TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 368, Number 5, May 2016, Pages 3243–3270
http://dx.doi.org/10.1090/tran/6590
Article electronically published on July 29, 2015

OPERATOR ALGEBRAS WITH CONTRACTIVE APPROXIMATE IDENTITIES: A LARGE OPERATOR ALGEBRA IN c_0

DAVID P. BLECHER AND CHARLES JOHN READ

ABSTRACT. We exhibit a singly generated, semisimple commutative operator algebra with a contractive approximate identity such that the spectrum of the generator is a null sequence and zero, but the algebra is not the closed linear span of the idempotents associated with the null sequence and obtained from the analytic functional calculus. Moreover the multiplication on the algebra is neither compact nor weakly compact. Thus we construct a 'large' operator algebra of orthogonal idempotents, which may be viewed as a dense subalgebra of c_0.

1. INTRODUCTION

There is an extensive history and theory of operator algebras on a Hilbert (or Banach) space that are generated by a family of idempotent operators which are orthogonal (that is, the product of any two of which is zero) and using such families in 'spectral resolutions' of operators. Related to this, it is well known that there exist exotic Banach algebras whose elements are sequences of scalars, with the multiplication being the obvious pointwise one. That is, there can be quite complicated Banach algebra norms on subalgebras of the C^*-algebra c_0 of null sequences. However examples of both of these kinds of algebras, of a certain interesting type described below and which have an approximate identity, seem to be missing from the Banach algebra and operator theory literature. Our main goal here is to provide an explicit, yet in some sense universal, example of this kind. We first discuss our goal from the operator-theoretic angle, and later in the Introduction we will mention the Banach algebra viewpoint.

Henceforth, by an *operator algebra*, we will mean a norm closed subalgebra of $B(H)$, where the latter denotes the bounded linear operators on a Hilbert space H. There is a large literature on operator algebras generated by a family of mutually orthogonal idempotents (see e.g. [3, 12, 16, 19, 22] and references therein). Such a family arises naturally when one considers for a bounded operator T on a Hilbert space H with $\mathrm{Sp}(T)$ (countable and) having no nonzero limit points the spectral idempotents obtained by the analytic functional calculus from the nonzero isolated points. These idempotents will be called *minimal spectral idempotents*, and they are

Received by the editors February 27, 2014.
2010 *Mathematics Subject Classification.* Primary 46B15, 47L30, 47L55; Secondary 43A45, 46B28, 46J10, 46J40.
Key words and phrases. Singly generated operator algebra, algebras generated by orthogonal idempotents, spectral idempotent, approximate identity, semisimple, set of synthesis, Banach sequence algebra, Banach function algebra, Tauberian, socle.
The first author was partially supported by a grant from the National Science Foundation.
The second author is grateful for support from UK Research Council grant EP/K019546/1.

nonzero by a basic property of the functional calculus (and even have the uniqueness property in the Shilov idempotent theorem; see e.g. 2.4.33 in [10]). In this case it is standard to try to use this family to analyze the structure of T (often with an eye to decomposing T in terms of these idempotents). We will henceforth assume that the norm closed algebra B_T generated by T in $B(H)$ is semisimple, which implies that these minimal spectral idempotents e are minimal in the sense that $eB_T = \mathbb{C}\, e$. It seems that certain specific questions about the algebra B_T that arise in this setting are essentially unaddressed in the literature, to the best of our knowledge. It is a simple exercise in matrix theory (and using the blanket assumption that B_T is semisimple) that if H above is finite dimensional, then T is the sum of the minimal spectral idempotents, each multiplied by the corresponding eigenvalue. So it is natural to ask if, for T as above, B_T is generated by the minimal spectral idempotents of T. (Saying that B is 'generated' by a subset will for us always mean, unless stated to the contrary, that B is the smallest norm closed subalgebra of B containing the subset.) Such questions become quite difficult if one adds the assumption that the operator algebras involved have approximate identities. We will give a counterexample to the question, with algebras possessing contractive approximate identities (or *cai*'s). In this example, B_T is 'large' and in particular is not *weakly compact*. We will define, for the purposes of this paper, a commutative Banach algebra A to be weakly compact (resp. compact) if multiplication on A by a is weakly compact (resp. compact) for every $a \in A$ (this is not the usual definition, but it is equivalent to it for algebras with a cai). Indeed it is the case that for an operator T with $\mathrm{Sp}(T)$ having no nonzero limit points, B_T *is* generated (in the norm topology) by the minimal spectral idempotents of T if B_T is semisimple, weakly compact, and its socle has an approximate identity (see the end of this Introduction for this and some related results, which we show there to be 'best possible' in some sense).

In algebraic language, B_T being generated by the minimal spectral idempotents is equivalent to B_T having *dense socle* (or being *Tauberian* [19]); such algebras can be considered to be not 'big'. However the point is that the examples in the literature of operator algebras generated by an operator and an associated sequence of mutually orthogonal minimal idempotents tend either to be 'small' in this sense, or to fall within the case where this sequence is uniformly bounded, or to not have approximate identities, and do not speak to the question we have mentioned. We remark that in joint work with Le Merdy, the first author studied some natural Banach sequence algebras which were shown to be operator algebras (see [5, Chapter 5]), but again these were not 'big' in the sense above and had no approximate identity.

We now discuss our goal from the Banach algebraic perspective, which goes back to Kaplansky (e.g. [17]). We recall that a *natural Banach sequence algebra* on $\mathbb{N}$ is a Banach algebra A of scalar sequences which contains the space c_{00} of finitely supported sequences and whose characters (i.e. nontrivial multiplicative linear functionals) are precisely the obvious ones: $\chi_n(\vec{a}) = a_n$ for $\vec{a} \in A$ (see [10, Section 4.1] and [11]). In our case the sequences in A will converge to 0, so that $c_{00} \subset A \subset c_0$ (we are not assuming of course that the norm on A is the c_0 norm). Natural Banach sequence algebras have been studied by many researchers (see e.g. [10, 11, 19] and references therein); however we are not aware of any such algebras in the literature which are 'big' in the previous sense, namely that the socle of A, which in this case is c_{00}, is not dense in A (that is, A is not Tauberian) or, more generally, that A is not weakly compact, and which also have a bounded approximate identity (or *bai*). An example due to Joel Feinstein of a natural Banach sequence algebra without a dense socle is given in [10, Section 4.1]; however it has no bai. The existence of an example of this kind with a bai was asked of us by Dales. We will construct here a 'big' example, which is a singly generated operator algebra on a Hilbert space and which has a cai. To relate the Banach sequence algebra setting to that of the previous paragraphs, note that if T is an operator on a Hilbert space H with $\mathrm{Sp}(T)$ having no nonzero limit points, and if the closed algebra B_T generated by T is semisimple, then the Gelfand transform makes B_T into a (semisimple) natural Banach sequence algebra in c_0 (by basic Gelfand theory, e.g. the standard ideas in the proof of Corollary 1.3 below).

In this paper we exhibit an operator algebra example with the desired features discussed above. In hopes of obtaining a tool useful for solving other questions in this area in the literature, we have deliberately built our example to be as 'large as possible', and this has probably added to the difficulty and complexity of our proofs. In particular we show:

Theorem 1.1. *There exists a semisimple operator algebra A which has a cai and a single generator g (and hence A is separable), with the following properties. The spectrum of the generator of A is a null sequence and zero, but A is strictly larger than $\overline{A_{00}}$, the norm closed linear span of the minimal spectral idempotents associated with this null sequence and obtained from the analytic functional calculus. Also, multiplication by the generator g on A is not a weakly compact operator (equivalently, A is not an ideal in A^{**} with the Arens product; see e.g.* [20, 1.4.13] *or* [1, Lemma 5.1]*). The algebra can be chosen further with $\overline{A_{00}}$ having a cai too and with either A contained in the strong operator closure of $\overline{A_{00}}$ or not, as desired.*

Our example (in particular our algebras A, $\overline{A_{00}}$, and their unitizations) will hopefully be useful in settling other open questions in the subject. As an illustration of how it can be used in that capacity, we mention that Joel Feinstein has pointed out to us that the unitization of our example solves an old question of his (see [13, 14], although he has informed us that the question goes back at least as far his thesis) and another similar question of Dales, namely whether a certain variant of the notion of peak sets for regular Banach function algebras are 'sets of synthesis'. We shall explain this application in more detail at the end of Section 6.

Concerning the layout of our paper, in the next section we turn to the construction of our algebras A and $\overline{A_{00}}$ described above. The development will become increasingly technical as the paper proceeds. However in Section 5 we will prove the key part of our main theorem with one lemma taken on faith, and in Section 6 we will pause and describe many properties that our algebras A and $\overline{A_{00}}$ possess. The material following Section 6 consists of the lengthy proof of the lemma just referred to.

We end this Introduction with some general positive results on the topics above, namely sufficient conditions for when B_T is generated by the minimal spectral idempotents of T. We remark that in [3, Proposition 1.1] it is shown that any closed algebra on a separable Hilbert space generated by a family of 'mutually orthogonal' idempotents is topologically singly generated.

Proposition 1.2. *Suppose that D is a Banach algebra, which is an essential ideal in a commutative Arens regular Banach algebra A ('essential' means that the canonical representation of A on D is one-to-one). Assume that A is weakly compact and D has a bai. Then $A = D$.*

Proof. By e.g. [20, 1.4.13] or [1, Lemma 5.1] we have $A^{**}A \subset A$. Let $e \in D^{\perp\perp}$ be the 'support projection' in A^{**} of D, an identity for $D^{\perp\perp}$, and write 1 for the identity of the unitization of A. Then $(1-e)D = 0$, and for any $a \in A$ we have $a(1-e)D = 0$. Since $eA \subset A$ and D is an essential ideal in A, we see that $a(1-e) = 0$. Thus e acts as an identity on A and therefore also on A^{**}, so that $A^{**} = eA^{**} \subset D^{\perp\perp}$. Hence $A = D$. ☐

Corollary 1.3. *Let T be an operator on a Hilbert space whose spectrum has no nonzero limit points, and let B_T (resp. B) be the closed algebra generated by T (resp. by the minimal spectral idempotents of T). If B_T is semisimple and weakly compact, and if B has a bai, then $B = B_T$.*

Proof. Suppose that $\mathrm{Sp}(T) \setminus \{0\} = \{\lambda_n\}$. By basic Gelfand theory, the set of characters of the unitization B_T^1 is $\{\chi_n : n \in \mathbb{N}_0\}$, where χ_0 annihilates B_T and $\chi_n(T) = \lambda_n$ for $n \in \mathbb{N}$. By the functional calculus $\chi_m(e_n) = \delta_{nm}$ for $n \in \mathbb{N}, m \in \mathbb{N}_0$, where e_n is the spectral idempotent in B_T^1 corresponding to λ_n. Hence $e_n \in B_T$, and $e_nT - \lambda_n e_n \in \cap_m \mathrm{Ker}(\chi_m) = (0)$. Thus $e_nB_T = \mathbb{C}\, e_n$ for all n, and $\chi_n(a)e_n = ae_n$ for all $a \in B_T$. Hence $BB_T \subset B$; that is, B is an ideal in B_T. Indeed B is an essential ideal in B_T, since the latter is semisimple (if $ae_n = 0$ for all n, then $\chi(a) = 0$ for all characters χ). Thus $B = B_T$ by Proposition 1.2. ☐

For operators on a Hilbert space whose spectrum has no nonzero limit points, the last result is sharp in the following sense:

Theorem 1.4. *In the last result not one of the following three hypotheses can simply be removed, in general: B_T is semisimple, or B_T is weakly compact, or B has a bai.*

Proof. To see that the semisimplicity condition cannot be removed, consider the long example in [6, Section 5] (or one could consider the Volterra operator or the direct sum of the Volterra operator and a generator for c_0). Theorem 1.1 in the present paper shows that the weak compactness condition cannot be removed, even if in addition B and B_T have cai.

Finally, we will show that the approximate identity condition cannot be removed, even if the algebra is 'compact'. Our algebra A will be the space c of convergent sequences with product $\vec{x} \cdot \vec{y} = (\frac{1}{2^n}\, x_n y_n)$ (an example also mentioned briefly by Mirkil in a Banach algebra context). Clearly A is generated by $T = (1, 1, 1 \cdots)$ and c_{00}. Since $\vec{x} \cdot \vec{y}$ is the usual product of $\vec{x}, \vec{y}$, and $(\frac{1}{2^n})$, A is a commutative operator algebra by [9, Remark 2 on p. 194]. The vectors $2^n \vec{e}_n$ are minimal idempotents in A, inducing characters χ_n on A, and it is clear now that A is semisimple. Conversely, since any character on c_0 must be induced by a sequence in ℓ^1, it is easy to see that such a character is the restriction of one of the χ_n. It is also then easy to see that any character on A is one of the χ_n. We leave it as an exercise that T generates c_{00}. Hence the spectrum of T is $\{\frac{1}{2^n}\} \cup \{0\}$ and the spectrum of A is homeomorphic to $\mathbb{N}$. If E_n is the minimal spectral idempotent of T corresponding to $\frac{1}{2^n}$ in the spectrum, then $E_n \in A$ by an argument in Corollary 1.3. Thus these minimal spectral idempotents are exactly the $2^n\vec{e}_n$ above, by e.g. [18, Theorem 1.2]. So $A = B_T$ has discrete spectrum, but it is clearly not Tauberian. To see that A is compact suppose that $(\vec{x}(n))_n$ is a bounded sequence in c. Then $T \cdot \vec{x}(n) = (x(n)_m/2^m)$, which is the product of a fixed sequence in c_0 with a bounded sequence in c_0. Since c_0 is compact, there is a convergent subsequence as desired. □

Remark 1.5. Theorem 1.4 suggests the question, if in Corollary 1.3 the condition that B has a bai may be replaced by B_T having a bai (or cai).

Corollary 1.6. *A semisimple, topologically singly generated operator algebra, whose socle has a bai, has a dense socle if and only if it is compact.*

Proof. We first prove that, more generally, a semisimple, weakly compact operator algebra with a topological single generator, whose socle has a bai and which has discrete spectrum, has a dense socle (equivalently, is Tauberian). This follows from Corollary 1.3. Indeed if T is a topological single generator for such an algebra A, then the spectrum of g (minus 0) is homeomorphic to the discrete spectrum of A, so has no nonzero limit points. The minimal idempotents in A, whose span defines the socle, define characters of A, so, as in the proof of Corollary 1.3, they must be the minimal spectral idempotents of T.

That a Banach algebra with dense socle is 'compact' is obvious (or see e.g. [20, Proposition 8.7.7]). Conversely, semisimple compact Banach algebras have discrete spectrum (see e.g. [20, Chapter 8]). Thus the first result follows from the last paragraph. □

Remark 1.7. We point out an error in [1] that momentarily led us astray early in this work. Namely, in the last assertion of [1, Theorem 5.10 (4)], to get a correct statement the characters there are not allowed to vanish on A. This led to a mistaken comment at the end of the first paragraph of the Remark after Proposition 5.6 there, concerning the spectrum of A^{**}. Fortunately these results have not been used elsewhere.

2. The general construction

Let A_0 be the dense subalgebra of c_0 generated by the unit vectors e_i and the vector $g = \sum_{i=1}^\infty 2^{-i}\, e_i = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$. (In passing we remark that this notation differs from the meaning of A_0 in [10, 11], and we apologize for any confusion to those familiar with that literature.) We seek to renorm A_0 so that its completion A is an operator algebra such that:

(1) A has a cai and is topologically generated by g,
(2) the spectrum of g is $\{2^{-n} : n \in \mathbb{N}\}$,
(3) $g \notin \overline{\text{lin}}\{e_i : i \in \mathbb{N}\}$, and
(4) A is semisimple.

Note that what forces us to change (increase!) the usual norm on c_0 is condition (3). In fact the norm will be increased in such a way that the unit vectors e_i, which will be the spectral idempotents for the generator g, are unbounded. For $n \in \mathbb{N}_0$, write $P_n = \sum_{i=1}^n e_i$, and let

$$g_n = 2^n g \wedge 1 = (1, 1, 1, \ldots, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots) = \sum_{i=1}^n e_i + \sum_{i=1}^\infty 2^{-i} e_{i+n}.$$

Note that $g_n \in 2^n g + \text{lin}\{e_1, \cdots, e_n\} \in A_0$.

Lemma 2.1. *Let $\|\cdot\|$ be any algebra norm on the algebra A_0. Suppose that for some strictly increasing sequence $(a_i)_{i=1}^{\infty} \subset \mathbb{N}$, we have*

$$\|g_{a_n}^{a_n}\| \leq 1 + \frac{1}{n} \quad \text{and} \quad \|g_{a_n}^{a_n} \cdot g - g\| \leq \frac{1}{n}.$$

Then the vectors $x_n = \frac{n}{n+1} g_{a_n}^{a_n}$ are a cai for $(A_0, \|\cdot\|)$.

Proof. The conditions we are given ensure that $\|x_n\| \leq 1$ and $x_n g \to g$. But $e_m = 2^m g e_m$, and so $x_n e_m \to e_m$ also. The vectors e_m and g generate A_0, and so $(x_n)_{n=1}^{\infty}$ is a cai. □

If A is the completion of A_0 in some algebra norm, we write $\overline{A_{00}}$ for the closed ideal $\overline{\text{lin}}\{e_i : i \in \mathbb{N}\}$ in A.

Lemma 2.2. *Once again, let $\|\cdot\|$ be any algebra norm on the algebra A_0, and let A denote the completion of $(A_0, \|\cdot\|)$. Suppose that for some strictly increasing sequence $(a_i)_{i=1}^{\infty} \subset \mathbb{N}$, we have*

$$\|g^{a_n}(I - P_{a_n})\| \leq n^{-a_n}.$$

Then the spectrum of $g \in A$ is precisely $\{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, and the spectral idempotents for the eigenvalues 2^{-n}, obtained from the analytic functional calculus for g, are precisely the unit vectors e_n.

Proof. For every $x \in A_0$ and $n \in \mathbb{N}$ we have $xe_n = \chi_n(x)\, e_n$ for a unique complex number $\chi_n(x)$. Even in the completion A this will be true, because for $y \in A$ the product ye_n is a limit of scalar multiples of e_n, and so is a multiple of e_n. We now see that χ_n is a character on A with $\chi_n(g) = 2^{-n}$. Thus $2^{-n} \in \text{Sp}(g)$.

Conversely, we claim that $\text{Sp}(g)$ does not contain any $\lambda \neq 0$ that is not a negative integer power of 2. If it does, it contains such a $\lambda \in \partial\, \text{Sp}(g)$, so λ is in the approximate point spectrum of g. Pick n so large that $1/n < |\lambda|$. For $y \in (I - P_{a_n})A$ with $\|y\| = 1$ we have by hypothesis that $\|g^{a_n} y\| \leq n^{-a_n} < |\lambda|^{a_n}$. Thus λ is not in the spectrum of the operator of multiplication by g on $(I - P_{a_n})A$. There is therefore an $\eta > 0$ such that $\|gy - \lambda y\| \geq \eta \|y\|$ for all $y \in (I - P_{a_n})A$. In the subalgebra

$$P_m A = (e_1 + e_2 + \ldots e_m)A = \text{lin}\{e_j : j \leq m\}$$

the spectrum of g is $\{2^{-j} : j \leq m\}$, so there is an η' such that $\|gy' - \lambda y'\| \geq \eta' \|y'\|$ for $y' \in P_m A$. Write $m = a_n$ and let $z \in A$. Then

$$\|gz - \lambda z\| \geq \frac{1}{\|P_m\|} \cdot \|gP_m z - \lambda P_m z\| \geq \eta' \|P_m z\| / \|P_m\|,$$

and also

$$\|gz - \lambda z\| \geq \frac{1}{1 + \|P_m\|} \cdot \|g(1 - P_m)z - \lambda(1 - P_m)z\| \geq \eta \|(1 - P_m)z\| / (1 + \|P_m\|).$$

Also $\|gz - \lambda z\| \geq C\|P_m z + (z - P_m z)\| = C\|z\|$, for some positive constant C. Therefore λ is not in the approximate point spectrum of g. This contradiction proves the claim; that is, $\text{Sp}(g) = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$.

To identify the spectral idempotent for the point 2^{-n}, we decompose the unitization A^1 as a direct sum of ideals $P_{a_m} A^1 \oplus (I - P_{a_m})A^1$, where m is much larger than 2^n. There is a corresponding decomposition of the spectral idempotent as a sum of the spectral idempotent in $P_{a_m} A^1$, which is easy to see is e_n, and the spectral idempotent in $(I - P_{a_m})A^1$. The latter is zero since by hypothesis,

$\|g^{a_m}(I-P_{a_m})\| \leq m^{-a_m}$, so that the spectral radius of $g(I-P_{a_m})$ is much smaller than 2^{-n}. (One may also use this in conjunction with the criteria in [18, Theorem 1.2]). □

Corollary 2.3. *If the conditions of Lemma* 2.2 *hold, then A is singly generated by g, and the space of characters of A is $\{\chi_n : n \in \mathbb{N}\}$, with χ_n as defined above.*

Proof. The spectral idempotents e_n are, by the functional calculus, in the closed algebra $\mathrm{oa}(g)$ generated by g (note that $e_n \in \mathrm{oa}(1,g)$ by e.g. [10, Theorem 2.4.4 (ii)], but $e_n = 2^n g e_n$, so that e_n is in $\mathrm{oa}(g)$). Together with g itself, these idempotents generate A_0 algebraically, so A is singly generated by g. As in the proof of Corollary 1.3 the characters χ_n constitute the character space of A. □

Remark 2.4. It is easy to find operator algebra norms so that (1)–(4) at the start of Section 2 hold with the exception of A having a cai. The last example in the proof of Theorem 1.4 is such. Also the operator algebra norm in [2, Example 4.30] can easily be seen to work (with the help of the last two results).

3. Maximal norms

Definition 3.1. Let a strictly increasing sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{N}$ be given. Set $a_0 = 1$, and define a subset $S_0 \subset c_0$ as follows:

$$S_0 = \{g, a_n^{-1}e_n, \frac{n}{n+1}g_{a_n}^{a_n}, n(g_{a_n}^{a_n}\cdot g - g), n^{a_n}g^{a_n}(1-P_{a_n}) : n \in \mathbb{N}\}. \tag{3.1}$$

Also, let S be the collection of all finite products of elements of S_0.

Note: Later on, we are going to impose growth conditions on this underlying sequence. Our main results will happen provided that the underlying sequence (a_n) increases sufficiently rapidly. But for now, we note that S_0 includes both g and nonzero multiples of the unit vectors e_n, so S_0 generates A_0 algebraically; indeed the linear span of S is A_0. Thus, we get a finite seminorm on A_0 if we define

$$\|x\|_{\max} = \inf\{\sum_{i=1}^{n}|\lambda_i| : x = \sum_{i=1}^{n}\lambda_i s_i : n \in \mathbb{N}, \lambda_i \in \mathbb{C}, s_i \in S\}. \tag{3.2}$$

It will eventually transpire that, given growth conditions on the a_n, the completion of A_0 in the norm $\|\cdot\|_{\max}$ gives a Banach algebra satisfying (1)–(3) at the start of Section 2 and that the quotient of this by the radical satisfies (1)–(4) there.

Lemma 3.2. *Let S, S_0 be defined as in Definition* 3.1. *Then the seminorm $\|\cdot\|_{\max}$ in* (3.2) *is a norm greater than or equal to the c_0-norm $\|\cdot\|_0$. Indeed $\|\cdot\|_{\max}$ is the largest seminorm on A_0 such that $\|s\|_{\max} \leq 1$ for all $s \in S$, and $\|\cdot\|_{\max}$ is equal to the largest algebra seminorm on A_0 such that $\|s\|_{\max} \leq 1$ for all $s \in S_0$. If $\|\cdot\|$ is any algebra norm on A_0 with $\|\cdot\|_0 \leq \|\cdot\| \leq \|\cdot\|_{\max}$, then writing A for the completion of $(A_0, \|\cdot\|)$, A has a cai, the spectrum of $g \in A$ is precisely $\{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, and the spectral idempotents for the eigenvalues 2^{-n}, obtained from the analytic functional calculus for g, are the unit vectors e_n. Finally, A is singly generated by g.*

Proof. Expression (3.2) is at least the c_0-norm if and only if every $x \in S$ has c_0-norm at most 1, if and only if every $x \in S_0$ has c_0-norm at most 1.

Looking at the definition of S_0, it is obvious that the vectors g and $a_n^{-1}e_n$ and $\frac{n}{n+1}g_{a_n}^{a_n}$ have c_0-norm at most 1. Also the l^∞-norm of $g_{a_n}^{a_n}-1$ is 1, but its jth entry is zero for $j \le a_n$; hence the c_0-norm of $(g_{a_n}^{a_n}\cdot g - g)$ is at most $\|g(1-P_{a_n})\|_0 = 2^{-(1+a_n)} < \frac{1}{n}$. So $\|n(g_{a_n}^{a_n}\cdot g-g)\|_0 < 1$ also. Finally the c_0-norm of $g^{a_n}(1-P_{a_n})$ is $2^{-a_n(1+a_n)}$, so since $a_n \ge n$, the c_0-norm of $n^{a_n}g^{a_n}(1-P_{a_n})$ is at most 1 also. Thus $\|x\|_0 \le 1$ for every $x \in S_0$, and $\|x\| \ge \|x\|_0$ for every $x \in A_0$.

Let $\|\cdot\|$ be any algebra seminorm on A_0 such that $\|s\| \le 1$ for all $s \in S_0$. Then plainly we have $\|s\| \le 1$ for all $s \in S$. Also, given $\|\cdot\|$ is a norm such that $\|s\| \le 1$ for all $s \in S$, plainly we must have $\|s\| \le \|s\|_{\max}$ as given in (3.2).

Every element in the set S has $\|\cdot\|_{\max}$ norm at most 1, so $\|\cdot\|_{\max}$ is indeed the maximal seminorm with this property, as claimed in the lemma. Also, if $\|\cdot\|$ is any algebra norm with $\|\cdot\|_0 \le \|\cdot\| \le \|\cdot\|_{\max}$, we have

$$\|g\| \le 1 \quad \text{and} \quad \|e_n\| \le a_n \tag{3.3}$$

for each $n \in \mathbb{N}$; also

$$\|g_{a_n}^{a_n}\| \le 1 + \tfrac{1}{n}; \quad \|g_{a_n}^{a_n}\cdot g - g\| \le \tfrac{1}{n}; \quad \text{and} \quad \|g^{a_n}(1-P_{a_n})\| \le n^{-a_n}.$$

By Lemma 2.1, the vectors $x_n = \frac{n}{n+1}g_{a_n}^{a_n}$ are a cai for A. By Lemma 2.2, the spectrum of $g \in A$ is precisely $\{2^{-n} : n \in \mathbb{N}\} \cup \{0\}$, and the spectral idempotents for the eigenvalues 2^{-n}, obtained from the analytic functional calculus for g, are the unit vectors e_n. By Corollary 2.3, A is topologically generated by g. ☐

Corollary 3.3. *Let $\|\cdot\|$ be an algebra norm on A_0 with $\|\cdot\|_0 \le \|\cdot\| \le \|\cdot\|_{\max}$, and let A be the completion of $(A_0, \|\cdot\|)$, and J the Jacobson radical of A. Let $q : A \to A/J$ be the quotient map. Then the norm $|||\cdot|||$ on A_0 with $|||x||| = \|q(x)\|$ also satisfies $\|\cdot\|_0 \le |||\cdot||| \le \|\cdot\|_{\max}$, and the conclusions of Lemma* 3.2 *are also satisfied when A is replaced by the semisimple Banach algebra A/J.*

Proof. Since A satisfies the conditions of the previous result, and therefore of Corollary 2.3 also, the characters on A are the χ_n mentioned there. Of course J is the intersection of the kernels of the characters of A; hence $A_0 \cap J = (0)$. Therefore the quantity $|||x||| = \|q(x)\|$ is a norm on A_0, and it is dominated by $\|\cdot\|_{\max}$ clearly. It is only necessary to show that $|||x||| \ge \|x\|_0$, for then A/J is the completion of $(A_0, |||\cdot|||)$, and the conclusions of Lemma 3.2 will follow for the norm $|||\cdot|||$. But $|||x|||$ dominates $\sup\{|\chi(x)| : \chi \text{ is a character of } A\} = \|x\|_0$. ☐

4. A general (Banach algebraic) theorem

Write $\Delta_n = P_{a_{n+1}} - P_{a_n}$ and

$$H_n = \text{lin}\{e_j : a_n < j \le a_{n+1}\} = \Delta_n A_0.$$

A basis for the dual of this vector space is the set of characters $\{\chi_j : j = a_n+1, \ldots, a_{n+1}\}$. For these j, we have $\chi_j = \chi_j \circ \Delta_n$.

We have the following general theorem, which is part of what we need, but it does not necessarily give an operator algebra, merely a Banach algebra. However we will use it later to get an operator algebra.

Theorem 4.1. *Let a strictly increasing sequence (a_n) be given, and let A be the completion of A_0 in an algebra norm $\|\cdot\|$, where $\|\cdot\|_0 \le \|\cdot\| \le \|\cdot\|_{\max}$, with $\|\cdot\|_0$ being the c_0-norm, and $\|\cdot\|_{\max}$ the maximal norm as defined in* (3.2)*. Suppose in* *addition that there is a bounded sequence* $(\psi_n)_{n=1}^{\infty} \in A^*$ *such that* $\psi_n(g) = 1$ *and* $\psi_n = \psi_n \circ \Delta_n$ *for each* n. *Let* $B = A/\text{rad}A$. *Then* B *has the following properties.*

(1) B *is a Banach algebra with cai, topologically generated by* g*;*
(2) g *is a contraction with spectrum* $\{2^{-n} : n \in \mathbb{N}\}$*;*
(3) $g \notin \overline{\text{lin}}\,\{e_i : i \in \mathbb{N}\}$ *and*
(4) B *is semisimple.*

Proof. That B has a cai, and the spectrum of g is as stated, and that B is topologically generated by g follow from Corollary 3.3. Certainly $B = A/\text{rad}A$ is semisimple, so it remains to show that $g \notin \overline{\text{lin}}\{e_i : i \in \mathbb{N}\}$. To this end, let $\psi \in A^*$ be any weak-* accumulation point of the (by hypothesis bounded) sequence ψ_n. Since $\psi_n = \psi_n \circ \Delta_n$, we have $\psi_n \in \text{lin}\{\chi_j : a_n < j \le a_{n+1}\}$. Hence each ψ_n annihilates rad$A$, so ψ annihilates rad$A$ and yields a well defined element of B^*. Since $\psi_n(e_j) = 0$ for $j \le a_n$ and $a_n \to \infty$, we have $\psi(e_j) = 0$ for all $j \in \mathbb{N}$. But $\psi_n(g) = 1$ for all n, so $\psi(g) = 1$. Therefore $g \notin \overline{\text{lin}}\{e_i : i \in \mathbb{N}\}$. $\square$

5. Representations on Hilbert space

It is time to construct norms which will presently turn A_0 into a nontrivial operator algebra. These norms will be somewhat universal, in the sense that they are defined so as to encode abstractly the critical hypotheses in the lemmas in Section 2, which ensure that conditions (1) and (2) at the start of that section hold (after which we will proceed to prove that (3) and (4) also hold).

For $k \in \mathbb{N}_0$, write $\gamma_k^{(n)} = \sum_{j=1+a_n}^{a_{n+1}} 2^{-jk} e_j$. In the rest of our paper we will very often silently use the relation

$$\gamma_k^{(n)} \gamma_i^{(n)} = \gamma_{k+i}^{(n)}.$$

The reader can check the following relations: We have $\Delta_n g = \gamma_1^{(n)}$, and $\Delta_n g_{a_j}^{a_j} = 2^{a_j^2} \gamma_{a_j}^{(n)}$ if $j \le n$, while $\Delta_n g_{a_j}^{a_j} = \gamma_0^{(n)}$ if $j > n$. So $\Delta_n g_{a_j}^{a_j} \cdot g - g = 2^{a_j^2} \gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}$ if $j \le n$ and is zero if $j > n$. Similarly, $\Delta_n g^{a_j}(1 - P_{a_j}) = \gamma_{a_j}^{(n)}$ if $j \le n$ and is zero if $j > n$. Let us also write

$$\Lambda_n = \{\sum_{i=1}^{n} t_i a_i : t_i \in \mathbb{N}_0, t_i \le a_i \text{ for } 1 = i, \ldots, n\}$$

and

$$\xi_n = \max \Lambda_n = \sum_{i=1}^{n} a_i^2.$$

We shall assume that the sequence (a_n) increases sufficiently fast that these sums are distinct for distinct sequences (t_i) and that they appear in "lexicographic order". So, we assume that, for $t_i \in \mathbb{Z}$, $|t_i| \le 2a_i$, we have $\sum_{i=1}^{n} t_i a_i > 0$ if and only if $t_r > 0$, where $r = \max\{j : t_j \ne 0\}$. This condition, slightly stronger than our immediate need, will also ensure that elements of the set $\Lambda_n - \Lambda_n = \{\sum_{i=1}^{n} t_i a_i : -a_i \le t_i \le a_i\}$ also appear in "lexicographic" order.

We note that any collection of m of the vectors $\gamma_k^{(n)}$ is linearly independent, provided that $m \le a_{n+1} - a_n$. So a linear functional $\phi \in \text{lin}\{\chi_j : a_n < j \le a_{n+1}\}$ is specified uniquely by its action on $\{\gamma_k^{(n)} : 0 \le k < a_{n+1} - a_n\}$. Let ϕ_n be the unique such functional such that for all $0 \le k < a_{n+1} - a_n$,

$$\phi_n(\gamma_k^{(n)}) = \begin{cases} \prod_{i=1}^n 2^{-t_i a_i^2}(1 - t_i/a_i) & \text{if } k = 1 + \sum_{i=1}^n t_i a_i \in 1 + \Lambda_n, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Here the $t_i \in \mathbb{N}_0$ with $t_i \le a_i$ of course, and $\phi_n(\gamma_1^{(n)}) = 1$. We may view ϕ_n as a functional on A satisfying $\phi_n = \phi_n \circ \Delta_n$. Therefore from (5.1) we have that

$$\phi_n(g) = \phi_n(\Delta_n g) = \phi_n(\gamma_1^{(n)}) = 1.$$

However, for $j \le a_n$ we have $\phi_n(e_j) = 0$. We could go on from here and prove directly that the ϕ_n are uniformly $\|\cdot\|_{\max}$-bounded, whereupon we could apply Theorem 4.1, but this would not get us an operator algebra. Instead we proceed as follows.

We have established that $\Delta_n g = \gamma_1^{(n)}$ and $\Delta_n g_{a_j}^{a_j} = 2^{a_j^2}\gamma_{a_j}^{(n)}$ if $j \le n$, while $\Delta_n g_{a_j}^{a_j} = \gamma_0^{(n)}$ if $j > n$. Also, $\Delta_n e_j = e_j$ if $a_n < j \le a_{n+1}$, and is zero otherwise. Now $\gamma_0^{(n)}$ is the identity of $\Delta_n A_0$, so referring to (3.1) and removing from $\Delta_n S_0$ positive scalar multiples of the identity or of other elements of $\Delta_n S_0$, we are left with the set

$$\{\gamma_1^{(n)}, a_i^{-1}e_i, \frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)}, j(2^{a_j^2}\gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}) : a_n < i \le a_{n+1}, 1 \le j \le n\}. \tag{5.2}$$

Definition 5.1. The set given by (5.2) will be called $S_0^{(n)}$. Let $\mathcal{I}^{(n)}$ denote the set of all "index functions" $\mathbf{i} : S_0^{(n)} \to \mathbb{N}_0$. For $\mathbf{i} \in \mathcal{I}^{(n)}$, write $\mathbf{s}^{\mathbf{i}}$ for the product $\prod_{s \in S_0^{(n)}} s^{\mathbf{i}(s)}$. Equip H_n with a Euclidean seminorm $\|\cdot\|_2^{(n)}$ as follows. For $x \in H_n$, we define

$$\|x\|_2^{(n)} = (\sum_{\mathbf{i} \in \mathcal{I}^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}}x)|^2)^{1/2}. \tag{5.3}$$

We shall establish that $\|x\|_2^{(n)}$ is finite, so that we do indeed have a Euclidean seminorm. Letting $\mathcal{H}_n$ denote the associated Euclidean space, we shall represent each $T \in A_0$ in $B(\mathcal{H}_n, \|\cdot\|_2^{(n)})$ by its compression $\Delta_n T$. This representation will be called $\rho_n : A_0 \to B(\mathcal{H}_n, \|\cdot\|_2^{(n)})$, and the operator norm $\|\rho_n(T)\|$ will be called $\|T\|_{\rm op}^{(n)}$. We then define on A_0 the quantity

$$\|T\| = \sup_{n \in \mathbb{N}_0} \|T\|_{\rm op}^{(n)}, \tag{5.4}$$

where $\|T\|_{\rm op}^{(0)} = \|T\|_0$. We shall show that this is a norm.

The basic lemma we shall prove is as follows:

Lemma 5.2. *For every $x \in H_n$, we have $\|x\|_2^{(n)} < \infty$, provided that our underlying sequences satisfy certain growth conditions. The representation ρ_n above on $\mathcal{H}_n$ is well defined, and the operator norm $\|T\|_{\rm op}^{(n)} \le 1$ for every $T \in S_0$.*

Proof. Consider H_n as an algebra with pointwise product. The spectral radius here is the c_0-norm, and for the various elements of $S_0^{(n)}$ it is as follows:

$$\left\|\gamma_1^{(n)}\right\|_0 = 2^{-1-a_n}, \quad \|a_i^{-1}e_i\|_0 = a_i^{-1}, \quad \left\|\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)}\right\|_0 = \frac{j}{j+1}2^{a_j(a_j-a_n-1)}$$

and

$$\left\| j(2^{a_j^2}\gamma_{1+a_j}^{(n)} - \gamma_1^{(n)})\right\|_0 \leq j(2^{a_j(a_j-a_n)} + 2^{-1-a_n}).$$

A mild growth condition on the (a_j) will ensure that for all n,

$$2^{-1-a_n} + \sum_{i=a_n+1}^{a_{n+1}} a_i^{-1} + \sum_{j=1}^{n}(j+1)(2^{a_j(a_j-a_n)} + 2^{-1-a_n}) < 1.$$

In particular, the spectral radii of the elements of $S_0^{(n)}$ are then all strictly less than 1. It is a nice exercise that on any commutative algebra with a finite set of generators of spectral radius < 1, we can pick an algebra norm such that $||s|| < 1$ for all generators s simultaneously. (Hint: Let G_0 be the set of generators, each multiplied by $1+\epsilon$, such that the semigroup G generated by G_0 is bounded. Renorm A in the usual way so that $G \subset \mathrm{Ball}(A)$; see e.g. [20, Proposition 1.1.9].) With respect to such a norm on H_n, the square of the sum in (5.3) is at most

$$\sum_{\mathbf{i}} \prod_{s\in S_0^{(n)}} ||s||^{2\mathbf{i}(s)} \cdot ||\phi_n||^2 = ||\phi_n||^2 \prod_s (1 - ||s||^2)^{-1} < \infty.$$

So the expression $\|x\|_2^{(n)}$ in (5.3) is finite.

It is clear from (5.3) that for $s \in S_0^{(n)}$ and $x \in H_n$, the terms in the sum defining $\|x\|_2^{(n)}$ include all those in the sum defining $\|sx\|_2^{(n)}$ (plus certain extra terms, namely $|\phi_n(\mathbf{s}^{\mathbf{i}}x)|^2$ for index functions $\mathbf{i}$ such that $\mathbf{i}(s) = 0$). Therefore $\|sx\|_2^{(n)} \leq \|x\|_2^{(n)}$, and $\|s\|_{\mathrm{op}}^{(n)} \leq 1$ for $s \in S_0^{(n)}$. Also, for $s \in S_0$, we have $\|s\|_{\mathrm{op}}^{(n)} = \|\Delta_n s\|_{\mathrm{op}}^{(n)}$, which is either a multiple of the identity $\gamma_0^{(n)}$ of magnitude less than 1 or an element of $S_0^{(n)}$ (again possibly multiplied by a positive scalar of magnitude less than 1). Thus $\|s\|_{\mathrm{op}}^{(n)} \leq 1$ for all $s \in S$, as required. The last few lines also show that ρ_n above is well defined. □

It will be much harder to prove the next result. In fact this proof will take up almost all of the rest of our paper.

Lemma 5.3. *Given growth conditions on our underlying sequences, we have* $0 < \left\|\gamma_0^{(n)}\right\|_2^{(n)} \leq 3 \cdot \left\|\gamma_1^{(n)}\right\|_2^{(n)}$ *for all* n. *We have* $\|g\|_{\mathrm{op}}^{(n)} \geq \frac{1}{3}$.

Taking this lemma on faith for now, the rest of our assertions follow rather easily:

Theorem 5.4. *Given growth conditions on the* a_n, *the operator algebra norm defined in* (5.4) *is at most* $\|\cdot\|_{\max}$. *The completion* A *of* A_0 *in this norm is an operator algebra satisfying all of the conditions* (1)–(4) *at the start of Section* 2.

Proof. The completion A of A_0 in the norm $\|\cdot\|$ defined in (5.4) will be an operator algebra whose norm lies between $\|\cdot\|_0$ and $\|\cdot\|_{\max}$ as in Lemma 3.2 (for $\|\cdot\|_{\max}$ is the largest norm on A_0 such that $\|s\|_{\max} \leq 1$ for all $s \in S_0$). Furthermore, since $\|g\|_{\mathrm{op}}^{(n)} \geq \frac{1}{3}$ for each n by Lemma 5.3, there is a linear functional $\psi_n \in (A_0, \|\cdot\|_{\mathrm{op}}^{(n)})^*$ such that $\|\psi_n\| \leq 3$ and $\psi_n(g) = 1$. Finally, A is semisimple (there is no need to quotient out by $\mathrm{rad}A$), because the operator norm $\|T\|_{\mathrm{op}}^{(n)}$ is zero unless one of the characters $\chi_j(T) \neq 0$ for some j with $\alpha_n < j \leq \alpha_{n+1}$. Thus $(A, \|\cdot\|)$ satisfies the requirements of Theorem 4.1, but it is clearly an operator algebra. □

Corollary 5.5. *If $A_{\max}$ is the completion of A in the maximal norm defined in* (3.2)*, and given growth conditions on the a_n, then $A_{\max}$ (resp. $A_{\max}/radA_{\max}$) is a Banach algebra satisfying the desired conditions* (1)–(3) *(resp.* (1)–(4)*) at the start of Section* 2*.*

Proof. By Lemma 3.2 and Corollary 3.3, $A_{\max}$ (resp. $A_{\max}/\mathrm{rad}A_{\max}$) satisfy (1)–(2) (resp. (1)–(2) and (4)). If ψ_n is as in the proof of Theorem 5.4, and if $i : A_{\max} \to A$ is the canonical contraction due to $\|\cdot\|_{\max}$ being a larger norm, then $\psi_n \circ i$ satisfies the conditions of Theorem 4.1 with A replaced by $A_{\max}$. Then Theorem 4.1 or its proof implies that (3) also holds. □

Remark 5.6. In a previous draft Corollary 5.5 was proved directly, but for reasons of space this (very lengthy) computation has been omitted.

6. Additional properties of our algebras A and $\overline{A_{00}}$

In this section we take Lemma 5.3 on faith, so that our main theorems above hold for our algebra A.

Let $H = \ell^2 \oplus (\bigoplus_{n\in\mathbb{N}}^2 \mathcal{H}_n)$, and write ρ for the representation of A that we have already constructed, namely

$$\rho(a) = \Gamma(a) \oplus (\bigoplus_{n\in\mathbb{N}} \rho_n(a)).$$

Here Γ is the Gelfand transform, mapping into c_0, but with c_0 viewed as 'diagonal' operators on ℓ^2 in the usual way. Note that ρ is initially defined on A_0 and is easily seen to be a nondegenerate representation of A_0. The norm on A was defined in such a way that ρ extends to a (completely) isometric representation of A, which we will continue to write as ρ, and which is still nondegenerate.

Lemma 6.1. *The algebra $\overline{A_{00}}$ has a contractive approximate identity. Also A is a subalgebra of the closure of $\overline{A_{00}}$ in the strong operator topology of $B(H)$ for H as above.*

Proof. Indeed $(\rho(P_{a_{k+1}}))$ is a cai for A_{00}, and hence also for $\overline{A_{00}}$, since $\rho_n(P_{a_{k+1}})$ is just $I_{\mathcal{H}_k}$ if $k \geq n$ and is zero for $k < n$.

Clearly $\rho(P_{a_{k+1}}) \to I$ strongly on H, and hence $\rho(gP_{a_{k+1}}) = \rho(g)\rho(P_{a_{k+1}}) \to \rho(g)$ strongly. □

Hence $\overline{A_{00}}$ is a (complete) M-ideal in its bidual by [5, Theorem 4.8.5], with all the consequences that this brings (see e.g. [15, Chapter 3] and [1, Theorem 5.10]). Note that the explicit representation ρ of $\overline{A_{00}}$ given above shows that $\overline{A_{00}}$ is a subalgebra of the compact operators on H (since each $\rho(e_n)$ is compact on H). The characters on $\overline{A_{00}}$ are again the χ_n above of course, since for any such character χ we must have $\chi(e_n) \neq 0$ for some n, and then $\chi(e_m) = 0$ for all other $m \in \mathbb{N}$, so that $\chi = \chi_n$. The spectrum of $\overline{A_{00}}$ is thus the same as the spectrum of A (and equals the spectrum of $\overline{A_{00}}^{**}$ by a point above). Note that the $\rho(P_{a_{k+1}})$ above is a (contractive) spectral resolution of the identity.

Corollary 6.2. *The operator algebra A constructed in Section* 5 *is not compact or weakly compact. Thus A is not an ideal in its bidual.*

Proof. See e.g. [20, 1.4.13] or [1, Lemma 5.1] for the well known equivalence between being weakly compact and being an ideal in the bidual.

The rest follows from results at the end of the Introduction, but we give a direct proof. Assume by way of contradiction that multiplication by g on A (or, for that matter, on $\overline{A_{00}}$) is weakly compact. Suppose that (f_k) is a bai for $\overline{A_{00}}$. Since $\overline{A_{00}}$ is weakly closed, there is a subsequence $gf_{n_k} \to a \in \overline{A_{00}}$ weakly, say. Thus $gf_{n_k}e_m \to ae_m$ in norm for each $m \in \mathbb{N}$. However $gf_{n_k}e_m \to ge_m$ in norm since (f_k) is a bai. Thus $(g-a)e_m = 0$ for every m, yielding the contradiction $g = a \in \overline{A_{00}}$. $\square$

The proof of Theorem 1.1 is now complete, except for Lemma 5.3 and the very last assertion about obtaining $g \notin \overline{A_{00}}^{\rm SOT}$. To see the latter, we will change the Hilbert space A acts on. Suppose that A generates a C^*-algebra B. Then $\overline{A_{00}}$ generates a proper C^*-subalgebra B_0 of B (since if $B_0 = B$, then the cai of $\overline{A_{00}}$ would be a cai for A by [5, Lemma 2.1.7 (2)], and this is false). Let $B \subset B(K)$ be the universal representation, so that B^{**} may be represented as a von Neumann algebra M on K. Then $M = \overline{B}^{w*} = \overline{B}^{\rm SOT}$ by von Neumann's double commutant theorem. If $A \subset \overline{A_{00}}^{\rm SOT}$, then

$$A \subset \overline{B_0}^{\rm SOT} = \overline{B_0}^{w*},$$

so that $B^{**} \cong \overline{B}^{w*} \subset \overline{B_0}^{w*}$. This implies that $B_0^{\perp\perp} = B^{**}$, and we obtain the contradiction $B = B_0$.

Remark 6.3. Probably a modification of our construction in Section 5 would produce a representation in which we would explicitly have $g \notin \overline{A_{00}}^{\rm SOT}$. We had a more complicated construction for Theorem 1.1 in a previous draft for which this perhaps may have been true.

We recall that for a commutative semisimple Banach algebra the following are equivalent: (i) A is a modular annihilator algebra; (ii) the Gelfand spectrum of A is discrete; (iii) no element of A has a nonzero limit point in its spectrum; and (iv) for every $a \in A$, multiplication on A by a is a Riesz operator (see [20, Theorem 8.6.4 and Proposition 8.7.8] and [19, p. 400]). Thus A and $\overline{A_{00}}$ are modular annihilator algebras. It follows that our algebra A is a commutative solution to a problem raised in [7]: is every semisimple modular annihilator algebra with a cai weakly compact? In [8] we found a much simpler (but still deep) noncommutative counterexample to the latter question, an example with some interesting noncommutative features.

Theorem 6.4. *The operator algebras A and $\overline{A_{00}}$ constructed above have the following additional properties:*

(a) *Every maximal ideal in $\overline{A_{00}}$ and every maximal modular ideal in A have a bounded approximate identity.*
(b) *A and $\overline{A_{00}}$ are regular natural Banach function algebras (in the sense of* [10, Section 4.1]*) on $\mathbb{N}$ or on $\{\frac{1}{2^n} : n \in \mathbb{N}\}$.*
(c) *A is not Tauberian, nor is it strongly regular or Ditkin, nor does it satisfy spectral synthesis (see* [10, 19]*) for definitions). On the other hand, $\overline{A_{00}}$ does have all these properties; indeed it is a strong Ditkin algebra.*
(d) *A is a semisimple modular annihilator algebra, while $\overline{A_{00}}$ is a dual algebra in the sense of Kaplansky (see e.g.* [20, Chapter 8]*).*
(e) *The closure of the socle of A (or of $\overline{A_{00}}$) is $\overline{A_{00}}$, and A is not an annihilator algebra in the sense of* [20, Chapter 8].

(f) *A is not nc-discrete in the sense of* [2]*: indeed the support projection of* $\overline{A_{00}}$ *in* A^{**} *is open but not closed.*
(g) *A and its multiplier algebra* $M(A)$ *may be identifed completely isometrically isomorphically with subalgebras of the multiplier algebra* $M(\overline{A_{00}})$*.*

Proof. We prove only some of these assertions, leaving the others as exercises.

(a) The maximal ideals are the annihilators of the e_m, which have as a bai $(x_n - x_n e_m)$, where (x_n) is a cai for A or A_0.

(b) These follow easily from the definitions and the identification of the characters of these algebras.

(c) First, A is not Tauberian in the sense of e.g. [19, Definition 4.7.9]), because $\overline{A_{00}} \neq A$. This implies failure of spectral synthesis by e.g. [19, p. 385]. Similar arguments show the other assertions for A. The statements for $\overline{A_{00}}$ are easy or follow from [10, p. 419].

(d) We have already observed this for A. For $\overline{A_{00}}$ this follows from [10, Proposition 4.1.35] or from the observation whose proof we omit that for a natural Banach sequence algebra, being 'dual' is equivalent to spectral synthesis holding or to having 'approximate units' [10, Definition 2.9.10].

(e) The assertion for $\overline{A_{00}}$ is clear. If f is a minimal idempotent in $A \setminus A_{00}$, then $fe_n = 0$ for all $n \in \mathbb{N}$ (since $fe_n \in \mathbb{C}\, f \cap \mathbb{C}\, e = (0)$), and so $f = 0$.

(f) This is almost identical to the proof of [8, Corollary 2.13], except that we work with e_n as opposed to the e_{ii}^n there. Note that $\tilde{\rho}(1-p)e_n \neq 0$ for some n because otherwise the (faithful) Gelfand transform of $\tilde{\rho}(1-p)a$ would be zero for all $a \in A$.

(g) In the explicit nondegenerate representation ρ of A given in Section 5 and the start of Section 6, it is clear that $\rho(A)\rho(A_{00}) \subset \rho(\overline{A_{00}})$. More generally, if $T\rho(A) \subset \rho(A)$, then

$$T\rho(\overline{A_{00}}) = T\rho(\overline{A_{00}})\rho(\overline{A_{00}}) \subset \rho(A)\rho(\overline{A_{00}}) \subset \rho(\overline{A_{00}}).$$

A similar assertion holds if $\rho(A)T \subset \rho(A)$. The results now follow from basic facts about multiplier algebras [5, Section 2.5]. □

Remark 6.5. 1) By [1, Theorem 5.10], the dual space of $\overline{A_{00}}$ has no proper closed subspace that norms $\overline{A_{00}}$. On the other hand, one can show that the closure of the span of the characters of A in A^* is a proper norming subspace for A.

2) For our algebra, the interested reader can easily identify the algebras $M_0(\overline{A_{00}})$ and $M_{00}(\overline{A_{00}})$ studied in [19].

3) It is easy to see that g^n generates A for every $n \in \mathbb{N}$. Also, $A/\overline{A_{00}}$ is an interesting commutative radical operator algebra with cai.

Finally we give the illustration promised after Theorem 1.1, of how our examples can be useful in settling open questions in the immediate area. The unitization A^1 of our algebra A solves an old question of Joel Feinstein (see e.g. [13, 14]), and another similar question of Dales. They asked (in language explained for example in [10, Section 4.1]): If A is a regular unital Banach function algebra on its compact character space X, and if E is a closed subset of X such that the ideal M_E of functions in A vanishing on E has a bai (for regular uniform algebras this is equivalent to E being a p-set or generalized peak set), then is E a set of synthesis? That is, is the ideal J_E of functions in A with a compact support which is disjoint from E dense in M_E? Feinstein's question was the case that E is a singleton $\{x\}$; in this case it is equivalent to ask if A is 'strongly regular' at x. (If A is a uniform algebra, then this is related to an even older important open problem, which would have remarkable consequences if true.) Feinstein also asked if M_x has a bai for every $x \in X$, then is A 'strongly regular'; that is, is every point in X a set of synthesis?

As we have said in Theorem 6.4, our algebra A is a separable regular Banach function algebra, and by facts in [10, Section 4.1] we have that the unitization A^1 is also a regular unital Banach function algebra on the one-point compactification $\{0\} \cup \{\frac{1}{2^n} : n \in \mathbb{N}\}$. It is clear that if E is the single adjoined point, then $M_E = A$ has a cai, but $J_E = A_{00}$ is not dense in $M_0 = A$, answering the questions in the last paragraph in the negative.

7. A lower estimate on $\left\|\gamma_1^{(n)}\right\|_2^{(n)}$

We now return to the task of proving Lemma 5.3.

Lemma 7.1. *Given growth conditions on our underlying sequences, the following is true: for all $n \in \mathbb{N}$, we have $(\left\|\gamma_1^{(n)}\right\|_2^{(n)})^2 \geq \frac{1}{2} \cdot \prod_{j=1}^n \frac{(j+1)^2}{2j+1}$.*

Proof. Consider index functions $\mathbf{i} \in \mathcal{I}^{(n)}$ such that $\mathbf{i}(s) = 0$ unless $s = s_j = \frac{j}{j+1} 2^{a_j^2} \gamma_{a_j}^{(n)}$ for some $j = 1, \ldots, n$, and for each such s_j, we have $\mathbf{i}(s_j) < a_j$. For such an $\mathbf{i}$, the product $\mathbf{s}^{\mathbf{i}} \gamma_1^{(n)}$ is given by the formula

$$\mathbf{s}^{\mathbf{i}} \gamma_1^{(n)} = \prod_{j=1}^n (\frac{j}{j+1} 2^{a_j^2})^{i_j} \gamma_{1+\sum_{j=1}^n a_j i_j}^{(n)},$$

where $i_j = \mathbf{i}(s_j)$. The value of $\phi_n(\gamma_{1+\sum_{j=1}^n a_j i_j}^{(n)})$ is given by (5.1), and its value is

$$\prod_{j=1}^n 2^{-a_j^2 i_j} (1 - i_j/a_j).$$

So the sum, for these $\mathbf{i} \in \mathcal{I}^{(n)}$, of $|\phi_n(\mathbf{s}^{\mathbf{i}} \gamma_1^{(n)})|^2$, is precisely

$$\sum_{\substack{i_j = 0, \ldots, a_j - 1 \\ j = 1, \ldots, n}} (\prod_{j=1}^n (\frac{j}{j+1})^{i_j} (1 - i_j/a_j))^2 = \prod_{j=1}^n \sum_{i=0,\ldots,a_j-1} (\frac{j}{j+1})^{2i} (1 - i/a_j)^2. \tag{7.1}$$

We may assume that a_j is very large indeed compared to j. We claim that growth conditions will ensure that this quantity will be, for each n, at least half of the sum

$$\prod_{j=1}^n \sum_{\substack{i \in \mathbb{N}_0 \\ j=1,\ldots,n}} (\frac{j}{j+1})^{2i} = \prod_{j=1}^n \frac{1}{1 - (j/j+1)^2} = \prod_{j=1}^n \frac{(j+1)^2}{2j+1}.$$

To see this, let us choose positive $h_k < 1$ such that $\prod_{k=1}^m h_k > 1/2$ for all $m \in \mathbb{N}$. Our product (7.1) will be at least half of the product of $(j+1)^2/(2j+1)$, provided that for every $j = 1, \cdots, n$, we have

$$\sum_{i=0}^{a_j - 1} (\frac{j}{j+1})^{2i} (1 - \frac{i}{a_j})^2 \geq h_j \cdot \sum_{i=0}^{\infty} (\frac{j}{j+1})^{2i}.$$

The sum $\sum_i (j/j+1)^{2i}$ converges, and we may think of it as an integral with respect to counting measure. If $f_a(i) = 1 - a/i$ for $i < a$ and is zero for $i \geq a$, then (f_a)

is uniformly bounded and converges to 1 pointwise. Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{a\to\infty}\sum_i (j/j+1)^{2i} f_a(i)^2 = \sum_i (j/j+1)^{2i}.$$

If each a_j is chosen large enough we therefore have for $j = 1, \ldots, n$ that

$$\sum_i (j/j+1)^{2i} f_{a_j}(i)^2 = \sum_i (j/j+1)^{2i}(1-\frac{i}{a_j})^2 \geq (1-h_j)\sum_i (j/j+1)^{2i}.$$

This proves the claim. Finally, $(\left\|\gamma_1^{(n)}\right\|_2^{(n)})^2 \geq \frac{1}{2}\cdot\prod_{j=1}^n \frac{(j+1)^2}{2j+1}$, as desired. □

8. Strategy for an upper estimate for $\left\|\gamma_0^{(n)}\right\|_2^{(n)}$

In the remainder of our paper we strive for an upper estimate for $\left\|\gamma_0^{(n)}\right\|_2^{(n)}$. Now

$$(\left\|\gamma_0^{(n)}\right\|_2^{(n)})^2 = (\left\|\gamma_1^{(n)}\right\|_2^{(n)})^2 + \sum_{\mathbf{i}\in\mathcal{I}_0^{(n)}} |\phi_n(\mathbf{s^i})|^2, \tag{8.1}$$

where $\mathcal{I}_0^{(n)} = \{\mathbf{i}\in\mathcal{I}^{(n)} : \mathbf{i}(\gamma_1^{(n)}) = 0\}$. Let us write

$$\mathcal{I}_0^{(n)} = \mathcal{I}_1^{(n)} \cup \mathcal{I}_2^{(n)} \cup \mathcal{I}_3^{(n)},$$

with

$$\mathcal{I}_1^{(n)} = \{\mathbf{i}\in\mathcal{I}^{(n)} : \mathbf{i}(\gamma_1^{(n)}) = \mathbf{i}(a_i^{-1}e_i) = 0 \text{ for all } i, \text{ and } |\mathbf{i}| = \sum_s \mathbf{i}(s) < \sqrt{a_{n+1}}\},$$

and

$$\mathcal{I}_2^{(n)} = \{\mathbf{i}\in\mathcal{I}^{(n)} : \mathbf{i}(\gamma_1^{(n)}) = \mathbf{i}(a_i^{-1}e_i) = 0 \text{ for all } i, \text{ but } |\mathbf{i}| \geq \sqrt{a_{n+1}}\},$$

and

$$\mathcal{I}_3^{(n)} = \{\mathbf{i}\in\mathcal{I}^{(n)} : \mathbf{i}(\gamma_1^{(n)}) = 0, \text{ but } \mathbf{i}(a_i^{-1}e_i) > 0 \text{ for some } i\}.$$

The main contribution towards the sum (8.1) that we must investigate is from the sum over $\mathbf{i}\in\mathcal{I}_1^{(n)}$. We will estimate this in the lengthy Section 9. In the much easier Sections 10 and 11 we estimate the contribution from $\mathcal{I}_2^{(n)}$ and $\mathcal{I}_3^{(n)}$ respectively, and finally in Section 12 we summarize why this proves our main result.

9. Bound on $\sum_{\mathbf{i}\in\mathcal{I}_1^{(n)}} |\phi_n(\mathbf{s^i})|^2$

Let $\mathbf{i}\in\mathcal{I}_1^{(n)}$. Write

$$E(\mathbf{i}) = \{j\in[1,n] : \mathbf{i}(\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)}) > a_j\}.$$

Let $\eta(\mathbf{i})$ be the set whose elements are of the form

$$\sum_{j\in E(\mathbf{i})}(\lambda_j a_j) + \sum_{j=1}^n(\mu_j + \nu_j a_j),$$

for integers

$$\lambda_j = \mathbf{i}(\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)}), \quad 0 \leq \nu_j \leq \mu_j = \mathbf{i}(j(2^{a_j^2}\gamma_{1+a_j}^{(n)} - \gamma_1^{(n)})), \nu_j\in\mathbb{N}_0.$$

Lemma 9.1. *Let* $\mathbf{i} \in \mathcal{I}_1^{(n)}$. *If* $\phi_n(\mathbf{s}^{\mathbf{i}}) \neq 0$, *then the set* $\eta(\mathbf{i})$ *defined above must contain a positive element of the set* $1 + \Lambda_n - \Lambda_n$.

Proof. When $0 \leq k < a_{n+1} - a_n$, we have $\phi_n(\gamma_k^{(n)}) = 0$ unless $k \in 1 + \Lambda_n$ by (5.1) (of course things are more complicated for larger k). Writing $\lambda_j = \mathbf{i}(\frac{j}{j+1} 2^{a_j^2} \gamma_{a_j}^{(n)})$, the product $\prod_{j \notin E(\mathbf{i})} (\frac{j}{j+1} 2^{a_j^2} \gamma_{a_j}^{(n)}))^{\lambda_j}$ is a multiple $\lambda \cdot \gamma_k^{(n)}$, where $k = \sum_{j \notin E(\mathbf{i})} \lambda_j a_j \in \Lambda_n$. The full product $\mathbf{s}^{\mathbf{i}}$ is equal to

$$\mathbf{s}^{\mathbf{i}} = \lambda \cdot \gamma_k^{(n)} \cdot \prod_{j \in E(\mathbf{i})} (\frac{j}{j+1} 2^{a_j^2} \gamma_{a_j}^{(n)})^{\lambda_j} \cdot \prod_{j=1}^{n} (j(2^{a_j^2} \gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}))^{\mu_j},$$

where again, $\mu_j = \mathbf{i}(j(2^{a_j^2} \gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}))$. This is a linear combination of vectors $\gamma_m^{(n)}$ for $m \in k + \eta(\mathbf{i})$. Furthermore, since $|\mathbf{i}| < \sqrt{a_{n+1}}$, given a growth condition asserting a_{n+1} large compared to a_n, there is never any vector $\gamma_m^{(n)}$ involved when $m \geq a_{n+1} - a_n$. So the set $k + \eta(\mathbf{i})$ must meet the set $1 + \Lambda_n$, hence the result. $\square$

Having got Lemma 9.1, we want to separate the cases when $1 \in \eta(\mathbf{i})$ from the cases when $\eta(\mathbf{i})$ only contains larger elements of the set

$$1 + \Lambda_n - \Lambda_n = \{1 + \sum_{i=1}^{n} t_i a_i : -a_i \leq t_i \leq a_i\}.$$

Let us write $m_0(\mathbf{i}) = \min(\eta(\mathbf{i}) \cap (1 + \Lambda_n - \Lambda_n))$, and let us begin with the more challenging case when $m_0 = m_0(\mathbf{i}) > 1$. We can then write $m_0 = 1 + \sum_{i=1}^{r} a_i t_i$ with $t_r > 0$ and $-a_j \leq t_j \leq a_j$ for all j. In particular, $m_0 \leq 1 + \xi_r$. With λ_j and μ_j as above, we can write

$$m_0 = \sum_{j \in E(\mathbf{i})} a_j \lambda_j + \sum_{j=1}^{n} (\mu_j + a_j \nu_j)$$

with $0 \leq \nu_j \leq \mu_j$. But then, we must have $\nu_j = 0$ for $j > r$; otherwise the value of m_0 will be too big. Indeed $\nu_r = 0$ too, or we can get a smaller element of $\eta(\mathbf{i}) \cap (1 + \Lambda_n - \Lambda_n)$ by considering $m_0 - a_r$. Again, we cannot have $j \in E(\mathbf{i})$ for any $j \geq r$; otherwise the value of m_0 is again too big (these are j such that $\lambda_j > a_j$). So, $E(\mathbf{i}) \subset [1, r)$ and

$$m_0 = \sum_{j \in E(\mathbf{i}) \subset [1,r)} \lambda_j a_j + \sum_{j=1}^{r-1} (\mu_j + \nu_j a_j) + \sum_{j=r}^{n} \mu_j. \tag{9.1}$$

Let us consider the vector

$$x = (\gamma_1^{(n)})^{\sum_{j=r}^{n} \mu_j} \cdot \prod_{j=1}^{r-1} (\frac{j}{j+1} 2^{a_j^2} \gamma_{a_j}^{(n)})^{\lambda_j} \cdot (j(2^{a_j^2} \gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}))^{\mu_j}. \tag{9.2}$$

Given a growth condition, we can certainly assume that $m_0 \in [(t_r - \frac{1}{4}) a_r, (t_r + \frac{1}{4}) a_r]$. For which $i \in [0, a_{n+1} - a_n)$ does x have a nonzero coefficient for $\gamma_i^{(n)}$? (We may refer to the set of such i as "the γ-support of x"). Where does the γ-support of x lie? From (9.2), the generic element of that support is a sum

$$m = \sum_{r=1}^{n} (\mu_j + \nu'_j a_j) + \sum_{j=1}^{r-1} \lambda_j a_j, \qquad 0 \leq \nu'_j \leq \mu_j.$$

The difference between m_0 (as given by (9.1)) and this expression is a sum of the form

$$\sum_{j=1}^{r-1}(\nu'_j - \nu_j)a_j, \qquad 0 \le \nu'_j \le \mu_j,$$

plus the sum $\sum_{j\in[1,r)\setminus E(\mathbf{i})} \lambda_j a_j$. The second sum cannot be negative, but in the worst case might be as large as ξ_{r-1}. Write $m'_0 = \sum_{r=1}^{n}(\mu_j + \nu'_j a_j) + \sum_{j\in E(\mathbf{i})} \lambda_j a_j$. The ratio m'_0/m_0 is in $[1/(1+a_{r-1}), 1+a_{r-1}]$, and we have $m'_0 \le m \le m'_0 + \xi_{r-1}$. So the γ-support of x is contained in

$$[m_0/(1+a_{r-1}), m_0(1+a_{r-1}) + \xi_{r-1}) \subset (a_r/2a_{r-1}, 2a_r^2 a_{r-1}),$$

given a growth condition.

Let us write

$$\tau = \sum_{i=1}^{r-1}\lambda_i a_i + \sum_{i=1}^{n}\mu_i, \tag{9.3}$$

noting that τ is the minimum of the γ-support of x.

Given that the vector x is γ-supported well to the right of zero, let us introduce a Banach algebra norm $\|\cdot\|_\gamma$ to make use of this. The l_1 version of this is

$$\left\| \sum_{i=0}^{a_{n+1}-a_n-1} y_i\gamma_i^{(n)} \right\|_\gamma = \sum_{i=0}^{a_{n+1}-a_n-1} |y_i|.$$

We have

$$\|x\|_\gamma \le \prod_{j=1}^{r-1}(\frac{j}{j+1}2^{a_j^2})^{\lambda_j}(j(2^{a_j^2}+1))^{\mu_j}.$$

For a reasonable bound on this, let us write $\sum_{j=1}^{r-1}\lambda_j + \mu_j = L$, so the sum of all the indices λ_j, μ_j involved in the last product is L. Since the largest possible power is $(r-1)(2^{a_{r-1}^2}+1)$ and $\|\cdot\|_\gamma$ is a Banach algebra norm, we get

$$\|x\|_\gamma \le ((r-1)(2^{a_{r-1}^2}+1))^L. \tag{9.4}$$

The vector $\mathbf{s^i}$ is equal to

$$\begin{aligned}
&\prod_{j=1}^{n}(\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)})^{\lambda_j}(j(2^{a_j^2}\gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}))^{\mu_j} \\
&\qquad = x\cdot\prod_{j=r}^{n}(\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)})^{\lambda_j}(j(2^{a_j^2}\gamma_{a_j}^{(n)} - \gamma_0^{(n)}))^{\mu_j} \\
&\qquad = x'\cdot\prod_{j=r+1}^{n}(\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)})^{\lambda_j}(j(2^{a_j^2}\gamma_{a_j}^{(n)} - \gamma_0^{(n)}))^{\mu_j},
\end{aligned} \tag{9.5}$$

where

$$x' = x\cdot(\frac{r}{r+1}2^{a_r^2}\gamma_{a_r}^{(n)})^{\lambda_r}(r(2^{a_r^2}\gamma_{a_r}^{(n)} - \gamma_0^{(n)}))^{\mu_r}. \tag{9.6}$$

Now $\sum_{j=1}^{n}\mu_j \le m_0 \le 1+\xi_r$ from (9.1) and the definitions of r and ξ_r. So for each $j = r+1,\dots,n$ we have $\mu_j \le 1+\xi_r$ and $\lambda_j + \mu_j \le a_j + 1 + \xi_r < 2a_j$, since $j \notin E(\mathbf{i})$. When $j = r$ we have $\lambda_j \le a_j$, again because $r \notin E(\mathbf{i})$, but we must be content with the estimate $\mu_r \leq 1+\xi_r$ above. So $\lambda_r+\mu_r \leq 1+a_r+\xi_r \leq 2a_r^2$, given a growth condition.

Lemma 9.2. *Given growth conditions, the following is true. For r as above, and any nonnegative integer $\lambda \leq 2a_r^2$, and any*

$$x \in lin\{\gamma_j^{(n)} : \lambda a_r + a_r/2a_{r-1} < j \leq (\lambda+1)a_r + a_r/2a_{r-1}\},$$

we have

$$|\phi_n(x)| \leq 2^{-(\lambda+1)a_r^2}\|x\|_\gamma. \tag{9.7}$$

Furthermore, if $\lambda > 1+a_r$, then $\phi_n(x)=0$.

Proof. Equation (9.7) is equivalent to

$$|\phi_n(\gamma_i^{(n)})| \leq 2^{-(1+\lambda)a_r^2}, \qquad i \in (\lambda a_r + a_r/2a_{r-1}, (\lambda+1)a_r + a_r/2a_{r-1}].$$

By (5.1), the left hand side is zero unless $i \in 1+\Lambda_n$. But, given a growth condition, we can assume $a_r/2a_{r-1} > \xi_{r-1} = \max\Lambda_{r-1}$, so the element of $1+\Lambda_n$ involved must be at least $1+(\lambda+1)a_r$. Equation (5.1) then gives $|\phi_n(\gamma_i^{(n)})| \leq 2^{-(1+\lambda)a_r^2}$ as required. If $\lambda > 1+a_r$ we have $i > 1+\xi_r$, so the least element of $1+\Lambda_n$ available would be $1+a_{r+1}$. But given a growth condition, we can certainly assume that the absolute upper bound $i \leq (2a_r^2+1)a_r + a_r/2a_{r-1}$ is less than a_{r+1}, so for $\lambda > 1+a_r$ we have $\phi_n(x)=0$. ☐

We now use the lemma to estimate $|\phi_n(x')|$, where x' is as in (9.6). Define t to be the nonnegative integer with

$$\tau \in [ta_r, (t+1)a_r), \tag{9.8}$$

where τ is the minimum of the support of x as in (9.3) and (9.2). We will have $0 \leq t \leq a_r$ because from (9.1),

$$\tau \leq m_0 + \sum_{i\in[1,r)\setminus E(\mathbf{i})} \lambda_r a_r \leq m_0 + \xi_{r-1},$$

and then

$$m_0 \leq 1+\xi_r = 1+\xi_{r-1}+a_r^2,$$

so $\tau \leq 1+2\xi_{r-1}+a_r^2 < (a_r+1)a_r$, given a growth condition.

The vector x given by (9.2) is γ-supported on $(\max(a_r/2a_{r-1}, ta_r), 2a_r^2 a_{r-1})$, so by applying (9.7) for various λ and summing the results, we find that

$$|\phi_n(x)| \leq 2^{-a_r^2\max(1,t)}\|x\|_\gamma \leq 2^{-a_r^2\max(1,t)} \cdot ((r-1)(2^{a_{r-1}^2}+1))^L, \tag{9.9}$$

where $L = \sum_{i=1}^{r-1}\lambda_i+\mu_i \leq (t+1)a_r$, by (9.4). For all $\lambda \leq a_r^2$ it is easy to argue that

$$|\phi_n(\gamma_{\lambda a_r}\cdot x)| \leq 2^{-a_r^2(\lambda+\max(1,t))} \cdot ((r-1)(2^{a_{r-1}^2}+1))^L. \tag{9.10}$$

Given a growth condition, we may replace the bounds on the right of (9.9) and (9.10) by $2^{-a_r^2\max(1/2,t-1/2)}$ and $2^{-a_r^2(\lambda+\max(1/2,t-1/2))}$ respectively. Accordingly the vector $x' = x\cdot(\frac{r}{r+1}2^{a_r^2}\gamma_{a_r}^{(n)})^{\lambda_r}(r(2^{a_r^2}\gamma_{a_r}^{(n)}-1))^{\mu_r}$ from (9.6) satisfies

$$|\phi_n(x')| \leq 2^{-a_r^2\max(1/2,t-1/2)} \cdot (\frac{r}{r+1})^{\lambda_r}(2r)^{\mu_r}.$$

Crudely, we may estimate $\mu_r \leq \tau \leq a_r(t+1)$ (from (9.8) and the definition of τ), and then we have

$$2^{-a_r^2\max(1/2,t-1/2)} \cdot (2r)^{\mu_r} \leq 2^M,$$

where

$$M = (1 + \log_2 r)a_r(t+1) - a_r^2 \max(1/2, t-1/2).$$

Now a growth condition on the sequence (a_r) will ensure that $(1+\log_2 r)a_r < a_r^2/12$ for every r, and since $(t+1) \leq 4\max(1/2, t-1/2)$ for any $t \in \mathbb{N}_0$, we will then have

$$(1 + \log_2 r)a_r(t+1) - a_r^2 \max(1/2, t-1/2) \leq -\frac{2}{3}a_r^2 \max(1/2, t-1/2).$$

But this equals

$$-a_r^2 \max(1/3, (2t-1)/3) \leq -a_r^2 \max(1/3, t/3).$$

So we have

$$|\phi_n(x')| \leq 2^{-a_r^2 \max(1/3,t/3)} \cdot (\frac{r}{r+1})^{\lambda_r}. \tag{9.11}$$

Lemma 9.3. *If $w \in lin\{\gamma_i^{(n)} : 0 \leq i < a_{r+1}\}$, and $\rho_j \in \mathbb{N}_0$, for $0 \leq \rho_j \leq 2a_j$ for $j = r+1, \ldots, n$, then writing $y = w \cdot \prod_{j=r+1}^n (\gamma_{a_j}^{(n)})^{\rho_j}$, we also have*

$$\phi_n(y) = \phi_n(w) \cdot \prod_{j=r+1}^{n} 2^{-\rho_j a_j^2}(1 - \rho_j/a_j)_+, \tag{9.12}$$

where as usual, t_+ denotes the maximum $\max(t, 0)$.

Proof. Since $w \in \text{lin}\{\gamma_i^{(n)} : 0 < i < a_{r+1}\}$, we refer to (5.1) to find $\phi_n(\gamma^{(n)}_{i+\sum_{j=r+1}^n \rho_j a_j})$ for such i (and $\rho_j \leq 2a_j$), and it is $\phi_n(\gamma_i^{(n)}) \cdot \prod_{j=r+1}^n 2^{-a_j^2\rho_j}(1-\rho_j/a_j)_+$, given the usual growth condition which ensures that, for $\rho_j \leq 2a_j$, $i + \sum_{j=r+1}^n \rho_j a_j$ is not in $1 + \Lambda_n$ unless $i \in \Lambda_n$ and all the $\rho_j \leq a_j$. Equation (9.12) follows. ∎

Equation (9.12) can also be written as

$$\phi_n(w \cdot \prod_{j=r+1}^{n} (2^{a_j^2}\gamma_{a_j}^{(n)})^{\rho_j}) = \phi_n(w) \cdot \prod_{j=r+1}^{n} (1 - \rho_j/a_j)_+.$$

So if $\lambda_j, \mu_j \leq a_j$, and we write

$$y' = w \cdot \prod_{j=r+1}^{n} (2^{a_j^2}\gamma_{a_j}^{(n)} - \gamma_0^{(n)})^{\mu_j}(2^{a_j^2}\gamma_{a_j}^{(n)})^{\lambda_j},$$

we will then have

$$\phi_n(y') = \sum_{\substack{\alpha_j = 0,\ldots,\mu_j \\ j=r+1,\ldots,n}} \phi_n\Bigg(w \cdot \prod_{j=r+1}^{n} \Bigg((2^{a_j^2}\gamma_{a_j}^{(n)})^{\lambda_j+\alpha_j}(-1)^{\mu_j-\alpha_j}\binom{\mu_j}{\alpha_j}\Bigg)\Bigg)$$

$$= \sum_{\substack{\alpha_j = 0,\ldots,\mu_j \\ j=r+1,\ldots,n}} \phi_n(w) \cdot \prod_{j=r+1}^{n} (-1)^{\mu_j-\alpha_j}\binom{\mu_j}{\alpha_j}(1 - \frac{\alpha_j + \lambda_j}{a_j})_+. \tag{9.13}$$

We next note that, given a growth condition, the fact that x (as in (9.2)) is γ-supported on $[0, 2a_r^2 a_{r-1})$ tells us that x' (as in (9.6)) is supported on $[0, a_{r+1})$. We also recall that $m_0 \leq 1 + \xi_r$, where m_0 is as in (9.1), so

$$\mu_j \leq m_0 \leq 1 + \xi_r < a_j, \qquad j > r,$$

given a growth condition. Let us therefore apply (9.13) to obtain $\phi_n(\mathbf{s}^{\mathbf{i}})$ as a multiple of $\phi_n(x')$, using equation (9.5):

$$\phi_n(\mathbf{s}^{\mathbf{i}}) = \phi_n\Bigg(x' \cdot \prod_{j=r+1}^{n} (j(2^{a_j^2}\gamma_{a_j}^{(n)} - \gamma_0^{(n)}))^{\mu_j} (\frac{j}{j+1} 2^{a_j^2}\gamma_{a_j}^{(n)})^{\lambda_j}\Bigg).$$

But this equals

$$\sum_{\substack{\alpha_j=0,\dots,\mu_j\\ j=r+1,\dots,n}} \phi_n(x') \cdot \prod_{j=r+1}^{n} (-1)^{\mu_j-\alpha_j}\binom{\mu_j}{\alpha_j}(1-\frac{\alpha_j+\lambda_j}{a_j})_+,$$

which in turn equals

$$\phi_n(x') \cdot \prod_{j=r+1}^{n} j^{\mu_j}(\frac{j}{j+1})^{\lambda_j}\eta_j, \quad \text{where } \eta_j = \sum_{\alpha=0}^{\mu_j}(-1)^{\mu_j-\alpha}\binom{\mu_j}{\alpha}(1-\frac{\alpha+\lambda_j}{a_j})_+.$$

Putting our estimate (9.11) into this equation, we have

$$|\phi_n(\mathbf{s}^{\mathbf{i}})| \le 2^{-a_r^2 \max(1/3,t/3)} \cdot (\frac{r}{r+1})^{\lambda_r} \cdot \prod_{j=r+1}^{n} j^{\mu_j}(\frac{j}{j+1})^{\lambda_j}|\eta_j|. \tag{9.14}$$

We can estimate $|\eta_j|$ as follows. If $\lambda_j + \mu_j \le a_j$, there is no need to estimate: we have $\eta_j = \sum_{\alpha=0}^{\mu_j}(-1)^{\mu_j-\alpha}\binom{\mu_j}{\alpha}(1-\frac{\alpha+\lambda_j}{a_j})$. This is $1-\lambda_j/a_j$ if $\mu_j = 0$, and is $-1/a_j$ if $\mu_j = 1$. It is zero if $\mu_j > 1$, by a binomial series argument, or because second and higher differences of the sequence $(1-(\alpha+\lambda_j)/a_j)_{\alpha=0}^{\infty}$ are zero. If $\lambda_j+\mu_j > a_j$, we estimate as follows: we must have $\lambda_j \ge a_j - \xi_r$ because $\mu_j \le 1+\xi_r$, so $(1-(\alpha+\lambda_j)/a_j)_+ \le \xi_r/a_j$ for all $\alpha \ge 0$, so $|\eta_j| \le 2^{\mu_j}\xi_r/a_j \le 2^{1+\xi_r}\xi_r/a_j \le a_j^{-2/3}$, given a growth condition. We are now in a position to prove:

Lemma 9.4. *Given growth conditions, one has*

$$\sum_{\substack{\mathbf{i}\in\mathcal{I}_1^{(n)}\\ m_0(\mathbf{i})>1}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le \prod_{j=1}^{n} \frac{(j+1)^2}{2j+1}, \qquad n \in \mathbb{N}.$$

Proof. The full sum over $\mathbf{i} \in \mathcal{I}_1^{(n)}, m_0(\mathbf{i}) > 1$ is a sum, from $r = 1$ to n, of contributions involving $\mathbf{i}$ with $m_0(\mathbf{i}) = 1 + \sum_{j=1}^{r} t_j a_j$, and $t_r > 0$. For a fixed r, we further consider contributions for fixed λ_j $(j = 1,\dots,n)$, μ_j $(j = 1,\dots,r)$ and fixed $s = \sum_{j=r+1}^{n}\mu_j$ (where as usual, $\lambda_j = \mathbf{i}(\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)})$ and $\mu_j = \mathbf{i}(j(2^{a_j^2\gamma_{a_j}^{(n)}-\gamma_0^{(n)}}))$. So let us write $\mathcal{I}_1^{(n,r,\lambda_1\dots\lambda_n,\mu_1\dots\mu_r,s)}$ for the set of $\mathbf{i} \in \mathcal{I}_1^{(n)}$ with $m_0(\mathbf{i}) = 1 + \sum_{j=1}^{r} t_j a_j$, and $t_r > 0$, and the given values λ_j $(j = 1,\dots,n)$, μ_j $(j = 1,\dots,r)$ and $s = \sum_{j=r+1}^{n}\mu_j$.

Once these are all fixed, we know the vectors x and x' as in (9.2) and (9.6), and also the constants τ, t and L as in (9.3) and (9.9). For each $j > r$, the values $|\eta_j|$ are determined by λ_j and μ_j, as described below in (9.14). Given λ_j, the product $j^{\mu_j}|\eta_j|$ can take the value $1-\lambda_j/a_j$ once (when $\mu_j = 0$), and the value j/a_j once (when $\mu_j = 1$), and values up to $j^{\mu_j}a_j^{-2/3}$ for any $\mu_j = 1,\dots,1+\xi_r$. The sum of the squares of all such values is at most

$$1 + (j/a_j)^2 + (1+\xi_r)j^{2+2\xi_r}a_j^{-4/3} \le 1 + a_j^{-1}$$

for all $j > r$, given a growth condition. So the sum of the products

$$\prod_{j=r+1}^{n} j^{2\mu_j}(\frac{j}{j+1})^{2\lambda_j}|\eta_j|^2,$$

for various μ_j $(j = r+1, \ldots, n)$ with $\sum_{j=r+1}^{n}\mu_j = s$, is at most

$$\prod_{j=r+1}^{n}(\frac{j}{j+1})^{2\lambda_j}(1 + a_j^{-1}).$$

Writing $\mathcal{I}_1^{(\cdot)}$ for $\mathcal{I}_1^{(n,r,\lambda_1\ldots\lambda_n,\mu_1\ldots\mu_r,s)}$, we then get

$$\sum_{\mathbf{i}\in\mathcal{I}_1^{(\cdot)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le 2^{-a_r^2\max(2/3,2t/3)} \cdot (\frac{r}{r+1})^{2\lambda_r} \cdot \prod_{j=r+1}^{n}(\frac{j}{j+1})^{2\lambda_j}(1 + a_j^{-1})$$

from (9.14). We can sum this over all possible λ_j $(j = r, \ldots, n)$: writing $\overline{\mathcal{I}}_1^{(\cdot)} = \overline{\mathcal{I}}_1^{(n,r,\lambda_1\ldots\lambda_{r-1},\mu_1\ldots\mu_r,s)}$ for the union of all sets $\mathcal{I}_1^{(n,r,\lambda_1\ldots\lambda_n,\mu_1\ldots\mu_r,s)}$ as λ_j varies for $j = r, \ldots, n$, we have

$$\sum_{\mathbf{i}\in\overline{\mathcal{I}}^{(\cdot)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le 2^{-a_r^2\max(2/3,2t/3)} \cdot \frac{(r+1)^2}{2r+1} \cdot \prod_{j=r+1}^{n}\frac{(j+1)^2}{2j+1}(1 + a_j^{-1}).$$

The number of ways we can choose $\lambda_1 \ldots \lambda_{r-1}, \mu_1 \ldots \mu_r, s$ in order to get a nonempty set $\overline{\mathcal{I}}_1^{(n,r,\lambda_1\ldots\lambda_{r-1},\mu_1\ldots\mu_r,s)}$ associated with the given value t is less than the number of ways we can pick $2r$ nonnegative integers adding up to an answer $\tau \in [ta_r, (t+1)a_r)$ as in (9.3). Very crudely, this number is no bigger than $((t+1)a_r)^{2r}$. So writing $\overline{\mathcal{I}}_1^{(n,r,t)}$ for the union of all sets $\overline{\mathcal{I}}_1^{(n,r,\lambda_1\ldots\lambda_{r-1},\mu_1\ldots\mu_r,s)}$ such that (9.3) holds, we have

$$\sum_{\mathbf{i}\in\overline{\mathcal{I}}^{(n,r,t)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le ((t+1)a_r)^{2a_r}2^{-a_r^2\max(2/3,2t/3)} \cdot \frac{(r+1)^2}{2r+1} \cdot \prod_{j=r+1}^{n}\frac{(j+1)^2}{2j+1}(1+a_j^{-1}).$$

This is dominated by

$$2^{-a_r^2\max(1/3,t/3)} \cdot \prod_{j=r+1}^{n}\frac{(j+1)^2}{2j+1}(1 + a_j^{-1}),$$

given another growth condition. We can of course assume that $\prod_{j=1}^{\infty}(1 + a_j^{-1}) \le 2$. So

$$\sum_{\mathbf{i}\in\overline{\mathcal{I}}^{(n,r,t)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le 2^{1-a_r^2\max(1/3,t/3)} \cdot \prod_{j=r+1}^{n}\frac{(j+1)^2}{2j+1}.$$

Summing over all $t \in \mathbb{N}_0$ and $r \in [1, n]$ we get

$$\sum_{\substack{\mathbf{i}\in\mathcal{I}_1^{(n)}\\ m_0(\mathbf{i})>1}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le \sum_{r=1}^{n}\sum_{t=0}^{\infty}2^{1-a_r^2\max(1/3,t/3)} \cdot \prod_{j=1}^{n}\frac{(j+1)^2}{2j+1} \le \prod_{j=1}^{n}\frac{(j+1)^2}{2j+1},$$

given another growth condition. Thus the lemma is proved. □

We can now finish this section by polishing off the case when $\mathbf{i} \in \mathcal{I}_1^{(n)}$ but $m_0(\mathbf{i}) = 1$. In that case, since $\sum_{j\in E(\mathbf{i})} \lambda_j a_j + \sum_{j=1}^{n} \mu_j \le m_0$, we must have $E(\mathbf{i}) = \emptyset$ and at most one $\mu_j = 1$. In fact we must have exactly one $\mu_j = 1$, since $0 \notin 1 + \Lambda_n$. So $\mathbf{s}^{\mathbf{i}} = \prod_{j=1}^{n}(\frac{j}{j+1} 2^{a_j^2}\gamma_{a_j}^{(n)})^{\lambda_j} r(2^{a_r^2}\gamma_{1+a_r}^{(n)} - \gamma_1^{(n)})$ for some $r \le n$, and so, when $\lambda_r < a_r$, the reader can check that (5.1) tells us that

$$\phi_n(\mathbf{s}^{\mathbf{i}}) = -\frac{r}{a_r}\prod_{j=1}^{n}(\frac{j}{j+1})^{\lambda_j} \prod_{\substack{j=1,\dots,n\\ j\neq r}} (1 - \lambda_j/a_j).$$

If $\lambda_r = a_r$, we can safely assume that $\phi_n(\gamma_{1+(a_r+1)a_r + \sum_{j\neq r}\lambda_j a_j}) = 0$, and then $\phi_n(\mathbf{s}^{\mathbf{i}}) = 0$ in this case. So we have

$$\sum_{\substack{\mathbf{i}\in\mathcal{I}_1^{(n)}\\ m_0(\mathbf{i})=1}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le \sum_{r=1}^{n} \sum_{\substack{\lambda_j=0,\dots,a_j\\ j=1,\dots,n}} (r/a_r)^2 \prod_{j=1}^{n}(\frac{j}{j+1})^{\lambda_j},$$

which is dominated by

$$\sum_{r=1}^{n}(r/a_r)^2 \cdot \prod_{j=1}^{n}\sum_{\lambda_j=0}^{\infty}(\frac{j}{j+1})^{2\lambda_j} = \sum_{r=1}^{n}(r/a_r)^2 \cdot \prod_{j=1}^{n}\frac{(j+1)^2}{2j+1}.$$

Now $\sum_{r=1}^{\infty}(r/a_r)^2 < 1$, given a mild growth condition, so

$$\sum_{\substack{\mathbf{i}\in\mathcal{I}_1^{(n)}\\ m_0(\mathbf{i})=1}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \le \prod_{j=1}^{n}\frac{(j+1)^2}{2j+1}, \qquad n \in \mathbb{N}.$$

Combining this equation with the previous lemma, we have the result:

Theorem 9.5. *Given growth conditions, we have*

$$\sum_{\mathbf{i}\in\mathcal{I}_1^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 < 2 \cdot \prod_{j=1}^{n}\frac{(j+1)^2}{2j+1},$$

for all $n \in \mathbb{N}$.

10. Bound on $\sum_{\mathbf{i}\in\mathcal{I}_2^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2$

To get this bound, we need to have a good estimate of $|\phi_n(\gamma_j^{(n)})|$ in cases when $\gamma_j^{(n)}$ is not one of the basis vectors for H_n with respect to which ϕ_n is defined directly in (5.1). That is, we need to know about $\phi_n(\gamma_j^{(n)})$ when $j \ge a_{n+1} - a_n$.

Let e_j^* $(a_n < j \le a_{n+1})$ denote the linear functional on H_n with $\langle e_i, e_j^*\rangle = \delta_{i,j}$. A general linear functional $\psi = \sum_{j=1+a_n}^{a_{n+1}} \lambda_j e_j^*$ will have

$$\psi(\gamma_k^{(n)}) = \sum_{j=1+a_n}^{a_{n+1}} \lambda_j 2^{-jk} = p(2^{-k}), \tag{10.1}$$

where $p(t) = \sum_{j=1+a_n}^{a_{n+1}} \lambda_j t^j$ is a polynomial of degree at most a_{n+1}, with t^{1+a_n} a factor of $p(t)$. We will have $p(2^{-k}) = \delta_{i,k}$ when $0 \le k < a_{n+1} - a_n$, if we choose

$p = p_{n,i}$, where

$$p_{n,i}(t) = (2^i t)^{1+a_n} \prod_{\substack{0\le j<a_{n+1}-a_n \\ j\neq i}} \frac{t-2^{-j}}{2^{-i}-2^{-j}}. \tag{10.2}$$

If we write $\psi_{n,i}$ for the corresponding linear functional, we have

$$x = \sum_{i=0}^{a_{n+1}-a_n-1} \langle x, \psi_{n,i}\rangle \gamma_i^{(n)} \tag{10.3}$$

for every $x \in H_n$. This may be seen by checking it on the basis $(\gamma_i^{(n)})_{i=0}^{a_{n+1}-a_n-1}$, using (10.1) and the fact above (10.2). Indeed,

$$\gamma_k^{(n)} = \sum_{i=0}^{a_{n+1}-a_n-1} p_{n,i}(2^{-k})\gamma_i^{(n)}. \tag{10.4}$$

Lemma 10.1. *Given suitable growth conditions on our underlying sequences, we will have* $|\phi_n(\gamma_l)| \le 2^{-l(1+a_n)-a_{n+1}^2/3}$ *for all* $l \in \mathbb{N}$, $l \ge 1+\xi_n$.

Proof. It is enough to show this for $l \ge a_{n+1}-a_n$, since $\phi_n(\gamma_l^{(n)}) = 0$ for $l \in (1+\xi_n, a_{n+1}-a_n)$ by (5.1). Note that ϕ_n is γ-supported on $[0, 1+\xi_n]$. When $k \le 1+\xi_n$, we have by (10.2) that

$$p_{n,k}(2^{-l}) = 2^{(k-l)(1+a_n)} \prod_{\substack{0\le j<a_{n+1}-a_n \\ j\neq k}} \frac{2^{-l}-2^{-j}}{2^{-k}-2^{-j}}.$$

When $j < k$ the factor $|\frac{2^{-l}-2^{-j}}{2^{-k}-2^{-j}}|$ is in $(1,2]$. When $j > k$ we have $|\frac{2^{-l}-2^{-j}}{2^{-k}-2^{-j}}| \le 2^{k-j+1}$. Thus

$$|p_{n,k}(2^{-l})| \le 2^{(k-l)(1+a_n)} \cdot 2^k \cdot \prod_{j=k+1}^{a_{n+1}-a_n-1} 2^{k-j+1}$$

$$= 2^{(k-l)(1+a_n)} \cdot 2^k \cdot 2^{-\frac{1}{2}(a_{n+1}-a_n-k-2)(a_{n+1}-a_n-k-1)} \le 2^{-1-l(1+a_n)-a_{n+1}^2/3}, \tag{10.5}$$

given a suitable growth condition. Now the nonzero coefficients $\phi_n(\gamma_k)$ in (5.1) are positive numbers at most $2^{-\sum_j t_j a_j^2}$, where $k = 1+\sum_j t_j a_j$, $0 \le t_j < a_j$. These are distinct nonnegative powers of 2, so the sum of all the coefficients is at most 2. So (5.1), (10.4), and (10.5) give us $|\phi_n(\gamma_l)| \le 2^{-l(1+a_n)-a_{n+1}^2/3}$. □

Theorem 10.2. *Given growth conditions, we have* $\sum_{\mathbf{i}\in\mathcal{I}_2^{(n)}} |\phi_n(\gamma_i^{(n)})|^2 \le 2^{-2a_{n+1}^2/3}$ *for all* $n \in \mathbb{N}$.

Proof. If we impose the convolution multiplication on $c_{00}(\mathbb{N}_0)$, the norm $\left\|\sum_{i=0}^N \beta_i e_i\right\| = \sum_{i=0}^N 2^{-i(1+a_n)}|\beta_i|$ is an algebra norm. One can define an algebra homomorphism θ from $c_{00}(\mathbb{N}_0)$ into H_n with $\theta(e_i) = \gamma_i^{(n)}$, and Lemma 10.1 can then be rephrased as follows: if $z \in c_{00}$ with $z \in \text{lin}\{e_j : j > \xi_n\}$, then $|\phi_n(\theta(z))| \le 2^{-a_{n+1}^2/3}\|z\|$.

We note that, among the elements of $S_0^{(n)}$, $\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)} = \theta(u_j)$ with

$$\|u_j\| \le 2^{a_j(a_j-1-a_n)} \le 2^{-a_j},$$

and $j(2^{a_j^2}\gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}) = \theta(v_j)$ where

$$\|v_j\| \le 2j \cdot 2^{-a_n} \le 2^{-a_n/2}, \qquad j \le n,$$

given a growth condition. If $\mathbf{i} \in \mathcal{I}_2^{(n)}$ (so $|\mathbf{i}| \ge \sqrt{a_{n+1}}$), we write (as usual)

$$\lambda_j = \mathbf{i}(\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)}), \qquad \mu_j = \mathbf{i}(j(2^{a_j^2}\gamma_{1+a_j}^{(n)} - \gamma_1^{(n)})).$$

Let $w = \prod_{j=1}^n u_j^{\lambda_j} v_j^{\mu_j}$. Then $\mathbf{s}^{\mathbf{i}} = \theta(w)$, and $\|w\| \le 2^{-\sum_{j=1}^n(\lambda_j a_j + \mu_j a_n/2)}$. Also,

$$w \in \operatorname{lin}\{e_i : i \ge \sqrt{a_{n+1}}\} \subset \operatorname{lin}\{e_i : i > 1 + \xi_n\}$$

(given a growth condition). So Lemma 10.1 applies and tells us (in its "rephrased" form) that

$$|\phi_n(\mathbf{s}^{\mathbf{i}})| = |\phi_n(\theta(w))| \le 2^{-a_{n+1}^2/3}\|w\| \le 2^{-a_{n+1}^2/3 - \sum_{j=1}^n(\lambda_j a_j + \mu_j a_n/2)}.$$

So

$$\sum_{\mathbf{i}\in\mathcal{I}_2^{(n)}} |\phi_n(\gamma_i^{(n)})|^2 \le 2^{-2a_{n+1}^2/3} \cdot \sum_{\substack{\lambda_1,\mu_1,\ldots,\lambda_n,\mu_n = 0,\ldots,\infty \\ \sum_j \lambda_j + \mu_j \ge \sqrt{a_{n+1}}}} 2^{-2\sum_{j=1}^n \lambda_j a_j + \mu_j a_n/2}.$$

A mild growth condition ensures that the right hand sum is at most 1 for any n, so we have the required result. ☐

11. An estimate for $\sum_{\mathbf{i}\in\mathcal{I}_3^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2$.

We now turn our attention to the set $\mathcal{I}_3^{(n)}$, which involves index functions $\mathbf{i}$ for which $\mathbf{i}(a_k^{-1}e_k) > 0$ for some $k \in (a_n, a_{n+1}]$. Since the product of distinct e_k is zero, we get $\mathbf{s}^{\mathbf{i}} = 0$ if $\mathbf{i}(a_k^{-1}e_k) > 0$ for two distinct k. If there is one such k, and if the index $\mathbf{i}(a_k^{-1}e_k) = m > 0$, we get

$$\mathbf{s}^{\mathbf{i}} = a_k^{-m} e_k \cdot \mathbf{s}^{\mathbf{i}'} = a_k^{-m} e_k^*(\mathbf{s}^{\mathbf{i}'}) e_k,$$

where $\mathbf{i}'$ is an element of $\mathcal{I}_1^{(n)} \cup \mathcal{I}_2^{(n)}$. Accordingly we get

$$\sum_{\mathbf{i}\in\mathcal{I}_3^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 = \sum_{k=1+a_n}^{a_{n+1}} \sum_{m=1}^{\infty} \sum_{\mathbf{i}'\in\mathcal{I}_1^{(n)}\cup\mathcal{I}_2^{(n)}} a_k^{-2m} |e_k^*(\mathbf{s}^{\mathbf{i}'})\phi_n(e_k)|^2.$$

This is equal to

$$\sum_{k=1+a_n}^{a_{n+1}} (a_k^2 - 1)^{-1} \sum_{\mathbf{i}'\in\mathcal{I}_1^{(n)}\cup\mathcal{I}_2^{(n)}} |e_k^*(\mathbf{s}^{\mathbf{i}'})\phi_n(e_k)|^2,$$

which is dominated (since the e_k^* are characters, hence contractive) by

$$\sum_{k=1+a_n}^{a_{n+1}} (a_k^2 - 1)^{-1} \sum_{\mathbf{i}'\in\mathcal{I}_1^{(n)}\cup\mathcal{I}_2^{(n)}} \left\|\mathbf{s}^{\mathbf{i}'}\right\|_0^2 |\phi_n(e_k)|^2.$$

The c_0 norms of the elements of $S_0^{(n)}$ are listed in the first paragraph of the proof of Theorem 5.4. Writing

$$\varepsilon_j = \left\|\frac{j}{j+1}2^{a_j^2}\gamma_{a_j}^{(n)}\right\|_0 \le 2^{-a_j} \quad \text{and} \quad \varepsilon_j' = \left\|j(2^{a_j^2}\gamma_{1+a_j}^{(n)} - \gamma_1^{(n)})\right\|_0,$$

we have that

$$\varepsilon_j' \leq j(2^{-a_j} + 2^{-1-a_n}) \leq 2j2^{-a_j} \leq 2^{-a_j/2},$$

given a growth condition. Now $\{\mathbf{s}^{\mathbf{i}'} : \mathbf{i}' \in \mathcal{I}_1^{(n)} \cup \mathcal{I}_2^{(n)}\}$ is equal to

$$\{\prod_{j=1}^{n} (\frac{j}{j+1} 2^{a_j^2} \gamma_{a_j}^{(n)})^{\lambda_j} (j(2^{a_j^2} \gamma_{1+a_j}^{(n)} - \gamma_1^{(n)}))^{\mu_j} : \lambda_j, \mu_j \in \mathbb{N}_0, j = 1, \ldots, n\},$$

and so

$$\sum_{\mathbf{i}' \in \mathcal{I}_1^{(n)} \cup \mathcal{I}_2^{(n)}} \left\| \mathbf{s}^{\mathbf{i}'} \right\|_0^2 \leq \sum_{\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n \geq 0} \prod_{j=1}^{n} \varepsilon_j^{2\lambda_j} (\varepsilon_j')^{2\mu_j} \leq \prod_{j=1}^{n} (1 - \varepsilon_j^2)^{-1} (1 - {\varepsilon_j'}^2)^{-1}.$$

By the estimates for $\varepsilon_j, \varepsilon_j'$ above, the latter is dominated by

$$\prod_{j=1}^{n} (1 - 2^{-2a_j})^{-1} (1 - 2^{-a_j})^{-1} < 2,$$

given a growth condition. So,

$$\sum_{\mathbf{i} \in \mathcal{I}_3^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \leq \sum_{k=1+a_n}^{a_{n+1}} 2(a_k^2 - 1)^{-1} |\phi_n(e_k)|^2. \tag{11.1}$$

It is easy to argue from (5.1) that $\sum_j |\phi_n(\gamma_j^{(n)})| \leq 2$. Each $e_k = \sum_{i=1+a_n}^{a_{n+1}} \beta_{k,i} \gamma_i^{(n)}$, where $\beta_{k,i} = \langle e_k, \psi_{n,i} \rangle$ as in (10.3). Now $\langle e_k, \psi_{n,i} \rangle$ is the coefficient of t^k in the polynomial $p_{n,k}(t)$ as in (10.2); a crude estimate is that no coefficient of this polynomial exceeds

$$2^k (1 + a_n) \cdot \prod_{\substack{0 \leq j < a_{n+1} - a_n \\ j \neq k}} \frac{2}{|2^{-k} - 2^{-j}|} \leq 2^k (1 + a_n) \cdot \prod_{j=0}^{a_{n+1} - a_n - 1} 2^{j+2} \leq 2^{k(1+a_n) + \frac{1}{2} a_{n+1}^2}.$$

But this is dominated by $2^{a_{n+1}^2}$, given a mild growth condition. Putting this estimate in (11.1), we have

$$\sum_{\mathbf{i} \in \mathcal{I}_3^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \leq \sum_{k=1+a_n}^{a_{n+1}} 2(a_k^2 - 1)^{-1} 2^{a_{n+1}^2}.$$

Lemma 11.1. *Given growth conditions, we have* $\sum_{\mathbf{i} \in \mathcal{I}_3^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \leq 1$ *for every* $n \in \mathbb{N}$.

Proof. Given the last inequality, all we need to do is demand the growth conditions $a_n \geq n + 1$ so that $a_k^2 > 1 + 2^{2 + a_{k-1}^2}$ for all k, for then, since the (a_k) are strictly increasing, we have $a_k^2 > 1 + 2^{r+1+a_{k-r}^2}$ for all $0 < r < k$. This implies that

$$\sum_{k=1+a_n}^{a_{n+1}} 2(a_k^2 - 1)^{-1} 2^{a_{n+1}^2} \leq \sum_{k=2+n}^{\infty} 2(a_k^2 - 1)^{-1} 2^{a_{n+1}^2} \leq \sum_{r=1}^{\infty} 2 \cdot 2^{-r-1-a_{n+1}^2} 2^{a_{n+1}^2},$$

where $r = k - n - 1$. But the last quantity equals $\sum_{r=1}^{\infty} 2^{-r} = 1$. ∎

12. Conclusions

Theorem 12.1. *Given growth conditions on the underlying sequence* $(a_k)_{k=1}^{\infty}$*, we have* $\sum_{\mathbf{i}\in\mathcal{I}_0^{(n)}} |\phi_n(\mathbf{s}^{\mathbf{i}})|^2 \leq 2 + 2\cdot\prod_{j=1}^{n}\frac{(j+1)^2}{2j+1}$ *for all* $n \in \mathbb{N}$*. Lemma* 5.3 *holds: one has* $\left\|\gamma_0^{(n)}\right\|_2^{(n)} \leq 3\cdot\left\|\gamma_1^{(n)}\right\|_2^{(n)}$ *for all* n*. The operator norm* $\|g\|_{\text{op}}^{(n)} \geq \frac{1}{3}$*. Theorem* 5.4 *is true, as is Theorem* 4.1*, for these choices of the underlying sequence.*

Proof. The first estimate is obtained by summing the estimates given in Theorem 9.5, Theorem 10.2 and Lemma 11.1. Substituting in (8.1), we have

$$(\left\|\gamma_0^{(n)}\right\|_2^{(n)})^2 \leq (\left\|\gamma_1^{(n)}\right\|_2^{(n)})^2 + 2 + 2\cdot\prod_{j=1}^{n}\frac{(j+1)^2}{2j+1}.$$

Applying the lower estimate Lemma 7.1 for $\left\|\gamma_1^{(n)}\right\|_2^{(n)}$, we see that $(\left\|\gamma_0^{(n)}\right\|_2^{(n)})^2$ is dominated by $9(\left\|\gamma_1^{(n)}\right\|_2^{(n)})^2$, and the second estimate follows. Of course the operator norm

$$\|g\|_{\text{op}}^{(n)} \;\geq\; \left\|\, g\ \gamma_0^{(n)}\right\|_2^{(n)} / \left\|\gamma_0^{(n)}\right\|_2^{(n)} = \left\|\gamma_1^{(n)}\right\|_2^{(n)} / \left\|\gamma_0^{(n)}\right\|_2^{(n)} \geq 1/3.$$

Theorem 4.1 and Theorem 5.4 now follow by the argument after Lemma 5.3. □

Acknowledgements

The first author wishes to thank the departments at the Universities of Leeds and Lancaster, and in particular the second author and Garth Dales, for their warm hospitality during a visit in April–May 2013. The authors also gratefully acknowledge support from UK Research Council grant EP/K019546/1 for that visit. The authors thank Garth Dales for a discussion on Banach sequence algebras at that time, and also for providing a copy of the related preprint [11] in October 2014 after their paper had been accepted for publication (although Dales had told them some of the contents of it earlier). After one translates these results into their language, as explained two paragraphs before Theorem 1.1, and compares, it seems that the overlap between this paper and that of Dales and Ülger is quite small: for example Corollary 1.3 is also a consequence of their Proposition 3.1, and the authors also have examples in Section 1 of weakly compact non-Tauberian Banach sequence algebras. The authors also thank Joel Feinstein and Garth Dales for pointing out the application of their result mentioned after Theorem 1.1 and at the end of Section 6, and for pointing out several small typos, grammatical errors, and small misstatements.

References

[1] Melahat Almus, David P. Blecher, and Charles John Read, *Ideals and hereditary subalgebras in operator algebras*, Studia Math. **212** (2012), no. 1, 65–93, DOI 10.4064/sm212-1-5. MR3004167

[2] Melahat Almus, David P. Blecher, and Sonia Sharma, *Ideals and structure of operator algebras*, J. Operator Theory **67** (2012), no. 2, 397–436. MR2928323

[3] E. Azoff and H. Shehada, *Algebras generated by mutually orthogonal idempotent operators*, J. Operator Theory **29** (1993), no. 2, 249–267. MR1284883 (95h:47063)

[4] David P. Blecher, Damon M. Hay, and Matthew Neal, *Hereditary subalgebras of operator algebras*, J. Operator Theory **59** (2008), no. 2, 333–357. MR2411049 (2009m:46074)

[5] David P. Blecher and Christian Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, vol. 30, The Clarendon Press, Oxford University Press, Oxford, 2004. MR2111973 (2006a:46070)

[6] David P. Blecher and Charles John Read, *Operator algebras with contractive approximate identities*, J. Funct. Anal. **261** (2011), no. 1, 188–217, DOI 10.1016/j.jfa.2011.02.019. MR2785898 (2012e:47226)

[7] David P. Blecher and Charles John Read, *Operator algebras with contractive approximate identities, II*, J. Funct. Anal. **264** (2013), no. 4, 1049–1067, DOI 10.1016/j.jfa.2012.11.013. MR3004957

[8] David P. Blecher and Charles John Read, *Operator algebras with contractive approximate identities: weak compactness and the spectrum*, J. Funct. Anal. **267** (2014), no. 6, 1837–1850, DOI 10.1016/j.jfa.2014.06.007. MR3237775

[9] David P. Blecher, Zhong-Jin Ruan, and Allan M. Sinclair, *A characterization of operator algebras*, J. Funct. Anal. **89** (1990), no. 1, 188–201, DOI 10.1016/0022-1236(90)90010-I. MR1040962 (91b:47098)

[10] H. G. Dales, *Banach algebras and automatic continuity*, London Mathematical Society Monographs. New Series, vol. 24, The Clarendon Press, Oxford University Press, New York, 2000. MR1816726 (2002e:46001)

[11] H. G. Dales and A. Ülger, *Approximate identities in Banach function algebras*, Studia Math. **226** (2015), no. 2, 155–187. MR3346077

[12] Nelson Dunford and Jacob T. Schwartz, *Linear operators. Part III, Spectral operators*, with the assistance of William G. Bade and Robert G. Bartle. Wiley Classics Library, John Wiley & Sons, Inc., New York, 1988. Reprint of the 1971 original. MR1009164 (90g:47001c)

[13] J. F. Feinstein, *A note on strong Ditkin algebras*, Bull. Austral. Math. Soc. **52** (1995), no. 1, 25–30, DOI 10.1017/S0004972700014428. MR1344258 (96f:46097)

[14] J. F. Feinstein, *Regularity conditions for Banach function algebras*, Function spaces (Edwardsville, IL, 1994), Lecture Notes in Pure and Appl. Math., vol. 172, Dekker, New York, 1995, pp. 117–122. MR1352226 (96h:46077)

[15] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics, vol. 1547, Springer-Verlag, Berlin, 1993. MR1238713 (94k:46022)

[16] Taqdir Husain, *Orthogonal Schauder bases*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 143, Marcel Dekker, Inc., New York, 1991. MR1109914 (92h:46071)

[17] Irving Kaplansky, *Normed algebras*, Duke Math. J. **16** (1949), 399–418. MR0031193 (11,115d)

[18] J. J. Koliha, *Isolated spectral points*, Proc. Amer. Math. Soc. **124** (1996), no. 11, 3417–3424, DOI 10.1090/S0002-9939-96-03449-1. MR1342031 (97a:46057)

[19] Kjeld B. Laursen and Michael M. Neumann, *An introduction to local spectral theory*, London Mathematical Society Monographs. New Series, vol. 20, The Clarendon Press, Oxford University Press, New York, 2000. MR1747914 (2001k:47002)

[20] Theodore W. Palmer, *Banach algebras and the general theory of *-algebras. Vol. I*, Algebras and Banach algebras, Encyclopedia of Mathematics and its Applications, vol. 49, Cambridge University Press, Cambridge, 1994. MR1270014 (95c:46002)

[21] C. J. Read, *On the quest for positivity in operator algebras*, J. Math. Anal. Appl. **381** (2011), no. 1, 202–214, DOI 10.1016/j.jmaa.2011.02.022. MR2796203 (2012g:47217)

[22] Werner Ricker, *Operator algebras generated by commuting projections: a vector measure approach*, Lecture Notes in Mathematics, vol. 1711, Springer-Verlag, Berlin, 1999. MR1718704 (2001b:47055)

Department of Mathematics, University of Houston, Houston, Texas 77204-3008
E-mail address: dblecher@math.uh.edu

Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, England
E-mail address: read@maths.leeds.ac.uk