TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 368, Number 4, April 2016, Pages 2763–2792
http://dx.doi.org/10.1090/tran/6607
Article electronically published on April 8, 2015

ON RANDOM HERMITE SERIES

RAFIK IMEKRAZ, DIDIER ROBERT, AND LAURENT THOMANN

ABSTRACT. We study integrability and continuity properties of random series of Hermite functions. We get optimal results which are analogues to classical results concerning Fourier series, like the Paley-Zygmund or the Salem-Zygmund theorems. We also consider the case of series of radial Hermite functions, which are not so well-behaved. In this context, we prove some L^p bounds of radial Hermite functions, which are optimal when p is large.

1. Introduction

In this paper we prove some optimal integrability and regularity results on the convergence of random Hermite expansions, i.e. on series of eigenfunctions of the harmonic oscillator with random coefficients.

Before we enter into the details, let us recall an old result on the 1-D torus $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let $u(x) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{inx}$ and define the Sobolev space $H^s(\mathbb{T})$ by the norm $\|u\|^2_{H^s(\mathbb{T})} = \sum_{n\in\mathbb{Z}} (1+|n|)^{2s}|c_n|^2$. By the usual Sobolev embeddings, if $u \in H^{1/2-1/p}(\mathbb{T})$ with $p \geq 2$, then $u \in L^p(\mathbb{T})$, but in general $u \notin \mathcal{C}(\mathbb{T})$. Paley and Zygmund (1930) have improved this result allowing random coefficients.

Theorem 1.1 (Paley-Zygmund). *Let $u^\omega(x) = \sum_{n\in\mathbb{Z}} \varepsilon_n(\omega) c_n \mathrm{e}^{inx}$ where $(\varepsilon_n)_{n\in\mathbb{Z}}$ is a sequence of independent Rademacher random variables. If $u \in L^2(\mathbb{T})$, then for all $2 \leq p < +\infty$, a.s $u^\omega \in L^p(\mathbb{T})$.*

Moreover, if for some $\alpha > 1$, $\sum_{n\in\mathbb{Z}} \ln^\alpha(1+|n|)|c_n|^2 < +\infty$, then a.s $u^\omega \in \mathcal{C}(\mathbb{T})$.

Many other results concerning random trigonometric series were obtained by Paley and Zygmund, as detailed in the book of J-P. Kahane [9]. The study has been extended to random Fourier series on Lie groups (see Marcus-Pisier [12]) and to Riemannian compact manifolds for orthonormal basis of eigenfunctions of the Laplace-Beltrami operator (see Tzvetkov [21] and references therein).

On the torus $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, there is a natural choice of the basis for the expansion, namely the $(\mathrm{e}^{in\cdot x})_{n\in\mathbb{Z}}$. In our context (or more generally, if one studies expansions on eigenfunctions of the Laplacian on a compact manifold) it is not clear

Received by the editors March 20, 2014.
2010 *Mathematics Subject Classification.* Primary 60G50.
Key words and phrases. Harmonic oscillator, Hermite functions, random series, harmonic analysis.
The second author was partly supported by the grant "NOSEVOL" ANR-2011-BS01019 01.
The third author was partly supported by the grant "HANDDY" ANR-10-JCJC 0109, and by the grant "ANAÉ" ANR-13-BS01-0010-03.

which basis to choose, and the convergence properties of the random series $u^\omega(x) = \sum c_n X_n(\omega)\varphi_n(x)$ might depend on the choice of the basis $(\varphi_n)_{n\geq 0}$. For instance, an analogous result to Theorem 1.1 has been obtained by Tzvetkov [21, Theorem 5] in compact manifolds with a condition depending on the L^∞ bound of the φ_n.

Here we show that by adding a squeezing condition (see condition (1.5) below), we can use the intrinsic estimates of the spectral function and obtain a convergence condition on the (c_n) which does not depend on the choice of the basis of Hermite functions. The idea to profit from the bounds of the spectral function and from the Weyl law comes from [3, 18] and has been fruitful in different contexts (see [15–17]), where results have been obtained for a large class of probability laws. Here we extend this approach by working in a space $\mathcal{Z}^s_\varphi$ (instead of using condition (1.5)), which also enables us to exploit the estimates of the spectral function and which is compatible with the Lévy contraction principle of random series. We refer to the next paragraph for more details.

Let us now briefly describe our main contribution in this paper.

We first study integrability properties of the random series u^ω. We then detail the case of series of radial Hermite functions, for which the situation is different than in the general case.

Secondly, we prove regularity results of the random series. We prove a Salem-Zygmund theorem which describes the behavior of partial sums. We are then able to obtain an analogous result to Theorem 1.1 in our context, and we show that the ln factor is optimal. Finally, we state in Theorem 2.7 some more precise regularity results. Notice that due to dispersive effects of the harmonic oscillator on $\mathbb{R}^d$, the randomization yields better estimates than on the torus.

In Proposition 2.4 we state some L^p bounds of radial Hermite functions which are optimal at least for $p \geq 2$ large enough. Even if the proof is elementary, using the well-known asymptotic estimates of Laguerre functions, we did not find the result in the literature. Therefore, we have written the details, since the estimates we obtain are better than the bounds of general Hermite functions.

Finally, we point out that the previous results have analogues for random series of eigenfunctions of the Laplacian on a Riemannian compact manifold or for the Laplacian on $\mathbb{R}^d$ with a confining potential. These results can be obtained with the same strategy by using the corresponding bounds of the spectral function.

1.1. **Functional analysis.**

1.1.1. *Some elements on the harmonic oscillator.* We consider the multidimensional harmonic oscillator $H := -\Delta + |x|^2$ on $L^2(\mathbb{R}^d)$ with $d \geq 1$. The spectrum of H is $d + 2\mathbb{N}$, and we consider the sequence of eigenvalues $(\lambda_n)_{n\geq 0}$ by counting multiplicities:

$$d = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots .$$

Fix any orthonormal basis $(\varphi_n)_{n\geq 0}$ of normalized eigenfunctions for the harmonic oscillator H such that $H\varphi_n = \lambda_n \varphi_n$. For $j \geq 1$ denote

$$I(j) = \big\{ n \in \mathbb{N},\ 2j \leq \lambda_n < 2(j+1) \big\}.$$

Observe that for all $j \geq d/2$, $I(j) \neq \emptyset$ and that $\#I(j) \sim C_d j^{d-1}$ when $j \longrightarrow +\infty$, and therefore $\lambda_n \sim c_d n^{1/d}$. Though $(\varphi_n)_{n\geq 0}$ is arbitrary, the vector space spanned by $\{\varphi_n, n \in I(j)\}$ is independent of the choice of the Hilbert basis.

Now, we recall the natural Sobolev spaces for H:

$$\forall s \geq 0 \quad \forall p \in [1, +\infty) \cup \{+\infty\} \quad \mathcal{W}^{s,p}(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d), \quad H^{s/2} u \in L^p(\mathbb{R}^d)\}.$$

Therefore, we define

$$\|u\|_{\mathcal{W}^{s,p}(\mathbb{R}^d)} := \|H^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^d)}. \tag{1.1}$$

It turns out (see [23, Lemma 2.4]) that a functional characterization of $\mathcal{W}^{s,p}(\mathbb{R}^d)$ for $1 \leq p < +\infty$ and $s \geq 0$ is given by

$$u \in \mathcal{W}^{s,p}(\mathbb{R}^d) \quad \Leftrightarrow \quad \|(I - \Delta)^{\frac{s}{2}} u\|_{L^p(\mathbb{R}^d)} + \|\langle x \rangle^s u\|_{L^p(\mathbb{R}^d)} < +\infty.$$

In the Hilbertian framework, we have

$$\mathcal{H}^s(\mathbb{R}^d) := \mathcal{W}^{s,2}(\mathbb{R}^d) = \{u \in H^s(\mathbb{R}^d), \quad \langle x \rangle^s u \in L^2(\mathbb{R}^d)\}$$

where $H^s(\mathbb{R}^d) = \mathrm{Dom}((I - \Delta)^{s/2})$ is the classical Sobolev space. Thus, up to an equivalence of norm, one can define

$$\|u\|_{\mathcal{H}^s(\mathbb{R}^d)} = \|H^{s/2} u\|_{L^2(\mathbb{R}^d)} = \|u\|_{H^s(\mathbb{R}^d)} + \|\langle x \rangle^s u\|_{L^2(\mathbb{R}^d)}. \tag{1.2}$$

Consequently, one can check that $\mathcal{H}^s(\mathbb{R}^d)$ is an algebra if $s > \frac{d}{2}$ and is included in $L^\infty(\mathbb{R}^d)$.

We will need the L^∞ estimate of the spectral function given by Thangavelu/ Karadzhov (see [16, Lemma 3.5]), which reads

$$\|\Pi_j\|^2_{L^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \sum_{n \in I(j)} |\varphi_n(x)|^2 \leq C j^{\gamma(d)}, \tag{1.3}$$

with $\gamma(1) = -1/6$ and $\gamma(d) = d/2 - 1$ for $d \geq 2$ and where Π_j is the spectral projector of H on the eigenspace associated to the unique eigenvalue which belongs to $I(j)$. It is classical that the function defined in (1.3) does not depend on the choice of the $(\varphi_n)_{n \geq 0}$. For $d = 1$, (1.3) comes from the simplicity of the spectrum of H and the classical estimate of the normalized Hermite functions: $\|\varphi_j\|_{L^\infty(\mathbb{R})} \lesssim j^{-\frac{1}{12}}$.

In the sequel we will also need the notation $\beta(d) = d - 1 - \gamma(d)$ as follows:

(1.4)

	$\gamma(d)$	$\beta(d)$
$d = 1$	$-\frac{1}{6}$	$\frac{1}{6}$
$d \geq 2$	$\frac{d}{2} - 1$	$\frac{d}{2}$

1.1.2. *The space* $\mathcal{Z}^s_\varphi(\mathbb{R}^d)$. Given a Hilbertian basis of Hermite functions $(\varphi_n)_{n \geq 0}$ and $s \in \mathbb{R}$, any $u \in \mathcal{H}^s(\mathbb{R}^d)$ can be written in a unique fashion:

$$u = \sum_{n \geq 0} c_n \varphi_n, \quad \sum_{n \geq 0} \lambda_n^s |c_n|^2 < +\infty.$$

We define the space $\mathcal{Z}^s_\varphi(\mathbb{R}^d)$ by the norm

$$\|u\|^2_{\mathcal{Z}^s_\varphi} = \sum_{j \geq 1} j^{s+d-1} \max_{n \in I(j)} |c_n|^2,$$

and we stress that this space depends on the choice of the basis (φ_n). It is clear that we have the strict embeddings

$$\mathcal{H}^{s+d-1}(\mathbb{R}^d) \subset \mathcal{Z}^s_\varphi(\mathbb{R}^d) \subset \mathcal{H}^s(\mathbb{R}^d).$$

In the works [15–17], the following assumption on the coefficients of $u \in \mathcal{H}^s(\mathbb{R}^d)$ was made:

$$|c_k|^2 \leq \frac{C}{\#I(j)} \sum_{n \in I(j)} |c_n|^2, \quad \forall k \in I(j), \quad \forall j \geq 1. \tag{1.5}$$

Let us explain why the condition $u \in \mathcal{Z}^s_\varphi(\mathbb{R}^d)$ is more natural. Firstly, observe that if the coefficients of $u \in \mathcal{H}^s(\mathbb{R}^d)$ satisfy (1.5), then $u \in \mathcal{Z}^s_\varphi(\mathbb{R}^d)$. Secondly, consider two functions $u, v \in \mathcal{H}^s(\mathbb{R}^d)$,

$$u = \sum_{n=0}^{+\infty} c_n \varphi_n, \quad v = \sum_{n=0}^{+\infty} \gamma_n c_n \varphi_n,$$

where (γ_n) is a real bounded sequence. The contraction principle for the random series (see Theorem 5.5) states roughly that if one can prove an almost sure convergence for the random series coming from u (see (1.9) below), then the same is true for v. But it is easy to see that condition (1.5) is not stable by multiplication by bounded sequences, whereas $u \in \mathcal{Z}^s_\varphi(\mathbb{R}^d)$ is the most general condition which is implied by (1.5) and stable by multiplication by bounded sequences.

Sometimes, we also need the stronger condition

$$\frac{C_1}{\#I(j)} \sum_{n \in I(j)} |c_n|^2 \leq |c_k|^2 \leq \frac{C_2}{\#I(j)} \sum_{n \in I(j)} |c_n|^2, \quad \forall k \in I(j), \quad \forall j \geq 1. \tag{1.6}$$

1.2. **Probabilistic setting.** Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(X_n)_{n\geq 0}$ be independent and identically distributed random variables which are not constant almost surely. All random variables are real valued. Throughout the paper (except in the annex 5), we will make two different assumptions depending on whether we study integrability or regularity results:

$$\mathbb{E}[X_1] = 0 \quad \text{and} \quad \forall k \geq 1, \quad \mathbb{E}[|X_1|^k] < +\infty; \tag{1.7}$$

$$\exists \sigma > 0, \quad \forall r \in \mathbb{R}, \quad \mathbb{E}[e^{rX_1}] \leq e^{\frac{1}{2}\sigma^2 r^2}. \tag{1.8}$$

One checks that (1.8) implies (1.7). The usual laws we have in mind fulfill (1.8): the real Gaussian law $\mathcal{N}_\mathbb{R}(0,1)$ or the Rademacher law (in that case, we will write $X_n = \varepsilon_n$). More generally, any centered and bounded r.v. satisfies (1.8).

We explain now the way we introduce randomness in Sobolev spaces. Let $(c_n)_{n\geq 0}$ be such that $\sum_{n\geq 0} \lambda_n^s |c_n|^2 < +\infty$. Then we can define a random variable u^ω by

$$u^\omega = \sum_{n=0}^{+\infty} X_n(\omega) c_n \varphi_n. \tag{1.9}$$

It is clear that we have

$$\mathbb{E}\left[\|u^\omega\|^2_{\mathcal{H}^s(\mathbb{R}^d)}\right] = \mathbb{E}\left[\sum_{n=0}^{+\infty} \lambda_n^s |c_n|^2 |X_n|^2\right] \leq \mathbb{E}[X_1^2] \sum_{n=0}^{+\infty} \lambda_n^s |c_n|^2 < +\infty.$$

In other words, $\omega \mapsto u^\omega$ belongs to $L^2(\Omega, \mathcal{H}^s(\mathbb{R}^d))$ and almost surely u^ω belongs to $\mathcal{H}^s(\mathbb{R}^d)$.

2. Main results of the paper

2.1. **Integrability results for random Hermite series.** We state here convergence results in the $L^p(\mathbb{R}^d)$ scale with $p \in [2, \infty)$. The following result (used in a slightly weaker form in [7]) will play a key role. It is a combination of results of Hoffman-Jorgensen, Maurey-Pisier [13] and the fact that $L^p(\mathbb{R}^d)$ has finite cotype.

Proposition 2.1. *Let $p \in [2, +\infty)$ and $(f_n)_{n\geq 0}$ be a sequence of $L^p(\mathbb{R}^d)$, and assume that the sequence $(X_n)_{n\geq 0}$ fulfills* (1.7)*. Then the following statements are equivalent:*

(i) the series $\sum \varepsilon_n f_n$ converges almost surely in $L^p(\mathbb{R}^d)$,
(ii) the series $\sum X_n f_n$ converges almost surely in $L^p(\mathbb{R}^d)$,
(iii) the function $\displaystyle\sum_{n\geq 0} |f_n|^2$ belongs to $L^{\frac{p}{2}}(\mathbb{R}^d)$.

This proposition is a synthesis of known results on the convergence of random series in Banach spaces. For the reader's convenience, we have gathered the elements of the proof in Section 5.

Here is our first result involving random Hermite series. Recall the definition (1.4) of γ and β.

Theorem 2.2. *Let $d \geq 1$ and $2 \leq p < +\infty$. We assume that the r.v. $(X_n)_{n\geq 0}$ fulfill* (1.7) *and that $u = \displaystyle\sum_{n\geq 0} c_n \varphi_n$ belongs to $\mathcal{Z}_\varphi^{-2\beta(d)\left(\frac{1}{2}-\frac{1}{p}\right)}(\mathbb{R}^d)$; i.e. the sequence $(c_n)_{n\geq 0}$ is such that*

$$\sum_{j=1}^{+\infty} j^{\gamma(d)+\frac{2\beta(d)}{p}} \max_{n\in I(j)} |c_n|^2 < +\infty. \tag{2.1}$$

Then $u^\omega = \displaystyle\sum_{n\geq 0} X_n c_n \varphi_n$ converges almost surely in $L^p(\mathbb{R}^d)$.

We will see in the proof that the exponent $\gamma(d) + \frac{2\beta(d)}{p}$ which appears in (2.1) is such that

$$\Big\| \sum_{n\in I(j)} |\varphi_n(x)|^2 \Big\|_{L^{p/2}(\mathbb{R}^d)} \leq C j^{\gamma(d)+\frac{2\beta(d)}{p}}.$$

We refer to [15, Proposition 2.1], where a result similar to Theorem 2.2 was given using the condition (1.5).

By considering radial functions as in Ayache-Tzvetkov [2] and in Grivaux [7], we introduce now a natural example for which the gain of integrability may not hold in all the spaces $L^p(\mathbb{R}^d)$. In this case condition (1.5) does not hold true, and we may have $u \in \mathcal{Z}^s(\mathbb{R}^d)$ and $u \in \mathcal{H}^{s+d-1}(\mathbb{R}^d)\backslash\mathcal{H}^{s+d-1+\varepsilon}(\mathbb{R}^d)$.

Let $d \geq 2$ and $L^2_{rad}(\mathbb{R}^d)$ be the subspace of $L^2(\mathbb{R}^d)$ invariant by the action of the rotation group $SO(d)$. One can prove that there exists a Hilbertian basis $(\psi_n)_{n\geq 0}$ of $L^2_{rad}(\mathbb{R}^d)$ of eigenfunctions of H. Indeed, we have $H\psi_n = (4n+d)\psi_n$, each eigenspace has dimension 1 and ψ_n may be expressed with Laguerre polynomials (see Section 3 for more details).

Theorem 2.3. *Let $d \geq 2$, and assume that $(X_n)_{n\geq 0}$ verifies* (1.7) *and that* $u_{rad} := \sum\limits_{n\geq 0} c_n\psi_n$ *belongs to* $\bigcap\limits_{\varepsilon>0} \mathcal{H}^{-\varepsilon}(\mathbb{R}^d)$. *The random series*

$$u^\omega_{rad}(x) = \sum_{n=0}^{+\infty} X_n(\omega) c_n \psi_n(x)$$

converges almost surely in $L^p(\mathbb{R}^d)$ for any $p \in]2, \frac{d}{\alpha_\star(c)}[$ and diverges almost surely for any $p > \frac{d}{\alpha_\star(c)}$ where

$$\alpha_\star(c) := \inf \big\{\alpha > 0 \,:\, \sum_{n=0}^{N} n^{\frac{d}{2}-1}|c_n|^2 = \mathcal{O}(N^\alpha)\big\}.$$

Let us give some examples:

- If $d = 1$, then by Theorem 2.2, the series u^ω_{rad} (defined in the obvious way) converges a.s. in $L^p(\mathbb{R}^d)$ for all $p < \infty$.
- If (c_n) is such that $\sum_{n\geq 0} n^{\frac{d}{2}-1}|c_n|^2 < +\infty$, then $\frac{d}{\alpha_\star(c)} = +\infty$. Therefore u^ω_{rad} converges a.s. in $L^p(\mathbb{R}^d)$ for all $p < \infty$.
- Assume that $c_n \sim n^{-\kappa}$ with $\kappa \geq 1/2$; then $\alpha_\star(c) = \max\big(\frac{d}{2} - 2\kappa, 0\big)$ and

$$\frac{d}{\alpha_\star(c)} = \begin{cases} \min\Big(\dfrac{2d}{d-4\kappa}, +\infty\Big), & \text{if} \quad \kappa < \frac{d}{4}, \\ +\infty, & \text{if} \quad \kappa \geq \frac{d}{4}. \end{cases}$$

An analogous result to Theorem 2.3, but with a different numerology, was first obtained in [2, 7] for the family of the radial eigenfunctions of the Laplacian on the unit disc in $\mathbb{R}^d$ where the analogue value of $\frac{d}{\alpha_\star(c)}$ is called the critical convergence exponent of c. We will follow the main lines of [7]; the difference in the proof involves the study of $L^p(\mathbb{R}^d)$ bounds of the radial Hermite functions.

Proposition 2.4. *Let $d \geq 2$. Consider the family $(\psi_n)_{n\geq 0}$ of the L^2-normalized radial Hermite functions which satisfies $H\psi_n = (4n + d)\psi_n$. Then:*

(i) Assume that $\frac{2d}{d-1} < p \leq +\infty$. Then

$$c_p n^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}} \leq \|\psi_n\|_{L^p(\mathbb{R}^d)} \leq C_p n^{\frac{d}{2}(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}.$$

(ii) Assume that $p = \frac{2d}{d-1}$. Then

$$\|\psi_n\|_{L^p(\mathbb{R}^d)} \leq C_p n^{-\frac{1}{4}} \ln^{\frac{1}{p}}(n).$$

(iii) Assume that $2 \leq p < \frac{2d}{d-1}$. Then

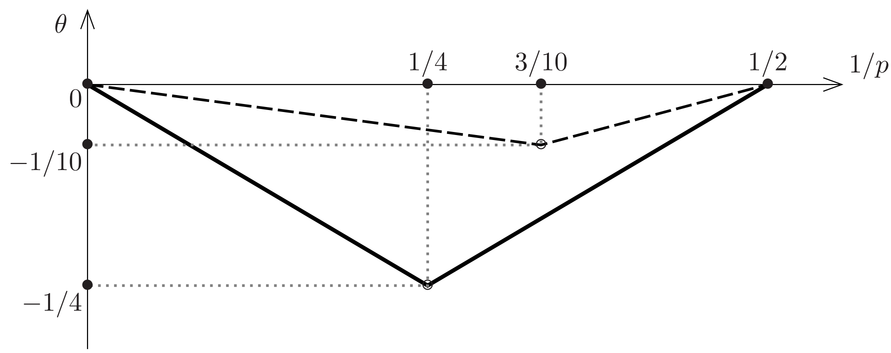
$$\|\psi_n\|_{L^p(\mathbb{R}^d)} \leq C_p n^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}.$$

The proof uses asymptotic estimates of Laguerre functions proved by Erdelyi (such a method has been used in [4, Lemma 3.1] for $d = 2$ and is indicated in [20, Chapter 1]).

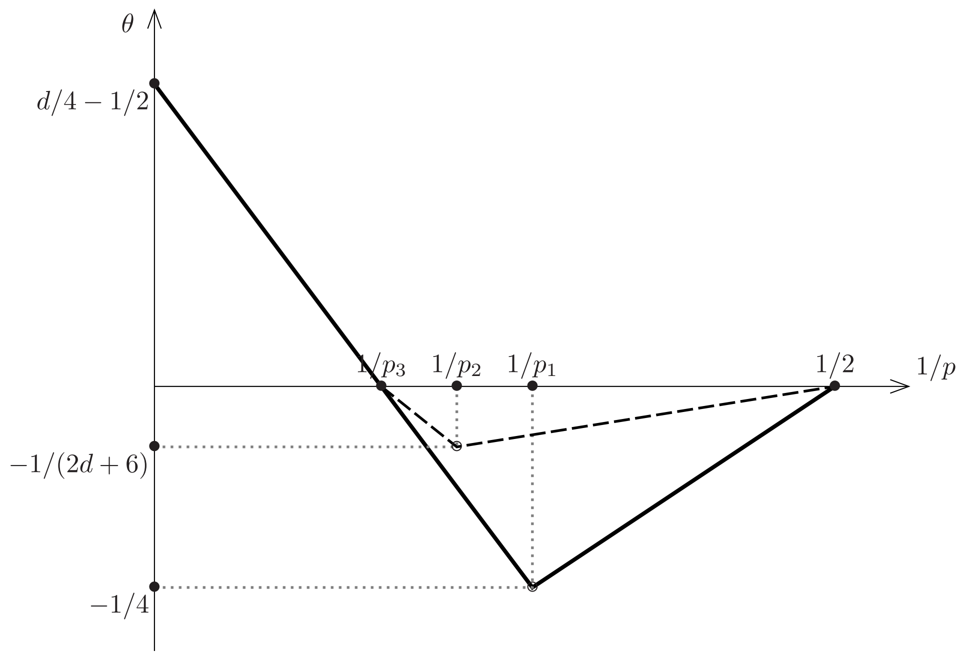
We do not know if the estimates stated in (ii) and (iii) are optimal or not. To get the lower bound in (i) we show that there exist $\varepsilon, c > 0$ such that for all $n \geq 1$ and all $|x| \leq \frac{\varepsilon}{\sqrt{n}}$, $|\psi_n(x)| \geq cn^{\frac{d}{4}-\frac{1}{2}}$, and the result follows by integrating this estimate.

In the figures below, we represent the estimates of Proposition 2.4. The dashed lines represent the bounds of Koch-Tataru [10, Corollary 3.2] obtained for general

Hermite functions as defined in Section 1.1. We see that in the range $2 < p < \frac{d-2}{2d}$ the radial functions enjoy better bounds than in the general case, but not in the regime $\frac{d-2}{2d} < p \leq +\infty$.



L^p estimates of radial Hermite functions: the case $d = 2$



L^p estimates of radial Hermite functions: the case $d \geq 3$

In the second figure we have set

$$2 < p_1 := \frac{2d}{d-1} \leq p_2 := \frac{2(d+3)}{d+1} \leq p_3 := \frac{2d}{d-2}.$$

2.2. **Continuity results for random Hermite series.** We are concerned with the random behavior of the partial sums of (1.9) in the space $L^\infty(\mathbb{R}^d)$. Let us define for any $\lambda \geq d$:

$$u_\lambda^\omega(x) = \sum_{\lambda_n \leq \lambda} c_n X_n(\omega)\varphi_n(x). \tag{2.2}$$

There is not an equivalent of Proposition 2.1 for the space $L^\infty(\mathbb{R}^d)$ (the reason is that $L^\infty(\mathbb{R}^d)$ is not a Banach space with finite cotype; see Annex 5). Hence, we will use other methods to get probabilistic results, like the following one which is in the spirit of the Salem-Zygmund inequality (see [9, Theorem, page 55]).

Theorem 2.5. *Assume that $(X_n)_{n\geq 0}$ is an i.i.d. family of r.v. which satisfies the subnormality condition* (1.8) *with a real number $\sigma > 0$. For any positive integer $N > 0$, there is $C := C(N, d, \sigma) > 0$ such that for any $\lambda \gg 1$ one has for any sequence $(c_n)_{n\geq 0}$:*

$$\mathbb{P}\left[\Big\|\sum_{\lambda_n \leq \lambda} c_n X_n \varphi_n\Big\|^2_{L^\infty(\mathbb{R}^d)} \leq C\ln(\lambda) \sum_{j\leq[\lambda/2]} j^{\gamma(d)} \max_{n\in I(j)} |c_n|^2\right] \geq 1 - \frac{1}{\lambda^N}, \tag{2.3}$$

where $\gamma(d)$ is defined in (1.4)*. Furthermore, if $d \geq 2$ holds and if the $(X_n)_{n\geq 0}$ are independent Gaussians $\mathcal{N}_\mathbb{R}(0,1)$, then one can find a sequence $(c_n)_{n\geq 0}$ such that we cannot replace the function $\lambda \mapsto \ln(\lambda)$ with a slower function of order $o(\ln(\lambda))$.*

In particular the previous result shows that there exists $C > 0$ such that almost surely we have

$$\limsup_{\lambda\to+\infty} \frac{\Big\|\sum_{\lambda_n\leq\lambda} c_n X_n \varphi_n\Big\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\ln(\lambda)}} \leq C\Big(\sum_{j\geq 0} j^{\gamma(d)} \max_{n\in I(j)} |c_n|^2\Big)^{1/2}.$$

Furthermore there exist a sequence $\{c_n\}$ and $c > 0$ such that

$$\sum_{j\geq 0} j^{\gamma(d)} \max_{n\in I(j)} |c_n|^2 = 1$$

and

$$\liminf_{\lambda\to+\infty} \frac{\Big\|\sum_{\lambda_n\leq\lambda} c_n X_n \varphi_n\Big\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\ln(\lambda)}} \geq c.$$

It is straightforward that if the coefficients $(c_n)_{n\geq 0}$ satisfy (1.5), then (2.3) implies

$$\mathbb{P}\left[\Big\|\sum_{\lambda_n \leq \lambda} c_n X_n \varphi_n\Big\|^2_{L^\infty(\mathbb{R}^d)} \leq C\ln(\lambda) \sum_{\lambda_n\leq\lambda} \lambda_n^{-\beta(d)} |c_n|^2\right] \geq 1 - \frac{1}{\lambda^N}, \tag{2.4}$$

with $\beta(1) = 1/6$ and $\beta(d) = d/2$ for $d \geq 2$. The Salem-Zygmund inequality in the classical case of random trigonometric polynomials is similar to (2.4) but holds for $\beta(d) = 0$. Thus, (2.4) shows that randomness for Hermite series has a much more smoothing effect than for Fourier series. Indeed, this is a consequence of better behavior of the spectral function (1.3) of H in the space $L^\infty(\mathbb{R}^d)$.

Let us add that the proof of the classical Salem-Zygmund inequality [9, Theorem 1, page 55] uses in an essential way that the torus $\mathbb{T}$ is compact. In our setting, the non-compactness of $\mathbb{R}^d$ is counterbalanced by the localization of (2.2) in any subset or $\mathbb{R}^d$ which contains strictly the ball $B(0, \sqrt{\lambda})$ (here we will choose the closed ball $\overline{B}(0, \lambda)$ which is much bigger than $B(0, \sqrt{\lambda})$).

Our next result gives a sufficient condition to get almost surely continuity as in Theorem 1.1.

Theorem 2.6. *Let $\gamma(d)$ be defined by* (1.4)*, and let $(c_n)_{n\in\mathbb{N}}$ be such that*

$$\exists\, \alpha > 1, \quad \sum_{j=1}^{+\infty} j^{\gamma(d)} (\ln j)^{\alpha} \max_{n\in I(j)} |c_n|^2 < +\infty. \tag{2.5}$$

Assume that $(X_n)_{n\geq 0}$ is an i.i.d. family of symmetric r.v. such that (1.8) *holds. Denote by*

$$u^{\omega}_{\lambda} = \sum_{\lambda_n \leq \lambda} c_n X_n(\omega) \varphi_n.$$

Then $u^{\omega}_{\lambda} \longrightarrow u^{\omega}$ in $L^{\infty}(\mathbb{R}^d)$ almost surely when $\lambda \longrightarrow +\infty$.

In particular for almost all $\omega \in \Omega$, u^{ω} is a bounded continuous function on $\mathbb{R}^d$.

In the particular case where $(c_n)_{n\geq 0}$ are such that (1.5) holds, then the assumption (2.5) becomes

$$\exists\, \alpha > 1, \quad \sum_{n=0}^{+\infty} \lambda_n^{-\beta(d)} (\ln \lambda_n)^{\alpha} |c_n|^2 < +\infty,$$

with $\beta(1) = 1/6$ and $\beta(d) = d/2$ for $d \geq 2$. This shows that for $d \geq 2$, u is in a slightly smaller space, denoted by $\mathcal{H}^{-d/2+}(\mathbb{R}^d)$ (with a log correction), than $\mathcal{H}^{-d/2}(\mathbb{R}^d)$. In other words, under condition (1.5) almost all series u^{ω} in the very irregular distribution space $\mathcal{H}^{-d/2+}(\mathbb{R}^d)$ is actually a continuous function on $\mathbb{R}^d$.

It is interesting to notice that if we forget the logarithmic term in the assumption (2.5), we find exactly the assumption (2.1) of Theorem 2.2 as p tends to infinity, although methods of proofs are different.

The symmetry assumption of the r.v. is only needed for the convergence of the partial sums, but the continuity result holds without this assumption.

We shall give two different proofs of Theorem 2.6: one is an application of the Salem-Zygmund inequality (Theorem 2.5), and the other relies on an entropy criterion (see Section 6).

From the Salem-Zygmund inequality we can also get a sufficient condition so that $u^{\omega}(x)$ satisfies a global Hölder continuity condition. Recall the definition of the modulus of continuity of $u : \mathbb{R}^d \to \mathbb{C}$:

$$m_u(h) = \sup_{\substack{|x-y|\leq h \\ x,y\in\mathbb{R}^d}} |u(x) - u(y)|, \quad h > 0.$$

Theorem 2.7. *Let $(c_n)_{n\geq 0}$ be such that there exists $C > 0$ such that*

$$\sum_{k=2^j}^{2^{j+1}} \max_{n\in I(k)} |c_n|^2 \leq C 2^{(-\gamma(d)-\mu)j} j^{2\nu}, \quad \forall j \geq 0, \tag{2.6}$$

with $C > 0$, ($\nu \in \mathbb{R}$ and $0 < \mu \leq 1$) or ($\nu < -1$ and $\mu = 0$). Assume that $(X_n)_{n\geq 0}$ is an i.i.d. family of r.v. such that (1.8) *holds. Then we have, almost surely in ω,*

$$m_{u^{\omega}}(h) = \mathcal{O}(h^{\mu} |\ln h|^{\theta})$$

where

- $\theta = \frac{1}{2} + \nu$ *if* $0 < \mu < 1$,
- $\theta = 1 + \nu$ *if* $\mu = 0$,
- *if* $\mu = 1$, *then*

$$\begin{cases} \theta = 1 + \nu \quad \textit{if} \quad \nu \geq -\frac{1}{2}, \\ \theta = \frac{1}{2} \quad \textit{if} \quad -1 \leq \nu \leq -\frac{1}{2}, \\ u^\omega \quad \textit{is a.e. differentiable if} \quad \nu < -1. \end{cases}$$

In particular, if $(c_n)_{n\geq 0}$ is a sequence which satisfies (1.5) and such that there exists $C > 0$ such that

$$\sum_{n\,:\,2^j \leq \lambda_n < 2^{j+1}} |c_n|^2 \leq C 2^{(\beta(d)-\mu)j} j^{2\nu}, \quad \forall j \geq 0,$$

then (2.6) is satisfied.

Remark 2.8. With a slight modification of the proof of Theorem 2.7 we can get the following extension of Theorem 2.6. If

$$\sum_{j=1}^{+\infty} j^{\gamma(d)+\mu} (\ln j)^\alpha \max_{n \in I(j)} |c_n|^2 < +\infty,$$

then almost surely in ω,

$$m_{u^\omega}(h) = \mathcal{O}(h^\mu |\ln h|^\theta)$$

where

- $\theta = \frac{1}{2} - \alpha + \varepsilon$ for all $\varepsilon > 0$ if $0 < \mu < 1$,
- $\theta = 1 - \alpha + \varepsilon$ for all $\varepsilon > 0$ if $\mu = 0$.

2.3. **Notation and plan of the paper.**

Notation. In this paper $c, C > 0$ denote constants, the value of which may change from line to line. These constants will always be universal or uniformly bounded with respect to the other parameters. We write $a \lesssim b$ if $a \leq Cb$ and $a \approx b$ if $ca \leq b \leq Ca$, for some $c, C > 0$.

The rest of the paper is organized as follows. In Section 3 we prove the integrability results on the Hermite series. Section 4 is devoted to the proof of the regularity results (Theorems 2.5, 2.6 and 2.7). In Section 5 we review some results we need about the convergence of random series in Banach spaces. Finally, in Section 6 we give an alternative proof of Theorem 2.6.

3. Proof of the integrability results

3.1. **Proof of Theorem 2.2.** We see Theorem 2.2 as a consequence of Proposition 2.1, and it is equivalent to check

$$\sum_{n\geq 0} |c_n|^2 |\varphi_n|^2 \in L^{\frac{p}{2}}(\mathbb{R}^d). \tag{3.1}$$

By interpolating the $L^{\frac{p}{2}}$ norm and using that $I(j) \sim C_j j^{d-1}$, we get

$$\begin{aligned} \Big\| \sum_{n \in I(j)} |\varphi_n|^2 \Big\|_{L^{\frac{p}{2}}(\mathbb{R}^d)} &\leq \Big\| \sum_{n \in I(j)} |\varphi_n|^2 \Big\|_{L^1(\mathbb{R}^d)}^{\frac{2}{p}} \Big\| \sum_{n \in I(j)} |\varphi_n|^2 \Big\|_{L^\infty(\mathbb{R}^d)}^{1-\frac{2}{p}} \\ &\lesssim j^{\frac{2}{p}(d-1)+\left(1-\frac{2}{p}\right)\gamma(d)} = j^{\gamma(d)+2\beta(d)/p}; \end{aligned}$$

as a consequence

$$\Big\| \sum_{n\geq 0} |c_n|^2 |\varphi_n|^2 \Big\|_{L^{\frac{p}{2}}(\mathbb{R}^d)} \leq \sum_{j\geq 1} \big(\max_{n\in I(j)} |c_n|^2 \big) \Big\| \sum_{n\in I(j)} |\varphi_n|^2 \Big\|_{L^{\frac{p}{2}}(\mathbb{R}^d)}$$
$$\lesssim \sum_{j\geq 1} j^{\gamma(d)+2\beta(d)/p} \max_{n\in I(j)} |c_n|^2 < +\infty.$$

We get (3.1) and hence conclude.

3.2. **Proof of Proposition 2.4.** Let us first recall some results concerning the Laguerre polynomials; see [20, Chapter 1] or [19]. For $\alpha > -1$, the Laguerre polynomial $L_n^{(\alpha)}$ of type α and degree $n \geq 0$ is defined by

$$\mathrm{e}^{-r} r^\alpha L_n^{(\alpha)}(r) = \frac{1}{n!} \frac{d^n}{dr^n} \big(\mathrm{e}^{-r} r^{n+\alpha} \big), \quad x \in \mathbb{R}. \tag{3.2}$$

We need the following identities (see [19, lines (5.1.1), (5.1.3), (5.1.7) and (5.1.14)]):

$$\int_0^{+\infty} L_n^{(\alpha)}(r) L_m^{(\alpha)}(r) \mathrm{e}^{-r} r^\alpha dr = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \delta_{nm}, \tag{3.3}$$

$$L_n^{(\alpha)}(0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \approx n^\alpha, \tag{3.4}$$

$$\forall n \geq 1, \qquad \frac{d}{dr} L_n^{(\alpha)}(r) = -L_{n-1}^{(\alpha+1)}(r), \tag{3.5}$$

$$r \frac{d^2 L_n^{(\alpha)}}{dr^2} + (\alpha + 1 - r) \frac{d L_n^{(\alpha)}}{dr} + n L_n^{(\alpha)} = 0. \tag{3.6}$$

We will need the following lemma.

Lemma 3.1. *For any $\alpha > -1$ there are $c, \varepsilon > 0$ such that*

$$\forall n \geq 1, \quad \forall r \in \Big(0, \frac{\varepsilon^2}{n}\Big), \quad |L_n^{(\alpha)}(r)| \geq c n^\alpha.$$

Proof. As in [19, p. 176], we introduce the function

$$r \mapsto n L_{n-1}^{(\alpha+1)}(r)^2 + r \left(\frac{d}{dr} L_{n-1}^{(\alpha+1)}(r) \right)^2,$$

whose derivative is $2\big(r - \frac{3}{2} - \alpha\big)\big(\frac{d}{dr} L_{n-1}^{(\alpha+1)}\big)^2$ thanks to (3.6). Thus, one has

$$\forall r \in \Big[0, \alpha + \frac{3}{2}\Big], \quad |L_{n-1}^{(\alpha+1)}(r)| \leq |L_{n-1}^{(\alpha+1)}(0)| \lesssim n^{\alpha+1}.$$

By using (3.5), we have

$$\forall r \in \Big[0, \alpha + \frac{3}{2}\Big], \quad |L_n^{(\alpha)}(r) - L_n^{(\alpha)}(0)| \lesssim r n^{\alpha+1}.$$

We can conclude by using (3.4). $\square$

Because of the orthogonality condition (3.3), it is usual to introduce the Laguerre functions normalized in $L^2(0, +\infty)$:

$$\mathcal{L}_n^\alpha(r) := \frac{\sqrt{n!}}{\sqrt{\Gamma(n+\alpha+1)}} L_n^{(\alpha)}(r) \mathrm{e}^{-r/2} r^{\alpha/2}, \quad \frac{\sqrt{n!}}{\sqrt{\Gamma(n+\alpha+1)}} \approx n^{-\alpha/2}. \tag{3.7}$$

These functions satisfy the following uniform estimates (see [1, 6, 14]).

Proposition 3.2. *For any $\alpha > -1$, there are $C = C(\alpha)$ and $\gamma = \gamma(\alpha) > 0$ such that, by denoting $\nu = 4n + 2\alpha + 2$, one has*

$$|\mathcal{L}_n^{(\alpha)}(r)| \leq \begin{cases} C(r\nu)^{\alpha/2} & \text{if } \ 0 \leq r \leq \frac{1}{\nu}, \\ C(r\nu)^{-1/4} & \text{if } \ \frac{1}{\nu} \leq r \leq \frac{\nu}{2}, \\ C\nu^{-1/4}(\nu^{1/3} + |\nu - r|)^{-1/4} & \text{if } \ \frac{\nu}{2} \leq r \leq \frac{3\nu}{2}, \\ Ce^{-\gamma r} & \text{if } \ \frac{3\nu}{2} \leq r. \end{cases}$$

Now, denote by ψ_n the nth $L^2(\mathbb{R}^d)$-normalized radial Hermite function for $d \geq 2$. One can prove that $H\psi_n = (4n + d)\psi_n$ holds and that ψ_n is proportional to $L_n^{(d/2-1)}(|x|^2)\mathrm{e}^{-|x|^2/2}$ (see for instance [20, Corollary 3.4.1]). By using the orthogonality of Laguerre functions $\mathcal{L}_n^{(\frac{d}{2}-1)}$, one easily gets

$$\psi_n(x) := c(d)\mathcal{L}_n^{(\frac{d}{2}-1)}(|x|^2)|x|^{-(\frac{d}{2}-1)} = c(d)\frac{\sqrt{n!}}{\sqrt{\Gamma(n+\frac{d}{2})}}L_n^{(\frac{d}{2}-1)}(|x|^2)\mathrm{e}^{-|x|^2/2} \tag{3.8}$$

with $c(d) := \frac{\sqrt{2}}{\sqrt{\mathrm{Vol}(S^{d-1})}}$ (see (3.9) for $p = 2$).

Let us estimate the $L^p(\mathbb{R}^d)$ norm of ψ_n for $p \geq 2$ by using Proposition 3.2 with $\nu \sim n$ and $\alpha = \frac{d}{2} - 1$.

The case $p = \infty$ is the easiest, and we get directly that $|\mathcal{L}_n^{(\alpha)}(r)|r^{-\frac{\alpha}{2}} \leq C\nu^{\frac{\alpha}{2}}$; in other words, $\|\psi_n\|_{L^\infty(\mathbb{R}^d)} \lesssim n^{\frac{d}{4}-\frac{1}{2}}$. To get the lower bound, it is sufficient to combine Lemma 3.1 with (3.8) and the equivalent in (3.7).

We now consider $p \in [2, +\infty)$. Then we have

$$\begin{aligned} \|\psi_n\|^p_{L^p(\mathbb{R}^d)} &= c(d)^p \mathrm{Vol}(S^{d-1}) \int_0^{+\infty} |\mathcal{L}_n^{(d/2-1)}(r^2)|^p r^{-p(\frac{d}{2}-1)} r^{d-1} dr \\ &= c(d)^{p-2} \int_0^{+\infty} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{-(\frac{p}{2}-1)(\frac{d}{2}-1)} dr. \end{aligned} \tag{3.9}$$

We begin with the following integrals:

$$\begin{aligned} \int_0^{\frac{1}{\nu}} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{-(\frac{p}{2}-1)(\frac{d}{2}-1)} dr &\lesssim n^{\frac{p}{2}(\frac{d}{2}-1)} \int_0^{\frac{1}{\nu}} r^{\frac{d}{2}-1} dr \lesssim n^{\frac{p}{2}(\frac{d}{2}-1)-\frac{d}{2}} \\ &= n^{\frac{pd}{2}(\frac{1}{2}-\frac{1}{p})-\frac{p}{2}}, \\ \int_{\frac{3\nu}{2}}^{+\infty} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{-(\frac{p}{2}-1)(\frac{d}{2}-1)} dr &\lesssim \int_{\frac{3\nu}{2}}^{+\infty} e^{-\gamma r} r^{-(\frac{p}{2}-1)(\frac{d}{2}-1)} dr = \mathcal{O}(n^{-\infty}). \end{aligned} \tag{3.10}$$

To study the integrals over the others intervals given by Proposition 3.2, we have to consider several subcases.

• If $p > \frac{2d}{d-1}$ holds, one has obviously $\frac{1}{2} - \frac{1}{p} > \frac{1}{2d}$, and the comparison of different exponents of n will rely on

$$\frac{-d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) < \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}. \tag{3.11}$$

Notice that one also has

$$\frac{p}{4} + \left(\frac{p}{2} - 1\right)\left(\frac{d}{2} - 1\right) > \frac{d}{2(d-1)} + \left(\frac{d}{d-1} - 1\right)\frac{d-2}{2} = 1, \tag{3.12}$$

which implies that the following integral is of interest near $r = 0$:

$$\int_{\frac{1}{\nu}}^{\frac{\nu}{2}} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} dr \lesssim n^{-\frac{p}{4}} \int_{\frac{1}{\nu}}^{\frac{\nu}{2}} r^{-\frac{p}{4}-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} dr$$
$$\lesssim n^{-\frac{p}{4}+\frac{p}{4}+\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)-1} \approx n^{\frac{pd}{2}\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{p}{2}}. \tag{3.13}$$

The integral over $[\frac{\nu}{2}, \frac{3\nu}{2}]$ is bounded by

$$\begin{aligned}
\int_{\frac{\nu}{2}}^{\frac{3\nu}{2}} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} dr \quad &\lesssim \quad n^{-\frac{p}{4}} \int_{\frac{\nu}{2}}^{\frac{3\nu}{2}} \frac{dr}{(\sqrt[3]{\nu} + |\nu - r|)^{\frac{p}{4}} r^{\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)}} \\
&\lesssim \quad n^{-\frac{p}{4}-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} \int_{\frac{\nu}{2}}^{\frac{3\nu}{2}} \frac{dr}{(\sqrt[3]{\nu} + |\nu - r|)^{\frac{p}{4}}} \\
&\lesssim \quad n^{\frac{p}{4}+\frac{d}{2}-\frac{pd}{4}-1} \int_0^{\frac{\nu}{2}} \frac{dr}{(\sqrt[3]{\nu} + r)^{\frac{p}{4}}} \\
&\lesssim \quad n^{\frac{p}{4}+\frac{d}{2}-\frac{pd}{4}-1} \nu^{\frac{1}{3}-\frac{p}{12}} \int_0^{\frac{1}{2}\nu^{\frac{2}{3}}} \frac{dr}{(1+r)^{\frac{p}{4}}} \\
&\lesssim \quad n^{\frac{p}{6}+\frac{d}{2}-\frac{pd}{4}-\frac{2}{3}} \int_0^{\frac{1}{2}\nu^{\frac{2}{3}}} \frac{dr}{(1+r)^{\frac{p}{4}}}.
\end{aligned}$$

We have to now use the following fact if $p > 4$ holds (which is necessary for $d = 2$):

$$\frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2} - \left(\frac{1}{6} + \frac{d}{2p} - \frac{d}{4} - \frac{2}{3p}\right) = \frac{d}{2} - \frac{2}{3} - \left(d - \frac{2}{3}\right)\frac{1}{p} > \frac{d}{4} - \frac{1}{2} \geq 0.$$

That brings us to

$$\int_{\frac{\nu}{2}}^{\frac{3\nu}{2}} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} dr \tag{3.14}$$
$$\lesssim \begin{cases} n^{\frac{p}{6}+\frac{d}{2}-\frac{pd}{4}-\frac{2}{3}} \leq n^{p\left[\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}\right]} & \text{if } p > 4, \\ n^{-\frac{d}{2}} \ln(n) \lesssim n^{\frac{d}{2}} = n^{-\frac{pd}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} & \text{if } p = 4, \\ n^{\frac{p}{6}+\frac{d}{2}-\frac{pd}{4}-\frac{2}{3}+\frac{2}{3}\left(1-\frac{p}{4}\right)} = n^{-\frac{pd}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} & \text{if } p < 4. \end{cases}$$

Thanks to (3.11), the comparison of exponents in (3.10), (3.13) and (3.14) gives $\|\psi_n\|_{L^p(\mathbb{R}^d)} \lesssim n^{\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p}\right)-\frac{1}{2}}$.

• If $p < \frac{2d}{d-1}$ holds, one has $\frac{1}{2} - \frac{1}{p} < \frac{1}{2d}$ and the opposites of (3.11) and (3.12) hold:

$$\frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2} < -\frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right), \tag{3.15}$$
$$\frac{p}{4} + \left(\frac{p}{2} - 1\right)\left(\frac{d}{2} - 1\right) < 1.$$

Hence, the integral over $\left[\frac{1}{\nu}, \frac{\nu}{2}\right]$ is of interest for $r \gg 1$:

$$\int_{\frac{1}{\nu}}^{\frac{\nu}{2}} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} dr \lesssim n^{-\frac{p}{4}} \int_{\frac{1}{\nu}}^{\frac{\nu}{2}} r^{-\frac{p}{4}-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} dr \tag{3.16}$$
$$\lesssim n^{-\frac{p}{4}+1-\frac{p}{4}-\left(\frac{p}{2}-1\right)\left(\frac{d}{2}-1\right)} \lesssim n^{-\frac{pd}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}.$$

We deal with the integral over $\left[\frac{\nu}{2}, \frac{3\nu}{2}\right]$ in the same way with the help of (3.14) and by noticing that $p < \frac{2d}{d-1} \leq 4$ holds. Hence we get

$$\int_{\frac{\nu}{2}}^{\frac{3\nu}{2}} |\mathcal{L}_n^{(d/2-1)}(r)|^p r^{\left(1-\frac{p}{2}\right)\left(\frac{d}{2}-1\right)} dr \lesssim n^{-\frac{pd}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}. \tag{3.17}$$

Once again, we compare the exponents in (3.10), (3.16) and (3.17) with the help of (3.15) and we get $\|\psi_n\|_{L^p(\mathbb{R}^d)} \lesssim n^{-\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}$.

• If $p = \frac{2d}{d-1}$ holds, we follow the previous analysis and we see that $\|\psi_n\|_{L^p}^p \lesssim n^{-\frac{p}{4}} \ln(n)$.

We have finished the proof of Proposition 2.4.

3.3. **Proof of Theorem 2.3.** We will use Proposition 2.1 and Proposition 2.4.

We consider $p > \frac{d}{\alpha_\star(c)}$ and we write

$$\begin{aligned}
\int_{\mathbb{R}^d} \Big(\sum_{n\geq 0} |c_n|^2 |\psi_n(x)|^2\Big)^{\frac{p}{2}} dx &\geq \sup_{N\geq 1} \int_{|x|\leq \frac{\varepsilon}{\sqrt{N}}} \Big(\sum_{n=1}^{N} |c_n|^2 |\psi_n(x)|^2\Big)^{\frac{p}{2}} dx \\
&\geq C \sup_{N\geq 0} \frac{\varepsilon^d}{N^{\frac{d}{2}}} \Big(\sum_{n=1}^{N} |c_n|^2 n^{\frac{d}{2}-1}\Big)^{\frac{p}{2}} \\
&\geq +\infty.
\end{aligned}$$

We consider $p < \frac{d}{\alpha_\star(c)}$ and we write

$$\Big\|\sum_{n\geq 0} |c_n|^2 |\psi_n|^2\Big\|_{L^{p/2}(\mathbb{R}^d)} \leq \sum_{n\geq 0} |c_n|^2 \|\psi_n^2\|_{L^{p/2}(\mathbb{R}^d)} = \sum_{n\geq 0} |c_n|^2 \|\psi_n\|^2_{L^p(\mathbb{R}^d)}.$$

If p belongs to $(2, \frac{2d}{d-1}]$, then $\|\psi_n\|_{L^p(\mathbb{R}^d)}$ is less than $n^{-\varepsilon}$ for some $\varepsilon > 0$ (see Proposition 2.4). By using that $\sum_{n\geq 0} c_n\psi_n$ belongs to $\bigcap_{\varepsilon>0} \mathcal{H}^{-\varepsilon}(\mathbb{R}^d)$, it is clear that the series $\sum |c_n|^2 \|\psi_n\|^2_{L^p(\mathbb{R}^d)}$ converges.

If p is greater than $\frac{2d}{d-1}$, we use first an Abel summation and then two times the inequality $\alpha_\star(c) < \frac{d}{p}$ to bound the sum of the series $\sum |c_n|^2 \|\psi_n\|^2_{L^p(\mathbb{R}^d)}$ by

$$\begin{aligned}
&C|c_0|^2 + \sum_{n\geq 1} |c_n|^2 n^{\frac{d}{2}-1} n^{-\frac{d}{p}} \\
&\quad \leq C|c_0|^2 + \Big(\lim_{N\to+\infty} N^{-\frac{d}{p}} \sum_{n=1}^{N} |c_n|^2 n^{\frac{d}{2}-1}\Big) \\
&\qquad + \sum_{N\geq 1} \Big(\sum_{n=1}^{N} |c_n|^2 n^{\frac{d}{2}-1}\Big) |N^{-\frac{d}{p}} - (N+1)^{-\frac{d}{p}}| \\
&\quad \lesssim C|c_0|^2 + 0 + \sum_{N\geq 1} \frac{1}{N^{1+\frac{d}{p}}} \Big(\sum_{n=1}^{N} |c_n|^2 n^{\frac{d}{2}-1}\Big) < +\infty.
\end{aligned}$$

Remark. If we define for any sequence $(c_n)_{n\geq 0}$,

$$\forall p > \frac{2d}{d-1} \qquad \|c\|_{d,p} := |c_0| + \sup_{N\geq 1} \frac{1}{N^{\frac{d}{2p}}} \left(\sum_{n=1}^{N} |c_n|^2 n^{\frac{d}{2}-1}\right)^{\frac{1}{2}},$$

then the previous proof shows that one has

$$\forall \nu > 0 \qquad C\|c\|_{d,p} \leq \left\| \sqrt{\sum_{n\geq 0} |c_n|^2 \psi_n^2} \right\|_{L^p} \leq C(\nu)\|c\|_{d,p+\nu}.$$

It is not clear if one can find a more precise norm on the sequence $(c_n)_{n\geq 0}$ which is equivalent to $\left\| \sqrt{\sum_{n\geq 0} |c_n|^2 \psi_n^2} \right\|_{L^p}$. Indeed, this is essentially equivalent to deciding whether or not the almost sure convergence in $L^p(\mathbb{R}^d)$ holds if p is the critical convergence exponent $\frac{d}{\alpha_\star(c)}$.

4. Proof of the regularity results

4.1. **Proof of Theorem 2.5.** Let us begin by introducing the following notation:

$$\forall \lambda > 0 \quad \mathcal{E}_H(\lambda) := \mathrm{Span}\{\varphi_j, \lambda_j \leq \lambda\},$$

and let us recall the following bound on the spectral function of H (see [16, Lemmas 3.1, 3.2 and 3.5]: there are constants $C, c > 0$ such that for any $\lambda \geq 1$ and $x \in \mathbb{R}^d$ one has

$$\forall u \in \mathcal{E}_H(\lambda), \qquad |u(x)| \leq C\lambda^{\frac{d}{4}} \exp\left(-c\frac{|x|^2}{2\lambda}\right) \|u\|_{L^2(\mathbb{R}^d)}. \tag{4.1}$$

The first tool we need to prove Theorem 2.5 is a Bernstein inequality for the harmonic oscillator. In the Hilbertian framework, it is easy to check that one has

$$\forall \lambda \geq 1 \quad \forall u \in \mathcal{E}_H(\lambda) \quad \|\partial_x u\|_{L^2(\mathbb{R}^d)} \leq C\sqrt{\lambda}\|u\|_{L^2(\mathbb{R}^d)}.$$

We need a version of the previous inequality by replacing the space $L^2(\mathbb{R}^d)$ with $L^\infty(\mathbb{R}^d)$.

Lemma 4.1. *For any dimension $d \geq 1$, there are $s(d) \geq 0$ and $C = C(d) > 0$ such that the following inequalities hold:*

$$\forall \lambda \geq 1 \quad \forall u \in \mathcal{E}_H(\lambda) \quad \|\nabla u\|_{L^\infty(\mathbb{R}^d)} \leq C\lambda^{s(d)}\|u\|_{L^\infty(\mathbb{R}^d)}. \tag{4.2}$$

Proof. For any real number $s > \frac{d}{2}$, the Sobolev embedding $\mathcal{H}^s(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$ allows us to write for each $\ell \in \{1, 2, \dots, d\}$:

$$\begin{aligned}
\|\partial_{x_\ell} u\|_{L^\infty(\mathbb{R}^d)} &\leq C\|\partial_{x_\ell} u\|_{\mathcal{H}^s(\mathbb{R}^d)} \leq C\|u\|_{\mathcal{H}^{s+1}(\mathbb{R}^d)} \\
&\leq C\left(\sum_{\lambda_j\leq\lambda} \lambda_j^{s+1} \left|\int_{\mathbb{R}^d} \varphi_j(x)u(x)dx\right|^2\right)^{\frac{1}{2}} \\
&\leq C\lambda^{s+1}\lambda^d\|u\|_{L^\infty(\mathbb{R}^d)} \sup_{\lambda_j\leq\lambda} \|\varphi_j\|_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

In order to get a bound of $\|\varphi_j\|_{L^1(\mathbb{R}^d)}$ we just use the Cauchy-Schwarz inequality:

$$\begin{aligned}
\int_{\mathbb{R}^d} |\varphi_j(x)|dx &= \int_{\mathbb{R}^d} \langle x\rangle^{-\frac{d+1}{2}} \langle x\rangle^{\frac{d+1}{2}} |\varphi_j(x)|dx \\
&\leq C\|\langle x\rangle^{\frac{d+1}{2}} \varphi_j(x)\|_{L^2(\mathbb{R}^d)} \\
&\leq C\|\varphi_j\|_{\mathcal{H}^{\frac{d+1}{2}}(\mathbb{R}^d)} \\
&\leq C\lambda_j^{\frac{d+1}{4}}.
\end{aligned}$$

Thus (4.2) is proved. $\square$

It is not clear to us if the exponent $s(d)$ can be chosen to be independent of d or if we can find the optimal value of $s(d)$.

Corollary 4.2. *If λ is large enough, there is a constant $c>0$ which is independent of λ such that for any $u \in \mathcal{E}_H(\lambda)$ there is $y \in \overline{B}(0,\lambda)$ for which we have*

(i) $\|u\|_{L^\infty(\overline{B}(0,\lambda))} = \|u\|_{L^\infty(\mathbb{R}^d)}$,

(ii) $\forall x \in \overline{B}(y, c\lambda^{-s(d)}) \cap \overline{B}(0,\lambda), \quad |u(x)| \geq \frac{1}{2}\|u\|_{L^\infty(\overline{B}(0,\lambda))}$,

(iii) by denoting by Vol *the volume function, we have*

$$\mathrm{Vol}\Big\{\overline{B}(y, c\lambda^{-s(d)}) \cap \overline{B}(0,\lambda)\Big\} \geq \frac{1}{3}\mathrm{Vol}\big\{\overline{B}(y, c\lambda^{-s(d)})\big\}.$$

Proof. By the same argument we used in the proof of Lemma 4.1, we claim that there is a constant $\nu>0$, independent of λ, such that

$$\forall u \in \mathcal{E}_H(\lambda) \quad \|u\|_{L^2(\mathbb{R}^d)} \leq C\lambda^\nu \|u\|_{L^\infty(\mathbb{R}^d)}. \tag{4.3}$$

By combining (4.1) and (4.3), we understand that if λ is large enough and if $|x|>\lambda$, holds, then we have

$$|u(x)| \leq C\lambda^{\frac{d}{4}+\nu} \exp\big(-\frac{c\lambda}{2}\big)\|u\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{2}\|u\|_{L^\infty(\mathbb{R}^d)}.$$

This proves point (i). Let us check point (ii). By a compactness argument, there is $y \in \overline{B}(0,\lambda)$ which maximizes u on the whole space $\mathbb{R}^d$. For any $x \in \overline{B}(0,\lambda)$, Lemma 4.1 gives us

$$|u(x)-u(y)| \leq C|x-y|\lambda^{s(d)}\|u\|_{L^\infty(\mathbb{R}^d)}.$$

If $|x-y| < \frac{C}{2}\lambda^{-s(d)}$ holds, then $|u(x)| \geq \frac{1}{2}\|u\|_{L^\infty(\mathbb{R}^d)}$.

Point (iii) is a consequence of a geometric fact. Indeed, it is quite clear that we have

$$\liminf_{R\to 0} \inf_{z\in\overline{B}(0,1)} \frac{\mathrm{Vol}\big\{\overline{B}(z,R)\cap\overline{B}(0,1)\big\}}{\mathrm{Vol}\big\{\overline{B}(z,R)\big\}} = \frac{1}{2}.$$

Consequently, if λ is large enough, then point (iii) holds. □

We can prove Theorem 2.5 by following [9, Theorem 1, page 55]. Our preliminaries allow us to deal with the non-compactness of $\mathbb{R}^d$. We define the random maximum

$$M_\lambda^\omega := \|u_\lambda^\omega(x)\|_{L_x^\infty(\mathbb{R}^d)} = \|u_\lambda^\omega(x)\|_{L_x^\infty(B(0,\lambda))}.$$

We apply Lemma 4.2 to the random function $u_\lambda^\omega \in \mathcal{E}_H(\lambda)$. If x belongs to the random set $A_\lambda^\omega := \overline{B}(y^\omega, c\lambda^{-s(d)}) \cap \overline{B}(0,\lambda)$, then we have

$$u_\lambda^\omega(x) \geq \frac{1}{2}M_\lambda^\omega \quad \text{or} \quad -u_\lambda^\omega(x) \geq \frac{1}{2}M_\lambda^\omega.$$

Thus, for any $r>0$, we get

$$\mathbb{E}\Big[\exp\big(\frac{1}{2}rM_\lambda^\omega\big)\Big] \leq \mathbb{E}\Big[\frac{1}{\mathrm{Vol}(A_\lambda^\omega)}\int_{A_\lambda^\omega} \exp\big(ru_\lambda^\omega(x)\big) + \exp\big(-ru_\lambda^\omega(x)\big)\,dx\Big].$$

From point (iii) of Lemma 4.2, we get

$$\begin{aligned}\mathbb{E}\Big[\exp\big(\tfrac{1}{2}rM_\lambda^\omega\big)\Big] &\leq C\lambda^{ds(d)}\mathbb{E}\Big[\int_{A_\lambda^\omega} \exp\big(ru_\lambda^\omega(x)\big) + \exp\big(-ru_\lambda^\omega(x)\big)\,dx\Big] \\ &\leq C\lambda^{ds(d)}\int_{\overline{B}(0,\lambda)} \mathbb{E}\Big[\exp\big(ru_\lambda^\omega(x)\big) + \exp\big(-ru_\lambda^\omega(x)\big)\Big]dx.\end{aligned}$$

By coming back to the definition (1.9) of u^ω_λ, we can use the independence of the random variables X_n:

$$\mathbb{E}\Big[\exp\big(ru^\omega_\lambda(x)\big)\Big] = \prod_{\lambda_n\le\lambda}\mathbb{E}\Big[\exp\big(rc_nX_n(\omega)\varphi_n(x)\big)\Big].$$

Now we use (1.8) and (1.3) to get

$$\begin{aligned}\mathbb{E}\Big[\exp\big(ru^\omega_\lambda(x)\big)\Big] &\le \exp\Big(Cr^2\sum_{\lambda_n\le\lambda}|c_n|^2|\varphi_n(x)|^2\Big)\\ &\le \exp\Big(Cr^2\sum_{j\le\lambda}\Big[\max_{n\in I(j)}|c_n|^2\sum_{n\in I(j)}|\varphi_n(x)|^2\Big]\Big)\\ &\le \exp\big(Cr^2\rho_\lambda\big),\end{aligned}\tag{4.4}$$

where we have set

$$\rho_\lambda := \sum_{j\le\lambda} j^{\gamma(d)}\max_{n\in I(j)}|c_n|^2.$$

Obviously, a similar argument gives the same bound for $\mathbb{E}\Big[\exp\big(-ru^\omega_\lambda(x)\big)\Big]$, and we have obtained

$$\mathbb{E}\Big[\exp\big(\frac{1}{2}rM^\omega_\lambda\big)\Big] \le C\lambda^{ds(d)+d}\exp\big(C\frac{r^2}{2}\rho_\lambda\big),$$

which is totally equivalent to

$$\forall L\ge 1\quad \forall r>0\quad \mathbb{E}\left[\exp\left(\frac{r}{2}\Big(M^\omega_\lambda - Cr\rho^2_\lambda - \frac{2}{r}\ln(C\lambda^{ds(d)+d}L)\Big)\right)\right] \le \frac{1}{L}.$$

From Markov's inequality, we get

$$\mathbb{P}\left[M^\omega_\lambda - Cr\rho^2_\lambda - \frac{2}{r}\ln(C\lambda^{ds(d)+d}L)\ge 0\right] \le \frac{1}{L}.$$

Now we just have to optimize in r by choosing $r^2 = \frac{1}{\rho^2_\lambda}\ln(C\lambda^{ds(d)+d}L)$. For another constant $C>0$, we have

$$\mathbb{P}\left[M^\omega_\lambda \ge C\rho_\lambda\sqrt{\ln(C\lambda^{ds(d)+d}L)}\right] \le \frac{1}{L}.$$

The conclusion comes with the choice $L=\lambda^N$.

Finally, we have to see that the term $\ln(\lambda)$ is optimal in (2.3) if $d\ge 2$ holds and when the $(X_n)_{n\ge0}$ are independent Gaussians $\mathcal{N}_{\mathbb{R}}(0,1)$.

Let us suppose the contrary and consider a function $\vartheta(\lambda)=o(\ln(\lambda))$ such that Theorem 2.5 holds true by replacing $\ln(\lambda)$ with $\vartheta(\lambda)$.

To see this implies a contradiction, let us recall a result proved in [16, Theorem 1.1][1] (with the sequence $d_j = \lambda_j^{-\frac{d}{4}}c_j$ and assuming (1.6)). There are real numbers $C_0>0$ and $c>0$ such that for any $j\gg 1$ one has

$$\begin{aligned}\mathbb{P}\left[C_0\ln(j)\Big\|\sum_{n\in I(j)}\lambda_n^{-\frac{d}{4}}c_nX_n\varphi_n\Big\|^2_{L^2(\mathbb{R}^2)} \le \Big\|\sum_{n\in I(j)}\lambda_n^{-\frac{d}{4}}c_nX_n\varphi_n\Big\|^2_{\mathcal{W}^{\frac{d}{2},\infty}(\mathbb{R}^d)}\right]&\\ \ge 1-\frac{1}{(j+2)^c}&.\end{aligned}$$

[1] [16, Theorem 1.1] is stated for complex Gaussians, but the result also holds for real r.v. (see [16, Assumption 1]).

From the definition (1.1) and Theorem 2.5 with the function ϑ and any chosen positive integer N, we have with probability greater than $1-\frac{1}{(j+2)^N}-\frac{1}{j^N}$:

$$\begin{aligned}\Big\|\sum_{n\in I(j)}\lambda_n^{-\frac{d}{4}}c_nX_n\varphi_n\Big\|_{\mathcal{W}^{\frac{d}{2},\infty}(\mathbb{R}^d)} &= \Big\|\sum_{n\in I(j)}c_nX_n\varphi_n\Big\|_{L^\infty(\mathbb{R}^d)}\\ &\leq \Big\|\sum_{\lambda_n<2j}c_nX_n\varphi_n\Big\|_{L^\infty(\mathbb{R}^d)}\\ &\quad+\Big\|\sum_{\lambda_n<2j+2}c_nX_n\varphi_n\Big\|_{L^\infty(\mathbb{R}^d)}\\ &\leq C\sqrt{\vartheta(2j)+\vartheta(2j+2)}\Big(\sum_{\lambda_n<2j+2}\lambda_n^{-\frac{d}{2}}|c_n|^2\Big)^{\frac12}.\end{aligned}$$

We now have to make use of the condition (1.6):

$$\begin{aligned}\Big(\sum_{n\in I(j)}\lambda_n^{-\frac{d}{2}}|c_n|^2\Big)\times\frac{1}{\#I(j)}\sum_{n\in I(j)}|X_n|^2 &\leq C\sum_{n\in I(j)}|\lambda_n^{-\frac{d}{4}}c_nX_n|^2\\ &= C\Big\|\sum_{n\in I(j)}\lambda_n^{-\frac{d}{4}}c_nX_n\varphi_n\Big\|^2_{L^2(\mathbb{R}^2)}.\end{aligned}$$

By combining these arguments, we have with probability greater than $1-\frac{1}{(j+2)^c}-\frac{1}{(j+2)^N}-\frac{1}{j^N}$:

$$\frac{1}{\#I(j)}\sum_{n\in I(j)}|X_n|^2\leq C\frac{\vartheta(2j)+\vartheta(2j+2)}{\ln(j)}\Big(\sum_{\lambda_n<2j+2}\lambda_n^{-\frac{d}{2}}|c_n|^2\Big)\Big(\sum_{n\in I(j)}\lambda_n^{-\frac{d}{2}}|c_n|^2\Big)^{-1}.$$

One can obviously choose the sequence $(c_n)_{n\geq 0}$ such that (1.6) and the two following properties hold:

$$u\in\mathcal{H}^{-\frac{d}{2}}(\mathbb{R}^d),\quad \sum_{j\geq1}\sum_{n\in I(j)}\lambda_n^{-\frac{d}{2}}|c_n|^2<+\infty,$$

$$\#\Big\{j\geq 1,\quad \sum_{n\in I(j)}\lambda_n^{-\frac{d}{2}}|c_n|^2\geq\Big(\frac{\vartheta(2j)+\vartheta(2j+2)}{\ln(j)}\Big)^{\frac12}\Big\}=+\infty.$$

Hence, we get for probability greater than $1-\frac{1}{(j+2)^c}-\frac{1}{(j+2)^N}-\frac{1}{j^N}$:

$$\frac{1}{\#I(j)}\sum_{n\in I(j)}|X_n|^2\leq\varepsilon(j)\tag{4.5}$$

where $\liminf_{j\to+\infty}\varepsilon(j)=0$. Since $\lim_{j\to+\infty}\#I(j)=+\infty$ holds, the Law of Large Numbers ensures that the left side of (4.5) converges almost surely to $\mathbb{E}\big[|X_1|^2\big]>0$. Since the almost sure convergence implies the convergence in probability, we understand that (4.5) cannot hold. This proves that Theorem 2.5 is optimal for the function $\ln(\lambda)$.

4.2. **Proof of Theorem 2.6.** We give here an argument which uses the Salem-Zygmund theorem. In Section 6 we will present an alternative proof which relies on an entropy argument.

4.2.1. *Proof of Theorem* 2.6 *using the Salem-Zygmund Theorem* 2.5. For any positive integer K, we introduce $J(K) := \big\{n \in \mathbb{N}, \quad \lambda_n \in [2^{2^K}, 2^{2^{K+1}} - 1]\big\}$ and

$$u_K^\omega := \sum_{n \in J(K)} c_n X_n(\omega)\varphi_n.$$

By using Theorem 2.5, we have

$$\mathbb{P}\Bigg[\|u_K^\omega\|_{L^\infty(\mathbb{R}^d)} \geq C2^{K/2}\Big(\sum_{j=2^{2^K-1}}^{2^{2^{K+1}-1}-1} j^{\gamma(d)} \max_{k\in I(j)} |c_n|^2\Big)^{\frac{1}{2}}\Bigg] \leq \frac{1}{2^{2^{K+1}-1}}.$$

The Borel-Cantelli lemma ensures that almost surely there is $C_\omega > 0$ such that

$$\begin{aligned} \|u_K^\omega\|_{L^\infty(\mathbb{R}^d)} &\leq C_\omega 2^{K/2}\Big(\sum_{j=2^{2^K-1}}^{2^{2^{K+1}-1}-1} j^{\gamma(d)} \max_{k\in I(j)} |c_n|^2\Big)^{\frac{1}{2}} \\ &\leq C_\omega \frac{1}{2^{K(\alpha-1)/2}}\Big(\sum_{j=2^{2^K-1}}^{2^{2^{K+1}-1}-1} (\ln j)^\alpha j^{\gamma(d)} \max_{k\in I(j)} |c_n|^2\Big)^{\frac{1}{2}}. \end{aligned}$$

Now by (2.5) and by the Cauchy-Schwarz inequality, since $\alpha > 1$ holds we get

$$\sum_{K\geq 1} \|u_K^\omega\|_{L^\infty(\mathbb{R}^d)} < +\infty \quad \text{a.s.}$$

As a consequence, we have shown that a sub-sequence of the partial sum converges uniformly, a.s. This implies that u^ω is a continuous and bounded function, a.s.

Now moreover if we assume that the (X_n) are symmetric, we can apply [11, Theorem II.5, p. 120], which yields that

$$u_\lambda^\omega = \sum_{\lambda_n \leq \lambda} c_n X_n(\omega)\varphi_n$$

also converges in $L^\infty(\mathbb{R}^d)$, a.s. for $\lambda \to +\infty$.

4.3. **Proof of Theorem 2.7.** The proof will follow the proof of J.-P. Kahane [9, Theorem 2, p. 66], with the necessary modifications in our context.

Let $\kappa \geq 1$ and let us introduce the notation:

$$\nu_j = \kappa 2^{j-1}, \; N_j = 2^{\nu_j},$$

$$\begin{aligned} u_0^\omega(x) &= \sum_{\lambda_n < N_1} c_n X_n(\omega)\varphi_n(x), \;\; \text{for} \;\; j \geq 1, \\ u_j^\omega(x) &= \sum_{N_j \leq \lambda_n < N_{j+1}} c_n X_n(\omega)\varphi_n(x). \end{aligned}$$

Using the triangle inequality and the fundamental calculus theorem we have

$$m_{u^\omega}(h) \leq h\|\nabla_x u_0^\omega\|_{L^\infty(\mathbb{R}^d)} + 2\sum_{1\leq j<+\infty} \|u_j^\omega\|_{L^\infty(\mathbb{R}^d)}.$$

From Theorem 2.5 we have for $j \geq 1$,

(4.6)

$$\mathbb{P}\Bigg[\Big\|u_j^\omega\|_{L^\infty(\mathbb{R}^d)} \geq C(\ln N_{j+1})^{1/2}\Big(\sum_{N_j/2\leq \ell < N_{j+1}/2} \ell^{\gamma(d)} \max_{n\in I(\ell)} |c_n|^2\Big)^{1/2}\Bigg] \leq \frac{1}{N_{j+1}^2}.$$

The $j = 0$ term satisfies the following.

Lemma 4.3. *There exists $C > 0$ large enough such that*

$$\mathbb{P}\left[\left\|\nabla_x u_0^\omega\right\|_{L^\infty(\mathbb{R}^d)} \geq C(\ln N_1)^{1/2}\Big(\sum_{\ell<N_1/2} \ell^{1+\gamma(d)} \max_{n\in I(\ell)} |c_n|^2\Big)^{1/2}\right] \leq \frac{1}{N_1^2}. \tag{4.7}$$

The lemma will be proved later.

Remark 4.4. More generally, we can get a similar bound for $a(x,D)u_0^\omega$, when $a(x,\xi)$ is a polynomial in $(x,\xi) \in \mathbb{R}^{2d}$. We leave the details to the reader.

Using this lemma we can prove Theorem 2.7. Let us denote by $\Omega_j(\kappa)$ the event in (4.6), by $\Omega_0(\kappa)$ the event in (4.7) and $\Omega^\infty(\kappa) = \bigcup_{j\geq 0} \Omega_j(\kappa)$. Using the definition of N_j we have

$$\mathbb{P}\Big[\Omega^\infty(\kappa)\Big] \leq 2^{1-2\kappa}.$$

Hence using the Borel-Cantelli lemma we get that

$$\mathbb{P}\Big[\limsup_{\kappa\to+\infty} \Omega^\infty(\kappa)\Big] = 0.$$

On the other side denote by

$$\begin{aligned} E_0 &= (\ln N_1)^{1/2}\Big(\sum_{\ell<N_1/2} \ell^{1+\gamma(d)} \max_{n\in I(\ell)} |c_n|^2\Big)^{1/2}, \\ E_j &= (\ln N_{j+1})^{1/2}\Big(\sum_{N_j/2\leq\ell<N_{j+1}/2} \ell^{\gamma(d)} \max_{n\in I(\ell)} |c_n|^2\Big)^{1/2}. \end{aligned}$$

Using assumption (2.6) we have

$$\begin{aligned} E_0 &\leq (\ln N_1)^{1/2}\Big(\sum_{\ell=1}^{\kappa-1} \sum_{k=2^{\ell-1}}^{2^\ell} k^{1+\gamma(d)} \max_{n\in I(k)} |c_n|^2\Big)^{1/2} \\ &\leq C\kappa^{1/2}\Big(\sum_{\ell=1}^{\kappa-1} 2^{\ell(1+\gamma(d))} \sum_{k=2^{\ell-1}}^{2^\ell} \max_{n\in I(k)} |c_n|^2\Big)^{1/2} \\ &\leq C\kappa^{1/2}\Big(\sum_{\ell=1}^{\kappa-1} 2^{\ell(1-\mu)} \ell^{2\nu}\Big)^{1/2}, \end{aligned}$$

and for all $j \geq 1$,

$$\begin{aligned} E_j &\leq C(\kappa 2^j)^{1/2}\Big(\sum_{\nu_j\leq\ell<\nu_{j+1}} \sum_{k=2^{\ell-1}}^{2^\ell} k^{\gamma(d)} \max_{n\in I(k)} |c_n|^2\Big)^{1/2} \\ &\leq C(\kappa 2^j)^{1/2}\Big(\sum_{\nu_j\leq\ell<\nu_{j+1}} 2^{\ell\gamma(d)} \sum_{k=2^{\ell-1}}^{2^\ell} \max_{n\in I(k)} |c_n|^2\Big)^{1/2} \\ &\leq C(\kappa 2^j)^{1/2}\Big(\sum_{\nu_j\leq\ell<\nu_{j+1}} 2^{-\mu\ell} \ell^{2\nu}\Big)^{1/2}. \end{aligned}$$

• Assume that $0 < \mu < 1$. We easily compute the estimates

$$E_0 \leq C\kappa^{\frac{1}{2}+\nu} 2^{(1-\mu)\kappa/2}, \qquad E_j \leq C(\kappa 2^j)^{\frac{1}{2}+\nu} 2^{-\mu\kappa 2^{j-2}}$$

and

$$\sum_{j\geq 1} E_j \leq C \sum_{j\geq 1} (\kappa 2^j)^{\frac{1}{2}+\nu} 2^{-\mu\kappa 2^{j-2}} \leq C\kappa^{\frac{1}{2}+\nu} 2^{-\kappa\mu/2}.$$

Now taking $h = h_\kappa = 2^{-\kappa}$ we have proved that for every $\omega \notin \limsup_{\kappa\to+\infty} \Omega^\infty(\kappa)$ and for every κ large enough

$$m_{u^\omega}(h_\kappa) \leq C h_\kappa^\mu |\ln(h_\kappa)|^{\frac{1}{2}+\nu}.$$

Using that $m_{u^\omega}(h)$ is non-increasing in h we have proved Theorem 2.7 for $0 < \mu < 1$.

• Assume that $\mu = 0$ and $\nu < -1$. Then in this case we get

$$E_0 \leq C\kappa^{\frac{1}{2}+\nu} 2^{\kappa/2}, \qquad E_j \leq C\kappa^{\nu+1} 2^{(1+\nu)j}, \quad \sum_{j\geq 1} E_j \leq C\kappa^{\nu+1},$$

and the end of the proof is similar.

• The other cases are proved in the same way (see [9]) except the last one ($\mu = 1$, $\nu < -1$), where the result is obtained by applying Theorem 2.6 to the partial derivatives $\partial_{x_j} u^\omega$, $1 \leq j \leq d$. □

Now we prove Lemma 4.3.

Proof of Lemma 4.3. It is more convenient here to index the Hermite basis by $\mathbb{N}^d$. So we have

$$u_0^\omega(x) = \sum_{2|\alpha|+d\leq N_1} c_\alpha X_\alpha(\omega)\varphi_\alpha(x)$$

where we have denoted $|\alpha| = \alpha_1 + \cdots + \alpha_d$. We have $H\varphi_\alpha = \lambda_\alpha\varphi_\alpha$, with $\lambda_\alpha = 2|\alpha| + d$. It is easier to consider first the tensor basis:

$$\varphi_\alpha(x) = \mathfrak{h}_\alpha(x) = \mathfrak{h}_{\alpha_1}(x_1)\cdots\mathfrak{h}_{\alpha_d}(x_d).$$

Recall that in 1-D the Hermite functions satisfy for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\mathfrak{h}_k(t) = 2^{-1/2}\left(\sqrt{k}\mathfrak{h}_{k-1}(t) - \sqrt{k+1}\mathfrak{h}_{k+1}(t)\right).$$

So we get

$$\begin{aligned}\sqrt{2}\partial_{x_1} u_0^\omega(x) = &\sum_{2|\alpha|+d\leq N_1} \sqrt{\alpha_1} c_\alpha X_\alpha(\omega)\mathfrak{h}_{\alpha-e_1}(x)\\ &- \sum_{2|\alpha|+d\leq N_1} \sqrt{\alpha_1+1} c_\alpha X_\alpha(\omega)\mathfrak{h}_{\alpha+e_1}(x)\end{aligned}$$

where $\{e_j\}_{1\leq j\leq d}$ is the canonical basis of $\mathbb{R}^d$. Applying Theorem 2.5 to each term of the sum we have proved Lemma 4.3 for the tensor basis $\mathfrak{h}_\alpha$.

For a general orthonormal basis $(\varphi_\alpha)_{\alpha\in\mathbb{N}^d}$ of Hermite functions, we write

$$\varphi_\alpha(x) = \sum_{|\alpha|=|\beta|} t_{\alpha,\beta}\mathfrak{h}_\beta(x)$$

where $\{t_{\alpha,\beta}\}$ is a unitary matrix. So we have

$$\sqrt{2}\partial_{x_1} u_0^\omega(x) = \sum_{2|\alpha|+d\leq N_1} \sqrt{\alpha_1} c_\alpha X_\alpha(\omega) \sum_{|\beta|=|\alpha|} t_{\alpha,\beta}\mathfrak{h}_{\beta-e_1}(x)$$
$$- \sum_{2|\alpha|+d\leq N_1} \sqrt{\alpha_1+1} c_\alpha X_\alpha(\omega) \sum_{|\beta|=|\alpha|} t_{\alpha,\beta}\mathfrak{h}_{\beta+e_1}(x).$$

Now we estimate separately the two sums by revisiting the proof of Theorem 2.5.

It is enough to consider the first sum denoted by $v_\lambda^\omega(x)$ where $\lambda = N_1$. We have to estimate

$$\mathbb{E}\Big[\exp(rv_\lambda^\omega)\Big] \leq \exp\Big(Cr^2 \sum_{\lambda_\alpha\leq\lambda} \alpha_1|c_\alpha|^2 \sum_{|\beta|=|\alpha|} |t_{\alpha,\beta}\mathfrak{h}_{\beta-e_1}|^2\Big).$$

For $\lambda_\alpha \in I(j)$ we have

$$\sum_{\lambda_\alpha\in I(j)} \alpha_1|c_\alpha|^2\Big|\sum_{|\beta|=|\alpha|} t_{\alpha,\beta}\mathfrak{h}_{\beta-e_1}(x)\Big|^2 \leq C(1+j) \max_{\lambda_\alpha\in I(j)} |c_\alpha|^2\Big|\sum_{\substack{|\beta|=|\alpha| \\ \lambda_\alpha\in I(j)}} t_{\alpha,\beta}\mathfrak{h}_{\beta-e_1}(x)\Big|^2.$$

Using that the matrix $\{t_{\alpha,\beta}\}$ is unitary we have

$$\Big|\sum_{\substack{|\beta|=|\alpha| \\ \lambda_\alpha\in I(j)}} t_{\alpha,\beta}\mathfrak{h}_{\beta-e_1}(x)\Big|^2 \leq \sum_{\lambda_\alpha\in I(j-1)} |\mathfrak{h}_\alpha(x)|^2 \leq Cj^{\gamma(d)}$$

and

$$\mathbb{E}\Big[\exp(rv_\lambda^\omega)\Big] \leq \exp\Big(Cr^2 \sum_{\lambda_\alpha\leq\lambda} \lambda_\alpha^{\gamma(d)+1}|c_\alpha|^2\Big).$$

This inequality has the same form as (4.4); hence we can conclude here as in the proof of Theorem 2.5. □

5. Annex: About random series in Banach spaces

We present here some elements in the theory of random series in Banach spaces. We refer the reader to the books [8], [5] and [11] for more elements on this subject.

Let B be a Banach space on the field of real or complex numbers. Let $(\varepsilon_n)_{n\geq 0}$ be a sequence of Rademacher i.i.d. random variables and let us define

$$\Sigma(B) := \Big\{(b_n)_{n\geq 0}, \quad \sum \varepsilon_n b_n \quad \text{converges a.s.}\Big\}. \tag{5.1}$$

It is clear that $\Sigma(B)$ is a vector subspace of $B^{\mathbb{N}}$. The following theorem is well-known in the theory of Banach random series (see for instance [11, Chapitre 3, IV.2]):

Theorem 5.1. *Let B be a Banach space and consider a sequence $(b_n)_{n\geq 0}$ in B. The following facts are equivalent:*

(i) the sequence $(b_n)_{n\geq 0}$ belongs to $\Sigma(B)$,
(ii) the random series $\sum \varepsilon_n(\omega)b_n$ converges in probability,
(iii) the random series $\sum \varepsilon_n(\omega)b_n$ converges in law,
(iv) there is some $p \geq 1$ such that the random series $\sum \varepsilon_n(\omega)b_n$ converges in $L^p(\Omega, B)$,
(v) for any $p \geq 1$, the random series $\sum \varepsilon_n(\omega)b_n$ converges in $L^p(\Omega, B)$.

For instance, if B is a Hilbert space, the previous theorem can be used to see that $\Sigma(B)$ is nothing other than $\ell^2(B)$ (see also [9, Chapter 3]).

A natural question is to study what happens for the almost sure convergence of $\sum X_n b_n$ if $(X_n)_{n\geq 0}$ is i.i.d. with another reference law. A part of this question is solved by the following result proved by Hoffman-Jorgensen.

Theorem 5.2 (Hoffman-Jorgensen)**.** *Let $(X_n)_{n\geq 0}$ be a sequence of real, non-constant and i.i.d. random variables and let $(b_n)_{n\geq 0}$ be a sequence which takes values in a general Banach space B. We assume that the series $\sum X_n(\omega) b_n$ converges almost surely in B. Then the series $\sum \varepsilon_n(\omega) b_n$ converges almost surely in B; in other words $(b_n)_{n\geq 0}$ belongs to $\Sigma(B)$.*

We emphasize the fact that no integrability assumption is made on the law of X_n. We do not know any published reference to Theorem 5.2 and we give below a proof we learned from Hervé Queffélec. The converse question is not easy and needs assumptions on the geometry of the Banach space B. It is worthwhile now to recall Kahane-Khintchine's inequalities. For any real numbers $q, p \geq 1$ and any finite sequence $(b_n)_{n\geq 0}$ in B there is a constant $K(p,q)$ which depends only on p and q such that

$$\mathbb{E}\Big[\Big\|\sum_{n\geq 0} \varepsilon_n b_n\Big\|^q\Big]^{1/q} \leq K(p,q)\mathbb{E}\Big[\Big\|\sum_{n\geq 0} \varepsilon_n b_n\Big\|^p\Big]^{1/p}. \tag{5.2}$$

For the specific case $B = \mathbb{R}$, these inequalities are called Khinthine's inequalities, and we have

$$\mathbb{E}\Big[\Big|\sum_{n\geq 0} \varepsilon_n b_n\Big|^2\Big]^{1/2} = \Big(\sum_{n\geq 0} |b_n|^2\Big)^{1/2}.$$

We can now define the notion of cotype of a Banach space.

Definition 5.3. A Banach space B has cotype $p \geq 2$ if there are real numbers $q \geq 1$ and $C_q > 0$ such that for any finite sequence $(b_n)_{n\geq 0}$ in B one has

$$\Big(\sum_{n\geq 0} \|b_n\|^p\Big)^{1/p} \leq C_q \mathbb{E}\Big[\Big\|\sum_{n\geq 0} \varepsilon_n b_n\Big\|^q\Big]^{1/q}. \tag{5.3}$$

Thanks to (5.2), notice that if (5.3) holds, then it holds for any $q \geq 1$. For instance, one can prove that for any $p \geq 1$ the Banach space $B := L^p(\mathbb{R}^d)$ has cotype $\max(2,p)$. To see this, we can make use of Kahane-Khintchine's inequalities for $q = p$:

$$\begin{aligned} \mathbb{E}\left[\Big\|\sum_{n=1}^N \varepsilon_n f_n\Big\|^p_{L^p(\mathbb{R}^d)}\right] &= \int_{\mathbb{R}^d} \mathbb{E}\left[\Big|\sum_{n=1}^N \varepsilon_n(\omega) f_n(t)\Big|^p\right] dt \\ &\sim C_p \int_{\mathbb{R}^d} \Big(\sum_{n=1}^N |f_n(t)|^2\Big)^{\frac{p}{2}} dt. \end{aligned}$$

In the case $p \leq 2$, by denoting by $\|\cdot\|_{2/p}$ the obvious norm of $\mathbb{R}^N$, we can write

$$\begin{aligned}
\int_{\mathbb{R}^d} \Big(\sum_{n=1}^{N} |f_n(t)|^2\Big)^{\frac{p}{2}} dt &= \int_{\mathbb{R}^d} \Big\|\big(|f_1(t)|^p, \dots, |f_N(t)|^p\big)\Big\|_{2/p} dt \\
&\geq \Big\|\int_{\mathbb{R}^d} (|f_1(t)|^p, \dots, |f_N(t)|^p) dt\Big\|_{2/p} \\
&\geq \Big(\sum_{n=1}^{N} \|f_n\|^2_{L^p(\mathbb{R}^d)}\Big)^{\frac{p}{2}}.
\end{aligned}$$

In the case $p \geq 2$, we write

$$\int_{\mathbb{R}^d} \Big(\sum_{n=1}^{N} |f_n(t)|^2\Big)^{\frac{p}{2}} dt \geq \int_{\mathbb{R}^d} \sum_{n=1}^{N} |f_n(t)|^p dt = \sum_{n=1}^{N} \|f_n\|^p_{L^p(\mathbb{R}^d)}.$$

As used in [7] for Gaussian random variables, we have the following astonishing result of Maurey and Pisier:

Theorem 5.4 (Maurey-Pisier)**.** *The following assertions are equivalent:*

(i) *The Banach space B has finite cotype (this means that there is $p \geq 2$ such that B has cotype p).*

(ii) *For any sequence $(b_n)_{n\geq 0}$ of B, the almost sure convergence of $\sum \varepsilon_n b_n$ implies the almost sure convergence of $\sum G_n b_n$, where $(G_n)_{n\geq 0}$ is a sequence of i.i.d. $\mathcal{N}_{\mathbb{R}}(0,1)$ Gaussian random variables.*

(iii) *For any sequence $(b_n)_{n\geq 0}$ of B the almost sure convergence of $\sum \varepsilon_n b_n$ implies the almost sure convergence of $\sum X_n b_n$ where $(X_n)_{n\geq 0}$ is any sequence of real, centered and i.i.d random variables with finite moments of any order.*

Proof. The equivalence $(i) \Leftrightarrow (ii)$ is done in [13, Corollaire 1.3]. Obviously, $(iii) \Rightarrow (ii)$ is true by choosing $X_n = G_n$. Let us explain arguments which are not explicitly written in [13, Corollaire 1.3]. To see $(i) \Rightarrow (iii)$, we begin by assuming that the random variables X_n are symmetric. The proof of [13, Corollaire 1.3, a) $\Rightarrow$ b), page 69] shows that there is a positive constant C which involves a moment $\mathbb{E}[|X_1|^q]$ (for some $q > 0$) such that for any sequence $(b_n)_{n\geq 0}$ we have

$$\forall k, \ell \geq 1 \quad \mathbb{E}\Bigg[\Big\|\sum_{n=k}^{\ell} X_n b_n\Big\|^2\Bigg] \leq C\mathbb{E}\Bigg[\Big\|\sum_{n=k}^{\ell} \varepsilon_n b_n\Big\|^2\Bigg].$$

Since the series $\sum \varepsilon_n b_n$ converges almost surely, it converges in $L^2(\Omega, B)$ (see Theorem 5.1), and so does $\sum X_n b_n$. Now assume that X_n are merely centered. Clearly, $Z_n(\omega, \omega') = X_n(\omega) - X_n(\omega')$ is symmetric on the probability space $\Omega \times \Omega'$. Therefore, the previous analysis shows that $\sum Z_n(\omega, \omega') b_n$ converges in $L^2(\Omega \times \Omega', B)$ and also in $L^1(\Omega \times \Omega', B)$. Now we use that random variables X_n are centered:

$$\forall \ell \geq k \quad \mathbb{E}_\omega\Bigg[\Big\|\sum_{n=k}^{\ell} X_n(\omega) b_n\Big\|\Bigg] \leq \mathbb{E}_{\omega,\omega'}\Bigg[\Big\|\sum_{n=k}^{\ell} X_n(\omega) b_n - X_n(\omega') b_n\Big\|\Bigg].$$

That means that $\sum X_n b_n$ converges in $L^1(\Omega, B)$, so it converges in probability and almost surely in B (see [11, Théorème II.3]). ∎

5.1. **Proof of Proposition 2.1.** Equivalence of (i) and (ii) comes from Theorem 5.2, Theorem 5.4 and the fact that $L^p(\mathbb{R}^d)$ has finite cotype. In order to check the link with (iii), it is necessary and sufficient to study convergence in $L^p(\Omega, L^p(\mathbb{R}^d))$ (see Theorem 5.1). Cauchy criterion leads to handling terms of the following form:

$$\int_\Omega \int_{\mathbb{R}^d} \Big| \sum_{n=k}^{\ell} \varepsilon_n(\omega) f_n(x) \Big|^p d\mathbb{P}(\omega) dx = \int_{\mathbb{R}^d} \mathbb{E}_\omega \left[\Big| \sum_{n=k}^{\ell} \varepsilon_n(\omega) f_n(x) \Big|^p \right] dx.$$

By Khintchine's inequalities (5.2), there exists $C_p \geq 1$ so that

$$\begin{aligned} \frac{1}{C_p} \int_{\mathbb{R}^d} \Big| \sum_{n=k}^{\ell} |f_n(x)|^2 \Big|^{p/2} dx &\leq \int_\Omega \int_{\mathbb{R}^d} \Big| \sum_{n=k}^{\ell} \varepsilon_n(\omega) f_n(x) \Big|^p d\mathbb{P}(\omega) dx \\ &\leq C_p \int_{\mathbb{R}^d} \Big| \sum_{n=k}^{\ell} |f_n(x)|^2 \Big|^{p/2} dx, \end{aligned}$$

and we conclude easily.

5.2. **Proof of Theorem 5.2.** We need the contraction principle (see for instance [11, Théorème III.1] or [9, Chapter 2.6, in the Rademacher framework]) and a few lemmas.

Theorem 5.5 (Contraction principle)**.** *Let $(X_n)_{n\geq 0}$ be a sequence of symmetric independent random variables which takes values in a Banach space B. If $\sum X_n$ converges almost surely in B, then for any bounded real sequence $(\lambda_n)_{n\geq 0}$, the series $\sum \lambda_n X_n$ converges almost surely in B.*

Let us recall a classical lemma in probability theory.

Lemma 5.6. *Let X be a real random variable. Then the following statements are equivalent:*

(i) X is not almost surely constant,
(ii) there is $\xi \in \mathbb{R}$ such that $|\mathbb{E}[\exp(i\xi X)]| < 1$ holds,
(iii) the set $\big\{\xi \in \mathbb{R}, |\mathbb{E}[\exp(i\xi X)]| = 1\big\}$ is countable.

Proof. The implications $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious. Suppose now (i) and let $\xi_0 \neq \xi_1 \in \mathbb{R}\backslash\{0\}$ be two numbers such that $|\mathbb{E}[\exp(i\xi_0 X)]| = |\mathbb{E}[\exp(i\xi_1 X)]| = 1$. Since $|\exp(i\xi_0 X)| \leq 1$ holds, the equality $|\mathbb{E}[\exp(i\xi_0 X)]| = 1$ ensures there is $\alpha_0 \in \mathbb{R}$ such that one has $e^{i\xi_0 x} = e^{i\alpha_0}$ for μ-almost all $x \in \mathbb{R}$ where μ is the law of X. Hence, $x \in \frac{\alpha_0}{\xi_0} + \frac{2\pi}{\xi_0}\mathbb{Z}$ for μ-almost all $x \in \mathbb{R}$. The same is true by replacing ξ_0 with ξ_1 and α_0 with α_1. Because X is not constant almost surely, there are at least two numbers $x \neq y$ which both belong to $\big\{\frac{\alpha_0}{\xi_0} + \frac{2\pi}{\xi_0}\mathbb{Z}\big\} \cap \big\{\frac{\alpha_1}{\xi_1} + \frac{2\pi}{\xi_1}\mathbb{Z}\big\}$. We notice that $x - y \neq 0$ belongs to $\frac{2\pi}{\xi_0}\mathbb{Z} \cap \frac{2\pi}{\xi_1}\mathbb{Z}$. Finally ξ_0/ξ_1 is rational and (iii) is proved. □

Lemma 5.7. *For any sequence of real, non-constant and i.i.d. random variables $(Y_\ell)_{\ell\geq 1}$ we have*

$$\lim_{N\to+\infty} \mathbb{P}\big[\,|Y_1 + \cdots + Y_N| \geq 1|\,\big] = 1.$$

Proof. Let μ be the law of Y_1 and $\varphi \in L^1(\mathbb{R})$ be a function such that $\widehat{\varphi}(x) \geq 1$ holds for any $x \in (-1,+1)$. Then we have

$$\begin{aligned}\mathbb{P}\big[\,|Y_1 + \cdots + Y_N| < 1\,\big] &= \int_{\mathbb{R}} \mathbf{1}_{(-1,1)}(x) d\overbrace{\mu \star \cdots \star \mu}^{N \text{ times}}(x) \\ &\leq \int_{\mathbb{R}} \widehat{\varphi}(x) d\mu \star \cdots \star \mu(x) = \int_{\mathbb{R}} \varphi(\xi)\widehat{\mu}(\xi)^N d\xi.\end{aligned}$$

Point (iii) of Lemma 5.6 ensures that $|\widehat{\mu}(\xi)| < 1$ holds for almost all ξ in the sense of Lebesgue. We conclude by the dominated convergence theorem if N tends to infinity. $\square$

Lemma 5.8. *Let G be a locally compact Abelian group, and consider a subgroup $G_0 \subset G$ which has a positive Haar measure and is everywhere dense. Then G_0 is the whole group G.*

Proof. It is sufficient to prove that G_0 is closed. The Steinhaus theorem states that $G_0 - G_0 \subset G_0$ contains an open neighbourhood of the origin. By using translations of G_0, it turns out that G_0 is an open subgroup of G. A classical argument from the theory of topological groups asserts that G_0 is also closed: we just write $G = \bigsqcup_{i \in I}(G_0 + g_i)$ where $(g_i)_{i \in I}$ is a family of elements of G and $g_i = 0$ for one $i \in I$. It appears that the complementary subset of G_0 is open. $\square$

We can now prove Theorem 5.2.

Proof of Theorem 5.2.

Step 1. It is well known that we can realize any sequence of independent real random variables on the probability space $[0,1]$ endowed with the Lebesgue measure [22] (p. 34 and p. 43). For any $n \geq 0$, we consider a sequence $(\widehat{Z}_{n,\ell})_{\ell \geq 1}$ of i.i.d. random variables on $[0,1]$ and such that $X_n = \widehat{Z}_{n,0}$ for any $n \geq 0$. The random variables

$$\begin{aligned} Z_{n,\ell} : [0,1]^{\mathbb{N}} &\to \mathbb{R}, \\ (\omega_0, \omega_1, \dots) &\mapsto \widehat{Z}_{n,\ell}(\omega_n) \end{aligned}$$

are i.i.d with the same law as the random variables X_n. The assumption of Theorem 5.2 ensures that the series $\sum_{n \geq 0} Z_{n,\ell}(\omega) b_n = \sum_{n \geq 0} \widehat{Z}_{n,\ell}(\omega_n) b_n$ converges in B almost surely in $\omega \in [0,1]^{\mathbb{N}}$. By combining Lemma 5.6 and the equations

$$\forall \ell \geq 1 \quad \mathbb{E}\big[\exp(i\xi Z_{n,2\ell-1} - i\xi Z_{n,2\ell})\big] = \big|E[\exp(i\xi X_1)]\big|^2,$$

we see that $Z_{n,2\ell-1} - Z_{n,2\ell}$ is not constant almost surely. By using Lemma 5.7 with the sequence $Y_\ell = Z_{n,2\ell-1} - Z_{n,2\ell}$, we see that there is an integer $N \geq 1$ which depends only on the law of X_1 such that

$$\frac{1}{2} \leq \mathbb{P}\left[|Z_{n,1} - Z_{n,2} + \cdots + Z_{n,2N-1} - Z_{n,2N}| \geq 1\right] \quad \text{and is independent of } n.$$

By setting $S_n := Z_{n,1} - Z_{n,2} + \cdots + Z_{n,2N-1} - Z_{n,2N}$, we have the three properties:

(i) the series $\sum_{n \geq 0} S_n b_n$ converges almost surely in B in $\omega \in [0,1]^{\mathbb{N}}$,

(ii) $(S_n)_{n \geq 0}$ is a sequence of real, non-constant, **symmetric** and i.i.d. random variables,

(iii) for any $n \geq 0$ one has $\mathbb{P}[|S_n| \geq 1] \geq \frac{1}{2}$.

By construction, $S_n(\omega) = \widehat{S_n}(\omega_n)$ with $\widehat{S_n} := \widehat{Z}_{n,1} - \widehat{Z}_{n,2} + \cdots + \widehat{Z}_{n,2N-1} - \widehat{Z}_{n,2N}$.

Step 2. On the probability space $[0,1]^{\mathbb{N}} \times [0,1]$, one checks that the sequence $(S_n(\omega)\varepsilon_n(\omega'))_{n\geq 0}$ is i.i.d. and has the same common law than S_1. From (i) and (ii), the series $\sum S_n(\omega)\varepsilon_n(\omega')b_n$ converges almost surely in $(\omega,\omega') \in [0,1]^{\mathbb{N}} \times [0,1]$. Fubini's theorem ensures that almost surely in $\omega \in [0,1]^{\mathbb{N}}$ the sequence $(S_n(\omega)b_n)_{n\geq 0}$ belongs to $\Sigma(B)$ (see definition (5.1)). Since $S_n(\omega) = \widehat{S}_n(\omega_n)$, we also have $\mathbb{P}(|\widehat{S}_n| \geq 1) = \mathbb{P}(|S_n| \geq 1) \geq \frac{1}{2}$. Thus, we can consider a Borel subset $A_n \subset [0,1]$ such that

$$\mathbb{P}(A_n) = \frac{1}{2} \quad \text{and} \quad A_n \subset \big\{\omega_n \in [0,1], \quad |\widehat{S_n}(\omega_n)| \geq 1]\big\}.$$

Let us define $\rho_n(\omega) := \mathbf{1}_{A_n}(\omega_n) \leq |S_n(\omega)|$ for each $\omega \in [0,1]^{\mathbb{N}}$. It is obvious that $(\rho_n)_{n\geq 0}$ is a sequence of i.i.d. random variables with the $\frac{1}{2}$-Bernoulli law. From the contraction principle (Theorem 5.5), we know that almost surely in ω the sequence $(\rho_n(\omega)b_n)_{n\geq 0}$ belongs to $\Sigma(B)$.

Step 3. Let us identify $\mathbb{Z}/2\mathbb{Z}$ with $\{0,1\}$ and introduce the compact group $G := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ which becomes now our reference probability space. It is clear that the maps $g \in G \mapsto g_n \in \{0,1\}$ seen as random variables are independent and identically distributed with a $\frac{1}{2}$-Bernoulli law. Let us define $G_0 \subset G$ as the subset of elements $(g_n)_{n\geq 0}$ such that $(g_nb_n)_{n\geq 0}$ belongs to $\Sigma(B)$. Since $\Sigma(B)$ is a vector space, G_0 is a subgroup of G. We directly get from the previous analysis in Step 2 that G_0 has a full Haar measure in G. Furthermore, G_0 contains obviously the everywhere dense subgroup of G of elements $(g_n)_{n\geq 0}$ which satisfy $g_n = 0$ for $n \gg 1$. We use Lemma 5.8 to conclude that $(1,1,\dots)$ belongs to G_0; in other words $(b_n)_{n\geq 0}$ belongs to $\Sigma(B)$. □

6. Annex: An alternative proof of Theorem 2.6 inspired by [21]

We give here a different proof of Theorem 2.6 we learnt from [21], which we decided to detail for pedagogical reasons.

Lemma 6.1. *Let $(\varphi_n)_{n\geq 0}$ be any Hilbertian basis of eigenfunctions for the harmonic oscillator H. Let $\gamma(1) = -1/6$ and $\gamma(d) = d/2 - 1$ for $d \geq 2$. Then for all $j \geq 1$ and $x, y \in \mathbb{R}^d$ we have*

$$\Big(\sum_{n\in I(j)} |\varphi_n(y) - \varphi_n(x)|^2\Big)^{1/2} \leq C|y-x|j^{\gamma(d)/2+1/2}.$$

Proof. By the Taylor formula and Cauchy-Schwarz we get, for $n \in I(j)$,

$$\begin{aligned} |\varphi_n(y) - \varphi_n(x)|^2 &\leq |y-x|^2\Big(\int_0^1 \big|\nabla\varphi_n\big(x+(y-x)t\big)\big|dt\Big)^2 \\ &\leq |y-x|^2\int_0^1 \big|\nabla\varphi_n\big(x+(y-x)t\big)\big|^2dt \\ &\leq Cj|y-x|^2\int_0^1 \big|\varphi_n\big(x+(y-x)t\big)\big|^2dt, \end{aligned} \tag{6.1}$$

where in the last line we used

$$\begin{aligned}\int_0^1 \big|\nabla\varphi_n\big(x+(y-x)t\big)\big|^2 dt &\le \int_0^1 \big|H^{1/2}\varphi_n\big(x+(y-x)t\big)\big|^2 dt\\ &= \lambda_n \int_0^1 \big|\varphi_n\big(x+(y-x)t\big)\big|^2 dt.\end{aligned}$$

Now we sum up the inequalities (6.1) and with (1.3) get

$$\sum_{n\in I(j)} |\varphi_n(y)-\varphi_n(x)|^2 \le Cj|y-x|^2 \sup_{z\in\mathbb{R}} \sum_{n\in I(j)} \big|\varphi_n\big(z\big)\big|^2 \le Cj^{\gamma(d)+1}|y-x|^2,$$

which was the claim. □

We follow the main lines of the proof of N. Tzvetkov [21, Theorem 5]. We define the pseudo-distance δ by

$$\delta(x,y) = \Big(\sum_{n\ge 0} |c_n|^2|\varphi_n(y)-\varphi_n(x)|^2\Big)^{1/2}.$$

For $\alpha>1$, we define the function $\Phi_\alpha : (0,+\infty) \longrightarrow (0,+\infty)$:

$$\Phi_\alpha(t) = \begin{cases} (-\ln t)^{\alpha/2} & \text{if} \quad 0<t<1/a,\\ \Phi_\alpha(1/a) & \text{if} \quad t\ge 1/a,\end{cases}$$

where $a>1$ is chosen in such a way that the function $t\mapsto t\Phi_\alpha(t)$ is increasing on $(0,+\infty)$. Observe also that $t\mapsto\Phi_\alpha(t)$ is non-increasing on $(0,+\infty)$. Then we have a result similar to [21, Theorem 5].

Lemma 6.2. *Assume that the coefficients* (c_n) *satisfy* (2.5)*; then*

$$\delta(x,y) \le \frac{C}{\Phi_\alpha(|y-x|)}.$$

Proof. We clearly have

$$(\delta(x,y))^2 \le C\sum_{j=1}^{+\infty}\big(\max_{k\in I(j)}|c_k|^2\big)\sum_{n\in I(j)}|\varphi_n(y)-\varphi_n(x)|^2.$$

We split the previous sum into two parts. Then, by Lemma 6.1:

$$\begin{aligned} I_1(x,y) \quad &:= \sum_{j:\, aj^{1/2}\le|y-x|^{-1}} \big(\max_{k\in I(j)}|c_k|^2\big)\sum_{n\in I(j)}|\varphi_n(y)-\varphi_n(x)|^2\\ &\le C\sum_{j:\, aj^{1/2}\le|y-x|^{-1}} j^{\gamma(d)+1}\max_{k\in I(j)}|c_k|^2|y-x|^2\\ &= \frac{C}{\Phi_\alpha^2(|y-x|)}\sum_{j:\, aj^{1/2}\le|y-x|^{-1}} j^{\gamma(d)+1}\max_{k\in I(j)}|c_k|^2\Big(|y-x|\Phi_\alpha(|y-x|)\Big)^2. \end{aligned} \tag{6.2}$$

Now we use that the function $t\longmapsto t\Phi_\alpha(t)$ is increasing; thus for $aj^{1/2}\le|y-x|^{-1}$ we have

$$\big(|y-x|\Phi_\alpha(|y-x|)\big)^2 \le j^{-1}(\ln j)^\alpha.$$

Therefore from (6.2) and the assumption (2.5) on the c_n, we get

$$I_1(x,y) \le C\Phi_\alpha^{-2}(|y-x|)\sum_{j=1}^{+\infty} j^{\gamma(d)}(\ln j)^\alpha \max_{k\in I(j)}|c_k|^2 \le C\Phi_\alpha^{-2}(|y-x|).$$

Next, by (1.3),

$$
\begin{aligned}
I_2(x,y) \quad &:= \sum_{j:\, aj^{1/2}>|y-x|^{-1}} \big(\max_{k\in I(j)} |c_k|^2\big) \sum_{n\in I(j)} |\varphi_n(y)-\varphi_n(x)|^2 \\
&\leq C \sum_{j:\, aj^{1/2}>|y-x|^{-1}} j^{\gamma(d)} \max_{k\in I(j)} |c_k|^2. \qquad (6.3)
\end{aligned}
$$

Now we use the fact that Φ_α is non-increasing, and for $aj^{1/2}>|y-x|^{-1}$ we get

$$\Phi_\alpha(|y-x|) \leq \Phi_\alpha(a^{-1}j^{-1/2}) \leq C(\ln j)^{\alpha/2}.$$

As a consequence, from (6.3) and the assumption (2.5) on the c_n, we deduce that

$$I_2(x,y) \leq C\Phi_\alpha^{-2}(|y-x|)\sum_{j=1}^{+\infty} j^{\gamma(d)}(\ln j)^\alpha \max_{k\in I(j)} |c_k|^2 \leq C\Phi_\alpha^{-2}(|y-x|),$$

which completes the proof. □

Proof of Theorem 2.6. It is enough to prove that on every compact set $K\subset\mathbb{R}^d$, a.e. in ω, u^ω is continuous on K. Hence we can follow the proof given in [21, Theorem 5] using an entropy argument (Dudley-Fernique criterion), together with the result of Lemma 6.2. □

Remark 6.3. Let's compare the two different proofs. This proof relies on both a decomposition in space and in frequencies, while in the other proof one only needs a decomposition in frequencies. Observe also that in the first proof one moreover gets that for almost all $\omega\in\Omega$, u^ω is bounded.

Acknowledgment

The authors would like to warmly thank Hervé Queffélec for very interesting discussions about random Banach series, and in particular for the proof of Theorem 5.2.

References

[1] Richard Askey and Stephen Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708. MR0182834 (32 #316)

[2] Antoine Ayache and Nikolay Tzvetkov, *L^p properties for Gaussian random series*, Trans. Amer. Math. Soc. **360** (2008), no. 8, 4425–4439, DOI 10.1090/S0002-9947-08-04456-5. MR2395179 (2009e:30006)

[3] Nicolas Burq and Gilles Lebeau, *Injections de Sobolev probabilistes et applications* (French, with English and French summaries), Ann. Sci. Éc. Norm. Supér. (4) **46** (2013), no. 6, 917–962. MR3134684

[4] Yu Deng, *Two-dimensional nonlinear Schrödinger equation with random radial data*, Anal. PDE **5** (2012), no. 5, 913–960. MR3022846

[5] Joe Diestel, Hans Jarchow, and Andrew Tonge, *Absolutely summing operators*, Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995. MR1342297 (96i:46001)

[6] A. Erdélyi, *Asymptotic forms for Laguerre polynomials*, J. Indian Math. Soc. (N.S.) **24** (1960), 235–250 (1961). MR0123751 (23 #A1073)

[7] Sophie Grivaux, *Almost sure convergence of some random series* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **348** (2010), no. 3-4, 155–159, DOI 10.1016/j.crma.2010.01.020. MR2600068 (2011b:60111)

[8] J. Hoffmann-Jørgensen, *Probability in Banach space*, École d'Été de Probabilités de Saint-Flour, VI-1976, Springer-Verlag, Berlin, 1977, pp. 1–186. Lecture Notes in Math., Vol. 598. MR0461610 (57 #1595)

[9] Jean-Pierre Kahane, *Some random series of functions*, 2nd ed., Cambridge Studies in Advanced Mathematics, vol. 5, Cambridge University Press, Cambridge, 1985. MR833073 (87m:60119)

[10] Herbert Koch and Daniel Tataru, *L^p eigenfunction bounds for the Hermite operator*, Duke Math. J. **128** (2005), no. 2, 369–392, DOI 10.1215/S0012-7094-04-12825-8. MR2140267 (2006d:35044)

[11] Daniel Li and Hervé Queffélec, *Introduction à l'étude des espaces de Banach* (French), Cours Spécialisés [Specialized Courses], vol. 12, Société Mathématique de France, Paris, 2004. Analyse et probabilités. [Analysis and probability theory]. MR2124356 (2006c:46012)

[12] Michael B. Marcus and Gilles Pisier, *Random Fourier series with applications to harmonic analysis*, Annals of Mathematics Studies, vol. 101, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1981. MR630532 (83b:60031)

[13] Bernard Maurey and Gilles Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach* (French), Studia Math. **58** (1976), no. 1, 45–90. MR0443015 (56 #1388)

[14] Benjamin Muckenhoupt, *Mean convergence of Hermite and Laguerre series. II*, Trans. Amer. Math. **147** (1970), 433–460. MR0256051 (41 #711)

[15] Aurélien Poiret, Didier Robert, and Laurent Thomann, *Probabilistic global well-posedness for the supercritical nonlinear harmonic oscillator*, Anal. PDE **7** (2014), no. 4, 997–1026, DOI 10.2140/apde.2014.7.997. MR3254351

[16] Aurélien Poiret, Didier Robert, and Laurent Thomann, *Random-weighted Sobolev inequalities on $\mathbb{R}^d$ and application to Hermite functions*, Ann. Henri Poincaré **16** (2015), no. 2, 651–689, DOI 10.1007/s00023-014-0317-5. MR3302609

[17] Didier Robert and Laurent Thomann, *Random Weighted Sobolev Inequalities and Application to Quantum Ergodicity*, Comm. Math. Phys. **335** (2015), no. 3, 1181–1209, DOI 10.1007/s00220-015-2335-7. MR3320309

[18] Bernard Shiffman and Steve Zelditch, *Random polynomials of high degree and Levy concentration of measure*, Asian J. Math. **7** (2003), no. 4, 627–646. MR2074895 (2005e:32037)

[19] Gabor Szegö, *Orthogonal Polynomials*, American Mathematical Society, New York, 1939. American Mathematical Society Colloquium Publications, v. 23. MR0000077 (1,14b)

[20] Sundaram Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes, vol. 42, Princeton University Press, Princeton, NJ, 1993. With a preface by Robert S. Strichartz. MR1215939 (94i:42001)

[21] N. Tzvetkov, *Riemannian analogue of a Paley-Zygmund theorem*, Séminaire EDP X, 2008-2009, Exposé no. XV.

[22] David Williams, *Probability with martingales*, Cambridge Mathematical Textbooks, Cambridge University Press, Cambridge, 1991. MR1155402 (93d:60002)

[23] Kenji Yajima and Guoping Zhang, *Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity*, J. Differential Equations **202** (2004), no. 1, 81–110, DOI 10.1016/j.jde.2004.03.027. MR2060533 (2005f:35266)

Institut de Mathématiques de Bordeaux, UMR 5251 du CNRS, Université de Bordeaux 1, 351, cours de la Libération F33405 Talence Cedex, France

E-mail address: `rafik.imekraz@math.u-bordeaux1.fr`

Laboratoire de Mathématiques J. Leray, UMR 6629 du CNRS, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France

E-mail address: `didier.robert@univ-nantes.fr`

Laboratoire de Mathématiques J. Leray, UMR 6629 du CNRS, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France

E-mail address: `laurent.thomann@univ-nantes.fr`