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MEASURE-VALUED DISCRETE BRANCHING MARKOV PROCESSES

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ABSTRACT. We construct and study branching Markov processes on the space of finite configurations of the state space of a given standard process, controlled by a branching kernel and a killing kernel. In particular, we may start with a superprocess, obtaining a branching process with state space the finite configurations of positive finite measures on a topological space. A main tool in proving the path regularity of the branching process is the existence of convenient superharmonic functions having compact level sets, allowing the use of appropriate potential theoretical methods.

1. INTRODUCTION

The description of a discrete branching process is as follows (cf. e.g., [32, p. 235] and [17]). An initial particle starts at a point of a set E and moves according to a standard Markov process with state space E (called the base process) until a random terminal time when it is destroyed and replaced by a finite number of new particles, its direct descendants. Each direct descendant moves according to the same right standard process until its own terminal time when it too is destroyed and replaced by second generation particles; the process continues in this manner. N. Ikeda, M. Nagasawa, S. Watanabe, and M. L. Silverstein (cf. [24], [26], and [32]) indicated the natural connection between discrete branching processes and nonlinear partial differential operators Λ of the type

$$\Lambda u := \mathcal{L}u + \sum_{k=1}^{\infty} q_k u^k,$$

where $\mathcal{L}$ is the infinitesimal generator of the given base process and the coefficients q_k are positive, Borelian functions with $\sum_{k=1}^{\infty} q_k = 1$. In this case each direct descendant starts at the terminal position of the parent particle and $q_k(x)$ is the probability that a particle destroyed at x has precisely k descendants. It is possible to consider a more general nonlinear part for the above operator, generated by a branching kernel B; the descendants start to move from their birthplaces which have been distributed according to B. Thus, these processes are also called "nonlocal

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branching" (cf. [17]); from the literature about branching processes we indicate the classical monographs [23], [2], [1], the recent one [28], and the lecture notes [27] and [18].

In this paper we construct discrete branching Markov processes associated to operators of the type Λ, using several analytic and probabilistic potential theoretical methods. The base space of the process is the set $\widehat{E}$ of all finite configurations of E.

The structure and main results of this paper are the following. In Section 2 we collect some preliminaries on the resolvents of kernels and basic notions of potential theory. We present in (2.1) and Lemma 2.3 a suitable result on the existence of a càdlàg, quasi-left continuous strong Markov process, given a resolvent of kernels, and imposing conditions on the resolvent. The branching kernels on the space of finite configurations are introduced in Section 3.

Section 4 is devoted to the construction of the measure-valued discrete branching processes. The first step is to solve the nonlinear evolution equation induced by Λ (see Proposition 4.1 and Remark 4.2 (ii) below). Then, using a technique of absolutely monotonic operators developed in [32], it is possible to construct (cf. Corollary 4.3) a Markovian transition function on $\widehat{E}$, formed by branching kernels. We follow the classical approach from [24] and [32], but we consider a more general frame, the given topological space E being a Lusin one and not more compact (see Ch. 5 in [1] for the locally compact space situation; a detailed comment is given in Remark 4.11 (i)).

The second step is to show that the transition function we constructed on $\widehat{E}$ is indeed associated to a standard process with state space $\widehat{E}$. The main result is Theorem 4.10; its proof involves the entrance space $\widehat{E}_1$, an extension of $\widehat{E}$ constructed by using a Ray type compactification method. We apply the mentioned results from Section 2, showing that the required imposed conditions from (2.1) are satisfied by the resolvent of kernels on $\widehat{E}$ associated with the branching semigroup constructed in the previous step. In Proposition 4.8 (ii) we emphasize relations between a class of excessive functions with respect to the base process X and two classes of excessive functions (defined on $\widehat{E}$) with respect to the forthcoming branching process: the linear and the exponential type excessive functions. A particular linear excessive function for the branching process becomes a function having compact level sets (called a compact Lyapunov function) and will lead to the tightness property of the capacity on $\widehat{E}$ (see Proposition 4.8 (iii)). It turns out that it is necessary to make a perturbation of $\mathcal{L}$ with a kernel induced by the given branching kernel, and we present it in Proposition 4.5.

The above-mentioned tools were useful in the case of the continuous branching processes too (cf. [4] and [11]), e.g., for the superprocesses in the sense of E. B. Dynkin (cf. [20]; see Section 5 for the basic definitions), such as the super-Brownian motion, processes on the space of all finite measures on E induced by operators of the form $\mathcal{L}u - u^\alpha$ with $1 < \alpha \leqslant 2$. We establish in Remark 4.4 several links with the nonlinear partial differential equations associated with the branching semigroups, and we point out connections between the continuous and discrete branching processes. Note that a cumulant semigroup (similar to the continuous branching case; see (5.2) below) is introduced in Corollary 4.3 for the discrete branching processes. In particular, when the base process X is the Brownian motion, then the cumulant semigroup of the induced discrete branching process formally satisfies a nonlinear

evolution equation involving the square of the gradient. Finally, recall that the method of finding a convenient compact Lyapunov function was originally applied in order to obtain martingale solutions for singular stochastic differential equations on Hilbert spaces (cf. [7] and [15]) and for proving the standardness property of infinite dimensional Lévy processes on Hilbert spaces (see [9]).

We complete the main result with an application as suggested in [23, p. 50], where T. E. Harris emphasized the interest for branching processes for which "each object is a very complicated entity; e.g., an object may itself be a population". More precisely, because we may consider base processes with general state space, it might be a continuous branching process playing this role. In Section 5, Corollary 5.1, we obtain in this way a branching Markov process, having the space of finite configurations of positive finite measures on E as the base space. Note that in [3] new branching processes are generated starting with a superprocess and using an appropriate subordination theory.

The proofs of Lemma 2.3, Proposition 4.1, and Proposition 4.5 are presented in the Appendix, which includes as well complements on excessive measures, Ray cones, and the extension of the base space E up to the entrance space E_1, using the energy functional.

2. Preliminaries on the resolvents of kernels and standard processes

Let E be a Lusin topological space (i.e., E is homeomorphic to a Borel subset of a compact metric space) with Borel σ-algebra $\mathcal{B}(E)$. Let further $\mathcal{U} = (U_\alpha)_{\alpha>0}$ be a sub-Markovian resolvent of kernels on $(E, \mathcal{B}(E))$.

We denote by $\mathcal{E}(\mathcal{U})$ the set of all $\mathcal{B}(E)$-measurable $\mathcal{U}$-*excessive functions*: $u \in \mathcal{E}(\mathcal{U})$ if and only if u is a nonnegative, numerical, $\mathcal{B}(E)$-measurable function, $\alpha U_\alpha u \leqslant u$ for all $\alpha > 0$, and $\lim_{\alpha\to\infty} \alpha U_\alpha u(x) = u(x)$, $x \in E$.

If $\beta > 0$, we denote by $\mathcal{U}_\beta$ the sub-Markovian resolvent of kernels $\mathcal{U}_\beta = (U_{\beta+\alpha})_{\alpha>0}$. If w is a $\mathcal{U}_\beta$-*supermedian function* (i.e., $\alpha U_{\beta+\alpha} w \leqslant w$ for all $\alpha > 0$), then its $\mathcal{U}_\beta$-excessive regularization $\widehat{w}$ is given by $\widehat{w}(x) = \sup_\alpha \alpha U_{\beta+\alpha} w(x)$, $x \in E$. Let $\mathcal{S}(\mathcal{U}_\beta)$ denote the set of all $\mathcal{B}(E)$-measurable $\mathcal{U}_\beta$-supermedian functions.

For a familiy $\mathcal{G}$ of real-valued functions on E we denote by $\sigma(\mathcal{G})$ (resp. by $\mathcal{T}(\mathcal{G})$) the $\sigma-$algebra (resp. the topology) generated by $\mathcal{G}$ and by $b\mathcal{G}$ (resp. $[\mathcal{G}]$, $\overline{\mathcal{G}}$) the subfamily of bounded functions from $\mathcal{G}$ (resp. the linear space spanned by $\mathcal{G}$, the closure in the supremum norm of $\mathcal{G}$).

We denote by $p\mathcal{B}(E)$ the set of all positive $\mathcal{B}(E)$-measurable functions on E.

Remark 2.1. The vector space $[b\mathcal{E}(\mathcal{U}_\beta)]$ does not depend on $\beta > 0$.

Indeed, if $\alpha > 0$, $\alpha < \beta$, then because $\mathcal{E}(\mathcal{U}_\alpha) \subset \mathcal{E}(\mathcal{U}_\beta)$ it is sufficient to prove that $b\mathcal{E}(\mathcal{U}_\beta) \subset [b\mathcal{E}(\mathcal{U}_\alpha)]$. For, if $v \in b\mathcal{E}(\mathcal{U}_\beta)$, then both $U_\alpha v$ and $v + (\beta - \alpha)U_\alpha v$ belong to $b\mathcal{E}(\mathcal{U}_\alpha)$.

We assume further in this section that for some $\beta > 0$ there exists a strictly positive constant k with $k \leqslant U_\beta 1$ (in particular, this happens if the resolvent $\mathcal{U}$ is Markovian, i.e., $\alpha U_\alpha 1 = 1$).

Recall that a right process $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space E is called *standard* if for every finite measure μ on $(E, \mathcal{B}(E))$ X has *càdlàg trajectories* under P^μ, i.e., it possesses left limits in E P^μ-a.e. on $[0, \zeta)$ and X is *quasi-left continuous up to* ζ P^μ-a.e., i.e., for every increasing sequence $(T_n)_n$ of stopping times with

$T_n \nearrow T$ we have $X_{T_n} \longrightarrow X_T$ P^μ-a.e. on $[T < \zeta]$, with ζ the life time of X; a *stopping time* is a map $T : \Omega \longrightarrow \overline{\mathbb{R}}_+$ such that the set $[T \leqslant t]$ belongs to $\mathcal{F}_t$ for all $t \geqslant 0$.

The next result is the convenient one for the construction of the discrete branching measure-valued processes we give in Theorem 4.10 below. It follows from [11, Theorem 2.1], and it is a consequence of [15, Theorem 5.2, Corollary 5.3 (ii), and Theorem 5.5 (i)].

(2.1) Suppose the following three conditions are satisfied by the sub-Markovian resolvent of kernels $\mathcal{U} = (U_\alpha)_{\alpha>0}$ on $(E, \mathcal{B}(E))$:

$(h1)$ $\sigma(\mathcal{E}(\mathcal{U}_\beta)) = \mathcal{B}(E)$ and all the points of E are nonbranch points with respect to $\mathcal{U}_\beta$, that is, $1 \in \mathcal{E}(\mathcal{U}_\beta)$, and if $u, v \in \mathcal{E}(\mathcal{U}_\beta)$, then for all $x \in E$ we have $\inf(u, v)(x) = \inf\widehat{(u, v)}(x)$.

$(h2)$ For every $x \in E$ there exists $v_x \in \mathcal{E}(\mathcal{U}_\beta)$ such that $v_x(x) < \infty$ and the set $[v_x \leqslant n]$ is relatively compact for all n; such a function v_x is called a *compact Lyapunov function.*

$(h3)$ There exists a countable subset $\mathcal{F}$ of $[b\mathcal{E}(\mathcal{U}_\beta)]$ generating the topology of E, $1 \in \mathcal{F}$, and there exists $u_o \in \mathcal{E}(\mathcal{U}_\beta)$, $u_o < \infty$, such that if ξ, η are two finite $\mathcal{U}_\beta$-excessive measures with $L_\beta(\xi + \eta, u_o) < \infty$ and such that $L_\beta(\xi, \varphi) = L_\beta(\eta, \varphi)$ for all $\varphi \in \mathcal{F}$, then $\xi = \eta$. Here L_β denotes the energy functional associated with $\mathcal{U}_\beta$; see $(A1)$ in Appendix.

Then there exists a càdlàg process X with state space E such that $\mathcal{U}$ is the resolvent of X, i.e., for all $\alpha > 0$, $f \in bp\mathcal{B}(E)$, and $x \in E$ we have $U_\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t) dt$.

Remark 2.2. (i) Condition $(h1)$ is necessary in order to deduce that $\mathcal{U}$ is the resolvent family of a (Borel) right process; see [31], [33], [5], and [8].

(ii) According to [29] and [6], condition $(h2)$ is necessary for proving that the process X has càdlàg trajectories. It is related to the tightness of the associated capacity; we give some details below.

(iii) The quasi-left continuity of the forthcoming measure-valued branching process will be deduced from the next lemma. Some arguments in its proof are classical, e.g., similar to the Ray resolvent case (see for example Theorem (9.21) from [31], the proof of Lemma IV.3.21 from [30], and the proof of Theorem 3.7.7 from [5]). However, none of the existing results covers our context; therefore for the reader's convenience, we give the proof of the lemma in Appendix $(A2)$.

Lemma 2.3. *Let X be a right process with state space E and càdlàg trajectories. Let $(p_t)_{t\geqslant 0}$ be its transition function, and assume that there exists a countable subset $\mathcal{F}$ of $[b\mathcal{E}(\mathcal{U}_\beta)]$ generating the topology of E and a family $\mathcal{K} \subset \overline{[\mathcal{F}]}$ which is multiplicative (i.e., if $f, g \in \mathcal{K}$, then $fg \in \mathcal{K}$) and separating the points of E. Then the process X is quasi-left continuous, hence standard, provided that $p_t f$ belongs to $\overline{[\mathcal{F}]}$ for all $f \in \mathcal{K}$ and $t > 0$. If $(p_t)_{t\geqslant 0}$ is Markovian, then it is enough to assume that $\mathcal{K}$ is a family of bounded, continuous, real-valued functions on E which is multiplicative, separating the points of E, and $p_t f$ is a continuous function for all $f \in \mathcal{K}$ and $t > 0$.*

We present some necessary preliminaries on the tightness of the capacity induced by a sub-Markovian resovent of kernels.

Assume that $\mathcal{U} = (U_\alpha)_{\alpha>0}$ is a sub-Markovian resolvent of kernels on $(E, \mathcal{B}(E))$ satisfying condition $(h1)$. If $M \in \mathcal{B}(E)$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$, then the *reduced function* (with respect to $\mathcal{U}_\beta$) *of u on M* is the function $R_\beta^M u$ defined by

$$R_\beta^M u := \inf\{v \in \mathcal{E}(\mathcal{U}_\beta) : v \geqslant u \text{ on } M\}.$$

The reduced function $R_\beta^M u$ is a universally $\mathcal{B}(E)$-measurable $\mathcal{U}_\beta$-supermedian function.

Let further $u_o := U_\beta f_o$, where f_o is a bounded, strictly positive $\mathcal{B}(E)$-measurable function, and fix a finite measure λ on $(E, \mathcal{B}(E))$. Consider the functional $M \longmapsto c_\lambda(M), M \subset E$, defined as

$$c_\lambda(M) := \inf\{\lambda(R_\beta^G u_o) : G \text{ open}, M \subset G\}.$$

By [5] c_λ is a Choquet capacity on E.

(2.2) The following assertions are equivalent (see Proposition 4.1 in [15] and Proposition 2.1.1 in [14]):

(i) The capacity c_λ is *tight*, i.e., there exists an increasing sequence $(K_n)_n$ of compact sets such that $\inf_n c_\lambda(E \setminus K_n) = 0$.

(ii) There exist a $\mathcal{U}_\beta$-excessive function v which is finite λ-a.e. and a bounded strictly positive $\mathcal{U}_\beta$-excessive function u such that the set $[\frac{v}{u} \leqslant \alpha]$ is relatively compact for all $\alpha > 0$.

(iii) There exists a $\mathcal{U}_\beta$-excessive function v which is λ-integrable and such that the set $[\frac{v}{u_o} \leqslant \alpha]$ is relatively compact for all $\alpha > 0$.

Remark 2.4. (i) If there exists a strictly positive constant k such that $k \leqslant u_o$ (in particular, this happens if the resolvent $\mathcal{U}$ is Markovian), then in the above assertion (ii) one can take $u = 1$.

(ii) If $\mathcal{U}$ is the resolvent of a right process X, then the following fundamental result of G. A. Hunt holds for all $A \in \mathcal{B}(E)$, $x \in E$, and $u \in \mathcal{E}(\mathcal{U}_\beta)$:

$$R_\beta^A u(x) = E^x(e^{-\beta D_A} u(X_{D_A})),$$

where D_A is the entry time of A, $D_A := \inf\{t \geqslant 0 : X_t \in A\}$; see, e.g., [19]. Consequently, the capacity c_λ on E is tight if and only if $P^\lambda(\lim_n D_{E\setminus K_n} < \zeta) = 0$.

3. Branching kernels on the space of finite configurations

The state space for the forthcoming discrete branching process will be the set $\widehat{E}$ of finite sums of Dirac measures on E, defined as

$$\widehat{E} := \left\{ \sum_{k \leq k_0} \delta_{x_k} : k_0 \in \mathbb{N}^*, x_k \in E \text{ for all } 1 \leq k \leq k_0 \right\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ denotes the zero measure. The set $\widehat{E}$ is identified with the union of all symmetric m-th powers $E^{(m)}$ of E (i.e., if $m \geq 1$, then $E^{(m)}$ is the factorization of the Cartesian product E^m by the equivalence relation induced by the permutation group σ^m; see, e.g., [12] for details), and hence

$$\widehat{E} = \bigcup_{m \geq 0} E^{(m)},$$

where $E^{(0)} := \{\mathbf{0}\}$ (see, e.g., [24]). The set $\widehat{E}$ is called the *space of finite configurations of* E and it is endowed with the topology of the disjoint union of topological spaces and the corresponding Borel σ-algebra $\mathcal{B}(\widehat{E})$; see [21].

Let $M(E)$ be the set of all positive finite measures on E, endowed with the weak topology, and let $\mathcal{M}(E)$ be its Borel σ-algebra. For a function $f \in p\mathcal{B}(E)$ we consider the mappings $l_f : M(E) \longrightarrow \overline{\mathbb{R}}_+$ and $e_f : M(E) \longrightarrow [0,1]$, defined as

$$l_f(\mu) := \langle \mu, f\rangle := \int_E f \mathrm{d}\mu,\ \mu \in M(E), \quad e_f := \exp\,(-l_f).$$

Note that the Borel σ-algebra $\mathcal{M}(E)$ of $M(E)$ is generated by $\{l_f\ :\ f \in bp\mathcal{B}(E)\}$, $\widehat{E}$ becomes a Borel subset of $M(E)$, and the trace of $\mathcal{M}(E)$ on $\widehat{E}$ is $\mathcal{B}(\widehat{E})$.

Recall that if p_1, p_2 are two finite measures on $\widehat{E}$, then their convolution $p_1 * p_2$ is the finite measure on $\widehat{E}$ defined for every $F \in bp\mathcal{B}(\widehat{E})$ by

$$\int_{\widehat{E}} p_1 * p_2(\mathrm{d}\nu)F(\nu) := \int_{\widehat{E}} p_1(\mathrm{d}\nu_1) \int_{\widehat{E}} p_2(\mathrm{d}\nu_2)F(\nu_1 + \nu_2).$$

In particular, if $f \in p\mathcal{B}(E)$, then $p_1 * p_2(e_f) = p_1(e_f)p_2(e_f)$.

According to [32], a kernel N on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ which is sub-Markovian (i.e., $N1 \leqslant 1$) is called *branching kernel* provided that for all $\mu, \nu \in \widehat{E}$ we have

$$N_{\mu+\nu} = N_\mu * N_\nu,$$

where N_μ denotes the measure on $\widehat{E}$ such that $\int g\, dN_\mu = Ng(\mu)$ for all $g \in bp\mathcal{B}(\widehat{E})$. Note that if N is a nonzero branching kernel on $\widehat{E}$, then $N_{\mathbf{0}} = \delta_{\mathbf{0}} \in M(\widehat{E})$.

A right (Markov) process with state space $\widehat{E}$ is called a *branching process* provided that its transition function is formed by branching kernels. For the probabilistic interpretation of this analytic branching property, see, e.g., [22], page 337.

Multiplicative functions and branching kernels on $\widehat{E}$. For every real-valued, $\mathcal{B}(E)$-measurable function φ define the *multiplicative function* $\widehat{\varphi} : \widehat{E} \longrightarrow \mathbb{R}_+$ as

$$\widehat{\varphi}(\mathbf{x}) = \begin{cases} \prod_k \varphi(x_k), & \text{if } \mathbf{x} = (x_k)_{k\geq 1} \in \widehat{E}, \mathbf{x} \neq \mathbf{0}, \\ 1, & \text{if } \mathbf{x} = \mathbf{0} \end{cases}$$

(cf. [32]; see also [12]). Observe that each multiplicative function $\widehat{\varphi}$, $\varphi \in p\mathcal{B}(E)$, $\varphi \leq 1$, is the restriction to $\widehat{E}$ of an exponential function on $M(E)$,

$$\widehat{\varphi} = e_{-\ln \varphi}.$$

In the harmonic analysis on configuration spaces the multiplicative function $\widehat{\varphi}$ is called a *coherent state*; see, e.g., [21].

If $\widehat{\varphi}$ is a multiplicative function on $\widehat{E}$, then $\widehat{\varphi}(\mu+\nu) = \widehat{\varphi}(\mu)\widehat{\varphi}(\nu)$ for all $\mu, \nu \in \widehat{E}$, and therefore $p_1 * p_2(\widehat{\varphi}) = p_1(\widehat{\varphi})\ p_2(\widehat{\varphi})$.

Let N be a sub-Markovian kernel on $(\widehat{E}, \mathcal{B}(\widehat{E}))$. By Remark 3.1 in [12] the following assertions are equivalent:

(i) N is a branching kernel.

(ii) For all $\varphi \in p\mathcal{B}(E)$, $\varphi \leqslant 1$,

$$N\widehat{\varphi} = (\widehat{N\widehat{\varphi})|_E}.$$

(iii) N maps multiplicative functions into multiplicative functions.

It is possible to construct branching kernels on $\widehat{E}$, using the above characterization, as follows:

(3.1) For every sub-Markovian kernel $B : p\mathcal{B}(\widehat{E}) \longrightarrow p\mathcal{B}(E)$ there exists a branching kernel $\widehat{B}$ on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ such that for every $\mathcal{B}(E)$-measurable function φ, $|\varphi| \leqslant 1$, we have

$$\widehat{B}\widehat{\varphi} = \widehat{B\widehat{\varphi}}.$$

The kernel $\widehat{B}$ is defined as

$$\widehat{B}_{\mathbf{x}} := \begin{cases} B_{x_1} * \ldots * B_{x_n}, & \text{if } \mathbf{x} = \delta_{x_1} + \ldots + \delta_{x_n},\ x_1, \ldots, x_n \in E, \\ \delta_{\mathbf{0}} & , \text{ if } \mathbf{x} = \mathbf{0}. \end{cases}$$

Conversely, if H is a branching kernel on $(\widehat{E}, \mathcal{B}(\widehat{E}))$, then there exists a unique sub-Markovian kernel $B : p\mathcal{B}(\widehat{E}) \longrightarrow p\mathcal{B}(E)$ such that $H = \widehat{B}$ (see Proposition 3.2 in [12]).

Example of a branching kernel on $\widehat{E}$. Let $q_k \in p\mathcal{B}(E)$, $k \geqslant 1$, satisfying $\sum_{k\geqslant 1} q_k \leqslant 1$. Consider the kernel $B : p\mathcal{B}(\widehat{E}) \longrightarrow p\mathcal{B}(E)$ defined as

$$Bg(x) := \sum_{k\geqslant 1} q_k(x) g_k(x, \ldots, x),\ g \in bp\mathcal{B}(\widehat{E}),\ x \in E, \tag{3.2}$$

where $g_k := g|_{E^{(k)}}$ for all $k \geqslant 1$. By (3.1) there exists a branching kernel $\widehat{B}$ on $\widehat{E}$ such that for all $\varphi \in p\mathcal{B}(E)$, $\varphi \leqslant 1$, we have

$$\widehat{B}\widehat{\varphi}|_E = \sum_{k\geqslant 1} q_k \varphi^k.$$

4. Discrete branching processes

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a fixed Borel right process with space E and suppose that the transition function $(T_t)_{t\geqslant 0}$ of X is Markovian.

Let $B : bp\mathcal{B}(\widehat{E}) \longrightarrow bp\mathcal{B}(E)$ be a sub-Markovian kernel such that

$$\sup_{x\in E} Bl_1(x) < \infty. \tag{4.1}$$

Note that (4.1) is precisely condition 4.1.2 from [32], (2.1) in [17], or (4.25) from [28], and if B is given by (3.2), then (4.1) is equivalent with

$$\sup_{x\in E} \sum_{k\geqslant 1} k q_k(x) < \infty.$$

We also fix a function $c \in bp\mathcal{B}(E)$ and we denote by $(T_t^c)_{t\geqslant 0}$ the transition function of the process obtained by killing X with the multiplicative functional $(e^{-\int_0^t c(X_s)ds})_{t\geqslant 0}$. It is given by the Feynman-Kac formula:

$$T_t^c f(x) = E^x(e^{-\int_0^t c(X_s)\,ds} f(X_t)), \quad f \in bp\mathcal{B}(E),\ x \in E.$$

Note that if $\mathcal{L}$ is the infinitesimal generator of X, then the above killed process has the generator $\mathcal{L} - c$.

We denote by $\mathcal{U} = (U_\alpha)_{\alpha>0}$ the resolvent of the process X and let $\mathcal{U}^c = (U_\alpha^c)_{\alpha>0}$ be the resolvent of kernels induced by $(T_t^c)_{t\geqslant 0}$, i.e., the resolvent family of the process killed with c.

Denote by $\mathcal{B}_{\mathsf{u}}$ the set of all functions $\varphi \in p\mathcal{B}(E)$ such that $\varphi \leqslant 1$.

Recall that a map $H : \mathcal{B}_{\mathsf{u}} \longrightarrow \mathcal{B}_{\mathsf{u}}$ is called *absolutely monotonic* provided that there exists a sub-Markovian kernel $\mathbf{H} : bp\mathcal{B}(\widehat{E}) \longrightarrow bp\mathcal{B}(E)$ such that $H\varphi = \mathbf{H}\widehat{\varphi}$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$. By (3.1) we have:

(4.2) *A map $H : \mathcal{B}_{\mathsf{u}} \longrightarrow \mathcal{B}_{\mathsf{u}}$ is absolutely monotonic if and only if there exists a branching kernel $\widehat{\boldsymbol{H}}$ on $\widehat{E}$ such that $\widehat{\boldsymbol{H}}\widehat{\varphi} = \widehat{H\varphi}$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$.*

We also have (cf. Lemma 2.2 and Theorem 1 from [32]):

(4.3) *If H, K are absolutely monotonic, then their composition HK is also absolutely monotonic and $\widehat{\boldsymbol{HK}} = \widehat{\boldsymbol{H}}\widehat{\boldsymbol{K}}$. The map $H \longmapsto \widehat{\boldsymbol{H}}$ is a bijection between the set of all absolutely monotonic mappings and the set of all branching kernels on $\widehat{E}$.*

In the next proposition we solve an appropriate integral equation (following the approach of [32]; see also [12]); we present its proof in Appendix ($A3$).

Proposition 4.1. *For any $\varphi \in \mathcal{B}_{\mathsf{u}}$ the equation*

$$h_t(x) = T_t^c\varphi(x) + \int_0^t T_{t-u}^c(cB\widehat{h_u})(x)du,\ t \geqslant 0,\ x \in E, \tag{4.4}$$

has a unique solution $(t, x) \longmapsto H_t\varphi(x)$ jointly measurable in (t, x) such that $H_t\varphi \in \mathcal{B}_{\mathsf{u}}$ for all $t > 0$ and the following assertions hold:

(i) For each $t > 0$ the mapping $\varphi \longmapsto H_t\varphi$ is absolutely monotonic and it is Lipschitz with the constant $\beta_o t$, where

$$\beta_o := \|c\|_\infty \|Bl_1\|_\infty.$$

(ii) The familiy $(H_t)_{t\geqslant 0}$ is a semigroup of (nonlinear) operators on $\mathcal{B}_{\mathsf{u}}$. If $B1 = 1$, then $H_t 1 = 1$ for all $t \geqslant 0$.

(iii) For each $x \in E$ the function $t \longmapsto H_t\varphi(x)$ is right continuous on $[0, \infty)$, provided that $t \longmapsto T_t^c\varphi(x)$ is right continuous.

Remark 4.2. (i) If $\varphi \in \mathcal{B}_{\mathsf{u}}$ and $t \geqslant 0$, then the sequence $(H_t^n\varphi)_{n\geqslant 0}$ defined by ($A3.4$) in the Appendix converges uniformly to the solution $H_t\varphi$ of (4.4). The assertion is a consequence of the following inequality which may be proved by induction:

$$\|H_t^{n+1}\varphi - H_t^n\varphi\|_\infty \leqslant \frac{(\beta_o t)^n}{n!}\|\varphi\|_\infty \quad \text{for all } n \geqslant 0.$$

(ii) Note that if $\mathcal{L}$ is the infinitesimal generator of the base process X, then (4.4) is formally equivalent to

$$\frac{d}{dt}h_t = (\mathcal{L} - c)h_t + cB\widehat{h_t},\ t \geqslant 0.$$

(iii) If B is given by (3.2), then the condition $B1 = 1$ is equivalent with

$$\sum_{k\geqslant 1} q_k(x) = 1 \ \text{ for all } x \in E.$$

Corollary 4.3. *For each $t \geqslant 0$ let $\widehat{\boldsymbol{H}_t}$ be the branching kernel on $\widehat{E}$ associated by* (4.2) *with the absolutely monotonic operator H_t from Proposition* 4.1*, with $H_t\varphi = \widehat{\boldsymbol{H}_t}\widehat{\varphi}|_E$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$. Then the following assertions hold:*

(i) The family $(\widehat{\boldsymbol{H}_t})_{t\geqslant 0}$ is a sub-Markovian semigroup of branching kernels on $(\widehat{E}, \mathcal{B}(\widehat{E}))$.

(*ii*) *For each* $t \geqslant 0$ *and* $f \in bp\mathcal{B}(E)$ *define the function* $V_t f \in p\mathcal{B}(E)$ *as*

$$V_t f := -\ln H_t(e^{-f}).$$

Then the family $(V_t)_{t\geqslant 0}$ *is a semigroup of (nonlinear) operators on* $bp\mathcal{B}(E)$ *and*

$$\widehat{\boldsymbol{H}}_t(e_f) = e_{V_t f} \text{ for all } f \in bp\mathcal{B}(E). \tag{4.5}$$

Proof. Assertion (*i*) follows from (4.3) and Proposition 4.1. To prove assertion (*ii*) it is enough to show that if $f \in bp\mathcal{B}(E)$, then $V_t f \in bp\mathcal{B}(E)$. If $f \leqslant M$, because $H_t(e^{-M}) \geqslant T_t^c(e^{-M})$ and since the transition function of X is Markovian, we have $V_t f \leqslant V_t M \leqslant -\ln(e^{-M} E^x(e^{-\int_0^t c(X_s)ds})) \leqslant M + t\|c\|_\infty$. □

Remark 4.4. (*i*) If $v \in \mathcal{B}_{\mathsf{u}}$ is such that $\widehat{v}$ is an invariant function with respect to the branching semigroup $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ on $\widehat{E}$, then v belongs to the domain of $\mathcal{L}$ (the infinitesimal generator of the base process X) and

$$(\mathcal{L} - c)v + cB\widehat{v} = 0. \tag{4.6}$$

In particular, if B is given by (3.2), then v is the solution of the nonlinear equation

$$(\mathcal{L} - c)v + c\sum_{k\geqslant 1} q_k v^k = 0.$$

It turns out that $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ is the main tool in the treatment of the (nonlinear) Dirichlet problem associated with equation (4.6); see Proposition 6.1 and Subsection 6.2 in [12] and Section 3 from [13].

(*ii*) The equation (4.4) and the equality (4.5) are analogous to (2.2) and respectively (2.4) from [17], where, however, the forthcoming branching process having $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ as its transition function is used. Assertion (*ii*) of Corollary 4.3 shows that the semigroup approach for the continuous branching (developed by E.B. Dynkin [20] and P.J. Fitzsimmons [22]; see also [4] and [28]) is analogous to the above construction of the branching semigroup in the discrete branching case, due to N. Ikeda, M. Nagasawa, S. Watanabe, and M.L. Silverstein. Recall that in the case of the continuous branching the family $(V_t)_{t\geqslant 0}$ is the so-called "cumulant semigroup"; for more details see Section 5 below.

(*iii*) Assume that E is a Euclidean open set, X is the Brownian motion on E, and B is given by (3.2). Then by (4.5) and Remark 4.2 (*ii*) one can see that the cumulant semigroup $(V_t)_{t\geqslant 0}$ is formally the solution of the following evolution equation:

$$\frac{d}{dt}V_t f = \Delta V_t f - |\nabla V_t f|^2 + c(1 - \sum_{k\geqslant 1} q_k e^{(1-k)V_t f}),\ t \geqslant 0.$$

This should be compared with the equation satisfied by the cumulant semigroup of a mesure-valued continuous branching process (cf. (2.2)′ from [22] and (5.1) below), in particular, in the case of the super-Brownian motion: $\frac{d}{dt}V_t f = \Delta V_t f - (V_t f)^2$, $t \geqslant 0$.

(*iv*) We refer to the survey article [11] for a version of assertion (*ii*) of Corollary 4.3 and for further connections between the continuous and discrete measure-valued processes.

We establish now a linear version of Proposition 4.1 which leads to a result on the perturbation with kernels of the sub-Markovian semigroups and resolvents; see Appendix (*A*4) for its proof.

Proposition 4.5. *Let K be a sub-Markovian kernel on $(E, \mathcal{B}(E))$. Then for any $f \in bp\mathcal{B}(E)$ the equation*

$$r_t(x) = T_t^c f(x) + \int_0^t T_{t-u}^c(cKr_u)(x)du,\ t \geqslant 0,\ x \in E, \tag{4.7}$$

has a unique solution $Q_t f \in bp\mathcal{B}(E)$, the function $(t,x) \longmapsto Q_t f(x)$ is measurable and the following assertions hold:

(i) The family $(Q_t)_{t\geqslant 0}$ is a semigroup of sub-Markovian kernels on $(E, \mathcal{B}(E))$ and it is the transition function of a Borel right process with state space E.

(ii) The function $t \longmapsto Q_t f(x)$ is right continuous on $[0,\infty)$ for each $x \in E$ if and only if the function $t \longmapsto T_t^c f(x)$ has the same property.

(iii) The resolvent of kernels $\mathcal{U}^o = (U_\alpha^o)_{\alpha>0}$ on $(E, \mathcal{B}(E))$ induced by $(Q_t)_{t\geqslant 0}$ ($U_\alpha^o = \int_0^\infty e^{-\alpha t} Q_t dt$, $\alpha > 0$) has the property $(h1)$. If $\beta > 0$, then $U_\beta^o = U_\beta^c + U_\beta^c cKU_\beta^o = U_\beta^c + G_\beta U_\beta^c$, where G_β is the bounded kernel defined as $G_\beta := \sum_{k\geqslant 1}(U_\beta^c cK)^k$. We have $\mathcal{E}(\mathcal{U}_\beta^o) \subset \mathcal{E}(\mathcal{U}_\beta^c)$, $[b\mathcal{E}(\mathcal{U}_\beta^o)] = [b\mathcal{E}(\mathcal{U}_\beta^c)] = [b\mathcal{E}(\mathcal{U}_\beta)]$, and $G_\beta(\mathcal{E}(\mathcal{U}_\beta^c)) \subset \mathcal{E}(\mathcal{U}_\beta^o)$.

If $M \in \mathcal{B}(E)$, $\beta > 0$, and $u \in \mathcal{E}(\mathcal{U}_\beta^c)$ (resp. $u \in \mathcal{E}(\mathcal{U}_\beta^o)$), let ${}^cR_\beta^M u$ (resp. ${}^oR_\beta^M u$) be the reduced function of u on M with respect to $\mathcal{U}_\beta^c$ (resp. $\mathcal{U}_\beta^o$). Let further $v_o := U_\beta^o 1 = U_\beta^c f_o$, where $f_o := 1 + cKU_\beta^o 1$, and fix a finite measure λ on $(E, \mathcal{B}(E))$. We denote by c_λ^c (resp. c_λ^o) the induced capacity: $c_\lambda^c(M) := \inf\{\lambda({}^cR_\beta^D v_o) : D \text{ open}, M \subset D\}$ (resp. $c_\lambda^o(M) := \inf\{\lambda({}^oR_\beta^D v_o) : D \text{ open}, M \subset D\}$).

A result related to the following corollary was stated in Proposition 3.7 from [16].

Corollary 4.6. *We have $c_\lambda^o \leqslant c_{\lambda'}^c \leqslant c_{\lambda'}$, where $\lambda' := \lambda + \lambda \circ G_\beta$. In particular, if the capacity $c_{\lambda'}$ is tight, then the capacities $c_{\lambda'}^c$ and c_λ^o are also tight.*

Proof. Because $\mathcal{E}(\mathcal{U}_\beta) \subset \mathcal{E}(\mathcal{U}_\beta^c)$ we get ${}^cR_\beta^M v_o \leqslant {}^cR_\beta^M u_o \leqslant R_\beta^M u_o$ for every $M \in \mathcal{B}(E)$ (where recall that $u_o = U_\beta f_o \geqslant v_o$), and therefore $c_{\lambda'}^c \leqslant c_{\lambda'}$. By assertion (iii) of Proposition 4.5 we may apply the result from [5], Proposition 5.2.5, to obtain the inequality of kernels ${}^oR_\beta^M \leqslant {}^cR_\beta^M + G_\beta {}^cR_\beta^M$ which leads to $c_\lambda^o(M) \leqslant c_{\lambda'}^c(M)$. $\square$

Let $\widehat{\mathcal{U}} = (\widehat{U}_\alpha)_{\alpha>0}$ (resp. $\widehat{\mathcal{U}}^c = (\widehat{U}_\alpha^c)_{\alpha>0}$) be the sub-Markovian resolvent of kernels on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ generated by the semigroup $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ (resp. by $(\widehat{T}_t^c)_{t\geqslant 0}$).

Let further

$$\beta_1 := ||Bl_1||_\infty,$$

and assume that $B1 = 1$; hence $\beta_1 \geqslant 1$. We suppose that $\beta_1 > 1$ and that the function c is such that $c < \frac{\beta_1}{\beta_1 - 1}$. Let $(Q_t)_{t\geqslant 0}$ be the semigroup given by Proposition 4.5, with the sub-Markovian kernel K on $(E, \mathcal{B}(E))$ defined as $Kf := \frac{c}{c+\beta_1} B(l_f)$ and with $c + \beta_1$ instead of c.

Lemma 4.7. *If B is given by (3.2) and c does not depend on $x \in E$, then*

$$Q_t f(x) = e^{-(c+\beta_1)t} E^x(e^{\int_0^t cq_o(X_s)\,ds} f(X_t)), \quad f \in bp\mathcal{B}(E), x \in E, t > 0,$$

where $q_o := \sum_{k\geqslant 1} kq_k$, and we have $[b\mathcal{E}(\mathcal{U}^o)] = [b\mathcal{E}(\mathcal{U}_\beta)]$ for all $\beta > 0$.

Proof. Observe first that in this case

$$B(l_f) = q_o f,\ f \in bp\mathcal{B}(E),$$

and equation (4.7) with $c+\beta_1$ instead of c and $Kf = \frac{c}{c+\beta_1}B(l_f)$ becomes

$$k_t(x) = T_t f(x) - \int_0^t T_{t-u}(bk_u)(x)du,\ t \geqslant 0,\ x \in E,$$

where $k_t := e^{(c+\beta_1)t}r_t$ and $b := -cq_o$. By Proposition 3.3 (i) from [4] the above equation has a unique solution $T_t^b f$, and consequently $Q_t = T_t^{c+\beta_1-cq_o}$ for all $t \geqslant 0$. The claimed expression of Q_t now also follows from [4], equality (3.3). Let $\beta_2 := c+\beta_1-c\beta_1$. From Proposition 3.3 (iii) from [4], since $0 < \beta_2 \leqslant c+\beta_1-cq_o \leqslant \beta_1$, we have $T_t^{\beta_1} \leqslant Q_t \leqslant T_t^{\beta_2}$, $\mathcal{E}(\mathcal{U}_{\beta_2}) \subset \mathcal{E}(\mathcal{U}^o) \subset \mathcal{E}(\mathcal{U}_{\beta_1})$. By Remark 2.1 we conclude that $[b\mathcal{E}(\mathcal{U}^o)] = [b\mathcal{E}(\mathcal{U}_{\beta_1})] = [b\mathcal{E}(\mathcal{U}_\beta)]$ for all $\beta > 0$. □

In the next proposition we prove relations between a set of excessive functions with respect to the base process X and excessive functions with respect to the forthcoming branching process on $\widehat{E}$. The key tool is the equality from assertion (i) below. Note that a similar equality was obtained in the case of continuous branching processes in [22], Proposition (2.7) (see also [4, Proposition 4.2]). It turns out that a perturbation of the generator of the base process was necessary in that case too, although a simpler one, with the perturbed semigroup $(Q_t)_{t\geqslant 0}$ being expressed with a Feynman-Kac formula, as in the particular case discussed in the above Lemma 4.7.

Proposition 4.8. *The following assertions hold:*

(i) If $f \in bp\mathcal{B}(E)$ and $t > 0$, then

$$e^{-\beta_1 t}\widehat{\boldsymbol{H}}_t(l_f) = l_{Q_t f}.$$

(ii) If $\beta > 0$ and $\beta' := \beta_1 + \beta$, then the following assertions are equivalent for every $u \in bp\mathcal{B}(E)$:

$(ii.a)$ $u \in b\mathcal{E}(\mathcal{U}_\beta^o)$.

$(ii.b)$ $l_u \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$.

$(ii.c)$ For every $\alpha > 0$ we have $1 - e_{\alpha u} \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$.

(iii) If $u \in \mathcal{E}(\mathcal{U}_\beta^o)$ is a compact Lyapunov function on E, then $l_{1+u} \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$ is a compact Lyapunov function on $\widehat{E}$.

Proof. (i) We show first that if $f \in bp\mathcal{B}(E)$ and N is a kernel on $\widehat{E}$ or from $\widehat{E}$ to E such that $N(l_1)$ is a bounded function, then

$$N(e_{\lambda f})'_{\lambda=0} := \lim_{\lambda\to 0}\frac{N(e_{\lambda f}) - N1}{\lambda} = -N(l_f). \tag{4.8}$$

Indeed, the assertion follows since $\frac{1-e_{\lambda f}}{\lambda} \nearrow l_f$ pointwise on $\widehat{E}$. The Lipschitz property of H_t (Proposition 4.1 (i)) implies

$$||H_t(e^{-f}) - 1||_\infty \leqslant \beta_o t||1-e^{-f}||_\infty \leqslant \beta_o t||f||_\infty, \quad \left|\frac{H_t(e^{-\lambda f}) - 1}{\lambda}\right| \leqslant \beta_o t||f||_\infty.$$

Applying (4.8) with $N = \widehat{\mathbf{H}}_t$ and since $\widehat{\mathbf{H}}_t(e_{\lambda f})|_E = H_t(e^{-\lambda f})$, we deduce that $\widehat{\mathbf{H}}_t(l_f)|_E \leqslant \beta_o t||f||_\infty$ and we claim that

$$\widehat{\mathbf{H}}_t(l_f) = -\widehat{\mathbf{H}}_t(e_{\lambda f})'_{\lambda=0} = l_{\widehat{\mathbf{H}}_t(l_f)|_E}. \tag{4.9}$$

By (4.8) it is sufficient to show the second equality. Let $\mu \in E^{(m)}$, $\mu = \sum_{k=1}^m \delta_{x_k}$. Again using (4.8) for the kernel $\widehat{\mathbf{H}_t}|_E$ and since $\widehat{\mathrm{H}_t}1 = 1$ we get

$$\widehat{\mathbf{H}_t}(e_{\lambda f})'_{\lambda=0}(\mu) = (\prod_{k=1}^m \widehat{\mathbf{H}_t}(e_{\lambda f})|_E(x_k))'_{\lambda=0}$$

$$= (\widehat{\mathbf{H}_t}(e_{\lambda f})|_E)'_{\lambda=0}(x_1) \cdot \widehat{\mathbf{H}_t}(e_0)(x_2) \cdot \ldots \cdot \widehat{\mathbf{H}_t}(e_0)(x_m) + \ldots$$

$$= -[\widehat{\mathbf{H}_t}(l_f)(x_1) + \ldots + \widehat{\mathbf{H}_t}(l_f)(x_m)] = -l_{\widehat{\mathbf{H}_t(l_f)|_E}}(\mu).$$

For each $t \geqslant 0$ define the function $\varphi_t : \mathbb{R}_+ \longrightarrow bp\mathcal{B}(E)$ by $\varphi_t(\lambda) := V_t(\lambda f)$. We clearly have $H_t(e^{-\lambda f}) = e^{-\varphi_t(\lambda)}$, and from Proposition 4.1 we obtain

$$e^{-\varphi_t(\lambda)} = T_t^c(e^{-\lambda f}) + \int_0^t T_{t-u}^c cB\widehat{\mathbf{H}_t}(e_{\lambda f})du, \quad t \geqslant 0.$$

We have $\varphi_t(0) = 0$, and using (4.9) we get $\varphi'_t(0) = \widehat{\mathbf{H}_t}(l_f)|_E$. By derivation of the above equation in $\lambda = 0$ and multiplying with $e^{-\beta_1 t}$, applying (4.8) for $N = T_{t-u}^c cB\widehat{\mathbf{H}_t}$, and again from (4.9) we conclude that

$$e^{-\beta_1 t}\varphi'_t(0) = T_t^{c+\beta_1} f + \int_0^t T_{t-u}^{c+\beta_1}(c+\beta_1)K(e^{-\beta_1 u}\varphi'_u(0))du, \quad t \geqslant 0,$$

where $Kf := \frac{c}{c+\beta_1}B(l_f)$ and $(T_t^{c+\beta_1})_{t\geqslant 0}$ is the transition function of the process obtained by killing X with the multiplicative functional $(e^{-\beta_1 t - \int_0^t c(X_s)ds})_{t\geqslant 0}$, $T_t^{c+\beta_1} = e^{-\beta_1 t}T_t^c$. Hence $e^{-\beta_1 t}\varphi'_t(0)$ verifies (4.7) with $c+\beta_1$ instead of c and the kernel K. Proposition 4.5 implies $e^{-\beta_1 t}\varphi'_t(0) = Q_t f$, and by (4.9) $e^{-\beta_1 t}\widehat{\mathbf{H}_t}(l_f) = l_{e^{-\beta_1 t}\widehat{\mathbf{H}_t(l_f)|_E}} = l_{Q_t f}$.

The proof of (ii) is similar to that of Corollary 4.3 from [4], using the above assertion (i).

(iii) Let $u \in \mathcal{E}(\mathcal{U}_\beta^o)$ be a Lyapunov function on E, and for each $n \in \mathbb{N}^*$ consider the compact set K_n such that $[u \leqslant n] \subset K_n$. Since $l_1 = m$ on $E^{(m)}$, $m \geqslant 1$, we conclude that $[l_{1+u} \leqslant n]$ is included in the compact set $E^{(0)} \cup K_n \cup (K_n)^{(2)} \cup \ldots \cup (K_n)^{(n)}$ of $\widehat{E}$. □

Let $\mathcal{A} := \overline{[b\mathcal{E}(\mathcal{U}_\beta)]}$ (= the closure in the supremum norm of $[b\mathcal{E}(\mathcal{U}_\beta)]$). Note that $\mathcal{A}$ is an algebra; see, e.g., Corollary 23 from [4]. By Remark 2.1 $\mathcal{A}$ does depend on $\beta > 0$, and also using Proposition 4.5 (iii) we get $\mathcal{A} = \overline{[b\mathcal{E}(\mathcal{U}_\beta^c)]} = \overline{[b\mathcal{E}(\mathcal{U}_\beta^o)]}$. Recall that $1 - e^{-u} \in \mathcal{E}(\mathcal{U}_\beta^o)$ provided that $u \in \mathcal{E}(\mathcal{U}_\beta^o)$, and therefore $\{e^{-u} : u \in \mathcal{E}(\mathcal{U}_\beta^o)\} \subset \mathcal{A} \cap \mathcal{B}_{\mathsf{u}}$. We need a supplementary hypothesis:

$(*)$ *There exist a countable subset $\mathcal{F}_o$ of $b\mathcal{E}(\mathcal{U}_\beta^o)$ which is additive, $0 \in \mathcal{F}_o$, and separates the finite measures on E, and a separable vector lattice $\mathcal{C} \subset \mathcal{A}$ such that $\{e^{-u} : u \in \mathcal{F}_o\} \subset \mathcal{C}$ and $T_t^c\varphi, T_t^c(cB\widehat{\varphi}) \in \mathcal{C}$ for all $\varphi \in \mathcal{C} \cap \mathcal{B}_{\mathsf{u}}$ and $t > 0$.*

Proposition 4.9. *The following assertions hold:*

(i) *If c does not depend on $x \in E$ and B is given by (3.2) with $\sum_{k\geqslant 1} ||q_k||_\infty < \infty$, then $(*)$ is verified by taking any countable subset $\mathcal{F}_o$ of $b\mathcal{E}(\mathcal{U}_\beta^o)$ which is additive, $0 \in \mathcal{F}_o$, and separates the finite measures on E, and as $\mathcal{C}$ the closure in the supremum norm of a separable vector lattice $\mathcal{C}_o \subset \mathcal{A}$ such that $\{e^{-u} : u \in \mathcal{F}_o\} \subset \mathcal{C}_o$, $(q_k)_{k\geqslant 1} \subset \mathcal{C}_o$ and $T_t(\mathcal{C}_o) \subset \mathcal{C}_o$ for all $t > 0$.*

(ii) Assume that E is a locally compact space, $(T_t^c)_{t\geqslant 0}$ a C_0-semigroup on $C_0(E)$, $c \in bC(E)$, and $B\widehat{\varphi}$ and $B(l_\varphi)$ also belong to $C_0(E)$ for all $\varphi \in C_0(E) \cap \mathcal{B}_{\mathsf{U}}$. Then $(*)$ holds by taking $\mathcal{C} = C_0(E) \oplus \mathbb{R}$ and for any countable subset $\mathcal{F}_o$ of $C_0(E) \cap b\mathcal{E}(\mathcal{U}_\beta^o)$ which is additive, $0 \in \mathcal{F}_o$, and separates the finite measures on E.

(iii) If condition $(*)$ holds then $V_t(\mathcal{F}_o) \subset \overline{\mathcal{C}}$ (the closure in the supremum norm of $\mathcal{C}$) for every $t \geqslant 0$.

Proof. By (3.2) $B\widehat{\varphi} = \sum_{k\geqslant 1} q_k \varphi^k$ and $\mathcal{C}$ is an algebra. Therefore $B\widehat{\varphi} \in \mathcal{C} \cap \mathcal{B}_{\mathsf{U}}$ provided that $\varphi \in \mathcal{C} \cap \mathcal{B}_{\mathsf{U}}$, so, assertion (i) holds.

Assertion (ii) is clear, observing that $C_0(E) \oplus \mathbb{R} \subset \mathcal{A}$. Note that by Remark 4.2 (i) we have $Q_t(C_0(E)) \subset C_0(E)$ for all $t > 0$, and using (4.7) one can see that $(Q_t)_{t\geqslant 0}$ is also a C_0-semigroup on $C_0(E)$.

(iii) Using condition $(*)$ it follows that $H_t^n(e^{-u}) \in \overline{\mathcal{C}} \cap \mathcal{B}_{\mathsf{U}}$ for all $n \geqslant 0$ and $u \in \mathcal{F}_o$, where H_t^n is given by $(A3.4)$. Since the sequence $(H_t^n(e^{-u}))_n$ converges uniformly (see Remark 4.2 (i)), $H_t(e^{-u})$ also belongs to $\overline{\mathcal{C}}$ which is an algebra and we conclude that $V_t u = -\ln H_t(e^{-u}) \in \overline{\mathcal{C}}$. □

We now state the main result of this paper, the existence of a discrete branching process associated with the base process X, the branching kernel B and the killing kernel c.

Theorem 4.10. *If the base process X is standard and condition $(*)$ holds, then there exists a branching standard process with state space $\widehat{E}$, having $(\widehat{\boldsymbol{H}}_t)_{t\geqslant 0}$ as its transition function.*

Proof. According to (2.1), in order to show that $(\widehat{\mathbf{H}}_t)_{t\geqslant 0}$ is the transition function of a càdlàg process with state space $\widehat{E}$, we have to verify conditions $(h1)$-$(h3)$ for $\widehat{\mathcal{U}}_{\beta'}$.

We show first that $(h1)$ is satisfied by $\widehat{\mathcal{U}}_{\beta'}$, in particular, all the points of $\widehat{E}$ are nonbranch points for $\widehat{\mathcal{U}}_{\beta'}$. We proceed as in the proof of Proposition 4.5 from [4]. According to Corollary 3.6 from [33], it will be sufficient to prove that the uniqueness of charges and the specific solidity of potentials properties hold for $\widehat{\mathcal{U}}_{\beta'} = (\widehat{U}_{\beta'+\alpha})_{\alpha>0}$, where we recall that $\widehat{U}_\alpha = \int_0^\infty e^{-\alpha t}\widehat{\mathbf{H}}_t\, dt$.

The uniqueness of charges property. We have to show that if μ, ν are two finite measures on $\widehat{E}$ such that $\mu \circ \widehat{U}_{\beta'} = \nu \circ \widehat{U}_{\beta'}$, then $\mu = \nu$. We get $\mu(1) = \nu(1)$, and by Hunt's approximation theorem $\mu(F) = \nu(F)$ for every $F \in [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$. We already observed that the multiplicative family of functions

$$\widehat{\mathcal{F}}_o := \{e_u : u \in \mathcal{F}_o\}$$

is a subset of $[b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$. Therefore $\mu(e_u) = \nu(e_u)$ for every $u \in \mathcal{F}_o$ and $\mathcal{B}(\widehat{E}) = \sigma(\widehat{\mathcal{F}}_o) = \sigma(\widehat{\mathcal{E}}(\mathcal{U}_{\beta'}))$. By a monotone class argument we conclude that $\mu = \nu$.

The specific solidity of potentials. We have to show that if $\xi, \mu \circ \widehat{U}_{\beta'} \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ and $\xi \prec \mu \circ \widehat{U}_{\beta'}$, then ξ is a potential. Here $\prec$ denotes the specific order relation on the convex cone $\mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ of all $\widehat{\mathcal{U}}_{\beta'}$-excessive measures: if $\xi, \xi' \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$, then $\xi \prec \xi'$ means that there exists $\eta \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ such that $\xi + \eta = \xi'$.

Let $\mathcal{A}_o$ be the additive semigroup generated by $\{V_t u : u \in \mathcal{F}_o, t \geqslant 0\}$ and $[\widehat{\mathcal{A}}_o]$ the vector space spanned by $\{e_v : v \in \mathcal{A}_o\}$. Then $[\widehat{\mathcal{A}}_o]$ is an algebra of functions on

$\widehat{E}$, $1 \in [\widehat{\mathcal{A}}_o]$, and since $\widehat{\mathcal{F}}_o \subset [\widehat{\mathcal{A}}]$ we have $\sigma([\widehat{\mathcal{A}}]) = \mathcal{B}(\widehat{E})$. We prove now that

$$[\widehat{\mathcal{A}}_o] \subset [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]. \tag{4.10}$$

Since $\widehat{\mathcal{F}}_o \subset [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$ we get from (4.5) that $e_{V_t u} \in [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$ for all $u \in \mathcal{F}_o$. By Corollary 2.3 from [4] the vector space $[b\mathcal{S}(\widehat{\mathcal{U}}_{\beta'})]$ is an algebra, and therefore $e_v \in [b\mathcal{S}(\widehat{\mathcal{U}}_{\beta'})]$ for all $v \in \mathcal{A}_o$. It remains to prove that the map $s \longmapsto \widehat{\mathbf{H}}_s(e_v)(\mu)$ is right continuous on $[0,\infty)$ for every $v \in \mathcal{A}_o$ and $\mu \in \widehat{E}$. Because $\widehat{\mathbf{H}}_s(e_v) = \widehat{H_s(e^{-v})}$, we have to prove the right continuity of the mapping $s \longmapsto H_s(e^{-v})(x)$, $x \in E$. According to Proposition 4.1 (iii) it will be sufficient to show that the map $s \longmapsto T_s^c(e^{-v})(x)$ is right continuous for every $v \in \mathcal{A}_o$ and $x \in E$. This last right continuity property is satisfied since by Proposition 4.9 (iii) the function e^{-v} belongs to the algebra $\mathcal{A}$.

Let $\xi, \mu \circ \widehat{U}_{\beta'} \in \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$, $\xi \prec \mu \circ \widehat{U}_{\beta'}$. We may suppose that $\mu(1) \leqslant 1$; if it is not the case, then $\mu = \sum_n \mu_n$ with $\mu_n(1) \leqslant 1$ for all n, and by Ch. 2 in [5] there exists a sequence $(\xi_n)_n \subset \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ such that $\xi = \sum_n \xi_n$ and $\xi_n \prec \mu_n \circ \widehat{U}_\beta$ for every n. Let $\varphi_\xi : \mathcal{E}(\widehat{\mathcal{U}}_{\beta'}) \longrightarrow \overline{\mathbb{R}}_+$ be the functional defined by $\varphi_\xi(F) := \widehat{L}_{\beta'}(\xi, F)$, $F \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$, where $\widehat{L}_{\beta'}$ denotes the energy functional associated with $\widehat{\mathcal{U}}_{\beta'}$. By (4.10) we may extend φ_ξ to an increasing linear functional on $[\widehat{\mathcal{A}}_o]$. Let $\mathcal{M}$ be the closure of $[\widehat{\mathcal{A}}_o]$ with respect to the supremum norm. Clearly, $\mathcal{M}$ is a vector lattice and we claim that φ_ξ extends to a positive linear functional on $\mathcal{M}$. Indeed, if $(F_n)_n \subset [\widehat{\mathcal{A}}_o]$ is a sequence converging uniformly to zero and we consider a sequence $(\nu_k \circ \widehat{U}_{\beta'})_k \subset \mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$, $\nu_k \circ \widehat{U}_{\beta'} \nearrow \xi$, then we have $|\varphi_\xi(F_n)| = \sup_k |\nu_k(F_n)| \leqslant \liminf_k \nu_k(|F_n|) \leqslant \varepsilon \liminf_k \nu_k(1) = \varepsilon \widehat{L}_{\beta'}(\xi, 1) \leqslant \varepsilon\mu(1) \leqslant \varepsilon$, provided that $n \geqslant n_0$ and $||F_n||_\infty < \varepsilon$ for all $n \geqslant n_0$. Since $\xi \prec \mu \circ \widehat{U}_{\beta'}$ we have $\varphi_\xi(F) \leqslant \mu(F)$ for every $F \in \mathcal{M}_+$. Consequently, if $(F_n)_n \subset \mathcal{M}_+$ is a sequence decreasing pointwise to zero, then $\varphi_\xi(F_n) \searrow 0$. By Daniell's Theorem there exists a measure ν on $(\widehat{E}, \mathcal{B}(\widehat{E}))$ such that $\varphi_\xi(F) = \nu(F)$ for all $F \in \mathcal{M}$. In particular, if $u \in \mathcal{F}_o$, then $\widehat{L}_{\beta'}(\xi, \widehat{\mathbf{H}}_t(e_u)) = \varphi_\xi(e_{V_t u}) = \nu(\widehat{\mathbf{H}}_t(e_u))$ and therefore $\widehat{L}_{\beta'}(\xi, \widehat{U}_{\beta'}(e_u)) = \lim_k \nu_k(\widehat{U}_{\beta'}(e_u)) = \int_0^\infty e^{-\beta' t} \lim_k \nu_k(\widehat{\mathbf{H}}_t(e_u))dt = \int_0^\infty e^{-\beta' t} \widehat{L}_{\beta'}(\xi, \widehat{\mathbf{H}}_t(e_u))dt = \nu(\widehat{U}_{\beta'}(e_u))$. We conclude that $\xi = \nu \circ \widehat{U}_{\beta'}$. Observe that we actually proved the following assertion:

(4.11) *If ξ, η are two finite measures from $\mathsf{Exc}(\widehat{\mathcal{U}}_{\beta'})$ and $\widehat{L}_{\beta'}(\xi, \widehat{\boldsymbol{H}}_t(e_u)) = \widehat{L}_{\beta'}(\eta, \widehat{\boldsymbol{H}}_t(e_u))$ for all $u \in \mathcal{F}_o$ and $t \geqslant 0$, then $\xi = \eta$.*

Because X is a Hunt process, Theorem (47.10) in [31] implies that X has càdlàg trajectories in any Ray topology (see $(A1.2)$ in the Appendix). Consider a Ray topology $\mathcal{T}(\mathcal{R})$ with respect to $\mathcal{U}^o$, which is finer than the original topology and is generated by a Ray cone $\mathcal{R} \subset b\mathcal{E}(\mathcal{U}_\beta^o)$ such that $\mathcal{F}_o \subset \mathcal{R}$. So, without loss of generality, we may assume in the sequel that the original topology of E is a Ray one.

We now check condition $(h2)$. Let $\lambda \in \widehat{E}$, $\lambda \neq \mathbf{0}$, and set as before $\lambda' = \lambda + \lambda \circ G_\beta$. Since the base process X on E has càdlàg trajectories the capacity $c_{\lambda'}$ is tight (see Remark 2.2), and by Corollary 4.6 the capacity c_λ^o is also tight. According to (2.2)

and Remark 2.4 (i) there exists a compact Lyapunov function $u \in \mathcal{E}(\mathcal{U}^o_\beta) \cap L^1(\lambda)$. Proposition 4.8 (iii) implies that $l_{1+u} \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$ is a compact Lyapunov function on $\widehat{E}$ and $l_{1+u}(\lambda) < \infty$; hence $(h2)$ holds.

We show that $(h3)$ also holds for $\widehat{\mathcal{U}}_{\beta'}$. We take l_1 as the function u_o; observe that by Proposition 4.8 (ii) we have $l_1 \in \mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$, and clearly l_1 is a real-valued function. Let $\mathcal{C}_o$ be a countable subset of $b\mathcal{E}(\mathcal{U}^o_\beta)$ such that $\mathcal{F}_o \subset \mathcal{C}_o$, $\mathcal{C}_o$ is additive, and $p\mathcal{C}$ is included in the closure in the supremum norm of $(\mathcal{C}_o - \mathcal{C}_o)_+$. Let further $\mathcal{R}_o$ be a countable dense subset of the Ray cone $\mathcal{R}$ such that $\mathcal{C}_o \subset \mathcal{R}_o$. We may consider $\widehat{\mathcal{R}}_o := \{e_u : u \in \mathcal{R}_o\}$ as the countable set $\mathcal{F}$ from $(h3)$. Note that since $\mathcal{R}_o$ generates the (Ray) topology on E, by Lemma 02 from [24] (see also the proof of Lemma 2.4 from [12]) one can see that $\widehat{\mathcal{R}}_o$ generates the topology of $\widehat{E}$. Let further ξ, η be two finite $\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})$-excessive measures such that $\widehat{L}_{\beta'}(\xi, e_u) = \widehat{L}_{\beta'}(\eta, e_u)$ for all $u \in \mathcal{R}_o$ and

$$\widehat{L}_{\beta'}(\xi + \eta, l_1) < \infty. \tag{4.12}$$

To show that $\xi = \eta$ we can now proceed as in the proof of Theorem 4.9 from [4], Step I, page 699; this procedure was also used in the proof of Theorem 3.5 from [11]. Because the σ-algebra $\mathcal{B}(\widehat{E})$ is generated by the multiplicative family $\widehat{\mathcal{F}}_o$, a monotone class argument implies that $\xi = \eta$ provided that

$$\xi(e_u) = \eta(e_u) \text{ for all } u \in \mathcal{F}_o.$$

By (4.11) the above equality holds if

$$\widehat{L}_{\beta'}(\xi, \widehat{\mathbf{H}}_t(e_u)) = \widehat{L}_{\beta'}(\eta, \widehat{\mathbf{H}}_t(e_u)) \text{ for all } u \in \mathcal{F}_o \text{ and } t \geqslant 0. \tag{4.13}$$

From (4.12) and $(A1.1.a)$ there exist two measures μ and ν on $\widehat{E}_1$ such that $\xi = \mu \circ \widehat{U}^1_{\beta'}$ and $\eta = \nu \circ \widehat{U}^1_{\beta'}$. Let further $\widetilde{\mathcal{C}}_o := \{\widetilde{e}_u : u \in \mathcal{C}_o\}$. Because $\widetilde{\mathcal{C}}_o$ is a multiplicative class of functions on $\widehat{E}_1$ and $\mu(\widetilde{e}_u) = \widehat{L}_{\beta'}(\xi, e_u) = \widehat{L}_{\beta'}(\eta, e_u) = \nu(\widetilde{e}_u)$ for every $u \in \mathcal{C}_o$, by the monotone class theorem we have

$$\mu(F) = \nu(F) \text{ for all } F \in \sigma(\widetilde{\mathcal{C}}_o). \tag{4.14}$$

If $u \in \mathcal{F}_o$, then by Proposition 4.9 (iii) there exists a sequence $(f_n)_n \subset (\mathcal{C}_o - \mathcal{C}_o)_+$ converging uniformly to $V_t u$. Note that if $f \in (\mathcal{C}_o - \mathcal{C}_o)_+$, then e_f has a finely continuous extension $\widetilde{e}_f$ from $\widehat{E}$ to $\widehat{E}_1$. Since $e_u \in [b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$, by using (4.5) we get that $e_{V_t u}$ belongs to $[b\mathcal{E}(\widehat{\mathcal{U}}_{\beta'})]$ and by $(A1.1.c)$ it has a unique extension $\widetilde{e}_f$ from $\widehat{E}$ to $\widehat{E}_1$ (by fine continuity too). As a consequence, and using $(A3.2)$, for every $\lambda \in \widehat{E}$ we have $|e_{f_n}(\lambda) - e_{V_t u}(\lambda)| \leqslant ||f_n - V_t u||_\infty \cdot l_1(\lambda)$; hence $|\widetilde{e}_{f_n} - \widetilde{e}_{V_t u}| \leqslant ||f_n - V_t u||_\infty \cdot \widetilde{l}_1$ on $\widehat{E}_1$. It follows that $(\widetilde{e}_{f_n})_n$ converges pointwise to $\widetilde{e}_{V_t u}$ on the set $[\widetilde{l}_1 < \infty] \in \sigma(\widetilde{\mathcal{C}}_o)$. From (4.12) we get $(\mu + \nu)(\widetilde{l}_1) = \widehat{L}_\beta(\xi + \eta, l_1) < \infty$; hence $\widetilde{l}_1 < \infty$ $(\mu + \nu)$-a.e. Therefore, $1_{[\widetilde{l}_1 < \infty]} \cdot \widetilde{e}_{V_t u}$ is $\sigma(\widetilde{\mathcal{C}}_o)$-measurable and by (4.14) we now deduce that $\mu(\widetilde{e}_{V_t u}) = \nu(\widetilde{e}_{V_t u})$ for all $u \in \mathcal{F}_o$. We conclude that (4.13) holds, so $\xi = \eta$. Applying (2.1), $\widehat{\mathcal{U}}$ is the resolvent of a standard process with state space $\widehat{E}$.

The quasi-left continuity follows by Lemma 2.3, taking $\widehat{\mathcal{F}}_o$ as the multiplicative set $\mathcal{K}$, since by Proposition 4.1 (ii) the semigroup $(\widehat{\mathbf{H}}_t)_{t \geqslant 0}$ is Markovian. □

Remark 4.11. (i) For constructions of branching Markov processes we refer to [1], [24], [25], [26], and [32]. In particular, for locally compact base space, Theorem 4.10 is very closely related to Theorems 2.2–2.5 and Theorems 3.3–3.5 in [25],

where the branching process is obtained by a path-wise piecing out procedure, starting from a canonical diagonal (branching) process on $\widehat{E}$. Because in [25] any Feller or $(*)$ condition is not assumed, it is of interest to show that the piecing out procedure carries over to a Lusin base space. We thank the anonymous referee for suggesting this comment. Note that the diagonal (branching) process on $\widehat{E}$ was already essentially used in [12] and [13], in the case of Lusin spaces.

(ii) The extra point $\mathbf{0}$ is a trap for the branching process X on $\widehat{E}$; see also Theorem 1 from [24]. Indeed, the assertion follows since, with the notation from Proposition 4.8, the mapping l_1 is $\widehat{\mathcal{U}}_{\beta'}$-excessive and we have $\{\mathbf{0}\} = [l_1 = 0]$, so the set $\{\mathbf{0}\}$ is absorbing.

5. Application: Continuous branching as base process

In this section we give an example of a branching Markov process, having as base space the set of all finite configurations of positive finite measures on a topological space. Note that an example of a branching type process on this space was given in [12], obtained by perturbing a diagonal semigroup with a branching kernel.

Let Y be a standard (Markov) process with state space a Lusin topological space F, called *spatial motion*. We fix a *branching mechanism*, that is, a function $\Phi : F \times [0,\infty) \longrightarrow \mathbb{R}$ of the form

$$\Phi(x,\lambda) = -b(x)\lambda - a(x)\lambda^2 + \int_0^\infty (1 - e^{-\lambda s} - \lambda s)N(x,ds),$$

where $a \geqslant 0$ and b are bounded $\mathcal{B}(F)$-measurable functions and $N : p\mathcal{B}((0,\infty)) \longrightarrow p\mathcal{B}(F)$ is a kernel such that $N(u \wedge u^2) \in bp\mathcal{B}(F)$. Examples of branching mechanisms are $\Phi(\lambda) = -\lambda^\alpha$ for $1 < \alpha \leq 2$.

We first present the measure-valued branching Markov process associated with the spatial motion Y and the branching mechanism Φ, the (Y,Φ)-superprocess, a standard process with state space $M(F)$, the space of all positive finite measures on $(F,\mathcal{B}(F))$, endowed with the weak topology (cf. [22] and [20]; see also [4]). For each $f \in bp\mathcal{B}(F)$ the equation

$$v_t(x) = P_t f(x) + \int_0^t P_s(x, \Phi(\cdot, v_{t-s}))ds, \quad t \geqslant 0, \quad x \in F,$$

has a unique solution $(t,x) \longmapsto N_t f(x)$ jointly measurable in (t,x) such that $\sup\limits_{0 \leq s \leq t} ||v_s||_\infty < \infty$ for all $t > 0$; we have denoted by $(P_t)_{t \geqslant 0}$ the transition function of the spatial motion Y. Assume that Y is conservative, that is, $P_t 1 = 1$. The mappings $f \longmapsto N_t f$, $t \geqslant 0$, form a nonlinear semigroup of operators on $bp\mathcal{B}(F)$ and the above equation is formally equivalent with

$$\text{(5.1))} \qquad \begin{cases} \frac{d}{dt} v_t(x) = \mathsf{L} v_t(x) + \Phi(x, v_t(x)), \\ v_0 = f, \end{cases}$$

where L is the infinitesimal generator of the spatial motion Y. For every $t \geqslant 0$ there exists a unique kernel T_t on $(M(F), \mathcal{M}(F))$ such that

$$T_t(e_f) = e_{N_t f}, \quad f \in bp\mathcal{B}(F), \tag{5.2}$$

where for a function $g \in bp\mathcal{B}(F)$ the exponential function e_g is defined on $M(F)$ as in Section 3. Since the family $(N_t)_{t \geqslant 0}$ is a (nonlinear) semigroup on $bp\mathcal{B}(F)$, $(T_t)_{t \geqslant 0}$ is a linear semigroup of kernels on $M(F)$. Suppose that F is a locally compact space,

$(P_t)_{t\geqslant 0}$ is a C_0-semigroup on $C_0(F)$, and a, b, and N do not depend on $x \in F$. We may assume that $b \geqslant 0$. Arguing as in the proof of Proposition 4.8 from [4] one can see that $N_t(C_0(F)) \subset C_0(F)$ and that $N_t(b\mathcal{E}(\mathcal{U}_{b'})) \subset b\mathcal{E}(\mathcal{U}_{b'})$ for every $t \geqslant 0$, where $b' := b+\beta$, with $\beta > 0$. Then $(T_t)_{t\geqslant 0}$ is the transition function of a standard process with state space $M(F)$, called (Y,Φ)-*superprocess*; see, e.g., [22], [4], and [11]. In addition, the (Y,Φ)-superprocess is a branching process on $M(F)$, i.e., T_t is a branching kernel on $M(F)$ for all $t > 0$. Recall that the nonlinear semigroup $(N_t)_{t\geq 0}$ is called the *cumulant semigroup* of this branching process.

We can now apply results from Section 4, starting with the (Y,Φ)-superprocess as a base process with state space $E := M(F)$.

Corollary 5.1. *Let c and $(q_k)_{k\geqslant 1}$ be positive real numbers such that $\sum_{k\geqslant 1} q_k = 1$, $\sum_{k\geqslant 1} kq_k =: q_o < \infty$, and $0 < \beta < c + q_o - cq_o$. Then there exists a discrete branching process with state space $\widehat{M(F)}$, the set of all finite configurations of positive finite measures on F, associated to c and $(q_k)_{k\geqslant 1}$, and with base process the (Y,Φ)-superprocess.*

Proof. We apply Theorem 4.10, so we have to check condition $(*)$. Let $\mathcal{R}$ be a Ray cone with respect to the resolvent $\mathcal{W} = (W_\alpha)_{\alpha>0}$ of the process Y on F, constructed as in the proof of Proposition 4.8 from [4], $\mathcal{R} \subset b\mathcal{E}(\mathcal{W}_{b'})$, such that $[\mathcal{R} \cap C_0(F)]$ is dense in $C_0(F)$. Let $\mathcal{R}_o$ be a countable, additive, dense subset of $\mathcal{R}$. Then $\{e_r : r \in \mathcal{R}_o\}$ is a multiplicative set of functions on E and separates the measures on E. Let further $\mathcal{C}$ be the closure in the supremum norm of the vector space spanned by $\{e_w : w \in b\mathcal{E}(\mathcal{W}_{b'}) \cap C_0(F)\}$ and denote by $\mathcal{U} = (U_\alpha)_{\alpha>0}$ the resolvent of the (Y,Φ)-superprocess on E. By Corollary 4.4 from [4] $1 - e_w \in \mathcal{E}(\mathcal{U}_\beta)$ for all $w \in b\mathcal{E}(\mathcal{W}_{b'})$. Therefore $\mathcal{C} \subset \mathcal{A}$, and we may take as $\mathcal{F}_o$ the additive semigroup generated by the set $\{1 - e_w : w \in \mathcal{R}_o\}$. From (5.2) and the above considerations $T_t(\mathcal{C}) \subset \mathcal{C}$, and since $\mathcal{C}$ is a Banach algebra we clearly have $\{e^{-u} : u \in \mathcal{F}_o\} \subset \mathcal{C}$; hence condition $(*)$ holds. $\square$

Appendix

(A1) Excessive measures, Ray cones. Let $\mathcal{U} = (U_\alpha)_{\alpha\geqslant 0}$ be a sub-Markovian resolvent of kernels on $(E,\mathcal{B}(E))$ such that condition $(h1)$ holds.

Let $\mathsf{Exc}(\mathcal{U})$ be the set of all $\mathcal{U}$-*excessive measures* on E: $\xi \in \mathsf{Exc}(\mathcal{U})$ if and only if it is a σ-finite measure on $(E,\mathcal{B}(E))$ such that $\xi \circ \alpha U_\alpha \leqslant \xi$ for all $\alpha > 0$. Recall that if $\xi \in \mathsf{Exc}(\mathcal{U})$, then actually $\xi \circ \alpha U_\alpha \nearrow \xi$ as $\alpha \to \infty$. We denote by $\mathsf{Pot}(\mathcal{U})$ the set of all *potential* $\mathcal{U}$-excessive measures: if $\xi \in \mathsf{Exc}(\mathcal{U})$, then $\xi \in \mathsf{Pot}(\mathcal{U})$ if $\xi = \mu \circ U$, where μ is a σ-finite on $(E,\mathcal{B}(E))$.

If $\beta > 0$, then the *energy functional* $L_\beta : \mathsf{Exc}(\mathcal{U}_\beta) \times \mathcal{E}(\mathcal{U}_\beta) \longrightarrow \overline{\mathbb{R}}_+$ is defined by

$$L_\beta(\xi, u) := \sup\{\nu(u) : \mathsf{Pot}(\mathcal{U}_\beta) \ni \nu \circ U_\beta \leqslant \xi\}.$$

$(A1.1)$ There exists a second Lusin measurable space $(E_1,\mathcal{B}_1)$ such that $E \subset E_1$, $E \in \mathcal{B}_1$, $\mathcal{B}(E) = \mathcal{B}_1|_E$, and there exists a resolvent of kernels $\mathcal{U}^1 = (U^1_\alpha)_{\alpha>0}$ on $(E_1,\mathcal{B}_1)$ satisfying $(h1)$ on this larger space, $U^1_\beta(1_{E_1\setminus E}) = 0$, and $\mathcal{U}$ is the restriction of $\mathcal{U}^1$ to E (i.e., $U_\beta(g) = U^1_\beta(g^1)$, where $g^1 \in p\mathcal{B}_1$ and $g^1|_E = g$). We clearly have $\mathsf{Exc}(\mathcal{U}_\beta) = \mathsf{Exc}(\mathcal{U}^1_\beta)$ and the following property holds (for one and therefore for all $\beta > 0$).

$(A1.1.a)$ every $\xi \in \mathsf{Exc}(\mathcal{U}^1_\beta)$ with $L_\beta(\xi, 1) < \infty$ is a potential on E_1 (with respect to $\mathcal{U}^1_\beta$).

One can take for E_1 the set of all extreme points of the set $\{\xi \in \mathsf{Exc}(\mathcal{U}_\beta) |\, L_\beta(\xi, 1) = 1\}$, endowed with the σ-algebra $\mathcal{B}_1$ generated by the functionals $\widetilde{u}$, $\widetilde{u}(\xi) := L_\beta(\xi, u)$ for all $\xi \in E_1$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$. Let $(E', \mathcal{B}')$ be a Lusin measurable space such that $E \subset E'$, $E \in \mathcal{B}'$, $\mathcal{B}(E) = \mathcal{B}'|_E$, and there exists a proper sub-Markovian resolvent of kernels $\mathcal{U}' = (U'_\alpha)_{\alpha>0}$ on $(E', \mathcal{B}')$ with $U'_\beta(1_{E'\setminus E}) = 0$, $\mathcal{U}'$ satisfies $(h1)$ on E', E' satisfies $(A1.1.a)$ with respect to $\mathcal{U}'$, and $\mathcal{U}$ is the restriction of $\mathcal{U}'$ to E. Then the map $x \longmapsto \delta_x \circ U'_\beta$ is a measurable isomorphism between $(E', \mathcal{B}')$ and the measurable space $(E_1, \mathcal{B}_1)$.

Extension of excessive functions from E to E_1. If $\xi = \mu \circ U_\beta \in \mathsf{Pot}(\mathcal{U}_\beta)$ and $u \in \mathcal{E}(\mathcal{U}_\beta)$, then by Theorem 1.4.5 from [5] we have

$$(A1.1.b) \qquad L_\beta(\xi, u) = \int u \, d\mu.$$

$(A1.1.c)$ For every $u \in \mathcal{E}(\mathcal{U}_\beta)$ we consider the function $\widetilde{u} : E_1 \longrightarrow \overline{\mathbb{R}}_+$ defined above,

$$\widetilde{u}(\xi) := L_\beta(\xi, u), \quad \xi \in E_1.$$

Then by $(A1.1.b)$ we have $\widetilde{u}(\delta_x \circ U_\beta) = u(x)$ for all $x \in E$, and therefore, by the embedding of E in E_1,

$$\widetilde{u}|_E = u.$$

In addition, $\widetilde{u}$ is $\mathcal{U}^1_\beta$-excessive and it is the (unique) extension by fine continuity of u from E to E_1.

$(A1.2)$ **Ray cones.** If $\beta > 0$, then a *Ray cone* associated with $\mathcal{U}_\beta$ is a cone $\mathcal{R}$ of bounded $\mathcal{U}_\beta$-excessive functions such that: $U_\alpha(\mathcal{R}) \subset \mathcal{R}$ for all $\alpha > 0$, $U_\beta((\mathcal{R} - \mathcal{R})_+) \subset \mathcal{R}$, $\sigma(\mathcal{R}) = \mathcal{B}(E)$, it is min-stable, separable in the supremum norm and $1 \in \mathcal{R}$. Such a Ray cone always exists. Below if we refer to a Ray cone it is always meant to be associated with one fixed resolvent $\mathcal{U}_\beta$. If $\mathcal{U}$ is transient (i.e., there exists a strictly positive function $f_o \in bp\mathcal{B}(E)$ with $\sup_\alpha U_\alpha f_o < \infty$), then one can take $\beta = 0$, that is, there is a Ray cone of $\mathcal{U}$-excessive functions. A *Ray topology* on E is a topology generated by a Ray cone; for more details see ch. 1 in [5] and also [8] for the nontransient case.

(A2) Proof of Lemma 2.3. As we already mentioned, we follow the classical approach. Compare, e.g., page 48 from [31], page 115 in [30]; see also pages 133-134 in [5] and the proof of Theorem 5.5 (ii) from [15].

We start with the construction of a convenient compactification of E, as in the proof of Theorem 5.2 from [15].

Let K be the compactification of E with respect to $\mathcal{F}$. Since for every real-valued function $u \in \mathcal{E}(\mathcal{U}^o_\beta)$ the real-valued process $(e^{-\beta t} u \circ X_t)_{t \geqslant 0}$ is a right continuous (P^x-integrable) supermartingale under P^x for all $x \in E$, it follows that this process has left limits P^x-a.s. and we conclude that X has left limits in K a.s.

Let $(T_n)_n$ be an increasing sequence of stopping times and $T = \lim_n T_n$. There is no loss of generality to assume that T is bounded. From the above considerations the limit $Z := \lim_n X_{T_n}$ exists in K a.s. and $Z(\omega) \in E$ if $T(\omega) < \zeta(\omega)$.

In order to prove that $Z = X_T$ a.s. on $[T < \zeta]$, it is enough to show that for every $x \in E$ and $G \in bp\mathcal{B}(K \times K)$,

$$E^x(G1_{E\times E}(Z, X_T)) = E^x(G1_{E\times E}(Z, Z)). \tag{A2.1}$$

Indeed, taking as G the indicator function of the diagonal of $K \times K$, from $(A2.1)$ we get $P^x([Z \in E, Z \neq X_T]) = 0$.

Note that every function f from $\overline{[\mathcal{F}]}$ has an extension by continuity from E to K, denoted by $\overline{f}$. Since $[b\mathcal{E}(\mathcal{U}_\beta)]$ is an algebra, we may assume that $\mathcal{F}$ is multiplicative. In order to prove $(A2.1)$ we first use the strong Markov property (clearly, $\overline{f}(Z) \in \mathcal{F}_T$) and then the P^x-a.s. equality $\lim_n f(X_{T_n})p_t g(X_{T_n}) = \overline{f}(Z)\overline{p_t g}(Z)$ (because we take $f \in [\mathcal{F}]$ and $p_t g$ belongs to $\overline{[\mathcal{F}]}$ provided that $g \in \mathcal{K}$):

$$\begin{aligned}
E^x(\overline{f}(Z)U_\alpha g(X_T)) &= E^x(\overline{f}(Z)E^{X_T}\int_0^\infty e^{-\alpha t}g(X_t)\,\mathrm{d}t)\\
&= E^x(\overline{f}(Z)e^{\alpha T}\int_T^\infty e^{-\alpha t}g(X_t)\,\mathrm{d}t)\\
&= \lim_n E^x(f(X_{T_n})e^{\alpha T_n}\int_{T_n}^\infty e^{-\alpha t}g(X_t)\,\mathrm{d}t) = \lim_n E^x(f(X_{T_n})U_\alpha g(X_{T_n}))\\
&= \lim_n E^x(f(X_{T_n})\int_0^\infty e^{-\alpha t}p_t g(X_{T_n})\,\mathrm{d}t) = E^x(\overline{f}(Z)\int_0^\infty e^{-\alpha t}\overline{p_t g}(Z)\,\mathrm{d}t).
\end{aligned}$$

By a monotone class argument we have for all $h \in bp\mathcal{B}(K)$

$$E^x(h1_E(Z)U_\alpha g(X_T)) = E^x(h1_E(Z)\int_0^\infty e^{-\alpha t}\overline{p_t g}(Z)\,\mathrm{d}t),$$

and therefore

$$\begin{aligned}
E^x(h1_E(Z)U_\alpha g(X_T)) &= E^x(h(Z)\int_0^\infty e^{-\alpha t}p_t g(Z)\,\mathrm{d}t; Z \in E)\\
&= E^x(h(Z)U_\alpha g(Z); Z \in E).
\end{aligned}$$

Because $\lim_{\alpha\to\infty} \alpha U_\alpha g = g$ (since g is continuous), multiplying by α and letting α tend to infinity we get

$$E^x(h1_E(Z)g(X_T)) = E^x((hg1_E)(Z)).$$

Again using monotone class arguments we first obtain

$$E^x(h1_E(Z)\cdot k1_E(X_T)) = E^x(h1_E(Z)\cdot k1_E(Z)) \quad \text{for all } h, k \in bp\mathcal{B}(K),$$

and then $(A2.1)$.

If the transition function $(p_t)_{t\geqslant 0}$ is Markovian, then the limit $Z = \lim_n X_{T_n}$ exists in E a.s. Therefore, in this case it is enough to show that $(A2.1)$ holds for every $G \in bp\mathcal{B}(E \times E)$. Note that the extensions by continuity of f and $p_t g$ from E to K are not longer necessary, in particular, $\lim_n f(X_{T_n})p_t g(X_{T_n}) = f(Z)p_t g(Z)$ P^x-a.s. □

(A3) Proof of Proposition 4.1. Let

$$K\varphi := B\widehat{\varphi}.$$

With this notation (4.4) becomes

$$h_t(x) = T_t^c\varphi(x) + \int_0^t T_{t-u}cKh_u(x)\,du, \quad t \geqslant 0,\ x \in E. \tag{A3.1}$$

We prove first the uniqueness. As in [32], the inequality (4.11), one can see that if $\varphi, \psi \in \mathcal{B}_{\mathsf{u}}$ and $\mu \in \widehat{E}$, then

$$(A3.2) \qquad |\widehat{\varphi}(\mu) - \widehat{\psi}(\mu)| \leqslant l_1(\mu)||\varphi - \psi||_\infty.$$

From (4.1) and the $(A3.2)$ we conclude that

$(A3.3)$ the mapping $\varphi \longmapsto cK\varphi$ is Lipschitz with the constant β_o.

If h_t and h'_t are two solutions of (4.4), then for all $t \geqslant 0$,

$$\|h_t - h'_t\|_\infty \leqslant \int_0^t \|T^c_{t-u}(\mid cKh_u - cKh'_u \mid)\|_\infty du \leqslant \beta_o \int_0^t \parallel h_u - h'_u \parallel_\infty du.$$

It follows by Gronwall's Lemma that $\parallel h_t - h'_t \parallel_\infty = 0$.

To prove the existence, define inductively the operators H^n_t, $n \geqslant 0$, as $H^0_t\varphi := T^c_t\varphi$:

$$(A3.4) \qquad H^{n+1}_t\varphi := T^c_t\varphi + \int_0^t T^c_{t-u} cKH^n_u\varphi \, du, \ \varphi \in \mathcal{B}_{\mathsf{u}}.$$

Clearly the function $(t,x) \longmapsto H^n_t\varphi(x)$ is measurable. We claim that the sequence $(H^n_t\varphi)_n$ is increasing. Indeed, $H^1_t\varphi = T^c_t\varphi + \int_0^t T^c_{t-u} cKH^0_u\varphi \, du \geqslant H^0_t\varphi$. If we suppose that $H^{n-1}_t\varphi \leqslant H^n_t\varphi$, then $H^{n+1}_t\varphi = T^c_t\varphi + \int_0^t T^c_{t-u} cKH^n_u\varphi \, du \geqslant T^c_t\varphi + \int_0^t T^c_{t-u} cKH^{n-1}_u\varphi \, du = H^n_t\varphi$. The last inequality holds because if $\varphi \leqslant \psi$, then $cK\varphi \leqslant cK\psi$, and in addition one can prove inductively that $H^n_t\varphi \leqslant H^n_t\psi$ for all n.

We claim now that

$$(A3.5) \qquad H^n_t 1 \leqslant 1 \quad \text{ for all } n \geqslant 0.$$

We proceed again by induction. The inequality holds for $n = 1$ because $(T^c_t)_{t\geqslant 0}$ is sub-Markovian and $H^0_t 1 = T^c_t 1$. If we assume that $H^n_t 1 \leqslant 1$, then $\widehat{H^n_t 1} \leqslant 1$ and therefore

$$\begin{aligned} H^{n+1}_t 1 &= T^c_t 1 + \int_0^t T^c_u(cB\widehat{H^n_t 1})du \leqslant T^c_t 1 + \int_0^t T^c_u c \, du \\ &= E^x(e^{-\int_0^t c(X_u)\,du} + \int_0^t e^{-\int_0^s c(X_u)\,du} c(X_s)\, ds) = 1. \end{aligned}$$

If $\varphi \in \mathcal{B}_{\mathsf{u}}$, then by $(A3.5)$ $H^n_t\varphi \in \mathcal{B}_{\mathsf{u}}$ for all $n \geqslant 0$. For $x \in E$, $t \geqslant 0$, and $\varphi \in \mathcal{B}_{\mathsf{u}}$ we set

$$H_t\varphi(x) := \sup_n H^n_t\varphi(x).$$

The function $(t,x) \longmapsto H_t\varphi(x)$ is measurable; by $(A3.5)$ we have $H_t 1 \leqslant 1$, $H_t(\mathcal{B}_{\mathsf{u}}) \subset \mathcal{B}_{\mathsf{u}}$, and passing to the pointwise limit in $(A3.4)$ it follows that $(H_t\varphi)_{t\geqslant 0}$ verifies $(A3.1)$.

(i) We show inductively that for all n the operator H^n_t is absolutely monotonic. If $n = 1$, then $H^1_t\varphi = T^c_t\varphi = \mathbf{T}_t\widehat{\varphi}$, where $\mathbf{T}_t : bp\mathcal{B}(\widehat{E}) \longrightarrow bp\mathcal{B}(E)$ is the kernel defined by $\mathbf{T}_t g = T^c_t(g|_E)$ for all $g \in bp\mathcal{B}(\widehat{E})$. Hence $H^1_t\varphi = \mathbf{T}_t\widehat{\varphi}$ for all $\varphi \in \mathcal{B}_{\mathsf{u}}$ and therefore H^1_t is absolutely monotonic. Suppose now that H^n_t is absolutely monotonic; $H^n_t\varphi = \mathbf{H}^n_t\widehat{\varphi}$. We have

$$H^{n+1}_t\varphi = \mathbf{T}_t\widehat{\varphi} + \int_0^t T^c_{t-u} cB\widehat{H^n_u\varphi} \, du = (\mathbf{T}_t + \int_0^t T^c_{t-u} cB\widehat{\mathbf{H}^n_u} \, du)\widehat{\varphi},$$

where $\widehat{\mathbf{H}_u^n}$ is the branching kernel on $\widehat{E}$ associated by (4.2) with $\mathbf{H}_u^n$. Taking

$$\mathbf{H}_t^{n+1} := \mathbf{T}_t + \int_0^t T_{t-u}^c cB\widehat{\mathbf{H}_u^n}\, du, \tag{A3.6}$$

it follows that H_t^{n+1} is also absolutely monotonic. One can deduce from $(A3.6)$ that for all $t \geqslant 0$ the sequence of kernels $(\mathbf{H}_t^n)_{n\geqslant 0}$ is increasing and therefore we may consider the kernel $\mathbf{H}_t$ defined as $\mathbf{H}_t := \sup_n \mathbf{H}_t^n$. From the above considerations for all $\varphi \in \mathcal{B}_{\mathsf{u}}$ we have $H_t\varphi = \sup_n H_t^n\varphi = \sup_n \mathbf{H}_t^n\widehat{\varphi} = \mathbf{H}_t\widehat{\varphi}$, and we conclude that H_t is absolutely monotonic.

We now prove the Lipschitz property of the mapping $\varphi \longmapsto H_t\varphi$. For, if $\varphi, \psi \in \mathcal{B}_{\mathsf{u}}$ and $t \geqslant 0$, then by $(A3.1)$ and $(A3.3)$

$$||H_t\varphi - H_t\psi||_\infty \leqslant ||\varphi - \psi||_\infty + \beta_o \int_0^t ||H_u\varphi - H_u\psi||_\infty du,$$

and by Gronwall's Lemma we conclude that $||H_t\varphi - H_t\psi||_\infty \leqslant \beta_o t||\varphi - \psi||_\infty$.

(ii) The semigroup property of $(H_t)_{t\geqslant 0}$ is a consequence of the uniqueness. Indeed, we have to show that $H_{t'+t}\varphi = H_t(H_{t'}\varphi)$, so it is enough to prove that the mapping $t \longmapsto H_{t'+t}\varphi$ verifies $(A3.1)$ with $H_{t'}\varphi$ instead of φ. We have

$$\begin{aligned} H_{t'+t}\varphi &= T_t^c T_{t'}^c\varphi + \int_0^{t'} T_t^c(T_{t'-u}^c cKH_u\varphi\, du) + \int_{t'}^{t'+t} T_{t'+t-u}^c cKH_u\varphi\, du \\ &= T_t^c(T_{t'}^c\varphi + \int_0^{t'} T_{t'-u}^c cKH_u\varphi\, du) + \int_0^t T_{t-s}^c cKH_{t'+s}\varphi\, ds \\ &= T_t^c H_{t'}\varphi + \int_0^t T_{t-s}^c cKH_{t'+s}\varphi\, ds. \end{aligned}$$

Suppose now that $B1 = 1$ and define inductively the operators $H_t'^n$, $n \geqslant 0$, as $H_t'^0\varphi := T_t^c\varphi + \int_0^t T_u^c cK\varphi du$:

$$H_t'^{n+1}\varphi := T_t^c\varphi + \int_0^t T_u^c cKH_{t-u}'^n\varphi\, du,\ \varphi \in \mathcal{B}_{\mathsf{u}}. \tag{A3.7}$$

We already observed that $T_t^c 1 + \int_0^t T_u^c c\, du = 1$; therefore $H_t'^0 1 = 1$ and by induction we get that $H_t'^n 1 = 1$ for all $n \in \mathbb{N}$. Using $(A3.3)$ as before we obtain

$$||H_t'^{n+1}\varphi - H_t'^n\varphi||_\infty \leqslant \beta_o \int_0^t ||H_u'^n\varphi - H_u'^{n-1}\varphi||_\infty du,$$

and because $||H_t'^1\varphi - H_t'^0\varphi||_\infty \leqslant \beta_o||\varphi||_\infty \int_0^t (2 + \beta_o u)du = ||\varphi||_\infty(2\beta_o t + \frac{(\beta_o t)^2}{2})$, again by induction

$$||H_t'^{n+1}\varphi - H_t'^n\varphi||_\infty \leqslant ||\varphi||_\infty \left(2\frac{(\beta_o t)^{n+1}}{(n+1)!} + \frac{(\beta_o t)^{n+2}}{(n+2)!}\right).$$

Consequently, if $t_o > 0$ is fixed, then

$$\sup_{\substack{x\in E\\ t\leqslant t_o}} \left|H_t'^{n+1}\varphi(x) - H_t'^n\varphi(x)\right| \leqslant \left(2\frac{(\beta_o t_o)^{n+1}}{(n+1)!} + \frac{(\beta_o t_o)^{n+2}}{(n+2)!}\right).$$

It follows that the sequence $(H_t'^n\varphi)_n$ is Cauchy in the supremum norm, and passing to the limit in $(A3.7)$, we deduce that the pointwise limit of this sequence verifies $(A3.1)$; hence it is $H_t\varphi$ by the uniqueness of the solution. In particular, $H_t 1 = \lim_n H_t'^n 1 = 1$.

(iii) Because the family $(H_t)_{t\geqslant 0}$ is a semigroup, it is enough to prove the right continuity in $t = 0$. Since $H_t\varphi(x)$ is a solution of $(A3.1)$ and the function $u \longmapsto T^c_{t-u}cKh_u(x)$ is bounded on $[0,\infty)$, by dominate convergence we get $\lim_{t\searrow 0}\int_0^t T^c_{t-u}cKh_u(x)\,du = 0$, and hence $t \longmapsto H_t\varphi(x)$ is right continuous in $t = 0$. □

(A4) Proof of Proposition 4.5. We may suppose that $f \leqslant 1$ and define the sub-Markovian kernel $\mathbf{K} : bp\mathcal{B}(\widehat{E}) \longrightarrow bp\mathcal{B}(E)$ by $\mathbf{K}g := K(g|_E)$ for all $g \in bp\mathcal{B}(\widehat{E})$. We apply Proposition 4.1 for $B := \mathbf{K}$ and observe that H^n_t extends to a kernel on $(E, \mathcal{B}(E))$ for each $t \geqslant 0$ and $n \in \mathbb{N}$. Since the limit of $(H^n_t)_n$ is increasing, we conclude that the solution of the equation (4.7) also extends to a kernel Q_t on $(E, \mathcal{B}(E))$. This proves (ii) and the first part of assertion (i).

(iii) The equality $U^o_\beta = U^c_\beta + U^c_\beta cKU^o_\beta$ follows from (4.7) by a straightforward calculation. Then by induction $U^o_\beta = U^c_\beta + (U^c_\beta cK)^2U^c_\beta + \ldots + (U^c_\beta cK)^nU^c_\beta + (U^c_\beta cK)^{n+1}U^o_\beta$ and letting n tend to infinity we have $U^o_\beta = U^c_\beta + G_\beta U^c_\beta$. The kernel G_β is bounded because $U^c_\beta cK1 \leqslant ||\frac{c}{c+\beta}||_\infty \lim_{t\to\infty}\int_0^t T^{c+\beta}_u(c+\beta)du \leqslant \frac{c_o}{c_o+\beta}$, where $c_o := ||c||_\infty$. If $u \in b\mathcal{E}(\mathcal{U}^o_\beta)$, then clearly $\alpha U^c_{\beta+\alpha}u \leqslant u$ for all $\alpha > 0$ because $U^c_\alpha \leqslant U^o_\alpha$. From $\lim_{t\to 0} Q_t u = u$ we get by (ii) that $\lim_{t\to 0} T^c_t u = u$; hence $u \in \mathcal{E}(\mathcal{U}^c_\beta)$, $b\mathcal{E}(\mathcal{U}^o_\beta) \subset \mathcal{E}(\mathcal{U}^c_\beta)$. The inequality $U^c_\beta \leqslant U^o_\beta$ for all $\beta > 0$ implies that the function $G_\beta U^c_\beta f = U^o_\beta f - U^c_\beta f$ is $\mathcal{U}^o_\beta$-excessive for every $f \in bp\mathcal{B}(E)$. If $v \in b\mathcal{E}(\mathcal{U}^c_\beta)$, then we take a sequence $(f_n)_n \subset bp\mathcal{B}(E)$ such that $U^c_\beta f_n \nearrow v$ and therefore $G_\beta U^c_\beta f_n \nearrow G_\beta v \in \mathcal{E}(\mathcal{U}^o_\beta)$, $U^o_\beta f_n \nearrow v + G_\beta v \in b\mathcal{E}(\mathcal{U}^o_\beta)$, so $v \in [b\mathcal{E}(\mathcal{U}^o_\beta)]$. We clearly have $\mathcal{E}(\mathcal{U}_\beta) \subset \mathcal{E}(\mathcal{U}^c_\beta) \subset \mathcal{E}(\mathcal{U}_{c_o+\beta})$, and by Remark 2.1 we get $[b\mathcal{E}(\mathcal{U}_\beta)] = [b\mathcal{E}(\mathcal{U}_{c_o+\beta})] = [b\mathcal{E}(\mathcal{U}^c_\beta)]$.

We now check $(h1)$ for $\mathcal{U}^o$. From $[b\mathcal{E}(\mathcal{U}^o_\beta)] = [b\mathcal{E}(\mathcal{U}^c_\beta)]$ and since $\mathcal{U}^c$ verifies $(h1)$ we conclude that $\mathcal{B}(E) = \sigma(\mathcal{E}(\mathcal{U}^c_\beta)) = \sigma(\mathcal{E}(\mathcal{U}^o_\beta))$. The constant function 1 is $\mathcal{U}^o$-supermedian and it belongs to $[b\mathcal{E}(\mathcal{U}^o_\beta)]$, therefore $\lim_{t\to 0} Q_t 1 = 1$, $1 \in \mathcal{E}(\mathcal{U}^o)$.

The fact that $(Q_t)_{t\geqslant 0}$ is the transition function of a right Markov process with state space E is a consequence of Proposition 5.2.4, Proposition 3.5.3, and Corollary 1.8.12 from [5]. □

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Note added in proof

Investigating the branching properties of the solution of a fragmentation equation for the mass distribution, the main result of this paper (Theorem 4.10) is used in [10] to construct a branching process corresponding to a rate of loss of mass greater than a given strictly positive threshold.

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