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THE PLANAR BUSEMANN-PETTY CENTROID INEQUALITY AND ITS STABILITY

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ABSTRACT. In *Centro-affine invariants for smooth convex bodies* [Int. Math. Res. Notices. DOI 10.1093/imrn/rnr110, 2012] Stancu introduced a family of centro-affine normal flows, p-flow, for $1 \leq p < \infty$. Here we investigate the asymptotic behavior of the planar p-flow for $p = \infty$, in the class of smooth, origin-symmetric convex bodies. First, we prove that the ∞-flow evolves appropriately normalized origin-symmetric solutions to the unit disk in the Hausdorff metric, modulo $SL(2)$. Second, using the ∞-flow and a Harnack estimate for this flow, we prove a stability version of the planar Busemann-Petty centroid inequality in the Banach-Mazur distance. Third, we prove that the convergence of normalized solutions in the Hausdorff metric can be improved to convergence in the $\mathcal{C}^\infty$ topology.

1. INTRODUCTION

The setting of this paper is the n-dimensional Euclidean space. A compact convex subset of $\mathbb{R}^n$ with non-empty interior is called a *convex body*. The set of convex bodies in $\mathbb{R}^n$ is denoted by $\mathcal{K}^n$. Write $\mathcal{K}^n_e$ for the set of origin-symmetric convex bodies and $\mathcal{K}^n_0$ for the set of convex bodies whose interiors contain the origin. Also write, respectively, $\mathcal{F}^n$, $\mathcal{F}^n_0$, and $\mathcal{F}^n_e$ for the set of smooth ($\mathcal{C}^\infty$-smooth), strictly convex bodies in $\mathcal{K}^n$, $\mathcal{K}^n_0$, and $\mathcal{K}^n_e$.

The support function of $K \in \mathcal{K}^n$, $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, is defined by

$$h_K(u) = \max_{x \in \partial K} \langle x, u \rangle,$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product of $\mathbb{R}^n$. For $K \in \mathcal{K}^n$, write $V(K)$ for its Lebesgue measure as a subset of $\mathbb{R}^n$. The centroid body $\Gamma K \in \mathcal{K}^n_e$ of convex body K is the convex body whose support function is given by

$$h_{\Gamma K}(u) = \frac{1}{V(K)} \int_K |\langle u, x \rangle| dx. \tag{1.1}$$

It was proved by Petty that $\Gamma \Phi K = \Phi \Gamma K$ for $\Phi \in GL(n)$ [60] (for newer references, see also [28, Theorem 9.1.3] and [57, Lemma 2.6]). Moreover, a very interesting theorem of Petty states that the centroid body of a convex body is always of class $\mathcal{C}^2_+$ (the class of convex bodies with two times continuously differentiable boundary hypersurface and positive principal curvatures everywhere) [60]. Geometrically, for an origin-symmetric convex body K, the boundary of ΓK is the locus of the

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centroids of all the halves of K obtained by cutting K with hyperplanes through the origin. This concept dates back at least to Dupin. The Busemann-Petty centroid inequality, which was conjectured by Blaschke [13] and established by Petty in his seminal work [60] using the Busemann random simplex inequality, states that

$$\frac{V(\Gamma K)}{V(K)} \geq \left(\frac{2\omega_{n-1}}{(n+1)\omega_n}\right)^n, \tag{1.2}$$

with equality if and only if K is an origin centered ellipsoid. See important extensions of this inequality by Campi, Gronchi, Haberl, Schuster, Lutwak, Yang, and Zhang in the context of the L_p Brunn-Minkowski theory and the Orlicz Brunn-Minkowski theory [24, 25, 36, 47, 55–57].

Let K be a body in $\mathcal{F}_0^n$ whose boundary ∂K is smoothly embedded in $\mathbb{R}^n$ by

$$X_K : \partial K \to \mathbb{R}^n.$$

We write $\nu : \partial K \to \mathbb{S}^{n-1}$ for the Gauss map of ∂K. That is, at each point $x \in \partial K$, $\nu(x)$ is the unit outwards normal at x. The support function of ∂K can also be described as

$$h_K(u) := \langle X_K(\nu^{-1}(u)), u\rangle.$$

The standard metric on $\mathbb{S}^{n-1}$ is denoted by $\bar{g}_{ij}$, and its standard Levi-Civita connection is denoted by $\bar{\nabla}$. Write $\mathcal{G}_K$ for the Gauss curvature of ∂K, and $\mathcal{S}_K$ for the reciprocal Gauss curvature. They are related to the support function by

$$\frac{1}{\mathcal{G}_K(\nu^{-1})} = \mathcal{S}_K := \det_{\bar{g}}(\bar{\nabla}_i\bar{\nabla}_j h_K + \bar{g}_{ij}h_K).$$

When there are no ambiguities, we will drop the subscripts K and K_t.

The flow whose definition will be given was first introduced by Stancu in [75]. In [75], Stancu introduces new equicentro-affine invariants and provides a geometric interpretation of the L_Φ affine surface area introduced by Ludwig and Reitzner [50]. Further applications are given by Stancu in [76] in connection to the Paouris-Werner invariant defined on convex bodies [59].

Let $K \in \mathcal{F}_0^n$. A family of convex bodies $\{K_t\}_t \subset \mathcal{F}_0^n$ given by the smooth map $X : \partial K \times [0, T) \to \mathbb{R}^n$ is said to be a solution to the p-flow if X satisfies the initial value problem

$$\partial_t X(x,t) = -\left(\frac{\mathcal{G}(x,t)}{\langle X(x,t), \nu(x,t)\rangle^{n+1}}\right)^{\frac{p}{p+n}-\frac{1}{n+1}} \mathcal{G}^{\frac{1}{n+1}}(x,t)\,\nu(x,t),\ \ X(\cdot,0) = X_K, \tag{1.3}$$

for a fixed $1 < p < \infty$. In this equation, $0 < T < \infty$ is the maximal time that the solution exists, and $\nu(x,t)$ is the unit normal to the hypersurface $X(\partial K, t) = \partial K_t$ at $X(x,t)$.

The long time behavior of the flow in $\mathbb{R}^n$ with $K_0 \in \mathcal{F}_e^n$ was investigated in [38, 44, 77]. It was proved that, for $1 < p < \frac{n}{n-2}$, the volume-preserving p-flow, which keeps the volume of the evolving bodies fixed and equal to the volume of the unit ball, evolves each body in $\mathcal{F}_e^n$ to the unit ball in $\mathcal{C}^\infty$, modulo $SL(n)$. Moreover, in $\mathbb{R}^3$ the aforementioned result holds for $p \in (1, \infty)$, [77]. Two applications arising from the tools developed in [38] to the L_{-2} Minkowski problem and to the stability of the p-affine isoperimetric inequality in $\mathbb{R}^2$ were given in [39, 40]. The case $p = 1$ corresponds to the well-known affine normal flow. This case in dimension two was addressed by Sapiro and Tannenbaum [64] and in higher dimensions by Andrews [4, 7]. It was proved by Andrews that the volume-preserving affine normal flow

evolves any convex initial bounded open set exponentially fast in the $\mathcal{C}^\infty$ topology to an ellipsoid. Ancient solutions and the existence and regularity of solutions to the affine normal flow on non-compact strictly convex hypersurfaces were treated in [49] by Loftin and Tsui.

It is easy to see from the definition of the support function that, as convex bodies K_t evolve by (1.3), their corresponding support functions satisfy the partial differential equation

$$\partial_t h(u,t) = -h(u,t)\left(\frac{1}{\mathcal{S}h^{n+1}}\right)^{\frac{p}{p+n}}(u,t),\ h(\cdot,0) = h_{K_0}(\cdot); \tag{1.4}$$

see also [75]. The short time existence and uniqueness of the solutions for a smooth and strictly convex initial body follow from the strict parabolicity of the equation as established in [75].

In this paper, we employ the flow (1.4) with $K_0 \in \mathcal{F}_e^2$ and, for the case $p = \infty$ and $n = 2$,

$$\begin{cases} \partial_t h(u,t) = -\frac{1}{h^2\mathcal{S}}(u,t) = -\frac{1}{h^2(u,t)(h_{\theta\theta}+h)(u,t)}, \\ h(\cdot,0) = h_{K_0}(\cdot). \end{cases} \tag{1.5}$$

Notice that the solution to (1.5) remains origin-symmetric. We identify the unit normal $u = (\cos\theta, \sin\theta)$ with θ. The short time existence and uniqueness of the solutions for a smooth and strictly convex initial body follow from the strict parabolicity of the equation.

Write B for the unit disk of $\mathbb{R}^2$. The first main result of the paper is contained in the following theorem.

Theorem A. *Let $K_0 \in \mathcal{F}_e^2$. Then there exists a unique solution $X : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ of the flow* (1.5) *with the initial data X_{K_0}. The solution remains smooth and strictly convex on $[0,T)$ for a finite time $T > 0$. The rescaled convex bodies given by $(V(B)/V(K_t))^{1/2} K_t$ converge sequentially in $\mathcal{C}^\infty$ to the unit disk, modulo $SL(2)$, as $t \to T$.*

It is probably worth mentioning that the method used to conclude the long time behavior in this paper is significantly different from the method of [38] and in particular relies on a number of affinely associated bodies from convex geometry, namely K^*, ΠK, and ΛK (see the next section for the definitions of these bodies). Essentially, the monotonicity of $V(\Gamma K_t)/V(K_t)$ along the flow (1.5) plays a key role. A byproduct of the monotonicity is a stability version of the planar Busemann-Petty centroid inequality.

Consider an inequality $\mathcal{I} : \mathcal{K}^n \to \mathbb{R}$, $\mathcal{I}(K) \geq 0$, for which the equality is precisely obtained for a family $\mathcal{M} \subseteq \mathcal{K}^n$. If for a convex body L and some $\varepsilon > 0$, $\mathcal{I}(L) \leq \varepsilon$ holds, what can be said about the distance of L from the objects in $\mathcal{M}$? Questions of this type investigate the stability of geometric inequalities and have appeared in the work of Minkowski and Bonnesen. See the beautiful survey [32] of Groemer for a wealth of information and references.

In recent times the stability of several significant inequalities has been addressed, and most of these geometric inequalities have balls, ellipsoids, or simplices as objects for the occurrence of the equality. To give examples, we mention stability versions of the Brunn-Minkowski inequality from Diskant [26] and Figalli, Maggi and Pratelli [27], the stability of the Orlicz-Petty projection inequality [20], the stability of the Rogers-Shephard inequality [18], the stability of the Blaschke-Santaló inequality

and the affine isoperimetric inequality in $\mathbb{R}^n$ [19] by Böröczky, the stability of the reverse Blaschke-Santaló inequality by Böröczky and Hug [17], the stability of the Prékopa-Leindler inequality by Ball and Böröczky [14, 15], and, more recently, the stability of the functional forms of the Blaschke-Santaló inequality by Barthe, Böröczky and Fradelizi [16]. The second aim of this paper is to prove a stability version of the planar Busemann-Petty centroid inequality using (1.5). Within the last few years, a substantial amount of research has been devoted to investigating applications of geometric flows to different areas of mathematics. In particular, there are several major contributions of geometric flows to convex geometry: a proof of the affine isoperimetric inequality by Andrews using the affine normal flow [4], the necessary and sufficient conditions for the existence of a solution to the discrete L_0-Minkowski problem using crystalline curvature flow by Stancu [71,72,74] and independently by Andrews [8], an application of the affine normal flow to the regularity of minimizers of Mahler volume by Stancu [73], and quermassintegral inequalities for k-convex star-shaped domains using a family of expanding flows [33].

To state our stability result, we recall the Banach-Mazur distance. A natural tool in affine geometry of convex bodies is the Banach-Mazur distance, which for two convex bodies $K, \bar{K} \in \mathcal{K}_e^n$ is defined by

$$d_{\mathcal{BM}}(K, \bar{K}) = \min\{\lambda \geq 1 : K - x \subseteq \Phi(\bar{K} - y) \subseteq \lambda(K - x),\ \Phi \in GL(n)\}.$$

It is clear that $d_{\mathcal{BM}}(K, \Phi\bar{K}) = d_{\mathcal{BM}}(K, \bar{K})$ for all $\Phi \in GL(n)$. Moreover, the Banach-Mazur distance is multiplicative. That is, for $K_1, K_2, K_3 \in \mathcal{K}_e^n$, the following inequality holds:

$$d_{\mathcal{BM}}(K_1, K_3) \leq d_{\mathcal{BM}}(K_1, K_2) d_{\mathcal{BM}}(K_2, K_3).$$

The second main result of the paper is contained in the following theorem.

Theorem B. *There exist $\varepsilon_0 > 0$ and $\gamma > 0$ such that the following holds: if $0 < \varepsilon < \varepsilon_0$ and K is a convex body in $\mathbb{R}^2$ such that $\frac{V(\Gamma K)}{V(K)} \leq \left(\frac{4}{3\pi}\right)^2 (1 + \varepsilon)$, then $d_{\mathcal{BM}}(K, B) \leq 1 + \gamma\varepsilon^{1/8}$. Furthermore, if K is origin-symmetric, then $d_{\mathcal{BM}}(K, B) \leq 1 + \gamma\varepsilon^{1/4}$.*

Since $\mathcal{I} : \mathcal{K}_0^2 \to \mathbb{R}$ defined by $\mathcal{I}(K) := \frac{V(\Gamma K)}{V(K)} - \left(\frac{4}{3\pi}\right)^2$ is a continuous functional in the Hausdorff distance, it suffices to prove Theorem B for bodies in $\mathcal{F}^2$. Moreover, in light of a theorem of Campi and Gronchi [24] that states that the volume of the centroid body is not increased after a Steiner symmetrization and Theorem 1.4 of Böröczky [19], it is enough to first prove Theorem B for bodies in $\mathcal{F}_e^2$. The *idea* to prove this result is as follows: let $K \in \mathcal{F}_e^2$ satisfy the assumption of Theorem B and let $\{K_t\}$ be the solution to the flow (1.5) with $K_0 = \Phi K$ for an appropriate $\Phi \in GL(2)$. It will be proved that $V(\Gamma K_t)/V(K_t)$ is non-increasing in time. Furthermore, calculating the evolution equation of $V(\Gamma K_t)/V(K_t)$, we prove that its time derivative is controlled by a stable area ratio that is zero only for ellipses. From this observation, we will conclude that, for some time $s > 0$ and close to zero, K_s must be close to the unit disk in the Banach-Mazur distance. Additionally, we can also control the distance between K and K_s in the Banach-Mazur distance (using a Harnack estimate) provided that ε is small enough. Putting these observations together, we can prove that K_0 is close to the unit disk in the Banach-Mazur distance, and so is K. This approach to the stability problem was employed by the

author to obtain the stability of the p-affine isoperimetric inequality for bodies in $\mathcal{K}_e^2$ [40].

The paper is structured as follows: In the next section, we recall some definitions and results from convex geometry. Section 3 focuses on establishing the basic properties of (1.5). We show that evolving bodies remain smooth and strictly convex and the area of the evolving convex bodies converge to zero in finite time. In section 4, we study the long time behavior of (1.5). To study the convergence of the solutions, we resort to the evolution equation of $V(\Gamma K_t)/V(K_t)$ along the flow. The crucial result is that $V(\Gamma K_t)/V(K_t)$ is non-increasing along the flow. This observation implies that $(V(B)/V(K_t))^{1/2} K_t$ converges in the Banach-Mazur distance to a limiting shape $\bar{K}_\infty$ with the property $\Lambda \bar{K}_\infty = \bar{K}_\infty$. It is here, using a theorem of Petty [62] on the latter equality in dimension two, where we conclude the convergence of solutions to the unit disk, modulo $SL(2)$. In section 5, we prove Theorem B. In the final section we prove the sequential convergence of the normalized solution in $\mathcal{C}^\infty$ to the unit disk, modulo $SL(2)$.

2. Background and notation

If K and L are convex bodies and $0 < a < \infty$, then the Minkowski sum $K + aL$ is defined by $h_{K+aL} = h_K + a h_L$ and the mixed volume $V_1(K, L)$ ($V(K, L)$ for planar convex bodies) of K and L is defined by

$$V_1(K, L) = \frac{1}{n} \lim_{a \to 0^+} \frac{V(K + aL) - V(K)}{a}.$$

A fundamental fact is that, corresponding to each convex body K, there is a unique Borel measure S_K on the unit sphere such that

$$V_1(K, L) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u) dS_K(u)$$

for each convex body L. The measure S_K is called the surface area measure of K. Recall that, if K is $\mathcal{C}_+^2$, then S_K is absolutely continuous with respect to σ, and the Radon-Nikodým derivative $dS_K(u)/d\sigma(u)$ defined on $\mathbb{S}^{n-1}$ is the reciprocal Gauss curvature of ∂K at the point of ∂K whose outer normal is u. For a body $K \in \mathcal{K}^n$,

$$V(K) = V_1(K, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) dS_K(u).$$

Of significant importance in convex geometry is the Minkowski mixed volume inequality. Minkowski's mixed volume inequality states that, for $K, L \in \mathcal{K}^n$,

$$V_1(K, L)^n \geq V(K)^{n-1} V(L).$$

In the class of origin-symmetric convex bodies, equality holds if and only if $K = cL$ for some $c > 0$.

In $\mathbb{R}^2$, a stronger version of Minkowski's inequality was obtained by Groemer [31]. We provide his result for bodies in $\mathcal{K}_e^2$.

Theorem (Groemer, [31]). *Let $K, L \in \mathcal{K}_e^2$ and set $D(K) = 2 \max_{\mathbb{S}^1} h_K$. Then*

$$\frac{V(K, L)^2}{V(K)V(L)} - 1 \geq \frac{V(K)}{4D^2(K)} \max_{u \in \mathbb{S}^1} \left| \frac{h_K(u)}{V(K)^{\frac{1}{2}}} - \frac{h_L(u)}{V(L)^{\frac{1}{2}}} \right|^2.$$

The projection body, $\Pi K \in \mathcal{K}_e^n$, of K is the convex body whose support function is given by

$$h_{\Pi K}(u) = \frac{1}{2}\int_{\partial K} |\langle u, \nu\rangle| dv, \tag{2.1}$$

where the integration is done with respect to the $(n-1)$-Hausdorff measure.

Remark 2.1. If $L \in \mathcal{K}_e^2$, then $\Pi L = L^{\pi/2} + L^{-\pi/2} = 2L^{\pi/2}$, where $h_{L^{\pi/2}}(\theta) = h_L(\theta + \pi/2)$ and $h_{L^{-\pi/2}}(\theta) = h_L(\theta - \pi/2)$; see [28, Theorem 4.1.4]. (Convex bodies $L^{\pi/2}$ and $L^{-\pi/2}$ are rotations of L counter-clockwise and clockwise through 90° respectively.)

The polar body, K^*, of $K \in \mathcal{K}_0^n$ is the convex body defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for all } y \in K\}.$$

A fundamental affine inequality is the Petty projection inequality, which states for $K \in \mathcal{K}^n$,

$$V(K)^{n-1} V((\Pi K)^*) \leq \left(\frac{\omega_n}{\omega_{n-1}}\right)^n, \tag{2.2}$$

with equality if and only if K is an ellipsoid. This inequality was proved by Petty [61], using the Busemann-Petty centroid inequality. Another proof of the Petty projection inequality using the Busemann-Petty centroid inequality is provided by Lutwak through a class reduction technique [53]. It is worth pointing out that the Petty projection inequality is a strengthened form of the classical isoperimetric inequality. Extensions of (2.2) are given in [51, 52].

For $x \in \operatorname{int} K$, let $K^x := (K - x)^*$. The Santaló point of K, denoted by s, is the unique point in $\operatorname{int} K$ such that

$$V(K^s) \leq V(K^x)$$

for all $x \in \operatorname{int} K$. The Blaschke-Santalò inequality [12, 63] states that

$$V(K^s)V(K) \leq \omega_n^2, \tag{2.3}$$

with equality if and only if K is an ellipsoid. The equality condition was proved by Saint-Raymond [65] in the symmetric case and Petty [62] in the general case. A proof of this inequality is also given via the affine normal flow by Andrews [4, 7].

The Santaló point of K is characterized by the property:

$$\int_{\mathbb{S}^{n-1}} \frac{u}{h_{K-s}^{n+1}(u)} d\sigma(u) = 0,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^{n-1}$. Thus, for an arbitrary convex body K, $h_{K-s}^{-(n+1)}$ satisfies the sufficient condition of Minkowski's existence theorem in $\mathbb{R}^n$. Thus, there exists a unique convex body (up to translation), denoted by ΛK, whose surface area measure, $S_{\Lambda K}$, satisfies

$$dS_{\Lambda K} = \frac{V(K)}{V(K^s)} h_{K-s}^{-(n+1)} d\sigma. \tag{2.4}$$

Moreover, $\Lambda\Phi K = \Phi\Lambda K$ (up to translation) for $\Phi \in GL(n)$; see [54, Lemma 7.12].

Remark 2.2. By the uniqueness of the solution to the even Minkowski problem, if $K \in \mathcal{K}_e^2$, then there is a unique origin-symmetric solution to (2.4). Indeed, for each solution ΛK, we have $\Lambda K + \vec{a} \in \mathcal{K}_e^2$, for some vector $\vec{a}$ (see, for example, [54, p. 370]). In the sequel, we always assume, without loss of generality, that if $L \in \mathcal{K}_e^2$, then $\Lambda L \in \mathcal{K}_e^2$. Also note that, for $K \in \mathcal{K}_e^2$, the property $\Lambda\Phi K = \Phi\Lambda K$ (up to translation), for $\Phi \in GL(2)$, implies that $(\Lambda K)^{\pi/2} = \Lambda K^{\pi/2}$.

A useful characterization of the centroid operator is given by Lutwak [53, Lemma 5]:

$$\Gamma(K - c) = \frac{2}{(n+1)V(K^c)}\Pi\Lambda K^c, \tag{2.5}$$

where c denotes the centroid of K.

Let K be an origin-symmetric convex body; then, the existence of John's ellipsoid implies $d_{\mathcal{BM}}(K, B) \leq \sqrt{n}$, [45] (see also [30] for a simple proof). In particular, this implies that, for each $K \in \mathcal{K}_e^2$, there is a linear transformation $\Phi \in GL(2)$ such that $1 \leq h_{\Phi K} \leq \sqrt{2}$.

Given a body $K \in \mathcal{K}^n$, the inner radius of K, $r_-(K)$, is the radius of the largest ball contained in K; the outer radius of K, $r_+(K)$, is the radius of the smallest ball containing K. For each $K \in \mathcal{K}_e^n$, the smallest and the largest balls will be centered at the origin of $\mathbb{R}^n$.

We conclude this section by mentioning that, for $K \in \mathcal{K}_e^n$, the Santaló point and the centroid coincide with the origin of $\mathbb{R}^n$.

3. Basic properties of the flow

Arguments in this section are standard. For completeness, we sketch their proofs. Recall that $V(K) = \frac{1}{2}\int_{\mathbb{S}^1} h_K \mathcal{S}_K \, d\theta$ and $V(K^*) = \frac{1}{2}\int_{\mathbb{S}^1} \frac{1}{h_K^2} d\theta$. The following evolution equations can be derived by direct computation.

Lemma 3.1. *Under the flow* (1.5)*, one has*

$$\frac{\partial}{\partial t}\mathcal{S} = -\left(h^{-2}\mathcal{S}^{-1}\right)_{\theta\theta} - h^{-2}\mathcal{S}^{-1}, \tag{3.1}$$

$$\frac{d}{dt}V(K_t) = -2V(K_t^*). \tag{3.2}$$

Proposition 3.2. *The time-dependent quantity* $\min\limits_{\theta\in\mathbb{S}^1}\left(h^{-2}\mathcal{S}^{-1}\right)(\theta, t)$ *increases in time under* (1.5).

Proof. Using the evolution equations (1.5) and (3.1), we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\left(h^{-2}\mathcal{S}^{-1}\right) =& \left(\frac{\partial}{\partial t}h^{-2}\right)\mathcal{S}^{-1} + h^{-2}\left(\frac{\partial}{\partial t}\mathcal{S}^{-1}\right)\\ =&\mathcal{S}^{-2}h^{-2}\left[\left(h^{-2}\mathcal{S}^{-1}\right)_{\theta\theta} + h^{-2}\mathcal{S}^{-1}\right]\\ &+ 2h^{-3}\mathcal{S}^{-1}\left(h^{-2}\mathcal{S}^{-1}\right).\end{aligned}$$

The standard parabolic maximum principle completes the proof. □

An immediate consequence is the preservation of the strict convexity.

Corollary 3.3. *The strict convexity of the evolving bodies is preserved as long as the flow exists.*

Proof. By Proposition 3.2, as long as the flow exists,

$$\min_{\theta\in\mathbb{S}^1}\left(\mathcal{G}/h^2\right)(\theta,t)\geq\min_{\theta\in\mathbb{S}^1}\left(\mathcal{G}/h^2\right)(\theta,0).$$

Therefore

$$\mathcal{G}(\theta,t)\geq h^2(\theta,t)\min_{\theta\in\mathbb{S}^1}\left(\mathcal{G}/h^2\right)(\theta,0)>0.$$

Thus, the claim easily follows. □

Lemma 3.4. *If there exists an $r>0$ such that $h_{K_t}\geq r$ on $[0,T)$, then $\mathcal{G}_{K_t}$ is uniformly bounded from above on $[0,T)$.*

Proof. We sketch a proof that is based on Tso's trick [78]. Define $\Omega(\theta,t):=\frac{h^{-2}\mathcal{S}^{-1}}{h-\rho}$, where $\rho=\frac{1}{2}r$. We may assume, without loss of generality, that the maximum of Ω occurs in $(0,T)$. At the point where the maximum of Ω occurs, we have

$$\Omega_\theta=0,\ \Omega_{\theta\theta}\leq 0$$

and

$$0\leq\frac{\partial}{\partial t}\Omega\leq\frac{1}{h-\rho}\left[h^{-2}\mathcal{S}^{-2}\left(-\frac{\rho h^{-2}\mathcal{S}^{-1}-h^{-2}}{h-\rho}\right)+\mathcal{S}^{-1}\frac{\partial}{\partial t}h^{-2}+\frac{(h^{-4}\mathcal{S}^{-2})}{h-\rho}\right].$$

This last inequality gives

$$-\rho\mathcal{G}-2\rho\frac{1}{h}+4\geq 0\Rightarrow 0<\mathcal{G}\leq\frac{4}{\rho}.$$

□

Proposition 3.5. *Let T be the maximal time of the existence of the solution to the flow* (1.5) *with $K_0\in\mathcal{K}_e^2$. Then, T is finite and $V(K_t)$ tends to zero as t approaches T.*

Proof. The first part of the claim follows from the comparison principle and the fact that the solution to (1.5) starting from a disk centered at the origin exists only on a finite time interval. Having established Corollary 3.3 and Lemma 3.4, bounds on higher derivatives of the support function follow from Schauder's theory [46] if $\lim_{t\to T}V(K_t)\neq 0$. This in turn contradicts the maximality of T. □

4. Long time behavior

In this section we calculate the time derivative of $V(\Gamma K_t)/V(K_t)$ and we deduce the asymptotic behavior of the normalized solution, $\left(V(B)/V(K_t)\right)^{1/2}K_t$. We shall begin by rewriting the integral representation of the centroid body of $K\in\mathcal{K}_0^n$ in terms of the support function of the polar body K^*. To this aim, we must first introduce the radial function of the convex body $K\in\mathcal{K}_0^n$.

The function $\rho_K:\mathbb{S}^{n-1}\to\mathbb{R}$ defined by

$$\rho_K(u):=\max\{\lambda\geq 0:\lambda u\in K\}$$

is called the radial function. This function parameterizes ∂K over the unit sphere by

$$X_K=\rho(u)u:\mathbb{S}^{n-1}\to\mathbb{R}^n.$$

It can be shown that ρ is a Lipschitz function. Moreover, $\rho_K(u) = 1/h_{K^*}(u)$; see [66, Theorem 1.7.6]. Thus, we may rewrite (1.1) for $K \in \mathcal{K}_0^n$ as

$$\begin{aligned} h_{\Gamma K}(u) =& \frac{1}{V(K)} \int_K |\langle u, x\rangle| dx = \frac{1}{(n+1)V(K)} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| \rho_K^{n+1}(v) d\sigma(v) \\ =& \frac{1}{(n+1)V(K)} \int_{\mathbb{S}^{n-1}} |\langle u, v\rangle| h_{K^*}^{-(n+1)}(v) d\sigma(v). \end{aligned}$$

From this last integral representation of $h_{\Gamma K}$, it is evident that, to calculate the time derivative of $V(\Gamma K_t)$, we must first calculate the time derivative of $h_{K_t^*}$.

Lemma 4.1. *Let K_t evolve by* (1.5)*. Then polar bodies, K_t^*, evolve according to*

$$\partial_t h_{K_t^*}(u) = h_{K_t^*}^4(u) \mathcal{S}_{K_t^*}(u).$$

Employing the evolution equation of polar bodies was first introduced by Stancu [75] in the context of centro-affine normal flows. In [41, 44], it was shown that the evolution equation of polar bodies combined with Tso's trick and the Salkowski-Kaltenbach-Hug identity (see [37, Theorem 2.8]) provides a useful tool for obtaining the regularity of solutions to a class of geometric flows. The proof of Lemma 4.1 is omitted because of its similarity to the proof of [41, Theorem 2.2]. We will now turn to the evolution equation of $h_{\Gamma K_t}$.

Lemma 4.2. *Let K_t evolve by* (1.5)*. Then, centroid bodies, ΓK_t, evolve according to*

$$\partial_t h_{\Gamma K_t}(u) = \frac{2V(K_t^*)}{V(K_t)} h_{\Gamma K_t}(u) - \frac{2}{V(K_t)} h_{\Pi K_t^*}(u).$$

Proof. We will use the evolution equation of $V(K_t)$ in Lemma 3.1 and the evolution equation of $h_{K_t^*}$ stated in Lemma 4.1. Since, for a small neighborhood of t, $\partial_t h_{K_t^*}$ are bounded and $h_{K_t^*}$ are bounded from above and from below by positive numbers, allowing ∂_t to commute with $\int_{\mathbb{S}^1}$ is justified.

$$\begin{aligned} &\partial_t h_{\Gamma K_t}(u) \\ &= \partial_t \left(\frac{1}{3V(K_t)} \int_{\mathbb{S}^1} |u, v| h_{K_t^*}^{-3}(v) d\sigma(v) \right) \\ &= \frac{-\frac{d}{dt} V(K_t)}{3V^2(K_t)} \int_{\mathbb{S}^1} |u, v| h_{K_t^*}^{-3}(v) d\sigma(v) + \frac{1}{3V(K_t)} \int_{\mathbb{S}^1} |u, v| \partial_t \left(h_{K_t^*}^{-3}(v) \right) d\sigma(v) \\ &= \frac{2V(K_t^*)}{3V^2(K_t)} \int_{\mathbb{S}^1} |u, v| h_{K_t^*}^{-3}(v) d\sigma(v) - \frac{1}{V(K_t)} \int_{\mathbb{S}^1} |u, v| \mathcal{S}_{K_t^*}(v) d\sigma(v) \\ &= \frac{2V(K_t^*)}{V(K_t)} h_{\Gamma K_t}(u) - \frac{2}{V(K_t)} h_{\Pi K_t^*}(u). \end{aligned}$$

In the last line, we used the fact that we may rewrite (2.1) for $K \in \mathcal{C}_+^2$ as follows:

$$h_{\Pi K}(u) = \frac{1}{2} \int_{\partial K} |\langle u, v\rangle| dv = \frac{1}{2} \int_{\mathbb{S}^1} |\langle u, v\rangle| \mathcal{S}(v) d\sigma(v).$$

□

Lemma 4.3. *As K_t evolves by* (1.5)*, centroid bodies, ΓK_t, evolve according to*

$$\frac{d}{dt} V(\Gamma K_t) = \frac{4V(K_t^*)}{V(K_t)} V(\Gamma K_t) - \frac{4}{V(K_t)} V(\Gamma K_t, \Pi K_t^*).$$

Proof. Recall that, if $K \in \mathcal{K}^n$ is $\mathcal{C}^2_+$, then h_K is two-times continuously differentiable (see [66, p. 106]). As $\Gamma K_t \in \mathcal{C}^2_+$ we can write

$$V(\Gamma K_t) = \frac{1}{2}\int_{\mathbb{S}^1} h_{\Gamma K_t}\left((h_{\Gamma K_t})_{\theta\theta} + h_{\Gamma K_t}\right) d\theta.$$

Furthermore, convex bodies $\Pi K_t^* = 2(K_t^*)^{\pi/2}$ are also $\mathcal{C}^2_+$. The claim now follows from Lemma 4.2, integration by parts, and the definition of the mixed volume. □

Corollary 4.4. *Along the flow* (1.5), $\frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)} \leq 0$ *with equality if and only if* K_t *is an origin-centered ellipse. In particular, if* $K \in \mathcal{F}^2_e$ *minimizes* $\frac{V(\Gamma K)}{V(K)}$, *then* K *must be an origin-centered ellipse.*

Proof. Lemmas 3.1 and 4.3 yield

$$\frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)} = \frac{6V(K_t^*)}{V^2(K_t)}V(\Gamma K_t) - \frac{4}{V^2(K_t)}V(\Gamma K_t, \Pi K_t^*).$$

Replacing ΓK_t on the right-hand side by its equivalent expression from (2.5) will lead us to

$$\frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)} = \frac{8}{3V^2(K_t)V(K_t^*)}\left(V(\Pi\Lambda K_t^*) - V(\Pi\Lambda K_t^*, \Pi K_t^*)\right). \tag{4.1}$$

By Remarks 2.1 and 2.2, we may rewrite (4.1) as

$$\begin{aligned}\frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)} &= \frac{32}{3V^2(K_t)V(K_t^*)}\left(V((\Lambda K_t^*)^{\pi/2}) - V(\Lambda(K_t^*)^{\pi/2}, (K_t^*)^{\pi/2})\right)\\ &= \frac{32}{3V^2(K_t)V(K_t^*)}\left(V((\Lambda K_t^*)^{\pi/2}) - V((K_t^*)^{\pi/2})\right)\\ &= \frac{32}{3V^2(K_t)V(K_t^*)}\left(V(\Lambda K_t^*) - V(K_t^*)\right). \qquad (4.2)\end{aligned}$$

Here we used the easily established identity $V(\Lambda L, L) = V(L)$ for $L \in \mathcal{K}^2$ (this identity follows from the definition of ΛL and the definition of the mixed volume; see [53, Lemma 3]). Note that, by the Minkowski inequality, $V^2(L) = V^2(\Lambda L, L) \geq V(L)V(\Lambda L)$. Therefore, $V(L) \geq V(\Lambda L)$ for all $L \in \mathcal{K}^2$ with equality if and only if ΛL is a translation of L. For $L \in \mathcal{K}^2_e$, equality is achieved only if $\Lambda L = L$, as $\Lambda L \in \mathcal{K}^2_e$. Moreover, [62, Lemma 8.1] states that $\Lambda L = L$ if and only if L is an origin-centered ellipse. □

Proposition 4.5. *As* K_t *evolves by* (1.5) *the following limit holds:*

$$\limsup_{t\to T}\frac{1}{V(K_t)V^2(K_t^*)}\left(V(\Lambda K_t^*) - V(K_t^*)\right) = 0.$$

Proof. Suppose to the contrary that there exist ε, $\delta > 0$ such that, for all $t \in (T-\delta, T)$, we have

$$\frac{(V(\Lambda K_t^*) - V(K_t^*))}{V(K_t)V^2(K_t^*)} \leq -\varepsilon.$$

Therefore, by Lemma 3.1 and inequality (4.2), we get

$$\frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)} \leq \frac{16}{3}\varepsilon\frac{d}{dt}\ln(V(K_t)).$$

Since $\lim_{t\to T} V(K_t) = 0$ by Proposition 3.5, we deduce that

$$\lim_{t\to T} \frac{V(\Gamma K_t)}{V(K_t)} = -\infty.$$

However, $V(\Gamma K_t)/V(K_t)$ is manifestly positive. □

4.1. **Convergence of a subsequence of the normalized solution in the Hausdorff metric.** We begin this section by recalling the weak continuity of surface area measures. If $\{K_i\}_i$ is a sequence in $\mathcal{K}_0^n$, then

$$\lim_{i\to\infty} K_i = K \in \mathcal{K}_0^2 \Rightarrow \lim_{i\to\infty} S_{K_i} = S_K, \text{ weakly.}$$

Weak convergence means that, for every continuous function f on $\mathbb{S}^{n-1}$, we have

$$\lim_{i\to\infty} \int_{\mathbb{S}^{n-1}} f dS_{K_i} = \int_{\mathbb{S}^{n-1}} f dS_K.$$

Lemma 4.6. *If $\{K_i\}$ is a sequence of bodies in $\mathcal{K}_e^2$ converging to a body $K \in \mathcal{K}_e^2$, then*

$$\lim_{i\to\infty} \Lambda K_i = \Lambda K.$$

Proof. On the one hand, by the definition of ΛK, for every continuous function f on $\mathbb{S}^1$, we have

$$\begin{aligned}\lim_{i\to\infty} \int_{\mathbb{S}^1} f dS_{\Lambda K_i} &= \lim_{i\to\infty} \left(\frac{V(K_i)}{V(K_i^*)} \int_{\mathbb{S}^1} f h_{K_i}^{-3} d\sigma \right) \\ &= \frac{V(K)}{V(K^*)} \int_{\mathbb{S}^1} f h_K^{-3} d\sigma = \int_{\mathbb{S}^1} f dS_{\Lambda K}.\end{aligned} \tag{4.3}$$

Note that, to go from the first line to the second line, we have used the bounded convergence theorem: indeed, $\lim_{i\to\infty} K_i = K$ implies that $\{h_{K_i}\}$ converges uniformly on $\mathbb{S}^1$ to h_K. Moreover, by the assumption $K \in \mathcal{K}_e^2$, we have $0 < m < h_K < M < \infty$, for some constants. Therefore, $\{1/h_{K_i}\}$ is uniformly bounded from above.

On the other hand, $\{\Lambda K_i\}$ is uniformly bounded: recall from the identity (2.5) that $\Pi\Lambda K_i = \frac{3}{2} V(K_i)\Gamma K_i^*$ and it is clear that $\{\Gamma K_i^*\}$ is uniformly bounded. Therefore, $\{\Pi\Lambda K_i = 2(\Lambda K_i)^{\pi/2}\}$ is uniformly bounded. Thus, every a priori chosen subsequence of $\{\Lambda K_i\}$ by Blaschke's selection theorem has a subsequence, denoted by $\{i_k\}_k$, such that $\{\Lambda K_{i_k}\}$ converges to a body $L \in \mathcal{K}_e^2$. Thus, $\lim_{k\to\infty} S_{\Lambda K_{i_k}} = S_L$, weakly. Putting these two observations together, we infer

$$\int_{\mathbb{S}^1} f dS_L = \lim_{k\to\infty} \int_{\mathbb{S}^1} f dS_{\Lambda K_{i_k}} = \int_{\mathbb{S}^1} f dS_{\Lambda K}, \tag{4.4}$$

for every continuous function f on $\mathbb{S}^1$. In particular, (4.4) implies that

$$V(L, Q) = V(\Lambda K, Q), \text{ for every } Q \in \mathcal{K}^2.$$

Therefore, $V^2(\Lambda K, L) = V^2(L) = V^2(\Lambda K) = V(L)V(\Lambda K)$. Thus, by the equality case in Minkowski's mixed volume inequality, $L = \Lambda K$. In particular, the limit L is independent of the subsequence we have chosen a priori. The proof is now complete. □

We now have all the necessary ingredients to prove the weak convergence (equivalently, convergence in the Hausdorff metric) of a subsequence of the normalized solution.

Proof. By Proposition 4.5, there exists a sequence of times, $\{t_k\}_k$, such that $t_k \to T$ and

$$\lim_{t_k\to T}\left(\frac{1}{V(K_{t_k})V(K^*_{t_k})}\left(\frac{V(\Lambda K^*_{t_k})}{V(K^*_{t_k})}-1\right)=:\chi(t_k)\right)=0. \tag{4.5}$$

Observe that $\chi(t_k)$, $V(\Lambda K^*_{t_k})/V(K^*_{t_k})$, and $V(K_{t_k})V(K^*_{t_k})$ are all invariant under $GL(2)$. Define $\bar{K}_t := (V(B)/V(K^*_t))^{1/2}\, K^*_t$ and, for each t, let $\Psi_t \in SL(2)$ be a linear transformation that minimizes the length of $\partial\bar{K}_t$. By John's ellipsoid theorem, the sequence $\{\Psi_{t_k}\bar{K}_{t_k}\}$ is uniformly bounded. Thus, by Blaschke's selection theorem and by passing to a subsequence of times denoted again by $\{t_k\}_k$, we deduce that there exists a body $\bar{K}_\infty \in \mathcal{K}^2_e$ such that $\lim_{t_k\to T}\Psi_{t_k}\bar{K}_{t_k} = \bar{K}_\infty$. In light of Lemma 4.6, we have $\lim_{t_k\to T}\Lambda\Psi_{t_k}\bar{K}_{t_k} = \Lambda\bar{K}_\infty$. Thus, from (4.5), we get

$$\lim_{t_k\to T}\frac{V(\Lambda\Psi_{t_k}\bar{K}_{t_k})}{V(\Psi_{t_k}\bar{K}_{t_k})} = \frac{V(\Lambda\bar{K}_\infty)}{V(\bar{K}_\infty)} = 1.$$

As previously mentioned, $V(\Lambda\bar{K}_\infty) = V(\bar{K}_\infty)$ implies that $\Lambda\bar{K}_\infty = \bar{K}_\infty$. Now, Lemma 8.1 of Petty [62] yields that $\bar{K}_\infty$ must be an origin-centered ellipse. Since the length of $\Psi_{t_k}\bar{K}_{t_k}$ is minimized and $V(\Psi_{t_k}\bar{K}_{t_k}) = \pi$, $\bar{K}_\infty$ must be the unit disk. Consequently, $\lim_{t_k\to T}\Psi_{t_k}\bar{K}_{t_k} = B$ and

$$\lim_{t_k\to T}(\Psi_{t_k}\bar{K}_{t_k})^* = B,\ \lim_{t_k\to T}V((\Psi_{t_k}\bar{K}_{t_k})^*) = \pi. \tag{4.6}$$

Let Φ_{t_k} be the inverse of the transpose of Ψ_{t_k}, for each t_k. Thus, we get

$$(\Psi_{t_k}\bar{K}_{t_k})^* = \left(\frac{V(B)}{V(K^*_{t_k})}\right)^{-\frac{1}{2}}\Phi_{t_k}K_{t_k} = \left(\frac{V^2(B)}{V(K_{t_k})V(K^*_{t_k})}\right)^{-\frac{1}{2}}\left(\frac{V(B)}{V(K_{t_k})}\right)^{\frac{1}{2}}\Phi_{t_k}K_{t_k}.$$

Combining (4.6) with $V\left(\left(\frac{V(B)}{V(K_{t_k})}\right)^{\frac{1}{2}}\Phi_{t_k}K_{t_k}\right) = \pi$ implies

$$\lim_{t_k\to T}\left(\frac{V^2(B)}{V(K_{t_k})V(K^*_{t_k})}\right)^{-\frac{1}{2}} = 1,$$

and, in turn,

$$\lim_{t_k\to T}\left(\frac{V(B)}{V(K_{t_k})}\right)^{\frac{1}{2}}\Phi_{t_k}K_{t_k} = B.$$

□

5. Stability of the planar Busemann-Petty centroid inequality

We will state several lemmas from [6, 38, 40, 42] to prepare the proof of Theorem B. The first lemma rewrites the identity (4.2).

Lemma 5.1. *Along the flow* (1.5)*, we have*

$$\frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)} = \frac{32V(\Lambda K^*_t)}{3V^2(K_t)V(K^*_t)}\left(1-\frac{V^2(\Lambda K^*_t, K^*_t)}{V(\Lambda K^*_t)V(K^*_t)}\right). \tag{5.1}$$

Lemma 5.2. *Under the evolution equation* (1.5)*, we have*

$$h_{K_t}(u) \le h_{K_0}(u) \le h_{K_t}(u)\left(1 + 2t\left(\frac{\mathcal{G}}{h^3}\right)(u,t)\right).$$

In particular,

$$d_{\mathcal{BM}}(K_0, K_t) \le \left(1 + 2t \max_{u\in\mathbb{S}^1}\left(\frac{\mathcal{G}}{h^3}\right)(u,t)\right).$$

To prove Lemma 5.2, one may first obtain a Harnack estimate using the method of [3], from which the right-hand side follows. The left-hand side holds trivially, as the flow is a shrinking flow. The details are given by the author for the p-centro-affine normal flows in [42, 43] (see also Lemmas 6.1 and 6.2).

Lemma 5.3. *For any smooth, strictly convex solution* $\{K_t\}$ *of the evolution equation* (1.5) *with* $0 < R_- \le h_{K_t} \le R_+ < \infty$*, for* $t \in [0, \delta]$*, and some positive numbers* $R_\pm$*, we have*

$$\mathcal{G}_{K_t} \le C_0 + C_1 t^{-1/2},$$

where C_0 *and* C_1 *are constants depending on* R_- *and* R_+*.*

Proof. We apply Tso's trick [78]. Consider the function

$$\Omega = \frac{h^{-2}\mathcal{S}^{-1}}{h - R_-/2}.$$

Using the maximum principle, we will show that Ω is bounded from above by a function of R_-, R_+, and time. At the point where the maximum of Ω occurs, we have

$$0 = \Omega_\theta = \left(\frac{h^{-2}\mathcal{S}^{-1}}{h - R_-/2}\right)_\theta \quad \text{and} \quad \Omega_{\theta\theta} \le 0.$$

Hence, we obtain

$$\frac{(h^{-2}\mathcal{S}^{-1})_\theta}{h - R_-/2} = \frac{(h^{-2}\mathcal{S}^{-1})h_\theta}{(h - R_-/2)^2}$$

and consequently,

$$\left(h^{-2}\mathcal{S}^{-1}\right)_{\theta\theta} + \left(h^{-2}\mathcal{S}^{-1}\right) \le \frac{h^{-2} - (R_-/2)h^{-2}\mathcal{S}^{-1}}{h - R_-/2}. \tag{5.2}$$

We calculate

$$\begin{aligned}\partial_t\Omega = {} & \frac{h^{-2}\mathcal{S}^{-2}}{h - R_-/2}\left[\left(h^{-2}\mathcal{S}^{-1}\right)_{\theta\theta} + \left(h^{-2}\mathcal{S}^{-1}\right)\right] \\ & + \frac{\mathcal{S}^{-1}}{h - R_-/2}\partial_t h^{-2} + \frac{h^{-4}\mathcal{S}^{-2}}{(h - R_-/2)^2}.\end{aligned}$$

Note that

$$\frac{\mathcal{S}^{-1}}{h - R_-/2}\partial_t h^{-2} = 2\Omega^2 - 2\frac{R_-}{2}\frac{h^{-5}\mathcal{S}^{-2}}{(h - R_-/2)^2} \le 2\Omega^2. \tag{5.3}$$

Thus, using inequalities (5.2) and (5.3), at the point where the maximum of Ω is reached, we have

$$\partial_t\Omega \le \Omega^2\left(4 - \frac{R_-}{2}\mathcal{S}^{-1}\right). \tag{5.4}$$

We can control $\mathcal{G}$ from below by a positive power of Ω:

$$\mathcal{S}^{-1} = \left(\frac{h - R_-/2}{h^{-2}\mathcal{S}^{-1}}\right)^{-1} \left(\frac{h^{-2}}{h - R_-/2}\right)^{-1} \geq \Omega \left(\frac{R_-^{-2}}{R_- - R_-/2}\right)^{-1}.$$

Therefore, we can rewrite the inequality (5.4) as

$$\partial_t \Omega \leq -\Omega^2 \left(\frac{R_-^4}{4}\Omega - 4\right).$$

Hence,

$$\Omega \leq C(R_-, R_+) t^{-\frac{1}{2}} + C'(R_-, R_+)$$

for some positive constants C and C' depending on R_- and R_+. The corresponding claim for $\mathcal{G}$ follows. □

Corollary 5.4. *For any solution $\{K_t\} \subset \mathcal{F}_e^2$ to (1.5) with $0 < R_- \leq h_{K_t} \leq R_+ < \infty$, for $t \in [0, \delta]$, and for some positive numbers $R_\pm$, we have*

$$h_{K_t}(u) \leq h_{K_0}(u) \leq h_{K_t}(u) \left(1 + \frac{2C_0}{R_-^3} t + \frac{2C_1}{R_-^3} t^{1/2}\right),$$

where C_0 and C_1 are constants depending on R_- and R_+. In particular, we have

$$d_{\mathcal{BM}}(K_0, K_t) \leq \left(1 + \frac{2C_0}{R_-^3} t + \frac{2C_1}{R_-^3} t^{1/2}\right).$$

Proof. The claim immediately follows from Lemmas 5.2 and 5.3. □

Lemma 5.5. *For any solution $\{K_t\} \subset \mathcal{F}_e^2$ to (1.5) with $0 < R_- \leq h_{K_t} \leq R_+ < \infty$, for $t \in [0, \delta]$, and for some positive numbers $R_\pm$, we have*

$$C_2 \leq h_{\Lambda K_t^*} \ \& \ h_{K_t^*} \leq C_3,$$

where C_2 and C_3 are constants depending on R_- and R_+.

Proof. The claim for K_t^* is trivial. To prove the claim for ΛK_t^*, first note that $(\Lambda K_t^*)^{\pi/2} = \frac{1}{2}\Pi\Lambda K_t^* = \frac{3}{4} V(K_t^*)\Gamma K_t$. Moreover, recall that $V(K^*) = \frac{1}{2}\int_{\mathbb{S}^1} h_K^{-2} d\theta$. Thus, uniform lower and upper bounds on the support functions of ΛK_t^* follow from the definition of $\Gamma(\cdot)$. □

Lemma 5.6 ([38])**.** *Suppose that $K \in \mathcal{F}_e^2$. If $m \leq \frac{h}{\mathcal{G}^{1/3}} \leq M$ for some positive numbers m and M, then there exist two origin-symmetric ellipses E_{in} and E_{out} such that $E_{in} \subseteq K \subseteq E_{out}$ and*

$$\left(\frac{V(E_{in})}{\pi}\right)^{2/3} = m, \ \left(\frac{V(E_{out})}{\pi}\right)^{2/3} = M.$$

Moreover, we have

$$d_{\mathcal{BM}}(K, B) \leq \left(\frac{M}{m}\right)^{\frac{3}{2}}.$$

Proof. A proof of the first part of the claim is given in [38]. To prove the second part of the claim, we may first apply a special linear transformation $\Phi \in SL(2)$ such that ΦE_{out} is a disk. Then, it is easy to see that $\Phi E_{out} \subseteq \frac{V(E_{out})}{V(E_{in})} \Phi E_{in}$. Therefore,

$$\Phi E_{in} \subseteq \Phi K \subseteq \frac{V(E_{out})}{V(E_{in})} \Phi E_{out}$$

and

$$d_{\mathcal{BM}}(K,B) \leq \frac{V(E_{out})}{V(E_{in})}.$$

□

A simple consequence of Lemma 5.6 is contained in the following corollary.

Corollary 5.7 ([6, 40])**.** *Let $K \in \mathcal{F}_e^2$ be of area π. Then,*

$$\min_{\mathbb{S}^1} \frac{h}{\mathcal{G}^{1/3}} \leq 1 \leq \max_{\mathbb{S}^1} \frac{h}{\mathcal{G}^{1/3}}.$$

For another proof of Corollary 5.7, see Andrews [6, Lemma 10], where he does not assume that K is origin-symmetric.

5.1. **Proof of Theorem B.** In this section, $C_4, C_5, \cdots$ are absolute positive constants. Moreover, we will repeatedly use Lemma 5.5 without further mention.

Let $K \in \mathcal{F}_e^2$ be a body such that

$$\frac{V(\Gamma K)}{V(K)} \leq \left(\frac{4}{3\pi}\right)^2 (1+\varepsilon),\ 0 < \varepsilon < \varepsilon_0.$$

The value of ε_0 will be determined later.

Let $\Phi \in GL(2)$ such that $1 \leq h_{\Phi K} \leq \sqrt{2}$, and let $\{K_t\}$ be the solution to the flow (1.5) with $K_0 = \Phi K$. It follows from the comparison principle that there exists $\delta > 0$, independent of K_0, such that $1/2 \leq h_{K_t} \leq \sqrt{2}$, for all $t \in [0,\delta]$. From Lemma 5.1, we have

$$\int_0^{f(\varepsilon)} \frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)}dt = \int_0^{f(\varepsilon)} \frac{32V(\Lambda K_t^*)}{3V^2(K_t)V(K_t^*)}\left(1 - \frac{V^2(\Lambda K_t^*, K_t^*)}{V(\Lambda K_t^*)V(K_t^*)}\right)dt,$$

where $f : [0,1] \to [0,\delta]$ is an increasing, continuous function such that $\lim_{\varepsilon\to 0} \varepsilon/f(\varepsilon) = 0$ and $\lim_{\varepsilon\to 0} f(\varepsilon) = 0$. Let $s \in [0, f(\varepsilon)]$ be the time that the integrand of the right-hand side is maximized. Therefore, we get

$$\int_0^{f(\varepsilon)} \frac{d}{dt}\frac{V(\Gamma K_t)}{V(K_t)}dt \leq f(\varepsilon)\left[\frac{32V(\Lambda K_s^*)}{3V^2(K_s)V(K_s^*)}\left(1 - \frac{V^2(\Lambda K_s^*, K_s^*)}{V(\Lambda K_s^*)V(K_s^*)}\right)\right]. \tag{5.5}$$

By the Busemann-Petty centroid inequality and the stability of Minkowski's mixed volume inequality, we get

$$\begin{aligned} -\left(\frac{4}{3\pi}\right)^2 \varepsilon &\leq -\frac{8}{3}\frac{f(\varepsilon)V(\Lambda K_s^*)}{V^2(K_s)D^2(K_s^*)} \max_{u\in\mathbb{S}^1}\left|\frac{h_{\Lambda K_s^*}(u)}{V(\Lambda K_s^*)^{1/2}} - \frac{h_{K_s^*}(u)}{V(K_s^*)^{1/2}}\right|^2 \\ &\leq -C_4 f(\varepsilon)\max_{u\in\mathbb{S}^1}\left|\frac{h_{\Lambda K_s^*}(u)}{h_{K_s^*}(u)} - \left(\frac{V(\Lambda K_s^*)}{V(K_s^*)}\right)^{1/2}\right|^2. \end{aligned} \tag{5.6}$$

By the definition of the operator Λ:

$$\frac{1}{\sqrt{2}} \leq \left(\mathcal{G}_{\Lambda K_s^*}\frac{V(K_s^*)}{V(K_s)}\right)^{\frac{1}{3}} = h_{K_s^*} \leq 2.$$

Combining this with the inequality (5.6), we conclude that

$$C_5\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2} \geq \max_{\mathbb{S}^1}\frac{h_{\Lambda K_s^*}}{\mathcal{G}_{\Lambda K_s^*}^{\frac{1}{3}}} - \min_{\mathbb{S}^1}\frac{h_{\Lambda K_s^*}}{\mathcal{G}_{\Lambda K_s^*}^{\frac{1}{3}}}.$$

As $V\left(\sqrt{\frac{\pi}{V(\Lambda K_s^*)}}\Lambda K_s^*\right)=\pi$, by Corollary 5.7, we get

$$C_5\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}\left(\frac{\pi}{V(\Lambda K_s^*)}\right)^{2/3}+1\geq\left(\frac{\pi}{V(\Lambda K_s^*)}\right)^{2/3}\max_{\mathbb{S}^1}\frac{h_{\Lambda K_s^*}}{\mathcal{G}_{\Lambda K_s^*}^{\frac{1}{3}}},$$

$$1-C_5\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}\left(\frac{\pi}{V(\Lambda K_s^*)}\right)^{2/3}\leq\left(\frac{\pi}{V(\Lambda K_s^*)}\right)^{2/3}\min_{\mathbb{S}^1}\frac{h_{\Lambda K_s^*}}{\mathcal{G}_{\Lambda K_s^*}^{\frac{1}{3}}}.$$

We take ε_0 small enough such that

$$1-C_5\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}\left(\frac{\pi}{V(\Lambda K_s^*)}\right)^{2/3}\geq 1-C_6\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}>0.$$

Thus, we have proved that, if $\varepsilon_0>0$ is small enough, then

$$\max_{\mathbb{S}^1}\frac{h_{\Lambda K_s^*}}{\mathcal{G}_{\Lambda K_s^*}^{\frac{1}{3}}}\leq\left(1+C_6\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}\right)\left(\frac{\pi}{V(\Lambda K_s^*)}\right)^{-2/3}$$

and

$$\min_{\mathbb{S}^1}\frac{h_{\Lambda K_s^*}}{\mathcal{G}_{\Lambda K_s^*}^{\frac{1}{3}}}\geq\left(1-C_6\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}\right)\left(\frac{\pi}{V(\Lambda K_s^*)}\right)^{-2/3}.$$

From these last inequalities and Lemma 5.6, we deduce that

$$d_{\mathcal{BM}}(\Lambda K_s^*,B)\leq\left(\frac{1+C_6\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}{1-C_6\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}\right)^{3/2}. \tag{5.7}$$

On the other hand, from (5.5) and the Busemann-Petty centroid inequality, we get

$$\begin{aligned}-\left(\frac{4}{3\pi}\right)^2\varepsilon&\leq f(\varepsilon)\left[\frac{32V(\Lambda K_s^*)}{3V^2(K_s)V(K_s^*)}\left(1-\frac{V^2(\Lambda K_s^*,K_s^*)}{V(\Lambda K_s^*)V(K_s^*)}\right)\right]\\&=f(\varepsilon)\left[\frac{32V(\Lambda K_s^*)}{3V^2(K_s)V(K_s^*)}\left(1-\frac{V(K_s^*)}{V(\Lambda K_s^*)}\right)\right].\end{aligned}$$

Thus,

$$1\leq\frac{V(K_s^*)}{V(\Lambda K_s^*)}\leq 1+C_7\frac{\varepsilon}{f(\varepsilon)}. \tag{5.8}$$

By the first line of inequality (5.6),

$$\begin{aligned}1-C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}&\leq\left(\frac{V(K_s^*)}{V(\Lambda K_s^*)}\right)^{1/2}-C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}\\&\leq\frac{h_{K_s^*}}{h_{\Lambda K_s^*}}\leq\left(\frac{V(K_s^*)}{V(\Lambda K_s^*)}\right)^{1/2}+C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}.\end{aligned}$$

We take ε_0 small enough such that $1 - C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2} > 0$. Thus, by inequalities (5.8), we obtain

$$d_{\mathcal{BM}}(K_s^*, \Lambda K_s^*) \leq \left(\frac{\left(1 + C_7\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2} + C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}{1 - C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}\right). \tag{5.9}$$

Combining (5.7) and (5.9), we get

$$d_{\mathcal{BM}}(K_s^*, B) \leq g(\varepsilon),$$

where

$$g(\varepsilon) := \left(\frac{1 + C_6\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}{1 - C_6\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}\right)^{3/2}\left(\frac{\left(1 + C_7\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2} + C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}{1 - C_8\left(\frac{\varepsilon}{f(\varepsilon)}\right)^{1/2}}\right).$$

This in turn implies that

$$d_{\mathcal{BM}}(K_s, B) \leq g(\varepsilon). \tag{5.10}$$

By Corollary 5.4, we have

$$d_{\mathcal{BM}}(K_0, K_s) \leq 1 + C_9 f(\varepsilon)^{\frac{1}{2}} + C_{10} f(\varepsilon). \tag{5.11}$$

Consequently, putting (5.10) and (5.11) together, we obtain

$$d_{\mathcal{BM}}(K, B) = d_{\mathcal{BM}}(K_0, B) \leq \left(1 + C_9 f(\varepsilon)^{\frac{1}{2}} + C_{10} f(\varepsilon)\right) g(\varepsilon).$$

In particular, with the choice $f(\varepsilon) = \varepsilon^{1/2}$, we get

$$d_{\mathcal{BM}}(K, B) \leq 1 + \gamma\varepsilon^{1/4},$$

for small enough ε_0 and $\gamma > 0$. Therefore, we have proved the claim for bodies in $\mathcal{F}_e^2$. An approximation argument will then upgrade the result for bodies in $\mathcal{F}_e^2$ to bodies in the larger class $\mathcal{K}_e^2$. To get the more general result, we will first need to recall Theorem 1.4 of Böröczky in [19] and a theorem of Campi and Gronchi in [24]:

Theorem (Böröczky, [19])**.** *For any convex body K in $\mathbb{R}^n$ with $d_{\mathcal{BM}}(K, B) \geq 1 + \varepsilon$ for some $\varepsilon > 0$, there exist an origin-symmetric convex body C and a constant $\gamma' > 0$ depending on n such that $d_{\mathcal{BM}}(C, B) \geq 1 + \gamma'\varepsilon^2$ and C results from K as a limit of subsequent Steiner symmetrization and linear transformations.*

Remark 5.8. In the statement of [19, Theorem 1.4], it is mentioned that, for an arbitrary convex body K, C results from K as a limit of subsequent Steiner symmetrization and affine transformations. However, no translation is needed. See Lemma 18 of [20].

By means of a shadow system, Campi and Gronchi proved the following theorem.

Theorem (Campi and Gronchi, [24])**.** *Let $K \in \mathcal{K}^n$. The ratio $\frac{V(\Gamma K)}{V(K)}$ is non-increasing after a Steiner symmetrization applied to K.*

Now, we give the proof in the general case. We prove by contraposition. Let $K \in \mathcal{K}^2$ be a convex body such that

$$d_{\mathcal{BM}}(K, B) > 1 + \left(\frac{\gamma}{\gamma'}\right)^{\frac{1}{2}} \varepsilon^{\frac{1}{8}}.$$

Then for an origin-symmetric convex body C, by the last two theorems, we have $\frac{V(\Gamma C)}{V(C)} \leq \frac{V(\Gamma K)}{V(K)}$ and $d_{\mathcal{BM}}(C,B) > 1+\gamma\varepsilon^{\frac{1}{4}}$. Therefore,

$$\frac{V(\Gamma K)}{V(K)} \geq \frac{V(\Gamma C)}{V(C)} > \left(\frac{4}{3\pi}\right)^2 (1+\varepsilon).$$

The proof of Theorem B is complete.

6. Higher order regularity

In section 4.1, we proved for a sequence of times $\{t_k\}_k$ that $\lim\limits_{t_k\to T} \frac{V(\Gamma K_{t_k})}{V(K_{t_k})} = \left(\frac{4}{3\pi}\right)^2$. Thus, by the monotonicity of $\frac{V(\Gamma K_t)}{V(K_t)}$, Corollary 4.4, we obtain $\lim\limits_{t\to T} \frac{V(\Gamma K_t)}{V(K_t)} = \left(\frac{4}{3\pi}\right)^2$. Hence, Theorem B implies that, for each time $t \in [0,T)$, there exists a linear transformation $\Phi_t \in SL(2)$ such that

$$\lim_{t\to T} \frac{r_+(\Phi_t K_t)}{r_-(\Phi_t K_t)} = 1. \tag{6.1}$$

To obtain higher order regularity, we closely follow [42] with minor modifications. To this aim, (6.1) plays a basic role.

We employ the technique of Andrews and McCoy in [9] and adapt it to the context of the centro-affine normal flow (1.5). An issue arises in trying to directly apply their method: Flow (1.5) is not invariant under Euclidean translations. So [9, Lemma 12.2], which is originally due to Smoczyk in the context of mean curvature flow [70], is not easily accessible. It was shown in [42] that such a lemma can also be obtained from a Harnack estimate. The proof of the next lemma is omitted, because of its similarity to the proof of Lemma 4.2 in [42] (see also [43, Section 2]); the proof employs the method of Andrews introduced in [3].

Lemma 6.1 (Harnack estimate)**.** *Under the evolution equation* (1.5)*, we have*

$$\partial_t \left(\frac{\mathcal{G}}{h^2} t^{\frac{1}{2}}\right) \geq 0, \ \textit{e.q.}, \ \partial_t \left(\frac{\mathcal{G}}{h^2}\right) \geq -\frac{1}{2t}\left(\frac{\mathcal{G}}{h^2}\right).$$

Lemma 6.2. *For each fixed $u \in \mathbb{S}^1$, define*

$$Q(t) := \frac{1}{2}(h_{K_t}(u) - h_{K_{t_0}}(u)) + (t-t_0)\left(\frac{\mathcal{G}}{h^2}\right)(u, K_t)$$

on the time interval $[t_0, T)$. Then, $Q(t) \geq 0$ on $[t_0, T)$.

Proof. In light of Lemma 6.1, after a time translation, we get

$$\partial_t \left(\frac{\mathcal{G}}{h^2}\right) \geq -\frac{1}{2(t-t_0)}\left(\frac{\mathcal{G}}{h^2}\right), \ \text{for all } t > t_0.$$

Thus, the time derivative of Q, given by $\partial_t Q = \frac{1}{2}\frac{\mathcal{G}}{h^2} + (t-t_0)\frac{\partial}{\partial t}\left(\frac{\mathcal{G}}{h^2}\right)$, is non-negative, and moreover $Q(t_0) = 0$. □

Next is an adjustment of the argument of Andrews and McCoy presented in section 12 of [9]. We will obtain a lower bound on $\mathcal{G}/h^3$, the centro-affine curvature. The lower bound of the next lemma then conveniently provides a uniform lower bound for the Gauss curvature of the normalized solution. In what follows, a key

property of $\mathcal{G}/h^3$ will repeatedly be used: for every $\Phi \in SL(2)$ and $K \in \mathcal{F}_0^2$, we have

$$\min_{u\in\mathbb{S}^1} \frac{\mathcal{G}}{h^3}(u,K) = \min_{u\in\mathbb{S}^1} \frac{\mathcal{G}}{h^3}(u,\Phi K) \text{ and } \max_{u\in\mathbb{S}^1} \frac{\mathcal{G}}{h^3}(u,K) = \max_{u\in\mathbb{S}^1} \frac{\mathcal{G}}{h^3}(u,\Phi K).$$

Lemma 6.3. *There exist an absolute constant $C > 0$ and a time $t_* < T$, such that for each $t \geq t_*$ we have*

$$\frac{\mathcal{G}}{h^3}(\cdot,t) := \frac{\mathcal{G}}{h^3}(\cdot,K_t) \geq \frac{C}{T-t}.$$

Proof. After a time shift, we may assume without loss of generality that, by (6.1), we have for a fixed $1 \leq \eta < 1.5^{\frac{1}{4}}$

$$\frac{r_+(\Phi_\tau K_\tau)}{r_-(\Phi_\tau K_\tau)} \leq \eta$$

for all $\tau \geq 0$. Fix a $\tau \geq 0$. Let $B_{r(t)}$ and $B_{R(t)}$ be solutions to the flow (1.5) starting with $B_{r_-(\Phi_\tau K_\tau)}$ and $B_{\eta r_-(\Phi_\tau K_\tau)}$ respectively. Radii $R(t)$ and $r(t)$ are explicitly given by

$$R(t) = \left[\eta r_-(\Phi_\tau K_\tau)^4 - 4(t-\tau)\right]^{\frac{1}{4}}, \tag{6.2}$$

$$r(t) = \left[r_-(\Phi_\tau K_\tau)^4 - 4(t-\tau)\right]^{\frac{1}{4}},$$

and by the comparison principle

$$B_{r(t)} \subseteq \Phi_\tau K_t \subseteq B_{R(t)}$$

for all $\tau \leq t \leq \tau + \frac{(r_-(\Phi_\tau K_\tau))^4}{4}$. Hence, we must have $T \geq \tau + \frac{(r_-(\Phi_\tau K_\tau))^4}{4}$. Take $\tau^* := \tau + \frac{(r_-(\Phi_\tau K_\tau))^4}{8}$ and an arbitrary $u \in \mathbb{S}^1$. By Lemma 6.2 and equation (6.2), we obtain:

$$\begin{aligned}
2\left[\eta^4 - 0.5\right]^{\frac{1}{4}} r_-(\Phi_\tau K_\tau)\frac{\mathcal{G}}{h^3}(u,\Phi_\tau K_{\tau^*}) &= 2R(\tau^*)\frac{\mathcal{G}}{h^3}(u,\Phi_\tau K_{\tau^*}) \\
&\geq \frac{2\mathcal{G}}{h^2}(u,\Phi_\tau K_{\tau^*}) \\
&\geq \frac{h_{\Phi_\tau K_\tau}(u) - h_{\Phi_\tau K_{\tau^*}}(u)}{\tau^* - \tau} \\
&\geq \frac{8(r_-(\Phi_\tau K_\tau) - R(\tau^*))}{(r_-(\Phi_\tau K_\tau))^4} \\
&= \frac{8\left(1 - \left[\eta^4 - 0.5\right]^{\frac{1}{4}}\right)}{(r_-(\Phi_\tau K_\tau))^3}.
\end{aligned}$$

Thus, we get

$$\frac{\mathcal{G}}{h^3}(u,\Phi_\tau K_{\tau^*}) \geq \frac{8C}{(r_-(\Phi_\tau K_\tau))^4},$$

for all $u \in \mathbb{S}^1$ and some positive absolute constant C. Recall that

$$T \geq \tau + \frac{(r_-(\Phi_\tau K_\tau))^4}{4} = \tau^* + \frac{(r_-(\Phi_\tau K_\tau))^4}{8}$$

and $\mathcal{G}/h^3$ is invariant under $SL(2)$. Therefore, we conclude that

$$\frac{\mathcal{G}}{h^3}(u,\tau^*) = \frac{\mathcal{G}}{h^3}(u,K_{\tau^*}) \geq \frac{8C}{(r_-(\Phi_\tau K_\tau))^4} \geq \frac{C}{T-\tau^*}.$$

Defining function f on the time interval $[t_*,T)$ by $f(\tau)=\tau+\frac{(r_-(\Phi_\tau K_\tau))^4}{8}-t$, and using Proposition 3.5 (which says $\lim\limits_{t\to T} V(\Phi_t K_t) = \lim\limits_{t\to T} V(K_t) = 0$), it is not difficult to see that each $t \geq t_* := \frac{(r_-(\Phi_0(K_0)))^4}{8}$ can be written as $t=\tau+\frac{(r_-(\Phi_\tau K_\tau))^4}{8}$ for a $\tau \geq 0$. □

Remark 6.4. The comparison principle implies that, for each $t \in [0,T)$,

$$\frac{(r_-(\Phi_t K_t))^4}{4} \leq T-t \leq \frac{(r_+(\Phi_t K_t))^4}{4} \leq \frac{(\eta r_-(\Phi_t K_t))^4}{4}. \tag{6.3}$$

Lemma 6.5. *There exists an absolute constant $C'>0$ such that, on the time interval $T/2 \leq t < T$, we have*

$$\left(\frac{\mathcal{G}}{h^3}\right)(\cdot,K_t) \leq \frac{C'}{T-t}.$$

Proof. For each fixed $t^* \in [T/2,T)$, the family of convex bodies defined by $\tilde{K}_t^{t^*} := \frac{1}{(T-t^*)^{\frac{1}{4}}}\Phi_{2t^*-T}K_{t^*+(T-t^*)t}$ is a solution of (1.5) on the time interval $[-1,0]$, with the initial data $\tilde{K}_{-1}^{t^*} = \frac{1}{(T-t^*)^{\frac{1}{4}}}\Phi_{2t^*-T}K_{2t^*-T}$. By inequalities (6.3), at time $t=-1$, we have

$$r_-(\tilde{K}_{-1}^{t^*}) = \frac{r_-(\Phi_{2t^*-T}K_{2t^*-T})}{(T-t^*)^{\frac{1}{4}}} \geq \frac{8^{\frac{1}{4}}}{\eta}$$

and

$$r_+(\tilde{K}_{-1}^{t^*}) = \frac{r_+(\Phi_{2t^*-T}K_{2t^*-T})}{(T-t^*)^{\frac{1}{4}}} \leq \eta 8^{\frac{1}{4}}.$$

From the assumption $1 \leq \eta < 1.5^{\frac{1}{4}}$, and again by the comparison principle, for each time $t \in [-1,0]$, we get

$$r_-(\tilde{K}_t^{t^*}) \geq \left(4\left(\frac{2}{\eta^4}-1\right)\right)^{\frac{1}{4}} \geq \left(\frac{4}{3}\right)^{\frac{1}{4}}$$

and

$$r_+(\tilde{K}_t^{t^*}) \leq \eta 8^{\frac{1}{4}} < 12^{\frac{1}{4}}.$$

These last two inequalities combined with Lemma 5.3 imply that the centro-affine curvature of $\frac{1}{(T-t^*)^{1/4}}\Phi_{2t^*-T}K_{t^*} = \tilde{K}_0^{t^*}$ is bounded by a positive constant C' independent of t^*. Therefore,

$$\frac{\mathcal{G}}{h^3}(\cdot,\Phi_{2t^*-T}K_{t^*}) \leq \frac{C'}{T-t^*} \Rightarrow \frac{\mathcal{G}}{h^3}(\cdot,K_{t^*}) \leq \frac{C'}{T-t^*}.$$

Taking into account that $t^* \in [T/2,T)$ is arbitrary and C' is independent of t^* completes the proof. □

6.1. **Proof of Theorem A: $\mathcal{C}^\infty$ convergence of the normalized solution.** For a fixed $t^* \in \left[\max\{3T/4, \frac{T+t_*}{2}\}, T\right)$, the family of convex bodies given by $\tilde{K}_t^{t^*} = \frac{1}{(T-t^*)^{\frac{1}{4}}}\Phi_{2t^*-T}K_{t^*+(T-t^*)t}$ is a solution of (1.5) on the time interval $[-1,0]$ with the initial data $\tilde{K}_{-1}^{t^*} = \frac{1}{(T-t^*)^{\frac{1}{4}}}\Phi_{2t^*-T}K_{2t^*-T}$ and with the properties

$$r_-(\tilde{K}_t^{t^*}) \geq \left(\frac{4}{3}\right)^{\frac{1}{4}}$$

and

$$r_+(\tilde{K}_t^{t^*}) < (12)^{\frac{1}{4}}.$$

Since $2t^* - T \geq \max\{T/2, t_*\}$, by Lemmas 6.3 and 6.5 we get

$$\frac{C}{2(T-t^*)} \leq \frac{\mathcal{G}}{h^3}(\cdot, 2t^*-T) \leq \frac{C'}{2(T-t^*)}.$$

Thus, the centro-affine curvature of $\tilde{K}_{-1}^{t^*}$ satisfies

$$\frac{C}{2} \leq \frac{\mathcal{G}}{h^3}(\cdot, \tilde{K}_{-1}^{t^*}) \leq \frac{C'}{2}.$$

To prove Theorem A, we mention two basic observations contained in the proofs of Corollary 3.3 and Lemma 3.4:

(1) $\min_{\mathbb{S}^1} \mathcal{G}/h^2(\cdot, t)$ is non-decreasing along the flow.

(2) $\max_{\mathbb{S}^1} \mathcal{G}/h^2(\cdot, t)$ remains bounded from above provided that the inner radius has a lower bound. This follows from Tso's trick, which is applying the maximum principle to the auxiliary function $\Omega(u,t) := \dfrac{\frac{1}{h^2\mathcal{S}}(u, \tilde{K}_t^{t^*})}{h(u, \tilde{K}_t^{t^*}) - \frac{1}{2}\left(\frac{4}{3}\right)^{\frac{1}{4}}}$. Furthermore, the upper bound on the speed depends only on $\max_{\mathbb{S}^1} \Phi(\cdot, 0)$, the outer radius of the initial convex body, and the lower bound on the inner radii of the evolving convex bodies.

Using observations (1) and (2), we conclude, for $t \in [-1,0]$, that

$$C_1 \leq \frac{\mathcal{G}}{h^2}(\cdot, \tilde{K}_t^{t^*}) \leq C_2,$$

for constants C_1 and C_2 independent of t^*. These constants are independent of t^* as they depend only on C and C'. Hence, each body $\tilde{K}_t^{t^*}$, for $t \in [-1,0]$, satisfies

$$C_3 \leq \mathcal{G}(\cdot, \tilde{K}_t^{t^*}) \leq C_4,$$

for some constants C_3 and C_4 independent of t^*. Therefore, by [46], we conclude that there are uniform bounds on higher derivatives of the curvature of $\tilde{K}_t^{t^*}$, for $t \in [-1/2, 0]$. In particular, the body

$$\tilde{K}_0^{t^*} = \frac{1}{(T-t^*)^{\frac{1}{4}}}\Phi_{2t^*-T}K_{t^*}$$

has uniform $\mathcal{C}^k$ bounds independent of t^*.

Consequently, for every given sequence of times $\{t_k\}_k$, we can find a subsequence, denoted again by $\{t_k\}_k$, such that as $t_k \to T$, the family $\left\{\frac{1}{(T-t_k)^{\frac{1}{4}}}\Phi_{2t_k-T}K_{t_k}\right\}$ approaches in $\mathcal{C}^\infty$ to a convex body $K_\infty \in \mathcal{F}_e^2$. Furthermore, we may use the monotonicity of $V(\Gamma K_t)/V(K_t)$ and the discussion in section 4.1 to conclude that

$V(\Gamma K_\infty)/V(K_\infty) = (4/3\pi)^2$. That is, K_∞ is a smooth minimizer of $V(\Gamma K)/V(K)$. Hence, by Corollary 4.4 or Theorem B, K_∞ is an origin-centered ellipse. Finally, note that $V(K_t)$ is comparable with $\frac{1}{(T-t)^{1/2}}$ which results in the convergence of $\sqrt{\frac{V(B)}{V(K_{t_k})}}\Phi_{2t_k-T}K_{t_k}$ in $\mathcal{C}^\infty$ to an origin-centered ellipse.

Remark 6.6. Let $K_0 \in \mathcal{F}_0^n$ be a convex body with its centroid at the origin, and let the family of convex bodies $\{K_t\} \in \mathcal{F}_0^n$ be the solution to

$$\begin{cases} \partial_t h(u,t) = -\frac{1}{h^n \mathcal{S}}(u,t), \\ h(\cdot,0) = h_{K_0}(\cdot). \end{cases} \tag{6.4}$$

Evolution equation (6.4) is a fully non-linear degenerate second-order parabolic differential equation with a high degree of homogeneity. In general, dealing with such flows is technically difficult [1, 2, 5, 9–11, 22, 23, 29, 34, 48, 67–69, 79]. It would be interesting to study the asymptotic behavior of (6.4) via the evolution equations of affinely associated bodies such as K^*, ΓK, and ΛK.

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