffiths neighborhood of $z$. We define

$$s_X(z) = \sup_U \{ s_{\tilde{U}}(\tilde{z}) \},$$

where the supremum is taken over all Griffiths neighborhoods $U$ of $z$, and $\tilde{z} \in \tilde{U}$ is an inverse image of $z$ under the covering map from $\tilde{U}$ to $U$. As $z$ varies on $X$, we get a function $s_X$ on $X$, which is also called the squeezing function of $X$. It is clear that $s_X$ is a positive function on $X$ which is invariant under holomorphic transformations. As in the case of bounded domains, we need to consider two key problems in the context of projective manifolds. We need to study which algebraic geometric properties of a projective manifold are encoded in its squeezing function, and develop methods to estimate squeezing functions on projective manifolds. In the special case that $X$ is a Riemann surface, it is clear that $s_X \equiv 1$, which is nothing but the Riemann Mapping Theorem. If $X$ is homogenous, then $s_X$ is a constant, which is a holomorphic invariant of $X$. Even in the case that $X = \mathbb{P}^n$, the projective space, it seems nontrivial to determine the exact value of $s_X$. If $X$ is a ball quotient, then $s_X = 1$. It is also interesting to consider possible gap phenomenon. Namely, for each fixed $n > 1$ and $0 < r < 1$, is there a projective manifold $X$ of dimension $n$ such that $r$ is the exact lower bound of $s_X$?

6.2. Holomorphic transformations of strongly pseudoconvex domains. Let $D$ be a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ and $p \in \partial D$. By definition, a peak function on $D$ at $p$ is a holomorphic function $h$ on $D$ (i.e., $h$ is holomorphic on some neighborhood of $D$) such that $h(p) = 1$ and $|h(z)| < 1, z \in D - \{p\}$.
Given the result of Diederich-Fornæss-Wold (Theorem 1.2), one can easily recover the well-known result that each boundary point of $D$ admits a peak function. We want to consider the theory of peak functions and globally strongly convexity more carefully. In each side, there are three levels:

- **Level One: pointwise existence.**
  
  **Assumption:** Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with $C^k(k \geq 2)$ boundary. Assume $p$ is an arbitrary boundary point of $D$.
  
  **Purpose:** to prove the existence of
  
  (1) a peak function on $D$ at $p$; and
  
  (2) a holomorphic injective map $f_p : \bar{D} \to \mathbb{C}^n$ such that $f_p(p)$ is a g.s.c. boundary point of $f_p(D)$.

- **Level Two: variation with respect to boundary points.**
  
  **Assumption:** Let $D \subset \mathbb{C}^n$ be a strongly pseudoconvex domain with $C^k(k \geq 2)$ boundary.
  
  **Purpose:** to prove the existence of
  
  (1) a $C^{k-2}$ map $H : \partial D \times \bar{D} \to \mathbb{C}$ such that, for each $p \in \partial D$, $h_p := H(p, \cdot) : \bar{D} \to \mathbb{C}$ is a peak function on $D$ at $p$; and
  
  (2) a $C^{k-2}$ map $F : \partial D \times \bar{D} \to \mathbb{C}^n$ such that, for each $p \in \partial D$, $f_p := F(p, \cdot) : \bar{D} \to \mathbb{C}^n$ is holomorphic and injective such that $f_p(p)$ is a g.s.c. boundary point of $f_p(D)$.

- **Level Three: variation over a family.**
  
  **Assumption:** Denote by $\Delta$ the unit disc. Let $\rho : \Delta \times \mathbb{C}^n \to \mathbb{R}$ be a $C^k$-smooth p.s.h. function $(k \geq 2)$ whose restriction $\rho_t$ on each fiber $\{t\} \times \mathbb{C}^n$ is strictly plurisubharmonic. Let $\mathcal{D} = \{(t, z) \in \Delta \times \mathbb{C}^n ; \rho(t, z) < 0\}$, and let $\pi : \mathcal{D} \to \Delta$ be the natural projection. Then $\mathcal{D}$ can be viewed as a family of strongly pseudoconvex domains over $\Delta$. We denote by $D_t$ the fiber over $t$ given by $\mathcal{D} \cap \{\{t\} \times \mathbb{C}^n\}$. Let $\partial^{\bar{t}} \mathcal{D} = \bigcup_{t \in \Delta} \partial D_t$, and $\bar{\partial}^{\bar{t}} \mathcal{D} = \mathcal{D} \cup \partial^{\bar{t}} \mathcal{D}$.
  
  **Purpose:** to prove the existence of
  
  (1) a $C^{k-2}$ map $\mathcal{H} : \partial^{\bar{t}} \mathcal{D} \times \pi \bar{\partial}^{\bar{t}} \mathcal{D} \to \mathbb{C}$ such that for each $p \in \partial D_t \subset \partial^{\bar{t}} \mathcal{D}$, $h_{p,t} := \mathcal{H}(p, \cdot) : \bar{D}_t \to \mathbb{C}$ is holomorphic and gives a peak function on $D_t$ at $p$; and
  
  (2) a $C^{k-2}$ map $\mathcal{F} : \partial^{\bar{t}} \mathcal{D} \times \pi \bar{\partial}^{\bar{t}} \mathcal{D} \to \mathbb{C}^n$ such that for each $p \in \partial D_t \subset \partial^{\bar{t}} \mathcal{D}$, $f_{p,t} := \mathcal{F}(p, \cdot) : \bar{D}_t \to \mathbb{C}^n$ is holomorphic and injective, and $f_{p,t}(p)$ is a g.s.c. boundary point of $f_{p,t}(D_t)$.

As we have seen, Level One was established by Diederich, Fornæss and Wold. The existence of $H$ in Level Two is also known, which plays an important role in the study of geometry of strongly pseudoconvex domains (see [10]).

**ACKNOWLEDGEMENTS**

The authors are grateful to J. E. Fornæss for showing them the preprint [6]. They would like to thank B. Y. Chen, K.-T. Kim, K. F. Liu, P. Pflug, S.-K. Yeung, W. P. Yin, and X. Y. Zhou for helpful discussions, and the referee for helpful comments.

**REFERENCES**


School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China

E-mail address: fshdeng@ucas.ac.cn

Beijing International Center for Mathematical Research, and School of Mathematical Sciences, Peking University, Beijing, 100871, People’s Republic of China

E-mail address: guanqian@math.pku.edu.cn

School of Mathematical Sciences, Capital Normal University, Beijing 100048, People’s Republic of China

E-mail address: zhangly@cnu.edu.cn