

NON-ASPHERICAL ENDS AND NON-POSITIVE CURVATURE

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ABSTRACT. We obtain restrictions on the boundary of a compact manifold whose interior admits a complete Riemannian metric of non-positive sectional curvature.

1. INTRODUCTION

In this paper all manifolds are smooth, all metrics are Riemannian, and the phrase “non-positive sectional curvature” is abbreviated to “ $K \leq 0$ ”. A manifold is *covered by \mathbb{R}^n* if its universal cover is diffeomorphic to \mathbb{R}^n , e.g., any complete n -manifold of $K \leq 0$ is covered by \mathbb{R}^n by the Cartan-Hadamard theorem. A boundary component B of a manifold N is π_i -*incompressible* if the inclusion $B \hookrightarrow N$ induces an injection of i th homotopy groups, and B is *incompressible* if it is π_i -incompressible for each i . Thus an incompressible boundary component of an aspherical manifold is aspherical. Here is the main result of this paper:

Theorem 1.1. *Let M be a closed connected m -manifold that is either infranil or locally symmetric irreducible of $K \leq 0$ and real rank ≥ 2 . If N is a connected n -manifold with compact connected boundary such that $\text{Int } N$ admits a complete metric g of $K \leq 0$ and $\pi_1(\partial N) \cong \pi_1(M)$, then:*

- (1) ∂N is incompressible if either
 - (1a) ∂N has a neighborhood U in N such that $\text{Vol}_g(U \cap \text{Int } N)$ is finite,
or
 - (1b) $\pi_1(M)$ has no proper torsion-free quotients and $n - m = 1$.
- (2) If $n - m \geq 3$, then ∂N is π_1 -incompressible, and ∂N is the total space of a spherical fibration over M .

Recall that a closed manifold is *infranil* if it is of the form G/Γ , where G is a connected, simply-connected, nilpotent Lie group, C is a maximal compact subgroup of $\text{Aut}(G)$, and Γ is a torsion-free lattice in $G \rtimes C$. Infranilmanifolds are aspherical, and their fundamental groups are precisely the finitely generated torsion-free virtually nilpotent groups.

The assumption “ $\pi_1(M)$ has no proper torsion-free quotients” in (1b) holds if M is not infranil (by the Margulis Normal Subgroup Theorem). There are also some infranilmanifolds whose fundamental group has no proper torsion-free quotients; such manifolds are known in dimensions 2, $p^2 - 1$ and $p^3 - p$ for every prime p (see Example 5.3).

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We actually prove somewhat stronger results as follows:

- The claim (1) holds when the group $\pi_1(\partial N)$ is reductive, where a group G is *reductive* if for any epimorphism of $G \rightarrow H$, where H is a discrete, non-cocompact, torsion-free isometry group of a Hadamard manifold, the group H stabilizes a horoball or acts cocompactly on a totally geodesic, proper submanifold. See Examples 2.6 and 3.2 for a list of reductive groups.
- (1a) is true when ∂N has a compact neighborhood U in N such that $U \cap \text{Int } N$ has $\text{Inj Rad} \rightarrow 0$; see Corollary 4.2.
- The fibration $\partial N \rightarrow M$ in (2) is not arbitrary, e.g., if the fibration admits *no* section, then it is isomorphic to a linear sphere bundle over M , and moreover M admits a metric of $K \leq 0$; see Theorem 3.7(2).

We say that B *bounds* N if B and N are connected (not necessarily compact) manifolds such that B is diffeomorphic to ∂N ; this terminology is non-standard.

Gromov [Gro03] and Izeki-Nayatani [IN05] constructed many a group with finite classifying space such that any isometric action of the group on a Hadamard manifold fixes a point. As we note in Section 3, every group with these properties is the fundamental group of a closed non-aspherical manifold that

- (i) bounds no manifold whose interior has a complete metric of $K \leq 0$, and
- (ii) bounds a compact manifold whose interior is covered by \mathbb{R}^n .

Theorem 1.1 also gives many closed non-aspherical manifolds satisfying (i)–(ii), e.g., if B is the total space of a linear S^k bundle over M , where $k \geq 2$ and M is as in Theorem 1.1, then B satisfies (ii) and the above discussion implies:

- If M is infranil and not flat, and if the S^k bundle $B \rightarrow M$ has no section, then B satisfies (i); see Theorem 3.7(2).
- B cannot bound a manifold that admits a finite volume complete metric of $K \leq 0$; see Corollary 4.2.

Finiteness of volume in the last example is essential: the interior of any linear disk bundle over a closed manifold of $K \leq 0$ admits a complete metric of $K \leq 0$ [And87].

Other ways to produce closed manifolds satisfying (i)–(ii) are described in Examples 3.8(2)–(3) and 5.5.

By contrast, one does not know whether a closed *aspherical* manifold B can satisfy (i). A basic difficulty is that $B \times (0, 1)$ could admit a complete metric of $K \leq 0$. Such a metric exists if e.g. B itself admits a metric of $K \leq 0$, or if B is infranil [BK05]. In fact, B might even bound a manifold whose interior admits a *finite volume* complete metric of $K \leq 0$, which happens in many cases; see [Ont, NP, Bela].

There seems to be no simple description of closed manifolds that bound aspherical ones and in Section 2 we review some obstructions and examples. Results generalizing the parts (2), (1a), and (1b) of Theorem 1.1, are discussed in Sections 3, 4, and 5, respectively. Section 6 contains a proof of Theorem 4.1, the main technical result of this paper, which implies (1).

2. BOUNDARIES OF ASPHERICAL MANIFOLDS

Before imposing any curvature restrictions we study topological properties of boundaries of aspherical manifolds. Allowing B to be non-compact almost makes this task meaningless: if B bounds N , then any open connected subset U of

B bounds $U \cup \text{Int } N$. As we see below aspherical manifolds with compact non-aspherical boundaries also exist in abundance.

Example 2.1. Fiber bundles of aspherical manifolds is aspherical, but their boundary is often not incompressible, e.g., this happens for the product of two compact aspherical manifolds with non-empty boundaries (as is immediate from the cohomological dimension count).

Example 2.2. Complete, locally symmetric manifolds of $K \leq 0$ that have finite volume and \mathbb{Q} -rank ≥ 3 are interiors of compact manifolds with non-aspherical boundary.

Example 2.3. The fundamental group of any countable, locally finite, finite dimensional, aspherical cell complex occurs as the fundamental group of the boundary of an aspherical manifold, namely, the boundary of the regular neighborhood of a codimension ≥ 3 embedding of the complex into some manifold (e.g. the Euclidean space). If the complex is a manifold, or if it is a finite complex, then the boundary is non-aspherical. (The former follows from the homotopy sequence of the normal sphere bundles, and the latter is true because the boundary and the complex have isomorphic fundamental groups, and if the boundary were aspherical, it would be a closed manifold homotopy equivalent to a complex of lower dimension.)

Example 2.4. There is many a complete manifold V of $K \leq 0$ that deformation retracts onto a compact locally convex submanifold S of codimension zero (they are usually called *convex-cocompact*). The normal exponential map to ∂S identifies $\text{Int } S$ with V , and ∂S is often not aspherical.

Example 2.5. Identifying two aspherical n -manifolds N_1, N_2 along diffeomorphic, aspherical, compact, π_1 -injectively embedded, proper, codimension zero submanifolds $D_i \subset \partial N_i$ results in an aspherical manifold N , see [SW79], whose boundary is a union of $\partial N_1 - \text{Int } D_1$ and $\partial N_2 - \text{Int } D_2$ along $\partial D_1 \cong \partial D_2$. Considering the case when D_i 's are disks we conclude that if two manifolds bound aspherical manifolds, then so does their connected sum. The boundary ∂N is often non-aspherical, e.g., if N_1, N_2 are compact, D_i is homotopy equivalent to a complex of dimension $\leq n - 3$, and the inclusions $D_i \hookrightarrow \partial N_i$ are not π_1 -surjective, then $\pi_1(\partial N)$ is a non-trivial amalgamated product over $\pi_1(D_i)$, so ∂N is not aspherical by a Mayer-Vietoris argument in group cohomology.

With plenty of examples, we now turn to obstructions. In order for B to bound an aspherical manifold, a certain covering space of B must bound a contractible manifold. In formalizing how this restricts the topology of B , the following definition is helpful: given a class of groups \mathcal{Q} , a group is *anti- \mathcal{Q}* if it admits no non-trivial homomorphism into a group in \mathcal{Q} . Clearly, the class of anti- \mathcal{Q} groups is closed under extensions, quotients, and any group generated by a family of anti- \mathcal{Q} subgroups is anti- \mathcal{Q} .

Example 2.6. Let \mathcal{A}_n denote the class of fundamental groups of aspherical n -manifolds. Here are some examples of anti- \mathcal{A}_n groups:

- (1) Any group generated by a set of finite order elements.
- (2) Any non-torsion-free simple group.
- (3) Any irreducible lattice in the isometry group of a symmetric space of rank ≥ 2 and dimension $> n$ [BF02].

- (4) Certain finitely presented groups with strong fixed point properties; see [ABJ⁺09, Theorem 1.5].
- (5) Any non-elementary finitely presented relatively hyperbolic group has a finitely presented anti- \mathcal{A}_n quotient, obtainable by adding finitely many relators [AMO07, Corollary 1.6].
- (6) Any finitely presented anti- \mathcal{A}_n group is a quotient of an anti- \mathcal{A}_n non-elementary hyperbolic group. (The Rips construction of [BO08] produces a relatively hyperbolic group G with $G/K \cong Q$, where Q is the given finitely presented anti- \mathcal{A}_n group, and K is a quotient of $\mathbb{Z}_3 * \mathbb{Z}_3$. Here G is anti- \mathcal{A}_n because so are Q and K .)

The following summarizes some obstructions that prevent a manifold from bounding an aspherical one.

Theorem 2.7. *If B bounds an aspherical, non-contractible n -manifold N , and $\pi_1(B)$ is anti- \mathcal{A}_n , then B is non-compact, parallelizable, its \mathbb{Z} -valued intersection form vanishes, and its \mathbb{Q}/\mathbb{Z} -valued torsion linking form vanishes.*

Proof. Since $\pi_1(B)$ is anti- \mathcal{A}_n , the manifold B also bounds a contractible manifold W , whose interior is the universal cover of $\text{Int } N$. Since W is parallelizable, B is stably parallelizable, and in particular orientable.

Let us show that B is non-compact. The long exact sequence of the pair gives an isomorphism $H_n(W, B) \rightarrow H_{n-1}(B)$, so if B were compact, Poincaré duality would imply non-triviality of $H_{n-1}(B)$, and hence non-triviality of $H_n(W, B) \cong H_c^0(W)$, so there would exist a compactly supported constant function on the 0-skeleton of W which is only possible if W were compact, but by assumptions $\pi_1(N)$ is non-trivial, and hence infinite, so W is non-compact.

Since B is non-compact, its stable parallelizability implies parallelizability.

Recall that the intersection form $H_k(B) \times H_{n-1-k}(B) \rightarrow \mathbb{Z}$ can be computed via algebraic intersection numbers. If α, β are simplicial cycles in B of complementary dimensions, then since B bounds a contractible manifold W , there are simplicial chains $\hat{\alpha}, \hat{\beta}$ in W with $\partial\hat{\alpha} = \alpha$ and $\partial\hat{\beta} = \beta$. Choosing $\hat{\alpha}, \hat{\beta}$ transverse, and subdividing if necessary, one observes that the intersection of $\hat{\alpha}, \hat{\beta}$ is a 1-chain \hat{c} , and the 0-homology class of $\partial\hat{c}$ equals the algebraic intersection number $\iota(\alpha, \beta)$ in B . Now $\iota(\alpha, \beta)$ is a sum of points labelled with \pm , and since it is a boundary, the points come in pairs and for each $+$ there is a $-$, so that $\iota(\alpha, \beta) = 0$.

If $\tau H_k(B)$ denotes the torsion subgroup of $H_k(B)$, then the linking form $\text{lk}: \tau H_k(B) \times \tau H_{n-2-k}(B) \rightarrow \mathbb{Q}/\mathbb{Z}$ is defined as follows: given two torsion cycles α, β the linking coefficient $\text{lk}([\alpha], [\beta])$ equals $\frac{1}{m}\iota(z, \beta)$ modulo 1, where z is any chain in B with $\partial z = m\alpha$. If $\hat{\beta}$ is a chain in W with $\partial\hat{\beta} = \beta$, then $\iota(z, \beta)$ equals the intersection number of z and $\hat{\beta}$ in W . Since $\alpha = \partial\hat{\alpha}$ in W , we conclude that $z - m\hat{\alpha}$ is a cycle in W , which is a boundary, as W is contractible. The intersection number of any boundary with $\hat{\beta}$ is zero, hence $\iota(z, \beta)$ is divisible by m , so $\text{lk}([\alpha], [\beta]) = 0$. □

Remark 2.8. If a closed $(n-1)$ -manifold B bounds a manifold W with $H_{n-1}(W; \mathbb{Z}_2) = 0$, then W is compact and $\partial W = B$. Indeed, B bounds $B \cup \text{Int}(W)$ which is compact by an argument in the second paragraph of the above proof giving a compactly supported constant function on the 0-skeleton of some triangulation of $B \cup \text{Int}(W)$.

Remark 2.9. The same proof gives that if B bounds a contractible manifold, then B is stably parallelizable, and its intersection form and torsion linking form vanish. By Remark 2.8 if B is compact, then it equals the boundary of a compact contractible manifold, so B is a homology sphere, and conversely, any homology sphere bounds a topological contractible manifold [Ker69, Fre82].

Example 2.10. The following manifolds do not bound aspherical ones:

- (1) The connected sum of lens spaces, because it is a closed manifold whose fundamental group is anti- \mathcal{A}_n .
- (2) The product of any manifold with CP^k with $k \geq 2$ because it contains a simply-connected open subset which is not parallelizable, namely, the tubular neighborhood of CP^k .
- (3) The connected sum of any manifold and the product of two closed manifolds whose fundamental groups are anti- \mathcal{A}_n , as the slice inclusions of the factors in the product have non-zero intersection number.
- (4) The product of a punctured 3-dimensional lens space and a closed manifold whose fundamental group is anti- \mathcal{A}_n (for if α represents a generator in the first homology of the lens space, then it links non-trivially with itself, see e.g. [PY03], so its product with the closed manifold factor links non-trivially with α).
- (5) Any manifold that contains the manifold in (2), (3), or (4) as an open subset.

3. REDUCTIVE GROUPS AND REGULAR NEIGHBORHOODS

Let \mathcal{NP}_n denote the class of the fundamental groups of complete n -manifolds of $K \leq 0$. By the Cartan-Hadamard theorem, \mathcal{NP}_n is a subset of \mathcal{A}_n . The two classes coincide for $n = 2$ by the uniformization and classification of surfaces. On the other hand, for each $n \geq 4$ there is a closed, locally CAT(0) and hence aspherical, n -manifold, whose fundamental group is not in \mathcal{NP}_n [DJ91, DJL12].

As mentioned in the introduction, examples of groups with strong fixed point properties due to Gromov and Izeki-Nayatani immediately imply:

Theorem 3.1. *There is a closed non-aspherical manifold that*

- (i) *bounds a manifold whose interior is covered by a Euclidean space;*
- (ii) *bounds no manifold whose interior has a complete metric of $K \leq 0$.*

Proof. Gromov [Gro03] (cf. [NS11, Theorem 1.1]) and Izeki-Nayatani [IN05] showed that there is many a finite aspherical simplicial complex Y such that $\pi_1(Y)$ is anti- \mathcal{NP}_n for all n . (In the work of Izeki-Nayatani Y is the quotient of a Euclidean building by a uniform lattice in a p -adic symmetric space, while in Gromov's example $\pi_1(Y)$ is (Gromov) hyperbolic; to date there is no explicit example of such a Y .) Now embed Y into a Euclidean space as a codimension ≥ 3 subcomplex and let N be its regular neighborhood. The closed manifold $B := \partial N$ is non-aspherical because the inclusion $B \hookrightarrow N$ is π_1 -injective and hence the cohomological dimension of $\pi_1(B)$ is bounded above by the cohomological dimension of $\pi_1(Y)$, which is $\leq \dim(Y)$ while $\dim(B) \geq \dim(Y) + 2$. One can choose the embedding of Y so that the universal cover of $\text{Int}(N)$ is simply-connected at infinity and hence diffeomorphic to a Euclidean space (in fact, all codimension ≥ 3 regular neighborhoods have this property). Since $\pi_1(B)$ is anti- \mathcal{NP}_n , Theorem 2.7 implies that B satisfies (i). \square

Reductive groups (defined in the introduction) give another source of manifolds satisfying (i)–(ii).

Example 3.2. (1) Clearly, the property of being reductive is inherited by quotients, and every group that is anti- \mathcal{NP}_n for all n is reductive.

(2) Any finitely generated, virtually nilpotent group is reductive; see [Belb] where this is deduced by combining results of Gromov [BGS85] with the Flat Torus Theorem.

(3) Any irreducible, uniform lattice in the isometry group of a symmetric space of $K \leq 0$ and real rank > 1 is reductive; see [Belb] where it is deduced from the harmonic map superrigidity.

Remark 3.3. If G is a group as in Examples 3.2(2)–(3), then any torsion-free, quotient of G is the fundamental group of a closed aspherical m -manifold, which is an infranil or irreducible, locally symmetric of $K \leq 0$ of rank ≥ 2 , respectively, where m is bounded above by the virtual cohomological dimension of G . If G is virtually nilpotent this follows from [DI94], and when G is a higher rank lattice one invokes Margulis's Normal Subgroup Theorem.

The following definition helps combine the above examples of reductive groups: Given groups I, J and a class of groups \mathcal{Q} we say that I reduces to J relative to \mathcal{Q} if every homomorphism $I \rightarrow Q$ with $Q \in \mathcal{Q}$ factors as a composite of an epimorphism $I \rightarrow J$ and a homomorphism $J \rightarrow Q$. Clearly if $\mathcal{Q} = \mathcal{NP}_n$, and J is reductive, then so is I , and moreover, I and J have the same quotients in \mathcal{NP}_n .

Example 3.4. The class of groups that reduce to J relative to \mathcal{Q} is sizable, e.g., if K is anti- \mathcal{Q} , then the direct product $K \times J$ reduces to J relative to \mathcal{Q} ; more generally, the same is true for any amalgamated product obtained by identifying a subgroup $A \leq J$ with the second factor of $K \times A$. Furthermore, a variant of the Rips construction established in [BO08] implies that if there is an anti- \mathcal{Q} non-elementary hyperbolic group, then for any finitely presented group J , there is a non-elementary hyperbolic group I that reduces to J relative to \mathcal{Q} . This result applies to $\mathcal{Q} = \mathcal{NP}_n$ since $\mathbb{Z}_2 * \mathbb{Z}_3$ is anti- \mathcal{NP}_n and non-elementary hyperbolic (if instead of $\mathbb{Z}_2 * \mathbb{Z}_3$ one uses the anti- \mathcal{NP}_n , torsion-free, hyperbolic groups of [Gro03], then I becomes torsion-free).

Theorem 3.5. *Let B be a closed $(n - 1)$ -manifold such that $\pi_1(B)$ is reductive and any non-trivial quotient of $\pi_1(B)$ in the class \mathcal{NP}_n has a finite classifying space of dimension $\leq n - 3$. If B bounds a manifold N such that $\text{Int } N$ admits a complete metric of $K \leq 0$, then N is compact, and the inclusion $B \rightarrow N$ is a π_1 -isomorphism.*

Proof. Let \hat{N} be the covering space of N that corresponds to the image of $\pi_1(B)$ under the inclusion $B \hookrightarrow N$. Then B bounds $B \cup \text{Int}(\hat{N})$ which by assumption has a classifying space of dimension $\leq n - 3$, so $B \cup \text{Int}(\hat{N})$ is compact by Remark 2.8. Now covering space considerations imply that $B = \partial\hat{N}$ and $\hat{N} = N$, so the inclusion $B \hookrightarrow N$ is π_1 -surjective.

Consider the $\pi_1(N)$ -action on the Hadamard manifold that covers $\text{Int } N$. Since $\pi_1(B)$ is reductive, the action either stabilizes a horoball or a totally geodesic submanifold S of dimension $\leq n - 3$ where the deck-transformation group acts

comactly. Hence $\text{Int } N$ is diffeomorphic to either the product of \mathbb{R} and another manifold, or to the normal bundle of $S/\pi_1(N)$, which is a compact submanifold of dimension $\leq n-3$. In the latter case N is obtained by attaching an h -cobordism to the tubular neighborhood of $S/\pi_1(N)$, so the inclusion $B \rightarrow N$ is a π_1 -isomorphism. The same is true in the former case by the main result of [Belb]. \square

Remark 3.6. The conclusion of Theorem 3.5 can be sharpened as follows. Stallings's embedding up to homotopy type theorem [Sta] implies that N of Theorem 3.5 deformation retracts to a regular neighborhood of a finite subcomplex of codimension $q \geq 3$, so an excision argument as in [Sie69, Theorem 2.1] shows that N is obtained by attaching an h -cobordism to the boundary of the regular neighborhood.

If N of Theorem 3.5 is homotopy equivalent to a closed manifold, one can say more, which requires the following background.

Let N be a compact n -manifold that is homotopy equivalent to a closed m -manifold M such that $q = n - m \geq 3$, and the inclusion $\partial N \rightarrow N$ is a π_1 -isomorphism. The Browder-Casson-Haefliger-Sullivan-Wall embedding theorem [Wal99, Corollary 11.3.4] shows that N is obtained by attaching an h -cobordism to the boundary of the regular neighborhood of a PL-embedded copy of M . Abstract regular neighborhoods of M of codimension $q \geq 3$ are q -block bundles over M , and they are classified by homotopy classes of maps of M into $B\widetilde{PL}_q$, the classifying space of the semisimplicial group \widetilde{PL}_q [RS68a]. Taking product with the $(q-k)$ -cube defines a stabilization map $B\widetilde{PL}_k \rightarrow B\widetilde{PL}_q$. Each q -block bundle R over M defines a spherical fibration over M with structure group in \widetilde{PL}_q and fiber S^{q-1} ; up to homotopy the fibration is the inclusion $\partial R \hookrightarrow R$ [RS68b]. Spherical fibrations over M with fiber S^{q-1} are classified by maps $M \rightarrow BG_q$, and we say that *the structure group of a fibration reduces to $\widetilde{PL}_{k \leq q}$* if its classifying map factors (up to homotopy) through $B\widetilde{PL}_k \rightarrow B\widetilde{PL}_q \rightarrow BG_q$. A spherical fibration is *linear* if it is isomorphic to a unit sphere bundle of a vector bundle.

Theorem 3.7. *Let M be a closed m -manifold that is either infranil or an irreducible, locally symmetric of $K \leq 0$ and real rank ≥ 2 . Let B be a closed $(n-1)$ -manifold such that $q = n - m \geq 3$, and $\pi_1(B)$ reduces to $\pi_1(M)$ relative to \mathcal{NP}_n . If B bounds a manifold whose interior admits a complete metric of $K \leq 0$, then*

(1) *B is the total space of a spherical fibration β over M whose structure group reduces to \widetilde{PL}_q .*

(2) *If $q \geq 4$ and the structure group of β does not reduce to \widetilde{PL}_{q-1} , or if $q = 3$ and β has no section, then M admits a metric of $K \leq 0$ and β is linear.*

Proof. Any quotient of $\pi_1(B)$ in \mathcal{NP}_n is also a quotient of $\pi_1(M)$ which by Remark 3.3 is the fundamental group of a closed aspherical manifold of dimension $\leq m$. Thus Theorem 3.5 applies.

If N is as in Theorem 3.5, then the isomorphism $\pi_1(B) \rightarrow \pi_1(N)$ factors through a surjection $\pi_1(B) \rightarrow \pi_1(M)$, so N is homotopy equivalent to M , and by the preceding discussion N is obtained by attaching an h -cobordism to the boundary of the regular neighborhood of a PL-embedded copy of M , which proves (1). (It is known that $\pi_1(M)$ has trivial Whitehead group, so the h -cobordism is smoothly trivial if $n \geq 6$ but we do not use this fact here.)

Let $H \cong \pi_1(M)$ be the deck-transformation isometry group of the non-positively curved metric on the universal cover of $\text{Int } N$.

If H stabilizes a horoball bounded by a horosphere S , then $\text{Int } N$ is diffeomorphic to $\mathbb{R} \times S/H$. In particular, the associated spherical fibration has a section obtained by sliding along the \mathbb{R} -factor. If $q \geq 4$, then again the homotopy equivalence $M \rightarrow S/H$ is homotopic to a PL-embedding [Wal99, Corollary 11.3.4], whose regular neighborhood is a block bundle with structure group in \widetilde{PL}_{q-1} , and hence the same is true for its product with \mathbb{R} , so that the structure group of the associated S^{q-1} -fibration reduces to \widetilde{PL}_{q-1} .

If H does not stabilize a horoball, then by superrigidity or the Flat Torus Theorem, H acts cocompactly on a totally geodesic subspace whose H -quotient is diffeomorphic to M (and in fact homothetic to M if it has higher rank, or affinely equivalent if M is flat). The normal exponential map to the subspace is an H -equivariant diffeomorphism, which identifies $\text{Int } N$ with the normal bundle to M . Thus the associated spherical fibration is linear. \square

Example 3.8. Let M be as in Theorem 3.7. In each of the following cases B bounds a manifold covered by a Euclidean space, but bounds no manifold whose interior admits a complete metric of $K \leq 0$:

(1) B is the total space of a linear S^{q-1} linear bundle over M with no section, where $q \geq 3$ and M is a non-flat infranilmanifold (e.g. if M is an orientable, even-dimensional, non-flat infranilmanifold, then the pullback of the unit tangent bundle under a degree one map $M \rightarrow S^m$ has non-zero Euler class, and hence no section).

(2) $B = M \times \partial C$ where C is a compact contractible manifold such that $\pi_1(\partial C)$ is a non-trivial group generated by finite order elements. (In each dimension ≥ 5 there are infinitely many such C 's, namely, given any finitely presented superperfect group K there is C with $\pi_1(\partial C) \cong K$ [Ker69], and the desired infinite family is obtained by varying K among the free products of finitely many superperfect finite groups.)

(3) B is the boundary of a q -block bundle over M such that $q \geq 4$, the associated spherical fibration is not linear, and its structure group does not reduce to \widetilde{PL}_{q-1} . (Such block bundles exist below metastable range. Here is a specific example that works whenever q is even and M has non-zero Betti numbers in degrees q and $4i$ for some $i > \frac{q}{2}$, as happens for example if M is a torus of dimension $\geq 4i > 2q$. It is well known but apparently not recorded in the literature (cf. [Kle]) that for $q \geq 3$ the classifying space \widetilde{BPL}_q is rationally equivalent to $BO \times BG_q$, which can be deduced by combining the following results: $(\widetilde{PL}, \widetilde{PL}_q) \rightarrow (G, G_q)$ is a weak homotopy equivalence [RS68b, Theorem 1.11], G is rationally contractible [MM79, Chapter 3A], and $BO \rightarrow \widetilde{BPL}$ is a rational equivalence; see [MM79, Chapter 4C] and [RS68b, Corollary 5.5]. Now BO is rationally the product of Eilenberg-MacLane spaces corresponding to Pontryagin classes, and if q is even, then BG_q is rationally $K(\mathbb{Q}, q)$, detected by the Euler class. By assumption there exist non-torsion cohomology classes $x \in H^q(M)$, $y \in H^{4i}(M)$ with $i > \frac{q}{2}$. As in [BK03, Appendix B] one can realize non-zero integer multiples of x , y as the Euler class e and the Pontryagin class p_i of a block bundle over M . Its structure group does not reduce to \widetilde{PL}_{q-1} as $e \neq 0$, so the associated spherical fibration has no section. The fibration is non-linear because for linear bundles $p_i = 0$ for $i > \frac{q}{2}$. Another example can be obtained if y has degree $2q$ and is not proportional to x^2 by realizing a multiple of y as $p_{q/2}$ and using that $e^2 = p_{q/2}$ for linear bundles.)

4. ENDS WITH INJECTIVITY RADIUS GOING TO ZERO

Following [BGS85], we say that a subset S of a Riemannian manifold *has Inj Rad* $\rightarrow 0$ if and only if for every $\varepsilon > 0$ the set of points of S with injectivity radius $\geq \varepsilon$ is compact; otherwise, S *has Inj Rad* $\not\rightarrow 0$. For example, by volume comparison any finite volume complete manifold of $K \leq 0$ has $\text{Inj Rad} \rightarrow 0$. Schroeder proved [BGS85, Appendix 2] that any complete manifold of $-1 \leq K \leq 0$ and $\text{Inj Rad} \rightarrow 0$ is the interior of a compact manifold with boundary provided it contains no sequence of totally geodesic, immersed, flat 2-tori whose diameters tend to zero.

Given a compact boundary component B of a manifold N , an *end* E of $\text{Int } N$ that corresponds to B is the intersection of $\text{Int } N$ with a closed collar neighborhood of B ; note that E is diffeomorphic to $[1, \infty) \times B$.

The *cohomological dimension* of a group G is denoted $\text{cd}(G)$.

Here is the main result of this paper whose proof is in Section 6.

Theorem 4.1. *Let N be a manifold with compact connected boundary B , let E be an end of $V := \text{Int } N$ that corresponds to B , and let H be the deck-transformation group of the universal cover $\tilde{V} \rightarrow V$ corresponding to the image of the inclusion induced homomorphism $\pi_1(E) \rightarrow \pi_1(V)$. If \tilde{V} admits a complete H -invariant metric g of $K \leq 0$ and \tilde{V} contains an H -invariant horoball, then*

(1) $\dim(B) = \text{cd}(H)$ if and only if B is incompressible in N .

(2) If $\text{Inj Rad} \rightarrow 0$ on E , then B is incompressible in N .

(3) If B is incompressible in N , then the universal cover of B is homeomorphic to a Euclidean space.

Corollary 4.2. *Let B be a closed manifold that bounds a manifold N , and let E be an end of $\text{Int } N$ corresponding to B . If $\pi_1(B)$ is reductive, and $\text{Int } N$ admits a complete metric of $K \leq 0$ and $\text{Inj Rad} \rightarrow 0$ on E , then B is incompressible in N .*

Proof. If H is as in Theorem 4.1, then since H is reductive, it either stabilizes a horoball or acts cocompactly on a totally geodesic proper submanifold S . In the former case B is incompressible by Theorem 4.1. The latter case cannot happen because the nearest point projection onto S is H -equivariant and distance non-increasing, and there is a lower bound for displacement of elements of H on S while the assumption $\text{Inj Rad} \rightarrow 0$ on E gives a sequence of points of V whose displacements under some elements of H tend to zero. \square

5. ENDS WITH FUNDAMENTAL GROUPS OF CODIMENSION ONE

A small variation on the proof of Theorem 4.1 yields:

Addendum 5.1. The part (1) of Theorem 4.1 holds if the assumption “ \tilde{V} contains an H -invariant horoball” is replaced by “ \tilde{V} contains an H -invariant totally geodesic submanifold of codimension one”.

Corollary 5.2. *Let B be a closed $(n-1)$ -manifold such that $\pi_1(B)$ is reductive and any non-trivial quotient of $\pi_1(B)$ in the class \mathcal{NP}_n has cohomological dimension $n-1$. If B bounds a manifold N such that $\text{Int } N$ admits a complete metric of $K \leq 0$, then B is incompressible in N .*

Proof. If H is as in Theorem 4.1, then H cannot be trivial by Theorem 2.7, hence $\text{cd}(H) = \dim(B)$, so B is incompressible in N by Theorem 4.1 and Addendum 5.1. \square

Example 5.3. Corollary 5.2 applies to the manifold M as in Theorem 3.7 provided $\pi_1(M)$ has no proper torsion-free quotient, which ensures that the cohomological dimension of any non-trivial torsion-free quotient of $\pi_1(M)$ equals $\dim(M)$. In fact if M is a higher rank, irreducible, locally symmetric manifold of $K \leq 0$, then $\pi_1(M)$ has no proper torsion-free quotients by the Margulis Normal Subgroup Theorem. There are also infranilmanifolds whose fundamental group has no proper torsion-free quotients. This includes all 3-dimensional infranilmanifolds with zero first Betti number, such as the infinite family 2 of [DIKL95, page 156] and the Hantzsche-Wendt flat 3-manifold, as well as some higher-dimensional flat manifolds in dimensions $p^2 - 1$ [GS99, Theorem 1(9)] and $p^3 - p$ [Cid02, page 29] for any prime p . The fundamental group of the Klein bottle also has no proper torsion-free quotients.

Remark 5.4. If one is not concerned with making sure that B bounds a manifold whose interior is covered by \mathbb{R}^n , there is a simple recipe for constructing B 's to which Corollaries 4.2 and 5.2 apply; for concreteness we focus on the latter. Start with $\pi_1(M)$ as in Example 5.3, consider any finitely presented group that reduces to $\pi_1(M)$ relative to \mathcal{NP}_n , and realize the group as the fundamental group of a closed $(n-1)$ -manifold B , which is always possible for $n \geq 5$. Now if B does bound a manifold whose interior admits a complete metric of $K \leq 0$, then Corollary 5.2 forces B to be incompressible, and hence aspherical with $\pi_1(B) \cong \pi_1(M)$. For instance, B is not incompressible if

- $\pi_1(B)$ has a non-trivial anti- \mathcal{NP}_n subgroup, or
- B is the connected sum $M \# S$, where S is simply-connected and not a homotopy sphere (if $M \# S$ were aspherical, its universal cover would contain a codimension one sphere bounding a punctured copy of S , so the latter would be contractible).

It takes more effort to find B that bounds a manifold whose universal cover is a Euclidean space but which cannot bound a manifold whose interior admits a complete metric of $K \leq 0$. Here is an infinite family of such examples.

Example 5.5. Let C be a compact contractible manifold as in Example 3.8(2). Fix a closed embedded disk $\Delta \subset \partial C$. Take M as in Example 5.3, and let α be a homotopically non-trivial embedded circle in M with a trivial normal bundle; the latter can always be arranged by replacing α with its “square” in $\pi_1(M)$. Attach $M \times [0, 1)$ to $S^1 \times C$ by identifying the tubular neighborhood of α in $M \times \{0\}$ with $S^1 \times \Delta$. The result is an aspherical manifold N with compact boundary $B := \partial N$. By Example 2.5 B is not aspherical. The arguments of [Belb, Theorem 6.1] based on a strong version of the Cantrell-Stallings hyperplane linearization theorem proved in [CKS12] imply $\text{Int } N$ is covered by \mathbb{R}^n . Since $\pi_1(B)$ is an amalgamated product of $\pi_1(M)$ and $\mathbb{Z} \times \pi_1(\partial C)$ along \mathbb{Z} , where $\pi_1(N)$ is generated by finite order elements, the only non-trivial quotient of $\pi_1(B)$ in \mathcal{NP}_n is $\pi_1(M)$, so B cannot bound a manifold whose interior admits a complete metric of $K \leq 0$ by Corollary 5.2.

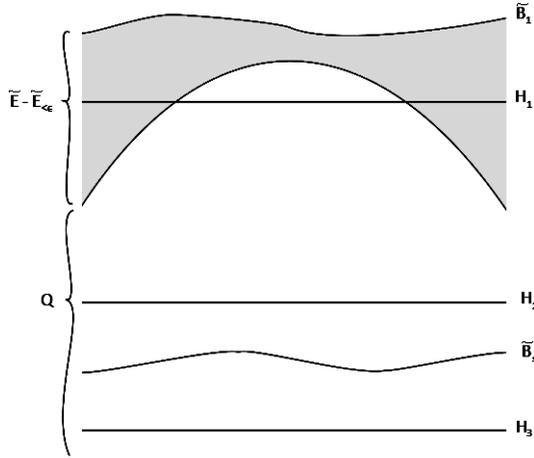


FIGURE 1. Schematic for the proof of Theorem 4.1(2)

6. PROOF OF THEOREM 4.1

Proof. Figure 1 is meant to accompany the proof of (2), but it also illustrates the proof of (1) if one ignores the labels on the left. Let \tilde{E} denote a connected H -invariant lift of E to \tilde{V} .

(1) The “only if” direction is trivial: the incompressibility and compactness of B implies $\text{cd}(H) = \dim(B)$, so we focus on the “if” direction.

Let B_t be the fiber of the projection $E \cong \partial E \times [1, \infty) \rightarrow [1, \infty)$ over $\{t\}$, and let $\tilde{B}_t \subset \tilde{E}$ be the lift of B_t .

Compactness of B_1 and the fact that H preserves a horoball implies that \tilde{B}_1 is in the r -neighborhood of a horosphere H_1 for some r . Let $N_{r+1}(H_1)$ be the closed $r + 1$ -neighborhood of H_1 .

We next show that \tilde{E} contains a component of $\tilde{V} - N_{r+1}(H_1)$. Otherwise, since \tilde{B}_1 separates \tilde{V} , both components lie in $\tilde{V} - \tilde{E}$, so that $\tilde{E} \subset N_{r+1}(H_1)$. Since $\text{cd}(H) = \dim(H_1)$ and H stabilizes H_1 , the H -action on H_1 is cocompact, and hence so is the H -action on $N_{r+1}(H_1)$. So E lies in a compact subset of V , which is a contradiction.

Denote the component of $\tilde{V} - N_{r+1}(H_1)$ that lies in \tilde{E} by U_2 ; then $H_2 := \partial U_2$ is a horosphere concentric to H_1 .

Since B_t separates E , the distance from B_1 and B_t is an increasing continuous function of t , and completeness of E implies that the function is unbounded. Hence the same is true for the distance from \tilde{B}_1 and \tilde{B}_t , and therefore there is s such that the distance between \tilde{B}_1 and \tilde{B}_s equals $2r + 3$. Note that U_2 contains \tilde{B}_s .

Since H acts cocompactly on $[1, s] \times \partial \tilde{E}$, there is R such that \tilde{B}_s lies in the R -neighborhood of \tilde{B}_1 , and hence in the $R + r$ -neighborhood of H_1 . Let H_3 be a horosphere in U_2 that bounds the $R + r + 1$ -neighborhood of H_1 , so that H_3 is disjoint from $[1, s] \times \partial \tilde{E}$.

By construction the inclusion of \tilde{B}_s into \tilde{E} , which is a homotopy equivalence, factors through the contractible region between H_2 and H_3 , so \tilde{B}_s is contractible.

It follows that B_s is aspherical and π_1 -injectively embedded into V , and hence ∂N is incompressible in N .

(2) Let $E_{<\varepsilon}$ denote the set of points of E with injectivity radius $< \varepsilon$. As ∂E is compact, ∂E and $E_{<\varepsilon}$ are disjoint for small enough ε , which we assume henceforth. Let $\tilde{E}_{<\varepsilon}$ be the preimage of $E_{<\varepsilon}$ under the covering $\tilde{E} \rightarrow E$. The set $\tilde{E}_{<\varepsilon}$ is open and hence locally convex (for if $d_\gamma(x) < \varepsilon$, then $d_\gamma(y) < \varepsilon$ for all y close to x , where d_γ denotes the displacement function for the deck-transformation γ). Since $\text{InjRad} \rightarrow 0$ on E , the H -action on $\tilde{E} - \tilde{E}_{<\varepsilon}$ is cocompact, so since H stabilizes a horoball, there is r such that $\tilde{E} - \tilde{E}_{<\varepsilon}$ lies in an r -neighborhood of an H -invariant horosphere, which we denote H_1 . Let H_2 be the horosphere that bounds the $r + 2$ -neighborhood of H_1 and lies in the horoball bounded by H_1 . Let U_2 be the horoball bounded by H_2 . The distance between U_2 and $\tilde{E} - \tilde{E}_{<\varepsilon}$ is ≥ 2 .

Next we show that $\tilde{E}_{<\varepsilon}$ is convex and $U_2 \subset \tilde{E}_{<\varepsilon}$. Let Q be an arbitrary component of $\tilde{E}_{<\varepsilon}$. Thus Q is convex (as a connected, locally convex subset of a Hadamard manifold) and hence its boundary $\partial Q \subset \tilde{E} - \tilde{E}_{<\varepsilon}$ is a topological, properly embedded submanifold that therefore separates \tilde{V} . Consider a ray σ that starts at a point $\sigma(0) \in Q$ in the 1-neighborhood of ∂Q , and that ends in the center $\sigma(\infty)$ of U_2 at infinity. As $\sigma(0)$ lies in the $r + 1$ -neighborhood of H_1 , the ray σ intersects H_2 . If $\gamma \in H$ whose displacement d_γ at $\sigma(0)$ is $< \varepsilon$, then since H fixes $\sigma(\infty)$, non-positivity of the curvature implies $d_\gamma(\sigma(t)) < \varepsilon$ for all t , so $\sigma \subset Q$. As ∂Q separates \tilde{V} and is disjoint from U_2 , we get $U_2 \subset Q$. Since Q is arbitrary and U_2 is connected, we conclude that $Q = \tilde{E}_{<\varepsilon}$.

The rest of the proof works verbatim as in (1), which is a purely topological argument and, in particular, the fact that U_2 need not be a horoball in (1), while it is a horoball in (2) does not matter.

(3) If B is incompressible, then B is homotopy equivalent to a horosphere quotient, which is a closed manifold whose universal cover is diffeomorphic to \mathbb{R}^{n-1} , where $n - 1 = \dim(B)$. Thus the universal cover of B is properly homotopy equivalent to \mathbb{R}^{n-1} , and hence is simply-connected at infinity, and so homeomorphic to \mathbb{R}^{n-1} by [Sta62, Edw63, Wal65, Fre82]. \square

Proof of Addendum 5.1. The proof of (1) actually works when H_1 is any codimension one properly embedded submanifold whose normal exponential map is a diffeomorphism. Then H_2, H_3 are hypersurfaces equidistant to H_1 that bound $r + 1, r + R + 1$ neighborhoods of H_1 , respectively. \square

Remark 6.1. In fact, if B is incompressible in N , then B is h-cobordant to a horosphere quotient: the region between \tilde{B}_1 and H_2 projects to an h-cobordism embedded in E .

Proof of Theorem 1.1. The assertions (1a), (1b), (2) follow from Corollaries 4.2, 5.2, and Theorem 3.7(1), respectively. \square

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REFERENCES

- [ABJ+09] Goulmira Arzhantseva, Martin R. Bridson, Tadeusz Januszkiewicz, Ian J. Leary, Ashot Minasyan, and Jacek Świątkowski, *Infinite groups with fixed point properties*, *Geom. Topol.* **13** (2009), no. 3, 1229–1263, DOI 10.2140/gt.2009.13.1229. MR2496045 (2010b:20069)
- [AMO07] G. Arzhantseva, A. Minasyan, and D. Osin, *The SQ-universality and residual properties of relatively hyperbolic groups*, *J. Algebra* **315** (2007), no. 1, 165–177, DOI 10.1016/j.jalgebra.2007.04.029. MR2344339 (2008g:20095)
- [And87] Michael T. Anderson, *Metrics of negative curvature on vector bundles*, *Proc. Amer. Math. Soc.* **99** (1987), no. 2, 357–363, DOI 10.2307/2046641. MR870801 (87k:53084)
- [Bela] Igor Belegradek, *An assortment of negatively curved ends*, *J. Topol. Anal.* **5** (2013), no. 4, 439–449, DOI 10.1142/S1793525313500180. MR3152210
- [Belb] Igor Belegradek, *Obstructions to nonpositive curvature for open manifolds*, to appear in *Proc. Lond. Math. Soc.*, arXiv:1208.5220.
- [BF02] Mladen Bestvina and Mark Feighn, *Proper actions of lattices on contractible manifolds*, *Invent. Math.* **150** (2002), no. 2, 237–256, DOI 10.1007/s00222-002-0239-6. MR1933585 (2004d:57042)
- [BGS85] Werner Ballmann, Mikhael Gromov, and Viktor Schroeder, *Manifolds of nonpositive curvature*, *Progress in Mathematics*, vol. 61, Birkhäuser Boston, Inc., Boston, MA, 1985. MR823981 (87h:53050)
- [BK03] Igor Belegradek and Vitali Kapovitch, *Obstructions to nonnegative curvature and rational homotopy theory*, *J. Amer. Math. Soc.* **16** (2003), no. 2, 259–284 (electronic), DOI 10.1090/S0894-0347-02-00418-6. MR1949160 (2003i:53043)
- [BK05] I. Belegradek and V. Kapovitch, *Pinching estimates for negatively curved manifolds with nilpotent fundamental groups*, *Geom. Funct. Anal.* **15** (2005), no. 5, 929–938, DOI 10.1007/s00039-005-0534-7. MR2221154 (2007b:53076)
- [BO08] Igor Belegradek and Denis Osin, *Rips construction and Kazhdan property (T)*, *Groups Geom. Dyn.* **2** (2008), no. 1, 1–12, DOI 10.4171/GGD/29. MR2367206 (2009c:20076)
- [Cid02] Carlos Frederico Cid, *Torsion-free metabelian groups with commutator quotient $C_{p^n} \times C_p^m$* , *J. Algebra* **248** (2002), no. 1, 15–36, DOI 10.1006/jabr.2001.9028. MR1879005 (2003a:20055)
- [CKS12] Jack S. Calcut, Henry C. King, and Laurent C. Siebenmann, *Connected sum at infinity and Cantrell-Stallings hyperplane unknotting*, *Rocky Mountain J. Math.* **42** (2012), no. 6, 1803–1862, DOI 10.1216/RMJ-2012-42-6-1803. MR3028764
- [DI94] Karel Dekimpe and Paul Igodt, *The structure and topological meaning of almost-torsion free groups*, *Comm. Algebra* **22** (1994), no. 7, 2547–2558, DOI 10.1080/00927879408824976. MR1271622 (95f:20057)
- [DIKL95] Karel Dekimpe, Paul Igodt, Suhyung Kim, and Kyung Bai Lee, *Affine structures for closed 3-dimensional manifolds with nil-geometry*, *Quart. J. Math. Oxford Ser. (2)* **46** (1995), no. 182, 141–167, DOI 10.1093/qmath/46.2.141. MR1333828 (96k:57039)
- [DJ91] Michael W. Davis and Tadeusz Januszkiewicz, *Hyperbolization of polyhedra*, *J. Differential Geom.* **34** (1991), no. 2, 347–388. MR1131435 (92h:57036)
- [DJL12] M. Davis, T. Januszkiewicz, and J.-F. Lafont, *4-dimensional locally CAT(0)-manifolds with no Riemannian smoothings*, *Duke Math. J.* **161** (2012), no. 1, 1–28, DOI 10.1215/00127094-1507259. MR2872552
- [Edw63] C. H. Edwards Jr., *Open 3-manifolds which are simply connected at infinity*, *Proc. Amer. Math. Soc.* **14** (1963), 391–395. MR0150745 (27 #732)
- [Fre82] Michael Hartley Freedman, *The topology of four-dimensional manifolds*, *J. Differential Geom.* **17** (1982), no. 3, 357–453. MR679066 (84b:57006)
- [Gro03] M. Gromov, *Random walk in random groups*, *Geom. Funct. Anal.* **13** (2003), no. 1, 73–146, DOI 10.1007/s000390300002. MR1978492 (2004j:20088a)
- [GS99] Narain Gupta and Said Sidki, *On torsion-free metabelian groups with commutator quotients of prime exponent*, *Internat. J. Algebra Comput.* **9** (1999), no. 5, 493–520, DOI 10.1142/S0218196799000308. MR1719723 (2001g:20038)
- [IN05] Hiroyasu Izeiki and Shin Nayatani, *Combinatorial harmonic maps and discrete-group actions on Hadamard spaces*, *Geom. Dedicata* **114** (2005), 147–188, DOI 10.1007/s10711-004-1843-y. MR2174098 (2006k:58024)

- [Ker69] Michel A. Kervaire, *Smooth homology spheres and their fundamental groups*, Trans. Amer. Math. Soc. **144** (1969), 67–72. MR0253347 (40 #6562)
- [Kle] J. Klein, *Characteristic classes for block bundles*, MathOverflow (2012), <http://mathoverflow.net/a/97484/1573>.
- [MM79] Ib Madsen and R. James Milgram, *The classifying spaces for surgery and cobordism of manifolds*, Annals of Mathematics Studies, vol. 92, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979. MR548575 (81b:57014)
- [NP] T. T. Nguyen Phan, *Nil happens. what about Sol?*, arXiv:1207.1734v1.
- [NS11] Assaf Naor and Lior Silberman, *Poincaré inequalities, embeddings, and wild groups*, Compos. Math. **147** (2011), no. 5, 1546–1572, DOI 10.1112/S0010437X11005343. MR2834732
- [Ont] P. Ontaneda, *Pinched smooth hyperbolization*, arXiv:1110.6374v1.
- [PY03] Józef H. Przytycki and Akira Yasukhara, *Symmetry of links and classification of lens spaces*, Geom. Dedicata **98** (2003), 57–61, DOI 10.1023/A:1024008222682. MR1988423 (2004g:57016)
- [RS68a] C. P. Rourke and B. J. Sanderson, *Block bundles. I*, Ann. of Math. (2) **87** (1968), 1–28. MR0226645 (37 #2234a)
- [RS68b] C. P. Rourke and B. J. Sanderson, *Block bundles. III. Homotopy theory*, Ann. of Math. (2) **87** (1968), 431–483. MR0232404 (38 #729)
- [Sie69] L. C. Siebenmann, *On detecting open collars*, Trans. Amer. Math. Soc. **142** (1969), 201–227. MR0246301 (39 #7605)
- [Sta] J. R. Stallings, *The embedding of homotopy types into manifolds*, 2000 version, <http://math.berkeley.edu/~stall/embkloz.pdf>.
- [Sta62] John Stallings, *The piecewise-linear structure of Euclidean space*, Proc. Cambridge Philos. Soc. **58** (1962), 481–488. MR0149457 (26 #6945)
- [SW79] Peter Scott and Terry Wall, *Topological methods in group theory*, Homological group theory (Proc. Sympos., Durham, 1977), London Math. Soc. Lecture Note Ser., vol. 36, Cambridge Univ. Press, Cambridge-New York, 1979, pp. 137–203. MR564422 (81m:57002)
- [Wal65] C. T. C. Wall, *Open 3-manifolds which are 1-connected at infinity*, Quart. J. Math. Oxford Ser. (2) **16** (1965), 263–268. MR0181993 (31 #6218)
- [Wal99] C. T. C. Wall, *Surgery on compact manifolds*, 2nd ed., Mathematical Surveys and Monographs, vol. 69, American Mathematical Society, Providence, RI, 1999. Edited and with a foreword by A. A. Ranicki. MR1687388 (2000a:57089)

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