TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 368, Number 6, June 2016, Pages 4439–4459
http://dx.doi.org/10.1090/tran/6474
Article electronically published on January 13, 2016

SPECTRAL FLOW IS A COMPLETE INVARIANT FOR DETECTING BIFURCATION OF CRITICAL POINTS

JAMES C. ALEXANDER AND PATRICK M. FITZPATRICK

Abstract. Given a one-parameter path of equations for which there is a trivial branch of solutions, to determine the points on the branch from which there bifurcate nontrivial solutions, there is the heuristic principle of linearization. That is to say, at each point on the branch, linearize the equation, and justify the inference that points on the branch that are bifurcation points for the path of linearized equations are also bifurcation points for the original path of equations. In quite general circumstances, for the bifurcation of critical points, we show that, at isolated singular points of the path of linearizations, a property of the path that is known to be sufficient to force bifurcation of nontrivial critical points is also necessary.

To be more precise, let I be an open interval of real numbers that contains the point λ_0 and B an open ball about the origin of a real, separable Hilbert space H. Let $\psi\colon I \times B \to R$ be a family of C^2 functions. For $\lambda \in I$, assume $\nabla_x \psi(\lambda, 0) = 0$, and set $\text{Hessian}_x\, \psi(\lambda, 0) \equiv L_\lambda$. Assume L_λ is invertible if $\lambda \neq \lambda_0$ and L_{λ_0} is Fredholm. It is known that if the spectral flow of $L\colon I \to \mathcal{L}(H)$ across λ_0 is nonzero, then in each neighborhood of $(\lambda_0, 0)$ there are pairs (λ, x), $x \neq 0$, for which $\nabla_x \psi(\lambda, x) = 0$. We prove that if $L\colon I \to \mathcal{L}(H)$ is a continuous path of symmetric operators for which L_λ is invertible for $\lambda \neq \lambda_0$, L_{λ_0} is Fredholm, and the spectral flow of $L\colon I \to \mathcal{L}(H)$ across λ_0 is zero, then there is an open interval J that contains the point λ_0, an open ball B about the origin, and a family $\psi\colon J \times B \to R$ of C^2 functions such that, for each $\lambda \in J$, $\nabla_x \psi(\lambda, 0) = 0$ and $\text{Hessian}_x\, \psi(\lambda, 0) = L_\lambda$, but $\nabla_x \psi(\lambda, x) \neq 0$ if $x \neq 0$. Therefore, at an isolated singular point of the path of linearizations of the gradient, under the sole further assumption that the linearization at the singular point is Fredholm, spectral flow is a complete invariant for the detection of bifurcation of nontrivial critical points.

Introduction

Throughout, I denotes an open interval of real numbers that contains the point λ_0, H denotes a real, separable Hilbert space, and B denotes an open ball about the origin in H.[1] A function $\psi\colon I \times B \to R$ is called a family of C^2 functions provided that, at each (λ, x) in $I \times B$, ψ has a first and second Fréchet derivative with respect to x and these derivatives depend continuously on (λ, x), so that the vector $\nabla_x \psi(\lambda, x)$ and the symmetric linear operator on H, $\text{Hessian}_x\, \psi(\lambda, x)$, depend continuously on (λ, x). We examine critical points of the functional $x \mapsto \psi(\lambda, x)$ that belong to B, for different values of λ. The variable λ acts as a parameter. Assume that for each $\lambda \in I$, 0 is a critical point of the function $x \mapsto \psi(\lambda, x)$.

Received by the editors May 15, 2013 and, in revised form, May 1, 2014.
2010 *Mathematics Subject Classification.* Primary 46T99, 47J15, 58E05, 58E07.

[1] $\mathcal{L}(H)$ denotes the Banach space of bounded linear operators on H, and $\mathcal{G}L(H)$ denotes the group, under composition, of invertible operators in $\mathcal{L}(H)$; $\mathcal{L}_{sym}(H)$ denotes the set of symmetric operators in $\mathcal{L}(H)$ and $\mathcal{G}L_{sym}(H)$ denotes the set of symmetric operators in $\mathcal{G}L(H)$.

We consider $I \times \{0\}$ as the trivial branch of critical points and say that nontrivial critical points bifurcate from $I \times \{0\}$ at $(\lambda_0, 0)$ provided that each neighborhood of $(\lambda_0, 0)$ contains points $(\lambda, x), x \neq 0$, at which $\nabla_x \psi(\lambda, x) = 0$. The implicit function theorem tells us that in order for $(\lambda_0, 0)$ to be a bifurcation point it is necessary that $\text{Hessian}_x\, \psi(\lambda_0, 0)$ be singular. But this is not sufficient to force bifurcation of critical points at $(\lambda_0, 0)$, even if the function ψ is C^∞.

In his classic book [13], Krasnosel'skii explicitly addressed the problem of justifying linearization as a criteria for determining bifurcation, by which he meant, in general terms, the following. Let $\psi\colon I \times B \to R$ be a suitably smooth function such that, for $\lambda \in I$, $\nabla_x \psi(\lambda, 0) = 0$. For $\lambda \in I$, define $L_\lambda = \text{Hessian}_x\, \psi(\lambda, 0)$, the linearization of $x \mapsto \nabla_x \psi(\lambda, x)$ at $x = 0$. Consider the path $L\colon I \to \mathcal{L}(H)$. In addition to requiring that L_{λ_0} be singular, Krasnosel'skii considered what further properties of the path $L\colon I \to \mathcal{L}(H)$ are sufficient to force bifurcation of critical points of ψ from $I \times \{0\}$ at $(\lambda_0, 0)$.

There is a large literature, dating back to the work of Euler (see [10]), regarding the provision of properties of the path of linearizations that are sufficient to force bifurcation of critical points of parametrized paths of functions: see, for instance, the papers [3], [5], [11], [15], [16], [20], [21], and for a wider perspective on bifurcation of critical points see, for instance, the books [2], [4], [14], [17], [22].

The concept of spectral flow was first introduced, by Atiyah, Patodi, and Singer [1], for paths of elliptic self-adjoint operators, and since then has been extended, in various degrees of generality, and applied to problems in analysis and geometry. Recall that a symmetric operator $T \in \mathcal{L}(H)$ is said to be Fredholm provided its kernel, $\ker T$, has finite dimension and its image, $\operatorname{Im} T$, is closed, in which case there is the orthogonal direct sum decomposition $H = \ker T \oplus \operatorname{Im} T$. The spectral flow of a path of symmetric Fredholm operators, tailored to the study of bifurcation of critical points of paths of strongly indefinite functionals, was introduced by Fitzpatrick, Pejsachowicz, and Recht ([8],[7]). There is the following justification of the linearization principle: Let $\psi\colon I \times B \to R$ be a family of C^2 functions for which $\nabla_x \psi(\lambda, 0) = 0$ for all $\lambda \in I$. For $\lambda \in I$, set $\text{Hessian}_x\, \psi(\lambda, 0) \equiv L_\lambda$. Assume L_λ is invertible for $\lambda \neq \lambda_0$, L_{λ_0} is Fredholm and

$$\text{the spectral flow of } L_\lambda \text{ across } \lambda_0 \text{ is nonzero.}$$

Then nontrivial critical points bifurcate from the branch $I \times \{0\}$ at the point $(\lambda_0, 0)$. This bifurcation theorem was first established in [8] in the case that ψ is C^2; its refinement for families of C^2 functions is due to Pejsachowicz and Waterstraat [19].

Our goal here is to prove the following complementary theorem, which shows that spectral flow is a complete invariant for the detection of bifurcation of critical points at an isolated singular point of the path of linearizations of the gradient.

Nonbifurcation Theorem. *Let $L\colon I \to \mathcal{L}(H)$ be a continuous path of symmetric operators such that L_λ is invertible for $\lambda \neq \lambda_0$ and L_{λ_0} is Fredholm. Assume*

$$\textit{the spectral flow of } L_\lambda \textit{ across } \lambda_0 \textit{ is zero.}$$

Then there is an open interval J containing λ_0, an open ball B about the origin in H, and a family $\psi\colon J \times B \to R$ of C^2 functions that possesses the following two properties: for $(\lambda, x) \in J \times B$,

$$\nabla_x \psi(\lambda, x) = 0 \quad \textit{if and only if} \quad x = 0, \tag{1}$$

and, for $\lambda \in J$,

$$\text{Hessian}_x\, \psi(\lambda, 0) = L_\lambda. \tag{2}$$

To describe some special cases of this theorem, a few observations regarding the computation of spectral flow are in order. For a continuous path $L\colon I \to \mathcal{L}(H)$ of bounded symmetric operators on a real, separable Hilbert space H such that L_λ is invertible if $\lambda \neq \lambda_0$ and L_{λ_0} is Fredholm, there is defined an integer, denoted by $\text{sf}(L,\, \lambda_0)$, called the spectral flow of L across λ_0.

In the finite dimensional case, so $H = \mathbf{R}^n$, if A belongs to $\mathcal{G}L_{sym}(\mathbf{R}^n)$, its Morse index, $\mu(A)$, is the dimension of the direct sum of the eigenspaces associated with negative eigenvalues of A. So $\mu(A)$ is the Morse index of 0, considered as a critical point of the quadratic form $x \mapsto \langle Ax,\, x\rangle$. If the continuous path $L\colon I \to \mathcal{L}_{sym}(\mathbf{R}^n)$ has L_λ invertible for $\lambda \neq \lambda_0$, let $\mu(L_\lambda, \lambda_0^+)$ be the constant Morse index of $L_\lambda, \lambda > \lambda_0$. Similarly for $\mu(L_\lambda, \lambda_0^-)$. There is the following simple expression for the integer spectral flow:

$$\text{sf}(L,\, \lambda_0) = \mu(L_\lambda, \lambda_0^-) - \mu(L_\lambda, \lambda_0^+). \tag{3}$$

A symmetric Fredholm operator $T \in \mathcal{L}(H)$ is said to be essentially positive provided that its negative spectrum comprises a finite number of eigenvalues, each of which has finite multiplicity. If the symmetric operator L_{λ_0} is essentially positive, then the Morse indices $\mu(L_\lambda, \lambda_0^+)$ and $\mu(L_\lambda, \lambda_0^-)$ are properly defined just as above and formula (3) continues to hold.

Specific computations of spectral flow for continuous paths of strongly-indefinite (meaning infinite dimensional positive and negative spectral subspaces) symmetric Fredholm operators that arise in the study of Hamiltonian systems are given in [9].

If the path of symmetric operators $L\colon I \to \mathcal{L}(H)$ is continuously differentiable and L_{λ_0} is Fredholm, spectral flow may be computed as follows: Let $V = \ker L_{\lambda_0}$ and define the quadratic form $Q = Q(L,\, \lambda_0)$ on V by

$$Q(x) \equiv \langle L'_{\lambda_0} x,\, x\rangle \text{ for } x \in V,$$

where L' denotes the derivative of L with respect to λ. The form $Q(L,\, \lambda_0)$ is called the crossing form of $L\colon I \to \mathcal{L}(H)$ at λ_0, and λ_0 is said to be a regular singular point of the path provided that L_{λ_0} is singular and the crossing form is nondegenerate. If λ_0 is a regular singular point of $L\colon I \to \mathcal{L}(H)$, then L_λ is invertible for $\lambda \neq \lambda_0$ near λ_0 and

$$\text{sf}(L,\, \lambda_0) = \text{sig}\, Q(L,\, \lambda_0), \tag{4}$$

where $\text{sig}\, Q$ is the signature of the quadratic form Q on the finite dimensional vector space V.

We have the following two corollaries of the Nonbifurcation Theorem, which follow from (3) and (4), respectively.

Corollary 1. *Let $L\colon I \to \mathcal{L}(H)$ be a continuous path of symmetric operators for which L_λ is invertible if $\lambda \neq \lambda_0$ and L_{λ_0} is both Fredholm and essentially positive. Assume*

$$\mu(L_\lambda, \lambda_0^+) = \mu(L_\lambda, \lambda_0^-).$$

Then there is an open interval J containing λ_0, an open B ball about the origin in H, and a family $\psi\colon J \times B \to R$ of C^2 functions for which (1) *and* (2) *hold.*

Corollary 2. *Let $L\colon I \to \mathcal{L}(H)$ be a continuously differentiable path of symmetric operators for which L_{λ_0} is Fredholm. Assume λ_0 is a regular singular point of $L\colon I \to \mathcal{L}(H)$ and*

$$\operatorname{sig} Q(L,\, \lambda_0) = 0.$$

Then there is an open interval J containing λ_0, an open ball B about the origin in H, and a family $\psi\colon J \times B \to R$ of C^2 functions for which (1) *and* (2) *hold.*

In [13, Theorem 2.2], Krasnosel'skii presented his solution, for a particular type of path, to the problem of justifying the heuristic principle of linearization for the detection of bifurcation. Indeed, for a C^2 function $\phi\colon H \to \mathbf{R}$ for which $\nabla\phi(0) = 0$, he considered critical points of the family ψ of C^2 functions defined by

$$\psi(\lambda,\, x) = x - \lambda\phi(x).$$

Then $\nabla_x\psi(\lambda,\, 0) = 0$ for all λ. The path of Hessians at the origin is

$$\lambda \mapsto L_\lambda = \operatorname{Id} - \lambda K, \text{ where } K = \operatorname{Hessian} \phi(0).$$

The assumptions Krasnosel'skii imposed on ϕ implied that the operator K is compact. Consequently, L is a continuously differentiable path of essentially positive, symmetric Fredholm operators. Krasnosel'skii proved that if λ_0 is any proper value of K, then there is bifurcation of nontrivial critical points from $\mathbf{R} \times \{0\}$ at $(\lambda_0,\, 0)$. This result is related to spectral flow by the following formula:

$$\operatorname{sf}(L, \lambda_0) = \mu(L, \lambda_0^-) - \mu(L, \lambda_0^+) = -\dim\ker(\operatorname{Id} - \lambda_0 K).$$

The spectral flow across a point is defined in terms of the spectral flow over an interval. We collect here the fundamental properties of spectral flow over an interval; these, and other properties of spectral flow that we need, are established in [8]. Denote by $\Phi_{sym}(H)$ the collection of symmetric Fredholm operators in $\mathcal{L}(H)$, and, for a compact interval $[a,\, b]$, call a continuous path $L\colon [a,\, b] \to \Phi_{sym}(H)$ admissible provided it has invertible end-points. For each real, separable Hilbert space H and admissible path $L\colon [a,\, b] \to \Phi_{sym}(H)$, an integer, denoted by $\operatorname{sf}(L,\, [a,\, b])$ and called the spectral flow of L over $[a,\, b]$, is defined. Spectral flow has the following properties:

Difference of Morse indices. In the case $H = \mathbf{R}^n$,

$$\operatorname{sf}(L,\, [a,\, b]) = \mu(L_a) - \mu(L_b).$$

Normalization. For a continuous path $L\colon [a,\, b] \to \mathcal{G}L(H)$,

$$\operatorname{sf}(L,\, [a,\, b]) = 0.$$

Concatenation. If $a < c < b$ and L_c is invertible, then

$$\operatorname{sf}(L,\, [a,\, b]) = \operatorname{sf}(L,\, [a,\, c]) + \operatorname{sf}(L,\, [c,\, b]).$$

Homotopy invariance. If $L\colon [0,\, 1] \times [a,\, b] \to \Phi_{sym}(H)$ is continuous and each $L(t, \cdot)$ is admisible, then

$$\operatorname{sf}(L(0, \cdot),\, [a,\, b]) = \operatorname{sf}(L(1, \cdot),\, [a,\, b]).$$

Additivity over direct sums. If each L_λ is reduced by the orthogonal direct sum $H = H_1 \oplus H_2$, then

$$\operatorname{sf}(L,\, [a,\, b]) = \operatorname{sf}(L|_{H_1},\, [a,\, b]) + \operatorname{sf}(L|_{H_2},\, [a,\, b]).$$

In Ciriza, Fitzpatrick, and Pejsachowicz [6], it is shown that spectral flow has the following uniqueness property: it is the only assignment of an integer to each admissible path of symmetric Fredholm operators on each real, separable Hilbert space for which the first four of the above properties hold.

Let I be an open interval containing the point λ_0 and $L\colon I \to \Phi_{sym}(H)$ a continuous path for which L_λ is invertible if $\lambda \neq \lambda_0$. By the concatenation and normalization properties of spectral flow, the spectral flow of $L\colon I \to \Phi_{sym}(H)$ across λ_0, denoted by $\mathrm{sf}(L,\, \lambda_0)$, is properly defined to be $\mathrm{sf}(L,\, [a,\, b])$, where $[a, b]$ is any closed interval contained in I for which λ_0 is an interior point.

In Section 1, we prove the Nonbifurcation Theorem in the case that $H = \mathbf{R}^n$ and $L_{\lambda_0} = 0$. In Section 2, we prove the general theorem.

1. The restricted finite dimensional case

In this section, we assume $H = \mathbf{R}^n$ and $L_{\lambda_0} = 0$. For notational convenience, we take $I = (-1,\, 1)$ and $\lambda_0 = 0$. Under the provisional assumption that $0 < ||L_\lambda|| < |\lambda|$ for $\lambda \neq 0$, we construct the function $\psi\colon I \times B \to \mathbf{R}$ having the required properties. We later remove this provisional assumption by making a change of variables in the parameter λ, which possibly requires replacing I by a smaller interval J.

We construct $\psi\colon [0,\, 1) \times B \to \mathbf{R}$ by a succession of extensions. First, for a suitably small $\epsilon > 0$, we define

$$\psi(\lambda, x) \equiv \frac{1}{2}\langle L_\lambda x,\, x\rangle \quad \text{for} \quad 0 < \lambda < 1, \|x\| < \epsilon \cdot \lambda. \tag{5}$$

Then ψ is successively extended to cones $\{(\lambda, x) \,|\, 0 < \lambda < 1, \|x\| < c \cdot \lambda\}$, on each of which, near the boundary, $\psi(\lambda, x) = \lambda/2 \cdot \langle A_\lambda x,\, x\rangle$, where the path $\lambda \to A_\lambda$ is first a path of orthogonal, symmetric operators, then a smooth path of orthogonal, symmetric operators, and finally a constant path, which takes the value I_0, I_0 being any orthogonal, symmetric operator which has Morse index equal to that of L_λ, for $\lambda > 0$. We then extend ψ to a family of C^2 functions on $[0, 1) \times B$, which takes values on $\{0\} \times B$ that only depend on the Morse index of L_λ, for $\lambda > 0$. Since the spectral flow of $L\colon I \to \mathcal{L}(\mathbf{R}^n)$ across λ_0 is 0, we can make a corresponding construction of ψ on $(-1, 0] \times B$, which agrees with the preceding construction on $\{0\} \times B$.

Of course, at each stage in the construction, we verify that

$$\nabla_x \psi(\lambda,\, x) \neq 0 \quad \text{if} \quad x \neq 0. \tag{6}$$

Moreover, in order to finally show that $\psi\colon [0,\, 1) \times B$ is a family of C^2 functions on a neighborhood of $(0, 0)$ for which $\nabla_x \psi(0, 0) = 0$ and $\mathrm{Hessian}_x(0, 0) = L_0 = 0$, at each stage of the construction we also verify that, for $1 \leq i, j \leq n$,

$$\lim_{(\lambda,x)\to(0,0)} \psi(\lambda, x) = \lim_{(\lambda,x)\to(0,0)} \frac{\partial \psi}{\partial x_i}(\lambda, x) = \lim_{(\lambda,x)\to(0,0)} \frac{\partial^2 \psi}{\partial x_i \partial x_j}(\lambda, x) = 0. \tag{7}$$

In what follows in this section, ϵ is a number in the interval $(0, 1/100)$. We establish three preliminary results. First we record the following lemma, whose proof follows by directly computing partial derivatives.

Lemma 1. *Let D' be a subset of $[0,\, 1) \times (0,\, 1)$ and $M\colon D' \to \mathcal{L}_{sym}(\mathbf{R}^n)$ a continuous map for which*

$$\frac{\partial M}{\partial r}(\lambda, r) \quad \textit{and} \quad \frac{\partial^2 M}{\partial r^2}(\lambda, r) \textit{ exist and depend continuously on } (\lambda, r).$$

Define

$$\psi(\lambda,\, x) \equiv \frac{1}{2}\langle M(\lambda, \|x\|)x,\, x\rangle \quad \textit{for} \quad (\lambda, x) \in D \equiv \{(\lambda,\, x) \mid\ (\lambda,\, \|x\|) \in D'\}\,.$$

Then ψ is a family of C^2 functions such that, for each $(\lambda,\, x) \in D$,

$$\nabla_x \psi(\lambda,\, x) = \frac{1}{2}\langle \frac{\partial M}{\partial r}(\lambda, \|x\|)x,\, x\rangle \frac{x}{\|x\|} + M(\lambda, \|x\|)x.$$

Moreover, if

$$\lim_{(\lambda,r)\to(0,0)} M(\lambda,\, r) = \lim_{(\lambda,r)\to(0,0)} r \cdot \frac{\partial M}{\partial r}(\lambda,\, r) = \lim_{(\lambda,r)\to(0,0)} r^2 \cdot \frac{\partial^2 M}{\partial r^2}(\lambda,\, r) = 0,$$

then, for $1 \le i, j \le n$, (7) holds.

Recall that an operator $T \in \mathcal{L}(\mathbf{R}^n)$ is said to be orthogonal provided that $\|Th\| = \|h\|$ for all h. A symmetric operator on $\mathbf{R}^n$ is orthogonal if and only if each of its eigenvalues is either 1 or -1.

Lemma 2. *Let I be an open interval and $h\colon I \to \mathcal{L}(\mathbf{R}^n)$ a continuously differentiable path of orthogonal symmetric operators. For $x \in \mathbf{R}^n$ with $\|x\| \in I$, define*

$$\phi(x) \equiv \frac{1}{2}\langle h(\|x\|)x,\, x\rangle.$$

Then

$$\nabla_x \phi(x) \neq 0 \quad \textit{if} \quad \|x\| \in I \quad \textit{and} \quad x \neq 0.$$

Proof. Let $\|x\|$ belong to I, $x \neq 0$. Since h is a differentiable path of symmetric operators, we compute

$$\nabla_x \phi(x) = \frac{1}{\|x\|} \cdot \langle h'(\|x\|)x,\, x\rangle x + h(\|x\|)x. \tag{8}$$

For $|t| < \epsilon$, define $s(t) \equiv h(\|x + tx\|)x$. Since each $h(\|x + tx\|)$ is orthogonal,

$$\langle s(t),\, s(t)\rangle = \|x\|^2 \quad \text{for} \quad |t| < \epsilon.$$

Differentiate each side to obtain

$$0 = \frac{d}{dt}\langle s(t),\, s(t)\rangle\big|_{t=0} = 2\langle s'(0),\, s(0)\rangle.$$

But $s'(0) = \|x\|h'(\|x\|)x$. Therefore

$$\langle h'(\|x\|)x,\, h(\|x\|)x\rangle = 0.$$

Take the inner-product of each side of (8) with $h'(\|x\|)x$ to obtain

$$\langle \nabla_x \phi(x),\, h'(\|x\|)x\rangle = \frac{1}{\|x\|} \cdot \|h'(\|x\|)x\|^2.$$

Assume $\nabla_x \phi(x) = 0$. Then $h'(\|x\|)x = 0$, and therefore we deduce from (8) that $h(\|x\|)x = 0$, which, since $x \neq 0$, contradicts the assumption that the operator $h(\|x\|)$ is orthogonal. □

Lemma 3. *Let I be an open interval and $g\colon I \to \mathbf{R}$ a differentiable function for which*

$$g(r) > 0 \quad \textit{and} \quad g'(r) \ge 0 \quad \textit{for} \quad r \in I. \tag{9}$$

Let A belong to $\mathcal{G}L_{sym}(\mathbf{R}^n)$. For $x \in \mathbf{R}^n$ with $\|x\| \in I$, define

$$\phi(x) \equiv \frac{1}{2} g(\|x\|)\langle Ax,\, x\rangle.$$

Then

$$\nabla_x \phi(x) \neq 0 \quad \textit{if} \quad \|x\| \in I \quad \textit{and} \quad x \neq 0.$$

Proof. Let $\|x\| \in I$ with $x \neq 0$. Since A is symmetric, we compute

$$\nabla_x \phi(x) = \frac{1}{\|x\|} \cdot g'(\|x\|)\langle Ax,\, x\rangle x + g(\|x\|)Ax.$$

Take the inner-product of each side with Ax to obtain

$$\langle \nabla_x \phi(x),\, Ax\rangle = \frac{1}{\|x\|} g'(\|x\|)[\langle Ax,\, x\rangle]^2 + g(\|x\|)\|Ax\|^2.$$

Since A is invertible, $Ax \neq 0$, and therefore, by (9),

$$\langle \nabla_x \phi(x),\, Ax\rangle > 0.$$

Hence $\nabla_x \phi(x) \neq 0$. □

1.1. **The extension of ψ to $\{(\lambda,\, x) \mid 0 < \lambda < 1, \|x\| < (1/4 + \epsilon) \cdot \lambda\}$.** For $A \in \mathcal{G}L_{sym}(\mathbf{R}^n)$, let $E^-(A)(E^+(A))$ denote the direct sum of the eigenspaces associated with the negative (respectively, positive) eigenvalues of A. Since A is symmetric and invertible, there is the orthogonal direct sum decomposition

$$\mathbf{R}^n = E^-(A) \oplus E^+(A).$$

Let $P(A)$ be the orthogonal projection of $\mathbf{R}^n$ onto $E^-(A)$. An orthogonal projection is symmetric. Define the operator $\mathcal{E}\colon \mathcal{G}L_{sym}(\mathbf{R}^n) \to \mathcal{G}L_{sym}(\mathbf{R}^n)$ by

$$\mathcal{E}(A) \equiv \mathrm{Id} - 2P(A) \quad \text{for} \quad A \in \mathcal{G}L_{sym}(\mathbf{R}^n). \tag{10}$$

Observe that the restricton of $\mathcal{E}(A)$ to $E^-(A)$ is $-\mathrm{Id}$, and its restriction to $E^+(A)$ is Id. Consequently, $\mathcal{E}(A)$ is orthogonal and symmetric. Moreover, since the operator $A \mapsto P(A)$ is smooth, so is the operator $A \mapsto \mathcal{E}(A)$.

Proposition 1. *Let $L\colon [0,\, 1) \to \mathcal{L}(\mathbf{R}^n)$ be a continuous path of symmetric operators such that $0 < \|L_\lambda\| < \lambda$ for $\lambda > 0$. Define*

$$D_1 \equiv \{(\lambda, x) \mid \ 0 < \lambda < 1, ||x|| < (1/4 + \epsilon) \cdot \lambda\}.$$

There is a family $\psi_1\colon D_1 \to \mathbf{R}$ of C^2 functions which possesses properties (5), (6), *and* (7), *and furthermore,*

$$\psi_1(\lambda,\, x) = \frac{\lambda}{2}\langle \mathcal{E}(L_\lambda)x,\, x\rangle \quad \textit{for} \quad 0 < \lambda < 1, (1/4 - \epsilon) \cdot \lambda < \|x\| < (1/4 + \epsilon) \cdot \lambda. \tag{11}$$

Proof. Choose a smooth, increasing function $v\colon \mathbf{R} \to [0, 1]$ such that

$$v(t) = 0 \quad \text{for} \quad t \leq \epsilon \quad \text{and} \quad v(t) = 1 \quad \text{for} \quad t \geq 1/4 - \epsilon.$$

For $0 < \lambda < 1$, define

$$T(\lambda) \equiv \lambda \mathcal{E}(L_\lambda) - L_\lambda.$$

Then, for $(\lambda,\, x) \in D_1$, define

$$\psi_1(\lambda,\, x) \equiv \frac{1}{2}\langle L_\lambda x,\, x\rangle + \frac{1}{2} v\Big(\frac{\|x\|}{\lambda}\Big)\langle T_\lambda x,\, x\rangle.$$

Since $v(t) = 0$ if $t < \epsilon$, ψ_1 satisfies (5), and since $v(t) = 1$ if $t > 1/4 - \epsilon$, ψ_1 satisfies (11).

For a point (λ, x) with $x = 0$, there is a neighborhood of this point on which $\psi_1(\lambda, x) = \frac{1}{2}\langle L_\lambda x,\, x\rangle$, and certainly restricted to this neighborhood ψ_1 is a family of C^2 functions. For a point (λ, x) with $x \neq 0$, since v is smooth, and the paths $\lambda \mapsto L_\lambda$ and $\lambda \mapsto \mathcal{E}(L_\lambda)$ are continuous, there is a neighborhood of (λ, x) restricted to which ψ_1 is a family of C^2 functions.

Now $\lim_{\lambda\to 0} L_\lambda = \lim_{\lambda\to 0} \lambda \cdot \mathcal{E}(L_\lambda) = 0$, and the functions $v(\|x\|/\lambda)$, $\|x\|/\lambda \cdot v'(\|x\|/\lambda)$, and $\|x\|^2/\lambda^2 \cdot v''(\|x\|/\lambda)$ are bounded on D_1. We therefore appeal to Lemma 1 to conclude that (7) holds.

It remains to verify (6). Fix $\lambda, 0 < \lambda < 1$. Since L_λ and $\mathcal{E}(L_\lambda)$ are symmetric, we compute

$$\nabla_x \psi_1(\lambda,\, x) = L_\lambda x + v\Big(\frac{\|x\|}{\lambda}\Big) T_\lambda x + \frac{1}{2\lambda \cdot \|x\|} v'\Big(\frac{\|x\|}{\lambda}\Big)\langle T_\lambda x,\, x\rangle x \quad \text{if} \quad x \neq 0.$$

Let $x \neq 0$ belong to $\mathbf{R}^n$ and suppose $\nabla_x \psi_1(\lambda,\, x) = 0$. Define

$$\alpha \equiv v\Big(\frac{\|x\|}{\lambda}\Big) \quad \text{and} \quad \beta \equiv \frac{1}{2\lambda \cdot \|x\|} v'\Big(\frac{\|x\|}{\lambda}\Big)\langle T_\lambda x,\, x\rangle$$

and

$$B \equiv L_\lambda + \alpha T_\lambda + \beta\, \mathrm{Id},$$

so that $\nabla_x \psi_1(\lambda,\, x) = Bx = 0$.

The orthogonal decomposition

$$\mathbf{R}^n = E^-(L_\lambda) \oplus E^+(L_\lambda) \tag{12}$$

reduces both L_λ and T_λ, and therefore also reduces B. By assumption,

$$\lambda > \|L_\lambda\| \geq \left|\frac{\langle L_\lambda h,\, h\rangle}{\langle h,\, h\rangle}\right| \quad \text{for} \quad h \neq 0.$$

Hence,

$$\langle T_\lambda h,\, h\rangle = \lambda\langle h,\, h\rangle - \langle L_\lambda h,\, h\rangle > 0 \quad \text{for} \quad h \in E^+(L_\lambda), h \neq 0$$

while

$$\langle T_\lambda h,\, h\rangle = -\lambda\langle h,\, h\rangle - \langle L_\lambda h,\, h\rangle < 0 \quad \text{for} \quad h \in E^-(L_\lambda), h \neq 0.$$

Consequently, since $\alpha \geq 0$, $L_\lambda + \alpha T_\lambda$ and T_λ are positive definite on $E^+(L_\lambda)$, and negative definite on $E^-(L_\lambda)$. In particular, $L_\lambda + \alpha T_\lambda$ is invertible.

We now establish a contradiction to $\nabla_x \psi_1(\lambda,\, x) = Bx = 0$ while $x \neq 0$.

Case 1 ($\beta = 0$). Then $(L_\lambda + \alpha T_\lambda)x = 0$. But $L_\lambda + \alpha T_\lambda$ is invertible. Consequently $x = 0$, which is a contradiction.

Case 2 ($\beta > 0$). Then B is positive definite on $E^+(L_\lambda)$, and hence, since B is reduced by the decomposition (12), x belongs to $E^-(L_\lambda)$. But T_λ is negative definite on $E^-(L_\lambda)$, a contradiction to the assumption that β, and hence $\langle T_\lambda x,\, x\rangle$, is positive.

Case 3 ($\beta < 0$). Therefore B is negative definite on $E^-(L_\lambda)$, and hence, since B is reduced by the decomposition (12), x belongs to $E^+(L_\lambda)$. But T_λ is positive definite on $E^+(L_\lambda)$, a contradiction to the assumption that β, and hence $\langle T_\lambda x,\, x\rangle$, is negative.

The proof is complete. □

1.2. **The extension of ψ to $\{(\lambda, x) \mid 0 < \lambda < 1, \|x\| < (1/2 + \epsilon) \cdot \lambda\}$.**

Proposition 2. *Let $L\colon [0, 1) \to \mathcal{L}(\mathbf{R}^n)$ be a continuous path of symmetric operators such that $0 < \|L_\lambda\| < \lambda$ for $\lambda > 0$. Define*

$$D_2 \equiv \{(\lambda, x) \mid \ 0 < \lambda < 1, ||x|| < (1/2 + \epsilon) \cdot \lambda\}.$$

There is a smooth path $\mathcal{S}\colon (0, 1) \to \mathcal{L}(\mathbf{R}^n)$ of orthoginal, symmetric operators and a family $\psi_2\colon D_2 \to \mathbf{R}$ of C^2 functions which possesses properties (5), (6), *and* (7), *and furthermore,*

$$\psi_2(\lambda, x) = \frac{\lambda}{2}\langle \mathcal{S}_\lambda x, x\rangle \quad \textit{for} \quad 0 < \lambda < 1, (1/2 - \epsilon) \cdot \lambda < \|x\| < (1/2 + \epsilon) \cdot \lambda. \tag{13}$$

Proof. By patching with the family of functions $\psi_1\colon D_1 \to \mathbf{R}$ defined in the preceding proposition, it suffices to define

$$D_2' \equiv \{(\lambda, x) \mid \ 0 < \lambda < 1, (1/4 - \epsilon) \cdot \lambda < \|x\| < (1/2 + \epsilon) \cdot \lambda\}$$

and find a family $\psi_2\colon D_2' \to \mathbf{R}$ of C^2 functions which satisfies (6), (7), and (13), and furthermore,

$$\psi_2(\lambda, x) = \frac{\lambda}{2}\langle \mathcal{E}(L_\lambda)x, x\rangle \quad \text{for} \quad 0 < \lambda < 1, (1/4 - \epsilon) \cdot \lambda < \|x\| < (1/4 + \epsilon) \cdot \lambda. \tag{14}$$

Define

$$b(x) \equiv \begin{cases} c \cdot \exp\bigl(-(x+1)^{-1}\bigr) \exp\bigl(-(x-1)^{-1}\bigr) & \text{for} \quad x \in [-1, 1], \\ 0 & \text{for} \quad x \notin [-1, 1], \end{cases}$$

where c is chosen so that $\int_{-1}^{1} h(x)\, dx = 1$. For $\epsilon > 0$, let $b_\lambda^\epsilon(x) = \epsilon^{-1} b\bigl(\epsilon^{-1}(x - \lambda)\bigr)$. Then the support of $b_\lambda^\epsilon(x)$ is $[\lambda - \epsilon, \lambda + \epsilon]$ and $\int_{\lambda-\epsilon}^{\lambda+\epsilon} b_\lambda^\epsilon(x)\, dx = 1$. Note that $b_\lambda^\epsilon(x)$ is C^∞ in its three variables. The functions b_λ^ϵ are used as mollifiers, integrated against continuous functions.

We deduce from the continuity of the path $\mathcal{E}(L)\colon (0, 1) \to \mathcal{L}(H)$ and the paracompactness and local compactness of the interval $(0, 1)$ that we may choose a smooth function $\epsilon\colon (0, 1) \to (0, \infty)$ such that, for $\lambda', \lambda'' \in (0, 1)$,

$$\|\mathcal{E}(L_{\lambda'}) - \mathcal{E}(L_{\lambda''})\| < \frac{1}{2} \quad \text{if} \quad |\lambda' - \lambda''| < \epsilon(\lambda'). \tag{15}$$

For $0 < \lambda < 1$, define the convolution

$$\mathcal{T}_\lambda \equiv b_\lambda^{\epsilon(\lambda)} * \mathcal{E}(L_\lambda) = \int_{\lambda-\epsilon(\lambda)}^{\lambda+\epsilon(\lambda)} b_\lambda^{\epsilon(\lambda)}(\lambda - \tilde{\lambda})\mathcal{E}(L_{\tilde{\lambda}})\, d\tilde{\lambda},$$

where multiplication of a matrix by a scalar means multiplication of each matrix entry. Thus $\lambda \mapsto \mathcal{T}_\lambda$ is C^∞, since convolution smooths continuous functions. We

estimate $\|\mathcal{T}_\lambda - \mathcal{E}(L_\lambda)\|$ as follows:

$$\begin{aligned}(16)\quad &\Big\|\int_{\lambda-\epsilon(\lambda)}^{\lambda+\epsilon(\lambda)} b_\lambda^{\epsilon(\lambda)}(\lambda-\tilde\lambda)\mathcal{E}(L_{\tilde\lambda})\,d\tilde\lambda - \mathcal{E}(L_\lambda)\Big\| \\ &\qquad = \Big\|\int_{\lambda-\epsilon(\lambda)}^{\lambda+\epsilon(\lambda)} b_\lambda^{\epsilon(\lambda)}(\lambda-\tilde\lambda)\big(\mathcal{E}(L_{\tilde\lambda}) - \mathcal{E}(L_\lambda)\big)\,d\tilde\lambda\Big\| \\ &\qquad \le \Big\|\int_{\lambda-\epsilon(\lambda)}^{\lambda+\epsilon(\lambda)} b_\lambda^{\epsilon(\lambda)}(\lambda-\tilde\lambda)|\mathcal{E}(L_{\tilde\lambda}) - \mathcal{E}(L_\lambda)|\,d\tilde\lambda\Big\| \\ &\qquad\qquad \le \frac{1}{2}\int_{\lambda-\epsilon(\lambda)}^{\lambda+\epsilon(\lambda)} b_\lambda^{\epsilon(\lambda)}(\lambda-\tilde\lambda)\,d\tilde\lambda = \frac{1}{2}.\end{aligned}$$

Now, for $0<\lambda<1$, $\|\mathcal{E}(L_\lambda)\| = 1$, since $\mathcal{E}(L_\lambda)$ is orthogonal and, by the above estimate, $\|\mathcal{T}_\lambda - \mathcal{E}(L_\lambda)\| < 1/2$. Consequently, for $0\le t\le 1, 0<\lambda<1$,

(17) $(1-t)\mathcal{E}(L_\lambda) + t\mathcal{T}_\lambda$ is invertible, and hence $\mathcal{E}((1-t)\mathcal{E}(L_\lambda) + t\mathcal{T}_\lambda)$ is defined.

Choose a smooth, increasing function $v\colon \mathbf{R}\to[0,1]$ such that

$$v(t) = 0 \quad\text{for}\quad t\le 1/4+\epsilon \quad\text{and}\quad v(t) = 1 \quad\text{for}\quad t\ge 1/2-\epsilon.$$

For $0<\lambda<1, (1/4-\epsilon)\cdot\lambda < r < (1/2+\epsilon)$, define

$$M(\lambda,\, r) \equiv \mathcal{E}\big((1-v(r/\lambda))\mathcal{E}(L_\lambda)\big) + v(r/\lambda)\mathcal{T}_\lambda,$$

and then define the function $\psi_2\colon D_2'\to\mathbf{R}$ by

$$\psi_2(\lambda,\, x) \equiv \frac{\lambda}{2}\langle M(\lambda,\, ||x||)x,\, x\rangle.$$

The function v takes values in [0, 1] and therefore we deduce from (17) that M, and hence ψ_2, is properly defined. For $0<\lambda<1$, define $\mathcal{S}_\lambda \equiv \mathcal{E}(\mathcal{T}_\lambda)$. The operator $\mathcal{E}$ is an idempotent, that is, $\mathcal{E}^2 = \mathcal{E}$. Hence, since $v(t) = 0$ if $t\le 1/4+\epsilon$, ψ_2 satisfies (13), and since $v(t) = 1$ if $t\ge 1/2-\epsilon$, ψ_2 satisfies (14).

Since the point $(\lambda, 0)$ does not belong to D_2', S is smooth, v is smooth, and the paths $\lambda\mapsto L_\lambda$ and $\lambda\mapsto\mathcal{E}(L_\lambda)$ are continuous, ψ_2 is a family of C^2 functions.

Now the set

$$\{((1-v(r/\lambda))\mathcal{E}(L_\lambda)) + v(r/\lambda)\mathcal{T}_\lambda \mid\ 0<\lambda<1, (1/4-\epsilon)\cdot\lambda < r < (1/2+\epsilon)\cdot\lambda\}$$

is a relatively compact subset of $\mathcal{G}L(H)$ since it is bounded and each operator in this set has norm at least 1/2. Therefore $\mathcal{E}$ and each derivative of $\mathcal{E}$ is bounded on this set. Furthermore, the functions $v(\|x\|/\lambda)$, $\|x\|/\lambda\cdot v'(\|x\|/\lambda)$, and $\|x\|^2/\lambda^2\cdot v''(\|x\|/\lambda)$ are bounded on D_2'. Consequently, because λ is coefficient of ψ_2, by direct calculation of partial derivatives we conclude that (7) holds for ψ_2.

Finally, each $M(\lambda, r)$ is orthogonal and symmetric and therefore we appeal to Lemma 2 to conclude that (6) holds for ψ_2. □

The above proposition enables the use of transversality in the next section. We observe however that this smoothing could be avoided at the expense of introducing the machinery of topological transversality.

1.3. **The extension of ψ to $\{(\lambda, x) \mid 0 < \lambda < 1, \|x\| < (3/4 + \epsilon) \cdot \lambda\}$.** For this step in the construction of ψ, we need some properties of the Grassmannian and its universal cover. For $0 \le d \le n$, define $\mathcal{O}S^d(\mathbf{R}^n)$ to be the collection of orthogonal, symmetric operators A on $\mathbf{R}^n$ for which the negative eigenspace $E^-(A)$ has dimension d. Define G_n^d to be the set of d-dimensional subspaces of $\mathbf{R}^n$. The set G_n^d is called the Grassmannian of d-dimensional subspaces of $\mathbf{R}^n$; it is a compact, connected, analytic manifold [18, chaps. 5-7], and the map $A \mapsto E^-(A)$ from $\mathcal{O}S^d(\mathbf{R}^n)$ to G_n^d is an analytic bijection. It is convenient to identify $\mathcal{O}S^d(\mathbf{R}^n)$ with G_n^d.

The universal cover of G_n^d, which we denote by $\hat{G}_n^d$, is the space of oriented d-planes in $\mathbf{R}^n$. It is a simply-connected, compact, symmetric space, being the quotient of $SO(n)$ by $SO(d) \times SO(n-d)$. Consider the covering map $\pi\colon \hat{G}_n^d \to G_n^d$. Every continuous path $\gamma'\colon (0,1) \to G_n^d$ lifts, through π, to a continuous path into $\hat{G}_n^d$, that is, there is a continuous path $\gamma\colon (0,1) \to \hat{G}_n^d$ for which $\gamma' = \pi \circ \gamma$.

For any point p belonging to a Riemannian manifold, the cut locus of p is the set of points for which there is not a unique shortest geodesic to p (for an exposition of cut loci, see [12]). Fix I_0 in $\mathcal{O}S^d(\mathbf{R}^n)$ and choose p_0 in $\hat{G}_n^d$ for which $\pi(p_0) = I_0$. Denote the cut locus of p_0 in $\hat{G}_n^d$ by $\mathcal{C}$. Let $(A, t) \mapsto \rho(A, t)$, $0 \le t \le 1$ be the geodesic deformation retraction from $\hat{G}_n^d \sim \mathcal{C}$ to p_0 along the unique shortest geodesic, suitably mollified so that each of its t derivatives is bounded. Given a path $T\colon (0,1) \to \mathcal{O}S^d(\mathbf{R}^n)$, lift it to a path $\gamma\colon (0,1) \to \hat{G}_n^d$. Assume that γ takes values in $\hat{G}_n^d \sim \mathcal{C}$, so that we may properly define

$$H(\lambda, t) \equiv \pi\big(\rho(\gamma(\lambda), t)\big) \quad \text{for} \quad 0 < \lambda < 1, 0 \le t \le 1.$$

Observe that each $H(\lambda, t)$ is orthogonal and symmetric,

$$H(\lambda, 0) = T_\lambda \quad \text{and} \quad H(\lambda, 1) = I_0 \quad \text{for} \quad 0 < \lambda < 1.$$

Proposition 3. *Let $L\colon [0, 1) \to \mathcal{L}(\mathbf{R}^n)$ be a continuous path of symmetric operators such that $0 < \|L_\lambda\| < \lambda$ for $\lambda > 0$. Define*

$$D_3 \equiv \{(\lambda, x) \mid \ 0 < \lambda < 1, ||x|| < (3/4 + \epsilon)\lambda\}.$$

Let d be the Morse index of L_λ for $\lambda > 0$. Let p_0 belong to $\hat{G}_n^d$. There is a smooth path $\gamma\colon (0, 1) \to \hat{G}_n^d$ which misses the cut locus of p_0 and a family $\psi_3\colon D_3 \to \mathbf{R}$ of C^2 functions which possesses properties (5), (6), *and* (7), *and furthermore,*

$$\psi_3(\lambda, x) = \frac{\lambda}{2}\langle \pi(\gamma(\lambda))x, x\rangle \text{ for } 0 < \lambda < 1,\ (3/4 - \epsilon) \cdot \lambda < \|x\| < (3/4 + \epsilon) \cdot \lambda. \tag{18}$$

Proof. By patching with the function ψ_2 defined in the preceding proposition, it suffices to define

$$D_3' \equiv \{(\lambda, x) \mid \ 0 < \lambda < 1, (1/2 - \epsilon) \cdot \lambda < \|x\| < (3/4 + \epsilon) \cdot \lambda\}$$

and find a path $\psi_3\colon D_3' \to \mathbf{R}$ of C^2 functions for which (6), (7), and (18) hold, and furthermore,

$$\psi_3(\lambda, x) = \frac{\lambda}{2}\langle \mathcal{S}_\lambda x, x\rangle \text{ for } 0 < \lambda < 1,\ (1/2 - \epsilon) \cdot \lambda < \|x\| < (1/2 + \epsilon) \cdot \lambda, \tag{19}$$

where $\lambda \mapsto \mathcal{S}_\lambda$ is the smooth path of orthogonal, symmetric operators on $\mathbf{R}^n$ from the preceding proposition.

We use the notation and assertions from the above discussion of Grassmannians. For certain manifolds, in particular for certain simply-connected, compact, symmetric manifolds like $\hat{G}_n^d$, which are quotients of semisimple Lie groups, the

cut locus of each point is an analytic stratified set of codimension at least 2 (see [24, Theorem 1.1] and [23, Theorem 5.3]).

We claim that that there is a smooth path $\gamma\colon (0,\,1) \to \hat{G}_n^d$ that misses the cut locus of p_0 and

$$\|\mathcal{S}_\lambda - \pi(\gamma(\lambda))\| < 1/2 \text{ for all } \lambda \text{ in } (0,\,1). \tag{20}$$

Indeed, consider the lift $\gamma'\colon (0,\,1) \to \hat{G}_n^d$ of the path $S\colon (0,1) \to \mathcal{O}S^d(\mathbf{R}^n)$. The space $\hat{G}_n^d$ is compact and π is continuous. Thus there is a $\delta > 0$ such that, for points p_1 and p_2 in $\hat{G}_n^d$, if $d(p_1, p_2) < \delta$, then $\|\pi(p_1) - \pi(p_2)\| < 1/2$; here d is the metric on $\hat{G}_n^d$. But the cut locus of p_0 is of codimension at least 2 in $\hat{G}_n^d$ and therefore, by the paracompactness and local compactness of the interval $(0,\,1)$, there is a smooth path $\gamma\colon (0,\,1) \to \hat{G}_n^d$ which misses the cut locus of p_0 and

$$d(\gamma(\lambda),\, \gamma'(\lambda)) < \delta \text{ for all } \lambda \text{ in } (0,\,1).$$

Therefore, by the choice of δ, since $S = \pi \circ \gamma'$, (20) holds.

Choose a smooth, increasing function $v\colon \mathbf{R} \to [0,1]$ such that

$$v(t) = 0 \quad \text{for} \quad t \le 1/2 + \epsilon \quad \text{and} \quad v(t) = 1 \quad \text{for} \quad t \ge 3/4 - \epsilon.$$

For $0 < \lambda < 1,\ (1/2 - \epsilon) \cdot \lambda < r < (3/4 + \epsilon) \cdot \lambda$, define

$$M(\lambda,\, r) \equiv \mathcal{E}\big((1 - v(r/\lambda))\mathcal{S}_\lambda + v(r/\lambda)(\pi \circ \gamma)\big),$$

and then define the function $\psi_3\colon D_3' \to \mathbf{R}$ by

$$\psi_3(\lambda,\, x) \equiv \frac{\lambda}{2}\langle M(\lambda,\, ||x||)x,\, x\rangle.$$

Now $\mathcal{E}$ only acts on invertible symmetric operators. We deduce from (20) that M is properly defined. The operator $\mathcal{E}$ is an idempotent, and therefore (19) holds since $v = 0$ for $t \le 3/4 - \epsilon$ and (18) holds since $v = 1$ for $t \ge 1/2 + \epsilon$. We argue as in the conclusion of the proof of Proposition 2 to verify that, in view of (20), property (7) holds for ψ_3. Finally, each $M(\lambda, r)$ is orthogonal and symmetric and therefore we appeal to Lemma 2 to conclude that (6) holds for ψ_3. □

1.4. The extension of ψ to $\{(\lambda,\, x) \mid 0 < \lambda < 1, \|x\| < (1+\epsilon)\cdot\lambda\}$.

Proposition 4. *Let $L\colon [0,\,1) \to \mathcal{L}(\mathbf{R}^n)$ be a continuous path of symmetric operators such that $0 < \|L_\lambda\| < \lambda$ for $\lambda > 0$. Choose any symmetric, orthogonal operator I_0 on $\mathbf{R}^n$ that has Morse index d equal to that of L_λ for $\lambda > 0$. Define*

$$D_4 \equiv \{(\lambda, x) \mid\ 0 < \lambda < 1, ||x|| < (1+\epsilon)\lambda\}\,.$$

There is a family $\psi_4\colon D_4 \to \mathbf{R}$ of C^2 functions which possesses properties (5), (6), *and* (7), *and furthermore,*

$$\psi_4(\lambda,\, x) = \frac{\lambda}{2}\langle I_0 x,\, x\rangle \quad \textit{for} \quad 0 < \lambda < 1,\ (1-\epsilon)\cdot\lambda < \|x\| < (1+\epsilon)\cdot\lambda. \tag{21}$$

Proof. Choose a point p_0 in $\hat{G}_n^d$ for which $\pi(p_0) = I_0$, where $\pi\colon \hat{G}_n^d \to \mathcal{O}S^d(\mathbf{R}^n)$ is the covering map. For this choice of p_0, let $\gamma\colon (0,\,1) \to \hat{G}_n^d$ be the smooth path which misses $\mathcal{C}$, the cut locus of p_0, and $\psi_3\colon D_3 \to \mathbf{R}$ be the function having the properties asserted in the statement of the preceding proposition. By patching with $\psi_3\colon D_3 \to \mathbf{R}$, it suffices to define

$$D_4' \equiv \{(\lambda,\, x) \mid\ 0 < \lambda < 1, (3/4 - \epsilon)\cdot\lambda < \|x\| < (1+\epsilon)\cdot\lambda\}$$

and find a family $\psi_4\colon D_4' \to \mathbf{R}$ of C^2 functions for which (6), (7), and (21) hold, and furthermore,

$$\psi_4(\lambda,\, x) = \frac{\lambda}{2}\langle \pi(\gamma(\lambda))x,\, x\rangle \text{ for } 0<\lambda<1,\ (3/4-\epsilon)\cdot\lambda < \|x\| < (3/4+\epsilon)\cdot\lambda. \tag{22}$$

Recall that we have the map $(A,t)\mapsto \rho(A,t)$, $0\le t\le 1$, the geodesic deformation retraction from $\hat{G}_n^d \sim \mathcal{C}$ to p_0 along the unique shortest geodesic, where $\mathcal{C}$ is the cut locus of p_0. Define

$$H(\lambda,t) \equiv \pi(\rho(\gamma(\lambda),t)) \quad \text{for} \quad 0<\lambda<1, 0\le t\le 1.$$

This is properly defined since the path γ misses $\mathcal{C}$. Observe that

$$H(\lambda,0) = (\pi\circ\gamma)(\lambda) \quad \text{and} \quad H(\lambda,1)\equiv I_0 \quad \text{for} \quad 0<\lambda<1. \tag{23}$$

Choose a smooth, increasing function $v\colon \mathbf{R}\to[0,1]$ such that

$$v(t)=0 \quad \text{for} \quad t\le 3/4+\epsilon \quad \text{and} \quad v(t)=1 \quad \text{for} \quad t\ge 1-\epsilon.$$

For $0<\lambda<1, (3/4-\epsilon)\cdot\lambda < r < (1+\epsilon)\cdot\lambda$, define

$$M(\lambda,\, r) = \pi[\rho(\gamma(\lambda), v(r/\lambda))]$$

and then define the function $\psi_4\colon D_4'\to\mathbf{R}$ by

$$\psi_4(\lambda,\, x) = \frac{\lambda}{2}\langle M(\lambda,\, ||x||)x,\, x\rangle \quad \text{for} \quad (\lambda,\, x) \text{ in } D_4'.$$

We deduce from (23) that, since $v(t)=0$ if $t\le 3/4-\epsilon$, ψ_4 satisfies (22), and since $v(t)=1$ if $t\ge 1/2+\epsilon$, ψ satisfies (21). The functions $v(\|x\|/\lambda)$, $\|x\|/\lambda\cdot v'(\|x\|/\lambda)$, and $\|x\|^2/\lambda^2\cdot v''(\|x\|/\lambda)$ are bounded on D_4', and ρ and all partials $\partial^{(n)}\rho/\partial t^{(n)}$ are bounded. Consequently, because λ is a coefficient of ψ_4, by direct calculation of partial derivatives we conclude that (7) holds for ψ_4. Finally, each $M(\lambda,r)$ is orthogonal and symmetric and therefore we may appeal to Lemma 2 to deduce that (6) holds for ψ_4. □

1.5. **The extension of ψ to $[0,\,1)\times B$.**

Lemma 4. *Let $D\equiv\{(\lambda,r)\,|\,0\le\lambda<1, (1-\epsilon)\lambda<r<1\}$. There is a continuous function $g\colon D\to\mathbf{R}$ which possesses the following four properties:*

$$g(\lambda,r)=\lambda \quad \textit{for} \quad 0\le\lambda<1, (1-\epsilon)\lambda<r\le\lambda; \tag{24}$$

$$\frac{\partial g}{\partial r}(\lambda,r) \quad \textit{and} \quad \frac{\partial^2 g}{\partial r^2}(\lambda,r) \textit{ exist and depend continuously on } (\lambda,r)\in D; \tag{25}$$

$$g(\lambda,r)>0 \quad \textit{and} \quad \frac{\partial g}{\partial r}\ge 0 \quad \textit{for} \quad 0\le\lambda<1, \lambda<r<1; \tag{26}$$

$$\lim_{(\lambda,r)\to(0,0)}\frac{\partial g}{\partial r}(\lambda,r) = \lim_{(\lambda,r)\to(0,0)}\frac{\partial^2 g}{\partial r^2}(\lambda,r) = 0. \tag{27}$$

Proof. Choose $W\colon\mathbf{R}\to\mathbf{R}$ to be a smooth function for which

$$W^{(n)}(0)=0 \quad \text{for all } n, \quad W(1)>1,$$

and

$$W \quad \text{and} \quad W' \quad \text{are positive on } (0,\,1).$$

We first claim that for each (λ, r), with $0 \le \lambda < 1, \lambda \le r < 1$, there is a unique $\eta(\lambda, r)$ for which

$$W\big(r - \eta(\lambda, r)\big) = \eta(\lambda, r) - \lambda, \ \lambda \le \eta(\lambda, r) < 1. \tag{28}$$

Then, once this first claim is verified, we define

$$g(\lambda, r) \equiv \begin{cases} \eta(\lambda, r) & \text{for} \quad 0 \le \lambda < 1, \lambda \le r < 1, \\ \lambda & \text{for} \quad 0 \le \lambda < 1, (1-\epsilon)\lambda < r \le \lambda. \end{cases} \tag{29}$$

The second claim is that this g has the required properties.

To establish the first claim, define w to be the inverse of the restriction of W to $[0, \infty)$. Then $[0, 1)$ is in the domain of w since $W(1) > 1$. Furthermore, since W and W' are positive on $(0, 1)$, we deduce, by implicit differentiation, that w is C^2. For $0 \le \lambda < 1$ and $\lambda \le r < 1$, since w and w' is positive on $(0, 1)$ and $w(0) = 0$, we deduce from the mean value theorem that there is a unique $\overline{\lambda}$ for which

$$r = \overline{\lambda} + w(\overline{\lambda} - \lambda), \ \lambda \le \overline{\lambda} < 1. \tag{30}$$

Consequently, if, for $0 \le \overline{\lambda} < 1$, we define the function $\rho_{\overline{\lambda}}\colon [0, \lambda] \to \mathbf{R}$ by

$$\rho_{\overline{\lambda}}(\lambda) \equiv \overline{\lambda} + w(\overline{\lambda} - \lambda) \quad \text{for} \quad 0 \le \lambda \le \overline{\lambda},$$

then the graphs of the functions $\rho_{\overline{\lambda}}$, parametrized by $\overline{\lambda} \in [0, 1)$, foliate the triangle $\{(\lambda, r) \,|\, 0 \le \lambda < 1, \lambda \le r < 1\}$.

For $0 \le \lambda < 1, \lambda \le r < 1$, define $\eta(\lambda, r)$ to be the $\overline{\lambda}$ for which (30) holds, that is,

$$\eta(\lambda, r) + w(\eta(\lambda, r) - \lambda) = r, \ \lambda \le \eta(\lambda, r) < 1. \tag{31}$$

Observe that (31) is equivalent to (28). Thus the first claim is verified. Also observe that, for each (λ, r), with $0 \le \lambda < 1, \lambda \le r < 1$, the point $(\eta(\lambda, r), \eta(\lambda, r))$ is the foot of the leaf of the foliation on which the point (λ, r) lies, and therefore the function g, defined by (29), is constant on each leaf of the foliation.

We now verify that g has the required properties. By definition, (24) holds. Observe that

$$r - \eta(\lambda, r) > 0 \quad \text{for} \quad 0 \le \lambda < 1, \lambda < r < 1. \tag{32}$$

Differentiate (28) with respect to r and regather terms to obtain

$$\frac{\partial \eta(\lambda, r)}{\partial r} = \frac{W'\big(r - \eta(\lambda, r)\big)}{1 + W'\big(r - \eta(\lambda, r)\big)} \quad \text{for} \quad 0 \le \lambda < 1, \lambda < r < 1. \tag{33}$$

Then differentiate the above with respect to r and again regather terms to obtain

$$\frac{\partial^2 \eta(\lambda, r)}{\partial r^2} = \frac{W''\big(r - \eta(\lambda, r)\big)}{\big[1 + W'\big(r - \eta(\lambda, r)\big)\big]^3} \quad \text{for} \quad 0 \le \lambda < 1, \lambda < r < 1. \tag{34}$$

Since W' and W'' are continuous,

$$W'(0) = W''(0) = 0, \quad \text{and} \quad \lim_{(\lambda, r) \to (\lambda_0, \lambda_0)} [r - \eta(\lambda, r)] = 0,$$

we deduce from (33) and (34) that, for $0 \le \lambda_0 < 1$,

$$\lim_{(\lambda, r) \to (\lambda_0, \lambda_0), r > \lambda} \frac{\partial \eta(\lambda, r)}{\partial r} = \lim_{(\lambda, r) \to (\lambda_0, \lambda_0), r > \lambda} \frac{\partial^2 \eta(\lambda, r)}{\partial r^2} = 0. \tag{35}$$

We now verify (25). Certainly (25) holds on D away from the diagonal $r = \lambda$. Now consider a point in $p \in D$ on the diagonal: $p = (\lambda_0, \lambda_0)$ for $0 < \lambda_0 < 1$. Since

$$\lim_{(\lambda,r)\to(\lambda_0,\lambda_0),r>\lambda} \eta(\lambda,\, r) = \lim_{(\lambda,r)\to(\lambda_0,\lambda_0),r<\lambda} \lambda = \lambda_0 = g(\lambda_0, \lambda_0),$$

the function g is continuous at p. Moreover, we deduce from (35) that

$$\lim_{r\to\lambda_0,r>\lambda_0} \frac{\partial \eta}{\partial r}(\lambda_0,\, r) = \lim_{r\to\lambda_0,r<\lambda_0} \frac{\partial}{\partial r}\lambda = 0,$$

and

$$\lim_{r\to\lambda_0,r>\lambda_0} \frac{\partial^2 \eta}{\partial r^2}(\lambda_0,\, r) = \lim_{r\to\lambda_0,r<\lambda_0} \frac{\partial^2}{\partial r^2}\lambda = 0,$$

and therefore $\partial g/\partial r$ and $\partial^2 g/\partial r^2$ exist at p and both depend continuously on (λ, r) at (λ_0, λ_0). So (25) is verified.

We deduce from (35) that (27) holds if $(\lambda, r) \to (0, 0)$ with $r > \lambda$ and since $g(\lambda, r) = \lambda$ if $(1 - \epsilon)\lambda < r \le \lambda$, (27) is verified.

Finally, we verify (26). By definition $g(\lambda, r) = \eta(\lambda, r) > 0$ for $r > \lambda$ and, since $W' > 0$ on $(0, 1)$, (33) tells us that $\frac{\partial g}{\partial r}(\lambda, r) > 0$ for $r > \lambda$. Thus (26) is verified. □

Proposition 5. *Let $L\colon [0, 1) \to \mathcal{L}(\mathbf{R}^n)$ be a continuous path of symmetric operators such that $L_0 = 0$ and L_λ is invertible for $\lambda > 0$. Let I_0 be an orthogonal, symmetric operator on $\mathbf{R}^n$ for which the Morse index equals that of L_λ for $\lambda > 0$. Let the function g have the properties of the function in Lemma 4. There is a $c > 0$, an open ball B about the origin in $\mathbf{R}^n$, and a family $\psi\colon [0,\, c) \times B \to \mathbf{R}$ of C^2 functions for which $\nabla\psi_x(\lambda, x) = 0$ if and only if $x = 0$, and, for $0 \le \lambda < c$,* $\operatorname{Hessian}_x \psi(\lambda, 0) = L_\lambda$. *Furthermore,*

$$\psi(0\,, x) = \frac{1}{2} g(0, \|x\|)\langle I_0 x,\, x\rangle \quad \textit{for all } x \textit{ in } B. \tag{36}$$

Proof. First consider the special case that $0 < \|L_\lambda\| < \lambda$ for $\lambda > 0$. Define

$$D_5 \equiv \{(\lambda, x) \mid\ 0 \le \lambda < 1, (1 - \epsilon) \cdot \lambda < \|x\| < 1\}\,,$$

and, for $(\lambda, x) \in D_5$, define

$$\psi_5(\lambda, x) \equiv \frac{1}{2} g(\lambda, \|x\|)\langle I_0 x,\, x\rangle,$$

where the function g has the properties stated in the preceding lemma.

We appeal to Lemma 1, together with property (25) of g, to deduce that $\psi_5\colon D_5 \to \mathbf{R}$ is a family of C^2 functions. We appeal to Lemma 3, together with property (26) of g, to deduce that, for all $(\lambda, x) \in D_5$, $\nabla_x \psi_5(\lambda, x) \ne 0$. Finally, we again appeal to Lemma 1, but now together with property (27) of g, to deduce that the function ψ_5 satisfies (7).

Let $\psi_4\colon D_4 \to \mathbf{R}$ be the function in the statement of Proposition 4, for the same choice of I_0. Observe that

$$[0, 1) \times B = D_4 \cup D_5 \cup \{(0, 0)\}.$$

Define $\psi\colon [0, 1) \times B \to \mathbf{R}$ by

$$\psi(\lambda, x) \equiv \begin{cases} \psi_4(\lambda, x) & \text{for} \quad (\lambda, x) \text{ in } D_4, \\ \psi_5(\lambda, x) & \text{for} \quad (\lambda, x) \text{ in } D_5, \\ 0 & \text{for} \quad (\lambda, x) = (0, 0). \end{cases}$$

By construction, ψ is unambiguously defined. Observe that (36) holds, since $\psi(0,0) = g(0,0) = 0$ and $\psi = \psi_5$ on D_5.

For $(\lambda, x) \neq (0,0)$, there is a neighborhood of (λ, x) on which ψ is a family of C^2 functions, since this property holds for both ψ_4 and ψ_5. Now $\nabla\psi_x(\lambda, x) \neq 0$ if $x \neq 0$, since both ψ_4 and ψ_5 possess this property. For $0 < \lambda < 1$, $\|x\| < \epsilon$, $\psi(\lambda, x) = \psi_4(\lambda, x) = \frac{1}{2}\langle L_\lambda x, x\rangle$. Consequently, for $0 < \lambda < 1$, $\nabla\psi_x(\lambda, 0) = 0$ and $\text{Hessian}_x\, \psi(\lambda, 0) = L_\lambda$.

It remains only to examine ψ near $(\lambda, x) = (0,0)$. However, the function ψ possesses property (7) since both ψ_4 and ψ_5 do. Moreover, by definition, $\psi(0,0) = 0$. From these we deduce that the function ψ is continuous at $(0,0)$, that both $\nabla_x\psi(0, x)$ and $\text{Hessian}_x\, \psi(0, x)$ exist at $x = 0$,

$$\nabla_x\psi(0, x)|_{x=0} = \text{Hessian}_x\, \psi(0, x)|_{x=0} = 0,$$

and both $\nabla_x\psi(\lambda, x)$ and $\text{Hessian}_x\, \psi(\lambda, x)$ depend continuously on (λ, x) in a neighborhood, in $[0, 1) \times B$, of $(0,0)$. In particular, $\nabla\psi_x(\lambda, 0) = 0$ if $\lambda = 0$ and since, by assumption, $L_0 = 0, \text{Hessian}_x\, \psi(\lambda, 0) = L_\lambda$ if $\lambda = 0$.

This completes the proof in the special case in which $0 < \|L_\lambda\| < \lambda$ for $\lambda > 0$, and, in this case, we have $c = 1$ and B the open unit ball about the origin in $\mathbf{R}^n$.

Now consider the general case, that is, $L_0 = 0$ and L_λ is invertible for $\lambda > 0$. Choose a strictly increasing, continuous function $f\colon [0, 1] \to \mathbf{R}$ such that $f(0) = 0$, $f(1) > 1$, and $f(\lambda) > \|L_\lambda\|$ for $\lambda > 0$. Since $f(1) > 1$ we may define $\hat{L}\colon [0, 1) \to \mathcal{L}(H)$ by

$$\hat{L}_\lambda = L_{f^{-1}(\lambda)} \quad \text{for} \quad 0 \leq \lambda < 1.$$

Observe that $\hat{L}\colon (0, 1) \to \mathcal{L}(\mathbf{R}^n)$ is a continuous path of symmetric operators such that $0 < \|\hat{L}_\lambda\| < \lambda$ for $\lambda > 0$. We appeal to the case just considered to conclude that there is a family $\hat{\psi}\colon [0, 1) \times B \to \mathbf{R}$ of C^2 functions for which $\nabla\hat{\psi}_x(\lambda, x) = 0$ if and only if $x = 0$, and, for $0 \leq \lambda < 1$, $\text{Hessian}_x\, \hat{\psi}(\lambda, 0) = \hat{L}_\lambda$. Furthermore, if ψ is substituted by $\hat{\psi}$, then (36) holds. Let $c = f^{-1}(1)$ and define

$$\psi(\lambda, x) = \hat{\psi}(f(\lambda), x) \quad \text{for} \quad 0 \leq \lambda < c \text{ and } x \text{ in } B.$$

This function ψ has the properties asserted in the statement of the proposition. □

1.6. **The final extension of ψ to $J \times B$.**

Proposition 6. *The Nonbifurcation Theorem holds if $H = \mathbf{R}^n$ and $L_{\lambda_0} = 0$.*

Proof. By assumption, the spectral flow of L_λ across $\lambda = 0$ is zero, that is,

$$\mu(L, 0^+) = \mu(L, 0^-).$$

Define the continuous path $S\colon [0, 1) \to \mathcal{L}(\mathbf{R}^n)$ of symmetric operators by

$$S_\lambda = L_{-\lambda} \quad \text{for} \quad 0 \leq \lambda < 1.$$

For each $\lambda > 0$, the Morse index d of S_λ equals the Morse index of T_λ. Let I_0 be an orthogonal, symmetric operator on $\mathbf{R}^n$ that has Morse index d.

We apply Proposition 5 twice, each time with the above choice of I_0, first for the path L and then for the path S, to choose $c > 0$ and two families of C^2 functions

$\psi^+\colon [0,\, c) \times B \to \mathbf{R}$ and $\psi^-\colon [0,\, c) \times B \to \mathbf{R}$, for each of which the gradient with respect to x equals 0 if and only if $x = 0$, and moreover:

$$\text{Hessian}_x\, \psi^+(\lambda,\, 0) = L_\lambda \quad \text{and} \quad \text{Hessian}_x\, \psi^-(\lambda,\, 0) = S_\lambda \quad \text{for} \quad 0 \le \lambda < c;$$
$$\psi^+(0\,, x) = \psi^-(0\,, x) = \frac{1}{2} g(0, \|x\|)\langle I_0 x,\, x\rangle \text{ for all } x \text{ in } B.$$

For $(\lambda,\, x) \in (-c,\, c) \times B$, define

$$\psi(\lambda,\, x) = \begin{cases} \psi^+(\lambda,\, x) & \text{if } \lambda \ge 0, \\ \psi^-(-\lambda,\, x) & \text{if } \lambda \le 0. \end{cases}$$

Since ψ^- and ψ^+ agree on $\{0\} \times B$, $\psi\colon [0,\, c) \times B \to \mathbf{R}$ is a family of C^2 functions for which $\nabla_x(\lambda,\, x) = 0$ if and only if $x = 0$, and, for $-c < \lambda < c$, $\text{Hessian}_x\, \psi(\lambda,\, 0) = L_\lambda$. □

2. Proof of the Nonbifurcation Theorem for general H

2.1. **Spectral flow.** In the introduction, we stated fundamental properties of the spectral flow of an admissible path of symmetric Fredholm operators over a compact interval. We need these properties in order to extend the Nonbifurcation Theorem to the case that H is infinite dimensional. We also need another property: the cogredient invariance of spectral flow.

Definition. Two continuous paths $L\colon [a,\, b] \to \mathcal{L}(H)$ and $T\colon [a,\, b] \to \mathcal{L}(H)$ are said to be cogrediant provided there is a continuous path $M\colon [a,\, b] \to \mathcal{G}L(H)$ such that, for $a \le \lambda \le b$,

$$M_\lambda^* L_\lambda M_\lambda = T_\lambda.$$

Spectral flow is cogredient invariant; if two admissible paths are cogredient, then their spectral flows are equal. Indeed, if H has finite dimension, this follows from the Morse index formula. As observed in [6], if H has infinite dimension, cogredient invariance of spectral flow follows from its homotopy invariance and Kuiper's Theorem, which tells us that $\mathcal{G}L(H)$ is contractible.

The following local reduction property of spectral flow, which was used in [6] to prove the bifurcation theorem, will also be an ingredient in the proof of the Nonbifurcation Theorem.

Lemma 5. *Let I be an open interval containing 0 and $L\colon I \to \Phi_{sym}(H)$ a continuous path. There is an open interval J containing 0 and a continuous path $T\colon J \to \Phi_{sym}(H)$ for which*

$$L_0 = T_0, \tag{37}$$

$$L\colon J \to \Phi_{sym}(H) \text{ is cogredient to } T\colon J \to \Phi_{sym}(H), \tag{38}$$

and

$$\text{the orthogonal decomposition } \ker L_0 \oplus \operatorname{Im} L_0 \text{ reduces each } T_\lambda. \tag{39}$$

2.2. **Proof of the theorem.**

Proof. For notational convenience assume $\lambda_0 = 0$, and denote $\ker L_0$ by H_0. According to the preceding lemma, there is an open interval J' containing 0 and a continuous path $T\colon J' \to \Phi_{sym}(H)$ for which (37), (38), and (39) hold.

By assumption, L_λ is invertible for $\lambda \neq 0$, and T_λ is cogredient to L_λ. Therefore T_λ is invertible for $\lambda \neq 0$. We therefore deduce from (39) that $T_\lambda\colon \operatorname{Im} L_0 \to \operatorname{Im} L_0$ is invertible for $\lambda \neq 0$. On the other hand, according to (39), $\ker L_0 \oplus \operatorname{Im} L_0$ reduces $T_0 = L_0$. Thus $T_0\colon \operatorname{Im} L_0 \to \operatorname{Im} L_0$ is invertible. Consequently,

$$T_\lambda\colon \operatorname{Im} L_0 \to \operatorname{Im} L_0 \text{ is invertible for all } \lambda \in J'. \tag{40}$$

For $\lambda \in J'$, define $\ell_\lambda \in \mathcal{L}(H_0)$ to be the restriction of T_λ to H_0. Spectral flow is invariant under cogradience, and therefore

$$\operatorname{sf}(L, 0) = \operatorname{sf}(T, 0).$$

Spectral flow is additive over orthogonal decompositions of H that reduce the path, and therefore

$$\operatorname{sf}(T, 0) = \operatorname{sf}(T|_{H_0}, 0) + \operatorname{sf}(T|_{H_0^\perp}, 0).$$

The spectral flow of a path of isomorphisms is zero, and therefore

$$\operatorname{sf}(T|_{H_0}, 0) + \operatorname{sf}(T|_{H_0^\perp}, 0) = \operatorname{sf}(T|_{H_0}, 0) \equiv \operatorname{sf}(\ell,\, 0).$$

By assumption, $\operatorname{sf}(L,\, 0) = 0$. Therefore

$$\operatorname{sf}(\ell,\, 0) = 0.$$

According to (37), $T_0 = L_0$, and hence, since $H_0 = \ker L_0$, $\ell_0 = 0$. We may therefore appeal to the finite dimensional nonbifurcation result, Proposition 6, for the path $\ell\colon J' \to \mathcal{L}(H_0)$, to conclude that there is an open interval $J'' \subseteq J'$ containing 0, an open ball B' about the origin in H, and a path $\phi\colon J'' \times (B' \cap H_0) \to \mathbf{R}$ of C^2 functions such that, for $(\lambda,\, x) \in J'' \times (B' \cap H_0)$,

$$\nabla_x \phi(\lambda,\, x) \neq 0 \quad \text{if and only if} \quad x = 0 \tag{41}$$

and, for $\lambda \in J''$,

$$\operatorname{Hessian}_x \phi(\lambda, x)|_{x=0} = \ell_\lambda. \tag{42}$$

For $(\lambda,\, x) \in J'' \times B'$, define

$$\varphi(\lambda, x) \equiv \phi(\lambda, P_0 x) + \frac{1}{2}\langle (\operatorname{Id} - P_0) T_\lambda (\operatorname{Id} - P_0) x,\, x\rangle.$$

Since $\|P_0\| \leq 1$, ψ is properly defined. Observe that $\varphi\colon J'' \times B' \to \mathbf{R}$ is a family of C^2 functions since ϕ is such a family. We claim that, for $(\lambda,\, x) \in J'' \times B'$,

$$\nabla_x \varphi(\lambda,\, x) = 0 \quad \text{if and only if} \quad x = 0 \tag{43}$$

and, for $\lambda \in J''$,

$$\operatorname{Hessian}_x \varphi(\lambda,\, x)|_{x=0} = T_\lambda. \tag{44}$$

Indeed, since the projection P_0 is orthogonal, it is symmetric, and T_λ is symmetric since it is cogredient to the symmetric operator L_λ. Hence $(\operatorname{Id} - P_0) T_\lambda (\operatorname{Id} - P_0)$ is symmetric. Therefore, for $(\lambda,\, x) \in J' \times B'$,

$$\nabla_x \varphi(\lambda,\, x) = P_0 \nabla_x \phi(\lambda,\, P_0 x) + (\operatorname{Id} - P_0) T_\lambda (\operatorname{Id} - P_0) x.$$

Consequently, $\nabla_x \varphi(\lambda, x) = 0$ if and only if

$$P_0 \nabla_x \phi(\lambda, P_0 x) = 0 \quad \text{and} \quad (\mathrm{Id} - P_0) T_\lambda (\mathrm{Id} - P_0) x = 0. \tag{45}$$

However, since $\nabla_x \varphi(\lambda, x)$ belongs to $P_0(H)$, $P_0 \nabla_x \phi(\lambda, P_0 x) = \nabla_x \phi(\lambda, P_0 x)$, and therefore, by (41), $P_0 \nabla_x \phi(\lambda, P_0 x) = 0$ if and only if $P_0 x = 0$. Furthermore, we deduce from (40) that $(\mathrm{Id} - P_0) T_\lambda (\mathrm{Id} - P_0) x = 0$ if and only if $(\mathrm{Id} - P_0) x = 0$. Consequently, (45) implies (41).

We compute

$$\mathrm{Hessian}_x\, \varphi(\lambda, x)|_{x=0} = P_0 T_\lambda P_0 + (\mathrm{Id} - P_0) T_\lambda (\mathrm{Id} + P_0) = T_\lambda.$$

Hence (44) holds.

To complete the proof, we must find an open interval J containing 0, an open ball B about the origin in H, and a family of C^2 functions $\psi\colon J \times B \to \mathbf{R}$ such that, for $(\lambda, x) \in J \times B$,

$$\nabla_x \psi(\lambda, x) = 0 \quad \text{if and only if} \quad x = 0 \tag{46}$$

and, for $\lambda \in J$,

$$\mathrm{Hessian}_x\, \psi(\lambda, x)|_{x=0} = L_\lambda. \tag{47}$$

To do so, we make a linear change of variables. The path $T\colon J' \to \mathcal{L}(H)$ was chosen to be cogredient to $L\colon J' \to \mathcal{L}(H)$. Therefore there is a continuous path $M\colon J' \to \mathcal{G}L(H)$ such that, for $\lambda \in J'$,

$$M_\lambda^* T_\lambda M_\lambda = L_\lambda. \tag{48}$$

Since $M\colon J' \to \mathcal{G}L(H)$ is continuous, there is a ball B about the origin and an interval $J \subseteq J''$ containing 0 such that, for each $\lambda \in J$, M_λ maps B into B'. For $(\lambda, x) \in J \times B$, define

$$\psi(\lambda, x) \equiv \varphi(\lambda, M_\lambda x).$$

We have, for $(\lambda, x) \in J \times B$,

$$\nabla_x \psi(\lambda, x) = M_\lambda^* \nabla_x \phi(\lambda, M_\lambda x), \tag{49}$$

and, for $\lambda \in J$,

$$\begin{aligned} \mathrm{Hessian}_x\, \psi(\lambda, x)|_{x=0} &= [M_\lambda]^* \,\mathrm{Hessian}_x\, \varphi(\lambda, M_\lambda x)|_{x=0} M_\lambda \\ &= M_\lambda^* T_\lambda M_\lambda \\ &= L_\lambda. \end{aligned}$$

Consequently, $\psi\colon J \times B \to \mathbf{R}$ is a family of C^2 functions for which (47) holds. Moreover, since M_λ and M_λ^* are invertible, we deduce from (49) and (43) that (46) holds. The proof of the general Nonbifurcation Theorem is complete. □

Acknowledgement

The authors express their gratitude to the referee for a careful reading of the paper.

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Department of Mathematics, University of Maryland, College Park, Maryland 20742

Department of Mathematics, University of Maryland, College Park, Maryland 20742