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ON SUSPENSIONS AND CONJUGACY OF HYPERBOLIC AUTOMORPHISMS

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Abstract. We remark that the conjugacy problem for pairs of hyperbolic automorphisms of a finitely presented group (typically a free group) is decidable. The solution that we propose uses the isomorphism problem for the suspensions, and the study of their automorphism group.

Introduction

Let F be a finitely presented group, $\mathrm{Aut}(F)$ be its automorphism group, and $\mathrm{Out}(F) = \mathrm{Aut}(F)/\mathrm{Inn}(F)$ be its outer automorphism group.

A way to consider the conjugacy problem in $\mathrm{Aut}(F)$, or $\mathrm{Out}(F)$, is to relate it to an isomorphism problem on semi-direct products of F with $\mathbb{Z}$.

Given two semi-direct products, $F \rtimes_\alpha \langle t\rangle$ and $F \rtimes_\beta \langle t'\rangle$, their structural automorphisms α and β are conjugated in $\mathrm{Aut}(F)$ if and only if there is an isomorphism $F \rtimes_\alpha \langle t\rangle \to F \rtimes_\beta \langle t'\rangle$ sending F on F, and t on t'. They are conjugated in $\mathrm{Out}(F)$ if and only if there is an isomorphism sending F on F, and t in $t'F$ (though it is a well-known fact with a standard proof, we refer to Lemma 3.1 for the version that we'll use).

By analogy with topology and dynamical systems, we wish to call such semi-direct products *suspensions* of F, in which F is the *fiber* and t is the choice of a *transverse direction*, and tF is the choice of a transverse *orientation*. The conjugacy problem in $\mathrm{Out}(F)$ can be expressed as the problem of determining whether suspensions are fiber-and-orientation-preserving isomorphic.

We carry out this approach for automorphisms of finitely presented groups producing word-hyperbolic suspensions.

Consider for instance $F = F_n$ a free group of finite rank n. In that case, a solution to the conjugacy problem in $\mathrm{Out}(F_n)$ was announced by Lustig [Lu1], [Lu2]. However, it might still be desirable to find short complete solutions for specific classes of elements in $\mathrm{Out}(F_n)$ (in the sense that the exposition is short; in this paper we ostensibly ignore algorithmic complexity, and to some extent conceptual complexity hidden in the tools that are used). For instance, consider the class of atoroidal automorphisms: those whose powers do not preserve any conjugacy class beside $\{1\}$. Since Brinkmann proved in [Br1] that an automorphism produces a hyperbolic suspension if and only if it is atoroidal, there is a conceptually simple (slightly brutal) way to algorithmically check whether a given automorphism is indeed atoroidal. It consists in two parallel procedures. The first one looks for a

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preserved conjugacy class, by enumerating elements, and their images by powers of the given automorphism; it halts when a non-trivial element is found such that its image by a non-trivial power of the automorphism is found to be conjugated in F_n. The second is Papasoglu's procedure [P] applied on the semi-direct product of F_n by $\mathbb{Z}$ (with given structural automorphism), that halts if and only if the semi-direct product is word-hyperbolic. Brinkmann's results says that exactly one of these two procedures will halt, and depending which one halts, we deduce whether or not the automorphism is atoroidal.

If two given automorphisms are found to be atoroidal with this procedure, our main result will allow us to decide whether they are conjugate in $\mathrm{Out}(F_n)$.

For hyperbolic groups, the isomorphism problem is solved [Sel, DGr, DGu2]. In several examples, the solution available can settle the conjugacy problem. Take two pseudo-Anosov diffeomorphisms of a hyperbolic surface. The mapping tori are closed hyperbolic 3-manifold, hence hyperbolic and rigid (in the sense that their outer-automorphism groups are finite). Sela's solution to the isomorphism problem of their fundamental groups [Sel, 0.3 , 7.3] provides all conjugacy classes of isomorphisms (there are finitely many), and from that point, it is possible to check whether one of them preserves the fiber and the orientation.

For automorphisms of a free group, the analogous situation is when the two automorphisms are atoroidal, fully irreducible (with irreducible powers), and for their conjugacy problem, see [Lo, Lu3, Sel], the latter (in Coro. 0.6 *loc. cit.*), using the same strategy as above. However, there are atoroidal automorphisms for which the suspension, though hyperbolic, is not rigid. In [Br2] Brinkmann gave several examples with different behaviors. In particular, the solution to the isomorphism problem of hyperbolic groups will not reveal all isomorphisms between suspensions, and since the fibers are exponentially distorted in the suspensions, the usual rational tools (see [D, DGu1]) do not work for solving the isomorphism problem with such a preservation constraint. One can thus merely detect the existence of one isomorphism (say ι), but for investigating the existence of an isomorphism with the aforementioned properties, one is led to consider an orbit problem of the automorphism group of $F \rtimes \langle t \rangle$: decide whether an automorphism sends $\iota(F)$ on F and $\iota(t)$ in $t'F$.

Orbit problems are not necessarily easier, especially if the group acting is large and complicated. In [BMV], for instance, Bogopolski, Martino and Ventura propose a subgroup of $GL(4, \mathbb{Z})$ whose orbit problem on $\mathbb{Z}^4$ is undecidable.

In this paper we prove that, if F is finitely generated and $F \rtimes \langle t \rangle$ hyperbolic, then $\mathrm{Out}(F \rtimes \langle t \rangle)$ contains a finite index abelian subgroup, whose action on $H_1(F \rtimes \langle t \rangle)$ is generated by transvections. This allows us to prove that the specific orbit problem above is solvable in that case, by reducing it to a system of linear Diophantine equations, read in $H_1(F \rtimes \langle t \rangle)$. This is explained in Section 2.1.

These are thus the key steps to produce what we see as a picturesque way of solving the conjugacy problem for automorphisms of finitely presented groups with hyperbolic suspension (Theorem 3.2).

The proof that $\mathrm{Out}(F \rtimes \langle t \rangle)$ is virtually abelian is the conjunction of Proposition 2.1 (together with Remark 2.2, which is not actually needed in the rest of the proof) and Proposition 2.14. The proof of the latter is done by considering the canonical JSJ decomposition of the hyperbolic group $F \rtimes \langle t \rangle$, and by proving that this graph-of-groups decomposition does not contain any surface vertex group. We cannot

resist sketching the proof of this key fact (that will be detailed in Section 2.2, and takes root in the way Brinkmann produces his examples in [Br2]). Consider the tree of the JSJ decomposition T, and $\mathbb{X}$ the graph of the group quotient of T by $G = F \rtimes \langle t \rangle$. Since F is normal in G, T is a minimal tree for F, and $\mathbb{Y} = F \backslash T$ is a graph-of-groups decomposition of F whose underlying graph is its own core, and since its genus is bounded by the rank of F, it is finite. It follows that every vertex group (resp. edge group) in $\mathbb{X}$ is the suspension of a vertex group (resp. edge group) in $\mathbb{Y}$: lift the vertex in T, where its $\langle t \rangle$-orbit passes twice on a pre-image of a vertex in $\mathbb{Y}$, thus yielding the suspension (see Lemma 2.6 for more details). Since edge groups in $\mathbb{X}$ are cyclic, edge groups in $\mathbb{Y}$ must be trivial. Therefore $\mathbb{Y}$ is a free decomposition of F, and its vertex groups are of finite type. Going back to $\mathbb{X}$ again, vertex groups of $\mathbb{X}$ are suspensions of infinite, finitely generated groups, hence cannot be free nor surface groups, because finitely generated normal subgroups of free groups (or surface groups) are of finite index or trivial.

To present these arguments, the formalism of automorphisms of graph-of-groups is to be recalled in a rather precise way in order to be useful. The confident reader may skip this part (Section 1), and only retain that the small modular group is the group of automorphisms generated by Dehn twists on edges of a graph-of-groups.

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1. Preliminary on automorphisms of graphs-of-groups

1.1. **Trees and splittings.** We fix our formalism for graphs and graphs-of-groups. This material is classical, and can be found in Serre's book [Ser]. A graph is a tuple $X = (V, E, \bar{\ }, o, t)$ where V is a set (of vertices), E is a set (of oriented edges) and $\bar{\ }: E \to E$, $o : E \to V$, $t : E \to V$ verify $\bar{(\circ)} = Id_E$ and $(o\circ) = t$. A graph-of-groups $\mathbb{X}$ consists of a graph X, for each vertex v of X, a group Γ_v, for each unoriented edge $\{e, \bar{e}\}$ of X, a group $\Gamma_{\{e,\bar{e}\}}$ (but we will write $\Gamma_e = \Gamma_{\bar{e}}$ for it), and for each oriented edge e of X, an injective homomorphism $i_e : \Gamma_e \to \Gamma_{o(e)}$, where $o(e)$ is the origin vertex of the oriented edge e.

The Bass group $B(\mathbb{X})$ is

$$B(\mathbb{X}) = \langle (\bigcup_{v \in V} \Gamma_v) \cup E \mid \forall e \in E, \forall g \in \Gamma_e,\ \bar{e} = e^{-1},\ e^{-1} i_{\bar{e}}(g) e = i_e(g) \rangle.$$

An element $g_0 e_1 g_1 e_2 \dots g_n e_n g_{n+1}$ is a *path element* from $o(e_1)$ to $t(e_n)$ if $g_{i-1} \in o(e_i)$ and $g_i \in t(e_i)$ for all i.

The *fundamental group* $\pi_1(\mathbb{X}, v_0)$ of the graph-of-groups $\mathbb{X}$ *at the vertex* v_0 is the subgroup of the Bass group consisting of path elements from v_0 to v_0.

Choose a maximal subtree τ of the graph X, and consider $\pi_1(\mathbb{X}, \tau) = B(\mathbb{X}) / \langle\langle e \in \tau \rangle\rangle$. Then the quotient map $B(\mathbb{X}) \to \pi_1(\mathbb{X}, \tau)$ is, in restriction to $\pi_1(\mathbb{X}, v_0)$, an isomorphism.

A *G-tree* is a simplicial tree with a simplicial action of G without inversion. It is minimal if there is no proper invariant subtree. It is reduced if the stabilizer of any edge is a proper subgroup of the stabilizer of any adjacent vertex.

The quotient of a G-tree by G is naturally marked by the family of conjugacy classes of stabilizers of vertices and edges, and inherits a structure of a graph-of-groups (whose fundamental group is isomorphic to G).

The *Bass-Serre tree* of a graph-of-groups is its universal covering in the sense of graphs-of-groups. It is a $\pi_1(\mathbb{X}, v_0)$-tree.

A *collapse* of a G-tree T is a G-tree S with an equivariant map from T to S that sends each edge on an edge or a vertex and no pair of edges of T in different orbits are sent in the same edge (but they may be sent on the same vertex). A collapse of an edge e in a graph-of-groups decomposition is a collapse of the corresponding Bass-Serre tree in which only the edges in the orbit of a pre-image of e are mapped to a vertex.

Given a group G, and a class of groups $\mathcal{C}$, a *splitting* of G over groups in $\mathcal{C}$ is an isomorphism between G and the fundamental group of a graph-of-groups whose edge groups are in $\mathcal{C}$. Equivalently, a splitting of G can be thought of as an action of G on a tree, with edge stabilizers in $\mathcal{C}$. A splitting is called reduced if, in the tree, there is no edge whose stabilizer equals that of an adjacent vertex.

1.2. **Automorphisms, and the small modular group $Mod_{\mathbb{X}}$.** Let $\mathbb{X}$ and $\mathbb{X}'$ be two graphs-of-groups. An isomorphism of graphs-of-groups $\Phi : \mathbb{X} \to \mathbb{X}'$ is a tuple $(\Phi_X, (\phi_v), (\phi_e), (\gamma_e))$ such that

- $\Phi_X : X \to X'$ is an isomorphism of the underlying graphs,
- for all vertex v, $\phi_v : \Gamma_v \to \Gamma_{\Phi_X(v)}$ is an isomorphism, for all edge e, $\phi_e = \phi_{\bar{e}} : \Gamma_e \to \Gamma_{\Phi_X(e)}$ is an isomorphism,
- and for each edge e, $\gamma_e \in \Gamma_{\Phi_X(t(e))}$ satisfying, for $v = t(e)$,
$$\phi_v \circ i_e = \mathrm{ad}_{\gamma_e} \circ i_{\Phi_X(e)} \circ \phi_e,$$
for $(\mathrm{ad}_{\gamma_e} : x \mapsto \gamma_e^{-1} x \gamma_e)$ the inner automorphism of $\Gamma_{\Phi_X(t(e))}$ defined by the conjugacy by γ_e.

The last point is the commutation of the following diagram (for each edge e):

$$\begin{array}{ccccc} \Gamma_v & \overset{\phi_v}{\to} & \Gamma_{\Phi_X(v)} & \overset{\mathrm{ad}_{\gamma_e}}{\leftarrow} & \Gamma_{\Phi_X(v)} \\ \uparrow & & & & \uparrow \\ \Gamma_e & & \xrightarrow{\phi_e} & & \Gamma_{\Phi_X(e)} \end{array} \tag{1}$$

When $\mathbb{X} = \mathbb{X}'$ the isomorphisms can be composed in a natural way (see [Ba, 2.11]:

$$\begin{aligned} &(\Phi_X, (\phi_v), (\phi_e), (\gamma_e)) \circ (\Psi_X, (\psi_v), (\psi_e), (\eta_e)) \\ &\quad = (\Phi_X \circ \Psi_X, (\phi_{\Psi_X(v)} \circ \psi_v), (\phi_{\Psi_X(e)} \circ \psi_e), (\phi_{\Psi_X(t(e))}(\eta_e)\gamma_{\Psi_X(e)}))), \end{aligned}$$

thus providing the automorphism group of the graph-of-groups $\mathbb{X}$, denoted by $\delta\mathrm{Aut}(\mathbb{X})$.

This group $\delta\mathrm{Aut}(\mathbb{X})$ naturally maps into the automorphism group of the Bass group $B(\mathbb{X})$ in the following way: for all edge ϵ, and all automorphism $\Phi = (\Phi_X, (\phi_v), (\phi_e), (\gamma_e)) \in \delta\mathrm{Aut}(\mathbb{X})$, one has $\Phi(\epsilon) = \gamma_{\bar{\epsilon}}^{-1} \Phi_X(\epsilon) \gamma_\epsilon$, and $\Phi|_{\Gamma_v} = \phi_v$. One can check that, for any Φ, the relations of the Bass group are preserved, and that the thus induced morphism is bijective.

Remark 1.1. For this argument, see [Ba, 2.1, 2.2], but notice that Bass chose to let conjugations act on the left, while we chose to let them act on the right (as in [DGu2]). This difference yields a few harmless inversions in the formulae (actually the attentive reader may have spotted them already in the relations of the Bass group, [Ba, 1.5 (1.2)]), and the only risk here is to mix both (incompatible) choices.

Each automorphism in $\delta\mathrm{Aut}(\mathbb{X})$ sends path elements to path elements, hence naturally provides an outer-automorphism of $\pi_1(\mathbb{X}, v_0)$ (and a genuine automorphism if $\Phi_X(v_0) = v_0$), and we will often implicitly make this identification.

Let us define the *small modular group* of $\mathbb{X}$, denoted by $Mod_{\mathbb{X}}$, to be the subgroup of $\delta\mathrm{Aut}(\mathbb{X})$ consisting of elements of the form $(Id_X, (\phi_v), (Id_{\Gamma_e}), (\gamma_e))$, for $\phi_v \in \mathrm{Inn}\Gamma_v$ inner automorphisms. One can check, using the composition rule, that this forms a subgroup of $\delta\mathrm{Aut}(\mathbb{X})$.

If we note $\phi_v = \mathrm{ad}_{\gamma_v}$, then the compatibility condition (1) imposes that $\gamma_v\gamma_e^{-1} \in Z_{\Gamma_{t(e)}}(i_e(\Gamma_e))$ for all edge e (where $Z_{\Gamma_{t(e)}}(i_e(\Gamma_e))$ denotes the centralizer of the attached edge group $i_e(\Gamma_e)$ in the vertex group $\Gamma_{t(e)}$).

If one chooses a generating set S_e for each group $Z_{\Gamma_{t(e)}}(i_e(\Gamma_e))$, then the small modular group is generated by the union of two collections of elements. First, the collection of *oriented Dehn twists* $\{D_{\epsilon,\gamma}, \epsilon \in E, \gamma \in S_\epsilon\}$, defined by $D_{e,\gamma} = (Id_X, (Id_v), (Id_e), (x_e))$ for $x_\epsilon = \gamma$ and $x_e = 1$ for all $e \neq \epsilon$. Second, the collection of *inert twists*, $\{(Id_X, (\mathrm{ad}_{\gamma_v})), (Id_e), (\gamma_e = \gamma_{t(e)})\}$ (note that it is the same family of elements involved as (γ_e) and in defining the (ϕ_v), and that this defines an element of $Mod_{\mathbb{X}}$).

The inert twists are not so interesting. They correspond to changing some choice of fundamental domain in the Bass-Serre tree. In particular:

Lemma 1.2. *Any inert twist vanishes in* $\mathrm{Out}(\pi_1(\mathbb{X}, v_0))$.

Proof. Consider an inert twist Φ. After conjugation over the whole group, we may assume that $\phi_{v_0} = Id_{\Gamma_{v_0}}$. Then take a path element in the Bass group, $g_0e_1 \dots g_ne_ng_{n+1}$ that is a loop from v_0 to v_0. Each g_i is in $\Gamma_{t(e_i)} = \Gamma_{t(\overline{e_{i+1}})}$. Thus, $\Phi(g_i) = \mathrm{ad}_{\gamma_i}(g_i)$, and $\Phi(e_i) = \gamma_{i-1}^{-1}e_i\gamma_i$. Concatenation makes everything collapse, and $\Phi(w) = w$. □

We record how oriented Dehn twists are realized as automorphisms of $\pi_1(\mathbb{X}, v_0)$. The following is an immediate consequence of the definitions.

Lemma 1.3. *Let $\epsilon \in E$ be an oriented edge of $\mathbb{X}$, and $\gamma \in Z_{\Gamma_{t(\epsilon)}}(i(\Gamma_\epsilon))$.*

Let e be an oriented edge different from ϵ and $\bar{\epsilon}$.

The oriented Dehn twist $D_{\epsilon,\gamma}$, seen as an automorphism of $B(\mathbb{X})$, is such that $D_{\epsilon,\gamma}(\epsilon) = \epsilon\gamma$, $D_{\epsilon,\gamma}(\bar{\epsilon}) = \gamma^{-1}\bar{\epsilon}$ and $D_{\epsilon,\gamma}(e) = e$.

Moreover, for all vertex v in X, and $\gamma_v \in \Gamma_v$, $D_{\epsilon,\gamma}(\gamma_v) = \gamma_v$.

One should be nonetheless cautious with the interpretation of $D_{\epsilon,\gamma}$ as an automorphism of $\pi_1(\mathbb{X}, v_0)$, since the identification of a preferred copy of Γ_v in $\pi_1(\mathbb{X}, v_0)$ is subject to a choice of path from v_0 to v. If this path contains a copy (or several) of the edge ϵ, then $D_{\epsilon,\gamma}$ is actually conjugating, on the right, Γ_v by γ^{-1} (or a power of it).

2. The $\mathrm{Aut}(G)$-orbit of the fiber of a suspension

2.1. **An orbit problem for the small modular group.** In this section, we discuss an orbit problem for $Mod_{\mathbb{X}}$ in $H_1(G)$, the abelianisation of G. For that, we note that $Mod_{\mathbb{X}}$ naturally maps into $\mathrm{Aut}(H_1(G)) \simeq GL(H_1(G))$. We will denote by $^- : G \to H_1(G)$ the abelianisation map. Let us also choose $E^+ \subset E$ as a set of representatives of unoriented edges.

Proposition 2.1. *Let $G = \pi_1(\mathbb{X}, v_0)$. The image of $Mod_{\mathbb{X}}$ in $GL(H_1(G))$ is abelian, generated by transvections.*

Proof. Since inert twists vanish in $\mathrm{Out}(G)$, the images of oriented Dehn twists generate the image of $Mod_{\mathbb{X}}$ in $GL(H_1(G))$. It suffice to show that oriented Dehn twists induce transvections on $H_1(G)$ that commute.

Any element γ of $\pi_1(\mathbb{X}, v_0)$ has an expression as a normal form coming from the ambient Bass group: $\gamma = g_0 e_0 g_1 e_1 \dots e_m g_{m+1}$ where for all i, $e_i \in E$ and $g_{i+1} \in \Gamma_{t(e_i)}$, $g_0, g_{m+1} \in \Gamma_{v_0}$.

The normal form of γ turns, in $H_1(G)$, into $\bar{\gamma} = g_0 \dots g_{m+1} \times \prod_{e \in E^+} e^{n(\gamma,e)}$ where $n(\gamma, e)$ is the number of occurrences of e in $(e_0, \dots, e_m)$ minus the number of occurrences of $\bar{e}$.

If $D_{\epsilon,h}$ is the Dehn twist of h on ϵ, the induced automorphism on $H_1(G)$ is denoted by $\overline{D_{\epsilon,h}}$. From the expression of $D_{\epsilon,h}(e)$ in Lemma 1.3, $\overline{D_{\epsilon,h}}$ is the following transvection:

$$(*) \qquad \overline{D_{\epsilon,h}}(\bar{\gamma}) = (g_0 g_1 \dots g_{m+1}) \times \left(\prod_{e \in E^+} e^{n(\gamma,e)} \right) \times \bar{h}^{n(\gamma,\epsilon)} \ = \ \bar{\gamma} \times \bar{h}^{n(\gamma,\epsilon)}.$$

Since $h \in \Gamma_{t(\epsilon)}$, the element $h^{n(\gamma,\epsilon)}$ is in a vertex group, and it is fixed (if seen in the Bass group) or conjugated (if seen in G) by all oriented Dehn twists on edges, thus oriented Dehn twists on edges commute in the abelianisation. □

Remark 2.2. In fact, if $Z_{\Gamma_{t(e)}}(i_e(\Gamma_e))$ is abelian (which is the case if G is torsion free hyperbolic), then $Mod_{\mathbb{X}}$ is abelian itself.

We keep the same notation. Note that the group $H_1(G)/\bar{F}$ is infinite cyclic, by assumption, and is generated by the image of $\bar{t}$. Define $\bar{=}: G \to H_1(G)/\bar{F}$ and for all $\gamma \in G$, define $\delta(\gamma)$ to be the unique integer such that $\bar{\bar{\gamma}} = \bar{\bar{t}}^{\delta(\gamma)}$.

Proposition 2.3. *Let G be a finitely generated group that can be expressed as a semi-direct product $F \rtimes \langle t \rangle$.*

Given a splitting $\mathbb{X}$ of G, and for each $e \in E$, a generating set $S_e \subset \Gamma_{t(e)}$ for $Z_{\Gamma_{t(e)}}(i_e(\Gamma_e))$ and a family $(\gamma_j)_{0 \le j \le j_0}$ of elements of G, one can decide whether there is an element $\eta \in Mod_{\mathbb{X}}$ whose image $\bar{\eta}$ in $\mathrm{Aut}(H_1(G))$ sends $\bar{\gamma}_j$ in $\bar{F}$ for all $j < j_0$ and $\bar{\gamma}_{j_0}$ inside $\bar{t}\bar{F}$.

More precisely, there is such an element η if and only if the explicit Diophantine linear system of equations

$$(2) \qquad \forall j, \sum_{e \in E^+ s_e \in S_e} r_{s_e} n(\gamma_j, e) \delta(s_e) = -\delta(\gamma_j) \ + \ \mathrm{dirac}_{j=j_0}$$

(with unknowns r_{s_e}) has a solution.

Proof. By $(*)$ (from the proof of Proposition 2.1), $\delta(D_{\epsilon,h}(\gamma)) = \delta(\gamma) + n(\gamma, \epsilon)\delta(h)$ and by induction, and the fact that $\bar{h}$ is fixed by all Dehn twists,

$$\delta(D_{\epsilon_1,h_1} D_{\epsilon_2,h_2} \dots D_{\epsilon_k,h_k}(\gamma)) = \delta(\gamma) + \sum_{i=1}^{k} n(\gamma, \epsilon_i)\delta(h_i).$$

Assume that there exists $\eta \in Mod_{\mathbb{X}}$ such that $\delta(\eta(\gamma_j)) = 0$ for $j < j_0$ and $= 1$ for $j = j_0$. Since oriented Dehn twists generate the image of $Mod_{\mathbb{X}}$ in $GL(H_1(G))$, this element η can be chosen as a product of oriented Dehn twists $D_{\epsilon_1,h_1} D_{\epsilon_2,h_2} \dots D_{\epsilon_k,h_k}$, which by commutation in $\mathrm{Aut}(H_1(G))$ can be chosen to be

$$\prod_{e \in E^+, s_e \in S_e} D_{e,s_e}^{r_{s_e}}.$$

Therefore, by the previous equation, $\forall e \in E, \forall s_e \in S_e, \exists r_{s_e} \in \mathbb{Z}$

$$\forall j, \sum_{e \in E^+ s_e \in S_e} r_{s_e} n(\gamma_j, e)\delta(s_e) = -\delta(\gamma_j) + \mathrm{dirac}_{(j=j_0)}$$

(in which $\mathrm{dirac}_{j=j_0}$ yields 0 if $j \neq j_0$ and 1 otherwise).

Conversely, if the system of equations

$$\forall j, \sum_{e \in E^+ s_e \in S_e} r_{s_e} n(\gamma_j, e)\delta(s_e) = -\delta(\gamma_j) + \mathrm{dirac}_{j=j_0}$$

has a solution (in unknowns r_{s_e}), then $\delta(\prod_{e \in E^+, s_e \in S_e} D_{e,s_e}^{r_{s_e}}(\gamma_j)) = 0$ for $j < j_0$ and $= 1$ for $j = j_0$.

We have thus reduced the orbit problem to an equivalent problem of satisfaction of a system of linear Diophantine equations that are explicitly computable. The problem is therefore solvable, by classical techniques of linear algebra. □

If we had the same statement with $\mathrm{Aut}(G)$ replacing $Mod_{\mathbb{X}}$, we would be very close to our conclusion. What we will show is that $Mod_{\mathbb{X}}$ has finite index image in $\mathrm{Out}(G)$.

2.2. **No surface vertex group in splittings of suspensions.** Our aim is Proposition 2.11, that states that in any splitting of a suspension, there is no vertex whose group is a surface group with boundary, where the boundary subgroups are the adjacent edge groups.

More precisely, if T is a G-tree, we say that a vertex stabilizer H is a hanging surface group if it is the fundamental group of a non-elementary compact surface with boundary components, and the adjacent edge groups are exactly the subgroups of the boundary components. We say that it is a hanging bounded Fuchsian group if there is a finite normal subgroup $K \lhd H$ such that H/K is isomorphic to the fundamental group of a non-elementary hyperbolic compact 2-orbifold with boundary, by an isomorphism sending the images of the adjacent edge stabilizers on the boundary subgroups of the orbifold group. We will need the following well-known fact about bounded Fuchsian groups that we give for completeness.

Lemma 2.4. *Any hanging bounded Fuchsian group is virtually free.*

Proof. Let H be a hanging bounded Fuchsian group, and K a finite subgroup of H as above. Because H/K is the fundamental group of a hyperbolic compact 2-orbifold, it is a subgroup of $PSL(2,\mathbb{R}) \rtimes \mathbb{Z}/2\mathbb{Z}$, and by Selberg's lemma, it is residually finite, and it has a finite index subgroup $\bar{H}_0$ without torsion. This subgroup is the fundamental group of a finite cover of the orbifold, which is therefore a surface, with boundary, since the orbifold has boundary. Thus, $\bar{H}_0$ is free. It lifts as a free subgroup of H, whose index in H is the product of indices $|K| \times [H/K : \bar{H}_0]$. □

We will show (Proposition 2.11) that a splitting of a suspension of a finitely generated group has no hanging bounded Fuchsian vertex group.

Let T be the Bass-Serre G-tree of a reduced splitting of G. Let us introduce $Y = F\backslash T$ and $X = G\backslash T$. Both graphs provide respectively graph-of-groups decompositions $\mathbb{Y}$ and $\mathbb{X}$ of F and of G.

Lemma 2.5. *Y is a finite graph, and the G-action on T induces, by factorisation, a $\langle \bar{t} \rangle$-action on Y.*

Proof. The second part of the statement is obvious since F is normal in G. Also, since F is normal in G, its minimal subtree in T is G-invariant, therefore it is T itself. In other words, Y is its own core. As for its genus, it is finite, since it bounds from below the rank of F (the fundamental group of $\mathbb{Y}$ at a vertex v_0 projects on the fundamental group of the underlying graph Y at v_0). It follows that Y is a finite graph, and it is endowed with an action of $\langle \bar{t} \rangle = G/F$ for which the quotient is X. □

Lemma 2.6. *Given any vertex in T, its stabilizer in G is a suspension of its stabilizer in F.*

Proof. Let ν be such a vertex, and consider v its image in X and $\tilde{v}$ its image in Y. Because Y is finite, there is a smallest $n > 0$ such that $\bar{t}^n \tilde{v}$ is $\tilde{v}$. Lifting in T, we obtain the existence of $\gamma \in F$ such that $\gamma t^n \nu = \nu$. Therefore, if $(F)_\nu$ is the stabilizer of ν in F, then it is normalized by γt^n and $(F)_\nu \rtimes \langle \gamma t^n \rangle$ fixes ν.

We claim that $(F)_\nu \rtimes \langle \gamma t^n \rangle$ is the stabilizer in G of ν.

If there is another element in it, it is not in F by definition of $(F)_\nu$, and therefore it is some ηt^m, $\eta \in F, m \neq 0$. By minimality of n, n divides m, and if $\gamma' t^m = (\gamma t^n)^k$, then $\gamma' t^m \nu = \eta t^m \nu$, and $t^{-m} \gamma'^{-1} \eta t^m$ fixes ν and is in F by normality of F. Thus it is in $(F)_\nu$, and $\eta t^m \in (F)_\nu \rtimes \langle \gamma t^n \rangle$. This ensures the claim. □

Lemma 2.7. *For any edge in T, its stabilizer in G is a suspension of its stabilizer in F.*

Proof. Subdivide such an edge (and all edges in its orbit, equivariantly) by inserting a vertex on its midpoint, and apply the previous lemma to this vertex. □

Up to now we have not used any assumption on the nature of edge groups. But now two remarks are of interest, and directly follow from the previous lemmas.

Lemma 2.8. *A reduced G-tree T never has a finite edge stabilizer.*

If T has cyclic edge stabilizers in G, then $\mathbb{Y}$ is a free splitting of F.

If T has virtually cyclic edge stabilizers in G, then $\mathbb{Y}$ is a splitting of F over finite subgroups.

Lemma 2.9. *Let $\mathbb{X}$ be a graph-of-groups decomposition of $F \rtimes \langle t \rangle$ with virtually cyclic edge groups. Then no vertex group of $\mathbb{X}$ is free non-abelian, or even infinitely many ended.*

Proof. By Lemma 2.8, T, as an F-tree, is the tree of a virtually free splitting of F. In particular, since F is finitely generated, each vertex stabilizer in F is finitely generated. It follows from Lemma 2.6 that a vertex stabilizer of T in G contains an infinite index normal subgroup, which is as we saw, of finite type.

Such a group cannot be free. It cannot be infinitely many ended either (for the same reason actually, that we recall in the following lemma).

Lemma 2.10. *Let H be a finitely generated group with infinitely many ends; then H has no finitely generated infinite index normal subgroup.*

Proof. By Stallings' theorem [Sta, Thm. 4.A.6.5, Thm. 5.A.9], H is the fundamental group of a reduced finite graph-of-groups with finite edge groups (with at least one edge). Let T be the associated Bass-Serre tree, and N be a normal subgroup of H. Then, the tree T is minimal for N, hence T/N equals its own core. If N is finitely generated, T/N is finite. Moreover, the action of H on T factorizes through

T/N, and if N has infinite index in H, there is an edge $\tilde{e}$ in T/N fixed by infinitely many different elements of H, $\{h_i, i \in I\}$, all in different N-coset. Let e be its image in T/H and $\tilde{\tilde{e}}$ a choice of lift in T. There are $n_i \in N$ such that for all i, $h_i n_i$ fix e. By finiteness of edge stabilizers, infinitely many of the elements $h_i n_i$ are equal, contradicting that the h_i were in different N-cosets. □

Proposition 2.11. *Let F be a finitely generated group, and $G = F \rtimes \langle t \rangle$ a suspension. Given any graph-of-groups decomposition $\mathbb{X}$ of G, no vertex group of $\mathbb{X}$ is a hanging surface group, or a hanging bounded Fuchsian group.*

Proof. Assume the contrary: let Γ_{v_0} be an alleged hanging bounded Fuchsian group or hanging surface group, which is, in particular, virtually free (Lemma 2.4), hence infinitely many ended because it is non-elementary. Because it is hanging, all the neighboring edges of v_0 carry virtually cyclic groups. Collapse all other edges in $\mathbb{X}$ in order to get $\mathbb{X}'$, whose edges are virtually cyclic. The image v_0' of v_0 carries the same group, since no adjacent edge has been collapsed, and this group is infinitely many ended, as we noticed. Apply Lemma 2.9 to $\mathbb{X}'$ to get the contradiction. □

2.3. **The $\mathrm{Aut}(G)$-orbit of the fiber in the hyperbolic case.** Let us recall that the canonical $\mathcal{Z}_{\max}$-JSJ decomposition of a one-ended hyperbolic group is a certain finite splitting $\mathbb{X}$ of G over certain virtually cyclic subgroups (maximal with infinite center), such that every automorphism of G induces an automorphism of graph-of-groups of $\mathbb{X}$ (see [DGu2, section 4.4]). In other words, the natural map $\delta\mathrm{Aut}(\mathbb{X}) \to \mathrm{Out}(G)$ is surjective.

Remark 2.12. The choice to use the rather technical $\mathcal{Z}_{\max}$-JSJ splitting instead of the more natural "virtually cyclic" JSJ splitting, is only suggested by our ability to algorithmically compute this decomposition. In principle, we could work with the classical JSJ splitting as well in the same way.

Let $\mathbb{X}$ be a graph-of-groups, v a vertex therein, and Γ_v the vertex group. The choice of an order on the oriented edges adjacent to v, and of a generating set of the edge groups, endows Γ_v with a *marked peripheral structure*, that is, the tuple of conjugacy classes of the images of these generating sets by the attaching maps. We denote by $\mathcal{T}$ this tuple, and $\mathrm{Out}_m(\Gamma_v, \mathcal{T})$ the subgroup of $\mathrm{Out}(\Gamma_v)$ preserving $\mathcal{T}$ (see also [DGu2]). In the following, the choices of order and generating sets are implicit, and done a priori.

The following is a typical feature of a JSJ decomposition of a hyperbolic group, whose proof, in this specific setting, is essentially contained in [DGu2].

Lemma 2.13. *Let G be a one-ended hyperbolic group. Let $(\Gamma_v, \mathcal{T})$ be a vertex group of the $\mathcal{Z}_{\max}$-JSJ decomposition $\mathbb{X}$ of G, with the marked peripheral structure induced by $\mathbb{X}$ (and some choice of finite generating sets of edge groups). If $\mathrm{Out}_m(\Gamma_v, \mathcal{T})$ is infinite, then G admits a splitting with a hanging bounded Fuchsian vertex group.*

Proof. By [DGu2, Prop. 3.1], such a vertex group Γ_v must have a further compatible splitting over a maximal virtually cyclic group with infinite center, which allows us to use [DGu2, Prop 4.17] to ensure that $(\Gamma_v, \mathcal{T})$ is a so-called hanging orbisocket, which by definition [DGu2, Def. 4.15] allows us to refine $\mathbb{X}$ in order to get a splitting of G whose one vertex group is a hanging bounded Fuchsian group. □

Proposition 2.14. *Let F be a finitely presented group, and $G = F \rtimes \langle t \rangle$ a suspension that is assumed to be hyperbolic.*

Then, the image in $\mathrm{Out}(G)$ *of the small modular group of the $\mathcal{Z}_{\max}$-JSJ decomposition of G has finite index in* $\mathrm{Out}(G)$*. Moreover, one can compute a set of right-coset representatives of $Mod_{\mathbb{X}}$ in* $\mathrm{Out}(G)$ *(in the form of automorphisms of G).*

Proof. Since $\delta\mathrm{Aut}(\mathbb{X})$ surjects on $\mathrm{Out}(G)$ in the case of the $\mathcal{Z}_{\max}$-JSJ decomposition, it suffices to show that the small modular group has finite index in $\delta\mathrm{Aut}\mathbb{X}$ and that coset representatives can be computed in $\delta\mathrm{Aut}\mathbb{X}$.

Once again, this is essentially done in [DGu2] (and probably in other places).

First, the splitting $\mathbb{X}$ can be effectively computed [DGu2, Prop. 6.3].

Consider the following three maps. First $q_X : \delta\mathrm{Aut}(\mathbb{X}) \to \mathrm{Aut}(X)$ where $\mathrm{Aut}(X)$ is the automorphism group of the underlying finite graph X. Second, the natural map $q_E : \ker q_X \to \prod_{e\in E} \mathrm{Aut}(\Gamma_e)$. Third, the natural map $q_V : \ker q_E \to \prod_{v\in V} \mathrm{Out}_m(\Gamma_v, \mathcal{T})$, where $\mathcal{T}$ is the marked peripheral structure induced by the ambient graph-of-groups $\mathbb{X}$.

The group $\ker q_V$ is the small modular group. The first two maps have finite image. By Lemma 2.13 and Proposition 2.11, the map q_V also has finite image.

Therefore the small modular group has finite index in $\delta\mathrm{Aut}(\mathbb{X})$. In order to reconstruct coset representatives of the small modular group in $\delta\mathrm{Aut}(\mathbb{X})$, it is enough to find coset representatives for the kernel of each of these maps.

In order to compute coset representatives of $\ker q_E$ in $\delta\mathrm{Aut}(\mathbb{X})$, we can make the finite list of all graph automorphisms of X for which $\Gamma_{\Phi_X(e)} \simeq \Gamma_e$. Let Φ_X be any of them. We can make the list of all isomorphisms $\Gamma_{\Phi_X(e)} \simeq \Gamma_e$ (there are finitely many such isomorphisms since these groups are virtually cyclic). Then we can apply [DGu2, Prop. 2.28] in order to reveal whether this automorphism Φ_X has a pre-image by q_X.

In order to compute coset representatives of $\ker q_E$ in $\ker q_X$, we consider a collection of automorphisms of edge groups, and apply [DGu2, Prop. 2.28] in order to reveal whether this collection has a pre-image by q_E.

Finally, in order to compute coset representatives of $\ker q_V$ in $\ker q_E$, one can make the list of all elements in $\mathrm{Out}(\Gamma_v, \mathcal{T})$ expressed as automorphisms (by [DGu2, Coro. 3.5], we can enumerate all of them) and for each choice of them (for each v), check whether the collection defines a graph-of-groups automorphism by solving the simultaneous conjugacy problem that allows the diagram (1) to commute. □

Corollary 2.15. *Let F be finitely presented, and $G = F \rtimes \langle t \rangle$ a suspension that is assumed to be hyperbolic. Given a splitting $\mathbb{X}$ of G, generating sets for the centralizers of adjacent edge groups in vertex groups, and a family $(\gamma_j)_{0\le j\le j_0}$ of elements of G, one can decide whether there is an element $\eta \in \mathrm{Aut}(G)$ whose image $\bar\eta$ in $\mathrm{Aut}(H_1(G))$ sends $\bar\gamma_j$ in $\bar F$ for all $j < j_0$ and $\bar\gamma_{j_0}$ inside $\bar t\bar F$.*

Proof. First we take note that $\eta \in \mathrm{Aut}(G)$ satisfies the conclusion if and only if any other automorphism in the same class in $\mathrm{Out}(F)$ satisfies it. Thus, let $\alpha_1, \dots, \alpha_k$ be the right-coset representatives of $Mod_{\mathbb{X}}$ in $\mathrm{Out}(G)$ computed by Proposition 2.14, in the form of automorphisms. For each i, we compute, for each j, $\alpha_i(\gamma_j)$, and we use Proposition 2.3 in order to decide whether there is $\eta_0 \in Mod_{\mathbb{X}}$ such that $\bar\eta_0\overline{(\alpha_i(\gamma_j))} \in \bar F$ for all $j < j_0$ and $\bar\eta_0\overline{(\alpha_i(\gamma_{j_0}))} \in \bar t\bar F$.

If there exists an index i such that the answer is positive, then $\overline{\eta_0 \circ \alpha_i}$ sends $\bar{\gamma}_j$ in $\bar{F}$ for all $j < j_0$ and $\bar{\gamma}_{j_0}$ inside $\bar{t}\bar{F}$.

If for all i the answer is negative, then no automorphism of G satisfies this property. □

3. Conjugacy and suspensions

The following observations elaborate on some well-known point of view (see for instance [Sel], [ALM]), and, as stated in the introduction, is our angle of attack of the conjugacy problem.

Lemma 3.1. *Let ϕ_1 and ϕ_2 be two automorphisms of F. The following assertions are equivalent:*

(1) *ϕ_1 and ϕ_2 are conjugate in* $\mathrm{Out}(F)$*;*
(2) *there is an isomorphism between their suspensions that preserves the fiber (in both directions) and the orientation;*
(3) *there is an isomorphism between their suspensions that preserves the orientation and sends the fiber inside the fiber;*
(4) *there is an isomorphism between their suspensions whose factorization through the abelianisations preserves the orientation, and sends the image of the fiber inside the image of the fiber.*

Proof. Of course, 2 implies 3, which implies 4. Assuming 4 we now show 3. If the given isomorphism $\psi : F \rtimes_{\phi_1} \langle t\rangle \to F \rtimes_{\phi_2} \langle t\rangle$ does not preserve the fiber, it sends some $f \in F$ on some $f't^k, f' \in F, k \neq 0$. Since the fibers are kernels of some cyclic quotient, the derived subgroups of the suspensions are contained in the fibers. Thus the image of $f't^k$ in the abelianisation of $F \rtimes_{\phi_2} \langle t\rangle$ is not in the image of F. Thus, the factorisation through abelianisations of ψ does not send the image of the fiber inside the image of the fiber. If ψ does not preserve the orientation, it sends t to some $f't^k, f' \in F, k \neq 1$, and the same argument shows that the factorisation through abelianisations of ψ does not preserve the orientation. Thus we obtain that 4 implies 3. Let us prove that 3 implies 2. Let α be the isomorphism given by 3. Then $\alpha(F)$ is normal in $(F \rtimes_{\phi_2} \langle t\rangle)$ and the quotient is infinite cyclic. Thus the image of the fiber F is trivial in this quotient (because the further quotient by this image is also infinite cyclic). It follows that $\alpha(F) = F$.

Let us prove that 1 is equivalent to 2. Assume that $\Psi : F \rtimes_{\phi_1} \langle t\rangle \to F \rtimes_{\phi_2} \langle t'\rangle$ sends F to F, and t to $f_0 t'$. Then write ψ for the restriction of Ψ to F.

In $F \rtimes_{\phi_1} \langle t\rangle$, for all $f \in F$, one has $t^{-1} f t = \phi_1(f)$. Passing through Ψ, one gets (in $F \rtimes_{\phi_2} \langle t'\rangle$) $t'^{-1} f_0^{-1} \psi(f) f_0 t' = \psi(\phi_1(f))$, that is, $\phi_2 \circ \mathrm{ad}_{f_0} \circ \psi = \psi \circ \phi_1$.

Thus, the classes of ϕ_1 and ϕ_2 are conjugate in $\mathrm{Out}(F)$ and furthermore, if $f_0 = 1$, ϕ_1 and ϕ_2 are conjugate in $\mathrm{Aut}(F)$.

Conversely, if $\phi_1 = \psi^{-1} \circ \mathrm{ad}_{f_0} \circ \phi_2 \circ \psi$ for some ψ in $\mathrm{Aut}(F)$, one can extend ψ to $\tilde{\Psi} : F * \langle t\rangle \to F \rtimes_{\phi_2} \langle t\rangle$ by setting $\Psi(t) = f_0 t \in F \rtimes_{\phi_2} \langle t\rangle$. The relation of the semi-direct product by ϕ_1 vanishes in the image, thus $\tilde{\Psi}$ factorizes through $F \rtimes_{\phi_1} \langle t\rangle$ producing a bijective morphism. □

Theorem 3.2. *There is an algorithm that, given F a finitely presented group, and two automorphisms ϕ, ϕ' of F such that the suspensions are word-hyperbolic, decides whether ϕ and ϕ' are conjugated in* $\mathrm{Out}(F)$.

Proof. By Lemma 3.1, it suffices to decide whether the associated semi-direct products of F with $\mathbb{Z}$ with structural automorphisms ϕ and ϕ' are isomorphic by an isomorphism satisfying characterization 4 in Lemma 3.1.

Let F' be another copy of F, with the same presentation. We read ϕ' as an automorphism of F'. Let us denote by G and G' the groups of the suspensions of F and F' by the given ϕ and ϕ' respectively. Provided with a presentation of F, we have presentations of G and G'.

By the main result of [DGu2], we can decide whether there is an isomorphism between G and G'. If there is none, we are done. If there is one, say $\psi : G \to G'$, any other isomorphism is in the orbit of ψ by $\mathrm{Aut}(G')$. Let $\{f_i, i = 1, \dots, j_0 - 1\}$ be a generating set of F. We apply our solution to the orbit problem in Corollary 2.15 to the elements $\gamma_i = \psi(f_i)$ for $i < j_0$, and $\gamma_{j_0} = \psi(t)$. By definition, the answer to this orbit problem is positive if and only if there is an automorphism η such that $\eta \circ \psi$ satisfies characterization 4 in Lemma 3.1. Since all isomorphisms $G \to G'$ are of this form (i.e. $\eta \circ \psi$ for some automorphism η), this decides whether there is an isomorphism satisfying the assertion 4 of Lemma 3.1, hence, whether ϕ and ϕ' are conjugated in $\mathrm{Out}(F)$. □

Theorem 3.2 covers the case of atoroidal automorphisms of free groups, by [Br1].

Corollary 3.3. *Let F be a free group. The conjugacy problem in* $\mathrm{Out}(F)$ *restricted to atoroidal automorphisms is solvable.*

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