

CODIMENSION ONE STRUCTURALLY STABLE CHAIN CLASSES

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ABSTRACT. The well-known stability conjecture of Palis and Smale states that if a diffeomorphism is structurally stable, then the chain recurrent set is hyperbolic. It is natural to ask if this type of result is true for an individual chain class, that is, whether or not every structurally stable chain class is hyperbolic. Regarding the notion of structural stability, there is a subtle difference between the case of a whole system and the case of an individual chain class. The latter is more delicate and contains additional difficulties. In this paper we prove a result of this type for the latter, with an additional assumption of codimension 1. Precisely, let f be a diffeomorphism of a closed manifold M and let p be a hyperbolic periodic point of f of index 1 or $\dim M - 1$. We prove if the chain class of p is structurally stable, then it is hyperbolic. Since the chain class of p is not assumed in advance to be locally maximal, and since the counterpart of it for the perturbation g is defined not canonically but indirectly through the continuation p_g of p , the proof is quite delicate.

1. INTRODUCTION

Let M be a compact C^∞ Riemannian manifold without boundary, and let $f : M \rightarrow M$ be a diffeomorphism. Denote by $\text{Diff}(M)$ the space of diffeomorphisms of M with the C^1 -topology.

It is understood that the main dynamics of a system appears in the part that exhibits certain recurrence, as it contains the long-run behavior of all orbits. The most general notion of recurrence is the so-called chain recurrence. Its definition is standard, but we include it here for completeness.

Let $\delta > 0$ be given. A finite sequence $\{x_i\}_{i=0}^n \subset M$ is called a δ -pseudo orbit of f if $d(f(x_i), x_{i+1}) < \delta$ for all $0 \leq i \leq n - 1$, where d is the distance on M induced by the Riemannian metric. For two points $x, y \in M$, we write $x \dashv y$ if, for any $\delta > 0$, there is a δ -pseudo orbit of f going from x to y , that is, there is a δ -pseudo orbit $\{x_i\}_{i=0}^n$, where n depends on δ , such that $x_0 = x$ and $x_n = y$. A point $x \in M$ is called *chain recurrent* if $x \dashv x$. Thus a chain recurrent point is one with a (very weak) recurrence in the sense of pseudo orbits. The set of chain recurrent points of f is called the *chain recurrent set* of f , denoted by $\text{CR}(f)$. It is easy to see that $\text{CR}(f)$ is closed and $f(\text{CR}(f)) = \text{CR}(f)$. Clearly,

$$\overline{\text{Per}(f)} \subset \Omega(f) \subset \text{CR}(f),$$

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where $\text{Per}(f)$ is the set of periodic points and $\Omega(f)$ is the non-wandering set of f . By Conley [Con], any point that is not chain recurrent must be in the basin of some attracting set subtracting the attracting set itself, hence exhibits no recurrence of any type. Thus chain recurrence is the most general version of recurrence.

An important notion in dynamical systems coming from Physics and Mechanics is the so-called structural stability. Precisely, a diffeomorphism f is *structurally stable* if there is a C^1 neighborhood \mathcal{U} of f in $\text{Diff}(M)$ such that, for every $g \in \mathcal{U}$, there is a homeomorphism $h : M \rightarrow M$ such that $h \circ f = g \circ h$. Since such a homeomorphism h preserves orbits, a structurally stable system is one that has robust dynamics, that is, one whose orbital structure remains unchanged under perturbations.

The non-recurrent part of dynamical systems is fairly robust with respect to perturbations. But the recurrent part is fragile. To survive from perturbations, it needs the condition of (various versions of) hyperbolicity. For instance, a single periodic orbit is structurally stable if and only if it is hyperbolic, meaning no eigenvalue of modulus 1. For the whole system f to be structurally stable, a crucial condition needed is that $\text{CR}(f)$, the set that captures all the recurrence, is a hyperbolic set. Recall that a compact invariant set $\Lambda \subset M$ of f is called *hyperbolic* if, for each $x \in \Lambda$, the tangent space $T_x M$ splits into $T_x M = E^s(x) \oplus E^u(x)$ such that

$$Df(E^s(x)) = E^s(f(x)), \quad Df(E^u(x)) = E^u(f(x))$$

and, for some constants $C \geq 1$ and $0 < \lambda < 1$,

$$|Df^n(v)| \leq C\lambda^n|v|, \quad \forall x \in \Lambda, v \in E^s(x), n \geq 0,$$

$$|Df^{-n}(v)| \leq C\lambda^n|v|, \quad \forall x \in \Lambda, v \in E^u(x), n \geq 0.$$

Briefly, a hyperbolic set is one at which tangent vectors split into two directions, contracting and expanding upon iterates, respectively, with uniform exponential rates. This definition extends the hyperbolicity condition from a single periodic orbit to a general compact invariant set. It is closely related to structural stability. Indeed, the following remarkable result, known as the stability conjecture of Palis and Smale [PS1], is fundamental to dynamical systems:

Theorem (Mañé [Man2]). *If a diffeomorphism f is structurally stable, then $\text{CR}(f)$ is hyperbolic.*

In this paper we consider a localized and more delicate version of structural stability. It is for an individual “basic piece” of the dynamics, rather than the whole system. Note that, restricted to $\text{CR}(f)$, the relation $x \sim y$ (meaning $x \dashv y$ and $y \dashv x$) is an equivalence relation. The equivalence classes are called *chain classes* of f , which are each compact and invariant under f . Any chain class cannot be decomposed into two disjoint compact invariant sets, hence is regarded as a basic piece of the system. Generally, a diffeomorphism may have infinitely many chain classes, a phenomenon that causes a great deal of complexity of the dynamics. For any periodic point of f , denote by $C_f(p)$ the (unique) chain class of f that contains p .

A hyperbolic periodic point has its natural “continuation” under perturbations. Precisely, let $p \in M$ be a hyperbolic periodic point of f of period k . Then there exist a compact neighborhood U of $\text{Orb}(p)$ in M and a C^1 -neighborhood $\mathcal{U}(f)$ of

f such that for any $g \in \mathcal{U}(f)$, the maximal invariant set

$$\bigcap_{n=-\infty}^{\infty} g^n(U)$$

of g in U consists of a single periodic orbit O_g of g of the same period as p , which is hyperbolic with $\text{Ind}(O_g) = \text{Ind}(p)$. Here $\text{Ind}(p)$ denotes the *index* of p , which is the dimension of the stable manifold of p . The neighborhood U can be chosen to be the union of k arbitrarily small disjoint balls, each containing exactly one point of $\text{Orb}(p)$ and one point of O_g . This identifies the *continuation* p_g of p under g . Thus the notion of continuation p_g of p is defined for g sufficiently close to f .

However, there is no “continuation” well-defined for a general compact invariant set. Indeed, for a general compact invariant set Λ of f (not a specific one such as $\Omega(f)$, $\text{CR}(f)$, etc.), there is no canonical way to define the “counterpart” of Λ for g that is near f . Consequently, there is no canonical way to define such a general Λ to be “structurally stable”. Nevertheless for the case of a chain recurrent class that contains a hyperbolic periodic point p , there is an indirect way as follows to define its structural stability, through the continuation p_g of p . Let $C_g(p_g)$ denote the (unique) chain class of g that contains p_g .

Definition 1.1. Let p be a hyperbolic periodic point of f . We say that $C_f(p)$ is *C^1 -structurally stable* if there is a neighborhood \mathcal{U} of f in $\text{Diff}(M)$ such that, for every $g \in \mathcal{U}$, there is a homeomorphism $h : C_f(p) \rightarrow C_g(p_g)$ such that $h \circ f|_{C_f(p)} = g \circ h|_{C_f(p)}$, where p_g is the continuation of p .

Note that, while h in this definition preserves periodic points of $C_f(p)$, it is not clear if it preserves individual continuations. For instance, it is not clear if $h(p) = p_g$. Indeed, such an “indirect” definition of structural stability makes the proof of the following main theorem of this paper quite delicate:

Theorem A. *Let f be a diffeomorphism of M and let p be a hyperbolic periodic point of f of index 1 or $\dim M - 1$. If the chain class $C_f(p)$ of p is structurally stable, then $C_f(p)$ is hyperbolic.*

This result is in the spirit of the stability conjecture, but more delicate as just indicated. In particular, $C_f(p)$ is not assumed in advance to be locally maximal (meaning being the maximal invariant set in a neighborhood of itself), hence periodic orbits that are proved to exist in a neighborhood of $C_f(p)$ are hardly identified to be actually inside $C_f(p)$. This is a serious difficulty that appears in the proof.

A special strategy we will use to prove Theorem A is first to prove the theorem for a generic f , that is, for f in a residual family of diffeomorphisms. Most of this paper will be devoted to this special case. Then, for such a generic f , $C_f(p)$ is shadowable, because hyperbolicity implies the shadowing property. Since a topological conjugacy, even one that is defined on a chain class only, preserves the shadowing property, by picking up a generic diffeomorphism near f , we see that a structurally stable chain class $C_f(p)$ is robustly shadowable. But, according to a previous result of X. Wen et al. [WGW], a robustly shadowable chain class must be hyperbolic. This will be how Theorem A is proved.

2. PERIODIC POINTS IN $C_f(p)$

For a hyperbolic periodic point p of f , denote by $H(p, f)$ the *homoclinic class* of p , that is, the closure of the set of transverse homoclinic points of $\text{Orb}(p)$. We

say that two hyperbolic periodic points p and q of f are *homoclinically related* if $W^s(\text{Orb}(p)) \pitchfork W^u(\text{Orb}(q)) \neq \emptyset$ and $W^s(\text{Orb}(q)) \pitchfork W^u(\text{Orb}(p)) \neq \emptyset$. Note that two hyperbolic periodic points that are homoclinically related have the same index. A homoclinic class $H(p, f)$ contains a dense subset of periodic points that are homoclinically related to p .

In the main body of this paper we will work with a generic diffeomorphism f with the following properties:

Proposition 2.1. *There is a residual subset $\mathcal{R} \subset \text{Diff}(M)$ such that every $f \in \mathcal{R}$ satisfies the following conditions:*

- (1) f is Kupka-Smale, meaning periodic points of f are each hyperbolic and their stable and unstable manifolds meet transversally (see [PM]).
- (2) Any chain class of f containing a hyperbolic periodic point p of f equals $H(p, f)$ (see [BC]).
- (3) For any pair of hyperbolic periodic points p and q of f , either $H(p, f) = H(q, f)$ or $H(p, f) \cap H(q, f) = \emptyset$.
- (4) If two hyperbolic periodic points p and q of f are in the same topologically transitive set and $\text{Ind}(p) \leq \text{Ind}(q)$, then $W^s(\text{Orb}(q), f) \pitchfork W^u(\text{Orb}(p), f) \neq \emptyset$ (see [GW]).
- (5) Every chain transitive set of f is a Hausdorff limit of periodic orbits of f (see [Cro3]).

Note that item (3) is a consequence of item (2). Also note that, throughout this paper, the letter \mathcal{R} will denote the residual set described in this proposition.

Proposition 2.2. *Let $f \in \mathcal{R}$, and let $p \in M$ be a hyperbolic periodic point of f . If $C_f(p)$ is structurally stable, then every periodic point $q \in C_f(p)$ of f is homoclinically related to p .*

Proof. We prove all periodic points of f in $C_f(p)$ have the same index. Suppose for contradiction there is a periodic point $p' \in C_f(p)$ with $\text{Ind}(p') \neq \text{Ind}(p)$. Let

$$k = \max\{\pi(p), \pi(p')\},$$

where π denotes the period of a periodic point. Since $C_f(p)$ is structurally stable, there is a C^1 neighborhood \mathcal{U} of f in $\text{Diff}(M)$ such that, for any $g \in \mathcal{U}$, there is a homeomorphism h that conjugates $C_f(p)$ and $C_g(p_g)$. For any $g \in \mathcal{U}$, denote by $\mathcal{P}_k(g, C_g(p_g))$ the set of (not necessarily hyperbolic) periodic orbits of g in $C_g(p_g)$ that has period less than or equal to k . Then

$$h(\mathcal{P}_k(f, C_f(p))) = \mathcal{P}_k(g, C_g(p_g)).$$

Since f is Kupka-Smale, $\mathcal{P}_k(f, C_f(p))$ is a finite set. Then $\mathcal{P}_k(g, C_g(p_g))$ has the same number of elements.

We need a topological version of heteroclinic cycle here. Precisely, for a (not necessarily hyperbolic) periodic orbit Q of g , define the *stable manifold* of Q to be

$$W^s(Q) = W^s(Q, g) = \{x \in M \mid d(g^n(x), Q) \rightarrow 0, n \rightarrow +\infty\}.$$

Likewise for the *unstable manifold* $W^u(Q)$. Since Q is not required to be hyperbolic, $W^s(Q)$ and $W^u(Q)$ are not necessarily differentiable manifolds. We say two not necessarily hyperbolic periodic orbits Q_1 and Q_2 of g form a *heteroclinic cycle*, denoted $Q_1 \sim Q_2$, if $W^s(Q_1) \cap W^u(Q_2) \neq \emptyset$ and $W^s(Q_2) \cap W^u(Q_1) \neq \emptyset$. Here it may not be meaningful to talk about transversality of these intersections. The notion of

heteroclinic cycle is standard, here we just relax the requirement for the differentiability and transversality. We will use this topological version of heteroclinic cycle in the proof of this proposition only. Note that Q_1 and Q_2 are in the same chain class because of the intersections $W^s(Q_1) \cap W^u(Q_2)$ and $W^s(Q_2) \cap W^u(Q_1)$, and every point in the intersections belongs to the same chain class. Thus the above topological conjugacy h preserves heteroclinic cycles. That is, if $Q_1, Q_2 \subset C_f(p)$ and $Q_1 \sim Q_2$, then $h(Q_1) \sim h(Q_2)$, and vice versa.

Although the intersections in the heteroclinic cycles are not necessarily transverse, we prove that \sim is an equivalence relation on $\mathcal{P}_k(g, C_g(p_g))$, for every $g \in \mathcal{U}$. (We could prove this for all periodic orbits in $C_g(p_g)$. Nevertheless we are interested only in those with period $\leq k$.) We first prove this for f . Let Q_1 and Q_2 be two periodic orbits of f in $\mathcal{P}_k(f, C_f(p))$. If Q_1 and Q_2 have the same index, by item (4) of Proposition 2.1 they form a (transverse) heteroclinic cycle. If Q_1 and Q_2 have different indices, since f is Kupka-Smale, either $W^s(Q_1, f) \cap W^u(Q_2, f) = \emptyset$ or $W^s(Q_1, f) \cap W^u(Q_2, f) = \emptyset$, hence Q_1 and Q_2 do not form a heteroclinic cycle. Thus $Q_1 \sim Q_2$ if and only if Q_1 and Q_2 have the same index, hence \sim is an equivalence relation on $\mathcal{P}_k(f, C_f(p))$. But h preserves heteroclinic cycles, hence \sim is an equivalence relation on $\mathcal{P}_k(g, C_g(p_g))$ too, and h maps equivalence classes of $\mathcal{P}_k(f, C_f(p))$ to equivalence classes of $\mathcal{P}_k(g, C_g(p_g))$.

Let

$$l_g = \max\{\#\mathcal{C} : \mathcal{C} \text{ is an equivalence class of } \mathcal{P}_k(g, C_g(p_g)) \text{ with respect to } \sim\},$$

where $\#\mathcal{C}$ denotes the number of elements of \mathcal{C} . Since h preserves this number, l_g is independent of $g \in \mathcal{U}$, and will be denoted l below. Now take an equivalence class \mathcal{C} of $\mathcal{P}_k(f, C_f(p))$ such that $\#\mathcal{C} = l$. Let $\mathcal{C} = \{Q_1, Q_2, \dots, Q_l\}$. Note that $\mathcal{C} \neq \mathcal{P}_k(f, C_f(p))$ as we have assumed for contradiction that periodic orbits of $\mathcal{P}_k(f, C_f(p))$ do not have the same index. There are two cases to consider:

Case 1. $\text{Orb}(p) \in \mathcal{C}$. In this case $\text{Orb}(p)$ forms a (transverse) heteroclinic cycle with every $Q_i \in \mathcal{C}$. Note that there is a periodic orbit Q outside \mathcal{C} . Then $\text{Orb}(p)$ does not form a heteroclinic cycle with Q . Note that, by item (4) of Proposition 2.1, either $W^s(\text{Orb}(p)) \cap W^u(Q) \neq \emptyset$ or $W^u(\text{Orb}(p)) \cap W^s(Q) \neq \emptyset$, according to which pairing of W^s and W^u for $\text{Orb}(p)$ and Q has adequate dimensions. Thus there is only one pairing that needs to be connected and, as $\text{Orb}(p)$ and Q are in the same chain (actually homoclinic) class $C_f(p)$, by the C^1 connecting lemma, they indeed can be connected. Precisely, there is an arbitrarily small C^1 perturbation g of f that creates a heteroclinic cycle of g associated with $\text{Orb}(p_g)$ and Q_g . (See [GW] for details of the perturbation.) That is, $\text{Orb}(p_g) \sim Q_g$. On the other hand, since the heteroclinic cycles formed by $\text{Orb}(p)$ and $Q_i, i = 1, \dots, l$, respectively, are each transverse, they survive if the perturbation is small enough. That is, $\text{Orb}(p_g) \sim (Q_i)_g$ for all $i = 1, \dots, l$. Of course $Q_g, (Q_1)_g, \dots, (Q_l)_g$ are distinct if the perturbation is small enough. Thus the equivalence class of $\text{Orb}(p_g)$ contains at least $l + 1$ elements. This contradicts the definition of l .

Case 2. $\text{Orb}(p) \notin \mathcal{C}$. In this case Q_1 forms a transverse heteroclinic cycle with every $Q_i \in \mathcal{C}$. Note that $\text{Orb}(p)$ is outside \mathcal{C} . As discussed above, by the C^1 connecting lemma, there is an arbitrarily small C^1 perturbation g of f that creates a heteroclinic cycle associated with $(Q_1)_g$ and $\text{Orb}(p_g)$. That is, $(Q_1)_g \sim \text{Orb}(p_g)$. If the perturbation is small enough, the transverse heteroclinic cycle formed between Q_1, \dots, Q_l survives. That is, $(Q_1)_g \sim (Q_i)_g$ for all $i = 1, \dots, l$. Of course

$\text{Orb}(p_g), (Q_1)_g, \dots, (Q_l)_g$ are distinct if the perturbation is small enough. Thus the equivalence class of $\text{Orb}(p_g)$, which is the equivalence class of $(Q_1)_g$, contains at least $l + 1$ elements, contradicting the definition of l . This proves that all periodic points in $C_f(p)$ have the same index.

Thus, by item (4) of Proposition 2.1, every periodic point $q \in C_f(p)$ is homoclinically related to p , proving Proposition 2.2. □

The next proposition asserts that, in a structurally stable chain class, eigenvalues of periodic orbits are uniformly and robustly away from the unit circle.

Proposition 2.3. *Let $f \in \mathcal{R}$, and let $p \in M$ be a hyperbolic periodic point of f . If $C_f(p)$ is structurally stable, then there are a constant $0 < \lambda < 1$ and a neighborhood \mathcal{U} of f such that, for any $g \in \mathcal{U}$ and any periodic point q of g homoclinically related to p_g , the derivative $D_q g^{\pi(q)}$ has no eigenvalue with modulus in (λ, λ^{-1}) , where $\pi(q)$ is the period of q .*

Proof. We prove by contradiction. Suppose there are a diffeomorphism g arbitrarily C^1 close to f and a periodic point $q \in C_g(p_g)$ homoclinically related to p_g such that $D_q g^{\pi(q)}$ has an eigenvalue arbitrarily close to 1. Denote by μ the eigenvalue which is closest to 1, i.e., $|\log \mu| \leq |\log \mu'|$ for all eigenvalues μ' of $D_q g^{\pi(q)}$. For explicitness we assume $|\mu| < 1$. The case $|\mu| > 1$ can be treated similarly. Note that the notion of being homoclinically related requires hyperbolicity of the periodic orbits and transversality between the stable and unstable manifolds, hence rules out the case $|\mu| = 1$. Also, since being homoclinically related is a robust property, while keeping q and p_g homoclinically related, by taking an arbitrarily C^1 small perturbation we can assume that μ has multiplicity 1 and g is “locally linear” near $g^i q$ in the sense that there is $r > 0$ such that

$$g|_{B_r(g^i q)} = \exp_{g^{i+1}q} \circ D_{g^i q} g \circ \exp_{g^i q}^{-1}$$

for any $0 \leq i < \pi(q)$. Let $E^c(q) \subset T_q M$ be the eigenspace of $D_q g^{\pi(q)}$ associated to μ . It is a line if μ is real or a plane if μ is complex. In the second case by taking another arbitrarily small perturbation we can assume that $D_q g^{\pi(q)}|_{E^c(q)}$ is a rational rotation of the plane. For $\eta > 0$, denote the ball in $E^c(q)$ of radius η about the origin to be $E^c(q, \eta)$.

We construct a perturbation \tilde{g} of g . Let $\alpha(x) : [0, +\infty) \rightarrow [0, +\infty)$ be a bump function which satisfies (1) $\alpha|_{[0, 1/3]} = 1$, (2) $\alpha|_{[2/3, +\infty)} = 0$, (3) $0 < \alpha|_{(1/3, 2/3)} < 1$ and (4) $0 \leq \alpha'(x) < 4$ for all $x \in [0, +\infty)$. For a small $\eta > 0$, define a real function $\beta : T_q M \rightarrow \mathbb{R}$ by

$$\beta(v) = |\mu|^{-1} \alpha(|v|/\eta) + (1 - \alpha(|v|/\eta)).$$

Thus $\beta(v) = |\mu|^{-1}$ for $|v| \leq \eta/3$, $1 < \beta(v) < |\mu|^{-1}$ for $\eta/3 < |v| < 2\eta/3$, and $\beta(v) = 1$ for $|v| \geq 2\eta/3$. We always assume η is much less than r . Define a perturbation \tilde{g} of g to be

$$\tilde{g}(x) = \exp_{g(q)}(\beta(v) \cdot D_q g(v)), \quad v = \exp_q^{-1}(x),$$

for $x \in B(q, \eta)$, and define $\tilde{g}(x) = g(x)$ for $x \notin B(q, \eta)$. Briefly, in addition to the act of the tangent map $D_q g$, the perturbation stretches vectors of length $\leq \eta/3$ by a constant factor $|\mu|^{-1}$, stretches vectors of length between $\eta/3$ and $2\eta/3$ by a variable factor $1 < \beta(v) < |\mu|^{-1}$, and leaves alone vectors of length $\geq \eta$. Then \tilde{g} is

C^1 close to g if $|\mu|$ is sufficiently close to 1. We take η small so that the $\pi(q)$ balls $B(g^i(q), \eta)$ are mutually disjoint. To simplify notation we regard p and q below as fixed points. We prove that $\exp_q(\overline{E^c(q, \eta/3)})$, which is an interval if μ is real or a 2-disc if μ is complex, is contained in $C_{\tilde{g}}(p_{\tilde{g}})$.

Since q and p_g are homoclinically related with respect to g , there is $x^* \in W^s(q, g) \cap W^u(p_g, g)$ and $y^* \in W^u(q, g) \cap W^s(p_g, g)$. Since g is locally linear, we may assume $x^* \in \exp_q(E_r^s)$ and $y^* \in \exp_q(E_r^u)$. Also, we may assume that the positive orbit of x^* and the negative orbit of y^* both remain in $B(q, r)$. If η is small enough, the negative orbit of x^* and the positive orbit of y^* will be unchanged under the perturbation \tilde{g} . Hence $\tilde{g}^{-n}(x^*) \rightarrow p_g$ and $\tilde{g}^n(y^*) \rightarrow p_g$ as $n \rightarrow +\infty$. Now we consider the positive orbit of x^* and the negative orbit of y^* under \tilde{g} .

Denote

$$G : T_qM \rightarrow T_qM,$$

$$G(v) = \beta(v) \cdot D_qg(v).$$

Note that for v near the origin,

$$G(v) = \exp_{g(q)}^{-1} \circ \tilde{g} \circ \exp_q(v).$$

We prove $G^{-n}(v) \rightarrow 0$ for every $v \in E^u(q)$. Since G differs from D_qg by a factor $\beta(v)$ only, G preserves $E^u(q)$. Moreover,

$$G^{-n}(v) = \beta^{-1}(G^{-n}v) \cdots \beta^{-1}(G^{-1}v) D_qg^{-n}(v)$$

for any $n \geq 1$. Since μ is the eigenvalue of D_qg closest to the unit circle, and since $1 \leq \beta(v) \leq |\mu|^{-1}$, the factor β is strictly weaker than any of the eigenvalues of $E^u(q)$. Then $G^{-n}(v) \rightarrow 0$. Taking $v = \exp^{-1}(y^*)$ then gives

$$\tilde{g}^{-n}(y^*) \rightarrow q$$

as $n \rightarrow +\infty$.

We prove $G^n(v) \rightarrow \overline{E^c(q, \eta/3)}$ for every $v \in E^s(q)$. Note that $E^s(q)$ splits into a direct sum $E^{ss}(q) \oplus E^c(q)$, hence we may write $v = v^{ss} + v^c$. Write

$$G^n(v) = v_n^{ss} + v_n^c.$$

Since $1 \leq \beta(v) \leq |\mu|^{-1}$,

$$|v_n^{ss}| = |\beta(G^{n-1}v) \cdots \beta(v) D_qg^n(v^{ss})| \leq |\mu|^{-n} |D_qg^n(v^{ss})|.$$

Since μ is strictly weaker than any of the eigenvalues of $E^{ss}(q)$, we get

$$\lim_{n \rightarrow +\infty} |v_n^{ss}| = 0.$$

Then we check E^c . First consider the case when μ is real. Then E^c is a line. By the definition of G , the closed interval $[-\eta/3, \eta/3]$ of E^c consists of fixed points of G . Since $D_qg(v) = \mu v$ and since $\beta(v) < |\mu|^{-1}$ for any $v \notin [-\eta/3, \eta/3]$,

$$G(v) = \beta(v) \cdot T_qg(v)$$

is a strictly decreasing function on $E^c \setminus [-\eta/3, \eta/3]$. Thus every $v \in E^c \setminus [-\eta/3, \eta/3]$ approaches under G along E^c to one of the two end points of the interval. Taking $v = \exp^{-1}(x^*)$ and writing $v = v^{ss} + v^c$ then gives

$$\tilde{g}^n(x^*) \rightarrow \exp_q[-\eta/3, \eta/3] = \exp_q(\overline{E^c(\eta/3)}).$$

Clearly, any interval of fixed points is a chain transitive set, meaning its points are mutually chain equivalent. Thus the whole interval $\exp_q(\overline{E^c(\eta/3)})$ is contained in $C_{\bar{g}}(p_{\bar{g}})$.

Since $C_f(p)$ conjugates $C_{\bar{g}}(p_{\bar{g}})$, $C_f(p)$ also contains an interval of fixed points. This contradicts that f is Kupka-Smale, proving Proposition 2.3 in the case when μ is real.

The case μ is complex is proved similarly. In this case E^c is a plane P and $\exp_q(\overline{E^c(\eta/3)})$ is a disc. Note that we have assumed that $D_q g^{\pi(q)}$ is conjugate to a rational rotation of P . Hence the disc $\exp_q(\overline{E^c(\eta/3)})$ consists of periodic points of G of the same period. Thus the proof goes the same as the case when μ is real. This proves Proposition 2.3. \square

Let $\Lambda \subset M$ be an invariant set of f . A splitting $T_\Lambda M = E \oplus F$ is called (m, λ) -dominated, where $m \geq 1$ and $0 < \lambda < 1$, if

$$Df(E(x)) = E(f(x)), \quad Df(F(x)) = F(f(x))$$

and

$$\|Df^m|_{E(x)}\| \cdot \|Df^{-m}|_{F(f^m x)}\| < \lambda$$

for every $x \in \Lambda$. Since the constants $m \geq 1$ and $0 < \lambda < 1$ are uniform, a dominated splitting over Λ always extends to the closure $\bar{\Lambda}$. (See [BDP].)

A dominated splitting demands relative rates between the two subbundles rather than individual rates of each, which is what a hyperbolic splitting demands. A hyperbolic splitting is automatically a dominated splitting, but not vice versa. Note that if the dimensions of the summands are fixed, the dominated splitting is unique. Thus, besides being interesting on its own, a dominated splitting often serves as a (unique) candidate for a possible hyperbolic splitting. Indeed, if there is ever a hyperbolic splitting, it must be this.

A fundamental tool that ensures the existence of a dominated splitting is the perturbation theory of periodic linear co-cycles developed by Liao [Liao1] and Mañé [Man]. Let $\pi : E \rightarrow \Lambda$ be a finite-dimensional vector bundle and let $f : \Lambda \rightarrow \Lambda$ be a homeomorphism. A continuous map $A : E \rightarrow E$ is called a *linear co-cycle* (or *bundle isomorphism*) if $\pi \circ A = f \circ \pi$, and if A restricted to every fiber is a linear isomorphism. (Note that the letter π here denotes the bundle projection, but not the period of a periodic point as used above and below. This is the only place in this paper where π is used in this way.) The topology of Λ is not relevant to our aim here, and we assume that Λ has the discrete topology. We say A is *bounded* if there is $N > 0$ such that $\max\{\|A(x)\|, \|A^{-1}(x)\|\} \leq N$ for every $x \in \Lambda$, where $A(x)$ denotes $A|_{E(x)}$. For two linear co-cycles A and B over the same base map $f : \Lambda \rightarrow \Lambda$, define

$$d(A, B) = \sup_{x \in \Lambda} \{\|A(x) - B(x)\|, \|A^{-1}(x) - B^{-1}(x)\|\}.$$

A periodic point $p \in \Lambda$ of f is called *hyperbolic* with respect to A if $A^{\pi(p)}$ has no eigenvalues of absolute value 1, where $\pi(p)$ is the period of p . As usual, we denote the contracting and expanding subspaces of p to be $E^s(p)$ and $E^u(p)$. Then $E(p) = E^s(p) \oplus E^u(p)$. If every point in Λ is periodic of f , then A is called a *periodic*

linear co-cycle. A bounded periodic linear co-cycle A is called a *star system* if there is $\epsilon > 0$ such that any B with $d(B, A) < \epsilon$ has no non-hyperbolic periodic orbits. (This notion corresponds to that of diffeomorphisms on the manifold M but, since perturbations on manifolds are less restrictive, the star condition on manifolds is stronger. In fact it implies Axiom A and no-cycle.) The next fundamental result of Liao and Mañé says that, if A is a star system, then the individual hyperbolic splittings $E^s(p) \oplus E^u(p)$ of $p \in \Lambda$, put together, form a dominated splitting. It also gives some estimates for rates on periodic orbits.

Theorem 2.4 ([Liao1], [Man]). *Let $A : E \rightarrow E$ be a bounded periodic linear co-cycle over $f : \Lambda \rightarrow \Lambda$. If A is a star system, then there is $\epsilon > 0$ and three constants $m > 0$, $C > 0$ and $0 < \lambda < 1$ such that, for any linear co-cycle B over f with $d(B, A) < \epsilon$ and any periodic point q of B , the following conditions are satisfied:*

- (1) $\|B^m|_{E^s(q)}\| \cdot \|B^{-m}|_{E^u(f^m q)}\| < \lambda$.
- (2) Let $k = \lceil \pi(q)/m \rceil$; then

$$\prod_{i=0}^{k-1} \|B^m|_{E^s(f^{im}(q))}\| < C\lambda^k,$$

$$\prod_{i=0}^{k-1} \|B^{-m}|_{E^u(f^{-im}(q))}\| < C\lambda^k.$$

The two inequalities in item (2) are usually referred to as “uniformly contracting (expanding) at the periods” for periodic orbits. We remark that Liao and Mañé did not use the term linear co-cycles. Liao worked (for flows) on tangent bundles of manifolds, and Mañé worked on periodic sequences of linear isomorphisms.

Via Franks’ lemma [Fra], Theorem 2.4 applies to the manifold M and ensures a dominated splitting for a certain set of periodic orbits of f . The classical application is the one in the proof of the stability conjecture by Mañé [Man2]. We do not state Franks’ lemma as we will need a refined Franks’ lemma that, briefly, preserves intersections of stable and unstable manifolds, because we have to always stay inside the chain class. This is the result of Gourmelon [Gou]. We take a simple form of his result that is enough for our purpose:

Proposition 2.5 ([Gou]). *Let f be a diffeomorphism of M . For any C^1 neighborhood \mathcal{U} of f , there is $\epsilon > 0$ such that, for any pair of hyperbolic periodic points $p, q \in M$ of f that are homoclinically related, any neighborhood U of $\text{Orb}(q)$ in M not touching $\text{Orb}(p)$, and any continuous path of linear isomorphisms $A_{k,t} : T_{f^k q}M \rightarrow T_{f^{k+1} q}M$ that satisfies the following three assumptions:*

- (1) $A_{k,0} = D_{f^k q}f$ for all $0 \leq k < \pi(q)$,
- (2) $\|A_{k,t} - D_{f^k(q)}f\| < \epsilon$ for all $0 \leq k < \pi(q)$ and any $t \in [0, 1]$,
- (3) $A_{\pi(q)-1,t} \circ A_{\pi(q)-2,t} \circ \dots \circ A_{0,t}$ has no eigenvalue on the unit circle for all $t \in [0, 1]$,

there exists a perturbation $g \in \mathcal{U}$ with the following three properties:

- (A) $g = f$ on $(M \setminus U) \cup \text{Orb}(q)$,
- (B) $D_{f^k q}g = A_{k,1}$ for all $0 \leq k < \pi(q)$,
- (C) p and q are homoclinically related with respect to g .

Proposition 2.6. *Let $f \in \mathcal{R}$, and let $p \in M$ be a hyperbolic periodic point of f . If $C_f(p)$ is structurally stable, then there are three constants $m > 0$, $C > 0$ and $0 < \lambda < 1$ such that, for any periodic point q of f that is homoclinically related to p , the following conditions are satisfied:*

- (1) $\|Df^m|_{E^s(q)}\| \cdot \|Df^{-m}|_{E^u(f^mq)}\| < \lambda$.
- (2) Let $k = [\pi(q)/m]$; then

$$\prod_{i=0}^{k-1} \|Df^m|_{E^s(f^{im}(q))}\| < C\lambda^k,$$

$$\prod_{i=0}^{k-1} \|Df^{-m}|_{E^u(f^{-im}(q))}\| < C\lambda^k.$$

Proof. Let \mathcal{U} and $0 < \lambda < 1$ be given as in Proposition 2.3. For this \mathcal{U} , let $\epsilon > 0$ be the number given in Proposition 2.5.

Let Λ be the union of periodic orbits of $C_f(p)$. By Proposition 2.2, every $q \in \Lambda$ is homoclinically related to p . The tangent map $Df : T_\Lambda M \rightarrow T_\Lambda M$ acts as a periodic linear co-cycle over f . We verify that it is a star system in the sense of linear co-cycles.

Suppose to the contrary there is a linear co-cycle $A : T_\Lambda M \rightarrow T_\Lambda M$ arbitrarily close to Df that has a periodic orbit $\text{Orb}(q)$ of f which is non-hyperbolic with respect to A . We join A with Df by a path A_t with $A_0 = Df$ and $A_1 = A$. Since A can be arbitrarily close to Df , we may assume $A_t|_{\text{Orb}(q)}$ satisfies assumption (2) of Proposition 2.5, for every $t \in [0, 1]$. Let $s \in (0, 1]$ be the first parameter that makes q non-hyperbolic, namely, q is non-hyperbolic with respect to A_s , but is hyperbolic with respect to A_t , for every $t \in [0, s)$. Take s' slightly less than s so that one of the eigenvalues μ of $A_{\pi(q)-1, s'} \circ A_{\pi(q)-2, s'} \circ \dots \circ A_{0, s'}$ (in absolute value) is within (λ, λ^{-1}) . Then the path A_t , $t \in [0, s']$, satisfies the three assumptions of Proposition 2.5, hence there is $g \in \mathcal{U}$ that preserves $\text{Orb}(q)$ and $\text{Orb}(p)$ and realizes $A_{s'}|_{\text{Orb}(q)}$ to be $Dg|_{\text{Orb}(q)}$, such that p and q are homoclinically related with respect to g . Such a weak eigenvalue μ contradicts Proposition 2.3. This verifies that $Df : T_\Lambda M \rightarrow T_\Lambda M$ is a star periodic linear co-cycle over f . Thus Proposition 2.6 follows from Theorem 2.4. □

3. MINIMALLY NON-CONTRACTING SETS

The following result is well known as Pliss' lemma.

Proposition 3.1 (Pliss [Pli]). *Let $K > 0$ and $\gamma_1 < \gamma_2$ be given. There is $c > 0$ such that for any sequence of real numbers a_0, \dots, a_{n-1} with $|a_i| \leq K$, if*

$$\frac{1}{n} \sum_{i=0}^{n-1} a_i < \gamma_1,$$

then there are $0 \leq n_1 < \dots < n_j \leq n - 1$ such that

$$\frac{1}{k} \sum_{i=n_m}^{n_m+k-1} a_i < \gamma_2$$

for all $1 \leq m \leq j$ and $1 \leq k \leq n - n_m$. Furthermore, $j \geq cn$.

Briefly, Pliss' lemma says that if a finite sequence a_0, \dots, a_n has total (from 0 to n) average less than γ_1 , then there are proportionally many intermediate times n_m such that the averages from n_m to all its successors $n_m + k$ are less than γ_2 . This elementary lemma will be frequently used below.

Let $\Lambda \subset M$ be a compact invariant set of f and let E be a continuous subbundle of $T_\Lambda M$, invariant under Df . For $x \in \Lambda$, denote

$$\phi(x) = \log \|Df|_{E(x)}\|.$$

Thus ϕ is a real function on Λ about exponential rates of Df on E under positive iterates. Here is a corollary of Pliss' lemma:

Lemma 3.2. (1) *If there is $x \in \Lambda$ with $\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) < 0$, then there is $y \in \Lambda$ with $\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i y) < 0$.*

(2) *If there is $x \in \Lambda$ with $\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) > 0$, then there is $y \in \Lambda$ with $\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i y) > 0$.*

Proof. If there is $x \in \Lambda$ such that

$$s = \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i x) < 0,$$

then there are positive integers $n_j \rightarrow +\infty$ such that

$$\sum_{i=0}^{n_j-1} \phi(f^i x) < n_j(s + \varepsilon)$$

for a small $\varepsilon \in (0, -s/2)$. By Pliss' lemma, there is m_j for every j such that $n_j - m_j \rightarrow +\infty$ and

$$\frac{1}{k} \sum_{i=m_j}^{m_j+k-1} \phi(f^i x) < s + 2\varepsilon$$

for any $k = 1, \dots, n_j - m_j$. By taking a subsequence, we may assume $f^{m_j} x \rightarrow y \in \Lambda$. One can verify that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i y) \leq s + 2\varepsilon < 0.$$

Item (2) can be proven similarly. This proves Lemma 3.2. □

We also need the following result known as Liao's selecting lemma; see [Liao]. There is an exhibition for this lemma in [Wen].

Proposition 3.3 (Liao). *Let Λ be a compact invariant set of f with (m, λ) -dominated splitting $E \oplus F$ with $\dim(E) = I$, $1 \leq I \leq d - 1$. Assume the following:*

(1) *There is a point $b \in \Lambda$ satisfying*

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^i b)}\| \geq 1$$

for all $n \geq 1$.

(2) *There are λ_1 and λ_2 with $\lambda < \lambda_1 < \lambda_2 < 1$ such that for any $x \in \Lambda$ satisfying*

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^i x)}\| \geq \lambda_2^n$$

for all $n \geq 1$, $\omega(x)$ contains a point $c \in \Lambda$ satisfying

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^{im}c)}\| \leq \lambda_1^n$$

for all $n \geq 1$.

Then for any λ_3 and λ_4 with $\lambda_2 < \lambda_3 < \lambda_4 < 1$, there is a sequence of hyperbolic periodic point q_n of f of index I such that

- (A) $\text{Orb}(q_n)$ converge to a subset of Λ in the Hausdorff metric;
- (B) $\text{Orb}(q_n)$ are mutually homoclinically related;
- (C) the periods $\pi(q_n)$ are multiples of m such that

$$\prod_{i=0}^{k-1} \|Df^m|_{E^s(f^{im}q_n)}\| \leq \lambda_4^k,$$

$$\prod_{i=k-1}^{\pi(q_n)/m-1} \|Df^m|_{E^s(f^{im}q_n)}\| \geq \lambda_3^{\pi(q_n)/m-k+1}$$

for all $k = 1, \dots, \pi(q_n)/m$. Here E^s denotes the stable subbundle of $\text{Orb}(q_n)$.

Similar assertions for F hold respecting f^{-1} .

Thus, by taking λ_3 close to 1, the E^s -rates at the periods for $\text{Orb}(q_n)$ could be arbitrarily weak. Note that (B) was not included in the statement of the selecting lemma in [Wen]. For convenience of application we have added (B) here. It is a consequence of (C). In fact, since $\text{Orb}(q_n)$ is periodic, applying Pliss' lemma to this special case, one can find a point $x_n \in \text{Orb}(q_n)$ such that the E^s rates of x_n from 0 to ∞ are all less than a slightly larger λ_4 . This guarantees a certain uniform size of $W_{loc}^s(x_n)$ and likewise for $W_{loc}^u(x_n)$. Taking subsequences we may assume the x_n converge to a point of Λ hence, for n large, x_n are mutually homoclinically related. This gives (B).

Let E be a continuous subbundle of $T_\Lambda M$. As usual, E is called *contracting* if there are $m \geq 1$ and $0 < \lambda < 1$ such that

$$\|Df^m|_{E(x)}\| \leq \lambda$$

for any $x \in \Lambda$. In the spirit of Liao [Liao] we call a compact invariant set $K \subset \Lambda$ of f *minimally non-contracting* of E if $E|_K$ is not contracting but $E|_{K'}$ is contracting for any compact invariant proper subset $K' \subset K$. By Zorn's Lemma, every non-contracting set Λ of E contains a minimally non-contracting subset of E .

Let $f \in \mathcal{R}$, and let $C_f(p)$ be a structurally stable chain class of f . By Proposition 2.1, $C_f(p) = H(p, f)$, hence periodic points are dense in $C_f(p)$. By Proposition 2.2, every periodic point in $C_f(p)$ is homoclinically related to p . Then the (m, λ) -dominated splittings on these periodic orbits, obtained by Proposition 2.6, extend to a dominated splitting

$$T_{C_f(p)}M = E \oplus F$$

on the whole set $C_f(p)$ with the same constants $m \geq 1$ and $0 < \lambda < 1$. Note that $\dim E = \text{Ind}(p)$. Restricted to periodic points $q \in C_f(p)$, one has $E(q) = E^s(q)$ and $F(q) = E^u(q)$. We will work with this splitting throughout and eventually prove it is hyperbolic, assuming $\text{Ind}(p)$ is 1 or $\dim M - 1$.

Now assume $\text{Ind}(p) = 1$ and hence $\dim E = 1$. The case $\text{Ind}(p) = \dim M - 1$ can be treated similarly. We prove that, in this case, any minimally non-contracting

sets of E must be partially hyperbolic. Recall a dominated splitting $E \oplus F$ is called *partially hyperbolic* if either E is contracting or F is expanding.

Proposition 3.4. *Let $f \in \mathcal{R}$, and let $C_f(p)$ be a structurally stable chain class of f . Let $T_{C_f(p)}M = E \oplus F$ be the dominated splitting as above, and assume $\dim E = 1$. Let $\Lambda \subset C_f(p)$ be a minimally non-contracting set of E . Then Λ is partially hyperbolic. Indeed, writing $T_\Lambda M = E^c \oplus E^u$, where $E^c = E|_\Lambda$ and $E^u = F|_\Lambda$, then E^u is expanding. Moreover,*

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m|_{E^c(f^{im}x)}\| = 0$$

for all $x \in \Lambda$, where m is the constant given in Proposition 2.6.

Proof. It suffices to prove the limit equality only, as it directly implies that $F|_\Lambda$ is expanding, by domination. We prove by contradiction. Abbreviate

$$\phi(x) = \log \|Df^m|_{E(x)}\|$$

for $x \in \Lambda$. Here $E = E^c$. Suppose there is $y \in \Lambda$ such that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}y) > 0.$$

By (a variant use of) Pliss' lemma, there are $\lambda_1 \in (0, 1)$ and a sequence of positive integers $n_1 < n_2 < \dots$ such that

$$\frac{1}{k} \sum_{i=n_j-k}^{n_j-1} \phi(f^{mi}y) > -\log \lambda_1$$

or, what is the same,

$$\prod_{i=n_j-k}^{n_j-1} \|Df^m|_{E(f^{mi}(y))}\| > \lambda_1^{-k}$$

for any $j \geq 1$ and any $1 \leq k \leq n_j$. Taking the inverse then gives

$$\prod_{i=n_j-k+1}^{n_j} \|Df^{-m}|_{E(f^{mi}(y))}\| < \lambda_1^k$$

for any $j \geq 1$ and any $1 \leq k \leq n_j$. Note that this is the place where we use the assumption $\dim E = 1$ (otherwise the inequality would be about mininorm instead of norm). Since $E \oplus F$ is a dominated splitting,

$$\prod_{i=n_j-k+1}^{n_j} \|Df^{-m}|_{F(f^{mi}(y))}\| < \lambda_1^k$$

for any $j \geq 1$ and any $1 \leq k \leq n_j$. Since the angles between $E(x)$ and $F(x)$ have a positive minimum for all $x \in C_f(p)$, switching to an equivalent norm if necessary, we may assume

$$\prod_{i=n_j-k+1}^{n_j} \|Df^{-m}(f^{mi}(y))\| < \lambda_1^k$$

for any $j \geq 1$ and any $1 \leq k \leq n_j$. Here $\|Df^{-m}(x)\|$ denotes (as usual) the norm of Df^{-m} on the whole tangent space $T_x M$. Briefly, n_j are ‘‘hyperbolic times’’ (or more

precisely, “contracting times”) of Df^{-m} . Taking a limit point z of $\{f^{n_j m}(y)\}_{j=1}^\infty$, it is standard to check that $\text{Orb}(z)$ is a periodic source of f . But $z \in \Lambda \subset C_f(p)$, contradicting that any chain class cannot contain a periodic source unless the class reduces to this source. This proves

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}x) \leq 0$$

for all $x \in \Lambda$.

Next suppose there is $y \in \Lambda$ such that

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}y) < 0.$$

By Lemma 3.2, the set

$$S = \{s < 0 : \text{there is } x \in \Lambda \text{ with } \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}x) = s\}$$

is nonempty. There are two possibilities: $\sup S = 0$ or $\sup S < 0$.

If $\sup S = 0$, then there is $z \in \Lambda$ such that

$$\log \lambda < \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}z) < 0.$$

Applying Theorem 2 of [WD], we obtain a hyperbolic periodic point q of f such that $z \in H(q, f)$ and

$$\prod_{i=0}^{\pi(q)-1} \|Df^m|_{E^s(f^{im}(q))}\| > \lambda^{\pi(q)}.$$

Moreover, we have $\Lambda \cap H(q, f) \neq \emptyset$ and hence $H(q, f) = C_f(p)$. Note that in the homoclinic classes, we can choose a periodic point with arbitrarily large period such that the above inequality is satisfied; this contradicts Proposition 2.6.

If $\sup S < 0$, we prove that Λ satisfies the two assumptions of Liao’s selecting lemma (Proposition 3.3). Note that, since $E|_\Lambda$ is not contracting, there is a point $b \in \Lambda$ such that

$$\prod_{i=1}^{k-1} \|Df|_{E(f^{im}b)}\| \geq 1$$

for any $k \geq 1$. Thus the first assumption of Liao’s lemma is verified.

Now take ξ_1 and ξ_2 with

$$\max\{\lambda, e^{\sup S}\} < \xi_1 < \xi_2 < 1.$$

To verify the second assumption of Liao’s selecting lemma, let $x \in \Lambda$ be a point with

$$\prod_{i=0}^{n-1} \|Df^m|_{E(f^{im}x)}\| \geq \xi_2^n$$

for all $n \geq 1$. We verify that $\omega(x)$ contains a point c with

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}c) \leq \log \xi_1$$

for all $n \geq 1$. If $\omega(x) = \Lambda$, there is of course a point $y \in \omega(x)$ such that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}y) \leq \sup S.$$

If $\omega(x) \neq \Lambda$, $E|_{\omega(x)}$ must be contracting as Λ is minimally non-contracting of E . Hence

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}y) < 0$$

for every $y \in \omega(x)$. Since $\sup S < 0$, by the definition of S ,

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}y) \leq \sup S$$

for every $y \in \omega(x)$. Hence in both cases there is a point $y \in \omega(x)$ such that

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}y) \leq \sup S.$$

Then, by Pliss' lemma, there is a point $c \in \omega(x)$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}c) \leq \log \xi_1$$

for all $n \geq 1$. This verifies the second assumption of Liao's selecting lemma. Thus, by conclusion (C) of the lemma, there is a hyperbolic periodic point q of f such that

$$\prod_{i=0}^{\pi(q)-1} \|Df^m|_{E^s(f^{im}(q))}\| > \lambda^{\pi(q)}.$$

Moreover, by conclusions (A) and (B) of the lemma, we may assume $\Lambda \cap H(q, f) \neq \emptyset$ and hence $H(q, f) = C_f(p)$. This also contradicts Proposition 2.6, and proves

$$\liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^{im}(x)) \geq 0$$

for all $x \in \Lambda$. Thus

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m|_{E(f^{im}(x))}\| = 0$$

for all $x \in \Lambda$. This proves Proposition 3.4. □

4. A DOUBLE EXISTENCE OF PERIODIC ORBITS

The next result asserts a “double” existence of a periodic orbit near a minimal set K , i.e., the existence of a periodic orbit which is, simultaneously, near K and inside the chain class of K .

Theorem 4.1. *Let $f \in \mathcal{R}$, and let K be a non-trivial minimal set with a partially hyperbolic splitting $T_K M = E^c \oplus E^u$ such that E^c is 1-dimensional and E^u is expanding. Then for any neighborhood U of K in M , there exists a periodic orbit $O \subset U$ such that O is in the chain class of K .*

To prove Theorem 4.1 we use the δ -interval argument taken from Pujals-Sambarino [PS2], combined with ideas from the more recent central model theory of Crovisier [Cro2].

Proof. Since E^u is expanding, the stable manifolds theorem guarantees a family of local unstable manifolds $W_{loc}^u(x)$ tangent to E^u at every $x \in K$. There is a neighborhood U_0 of K such that $W_{loc}^u(x)$ is defined for every $x \in \bigcap_{n \leq 0} f^n(U_0)$. For any point $x \in \bigcap_{n \leq 0} f^n(U_0)$ and $y \in W_{loc}^u(x)$, the distance $d(f^{-n}(x), f^{-n}(y))$ converges exponentially to 0.

There is a family of central manifolds $W_{loc}^c(x)$ tangent to E^c at every $x \in K$. The definition is more delicate. Indeed, by Hirsch-Pugh-Shub [HPS] (also see [PS2]), there is (not uniquely) a continuous map

$$\phi^c : \bigcap_{n \geq 0} f^n(U_0) \rightarrow Emb([-1, 1], M)$$

that gives a family of central manifolds

$$W_{loc}^c(x) = \phi^c(x)[-1, 1]$$

such that

$$\begin{aligned} \phi^c(x)(0) &= x, \\ T_x W_{loc}^c(x) &= E^c(x). \end{aligned}$$

The family is invariant in the sense that, for any $\epsilon > 0$, there exists $\epsilon' > 0$ such that

$$f(W_{\epsilon'}^c(x)) \subset W_{\epsilon}^c(fx),$$

where $W_{\epsilon}^c(x) = \phi^c(x)[- \epsilon, \epsilon]$. The embedded disk $W_{loc}^c(x)$ is called the *center manifold* through x . We fix such a map ϕ^c in this section, and hence fix a family $W_{loc}^c(x)$ of central manifolds for $x \in K$. There exists a neighborhood U_0 of K such that the center manifolds $W_{loc}^c(x)$ are defined for every $x \in \bigcap_{n \geq 0} f^n(U_0)$. We often call a center manifold or a subinterval of it a *central segment*.

For any central segment $I = \phi^c(x)[0, \epsilon]$ or $I = \phi^c(x)[- \epsilon, 0]$, we say I has length ϵ , denoted by $l(I) = \epsilon$. We will say a central segment I is *based on* $x \in K$ if x is an end point of I . As in Pujals-Sambarino [PS2], a non-trivial central segment I based on $x \in K$ is called a δ - E^c segment if

$$l(f^{-n}(I)) \leq \delta$$

for any $n \geq 0$. If I is a δ - E^c segment, so is $f^{-n}(I)$ for any $n \geq 0$. If $0 < \delta_1 < \delta_2$, then a δ_1 - E^c segment is automatically a δ_2 - E^c segment. Note that if I is a δ - E^c segment, some bigger $I' \supset I$ could also be a δ - E^c segment. But one can always extend I to a biggest δ - E^c segment.

Case 1. For any $\delta > 0$, there is a δ - E^c segment I based on some point of K .

This condition is weaker than saying there is a Lyapunov stable point $x \in K$ which means, for any $\delta > 0$, there is a δ - E^c segment I based on the same $x \in K$. (Here we consider negative iterates, and consider a one-sided neighborhood I of x only.)

Let $\delta > 0$ be arbitrarily given. We fix δ until the end of Case 1. Let I be a δ - E^c segment based on some point $z \in K$. Denote by $I_{f^{-n}z}$ the biggest δ - E^c segment containing $f^{-n}(I)$. Note that

$$f^{-1}(I_{f^{-n}z}) \neq I_{f^{-n-1}z}$$

in general. Nevertheless

$$f^{-k}(I_{f^{-n}z}) \subset I_{f^{-n-k}z}$$

for all $k \geq 0$.

Claim 1. There is a subsequence $n_k \rightarrow +\infty$ such that $l(I_{f^{-n_k}z}) \rightarrow 0$ as $k \rightarrow \infty$.

In fact, suppose for contradiction

$$\inf\{l(I_{f^{-n}z}) : n \geq 0\} > 0.$$

Since K is minimal, there exist positive integers $m_1 > m_2$ such that

$$W_{loc}^u(f^{-m_1}z) \cap I_{f^{-m_2}z} \neq \emptyset.$$

Note that $I_{f^{-m_2}z}$ is also a δ - E^c segment and $f^{-m_2}I \subset I_{f^{-m_2}z}$. By Theorem 3.1 of [PS3], the α -limit set

$$\alpha(I_{f^{-m_2}z}) = \bigcup_{x \in I_{f^{-m_2}z}} \alpha(x)$$

falls into one of the following four cases:

- (1) $\alpha(I_{f^{-m_2}z}) \subset C$, where C is a periodic simple closed curve normally contracting for f^{-m} , where m is the period of C such that $f^{-m}|_C$ has no periodic points.
- (2) There exists a normally attracting periodic arc J such that $I_{f^{-m_2}z} \subset W^u(J)$, and f^k restricted to J (k being the period of J) is the identity map on J .
- (3) $\alpha(I_{f^{-m_2}z}) \subset \text{Per}(f)$. Moreover, one of the periodic points is either a semi-expanding periodic point or an expanding one.
- (4) $I_{f^{-m_2}z}$ is wandering.

Since, as mentioned above, there exist positive integers $m_1 > m_2$ such that

$$W_{loc}^u(f^{-m_1}z) \cap I_{f^{-m_2}z} \neq \emptyset,$$

case (4) is ruled out. We verify that each of the other three cases leads to a contradiction. In case (1) $\text{Orb}(C)$ is normally expanding hence locally maximal. By item (5) of Proposition 2.1, there is a periodic orbit P of f in any small neighborhood of $\text{Orb}(C)$. Then $P \subset \text{Orb}(C)$. Thus $f^m|_C$ has periodic points, ruling out case (1). Case (2) is directly ruled out because f is Kupka-Smale. In case (3) $\alpha(z)$ is a periodic orbit. This contradicts that K is a non-trivial minimal set because $z \in K$. This proves Claim 1.

Claim 2. There is a δ - E^c segment J based on some point $a \in K$ such that J is contained in the chain class of K .

Let n_k be the sequence given in Claim 1. Take I_k to be an E^c -segment based on $f^{-n_k}z$ that is slightly larger than $I_{f^{-n_k}z}$. Since $I_{f^{-n_k}z}$ is a biggest δ - E^c segment and I_k is strictly larger, one of the (negative) iterates of I_k has length near δ . Since $l(I_{f^{-n_k}z}) \rightarrow 0$ by Claim 1, we may assume $l(I_k) \rightarrow 0$ hence there is an integer t_k such that $l(f^{-t_k}I_k) \geq \delta/2$ but $l(f^{-i}I_k) < \delta/2$ for all $0 \leq i < t_k$. By taking subsequences, we may assume $f^{-t_k}I_k$ accumulate to a non-trivial central segment I' based on a point $b \in K$. Since $l(I_k) \rightarrow 0$, the segment I' goes into K in the sense of chains, i.e., for any point $y \in I'$ and any $\varepsilon > 0$, there is an ε -pseudo orbit $y = x_0, x_1, \dots, x_n$ such that $x_n \in K$.

Let $I'_0 = I'$ and

$$I'_n = f^{-1}(I'_{n-1}) \cap W_{loc}^c(f^{-n}(b)).$$

Being preimages of I' , I'_n also goes into K in the sense of chains.

We search for a non-trivial E^c -segment that not only goes into K , but also “comes from” K , in the sense of chains. If

$$\inf\{l(I'_n) : n \geq 0\} = 0,$$

then one can take positive integers n_k and m_k such that $l(I'_{n_k}) \geq \delta/2$, $l(f^{-m_k}(I'_{n_k})) \rightarrow 0$, and $l(f^{-i}(I'_{n_k})) \leq \delta$ for all $0 \leq i \leq m_k$. Let J be an accumulation segment of I'_{n_k} . It is easy to see J comes from and goes into K in the sense of chains, i.e., for any $y \in J$ and any $\varepsilon > 0$, there exists an ε -pseudo orbit starting from K and ending at y , and an ε -pseudo orbit starting from y and ending at K . In other words, J is contained in the chain class of K .

On the other hand, if

$$\inf\{l(I'_n) : n \geq 0\} > 0,$$

then by the minimality of K , we can find a subsequence of I'_n accumulating to some central segment J such that $J \cap I$ is a non-trivial interval, where I is the δ - E^c segment based on $z \in K$ given at the beginning of the proof for Case 1. (More detailed discussion on the orientations of the central models of Crovisier [Cro2] ensures that J can be chosen so that J and I are on the same side of z so that $J \cap I$ is not a single point z .) Then J still goes into K in the sense of chains. But $l(f^{-n_k}I) \rightarrow 0$, hence I comes from K in the sense of chains. Thus $J \cap I$ comes from and goes into K in the sense of chains. In other words, $J \cap I$ is contained in the chain class of K . This proves Claim 2.

Now let J be a δ - E^c segment based on $a \in K$ that meets the requirement of Claim 2. Note that $\bigcup_{y \in J} W_\delta^u(y)$ forms a neighborhood of J in M . Since $f \in \mathcal{R}$, and since J is contained in the chain class of K , by item (5) of Proposition 2.1, there is a periodic point $p \in \bigcup_{y \in \text{int}(J)} W_\delta^u(y)$. Thus there exists a point $y_0 \in \text{int}(J)$ such that

$$d(f^{-n}y_0, f^{-n}(p)) \rightarrow 0, \quad n \rightarrow +\infty,$$

hence p is contained in the chain class of K . Moreover,

$$d(f^{-n}y_0, f^{-n}(p)) \leq \delta, \quad d(f^{-n}y_0, f^{-n}(a)) \leq \delta$$

for all $n \geq 0$. Hence $\text{Orb}(p)$ is contained in the 2δ -neighborhood of K . Since δ can be taken arbitrarily from the very beginning of the proof for Case 1, this proves Theorem 4.1 in Case 1.

Case 2. For some $\delta > 0$, there is no δ - E^c segment based on a point of K .

This condition is sometimes referred to as “sensitive dependence on initial conditions” (see [B]) which, in our case, means there is $\delta > 0$ such that, for any $x \in K$ and any non-trivial E^c -segment I based on x , there is $m \geq 1$ such that $l(f^m(I)) > \delta$.

For any $x \in K$, denote

$$W_\gamma^{+c}(x) = \phi^c(x)[0, \gamma], \quad W_\gamma^{-c}(x) = \phi^c(x)[- \gamma, 0].$$

Claim 3. There is $\gamma \in (0, \delta)$ such that, for any $x \in K$, $l(f^n(W_\gamma^{\pm c}(x))) \rightarrow 0$ as $n \rightarrow +\infty$, where $W_\gamma^{\pm c}(x)$ means “ $W_\gamma^{+c}(x)$ and $W_\gamma^{-c}(x)$ ”.

This means $W_\gamma^c(x) \subset W^s(x)$ for any $x \in K$. (Recall $W_\gamma^c(x) = W_\gamma^{+c}(x) \cup W_\gamma^{-c}(x)$.) Thus Claim 3 says that, in dimension 1, sensitive dependence on initial conditions for f^{-1} with one-sided neighborhoods implies uniform size of stable manifolds for f with two-sided neighborhoods. The converse is obvious.

We first prove there is $\gamma \in (0, \delta)$ such that

$$l(f^n(W_\gamma^{\pm c}(x))) \leq \delta$$

for any $x \in K$ and $n \geq 0$. Suppose to the contrary, for any $\gamma > 0$ there exists a point $x \in K$ and a positive integer n_x such that

$$l(f^{n_x}(W_\gamma^{+(or-)^c}(x))) > \delta.$$

Without loss of generality, we assume that

$$l(f^{n_x}(W_\gamma^{+(or-)^c}(x))) = \delta$$

but

$$l(f^k(W_\gamma^c(x))) < \delta$$

for any $0 \leq k < n_x$. It is easy to see that $n_x \rightarrow +\infty$ as $\gamma \rightarrow 0$. We may assume $f^{n_x}(W_\gamma^{+(or-)^c}(x))$ accumulates to an arc I and $f^{n_x}(x)$ accumulates to some point $y \in K$. It is easy to check that I is a δ - E^c segment, contradicting the assumption.

Now we prove for this $\gamma \in (0, \delta)$ and any $x \in K$,

$$l(f^n(W_\gamma^{\pm c}(x))) \rightarrow 0$$

as $n \rightarrow +\infty$. Suppose there is $x \in K$ such that

$$l(f^n(W_\gamma^{+(or-)^c}(x))) \not\rightarrow 0.$$

Then there exist $\eta > 0$ and a sequence of positive integers n_k such that

$$l(f^{n_k}(W_\gamma^{+(or-)^c}(x))) > \eta.$$

We may assume $f^{n_k}(W_\gamma^{+(or-)^c}(x)) \rightarrow J$ and $f^{n_k}x \rightarrow z \in K$. Then J is non-trivial. Since

$$l(f^n(W_\gamma^{\pm c}(x))) \leq \delta$$

for any $n \geq 0$, it is easy to see that J is a δ - E^c segment, contradicting the assumption. This ends the proof of Claim 3. \square

Thus $W_\gamma^c(x) \subset W^s(x)$ for any $x \in K$. Since K is a minimal set, by item (5) of Proposition 2.1, there are periodic orbits P_n that approach K in the Hausdorff metric. Take $p_n \in P_n$ such that $p_n \rightarrow y \in K$. We may assume that $W_{loc}^u(p_n)$ and $W_{loc}^u(y)$ both have size at least γ and we simply denote them to be $W_\gamma^u(p_n)$ and $W_\gamma^u(y)$. Note that, while $W_\gamma^c(y) \subset W^s(y)$, it is not known if $W_\gamma^c(p_n) \subset W^s(p_n)$. What is known is that, being a periodic interval of a Kupka-Smale system f , $W_\gamma^c(p_n)$ has at most finitely many periodic points of the same (or doubled, depending on whether $f^{\pi(p_n)}$ preserves or flips the orientation of the interval) period as that of p_n . Since E^u is expanding, these periodic points have index 0 and 1, alternately. Now we assume p_n and y are close enough so that

$$W_\gamma^u(p_n) \cap W_\gamma^c(y) \neq \emptyset, \quad W_\gamma^u(y) \cap W_\gamma^c(p_n) \neq \emptyset.$$

We may assume γ was chosen small enough so that each intersection contains a single point. Let z be the unique point in $W_\gamma^u(y) \cap W_\gamma^c(p_n)$. On the E^c -interval $[p_n, z]$, the periodic point q that is closest to z cannot be a source, because the unstable manifolds of y and q cannot intersect. That is, q must be a saddle of index 1 and $[q, z] \subset W^s(q)$. Note that $W_\gamma^u(q) \cap W_\gamma^c(y) \neq \emptyset$. In summary,

$$W_\gamma^u(q) \cap W_\gamma^s(y) \neq \emptyset, \quad W_\gamma^u(y) \cap W^s(q) \neq \emptyset.$$

Thus y and q are in the same chain class. This proves Theorem 4.1 in Case 2, hence ends the proof of Theorem 4.1.

5. PROOF OF THEOREM A

In this section we complete the proof of Theorem A. First we prove it for a generic f .

Proposition 5.1. *Let $f \in \mathcal{R}$, and let $C_f(p)$ be a structurally stable chain class of f . If $\text{Ind}(p) = 1$ or $\dim M - 1$, then $C_f(p)$ is hyperbolic.*

Proof. Let

$$T_{C_f(p)}M = E \oplus F$$

be the dominated splitting given right before Proposition 3.4. We take the case $\text{Ind}(p) = 1$ and hence $\dim E = 1$. The case $\text{Ind}(p) = \dim M - 1$ can be treated similarly. We prove E is contracting.

Suppose E is not contracting. Then there is a minimally non-contracting set $\Lambda \subset C_f(p)$ of E . By Proposition 3.4, $E \oplus F$ restricted to Λ is partially hyperbolic. More precisely, write

$$T_\Lambda M = E^c \oplus E^u,$$

where $E^c = E|_\Lambda$ and $E^u = F|_\Lambda$; then E^u is expanding, and

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m|_{E^c(f^i x)}\| = 0$$

for any $x \in \Lambda$. Take any minimal set $K \subset \Lambda$. K must be non-trivial because otherwise, by the limit equality, K reduces to a non-hyperbolic periodic orbit, contradicting $f \in \mathcal{R}$. By Theorem 4.1, there exist periodic orbits $Q_n \subset C_f(p)$ such that $Q_n \rightarrow K$ in the Hausdorff metric. By Proposition 2.2, each Q_n is homoclinically related to $\text{Orb}(p)$. By Proposition 2.6, there exist $\lambda \in (0, 1)$ and a positive integer m such that

$$\prod_{i=0}^{k-1} \|Df^m|_{E^c(f^{im}(q))}\| < \lambda^k$$

for $q \in Q_n$, where $k = \lceil \pi(q)/m \rceil$. Note that since K is non-trivial and hence $\pi(Q_n) \rightarrow \infty$, by slightly enlarging λ if necessary we have put $C = 1$ in the inequality. Take $\lambda' \in (\lambda, 1)$. By Pliss' lemma, there are $q_n \in Q_n$ such that

$$\prod_{i=0}^{j-1} \|Df^m|_{E^c(f^{im}(q_n))}\| < (\lambda')^j$$

for all $j \geq 1$ (note that, since Q_n is periodic, q_n can be taken so that j runs over all positive integers). Taking a subsequence if necessary we assume $q_n \rightarrow x \in K$. Then

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m|_{E^c(f^{im}(x))}\| < \log \lambda'.$$

This contradicts the above limit equality. Thus E is contracting.

Note that, by Proposition 2.6, F is uniformly expanding at the periods for all periodic points q homoclinically related to p . (Here the phrase ‘‘uniformly expanding at the periods’’ means the inequality in Theorem 2.4, as remarked after Theorem 2.4.) Thus Proposition 5.1 follows directly from the following proposition. \square

Proposition 5.2 ([BGY]). *Let f be a diffeomorphism and let p be a hyperbolic periodic point of f . Assume the homoclinic class $H(p) = H(p, f)$ admits a dominated splitting $T_{H(p)}M = E \oplus F$ such that E is contracting and $\dim(E) = \text{Ind}(p)$. If F is uniformly expanding at the periods for all periodic points q homoclinically related to p , then F is uniformly expanding on $H(p)$.*

Now we prove Theorem A without assuming f is generic. We argue by the shadowing property. We say that $C_f(p)$ is C^1 -robustly shadowable (in [WGW] it is called *stably shadowable*) if there exists a neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $C_g(p_g)$ has the shadowing property, where p_g is the continuation of p .

Proposition 5.3. *Let $f \in \text{Diff}(M)$. If $C_f(p)$ is structurally stable, then $C_f(p)$ is C^1 -robustly shadowable.*

Proof. Fix a C^1 neighborhood \mathcal{U} of f in $\text{Diff}(M)$ such that, for any $g \in \mathcal{U}$, there is a homeomorphism h that conjugates $C_f(p)$ and $C_g(p_g)$. Take $f_0 \in \mathcal{R} \cap \mathcal{U}$. Since $C_f(p)$ is structurally stable, so is $C_{f_0}(p_{f_0})$. By Proposition 5.1, $C_{f_0}(p_{f_0})$ is hyperbolic, hence has the shadowing property. For every $g \in \mathcal{U}$, since $C_g(p_g)$ is conjugate to $C_{f_0}(p_{f_0})$, $C_g(p_g)$ has the shadowing property too. Thus $C_f(p)$ is C^1 -robustly shadowable, proving the proposition. \square

Thus our main result, Theorem A, follows from the following result:

Theorem 5.4 ([WGW]). *If $C_f(p)$ is C^1 -robustly shadowable, then $C_f(p)$ is hyperbolic.*

REFERENCES

- [B] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, *On Devaney's definition of chaos*, Amer. Math. Monthly **99** (1992), no. 4, 332–334, DOI 10.2307/2324899. MR1157223 (93d:54059)
- [BC] Christian Bonatti and Sylvain Crovisier, *Réurrence et généricité* (French, with English and French summaries), Invent. Math. **158** (2004), no. 1, 33–104, DOI 10.1007/s00222-004-0368-1. MR2090361 (2007b:37036)
- [BDP] C. Bonatti, L. J. Díaz, and E. R. Pujals, *A C^1 -generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources* (English, with English and French summaries), Ann. of Math. (2) **158** (2003), no. 2, 355–418, DOI 10.4007/annals.2003.158.355. MR2018925 (2007k:37032)
- [BGY] Christian Bonatti, Shaobo Gan, and Dawei Yang, *On the hyperbolicity of homoclinic classes*, Discrete Contin. Dyn. Syst. **25** (2009), no. 4, 1143–1162, DOI 10.3934/dcds.2009.25.1143. MR2552132 (2011b:37046)
- [Con] Charles Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, vol. 38, American Mathematical Society, Providence, R.I., 1978. MR511133 (80c:58009)
- [Cro1] Sylvain Crovisier, *Birth of homoclinic intersections: a model for the central dynamics of partially hyperbolic systems* (English, with English and French summaries), Ann. of Math. (2) **172** (2010), no. 3, 1641–1677, DOI 10.4007/annals.2010.172.1641. MR2726096 (2012k:37068)
- [Cro2] Sylvain Crovisier, *Partial hyperbolicity far from homoclinic bifurcations*, Adv. Math. **226** (2011), no. 1, 673–726, DOI 10.1016/j.aim.2010.07.013. MR2735772 (2012b:37087)
- [Cro3] Sylvain Crovisier, *Periodic orbits and chain-transitive sets of C^1 -diffeomorphisms*, Publ. Math. Inst. Hautes Études Sci. **104** (2006), 87–141, DOI 10.1007/s10240-006-0002-4. MR2264835 (2008f:37047)
- [Fra] John Franks, *Necessary conditions for stability of diffeomorphisms*, Trans. Amer. Math. Soc. **158** (1971), 301–308. MR0283812 (44 #1042)
- [HPS] M. W. Hirsch, C. C. Pugh, and M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, Vol. 583, Springer-Verlag, Berlin-New York, 1977. MR0501173 (58 #18595)

- [Gou] N. Gourmelon, *A Franks' lemma that preserves invariant manifolds*, preprint, 2008.
- [GW] Shaobo Gan and Lan Wen, *Heteroclinic cycles and homoclinic closures for generic diffeomorphisms*, Special issue dedicated to Victor A. Pliss on the occasion of his 70th birthday, *J. Dynam. Differential Equations* **15** (2003), no. 2-3, 451–471, DOI 10.1023/B:JODY.0000009743.10365.9d. MR2046726 (2005b:37029)
- [Liao1] Shan Tao Liao, *A basic property of a certain class of differential systems* (Chinese, with English summary), *Acta Math. Sinica* **22** (1979), no. 3, 316–343. MR549216 (81c:58045)
- [Liao] Shan Tao Liao, *Obstruction sets. II* (Chinese, with English summary), *Beijing Daxue Xuebao* **2** (1981), 1–36. MR646519 (83c:58065)
- [Man] Ricardo Mañé, *An ergodic closing lemma*, *Ann. of Math. (2)* **116** (1982), no. 3, 503–540, DOI 10.2307/2007021. MR678479 (84f:58070)
- [Man2] Ricardo Mañé, *A proof of the C^1 stability conjecture*, *Inst. Hautes Études Sci. Publ. Math.* **66** (1988), 161–210. MR932138 (89e:58090)
- [PPSV] M. J. Pacifico, E. R. Pujals, M. Sambarino, and J. L. Vieitez, *Robustly expansive codimension-one homoclinic classes are hyperbolic*, *Ergodic Theory Dynam. Systems* **29** (2009), no. 1, 179–200, DOI 10.1017/S0143385708000175. MR2470632 (2009k:37070)
- [PS1] J. Palis and S. Smale, *Structural stability theorems*, *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)*, Amer. Math. Soc., Providence, R.I., 1970, pp. 223–231. MR0267603 (42 #2505)
- [Pli] V. A. Pliss, *On a conjecture of Smale* (Russian), *Differencial'nye Uravnenija* **8** (1972), 268–282. MR0299909 (45 #8957)
- [PS2] Enrique R. Pujals and Martín Sambarino, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*, *Ann. of Math. (2)* **151** (2000), no. 3, 961–1023, DOI 10.2307/121127. MR1779562 (2001m:37057)
- [PS3] Enrique R. Pujals and Martin Sambarino, *Integrability on codimension one dominated splitting*, *Bull. Braz. Math. Soc. (N.S.)* **38** (2007), no. 1, 1–19, DOI 10.1007/s00574-007-0032-7. MR2302745 (2008a:37037)
- [PM] Jacob Palis Jr. and Welington de Melo, *Geometric theory of dynamical systems*, An introduction; Translated from the Portuguese by A. K. Manning, Springer-Verlag, New York-Berlin, 1982. MR669541 (84a:58004)
- [Sak] Kazuhiro Sakai, *C^1 -stably shadowable chain components*, *Ergodic Theory Dynam. Systems* **28** (2008), no. 3, 987–1029. MR2422024 (2009k:37053)
- [Wen] Lan Wen, *The selecting lemma of Liao*, *Discrete Contin. Dyn. Syst.* **20** (2008), no. 1, 159–175, DOI 10.3934/dcds.2008.20.159. MR2350064 (2008i:37058)
- [WD] Xiao Wen and Xiongping Dai, *A note on a sifting-type lemma*, *Discrete Contin. Dyn. Syst.* **33** (2013), no. 2, 879–884. MR2975139
- [WGW] Xiao Wen, Shaobo Gan, and Lan Wen, *C^1 -stably shadowable chain components are hyperbolic*, *J. Differential Equations* **246** (2009), no. 1, 340–357, DOI 10.1016/j.jde.2008.03.032. MR2467027 (2010a:37041)

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