A NEW MEAN VALUE PROPERTY FOR HARMONIC FUNCTIONS RELATIVE TO THE DUNKL-LAPLACIAN OPERATOR AND APPLICATIONS

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Abstract. For a root system in $\mathbb{R}^d$ furnished with its Coxeter-Weyl group $W$ and a multiplicity nonnegative function $k$, we consider the associated commuting system of Dunkl operators $D_1, \ldots, D_d$ and the Dunkl-Laplacian $\Delta_k = D_1^2 + \ldots + D_d^2$. This paper studies the properties of the functions $u$ defined on an open $W$-invariant set $\Omega \subset \mathbb{R}^d$ and satisfying $\Delta_k u = 0$ on $\Omega$ (D-harmonicity). In particular, we introduce and give a complete study of a new mean value operator which characterizes D-harmonicity. As applications we prove a strong maximum principle, a Harnack’s type theorem and a Bôcher’s theorem for D-harmonic functions.

1. Introduction

We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle \cdot, \cdot \rangle$ and its associated norm $\|x\| = \sqrt{\langle x, x \rangle}$. For $\alpha \in \mathbb{R}^d \setminus \{0\}$, the reflection $\sigma_\alpha$ with respect to the hyperplane $H_\alpha$ orthogonal to $\alpha$ is given by

$$\forall x \in \mathbb{R}^d, \quad \sigma_\alpha(x) = x - 2 \frac{\langle x, \alpha \rangle}{\|\alpha\|^2} \alpha.$$ 

A finite set $R \subset \mathbb{R}^d \setminus \{0\}$ is called a (reduced) root system if $R \cap \mathbb{R} \alpha = \{ \pm \alpha \}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$ (see [9] for details on root systems). The finite group $W$ generated by the reflections $\sigma_\alpha$, $\alpha \in R$, is called the Coxeter-Weyl group (or the reflection group) of the root system. Then, we fix a $W$-invariant function $k : R \longrightarrow \mathbb{C}$ called the multiplicity function of the root system and we consider the family of commuting operators $D_j$ $(j = 1, \ldots, d)$ defined for $f \in C^1(\mathbb{R}^d)$ by

$$D_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in R^+_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle},$$

where $R_+$ is a positive subsystem. These operators, defined by Dunkl in [3], are of fundamental importance in various areas of mathematics and mathematical physics (see [17] and its references for details).
Throughout the paper, we will assume that $k \geq 0$ and we will need the weight function defined by
\[
\forall x \in \mathbb{R}^d, \quad \omega_k(x) := \prod_{\alpha \in \mathbb{R}^+} |\langle \alpha, x \rangle|^{2k(\alpha)}.
\]
The function $\omega_k$ is $W$-invariant and homogeneous of degree $2\gamma$, where $\gamma := \sum_{\alpha \in \mathbb{R}^+} k(\alpha)$. But the main tool, as far as we are concerned, is the Dunkl intertwining operator $V_k$ which is the unique isomorphism from the space $\mathcal{P}$ of polynomials on $\mathbb{R}^d$ onto itself satisfying (see [5])
\[
\forall j = 1, \ldots, d, \quad D_j V_k = V_k \frac{\partial}{\partial x_j} \quad \text{and} \quad V_k(1) = 1.
\]
This operator has been extended by Trimèche (see [18]) to an isomorphism from $C^\infty(\mathbb{R}^d)$ (carrying its usual Fréchet topology) onto itself satisfying the intertwining relations (1.1). Rösler (see [15]) has obtained the following fundamental integral representation:
\[
\forall f \in C^\infty(\mathbb{R}^d), \quad \forall x \in \mathbb{R}^d, \quad V_k(f)(x) = \int_{\mathbb{R}^d} f(y) d\mu_x(y),
\]
where $\mu_x$ is a probability measure on $\mathbb{R}^d$ with compact support contained in
\[
C(x) := \text{co}\{g x, g \in W\},
\]
the convex hull of the orbit of $x$ under $W$.
Moreover, the Dunkl intertwining operator $V_k$ commutes with the $W$-action (see [17])
\[
\forall f \in C^\infty(\mathbb{R}^d), \quad \forall g \in W, \quad g^{-1} V_k(g.f) = V_k(f),
\]
where $g.f(x) = f(g^{-1} x)$. In terms of Rösler’s measures, this property means that for all $y \in \mathbb{R}^d$ and $g \in W$, $\mu_y$ is the image measure of $\mu_y$ by the map $x \mapsto g^{-1} x$.
In [18], Trimèche introduced the dual Dunkl intertwining operator $^{t}V_k$ defined on $\mathcal{D}(\mathbb{R}^d)$ (the space of $C^\infty$-functions on $\mathbb{R}^d$ with compact support) by
\[
^{t}V_k(f) = \mathcal{F}^{-1}[\mathcal{F}_D(f)], \quad f \in \mathcal{D}(\mathbb{R}^d),
\]
where $\mathcal{F}$ is the classical Fourier transform and $\mathcal{F}_D$ is the Dunkl transform whose precise definition will be recalled in section 2.
This operator is an isomorphism from $\mathcal{D}(\mathbb{R}^d)$ onto itself satisfying
\[
\int_{\mathbb{R}^d}^{t}V_k(f)(x)g(x)dx = \int_{\mathbb{R}^d} f(x)V_k(g)(x)\omega_k(x)dx,
\]
for all $f \in \mathcal{D}(\mathbb{R}^d)$ and $g \in C^\infty(\mathbb{R}^d)$. Note that by replacing $f$ by $(^{t}V_k)^{-1}(f)$ and $g$ by $V_k^{-1}(g)$ in (1.6) we have the following equivalent relation:
\[
\int_{\mathbb{R}^d} f(x)V_k^{-1}(g)(x)dx = \int_{\mathbb{R}^d} (^{t}V_k)^{-1}(f)(x)g(x)\omega_k(x)dx.
\]
Moreover, it is a consequence of the Paley-Wiener theorem for the Dunkl transform that $^{t}V_k$ is support preserving (see [19]), i.e.
\[
\text{supp} \ (f) \subset B(0, a) \iff \text{supp} \ (^{t}V_k(f)) \subset B(0, a).
\]
Throughout the paper we will suppose that the root system is normalized\(^2\) in the sense that \((\alpha, \alpha) = 2\), and the notation \(B(\xi, r)\) will denote the closed ball in \(\mathbb{R}^d\) with radius \(r\) centered at \(\xi \in \mathbb{R}^d\).

Let us now introduce the Dunkl-Laplacian operator (\cite{2} and \cite{6}, p.156) \(\Delta_k := \sum_{j=1}^{d} D_j^2\), which is known to act on \(\mathcal{C}^2(\mathbb{R}^d)\) functions as

\[
\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left( \langle \nabla f(x), \alpha \rangle - \langle f(x) - f(\sigma_{\alpha}(x)), x \rangle^2 \right),
\]

where \(\Delta\) is the classical Laplacian operator on \(\mathbb{R}^d\). When \(\Delta_k\) acts on \(\mathcal{C}^2\)-functions defined on an open subset \(\Omega\) of \(\mathbb{R}^d\), we obviously assume that \(\Omega\) is \(W\)-invariant and a function \(u \in \mathcal{C}^2(\Omega)\) is called Dunkl harmonic (D-harmonic) on \(\Omega\) if

\[
\forall x \in \Omega, \quad \Delta_k u(x) = 0.
\]

To our knowledge, up to now, D-harmonic functions have been studied only for \(\Omega = \mathbb{R}^d\) (see \cite{10}), for \(\Omega = B(0, 1)\) (the open unit ball of \(\mathbb{R}^d\)) (see \cite{12}) or for \(\Omega\) an ellipsoidal domain centered at the origin (see \cite{20} and \cite{21}). In \cite{10}, Trimèche and Mejjaoli proved that a function \(u \in \mathcal{C}^\infty(\mathbb{R}^d)\) is D-harmonic on \(\mathbb{R}^d\) if and only if \(u\) satisfies the following generalized spherical mean value property:

\[
\forall x \in \mathbb{R}^d, \forall r > 0, \quad u(x) = M^r_{\xi}(u)(x) := \frac{1}{d_k} \int_{S^{d-1}} \tau_x u(r\xi)\omega_k(\xi) d\sigma(\xi),
\]

where \(d\sigma(\xi)\) is the surface measure of the unit sphere \(S^{d-1}\) of \(\mathbb{R}^d\), \(d_k\) is the constant given by

\[
d_k = \int_{S^{d-1}} \omega_k(\xi) d\sigma(\xi)
\]

and \(\tau_x\) is the Dunkl translation operator acting on \(\mathcal{C}^\infty(\mathbb{R}^d)\) functions and whose precise expression is given in section 2. In the case of the ball \(B(0, 1)\), Maslouhi and Youssfi \cite{12} obtained a similar result under the condition that the function \(u\) can be extended to a \(\mathcal{C}^\infty\) function on \(\mathbb{R}^d\).

Our aim in this paper is to introduce and to study a new mean value operator which characterizes D-harmonicity for functions defined on an arbitrary open and \(W\)-invariant set \(\Omega \subset \mathbb{R}^d\).

In section 2 we recall some important facts about the Dunkl transform and Dunkl translation operators and we prove a duality formula for these translations which is used in the sequel as a very important technical tool.

Section 3 is the core of the paper. We introduce a nonnegative kernel \(h_k(r, x, y)\) (see \cite{3.1} for the explicit formula) such that for \(r > 0\) and \(x \in \mathbb{R}^d\) fixed, the function \(y \mapsto h_k(r, x, y)\) has compact support contained in \(\bigcup_{g \in W} B(gx, r)\). For a continuous function \(u\) on a \(W\)-invariant open set \(\Omega\), we define the volume mean value of \(u\) relative to \((x, r)\) as

\[
M_B^r(u)(x) = \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} u(y)h_k(r, x, y)\omega_k(y)dy,
\]

where \(x \in \Omega\), \(r > 0\) is such that \(B(x, r) \subset \Omega\) and \(m_k(B(0, r)) = \int_{B(0, r)} \omega_k(y)dy\) is the \(\omega_k\)-volume of the ball \(B(0, r)\).

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\(^1\)This simplifies the formulas in particular for the reflections; we have \(\sigma_{\alpha} x = x - \langle x, \alpha \rangle\).
We call $h_k$ the harmonic kernel. If $k \equiv 0$ (in the classical case of the Laplace operator $\Delta$), we have $h_0(r, x, y) = 1_{B(x, r)}(y)$ and $M_B^*(u)(x)$ is the usual volume mean value of $u$ at $x$ on the ball $B(x, r)$.

For a general root system and multiplicity function $k \geq 0$, the harmonic kernel $h_k(r, x, y)$ has some specific properties which we study in detail. It is interesting to note that if $u \in C^\infty(\mathbb{R}^d)$, we have a Gauss type formula relating the function $u$, the spherical means $M^*_R(u)$ of $u$ and the volume means $M^*_B(u)$ of the function $\Delta_k u$:

$$\forall r > 0, \forall x \in \mathbb{R}^d, \quad M^*_R(u)(x) = u(x) + \frac{1}{2\gamma + d} \int_0^r M^*_B(\Delta_k u)(x) t dt.$$  

But in the sequel we will concentrate on the properties of the volume mean which is particularly suitable to functions not necessarily defined on the whole $\mathbb{R}^d$. The main theorem of section 3 asserts that for a function $u \in C^2(\Omega)$ ($\Omega$ a $W$-invariant open set of $\mathbb{R}^d$), for all $x \in \Omega$ and $\rho > 0$ such that $B(x, \rho) \subset \Omega$, we have

$$M^*_B(u)(x) = u(x) + \frac{1}{R^{2\gamma + d}} \int_0^R \int_0^r M^*_B(\Delta_k u)(x) t dt r^{2\gamma + d - 1} dr,$$

for all $R \leq \rho/3$. As a corollary, we obtain the fundamental characterization that $u$ is $D$-harmonic in $\Omega$ if and only if it satisfies the volume mean value property. That is: for all $x \in \Omega$ and $\rho > 0$ such that $B(x, \rho) \subset \Omega$, we have

$$u(x) = M^*_B(u)(x),$$

for all $R \in [0, \frac{\rho}{3}]$. As another corollary we obtain a Liouville type theorem: Every positive $D$-harmonic function on $\mathbb{R}^d$ is a constant.

The main results of section 4 are the strong maximum principle for the Dunkl Laplacian operator and Harnack’s theorem for $D$-harmonic functions. We prove that if $\Omega$ is a $W$-invariant connected open subset of $\mathbb{R}^d$, every $D$-harmonic function on $\Omega$ which attains a maximum at $x_0 \in \Omega$ is constant. This is the so-called strong maximum principle. Under the same assumptions on $\Omega$, we have a generalization of the famous Harnack’s inequality: for each compact set $K \subset \Omega$, there exists a universal constant $C_K \geq 1$ such that the inequality

$$u(x) \leq C_K u(y)$$

holds for all $x, y \in K$ and all nonnegative $D$-harmonic functions $u$ in $\Omega$. The crucial tool to prove these results is a rather delicate comparison result involving the harmonic kernels at different quite close points. Precisely: Let $r > 0$ and $x_1, x_2 \in \mathbb{R}^d$ such that $\|x_1 - x_2\| \leq 2r$. Then

$$\forall y \in \mathbb{R}^d, \quad h_k(r, x_2, y) \leq h_k(r \sqrt{10}, x_1, y).$$

As in the classical case, Harnack’s principle for $D$-harmonic functions follows immediately from Harnack’s theorem: every increasing sequence of nonnegative $D$-harmonic functions on $\Omega$ either converge to a $D$-harmonic function or to $+\infty$.

Finally in section 5, we give an application of Harnack’s theorem and the strong maximum principle to a result which is a generalization to $D$-harmonic functions of the so-called Böcher’s theorem. Precisely, if $d \geq 3$ or if $d = 2$ and $k \neq 0$, we show that if $u$ is a positive function which is $D$-harmonic in the punctured open ball $\tilde{B}(0, 1) \setminus \{0\}$, then it is of the form:

$$u(x) = a \|x\|^{2-d-2\gamma} + v(x), \quad x \in \tilde{B}(0, 1) \setminus \{0\},$$
where \( a \) is a constant and \( v \) a D-harmonic function on \( \tilde{\mathcal{B}}(0,1) \). As a corollary, we obtain that a positive D-harmonic function on the punctured space \( \mathbb{R}^d \setminus \{0\} \) is of the form
\[
u(x) = a ||x||^{2-d-2\gamma} + b \quad (x \in \mathbb{R}^d \setminus \{0\})
\]
with constants \( a, b \geq 0 \).

### 2. The Dunkl transform and Dunkl’s translation operators

In this section we recall some properties of the Dunkl transform (see [11] and [17]) and the Dunkl translation operators (see [19]).

The Dunkl transform of a function \( f \in L^1(\mathbb{R}^d, \omega_k(x) dx) \) is defined by
\[
\mathcal{F}_D(f)(\lambda) := \int_{\mathbb{R}^d} f(x) E_k(-i\lambda, x) \omega_k(x) dx, \quad \lambda \in \mathbb{R}^d,
\]
where
\[
E_k(x, y) := V_k(\varepsilon^{(x,-)})(y), \quad x, y \in \mathbb{R}^d,
\]
is the Dunkl kernel (see [6] and [17]) analytically extendable to \( \mathbb{C}^d \times \mathbb{C}^d \) and in particular satisfying the following exchanging constants property:
\[
\forall a \in \mathbb{C}, \forall x, y \in \mathbb{C}^d, E_k(x, ay) = E_k(ax, y).
\]
It is well known (see [11]) that the Dunkl transform \( \mathcal{F}_D \) is an isomorphism of \( S(\mathbb{R}^d) \) (the Schwartz space of rapidly decreasing function \( f \in \mathcal{C}^{\infty}(\mathbb{R}^d) \)) onto itself and its inverse is given by
\[
\mathcal{F}_D^{-1}(f)(x) = \frac{1}{c_k^2} \int_{\mathbb{R}^d} f(\lambda) E_k(i\lambda x, \lambda) \omega_k(\lambda) d\lambda, \quad x \in \mathbb{R}^d,
\]
with
\[
c_k := \int_{\mathbb{R}^d} e^{-\frac{||x||^2}{2}} \omega_k(x) dx,
\]
the Macdonald-Mehta constant (see [13] and [7]).

It is useful to note that if \( f \in L^1(\mathbb{R}^d, \omega_k(x) dx) \) is radial (i.e. \( f(x) = F(||x||) \), with \( F \) a function defined on \([0, +\infty)\)), \( \mathcal{F}_D(f) \) is also radial. Precisely, using spherical coordinates and Corollary 2.5 of [16], we have
\[
\mathcal{F}_D(f)(\lambda) = d_k \int_0^{+\infty} F(r) j_{\gamma + \frac{d}{2} - 1}(r ||\lambda||) r^{2\gamma + d - 1} dr, \quad \lambda \in \mathbb{R}^d,
\]
where \( d_k \) is defined by the relation (1.11) and, for \( \alpha \geq -1/2 \), \( j_\alpha \) is the normalized Bessel function given by
\[
j_\alpha(z) = \Gamma(\alpha + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(n + \alpha + 1)} \left( \frac{z}{2} \right)^{2n}.
\]
Now, the Dunkl translation operators \( \tau_x, x \in \mathbb{R}^d \), are defined on \( \mathcal{C}^{\infty}(\mathbb{R}^d) \) by
\[
\forall y \in \mathbb{R}^d, \quad \tau_x f(y) = \int_{\mathbb{R}^d} V_k \circ T_z \circ V_k^{-1}(f) dy \mu_x(z)
\]
where \( T_x \) is the classical translation operator given by \( T_x f(y) = f(x + y) \). For brevity, we can also write
\[
\forall y \in \mathbb{R}^d, \quad \tau_x f(y) = (V_k)_x (V_k)_y [V_k^{-1}(f)(x + y)],
\]
where \( (V_k)_x \) denotes the operator acting on the \( x \)-variable.
If $f \in S(\mathbb{R}^d)$, then $\tau_x f \in S(\mathbb{R}^d)$ and using the Dunkl transform we have (see [19]):

$$\forall \ y \in \mathbb{R}^d, \ \tau_x f(y) = \mathcal{F}_D^{-1}[E_k(ix, \lambda)\mathcal{F}_D(f)](y)$$

$$= \frac{1}{c_k^d} \int_{\mathbb{R}^d} \mathcal{F}_D(f)(\lambda)E_k(ix, \lambda)E_k(iy, \lambda)\omega_k(\lambda)d\lambda.$$  \hspace{1cm} (2.7)

For $f \in \mathcal{D}(\mathbb{R}^d)$, the function $\tau_{-x}f$ can be expressed by using the dual intertwining operator as follows (see [19], formulas (87) and (88), p. 34):

$$\forall \ y \in \mathbb{R}^d, \ \tau_{-x}f(y) = \int_{\mathbb{R}^d} (iV_k)^{-1} \circ T_{-z} \circ iV_k(f)(y)d\mu_x(z)$$

$$= (V_k)_x (iV_k^{-1})_y [iV_k(f)(y-x)].$$  \hspace{1cm} (2.9)

The operators $\tau_x, x \in \mathbb{R}^d$, satisfy the following properties:

1) for all $x \in \mathbb{R}^d$, the operator $\tau_x$ is continuous from $C^\infty(\mathbb{R}^d)$ into itself,

2) for all $f \in C^\infty(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$, the function $x \mapsto \tau_x f(y)$ is of class $C^\infty$ on $\mathbb{R}^d$,

3) for all $f \in C^\infty(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$, we have

$$\tau_x f(0) = f(x), \ \tau_x f(y) = \tau_y f(x),$$

and

$$D_j(\tau_x f) = \tau_x (D_j f), \quad j = 1, \ldots, d,$$  \hspace{1cm} (2.12)

$$D_j(\tau_x f) = \tau_x (D_j f), \quad j = 1, \ldots, d,$$  \hspace{1cm} (2.13)

$$\tau_x (\Delta_k f) = \Delta_k (\tau_x f),$$  \hspace{1cm} (2.14)

where $D_j$ (resp. $\Delta_k$) are Dunkl’s operators (resp. Dunkl-Laplacian’s operator),

4) for all $f \in \mathcal{D}(\mathbb{R}^d)$, we have

$$\forall \ y \in \mathbb{R}^d, \ \int_{\mathbb{R}^d} \tau_x f(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\omega_k(y)dy,$$  \hspace{1cm} (2.15)

5) if $f \in C^\infty(\mathbb{R}^d)$ is radial (i.e. $f(x) = F(\|x\|)$), Rösler ([16]) has proved the useful formula

$$\forall \ x \in \mathbb{R}^d, \ \tau_x f(y) = \int_{\mathbb{R}^d} F(\sqrt{\|x\|^2 + \|y\|^2 + 2 \langle x, z \rangle})d\mu_y(z),$$  \hspace{1cm} (2.16)

where $\mu_y$ is Rösler’s measure introduced in (1.2).

In the sequel we will need the following crucial duality result:

**Proposition 2.1.** Let $f \in C^\infty(\mathbb{R}^d)$ and $g \in \mathcal{D}(\mathbb{R}^d)$. Then, for all $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \tau_x f(y)g(y)\omega_k(y)dy = \int_{\mathbb{R}^d} f(y)\tau_{-x} g(y)\omega_k(y)dy.$$  \hspace{1cm} (2.17)
Proof. Fix $x \in \mathbb{R}^d$. From the relations (1.6) and (1.7), we deduce that for all $z \in \mathbb{R}^d$,

$$
\int_{\mathbb{R}^d} V_k \circ T_z \circ (V_k)^{-1}(f)(y) g(y) \omega_k(y) dy
= \int_{\mathbb{R}^d} T_z \circ (V_k)^{-1}(f)(y) V_k(g)(y) dy
= \int_{\mathbb{R}^d} f(y)(\mathcal{I}V_k)^{-1} \circ T_{-z} \circ \mathcal{I}V_k(g)(y) \omega_k(y) dy.
$$

Integrating both sides of this equality with respect to the measure $d\mu_x(z)$ (whose support is compact by (1.3)), we obtain

$$(2.18) \int_{\mathbb{R}^d} \int_{\text{supp}(g)} V_k \circ T_z \circ (V_k)^{-1}(f)(y) g(y) \omega_k(y) dy d\mu_x(z)
= \int_{\mathbb{R}^d} \int_{B(0,a)} f(y)(\mathcal{I}V_k)^{-1} \circ T_{-z} \circ \mathcal{I}V_k(g)(y) \omega_k(y) dy d\mu_x(z),$$

where $B(0,a)$ is a closed ball such that

$$\forall z \in \text{supp}\mu_x, \text{ supp}(T_{-z} \circ \mathcal{I}V_k(g)) \subset \text{supp}\mu_x + \text{supp}(\mathcal{I}V_k(g)) \subset B(0,a),$$

and which, by (1.8), is also such that supp$(\mathcal{I}V_k)^{-1} \circ T_{-z} \circ \mathcal{I}V_k(g)) \subset B(0,a)$. This implies the desired result if interchanging the integrals is permissible in both sides of (2.18). Let’s now justify this:

- Using the relation (1.5), we deduce that the function

$$(z,y) \mapsto f(y)(\mathcal{I}V_k)^{-1} \circ T_{-z} \circ \mathcal{I}V_k(g)(y) = f(y)\mathcal{F}_D^{-1}[e^{-iz\cdot \cdot}]\mathcal{F}_D(g)(y)$$

is continuous on the compact set $\text{supp}\mu_x \times B(0,a)$. Thus, we can apply Fubini’s theorem for the right side in (2.18).

- The function $(z,y) \mapsto V_k \circ T_z \circ (V_k)^{-1}(f)(y)$ is measurable as we can see easily if $f$ is a polynomial and in general by approximating $f$ by a sequence of polynomials. Furthermore, using the relations (1.2) and (1.3) and the continuity of the function $(z,\xi) \mapsto (V_k)^{-1}(f)(z + \xi)$, there exists a positive constant $C > 0$ such that

$$(2.19) \forall (z,y) \in \text{supp}\mu_x \times B(0,b), \quad |V_k \circ T_z \circ (V_k)^{-1}(f)(y)|
\leq \int_{B(0,b)} |V_k^{-1}f(z + \xi)| d\mu_y(\xi) \leq C,$$

where $B(0,b)$ is a closed ball such that

$$\forall \ y \in \text{supp}g, \text{ supp}\mu_y \subset B(0,b).$$

Finally, (2.19) shows that we can also use Fubini’s theorem for the left side of (2.18). This completes the proof.

Remark 2.1. For $f$ and $g$ in $\mathcal{D}(\mathbb{R}^d)$, the result of Proposition 2.1 is much easier to prove and was already known (see [19]). It can also be obtained by Fourier-Dunkl transform.
3. The Volume Mean Value Property

In this section we study the notion of D-harmonicity on an arbitrary open \( W \)-invariant subset \( \Omega \) of \( \mathbb{R}^d \). This requires a generalization of the classical volume mean value operator of a function \( u \) defined on \( \Omega \). For this we introduce the following kernel.

**Definition 3.1.** For \( r > 0 \) and \( x, y \in \mathbb{R}^d \), we define the harmonic kernel \( h_k(r, x, y) \) as follows:

\[
h_k(r, x, y) := \int_{\mathbb{R}^d} 1_{[0,r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle})d\mu_y(z).
\]

**Example 3.1.** (1) For \( k = 0 \) (i.e. in the case of the classical Laplacian operator), we have \( \mu_y = \delta_y \) and \( h_0(r, x, y) = 1_{B(x, r)}(y) \).

(2) When \( d = 1 \), \( R = \{1, -1\} \), \( W = \mathbb{Z}_2 = \{id, -id\} \) and the multiplicity function is a constant \( k > 0 \), the intertwining operator is of the form \( V_k(f)(y) = \int_{-1}^{1} f(yt)\phi_k(t)dt \), where

\[
\phi_k(t) = \frac{\Gamma(k + 1/2)}{\sqrt{\pi}\Gamma(k)} (1 - t)^{k-1}(1 + t)^{k}1_{[-1,1]}(t)
\]
is the \( \mathbb{Z}_2 \)-Dunkl density function of parameter \( k \) (see [4] or [17], p.104). In this case, we have

\[
h_k(r, x, y) = \int_{-1}^{1} 1_{[0,r]}(\sqrt{x^2 + y^2 - 2txy})\phi_k(t)dt.
\]

It is easy to see that

\[
h_k(r, x, 0) = 1_{[-r,r]}(x), \quad h_k(r, -x, -y) = h_k(r, x, y)
\]

and

\[
h_k(r, -x, y) = h_k(r, x, -y).
\]

From these relations, it is enough to compute \( h_k(r, x, y) \) for \( x > 0 \) and \( y \in \mathbb{R}\setminus\{0\} \). For this, define

\[
\vartheta := \vartheta_{r,x,y} = \frac{x^2 + y^2 - r^2}{2xy}.
\]

- If \( y > 0 \), we have

\[
h_k(r, x, y) = \begin{cases} 1 & \text{if } \vartheta \leq -1, \\ \int_{\vartheta}^{1} \phi_k(t)dt & \text{if } -1 \leq \vartheta \leq 1, \\ 0 & \text{if } \vartheta \geq 1. \end{cases}
\]

For brevity, we can write

\[
h_k(r, x, y) = 1_{[-\infty,-1]}(\vartheta) + (\int_{\vartheta}^{1} \phi_k(t)dt)1_{[-1,1]}(\vartheta).
\]

- If \( y < 0 \), in the same way we obtain

\[
h_k(r, x, y) = 1_{[1,\infty]}(\vartheta) + (\int_{-1}^{\vartheta} \phi_k(t)dt)1_{[-1,1]}(\vartheta).
\]
We note that for $x$ and $r$ fixed, the function $y \mapsto h_k(r, x, y)$ has compact support equal to $I_{x, r} \cup I_{-x, r}$, where $I_{x, r} = [x - r, x + r]$. For example, for $y > 0$ and

- if $0 < r < x$, i.e. $I_{x, r} \cap I_{-x, r} = \emptyset$, we have
  \[
h_k(r, x, y) = \left( \int_0^1 \phi_k(t) dt \right) \mathbf{1}_{I_{x, r}}(y),
  \]

- if $0 < x \leq r$, i.e. $I_{x, r} \cap I_{-x, r} = \emptyset$, we have
  \[
h_k(r, x, y) = \mathbf{1}_{I_{x, r} \cap I_{-x, r}}(y) + \left( \int_0^1 \phi_k(t) dt \right) \mathbf{1}_{I_{x, r} \cap I_{-x, r}}(y).
  \]

(3) In the case of the root system $R = \{ \pm e_1 \}$ in $\mathbb{R}^2$ (where $e_1 = (1, 0)$), the Coxeter-Weyl group is $\mathbb{Z}_2 \times \{ id \}$, the multiplicity function reduces to the parameter $k = k(e_1) > 0$ and Rösler’s measure $\mu_y = \mu_{(y_1, y_2)}$ is of the form $\mu_{(y_1, y_2)} = \mu_{y_1} \otimes \delta_{y_2}$, where $\mu_{y_1}$ is Rösler’s measure on $\mathbb{R}$ associated to the Coxeter-Weyl group $\mathbb{Z}_2$ and $\delta_{y_2}$ is the Dirac measure at point $y_2 \in \mathbb{R}$. Therefore, for $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R}^2$, the harmonic kernel is given by

\[
(3.3) \quad h_k(r, x, y) = \int_{\mathbb{R}^2} 1_{[0, r]} \left( \sqrt{||x||^2 + ||y||^2 - 2(x_1 z_1 + x_2 z_2)} \right) d\mu_{y_1}(z_1) d\delta_{y_2}(z_2)
= \int_{-1}^1 1_{[0, r]} \left( \sqrt{||x||^2 + ||y||^2 - 2t x_1 y_1 - 2 x_2 y_2} \right) \phi_k(t) dt,
\]

where $\phi_k$ is the $\mathbb{Z}_2$-Dunkl density function of parameter $k$ defined by (3.2).

(4) We consider $\mathbb{R}^d$ with the root system $R = \{ \pm e_i \}$, where $(e_i)_{1 \leq i \leq d}$ is the canonical basis of $\mathbb{R}^d$. Then, the Coxeter-Weyl group is $\mathbb{Z}_2^d$, the multiplicity function can be represented by a multidimensional parameter $k = (k_1, \ldots, k_d)$, $k_i = k(e_i) > 0$, and the harmonic kernel is given by

\[
(3.4) \quad h_k(r, x, y)
= \int_{\mathbb{R}^d} 1_{[0, r]} \left( \sqrt{||x||^2 + ||y||^2 - 2(x_1 z_1 + \cdots + x_d z_d)} \right) d\mu_{y_1}(z_1) \otimes \cdots \otimes d\mu_{y_d}(z_d)
= \int_{[-1, 1]d} 1_{[0, r]} \left( \sqrt{||x||^2 + ||y||^2 - 2(t_1 x_1 y_1 + \cdots + t_d x_d y_d)} \right)
\times \phi_{k_1}(t_1) \cdots \phi_{k_d}(t_d) dt_1 \cdots dt_d,
\]

where $\phi_{k_i}$ is the $\mathbb{Z}_2$-Dunkl density function of parameter $k_i$ $(1 \leq i \leq d)$.

**Proposition 3.1.** The harmonic kernel satisfies the following properties:

1. For all $r > 0$ and $x, y \in \mathbb{R}^d$, $0 \leq h_k(r, x, y) \leq 1$.
2. For all fixed $x, y \in \mathbb{R}^d$, the function $r \mapsto h_k(r, x, y)$ ($r > 0$) is right-continuous and nondecreasing.
3. For all fixed $r > 0$ and $x \in \mathbb{R}^d$, the function $h_k(r, x, \cdot) : y \mapsto h_k(r, x, y)$ has compact support and

\[
(3.5) \quad \text{supp } h_k(r, x, \cdot) \subset \bigcup_{g \in W} B(gx, r).
\]

Moreover, if $r \geq ||x||$, we have

\[
\forall \ y \in \bigcap_{g \in W} B(gx, r), \quad h_k(r, x, y) = 1.
\]
(4) Let $r > 0$ and $x \in \mathbb{R}^d$. For any sequence $(\varphi_{\varepsilon})_{\varepsilon > 0} \subset D(\mathbb{R}^d)$ of radial functions satisfying

\[\forall \varepsilon > 0, \ 0 \leq \varphi_{\varepsilon} \leq 1 \quad \text{and} \quad \forall y \in \mathbb{R}^d, \quad \lim_{\varepsilon \to 0} \varphi_{\varepsilon}(y) = 1_{B(0,r)}(y),\]
we have

\[\forall y \in \mathbb{R}^d, \quad h_k(r, x, y) = \lim_{\varepsilon \to 0} \tau_{-x} \varphi_{\varepsilon}(y).\]

(5) For all $r > 0$ and $x, y \in \mathbb{R}^d$, we have

\[h_k(r, x, y) = h_k(r, y, x).\]

(6) For all $r > 0$ and $x \in \mathbb{R}^d$, we have

\[\|h_k(r, x, .)\|_{k, 1} := \int_{\mathbb{R}^d} h_k(r, x, y)\omega_k(y)dy = m_k(B(0, r)) = \frac{d_{k, r}^{2\gamma + d}}{2\gamma + d},\]

where $d\mu_k(y) = \omega_k(y)dy$ and $d_k$ is the constant defined in (11.11).

(7) Let $r > 0$ and $x, y \in \mathbb{R}^d$. Then, for all $g \in W$, we have

\[h_k(r, gx, gy) = h_k(r, x, y) \quad \text{and} \quad h_k(r, gx, y) = h_k(r, x, g^{-1}y).\]

(8) For all $r > 0$ and $x \in \mathbb{R}^d$, the function $h_k(r, x, .)$ is upper semi-continuous on $\mathbb{R}^d$.

Proof. Property (1) is clear.

(2) It is easy to see that $r \mapsto h_k(r, x, y)$ is nondecreasing. Let $r > 0$ be fixed. We will show that $h(., x, y)$ is continuous at $r^+$. Indeed, for all $t > 0$ small enough, we have

\[h_k(r + t, x, y) - h_k(r, x, y) = \int_{\mathbb{R}^d} 1_{[r, r+t]}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle})d\mu_y(z).\]

Using the dominated convergence theorem, we deduce that

\[\lim_{t \to 0} h_k(r + t, x, y) = h_k(r, x, y).\]

(3) Let $z \in \text{supp}\mu_y$. From (1.3) we can write

\[z = \sum_{g \in W} \lambda_g(z)gy,\]

where $\lambda_g(z) \in [0, 1]$ are such that $\sum_{g \in W} \lambda_g(z) = 1$. Then, we have

\[\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle = \sum_{g \in W} \lambda_g(z)\|g^{-1}x - y\|^2.\]

Thus if $y \notin \bigcup_{g \in W} B(gx, r)$, then $\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle > r^2$ and $h_k(r, x, y) = 0$. This proves (3.5).

Furthermore, if $y \in B(gx, r)$ for all $g \in W$, we deduce from (3.12) that

\[\forall \ z \in \text{supp}\mu_y, \quad \|x\|^2 + \|y\|^2 - 2\langle x, z \rangle \leq r^2.\]

Then, as $\mu_y$ is a probability measure, from (3.1) and (1.2) we obtain $h_k(r, x, y) = 1$.

(4) Let $(\varphi_{\varepsilon})$ be as in (3.6) and $\varphi_{\varepsilon}$ such that $\varphi_{\varepsilon}(\xi) = \phi_{\varepsilon}(\|\xi\|)$. By (2.16), for $y \in \mathbb{R}^d$ we have

\[\tau_{-x} \varphi_{\varepsilon}(y) = \int_{\mathbb{R}^d} \phi_{\varepsilon}(\sqrt{\|x\|^2 + \|y\|^2 - 2\langle x, z \rangle})d\mu_y(z).\]
Using the dominated convergence theorem and (3.5), we deduce that

\[
\lim_{\varepsilon \to 0} \tau_{-x} \varphi_{\varepsilon}(y) = \int_{\mathbb{R}^d} 1_{[0,r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle}) d\mu_y(z) = h_k(r, x, y).
\]

(5) We deduce the result from (3.7) and from the following lemma:

**Lemma 3.1.** Let \( f \in S(\mathbb{R}^d) \) be radial. Then, we have

\[
\tau_{-x} f(y) = \tau_{-y} f(x).
\]

**Proof of Lemma 3.1.** By (2.8), we have

\[
\tau_{-x} f(y) = \frac{1}{c_k} \int_{\mathbb{R}^d} F_D(f)(\lambda) E_k(\lambda) E_k(iy, \lambda) \omega_k(\lambda) d\lambda.
\]

As \( F_D(f) \) is radial by (2.4), the change of variables \( \xi = -\lambda \) and (2.2) give immediately (3.15).

(6) We deduce (3.9) from (2.15), (3.6), (3.7) and from the dominated convergence theorem.

(7) Let \( g \in W \). It is enough to prove that \( h_k(r, gx, gy) = h_k(r, x, y) \). We have

\[
h_k(r, gx, gy) := \int_{\mathbb{R}^d} 1_{[0,r]}(\sqrt{\|x\|^2 + \|y\|^2 - 2 \langle x, g^{-1}z \rangle}) d\mu_{gy}(z).
\]

Then, the relations (1.4) and (1.2) imply the desired result.

(8) Let \( \theta \) be the \( C^\infty \)-function on \( \mathbb{R}^d \) defined by

\[
\theta(t) = \begin{cases} 
\exp(-\frac{1}{1-|t|}) & \text{if } |t| < 1, \\
0 & \text{elsewhere.}
\end{cases}
\]

For \( \varepsilon > 0 \), we consider the function

\[
\psi_{\varepsilon}(t) = \begin{cases} 
\frac{1}{\theta(0)} \theta(t/\varepsilon) & \text{if } -\varepsilon < t < 0, \\
1 & \text{if } 0 \leq t \leq r^2, \\
\frac{1}{\theta(0)} \theta((t-r^2)/\varepsilon) & \text{if } r^2 < t < r^2 + \varepsilon, \\
0 & \text{elsewhere.}
\end{cases}
\]

It is easy to see that \( \psi_{\varepsilon} \in C^\infty(\mathbb{R}) \). Moreover, we have \( \psi_{\varepsilon}(t) \downarrow 1_{[0,r^2]}(t) \) as \( \varepsilon \downarrow 0 \).

Now, we define the radial function \( \varphi_{\varepsilon} \) on \( \mathbb{R}^d \) by

\[
\varphi_{\varepsilon}(y) = \psi_{\varepsilon}(\|y\|^2).
\]

We have \( \varphi_{\varepsilon} \in D(\mathbb{R}^d) \), \( \text{supp}(\varphi_{\varepsilon}) \subset B(0, \sqrt{r^2 + \varepsilon}) \) and, for all \( y \in \mathbb{R}^d \),

\[
\varphi_{\varepsilon}(y) \downarrow 1_{[0,r^2]}(\|y\|^2) = 1_{[0,r]}(\|y\|) = 1_{B(0,r)}(y), \quad \text{as } \varepsilon \downarrow 0.
\]

Furthermore by (2.10), we have for fixed \( x \in \mathbb{R}^d \)

\[
\tau_{-x} \varphi_{\varepsilon}(y) = \int_{\mathbb{R}^d} \phi_{\varepsilon}(\sqrt{\|x\|^2 + \|y\|^2 - 2 \langle x, z \rangle}) d\mu_y(z) \downarrow h_k(r, x, y), \quad \text{as } \varepsilon \downarrow 0,
\]

where \( \varphi_{\varepsilon}(y) = \phi_{\varepsilon}(\|y\|) \). This shows that \( h_k(r, x, .) \) is upper semi-continuous as a decreasing limit of continuous functions.

□
We now give another important aspect of the harmonic kernel which shows that for fixed \( x \), the function \( h_k(r, x, \cdot) \) concentrates in the neighborhood of \( x \) when \( r \to 0 \) and not on the other points \( g_x \) of the orbit \( Wx \) as we could think in view of (3.5).

**Proposition 3.2.** Let \( x \in \mathbb{R}^d \). The family of probability measures
\[
d\eta^x_r(y) = \frac{1}{m_k(B(0, r))} h_k(r, x, y) \omega_k(y) dy, \quad r > 0,
\]
is an approximation of the Dirac measure \( \delta_x \) as \( r \to 0 \). More precisely

1. For all \( \alpha > 0 \), \( \lim_{r \to 0} \int_{||x-y|| > \alpha} d\eta^x_r(y) = 0 \).
2. Let \( f \) be a locally bounded and measurable function defined on a \( W \)-invariant open set \( \Omega \subset \mathbb{R}^d \) and let \( x \in \Omega \). If \( f \) is continuous at point \( x \), then
\[
\lim_{r \to 0} \int_{\mathbb{R}^d} f(y) d\eta^x_r(y) = f(x).
\]

**Proof.** Let \( f \in \mathcal{C}^\infty(\mathbb{R}^d) \) and let \( (\varphi_\varepsilon) \) be as in (3.6). From (2.17), we have
\[
\int_{\mathbb{R}^d} f(y) \tau_{-x} \varphi_\varepsilon(y) \omega_k(y) dy = \int_{\mathbb{R}^d} \tau_x f(y) \varphi_\varepsilon(y) \omega_k(y) dy.
\]
Using (3.7) and the dominated convergence theorem, by letting \( \varepsilon \to 0 \) in the previous relation and dividing by \( m_k(B(0, r)) \), we get
\[
(3.16) \quad \int_{\mathbb{R}^d} f(y) d\eta^x_r(y) = \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} \tau_x f(y) 1_{B(0, r)}(y) \omega_k(y) dy.
\]
Then noting that the measures \( d\nu_r(y) = \frac{1}{m_k(B(0, r))} 1_{B(0, r)}(y) \omega_k(y) dy \) \( (r > 0) \) are an approximate identity when \( r \to 0 \), we obtain
\[
(3.17) \quad \lim_{r \to 0} \int_{\mathbb{R}^d} f(y) d\eta^x_r(y) = \tau_x f(0) = f(x).
\]
Now for \( \alpha > 0 \), it is easy to find a \( \mathcal{C}^\infty(\mathbb{R}^d) \) function \( g \) such that \( 1_{B(x, \alpha)\varepsilon} \leq g \) and \( g = 0 \) on \( B(x, \alpha/2) \). Then by (3.17) we get
\[
0 \leq \int_{||x-y|| > \alpha} d\eta^x_r(y) \leq \int_{\mathbb{R}^d} g(y) d\eta^x_r(y) \to 0 \quad \text{as} \quad r \to 0,
\]
which proves assertion (1) of the proposition. Assertion (2) is now a classical exercise consisting of the decomposition of the integral \( \int_{\mathbb{R}^d} (f(y) - f(x)) d\eta^x_r(y) \) into two integrals, the first one on a ball \( B(x, \alpha) \) adapted to the continuity of \( f \) at point \( x \) and the other on its complement \( B(x, \alpha)^c \), where we use the compactness of the support of the measure \( \eta^x_r \), the local boundedness of \( f \) and the first assertion. \( \square \)

**Definition 3.2.** Let \( u \) be a continuous function on a \( W \)-invariant open set \( \Omega \subset \mathbb{R}^d \), and let \( x \in \Omega \) and \( r > 0 \) be such that \( B(x, r) \subset \Omega \). We define the volume mean value of \( u \) relative to \( (x, r) \) as
\[
(3.18) \quad M_B^r(u)(x) := \frac{1}{m_k(B(0, r))} \int_{\mathbb{R}^d} u(y) h_k(r, x, y) \omega_k(y) dy.
\]

**Remark 3.1.** We note that by (3.5) the integration domain is in fact \( \text{supp} h_k(r, x, \cdot) \subset \Omega \).

Let \( u, x \) and \( r \) be as in the previous definition. By Proposition 3.1 (properties (5) and (2)), relation (3.9), Fubini’s theorem and the dominated convergence theorem,
we can see that the function \( t \mapsto M_B^t(u)(x) \) is continuous on \([0,r]\). Moreover, by Proposition 3.2, it is extendable to a continuous function at \( t = 0 \) such that \( M_B^0(u)(x) = u(x) \).

When \( \Omega = \mathbb{R}^d \) and \( u \in \mathcal{C}^\infty(\mathbb{R}^d) \), we have the following link between the volume mean value and the spherical mean value introduced in (1.10).

**Proposition 3.3.** Let \( u \in \mathcal{C}^\infty(\mathbb{R}^d) \). For all \( r > 0 \) and \( x \in \mathbb{R}^d \), we have
\[
M_B^r(u)(x) = \frac{1}{m_k(B(0,r))} \int_{B(0,r)} \tau_xu(y)\omega_k(y)dy
\]
and
\[
M_B^r(u)(x) = \frac{2\gamma + d}{r^{2\gamma+d}} \int_0^r M_S^t(u)(x)t^{2\gamma+d-1}dt,
\]
where \( M_S^t(u)(x) \) is the spherical mean value at \((x,t)\) defined by formula (1.10).

**Proof.** Formula (3.19) has already been proved in (3.16). We deduce (3.20) from (3.19) and integration in spherical coordinates. \( \blacksquare \)

In the following, we prove a Gauss type formula which gives a relation between a function \( u \) and its volume and spherical value means. Recall first Green’s formula associated to the Dunkl-Laplacian operator, given in [10].

**Proposition 3.4.** If \( \Omega \subset \mathbb{R}^d \) is a \( W \)-invariant regular open set and if \( u,v \in \mathcal{C}^2(\bar{\Omega}) \), then
\[
\int_{\Omega} (u\Delta_k v - v\Delta_k u)\omega_k(x)dx = \int_{\partial\Omega} (u \frac{\partial v}{\partial \eta} - v \frac{\partial u}{\partial \eta})(\xi)\omega_k(\xi)d\sigma_{\partial\Omega}(\xi),
\]
where \( \eta \) is the outer unit normal to the surface \( \partial\Omega \) and \( d\sigma_{\partial\Omega} \) is the surface measure on \( \partial\Omega \).

**Proposition 3.5.** Let \( u \in \mathcal{C}^\infty(\mathbb{R}^d) \). Then, for all \( r > 0 \) and \( x \in \mathbb{R}^d \), we have
\[
M_S^r(u)(x) = u(x) + \frac{1}{2\gamma + d} \int_0^r M_B^t(\Delta_k u)(x)tdt.
\]

**Proof.** Let \( t > 0 \). Using (3.19), (2.14), (3.9) and the change of variables \( y = tz \), we deduce that
\[
M_B^t(\Delta_k u)(x) = \frac{1}{m_k(B(0,t))} \int_{B(0,t)} \Delta_k[\tau_xu](y)\omega_k(y)dy = \frac{2\gamma + d}{d_k} \int_{B(0,1)} \Delta_k[\tau_xu](tz)\omega_k(z)dz,
\]
where \( d_k \) is the constant given in formula (1.11). But, from the homogeneity property of the Dunkl-Laplacian operator
\[
[\Delta_k f](rx) = \frac{1}{r^2} \Delta_k[f(r\cdot)](x) \quad (r > 0, \ f \in \mathcal{C}^2(\mathbb{R}^d))
\]
and Green’s formula (3.21), we have
\[ M_B^t(\Delta_k u)(x) = \frac{2\gamma + d}{dk t^2} \int_{B(0,1)} \Delta_k [\tau x u(t.)](z)\omega_k(z)dz \]
\[ = \frac{2\gamma + d}{dk t^2} \int_{S^{d-1}} \frac{\partial}{\partial \eta} [\tau x u(t.)](\xi)\omega_k(\xi)d\sigma(\xi) \]
\[ = \frac{2\gamma + d}{dk t^2} \int_{S^{d-1}} \langle \nabla [\tau x u(t.)](\xi), \xi \rangle \omega_k(\xi)d\sigma(\xi). \]

Now, by the classical relations
\[ \nabla [f(t.)(tx)] = t[\nabla f(tx)] \quad \text{and} \quad \langle \nabla f(tx), x \rangle = \frac{d}{dt} [f(tx)], \]
we can write
\[ M_B^t(\Delta_k u)(x) = \frac{2\gamma + d}{dk t^2} \int_{S^{d-1}} \frac{d}{dt} [\tau x u(t\xi)]\omega_k(\xi)d\sigma(\xi), \]
\[ = (2\gamma + d)(M_S^t(\Delta_k u)(x) - u(x)). \]

Finally, using Fubini’s theorem and relation (2.11), we deduce that
\[ \int_0^r M_B^t(\Delta_k u)(x)tdt = \frac{2\gamma + d}{dk} \int_{S^{d-1}} \int_0^r \frac{d}{dt} [\tau x u(t\xi)]\omega_k(\xi)d\sigma(\xi) \]
\[ = \frac{2\gamma + d}{dk} \int_{S^{d-1}} [\tau x u(r\xi) - \tau x u(0)]\omega_k(\xi)d\sigma(\xi) \]
\[ = (2\gamma + d)(M_S^t(\Delta_k u)(x) - u(x)). \]

Now, we can give another proof of the spherical mean value property theorem for D-harmonic functions when \( \Omega = \mathbb{R}^d \) (see [10]).

**Corollary 3.1.** Let \( u \in C^\infty(\mathbb{R}^d) \). Then \( u \) is D-harmonic if and only if for all \( x \in \mathbb{R}^d \) and \( r > 0 \) we have
\[ u(x) = M_S^r(u)(x). \]

**Proof.** By the relation (3.22) it is enough to prove that if \( u \) has the spherical mean value property, then \( u \) is D-harmonic.

Fix \( x \in \mathbb{R}^d \). Using relation (3.22) and differentiating with respect to \( r \), we obtain:
\[ \forall \ r > 0, \ M_B^r(\Delta_k u)(x) = 0. \]

Using the relation (3.19), the fact that the sequence of measures
\[ \mu_r(dy) := \frac{1}{m_k(B(0, r))} 1_{\mathbb{B}(0, r)}(y)\omega_k(y)dy \quad (r > 0) \]
is an approximate identity when \( r \to 0 \) and letting \( r \to 0 \), we deduce from (2.11) and (2.14) that
\[ \tau x \Delta_k u(0) = \Delta_k u(x) = 0. \]

This completes the proof. \( \square \)

**Corollary 3.2.** Let \( u \in C^\infty(\mathbb{R}^d) \). Then, for all \( x \in \mathbb{R}^d \) and \( R > 0 \), we have
\[ M_B^R(u)(x) = u(x) + \frac{1}{R^{2\gamma + d}} \int_0^R \int_0^r M_B^t(\Delta_k u)(x) t dt r^{2\gamma + d - 1}dr. \]
Proof. In formula (3.20) we replace \( M^k_u(x) \) by its value given in formula (3.22) and we obtain the result. \( \square \)

We will now study the volume mean value of a function defined on a \( W \)-invariant open subset of \( \mathbb{R}^d \). We begin by a result we will need in the sequel:

**Lemma 3.2.** Let \( f \) be a \( C^2 \)-function on an open set \( \Omega \subset \mathbb{R}^d \) and let \( K \) be a compact subset of \( \Omega \). Then there exists a sequence \( (p_n) \) of polynomial functions such that for all \( i, j = 1, \ldots, d \), \( p_n \rightarrow f \), \( \partial_i p_n \rightarrow \partial_i f \) and \( \partial_i \partial_j p_n \rightarrow \partial_i \partial_j f \) as \( n \rightarrow +\infty \), uniformly on \( K \).

**Proof.** This result is more or less known. For lack of reference, we give a proof.

**First case.** Let \( Q_n \) be defined by

\[
Q_n(x) = (1 - \|x\|^2)^n \quad \text{if} \quad x \in B(0,1),
\]

and \( Q_n(x) = 0 \) if \( \|x\| > 1 \). The sequence of functions \( (\phi_n) \) defined on \( \mathbb{R}^d \) by \( \phi_n(x) = \frac{1}{a_n} Q_n(x) \), where \( a_n = \int_{B(0,1)} Q_n(x) dx \), is an approximate identity as \( n \rightarrow +\infty \).

Let \( f \) be a \( C^2 \)-function on \( B(0,1/2) \). Then the functions defined by

\[
p_n(x) = f_n * \phi_n(x) = \frac{1}{a_n} \int_{B(0,1/2)} (1 - \|x - y\|^2)^n f(y) dy
\]

are polynomial functions on \( B(0,1/2) \) and for all \( i, j = 1, \ldots, d \), they clearly satisfy \( p_n \rightarrow f \), \( \partial_i p_n \rightarrow \partial_i f \) and \( \partial_i \partial_j p_n \rightarrow \partial_i \partial_j f \) uniformly on \( B(0,1/2) \) as \( n \rightarrow +\infty \).

**General case.** Let \( f \) and \( K \) as in the Lemma 3.2. We can find bounded open neighborhoods \( O^1 \) and \( O^2 \) of \( K \) such that

\[
K \subset O^2 \subset \overline{O^2} \subset O^1 \subset \overline{O^1} \subset \Omega,
\]

where \( \overline{O^i} (i = 1, 2) \) is the compact closure of \( O^i \).

Clearly, there exists \( t > 0 \) such that \( \overline{O^1} \subset B(0,1/2) \), where for a set \( E \subset \mathbb{R}^d \), we denote by \( E_t := \{tx, \ x \in E\} \) the image of \( E \) by the dilation \( x \mapsto tx \). In particular, we have

\[
K_t \subset O^2_t \subset \overline{O^2_t} \subset O^1_t.
\]

Now, define the function \( \delta_t f \) on \( O^1_t \) by \( \delta_t f(x) := f(t^{-1}x) \) and let \( g \) be a \( C^2 \)-function on \( \mathbb{R}^d \) such that

\[
g = 1 \text{ on } \overline{O^2_t} \quad \text{and} \quad supp g \subset O^1_t.
\]

Then we can see that the function \( (\delta_t f)g \) is of class \( C^2 \) on \( O^1_t \) and is extendable to a \( C^2 \)-function on \( \mathbb{R}^d \) by taking the value 0 in \( \mathbb{R}^d \setminus O^1_t \). We will denote it also by \( (\delta_t f)g \).

Moreover, for every \( i, j = 1, \ldots, d \), we have

\[
(3.25) \quad \partial_j [(\delta_t f)g] = \partial_j [(\delta_t f)] \quad \text{and} \quad \partial_i \partial_j [(\delta_t f)g] = \partial_i \partial_j [(\delta_t f)] \quad \text{on } O^2_t \supset K_t.
\]

By the first case, there exists a sequence of polynomial functions \( (p_n) \) such that \( p_n \rightarrow (\delta_t f)g \), \( \partial_j p_n \rightarrow \partial_j [(\delta_t f)g] \) and \( \partial_i \partial_j p_n \rightarrow \partial_i \partial_j [(\delta_t f)g] \) uniformly on \( B(0,1/2) \).

Consequently, from (3.25), we deduce that \( p_n \rightarrow \delta_t f \), \( \partial_j p_n \rightarrow \partial_j (\delta_t f) \) and \( \partial_i \partial_j p_n \rightarrow \partial_i \partial_j (\delta_t f) \) uniformly on \( K_t \). This implies

\[
\sup_{x \in K} |f(x) - (\delta_{t^{-1}} p_n)(x)| = \sup_{\xi \in K_t} |(\delta_t f)(\xi) - p_n(\xi)| \rightarrow 0, \text{ as } n \rightarrow +\infty.
\]
Furthermore, as $\partial_j(\delta_t f)(x) = t^{-1}[\delta_t(\partial_j f)](x)$, we can see that
\[
\sup_{x \in K} |\partial_j f(x) - \partial_j[\delta_{t-1}p_n](x)| = t \sup_{\xi \in K_t} |\partial_j(\delta_t f)(\xi) - \partial p_n(\xi)| \longrightarrow 0, \text{ as } n \longrightarrow +\infty.
\]
In the same way, we show that $\partial_i \partial_j [\delta_{t-1}p_n] \longrightarrow \partial_i \partial_j f$ uniformly on $K$. As $\delta_{t-1}p_n$ is a polynomial function, this completes the proof of the lemma. \hfill \qed

**Theorem 3.1.** Let $u \in \mathcal{C}^2(\Omega)$. Then, for all $x \in \Omega$ and $\rho > 0$ such that $B(x, \rho) \subset \Omega$, we have
\[
(3.26) \quad \forall \ 0 < R \leq \rho/3, \quad M^R_B(u)(x) = u(x) + \frac{1}{R^{2\gamma+d}} \int_0^R \int_0^r M^t_B(\Delta_k u)(x) \ dt \ dr.
\]

*Proof.* Let $u \in \mathcal{C}^2(\Omega)$. Fix $x, \rho$ and $R$ as in Theorem 3.1. We take a sequence $(p_n)$ approximating $u$ up to the second derivatives as in Lemma 3.2 for the compact set $K_1 = \bigcup_{g \in W} B(gx, \rho)$. We will use the following crucial approximation result:

**Lemma 3.3.** We have $\Delta_k p_n \longrightarrow \Delta_k u$ as $n \longrightarrow +\infty$ uniformly on the compact set $K_2 = \bigcup_{g \in W} B(gx, R)$.

Assume the result of the lemma for the moment. By Corollary 3.2, we have for all $n$
\[
(3.27) \quad M^R_B(p_n)(x) = p_n(x) + \frac{1}{R^{2\gamma+d}} \int_0^R \int_0^r M^t_B(\Delta_k p_n)(x) \ dt \ dr.
\]

By the compactness of the support of the harmonic kernel and we deduce that $|M^R_B(p_n - u)(x)| \leq \sup_{y \in K_2} |(p_n - u)(y)|$ and
\[
\left| \int_0^R \int_0^r M^t_B(\Delta_k(p_n - u))(x) t \ dt \ dr \right| \leq \frac{R^{2\gamma+d+2}}{2(2\gamma + d + 2)} \sup_{y \in K_2} |\Delta_k(p_n - u)(y)|.
\]

Using these inequalities, Lemma 3.3 and letting $n \longrightarrow +\infty$ in (3.27) the result of the theorem follows. \hfill \qed

*Proof of Lemma 3.3.* For all $f \in \mathcal{C}^2(\Omega)$, we put
\[
(3.28) \quad \delta_{n}(f)(x) = \langle \nabla f(x), \alpha \rangle \frac{f(x) - f(\sigma_n(x))}{\langle \alpha, x \rangle^2}.
\]

Denote $f_n = u - p_n$. From (1.9) and Lemma 3.2, it is enough to prove that
\[
\sum_{\alpha \in R_+} k(\alpha) \delta_{n}(f_n) \longrightarrow 0
\]
as $n \longrightarrow +\infty$ uniformly on $K_2$. We have
\[
(3.29) \quad \sup_{y \in K_2} \left| \sum_{\alpha \in R_+} k(\alpha) \delta_{n}(f_n)(y) \right| \leq \sum_{g \in W} \sum_{\alpha \in R_+} k(\alpha) \sup_{y \in B(gx, R)} |\delta_{n}(f_n)(y)|.
\]

Now, fix $g \in W$ and $\alpha \in R_+$. We will distinguish two cases:

*First case.* Suppose that $B(gx, R) \cap H_\alpha = \emptyset$. 

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Using the relation (3.28) and the Cauchy-Schwarz inequality, we deduce that for all \( y \in B(gx, R) \)
\[
|\delta_\alpha(f_n)(y)| \leq \left| \frac{\langle \nabla f_n(y), \alpha \rangle}{\langle \alpha, y \rangle} \right| \leq \frac{2\|\nabla f_n(y)\|}{\epsilon} + \frac{|f_n(y)| + |f_n(\sigma_\alpha(y))|}{\epsilon^2},
\]
where \( \epsilon = \inf_{y \in B(gx, R)} |\langle \alpha, y \rangle| > 0. \)

Using Lemma 3.2 and the fact that \( K_2 \) is \( W \)-invariant, we deduce that the second side in the previous relation converges to zero as \( n \to +\infty. \) Thus
\[
(3.30) \quad \sup_{y \in B(gx, R)} |\delta_\alpha(f_n)(y)| \to 0 \quad \text{as} \quad n \to +\infty.
\]

**Second case.** Suppose that \( B(gx, R) \cap H_\alpha \neq \emptyset. \) We denote by \( x_{g,\alpha} \) the orthogonal projection of \( gx \) onto \( H_\alpha. \) Then we can see that
\[
B(gx, R) \subset B(x_{g,\alpha}, 2R) \subset B(gx, \rho) \subset K_1.
\]

From these inclusions, we deduce that
\[
(3.31) \quad \sup_{y \in B(gx, R)} |\delta_\alpha(f_n)(y)| \leq \sup_{y \in B(x_{g,\alpha}, 2R)} |\delta_\alpha(f_n)(y)|.
\]

Moreover, we have \( \sigma_\alpha(y) \in B(x_{g,\alpha}, 2R) \) for \( y \in B(x_{g,\alpha}, 2R). \) Thus, by Taylor’s formula we can see that
\[
(3.32) \quad \forall \ y \in B(x_{g,\alpha}, 2R), \quad \delta_\alpha(f_n)(y) = t_\alpha D^2 f_n(\xi)y, \quad \text{for some} \ \xi \text{on the line segment between} \ y \text{and} \ \sigma_\alpha(y), \ \text{where} \ D^2 f_n(\xi) \text{is the Hessian matrix of} \ f \text{evaluated at point} \ \xi. \ \text{Using Lemma 3.2 and the relations (3.31) and (3.32), we obtain}
\]
\[
(3.33) \quad \sup_{y \in B(gx, R)} |\delta_\alpha(f_n)(y)| \to 0 \quad \text{as} \quad n \to +\infty.
\]

The relations (3.29), (3.30) and (3.33) show the desired result. This completes the proof of Lemma 3.3. \( \square \)

In the following result, we prove the volume mean value theorem.

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^d \) be an open and \( W \)-invariant set and \( u \in C^2(\Omega). \) Then \( u \) is \( D \)-harmonic in \( \Omega \) if and only if \( u \) has the mean value property, i.e. for all \( x \in \Omega \) and \( \rho > 0 \) such that \( B(x, \rho) \subset \Omega, \) we have:
\[
(3.34) \quad \forall \ 0 < R \leq \rho/3, \quad u(x) = M^R_B(u)(x).
\]

**Proof.** The relation (3.26) proves that if \( u \) is \( D \)-harmonic on \( \Omega, \) then \( u \) satisfies (3.34).

Now, we suppose that \( u \) satisfies the mean value property. Let \( B(x, \rho) \subset \Omega. \) From (3.26), we have
\[
\forall \ 0 < R \leq \rho/3, \quad \int_0^R \int_0^r M^r_B(\Delta_k u)(x) t \, dt \, r^{2\gamma+d-1} \, dr = 0.
\]
Differentiating two times with respect to \( R, \) we deduce that
\[
\forall \ R > 0, \quad M^R_B(\Delta_k u)(x) = \frac{1}{m_k(B(0, R))} \int_{\mathbb{R}^d} \Delta_k u(y) h(R, x, y) \omega_k(y) \, dy = 0.
\]
Finally by letting \( R \to 0 \) and using Proposition 3.2 we get \( \Delta_k u(x) = 0. \) \( \square \)
Example 3.2. We know from [6] that, for \( d = 2 \) and for the root system considered in Example 3.4, the polynomials
\[
P_n(x) := r^n C_n^{(k_2, k_1)}(\cos \theta), \quad x = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2, \quad n \in \mathbb{N} \setminus \{0\},
\]
where \( C_n^{(\lambda, \mu)} \), \( n \in \mathbb{N} \), are the generalized Gegenbauer polynomials of index \((\lambda, \mu)\) (with \( \lambda, \mu \geq 0 \)) (see [6], p. 26), are D-harmonic on \( \mathbb{R}^2 \). Then, by using the mean value property (3.34), for arbitrary fixed \( R > 0 \), we can write
\[
P_n(x) = \frac{1}{m_k(B(0, R))} \int_{\mathbb{R}^2} h_{k_1, k_2}(R, x, y) P_n(y) |y_1|^{2k_1} |y_2|^{2k_2} dy_1 dy_2
\]
and using polar coordinates and the relations (3.4) and (3.9), we obtain
\[
(3.35) \quad r^n C_n^{(k_2, k_1)}(\cos \theta) = \int_0^\infty \int_0^{2\pi} H_{k_1, k_2}(R, r, \theta, \rho, \phi) \rho^n C_n^{(k_2, k_1)}(\cos \phi) d\rho d\phi,
\]
where
\[
(3.36) \quad H_{k_1, k_2}(R, r, \theta, \rho, \phi) := \frac{2 + 2k_1 + 2k_2}{d_k R^{2k_1 + 2k_2 + 2}} \rho^{2k_1 + 2k_2 + 1} |\cos \phi|^{2k_1} |\sin \phi|^{2k_2}
\]
\[
\quad \times \int_{[0, R]} 1_{[0, R]} \left( \sqrt{r^2 + \rho^2 - 2tr \rho \cos \theta \cos \phi - 2t_2 r \rho \sin \theta \sin \phi} \right)
\]
\[
\quad \times \phi_{k_1}(t_1) \phi_{k_2}(t_2) dt_1 dt_2
\]
and \( d_k = d_{k_1, k_2} \) is the constant (1.11) associated to the Coxeter-Weyl group \( \mathbb{Z}_2^2 \).

Note that for any \((r, \theta)\), the function \((\rho, \phi) \mapsto H_{k_1, k_2}(R, r, \theta, \rho, \phi)\) is a probability density function with compact support contained in \([0, 2\pi] \times [0, R + r]\).

The mean value theorem implies immediately the following result:

Corollary 3.3. If \((u_n)\) is a sequence of D-harmonic functions on \( \Omega \) (a \( W \)-invariant open set of \( \mathbb{R}^d \)) such that \((u_n)\) converges uniformly to a function \( u \) on each compact subset of \( \Omega \), then \( u \) is D-harmonic on \( \Omega \).

As another application of the mean value theorem we show Liouville’s theorem for nonnegative Dunkl harmonic functions on all \( \mathbb{R}^d \).

Corollary 3.4. If \( u \) is D-harmonic and bounded from below on \( \mathbb{R}^d \), then \( u \) is a constant.

Proof. By eventually adding a constant, we can suppose \( u \geq 0 \) on \( \mathbb{R}^d \). By Theorem 3.2 we have for all \( x \in \mathbb{R}^d \) and \( R > 0 \)
\[
u(x) = \frac{1}{m_k(B(0, R))} \int_{\mathbb{R}^d} u(y) h_k(R, x, y) \omega_k(y) dy.
\]

Fix \( R \) and \( x \) such that \( R > \|x\| \) and let \( y \in \text{supp} \ h_k(R, x, .) \). From (3.5), \( y \in B(gx, R) \), for some \( g \in W \). In particular \( y \in B(0, 2R) \).

As \( 0 \leq h_k(R, x, y) \leq 1 \), we have
\[
H_{k_1, k_2}(R, x, y) \leq 1_{B(0, 2R)}(y) = h_k(2R, 0, y).
\]

Thus, using Theorem 3.2 and formula (3.9), we deduce that
\[
0 \leq u(x) \leq \frac{m_k(B(0, 2R))}{m_k(B(0, R))} u(0) = 2^{2r+d} u(0).
\]

That is, \( u \) is bounded. Then the classical Liouville’s theorem (see [8]) proves that \( u \) is a constant.
Remark 3.2. Let \( r > 0 \) and \( x \in \mathbb{R}^d \). In [10], Rösler proved that there exists a compactly supported probability measure \( \sigma_{x,r}^k \) on \( \mathbb{R}^d \) which represents the spherical mean operator. More precisely, for \( u \in C^\infty(\mathbb{R}^d) \), we have

\[
M_r^S(u)(x) = \int_{\mathbb{R}^d} u(y) d\sigma_{x,r}^k(y),
\]

with

\[
\text{supp} \sigma_{x,r}^k \subset \bigcup_{g \in W} B(gx, r).
\]

Then, using the relations (3.37), (3.38) and Lemma 3.3, the relation (3.22) can be extended in the same way to a function of class \( C^2 \) on an arbitrary open and \( W \)-invariant set \( \Omega \subset \mathbb{R}^d \) and we obtain the analogue of Theorem 3.2 where the volume mean \( M_r^B(u) \) is replaced by the spherical mean \( M_r^S(u) \).

Moreover, the relation (3.22) shows that the action of the measure \( \sigma_{x,r}^k - \delta_x \) on a function \( f \in C^2(\Omega) \) is given by

\[
\langle \sigma_{x,r}^k - \delta_x, f \rangle = \frac{1}{2^\gamma + d} \int_{\mathbb{R}^d} \int_0^r \widetilde{h}_k(t, x, y) t dt \Delta_k f(y) dy,
\]

where \( \delta_x \) is the Dirac measure at \( x \) and

\[
\widetilde{h}_k(t, x, y) = \frac{1}{m_k(B(0, t))} h_k(t, x, y) \omega_k(y).
\]

4. Harnack’s inequality and the strong maximum principle

In this section, we will prove the strong maximum principle and Harnack’s inequality for D-harmonic functions. Throughout the section, \( \Omega \) will always denote a \( W \)-invariant open subset of \( \mathbb{R}^d \) and we will denote by \( \mathcal{H}_+^D(\Omega) \) the set of D-harmonic and positive functions on \( \Omega \).

Lemma 4.1. Let \( r > 0 \) and \( x_1, x_2 \in \mathbb{R}^d \) such that \( \|x_1 - x_2\| \leq 2r \). Then

\[
\forall y \in \mathbb{R}^d, \quad h_k(r, x_2, y) \leq h_k(r \sqrt{10}, x_1, y).
\]

Proof. Let \( y \in \mathbb{R}^d \) and \( z \in \text{supp} \mu_y \). Using (3.1), it suffices to show that

\[
1_{[0, r^2]}(\|x_2\|^2 + \|y\|^2 - 2 \langle x_2, z \rangle) \leq 1_{[0, 10r^2]}(\|x_1\|^2 + \|y\|^2 - 2 \langle x_1, z \rangle).
\]

From (3.11) and (3.12), we have

\[
\|x_1\|^2 + \|y\|^2 - 2 \langle x_1, z \rangle = \sum_{g \in W} \lambda_g(z) \|g^{-1}x_1 - y\|^2
\]

\[
\leq \sum_{g \in W} \lambda_g(z) \left( \|g^{-1}x_1 - g^{-1}x_2\| + \|g^{-1}x_2 - y\| \right)^2
\]

\[
\leq 2\|x_1 - x_2\|^2 + 2 \sum_{g \in W} \lambda_g(z) \|g^{-1}x_2 - y\|^2
\]

\[
\leq 8r^2 + 2(\|x_2\|^2 + \|y\|^2 - 2 \langle x_2, z \rangle).
\]

This implies that the inequality (4.2) holds. \( \square \)
Lemma 4.2. Let $x \in \Omega$ and $r > 0$ such that $B(x, 13r) \subset \Omega$. Then there exists a constant $C \geq 1$ such that the inequality
\begin{equation}
(4.3) 
    u(x_2) \leq Cu(x_1)
\end{equation}
holds for all $x_1, x_2 \in B(x, r)$ and for all nonnegative and $D$-harmonic functions in $\Omega$.

Proof. We fix $u \geq 0$ D-harmonic in $\Omega$. Applying Lemma 4.1 for $x_1, x_2 \in B(x, r)$ and using property (2) of Proposition 3.1 we see that
\[
\int_{\mathbb{R}^d} u(y) h_k(r, x_2, y) \omega_k(y) dy \leq \int_{\mathbb{R}^d} u(y) h_k(r\sqrt{10}, x_1, y) \omega_k(y) dy.
\]
Now, as the two balls $B(x_1, 12r)$ and $B(x_2, 3r)$ are in $\Omega$, we can apply the volume mean value Theorem 3.2 and use (3.9) to obtain
\[
u(x_2) \leq \frac{m_k(B(0,4r))}{m_k(B(0,r))} u(x_1) = 4^{2\gamma +d} u(x_1).
\]

In the following result, we extend the strong maximum principle to $D$-harmonic functions.

Theorem 4.1. Suppose that $\Omega$ is connected. Let $u$ be a $D$-harmonic function on $\Omega$. If $u$ has a maximum in $\Omega$, then $u$ is constant.

Proof. Let $M := \max_{\Omega} u(x)$, $v := M - u$ and $F := \{ x \in \Omega : v(x) = 0 \}$. It is clear that $F$ is a nonempty closed set in $\Omega$. Let $x \in F$ and $r > 0$ such that $B(x, 13r) \subset \Omega$. Since the function $v$ is nonnegative and $D$-harmonic in $\Omega$, we can apply Lemma 4.2 to obtain
\[
0 \leq v(a) \leq Cv(x) = 0,
\]
for all $a \in B(x, r)$. That is, $B(x, r) \subset F$ and $F$ is an open set in $\Omega$. By connectivity, $F$ must coincide with $\Omega$. Then $u$ is constant as asserted.

Remark 4.1. (1) If we replace $u$ by $-u$, we obtain the strong minimum principle for $D$-harmonic functions.

(2) Clearly Theorem 4.1 implies the weak maximum principle obtained by Rösler: if $\Omega$ is bounded and $u$ is $D$-harmonic in $\Omega$ and continuous on $\overline{\Omega}$, then $\max_{\Omega} u = \max_{\partial \Omega} u$ (see [14], p. 533).

(3) A particular case of the strong maximum principle was obtained by Dunkl for $D$-harmonic polynomials on the unit ball of $\mathbb{R}^d$ (see [2]).

Now, we will show the second main result in this section. First, we will establish the following lemma:

Lemma 4.3. Suppose that $\Omega$ is connected. Then, for any finite set $E \subset \Omega$, there exists a constant $C_E \geq 1$ such that the inequality
\begin{equation}
(4.4) 
    u(x) \leq C_E u(y)
\end{equation}
holds for all $x, y \in E$ and $u \in H_D^+ (\Omega)$. 
Proof. For $x, y \in \Omega$, define the function
\[ \beta(x, y) := \sup \left\{ \frac{u(x)}{u(y)} : \ u \in H^D_+(\Omega) \right\}. \]

We fix $x_0 \in \Omega$ and we put
\[ \Omega_{x_0} := \{ y \in \Omega : \beta(x_0, y) < +\infty \}. \]

It is clear that $x_0 \in \Omega_{x_0}$.

We will show that $\Omega_{x_0} = \Omega$. For this purpose, it is enough to prove that $\Omega_{x_0}$ is an open and closed set in $\Omega$.

• Let $y \in \Omega_{x_0}$ and $r > 0$ such that $B(y, r) \subset \Omega$. For any $u \in H^D_+(\Omega)$ and $z \in B(y, r)$, by Lemma 4.2 with $x_1$ (resp. $x_2$) replaced by $y$ (resp. $z$), we have
\[ u(y) \leq C u(z). \]
Thus, for all $z \in B(y, r)$, we have
\[ u(x_0) u(y) \leq C u(x_0) u(y) \leq C \beta(x_0, y) < +\infty. \]
This shows that $\beta(x_0, z) < \infty$ for all $z \in B(y, r)$. Thus, $\Omega_{x_0}$ is an open set.

• Let $(y_n) \subset \Omega_{x_0}$ be a sequence such that $y_n \rightarrow y \in \Omega$ as $n \rightarrow \infty$. Let $\varepsilon > 0$ such that $B(y, 13\varepsilon) \subset \Omega$. There exists $N \in \mathbb{N}$ such that $y_N \in B(y, \varepsilon)$. Again by Lemma 4.2 (with $x_1$ (resp. $x_2$) replaced by $y$ (resp. $z$)), we deduce that
\[ \frac{u(x_0)}{u(y)} \leq C \frac{u(x_0)}{u(y_N)} \leq C \beta(x_0, y_N) < +\infty. \]
Thus, $y \in \Omega_{x_0}$ and $\Omega_{x_0}$ is a closed set.

Now, we take $C_E := \max \{ \beta(x, y) : (x, y) \in E^2 \} < \infty$. Clearly $C_E \geq 1$ and for all $x, y \in E$ and $u \in H^D_+(\Omega)$, we have
\[ u(x) = \frac{u(x)}{u(y)} u(y) \leq C_E u(y). \]
This completes the proof. \[ \square \]

Now, we can prove Harnack’s inequality:

**Theorem 4.2.** We suppose that $\Omega$ is connected. For each compact set $K \subset \Omega$, there exists a constant $C_K \geq 1$ such that the inequality
\[ (4.5) \quad \sup_K u \leq C_K \inf_K u \]
holds for all $u \in H^D_+(\Omega)$.

**Proof.** We have $K \subset \bigcup_{x \in K} B(x, r)$, where $0 < r < \frac{1}{13} d(K, \partial \Omega)$. By compactness, we can write
\[ K \subset \bigcup_{i=1}^p B(x_i, r), \]
for some $x_1, \ldots, x_p \in K$.

By Lemma 4.2 for all $i = 1, \ldots, p$, there is a constant $C_i \geq 1$ such that
\[ (4.6) \quad \forall \ y, z \in B(x_i, r), \ u(y) \leq C_i u(z). \]
Now we take $C = \max_{1 \leq i \leq p} C_i$ and $E = \{ x_1, \ldots, x_p \}$.
Let \( x, y \in K \). There exists \( i, j \) such that \( x \in B(x_i, r) \) and \( y \in B(x_j, r) \). The relations (4.6) and (4.4) imply that:

\[
    u(x) \leq C_i u(x_i) \leq CE u(x_j) \leq C^2 E u(y).
\]

Then the theorem is proved with \( C_K = C^2 C_E \). \( \square \)

**Corollary 4.1** (Harnack’s principle). Suppose that \( \Omega \) is connected. Let \((u_n)\) be a point-wise increasing sequence of D-harmonic functions on \( \Omega \). Then either \((u_n)\) converges uniformly on compact subsets of \( \Omega \) to a D-harmonic function or \( u_n(x) \to +\infty \) for all \( x \in \Omega \).

**Proof.** As in the classical case (see [1], p. 50), the result follows using Harnack’s inequality and Corollary 3.3. \( \square \)

### 5. Böcher’s theorem

In this section \( \hat{B} \) denotes the open unit ball of \( \mathbb{R}^d \). The aim of this section is to prove the following version of Böcher’s theorem:

**Theorem 5.1.** Suppose \( d \geq 2 \) and let \( u \) be a D-harmonic and positive function on \( \hat{B}\{0\} \). Then there exists a D-harmonic function \( v \) on \( \hat{B} \) and a constant \( a \geq 0 \) such that

\[
    \forall x \in \hat{B}\{0\}, \quad u(x) = \begin{cases} 
    a \ln(||x||^{-1}) + v(x) & \text{if } d = 2 \text{ and } \gamma = 0, \\
    a||x||^{2-2\gamma-d} + v(x) & \text{if } d \geq 3 \text{ or if } d = 2 \text{ and } \gamma > 0.
    \end{cases}
\]

**Remark 5.1.** When \( d \geq 2 \), the previous result implies that if \( u \) is D-harmonic and positive on \( \hat{B}\{0\} \), then either \( u \) can be extended to a harmonic function on the ball \( \hat{B} \) or \( \lim_{x \to 0} u(x) = +\infty \). In other words a singularity at \( x = 0 \) of a nonnegative and bounded D-harmonic function is always removable. But if \( d = 1 \) we will see at the end of this section that the situation is quite different.

The case \( d = 2 \) and \( \gamma = 0 \) is the classical Böcher’s theorem in the two dimensional Euclidean space. We will then suppose that \( d \geq 2 \) and \( d + 2\gamma > 2 \). The idea is to adapt the scheme of the classical proof given by Axler, Bourdon and Ramey (see [II]) to the situation of D-harmonic functions. For our purpose, we introduce the following definition.

**Definition 5.1.** Let \( u \) be a continuous function on \( \hat{B}\{0\} \). Define the Dunkl-average of \( u \) over the sphere of radius \( ||x|| \) by

\[
    A[u](x) := \frac{1}{dk} \int_{S^{d-1}} u(||x||\xi)\omega_k(\xi)d\sigma(\xi), \quad x \in \hat{B}\{0\}.
\]

**Lemma 5.1.** Suppose \( d \geq 2 \) and \( d + 2\gamma > 2 \) and let \( u \) be a D-harmonic function on \( \hat{B}\{0\} \). Then there are real constants \( a \) and \( b \) such that

\[
    \forall x \in \hat{B}\{0\}, \quad A[u](x) = a||x||^{2-2\gamma-d} + b.
\]

In particular, \( A[u] \) is D-harmonic in \( \hat{B}\{0\} \).
Proof. Define the function $f$ on $]0, 1[$ by

$$f(r) = \frac{1}{d_k} \int_{S^{d-1}} u(r\xi)\omega_k(\xi)d\sigma(\xi).$$

As $u$ is continuously differentiable on $B_0 \setminus \{0\}$, we can differentiate under the integral sign and we obtain

$$f'(r) = \frac{1}{d_k} \int_{S^{d-1}} \langle \nabla u(r\xi), \xi \rangle \omega_k(\xi)d\sigma(\xi) = \frac{r^{-(2\gamma+d)}}{d_k} \int_{S(0,r)} \langle \nabla u(\xi), \xi \rangle \omega_k(\xi)d\sigma_r(\xi),$$

where $d\sigma_r$ is the surface measure of the sphere $S(0, r)$ given by $d\sigma_r = r^{d-1}(\varphi_\gamma d\sigma)$ and $\varphi, d\sigma$ is the image measure of $d\sigma(= d\sigma_1)$ by the dilation $\varphi: \xi \mapsto r\xi$.

We put

$$g(r) = \frac{1}{d_k} \int_{S(0,r)} \langle \nabla u(\xi), \frac{\xi}{r} \rangle \omega_k(\xi)d\sigma_r(\xi).$$

We see that $g(r) = r^{2\gamma+d-1}f'(r)$. Then it suffices to prove that $g$ is constant on $]0, 1[$. For this purpose, we introduce the open set $\Omega = \{ r_1 < \|y\| < r_2 \}$, where $0 < r_1 < r_2 < 1$.

Using the Green formula (3.21) and the fact that $u$ is D-harmonic, we deduce that

$$0 = \int_{\Omega} \Delta_k u(y)\omega_k(y)dy = \int_{\partial \Omega} \frac{\partial u}{\partial \eta}(\xi)\omega_k(\xi)d\sigma_{\partial \Omega}(\xi),$$

where $\eta$ denotes the outward unit normal on $\partial \Omega$. The above equation implies that

$$\int_{S(0, r_1)} \langle \nabla u(\xi), \eta \rangle \omega_k(\xi)d\sigma_{r_1}(\xi) = \int_{S(0, r_2)} \langle \nabla u(\xi), \eta \rangle \omega_k(\xi)d\sigma_{r_2}(\xi).$$

But $\eta = \xi/r_1$ on $S(0, r_1)$ and $\eta = \xi/r_2$ on $S(0, r_2)$. Then $g(r_1) = g(r_2)$, for all $r_1, r_2$ in $]0, 1[$. This shows the relation (5.2).

Finally, we note that the function $x \mapsto \|x\|^{2-2\gamma-d}$ is D-harmonic on $\mathbb{R}^d \setminus \{0\}$ using the fact that if $f$ is a radial function, i.e. $f(x) = F(r), r = \|x\|$, then $\Delta_k f(x) = L_{\gamma+d/2-1}F(r)$, with

$$L_{\gamma+d/2-1}F(r) = \frac{d^2}{dr^2} + \frac{2\gamma + d - 1}{r} \frac{d}{dr},$$

is the Bessel operator of order $\gamma + d/2 - 1$ (see [10]).

Lemma 5.2. There exists a positive constant $\alpha \in ]0, 1[$ such that for every positive D-harmonic function on $B_0 \setminus \{0\}$,

$$\alpha u(y) < u(x), \quad \text{whenever} \quad 0 < \|x\| = \|y\| \leq 1/2.$$

Proof. By Theorem 1.2 there exists a constant $\beta \in ]0, 1[$ such that

$$\beta u(y) \leq u(x),$$

for all positive D-harmonic functions on $B_0 \setminus \{0\}$ and $\|x\| = \|y\| \leq \frac{1}{2}$.

For $r \in ]0, 1[$, we define the function $u_r(x) := u(rx)$. From (3.23), we see that $u_r$ is D-harmonic on $B(0, 1/r) \setminus \{0\}$. Applying the previous result to $u_r$, for all $r \in ]0, 1[$, we obtain $\beta u(y) \leq u(x)$ for all $x, y$ such that $\|x\| = \|y\| = \frac{r}{2}$. Taking $\alpha = \frac{\beta}{2}$, the result follows.
Lemma 5.3. Suppose $d \geq 2$ and $d + 2\gamma > 2$. Let $u$ be a $D$-harmonic and positive function on $\mathring{B}\setminus\{0\}$ such that $u(x) \to 0$ as $\|x\| \to 1$. Then there is a constant $a$ such that:

\[ \forall \, x \in \mathring{B}\setminus\{0\}, \quad u(x) = a\|x\|^{2-2\gamma-d} - a. \]

Proof. By Lemma 5.1, it is enough to prove that $u = A[u]$ on $\mathring{B}\setminus\{0\}$.

- First, we will show that if $u \geq A[u]$ on $\mathring{B}\setminus\{0\}$, then $u = A[u]$ on $\mathring{B}\setminus\{0\}$. Let $x \in \mathring{B}\setminus\{0\}$. As $A[u]$ is radial, we have $A[A[u]](x) = A[u](x)$. That is,

\[ \forall \, x \in \mathring{B}\setminus\{0\}, \quad \int_{S^{d-1}} \left( u(\|x\|\xi) - A[u](\|x\|\xi) \right) \omega_k(\xi)d\sigma(\xi) = 0. \]

As the function $\xi \mapsto u(\|x\|\xi) - A[u](\|x\|\xi)$ is continuous and nonnegative on $S^{d-1}$, we deduce that

\[ \forall \, \xi \in S^{d-1}, \quad u(\|x\|\xi) - A[u](\|x\|\xi) = 0. \]

Taking $\xi = \frac{x}{\|x\|^2}$, we obtain $u(x) = A[u](x)$.

- To prove $u \geq A[u]$ on $\mathring{B}\setminus\{0\}$, we will consider two steps.

**Step 1.** We will prove that $u - \alpha A[u] > 0$ on $\mathring{B}\setminus\{0\}$, where $\alpha$ is the constant of Lemma 5.2. By Lemma 5.2 we have

\[ \forall \, \xi \in S^{d-1}, \quad \alpha u(\|x\|\xi) < u(x), \]

for all $x$ such that $0 < \|x\| \leq 1/2$. Then

\[ \alpha A[u](x) - u(x) = -\frac{1}{d}\int_{S^{d-1}} \left( u(x) - \alpha u(\|x\|\xi) \right) \omega_k(\xi)d\sigma(\xi) < 0, \]

for all $x$ such that $0 < \|x\| \leq 1/2$.

Moreover, because $u(x) \to 0$ as $\|x\| \to 1$, we have $A[u](x) \to 0$ as $\|x\| \to 1$. Thus, $\alpha A[u](x) - u(x) \to 0$ as $\|x\| \to 1$. Using the fact that $\alpha A[u] - u$ is D-harmonic on $\mathring{B}\setminus\{0\}$, the strong maximum principle (Theorem 4.1) shows that $\alpha A[u] - u < 0$ on $\mathring{B}\setminus\{0\}$. That is, $u - \alpha A[u] > 0$ on $\mathring{B}\setminus\{0\}$ as desired.

**Step 2.** We will show that if

\[ (5.3) \quad w = u - tA[u] > 0 \]

for some $t \in [0, 1]$, then $u - A[u] \geq 0$ in $\mathring{B}\setminus\{0\}$. For this, we consider the function

$\psi(t) = \alpha + t(1 - \alpha), \quad t \in [0, 1]$. We have $w(x) \to 0$ as $\|x\| \to 1$. Then, by step 1, we have

\[ w - \alpha A[w] = u - \psi(t)A[u] > 0, \quad \text{on} \quad \mathring{B}\setminus\{0\}. \]

By induction, we deduce that

\[ (5.4) \quad \forall \, n \in \mathbb{N}^*, \quad u - \psi^{(n)}(t)A[u] > 0, \quad \text{on} \quad \mathring{B}\setminus\{0\}, \]

where $\psi^{(n)} = \psi \circ \psi \circ \cdots \circ \psi$ ($n$ times).

But, $\psi^{(n)}(t) = 1 - (1 - \alpha)^n + t(1 - \alpha)^n$. Then, $\psi^{(n)}(t) \to 1$ as $n \to \infty$ for all $t \in [0, 1]$. Thus the relation (5.4) implies that $u - A[u] \geq 0$ in $\mathring{B}\setminus\{0\}$.

Since (5.3) holds when $t = \alpha$, we have $u - A[u] \geq 0$ in $\mathring{B}\setminus\{0\}$ and Lemma 5.3 follows.\[\square\]
Proof of Theorem 5.1. First, we suppose that \( u \) is D-harmonic and positive on a neighborhood of \( B(0,1) \backslash \{0\} \). For \( x \in B \backslash \{0\} \), define

\[
w(x) = u(x) - P[u_{S^{-1}}](x) + \|x\|^{2-2\gamma-d} - 1,
\]

where

\[
P[u_{S^{-1}}](x) = \frac{1}{d_k} \int_{S^{-1}} u(\xi) P(x,\xi) \omega_k(\xi) d\sigma(\xi)
\]

is the Poisson integral of \( u_{S^{-1}} \) (see [6], pp. 189-190, and [12], Theorem A).

We have \( w(x) \to 0 \) if \( \|x\| \to 1 \) and as \( P[u_{S^{-1}}] \) is bounded, \( w(x) \to \infty \) as \( \|x\| \to 0 \). Then, by the strong minimum principle, the D-harmonic function \( w \) is positive in \( B \backslash \{0\} \). By Lemma 5.3 we deduce that

\[
\forall \, x \in B \backslash \{0\}, \quad w(x) = c\|x\|^{2-2\gamma-d} - c,
\]

where \( c \) is a constant. Thus,

\[
\forall \, x \in B \backslash \{0\}, \quad u(x) = a\|x\|^{2-2\gamma-d} + v_1(x),
\]

where \( a = c + 1 \geq 0 \) (otherwise \( u(x) \to -\infty \) as \( \|x\| \to 0 \) is in contradiction with the positivity of \( u \)) and \( v_1 = P[u_{S^{-1}}](x) - c + 1 \) is D-harmonic in \( B \) (see [12], Theorem A).

Now, we suppose that \( u \) is D-harmonic and positive in \( B \backslash \{0\} \). We apply the above result to the function \( u_{1/2} \) defined by

\[
u_{1/2}(x) = u(x/2), \quad x \in B \backslash \{0\}.
\]

We have

\[
u(x/2) = a\|x\|^{2-2\gamma-d} + v_1(x), \quad x \in B \backslash \{0\}.
\]

Thus

\[
u(x) = a2^{2-2\gamma-d}\|x\|^{2-2\gamma-d} + v_1(2x), \quad x \in B(0,1/2) \backslash \{0\}.
\]

We define the function \( v \) on \( B \) by

\[
v(x) = \begin{cases} 
  v_1(2x), & \text{if } x \in B(0,1/2), \\
  u(x) - a2^{2-2\gamma-d}\|x\|^{2-2\gamma-d}, & \text{if } x \in B \backslash B(0,1/2).
\end{cases}
\]

It is easy to see that \( v \) is D-harmonic in \( B \) and we have

\[
u(x) = a2^{2-2\gamma-d}\|x\|^{2-2\gamma-d} + v(x), \quad x \in B \backslash \{0\}.
\]

\[ \square \]

Corollary 5.1. If \( d \geq 2 \) and if \( u \) is a positive and D-harmonic function in \( \mathbb{R}^d \backslash \{0\} \), then

\[
\forall \, x \in \mathbb{R}^d \backslash \{0\}, \quad u(x) = \begin{cases} 
  a\|x\|^{2-2\gamma-d} + b, & \text{if } d = 2 \text{ and } \gamma = 0, \\
  a\|x\|^{2-2\gamma-d}, & \text{if } d \geq 3 \text{ or if } d = 2 \text{ and } \gamma > 0,
\end{cases}
\]

for some constants \( a, b \geq 0 \).
Proof. The case $d = 2$ and $\gamma = 0$ is known ([1], p. 46). Let’s suppose $d \geq 2$ and $d + 2\gamma > 2$ and let $u$ be a positive and D-harmonic function in $\mathbb{R}^d \setminus \{0\}$. By Böcher’s theorem, we have

$$u(x) = a\|x\|^{2 - 2\gamma - d} + v(x), \quad x \in \overset{\circ}{B} \setminus \{0\},$$

where $a$ is a positive constant.

The function $v$ extends D-harmonically to all of $\mathbb{R}^d$ by setting

$$v(x) = u(x) - a\|x\|^{2 - 2\gamma - d}, \quad x \in \mathbb{R}^d \setminus B.$$

Using the minimum principle and the positivity of $u$, we obtain for all $r > 1$ and all $x \in B(0, r)$

$$v(x) \geq \min \{v(y), \|y\| = r\} > -ar^{2 - 2\gamma - d}.$$ 

Letting $r \to \infty$, we see that $v$ is nonnegative in $\mathbb{R}^d$. Then, by Liouville’s theorem (Corollary 3.4), $v$ is constant. \hfill \Box

Remark 5.2. In the case $d = 1$, we have $\Delta_k f(x) = f''(x) + k \frac{f'(x)}{x} - k \frac{f(x) - f(-x)}{2x}$, where $k \geq 0$. If $k = 0$, the general solution of $\Delta_k f(x) = 0$ is $f(x) = ax + b$, where $a$ and $b$ are constants. If $k > 0$, $x = 0$ is a singularity for the difference-differential equation $\Delta_k f(x) = 0$. But by writing $f = f_e + f_o$, where $f_e$ (resp. $f_o$) is the even part (resp. the odd part) of $f$, the functions $f_e$ and $f_o$ satisfy ordinary second order differential equations singular at $x = 0$ but easily solvable and we can show that for $x \neq 0$, we have

$$f(x) = \begin{cases} C_1 + C_2 x + C_3 \text{sg}(x) |x|^{-k} + C_4 |x|^{-k} & \text{if } k \neq 1, \\ C_1 + C_2 x + C_3 \text{sg}(x) |x|^{-1} + C_4 \ln(|x|) & \text{if } k = 1, \end{cases}$$

where $C_i$ $(i = 1, \ldots, 4)$ are arbitrary constants and $\text{sg}(x) = 1$ (resp. $-1$) if $x > 0$ (resp. $x < 0$). This gives the explicit form of the singularities of $f$ at $x = 0$ and shows that if $f$ is bounded, the singularity $x = 0$ is removable if $k \geq 1$ but this is not true if $0 < k < 1$.

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References


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