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NONNEGATIVITY CONSTRAINTS FOR STRUCTURED COMPLETE SYSTEMS

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ABSTRACT. We investigate pointwise nonnegativity as an obstruction to various types of structured completeness in $L^p(\mathbb{R})$. For example, we prove that if each element of the system $\{f_n\}_{n=1}^\infty \subset L^p(\mathbb{R})$ is pointwise nonnegative, then $\{f_n\}_{n=1}^\infty$ cannot be an unconditional basis or unconditional quasibasis (unconditional Schauder frame) for $L^p(\mathbb{R})$. In particular, in $L^2(\mathbb{R})$ this precludes the existence of nonnegative Riesz bases and frames. On the other hand, there exist pointwise nonnegative conditional quasibases in $L^p(\mathbb{R})$, and there also exist pointwise nonnegative exact systems and Markushevich bases in $L^p(\mathbb{R})$.

1. INTRODUCTION

Complete systems in Banach spaces can have varying degrees of structure with respect to how they approximate or represent signals. A general complete system simply has the ability to provide arbitrarily precise approximations of any signal in the space via finite linear combinations of the system elements. At the more structured extreme, an orthonormal basis provides unique unconditionally convergent signal expansions in terms of the basis elements. Other structural considerations might include whether a complete system is minimal or redundant, or whether a basis gives conditionally or unconditionally convergent expansions.

It is a common theme that there are fundamental tradeoffs between structured completeness and desirable concrete properties such as smoothness or rapid decay. For example, strong versions of the uncertainty principle restrict the collective time-frequency localization of orthonormal bases and Riesz bases for $L^2(\mathbb{R})$ [5, 6, 18]. The tradeoffs between spanning structure and time-frequency localization are perhaps best illustrated by the Balian-Low theorems for Gabor systems. The classical Balian-Low theorem [2, 11] shows that if a Gabor system $\mathcal{G}(g,1,1)$ is an orthonormal basis or Riesz basis for $L^2(\mathbb{R})$, then the window function $g \in L^2(\mathbb{R})$ and its Fourier transform $\widehat{g}$ must satisfy

$$\int |t|^2|g(t)|^2dt \int |\xi|^2|\widehat{g}(\xi)|^2d\xi = \infty. \tag{1.1}$$

A different version of the Balian-Low theorem [12] shows that if a Gabor system $\mathcal{G}(g,1,1)$ is merely exact, i.e., complete and minimal, then the window function $g \in L^2(\mathbb{R})$ must satisfy

$$\int |t|^4|g(t)|^2dt \int |\xi|^4|\widehat{g}(\xi)|^2d\xi = \infty. \tag{1.2}$$

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Moreover, there is an entire scale of results [29] between (1.1) and (1.2), with a continuous range of weight pairs $(|t|^r, |\xi|^s)$ that become increasingly restrictive as one moves from exact systems (1.2) to Riesz bases (1.1) along intermediate classes of exact Bessel (C_q)-systems, and all of these results are sharp.

Shift invariance and translation structure are another type of obstruction to structured completeness that was studied in [14, 30]. There the authors considered systems generated by translates of a given $f \in L^p(\mathbb{R}^d)$ and studied the extent to which such translation structure constrains the system's completeness structure.

Of course, time-frequency localization and translation invariance are not the only obstructions to structured completeness. In this work we shall investigate the obstruction of *pointwise nonnegativity*. To illustrate the general direction of the paper, we begin with the following two elementary observations.

Observation. There does not exist an orthonormal basis $\{f_n\}_{n=1}^\infty$ for $L^2[0,1]$ such that each f_n is nonnegative almost everywhere.

Indeed, since the nonnegative functions f_n are orthogonal, they must all be disjointly supported. Using this it is straightforward to construct a function that is not identically zero and is orthogonal to all of the f_n, which means that $\{f_n\}_{n=1}^\infty$ cannot be an orthonormal basis.

On the other hand, characteristic functions of dyadic intervals give the following.

Observation. There exist complete systems $\{f_n\}_{n=1}^\infty$ in $L^p[0,1], 1 \le p < \infty$, such that each f_n is nonnegative almost everywhere.

The main aim of this article is to understand what is possible "between" these two observations in $L^p(S)$ when $1 \le p < \infty$ and S is either the real line $\mathbb{R}$ or the unit-interval $[0,1]$. The results in this paper will determine types of structured complete systems for which pointwise nonnegativity is or is not an obstruction. Our results are purely mathematical, but it is interesting to note that the initial motivation for this work was a question from an engineer indirectly related to vanishing moment conditions for wavelet bases (where vanishing moments are closely related to approximation order) and whether nonnegativity plays a role in obstructing properties of more general signal representations.

Let L^p denote either $L^p[0,1]$ or $L^p(\mathbb{R})$. Suppose that $\{f_n\}_{n=1}^\infty \subset L^p$. Our main results are summarized as follows:

(1) We prove that if each f_n is nonnegative a.e., then $\{f_n\}_{n=1}^\infty$ cannot be any of the following: a monotone basis for $L^p, 1 < p < \infty$, an unconditional basis or unconditional quasibasis (unconditional Schauder frame) for $L^p, 1 \le p < \infty$, a Riesz basis for L^2, or a frame for L^2.

(2) We constructively prove that it is possible for each f_n to be nonnegative a.e. and for the system $\{f_n\}_{n=1}^\infty$ to be any of the following: a Markushevich basis for $L^p, 1 \le p < \infty$, or a conditional quasibasis for $L^p, 1 \le p < \infty$.

The paper is organized as follows. In Section 2 we give necessary background on various types of structured complete systems. In Section 3 we prove that there do not exist pointwise nonnegative monotone bases for $L^p[0,1], 1 < p < \infty$; see Theorem 3.2. In Section 4 we prove that there do not exist pointwise nonnegative unconditional bases for $L^p[0,1]$ and there do not exist pointwise nonnegative frames and Riesz bases for $L^2[0,1]$; see Theorem 4.1 and Corollaries 4.2 and 4.3. In Section 5, we prove that there do not exist pointwise nonnegative unconditional quasibases

for $L^p[0,1]$, see Theorem 5.5, and we also construct simple examples of pointwise nonnegative (conditional) quasibases for $L^p[0,1]$; see Theorem 5.2. In Section 6, we prove that there exist pointwise nonnegative Markushevich bases for $L^p[0,1], 1 \leq p < \infty$; see Theorem 6.1. In Section 7, we show how results from the earlier sections can be extended from $L^p[0,1]$ to $L^p(\mathbb{R})$. In Section 8 we show that there exist a.e. nonnegative Hamel bases for L^p. We conclude with some open questions in Section 9. The Appendix in Section 10 briefly sketches proofs of some background lemmas stated in Section 2.2.

2. Background on structured complete systems

In this section we recall some necessary definitions and background on various types of structured complete systems. We will later mainly work in the L^p spaces, but we present the results of this section for a separable Banach space X over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The dual space of X is denoted by X^* and the action of $g \in X^*$ on $f \in X$ is denoted by the sesquilinear form $\langle f, g\rangle$. Recall that if X is reflexive, then X^* is separable (since X is assumed separable).

2.1. **Exact systems.** Recall that a system $\{f_n\}_{n=1}^\infty \subset X$ is *complete* in X if the closed linear span of $\{f_n\}_{n=1}^\infty$ equals X. The system $\{f_n\}_{n=1}^\infty \subset X$ is *minimal* if for every $N \in \mathbb{N}$, f_N is not in the closed linear span of the remaining elements of the system, namely

$$\forall N \in \mathbb{N}, \quad f_N \notin \overline{\operatorname{span}}\{f_n : n \neq N\}. \tag{2.1}$$

If the system $\{f_n\}_{n=1}^\infty \subset X$ is both complete and minimal, then it is said to be *exact*.

Minimality and exactness can be characterized in terms of biorthogonality. The system $\{g_n\}_{n=1}^\infty \subset X^*$ is *biorthogonal* to $\{f_n\}_{n=1}^\infty \subset X$ if $\langle f_n, g_n\rangle = 1$ holds for all n, and $\langle f_n, g_m\rangle = 0$ holds for all $m \neq n$. It is well known, e.g., [21], that $\{f_n\}_{n=1}^\infty \subset X$ is minimal if and only if there exists an associated biorthogonal system $\{g_n\}_{n=1}^\infty \subset X^*$. Similarly, $\{f_n\}_{n=1}^\infty \subset X$ is exact if and only if there exists a unique associated biorthogonal system.

An exact system $\{f_n\}_{n=1}^\infty \subset X$ is said to be a *Markushevich basis* or *M-basis*, e.g., see [36], if the associated biorthogonal system $\{g_n\}_{n=1}^\infty \subset X^*$ is total. Recall that $\{g_n\}_{n=1}^\infty \subset X^*$ is *total* if

$$f \in X \ \text{ and } \ \langle f, g_n\rangle = 0 \ \text{ for all } \ n \in \mathbb{N} \quad \Longrightarrow \quad f = 0. \tag{2.2}$$

Note that if X is reflexive, then $\{g_n\}_{n=1}^\infty \subset X^*$ is total if and only if it is complete in X^*. In particular, if X is reflexive, then $\{f_n\}_{n=1}^\infty \subset X$ is a Markushevich basis if and only if $\{f_n\}_{n=1}^\infty$ is exact in X and $\{g_n\}_{n=1}^\infty$ is exact in X^*. As a terminological caution it is worth remarking that a Markushevich basis need not be a Schauder basis, e.g., see Theorem 6.2.

2.2. **Quasibases and pseudobases.** Every orthonormal basis in a Hilbert space is an example of an exact system and Markushevich basis, but there is no guarantee that an exact system or Markushevich basis can provide basis-type expansions. For this we need to discuss Schauder bases, quasibases, and pseudobases.

A system $\{f_n\}_{n=1}^\infty \subset X$ is a *pseudobasis*, see [15], if for every $f \in X$ there exist scalars $\{c_n\}_{n=1}^\infty \subset \mathbb{K}$ such that

$$f = \lim_{N\to\infty} \sum_{n=1}^N c_n f_n, \tag{2.3}$$

with norm convergence in X. The choice of scalar sequence need not be unique. It is often desirable for the scalars $\{c_n\}_{n=1}^\infty$ to be obtained from linear functionals acting on f, and this leads to the more structured case of quasibases.

A system $\{f_n\}_{n=1}^\infty \subset X$ is a *quasibasis*, see [15], with associated *dual system* $\{g_n\}_{n=1}^\infty \subset X^*$ if

$$\forall f \in X, \quad f = \lim_{N\to\infty} \sum_{n=1}^N \langle f, g_n\rangle f_n, \tag{2.4}$$

with norm convergence in X. The dual system need not be unique. We say that $\{f_n\}_{n=1}^\infty \subset X$ is an *unconditional quasibasis* if there exists a dual system $\{g_n\}_{n=1}^\infty \subset X^*$ for which the quasibasis expansion (2.4) converges unconditionally (i.e., any rearrangement of the sum still converges to f); any such $\{g_n\}_{n=1}^\infty$ will be referred to as an *unconditional dual system*. The literature on quasibases and pseudobases goes at least as far back as [15], but quasibases have also been studied more recently under the name *Schauder frame*, e.g., [10, 14, 20, 28, 30].

The next result addresses boundedness of partial sum operators associated to a quasibasis. Such results are standard for Schauder bases, e.g., Theorem 5.12 in [21], and have also been noted without proof for quasibases, e.g., [26]. We include a sketch of the details in the Appendix.

Lemma 2.1. *Let $\{f_n\}_{n=1}^\infty \subset X$ and $\{g_n\}_{n=1}^\infty \subset X^*$ and consider the partial sum operators $S_N : X \to X$ defined by*

$$S_N(f) = \sum_{n=1}^N \langle f, g_n\rangle f_n.$$

The following are equivalent:

1. *$\{f_n\}_{n=1}^\infty$ is a quasibasis for X with dual system $\{g_n\}_{n=1}^\infty$.*
2. *$\sup_{N\in\mathbb{N}} \|S_N\|_{X\to X} < \infty$ and there exists a dense subset $D \subset X$ such that if $f \in D$, then $\lim_{N\to\infty} \|f - S_N(f)\|_X = 0$.*

The following result is well known for unconditional bases, see [21], and very similarly extends to unconditional quasibases. We include a sketch of the details in the Appendix.

Lemma 2.2. *Suppose that $\{f_n\}_{n=1}^\infty \subset X$ is an unconditional quasibasis for X and $\{g_n\}_{n=1}^\infty \subset X^*$ is an associated unconditional dual system. Given any scalar sequence $\mathcal{U} = \{u_n\}_{n=1}^\infty \subset \mathbb{K}$ with $|u_n| \le 1$ consider modified partial sum operators $S_{\mathcal{U},N} : X \to X$ defined by*

$$S_{\mathcal{U},N}(f) = \sum_{n=1}^N u_n \langle f, g_n\rangle f_n.$$

Then

$$\sup_{\mathcal{U},N} \|S_{\mathcal{U},N}\|_{X\to X} < \infty,$$

where the supremum is taken over all $N \in \mathbb{N}$ *and all* $\mathcal{U} = \{u_n\}_{n=1}^{\infty} \subset \mathbb{K}$ *with* $|u_n| \leq 1$.

The following result shows that a sufficiently small perturbation of a quasibasis or unconditional quasibasis is still a respective quasibasis or unconditional quasibasis. A proof of this result for pseudobases appears in [38] and the analog for quasibases is also mentioned there without proof. We include a sketch of the details in the Appendix.

Lemma 2.3. *Suppose that* $\{f_n\}_{n=1}^{\infty} \subset X$ *is a quasibasis for* X *and* $\{g_n\}_{n=1}^{\infty} \subset X^*$ *is an associated dual system. If* $0 < \epsilon < 1$ *and* $\{\varphi_n\}_{n=1}^{\infty}$ *satisfies*

$$\forall n \in \mathbb{N}, \quad \|\varphi_n - f_n\|_X \leq \frac{\epsilon}{2^{n+1}\|g_n\|_{X^*}}, \tag{2.5}$$

then $\{\varphi_n\}_{n=1}^{\infty}$ *is also a quasibasis for* X. *Moreover, if* $\{f_n\}_{n=1}^{\infty} \subset X$ *is assumed to be an unconditional quasibasis and* $\{g_n\}_{n=1}^{\infty} \subset X^*$ *is an associated unconditional dual system, then one may similarly conclude that* $\{\varphi_n\}_{n=1}^{\infty}$ *is also an unconditional quasibasis for* X.

2.3. **Schauder bases and unconditional bases.** If a pseudobasis is minimal (hence the choice of scalar sequence in (2.3) is unique), then it is said to be a Schauder basis. It turns out that every minimal pseudobasis in a Banach space is a minimal quasibasis, and Schauder bases can be equivalently defined (see Theorem 5.12 in [21]) as follows: $\{f_n\}_{n=1}^{\infty} \subset X$ is a *Schauder basis* for X if there exists a biorthogonal system $\{g_n\}_{n=1}^{\infty} \subset X^*$ such that

$$\forall f \in X, \quad f = \lim_{N\to\infty} \sum_{n=1}^{N} \langle f, g_n\rangle f_n, \tag{2.6}$$

with norm convergence in X. Moreover, if X is reflexive, then the dual system $\{g_n\}_{n=1}^{\infty} \subset X^*$ is also a Schauder basis for X^* and is referred to as the *dual basis*.

The following lemma provides control on the norms of the dual basis elements, e.g., see Theorem 4.13 in [21].

Lemma 2.4. *If* $\{f_n\}_{n=1}^{\infty} \subset X$ *is a Schauder basis for* X *with dual system* $\{g_n\}_{n=1}^{\infty} \subset X^*$, *then*

$$1 \leq \sup_{n\in\mathbb{N}} \|f_n\|_X \|g_n\|_{X^*} < \infty.$$

If the Schauder basis expansions (2.6) converge unconditionally, then $\{f_n\}_{n=1}^{\infty}$ is referred to as an *unconditional basis* for X. In $X = L^p[0,1]$ unconditional bases satisfy the following square function inequality, e.g., see equations (1.7) and (1.8) in [1], which may be viewed as a relative of the Parseval equality.

Lemma 2.5. *Fix* $1 < p < \infty$ *and let* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $\{f_n\}_{n=1}^{\infty}$ *is an unconditional basis for* $L^p[0,1]$ *with dual basis* $\{g_n\}_{n=1}^{\infty}$ *for* $L^q[0,1]$, *then there exist constants* $0 < A \leq B < \infty$ *such that*

$$\forall g \in L^q[0,1], \quad A\|g\|_q^q \leq \int_0^1 \left(\sum_{n=1}^{\infty} |\langle g, f_n\rangle|^2 |g_n(t)|^2\right)^{q/2} dt \leq B\|g\|_q^q. \tag{2.7}$$

The proof of Lemma 2.5 follows from Khinchine's inequality which we briefly recall here. Let $\{R_n\}_{n=0}^{\infty}$ be the *Rademacher functions* defined by

$$\forall n \geq 0, \quad R_n(t) = \operatorname{sign}[\sin(2^n \pi t)]. \tag{2.8}$$

Fix $1 \leq p < \infty$. *Khinchine's inequality* says that there exist constants $0 < a_p \leq b_p < \infty$ such that for any scalars $\{c_n\}_{n=1}^{\infty} \subset \mathbb{K}$ and any $N \in \mathbb{N}$ there holds

$$a_p \left(\sum_{n=1}^{N} |c_n|^2 \right)^{1/2} \leq \left\| \sum_{n=1}^{N} c_n R_n \right\|_{L^p[0,1]} \leq b_p \left(\sum_{n=1}^{N} |c_n|^2 \right)^{1/2}, \tag{2.9}$$

for example, see Theorem 9.1 in [8].

Example 2.6 (The Haar system)**.** The Haar system $\{h_n\}_{n=1}^{\infty}$ is an unconditional basis for $L^p[0,1]$ whenever $1 < p < \infty$, and is a conditional Schauder basis for $L^1[0,1]$; see [35] or Section 5.5 in [21]. Moreover, it is an orthonormal basis for $L^2[0,1]$.

Example 2.7 (The Fourier system)**.** The Fourier system or trigonometric system is a Schauder basis for $L^p[0,1]$ whenever $1 < p < \infty$, but is an unconditional basis only when $p = 2$ (and it is an orthonormal basis for $L^2[0,1]$). The Fourier system is not a Schauder basis for $L^1[0,1]$, but it is a Markushevich basis for $L^1[0,1]$ (and the biorthogonal system in $L^\infty[0,1]$ is itself). For further details see Theorem 4.25 and Section 14.2 in [21].

Example 2.8 (The Walsh system)**.** The Walsh functions $\{w_n\}_{n=1}^{\infty}$ are defined in terms of the Rademacher functions by

$$w_1(t) = 1 \qquad \text{and} \qquad w_{k+1}(t) = \prod_{j=1}^{\nu} R_{n_j+1}(t), \tag{2.10}$$

where the integers $\{n_j\}_{j=1}^{\nu}$ determine the binary expansion of k by

$$k = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_\nu}, \quad n_1 > n_2 > \cdots > n_\nu \geq 0. \tag{2.11}$$

In particular, the Walsh system $\{w_n\}_{n=1}^{\infty}$ contains the Rademacher system $\{R_n\}_{n=0}^{\infty}$ since $w_1 = R_0$ and

$$\forall m \geq 0, \quad w_{2^m+1}(t) = \operatorname{sign}[\sin(\pi 2^{m+1} t)] = R_{m+1}(t). \tag{2.12}$$

It is worth cautioning that there are different conventions regarding the enumeration of the Walsh functions. Definition (2.10) follows the conventions of Chapter II, §14 in [35], but see [34] for further discussion of this and other approaches.

The Walsh system $\{w_n\}_{n=1}^{\infty}$ is a Schauder basis for $L^p[0,1]$ whenever $1 < p < \infty$; see Section 5.3 in [34]. The Walsh system is an unconditional basis for $L^p[0,1]$ only when $p = 2$ (and it is an orthonormal basis for $L^2[0,1]$); this is a consequence of Orlicz's theorem; see Theorem 3.28 in [21]. The Walsh system is not a Schauder basis for $L^1[0,1]$, e.g., see Theorem 2 in Chapter 4 of [34]. However, the Walsh system is a Markushevich basis for $L^1[0,1]$ (and the biorthogonal system in $L^\infty[0,1]$ is itself). This holds since Lemma 14.2 of Chapter II in [35] relates Haar and Walsh dyadic partial sum operators and yields

$$\forall f \in L^1[0,1], \quad f = \lim_{N\to\infty} \sum_{n=1}^{2^N} \langle f, h_n \rangle h_n = \lim_{N\to\infty} \sum_{n=1}^{2^N} \langle f, w_n \rangle w_n, \tag{2.13}$$

with norm convergence in $L^1[0,1]$. Property (2.13) simultaneously implies that $\{w_n\}_{n=1}^{\infty}$ is complete in $L^1[0,1]$ and total in $L^\infty[0,1]$. For a direct proof that $\{w_n\}_{n=1}^{\infty}$ is total in $L^1[0,1]$, see [31].

It will be useful to recall the following analog of the Riemann-Lebegue lemma for Walsh functions; see Section 1.5 in [34].

Lemma 2.9. *If* $f \in L^1[0,1]$, *then* $\lim_{n\to\infty} |\langle f, w_n\rangle| = 0$. *In particular, if* $1 \le p \le \infty$ *and* $f \in L^p[0,1]$, *then* $\lim_{n\to\infty} |\langle f, R_n\rangle| = 0$.

2.4. **Orthogonality and monotone bases.** Orthogonality and orthonormal bases are particularly nice Hilbert space concepts which also admit generalizations to the setting of Banach spaces, e.g., [23]. Given $f_1, f_2 \in X$, we say that f_1 is *orthogonal* to f_2 if

$$\forall\, \lambda \in \mathbb{K}, \quad \|f_1\|_X \le \|f_1 + \lambda f_2\|_X, \tag{2.14}$$

and we denote this by $f_1 \perp f_2$. In Hilbert spaces, (2.14) is equivalent to the usual notion of orthogonality involving inner products, but orthogonality in Banach spaces can behave quite differently than in Hilbert spaces. For example, it is not true that $f_1 \perp f_2$ holds if and only if $f_2 \perp f_1$; see Example 3.3.

A Schauder basis $\{f_n\}_{n=1}^\infty \subset X$ is said to be a *monotone basis* if for any scalars $\{c_n\}_{n=1}^\infty \subset \mathbb{K}$ there holds

$$\forall N \le M, \quad \left\| \sum_{n=1}^N c_n f_n \right\|_X \le \left\| \sum_{n=1}^M c_n f_n \right\|_X . \tag{2.15}$$

Equivalently, a basis is monotone if and only if its basis constant $\mathcal{C} = 1$, where $\mathcal{C} = \sup_{N\in\mathbb{N}} \|S_N\|_{X\to X}$ and $S_N(f) = \sum_{n=1}^N \langle f, g_n\rangle f_n$ is the Nth partial sum operator associated to $\{f_n\}_{n=1}^\infty$ and its dual basis $\{g_n\}_{n=1}^\infty$. Note that if $\{f_n\}_{n=1}^\infty$ is a monotone basis, then $n < m$ implies $f_n \perp f_m$.

For the special case of orthogonality in $X = L^p$ spaces, we use the notation $f_1 \perp_p f_2$ to emphasize the dependence on p. It is known that in $X = L^p[0,1]$ with $1 < p < \infty$, every monotone basis is unconditional [13]. The Haar system is an example of a monotone basis for $L^p[0,1]$ for each $1 \le p < \infty$; see Theorem 5.18 in [21].

2.5. **Frames and Riesz bases.** Some of the previously defined spanning systems have especially nice formulations in Hilbert spaces and are worth commenting on briefly. Let H be a separable Hilbert space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

A collection $\{f_n\}_{n=1}^\infty \subset H$ is a *frame* for H if there exist constants $0 < A \le B < \infty$ for which

$$\forall f \in H, \quad A\|f\|_H^2 \le \sum_{n=1}^\infty |\langle f, f_n\rangle|^2 \le B\|f\|_H^2. \tag{2.16}$$

A main fact about frames is that there exists a *dual frame* $\{\widetilde{f}_n\}_{n=1}^\infty \subset H$ for which

$$\forall f \in H, \quad f = \lim_{N\to\infty} \sum_{n=1}^N \langle f, \widetilde{f}_n\rangle f_n = \lim_{N\to\infty} \sum_{n=1}^N \langle f, f_n\rangle \widetilde{f}_n, \tag{2.17}$$

with unconditional convergence in H. If $A = B$ in (2.16), then frame is said to be *tight*, and in this case (2.17) holds with $\widetilde{f}_n = (1/A) f_n$. The choice of dual frame is generally not unique (even for tight frames). The possibility for frames to be nonexact (hence redundant) can provide robustness against noise and numerical stability in applications such as transmission of data over erasure channels [16, 17],

oversampled analog-to-digital conversion [3, 32], wavelet-based image processing [7], and phase retrieval problems [4].

In the special case when a frame is minimal, it is referred to as a *Riesz basis*. In other words, a Riesz basis is an exact frame. For example, every orthonormal basis for H is a Riesz basis and frame. For further background on frames, see [9, 21].

2.6. **Completeness structure summary.** Some relations between the various structured complete systems discussed in this section may be summarized as follows.

In a Banach space X there holds

$$\{\text{Unconditional bases}\} \subset \{\text{Schauder bases}\} \subset \{\text{Markushevich bases}\} \subset \{\text{Exact systems}\},$$

$$\{\text{Schauder bases}\} \subset \{\text{Quasibases}\} \subset \{\text{Pseudobases}\},$$

and in a Hilbert space H there holds

$$\{\text{Orthonormal bases}\} \subset \{\text{Riesz bases}\} \subset \{\text{Unconditional bases}\},$$

$$\{\text{Riesz bases}\} \subset \{\text{Frames}\} \subset \{\text{Unconditional quasibases}\}.$$

Moreover, in L^p with $1 < p < \infty$, there holds

$$\{\text{Monotone bases}\} \subset \{\text{Unconditional bases}\}.$$

3. Monotone bases

In this section we begin to show how pointwise nonnegativity is an obstruction to some of the structured complete systems considered in Section 2. Motivated by Observation 1 for $L^2[0,1]$, we begin with an analog for monotone bases in $L^p[0,1]$.

Lemma 3.1. *Fix $1 < p < \infty$. If $f, g \in L^p[0,1]$ are a.e. nonnegative and $f \perp_p g$, then f and g are disjointly supported.*

Proof. Proceed by contradiction and suppose there exists a set of positive measure $E \subset [0,1]$ such that $f(t) > 0$ and $g(t) > 0$ for a.e. $t \in E$. Define

$$T(\lambda) = \int_0^1 |f(t) + \lambda g(t)|^p dt.$$

By Lemma 11.19 in [8],

$$T'(0) = \lim_{h \to 0} \frac{T(h) - T(0)}{h} = \int_0^1 p|f(t)|^{p-1} g(t) \operatorname{sign}(f(t)) dt.$$

Note that $|T'(0)| < \infty$ by Hölder's inequality. Also, $T'(0) \geq p \int_E |f(t)|^{p-1} g(t) dt > 0$. Hence there must exist some $\lambda_0 < 0$ such that $\|f + \lambda_0 g\|_p^p = T(\lambda_0) < T(0) = \|f\|_p^p$. This contradicts the hypothesis that $f \perp_p g$. □

Theorem 3.2 (Monotone bases)**.** *Fix $1 < p < \infty$. There does not exist a monotone basis $\{f_n\}_{n=1}^\infty$ for $L^p[0,1]$ such that each f_n is a.e. nonnegative.*

Proof. Proceed by contradiction and suppose such a basis exists. Since $\{f_n\}_{n=1}^\infty$ is a monotone basis, it follows that $f_n \perp_p f_m$ whenever $n \leq m$. Thus by Lemma 3.1, all of the $\{f_n\}_{n=1}^\infty$ are disjointly supported. This gives a contradiction since it allows one to produce $\varphi \in L^p[0,1]$ that is not in the closed linear span of $\{f_n\}_{n=1}^\infty$. For example, if $\operatorname{supp}(\varphi) \subsetneq \operatorname{supp}(f_1)$ and $0 < |\operatorname{supp}(\varphi)| < |\operatorname{supp}(f_1)|$, then φ is not spanned by $\{f_n\}_{n=1}^\infty$. □

Example 3.3. Note that Lemma 3.1 is not true when $p = 1$. For example, if $f = \chi_{[0,1/2]}$ is the characteristic function of $[0, 1/2]$ and $g = \chi_{[1/8,1]}$, then it can be verified that $f \perp_1 g$, but $g \not\perp_1 f$ and f, g are clearly not disjointly supported.

4. Unconditional bases and frames

In this section we show that pointwise nonnegativity is an obstruction to unconditional bases and frames. The following theorem uses a general square function condition (4.1), but will later be used to obtain results for unconditional bases and frames as corollaries.

Theorem 4.1. *Fix $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Fix constants $0 < A \le B < \infty$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset L^p[0,1]$ and $\{g_n\}_{n=1}^{\infty} \subset L^q[0,1]$ satisfy:*

$$\forall g \in L^q[0,1], \quad A\|g\|_q^q \le \int_0^1 \left(\sum_{n=1}^{\infty} |\langle g, f_n\rangle|^2 |g_n(t)|^2 \right)^{q/2} dt \le B\|g\|_q^q. \tag{4.1}$$

Then f_n cannot be a.e. nonnegative for each $n \ge 1$.

Proof. Step I. Proceed by contradiction and suppose that $f_n(t) \ge 0$ a.e. for each $n \ge 1$. For each $N \ge 0$, let $R_N(t) = \operatorname{sign}(\sin(2^N \pi t))$ be the Nth Rademacher function. Since $R_0(t) = 1$, the a.e. nonnegativity of f_n implies that

$$\forall N \ge 0, n \ge 1, \quad |\langle R_N, f_n\rangle| \le |\langle R_0, f_n\rangle|. \tag{4.2}$$

Since $\|R_N\|_q = 1$, applying (4.1) with $g = R_N$ gives

$$\forall N \ge 1, \quad 0 < A \le \int_0^1 \left(\sum_{n=1}^{\infty} |\langle R_N, f_n\rangle|^2 |g_n(t)|^2 \right)^{q/2} dt \le B < \infty. \tag{4.3}$$

Step II. Let

$$\Psi_N(t) = \left(\sum_{n=1}^{\infty} |\langle R_N, f_n\rangle|^2 |g_n(t)|^2 \right)^{q/2}.$$

We will show that $\lim_{N\to\infty} \|\Psi_N\|_1 = 0$, thereby contradicting the lower bound in (4.3).

By (4.3), we have that $\Psi_N \in L^1[0,1]$ for each $N \ge 0$. Also, by (4.2)

$$\forall N \ge 1, \quad |\Psi_N(t)| \le |\Psi_0(t)| \quad \text{a.e.} \tag{4.4}$$

If we show that

$$\lim_{N\to\infty} |\Psi_N(t)| = 0 \quad \text{a.e.}, \tag{4.5}$$

then this along with (4.4) and the Lebesgue dominated convergence theorem will yield $\lim_{N\to\infty} \|\Psi_N\|_1 = 0$ as desired.

So it remains to show (4.5). For this, suppose $N \ge 0$ and $t \in [0,1]$ are fixed and define the sequence $\Phi_{N,t} = \{\Phi_{N,t}(n)\}_{n=1}^{\infty}$ by

$$\Phi_{N,t}(n) = |\langle R_N, f_n\rangle|^2 |g_n(t)|^2. \tag{4.6}$$

Since $\Psi_N \in L^1[0,1]$ we have that $\Phi_{N,t} \in \ell^1(\mathbb{N})$ for every $N \ge 0$ and a.e. $t \in [0,1]$. Moreover, by (4.2), we have

$$|\Phi_{N,t}(n)| \le |\Phi_{0,t}(n)|, \quad \text{for all } N \ge 0, n \ge 1 \text{ and a.e. } t \in [0,1]. \tag{4.7}$$

Recalling that by Lemma 2.9

$$\forall g \in L^q[0,1], \quad \lim_{N\to\infty} |\langle R_N, g\rangle| = 0, \tag{4.8}$$

we have

$$\lim_{N\to\infty} |\Phi_{N,t}(n)| = 0, \quad \text{for all } n \geq 1 \text{ and a.e. } t \in [0,1]. \tag{4.9}$$

Thus, (4.7), (4.9), and the Lebesgue dominated convergence theorem for series imply that

$$\lim_{N\to\infty} \|\Phi_{N,t}\|_{\ell^1} = 0, \quad \text{for a.e. } t \in [0,1]. \tag{4.10}$$

Since $|\Psi_N(t)| = \|\Phi_{N,t}\|_{\ell^1}^{q/2}$, this yields (4.5) as required. □

Corollary 4.2 (Unconditional bases)**.** *Fix $1 < p < \infty$. There does not exist an unconditional basis $\{f_n\}_{n=1}^\infty$ for $L^p[0,1]$ such that each f_n is a.e. nonnegative.*

Proof. This follows from Theorem 4.1 and Lemma 2.5. □

Corollary 4.3 (Frames and Riesz bases)**.** *There does not exist a frame or Riesz basis $\{f_n\}_{n=1}^\infty$ for $L^2[0,1]$ such that each f_n is a.e. nonnegative.*

Proof. This follows from Theorem 4.1 since frames and Riesz bases for $L^2[0,1]$ satisfy (4.1) with $q = 2$ and $g_n(t) = 1$ for all $n \geq 1$. □

Theorem 4.1 and Corollary 4.2 give an alternate, albeit less direct, proof of Theorem 3.2 since every monotone basis in $L^p[0,1], 1 < p < \infty$, is an unconditional basis [13].

5. Quasibases

Example 5.1 (The Schauder system)**.** The Schauder system is a classical example of a Schauder basis for $C[0,1]$, e.g., see Section 4.5 in [21]. Each element of the Schauder system is nonnegative. It was shown in Theorem 1 of [27] that the Schauder system is a quasibasis for $L^p[0,1]$ when $1 \leq p < \infty$, but is not an unconditional quasibasis. For some discussion of pseudobasis properties and unconditionally convergent representations related to the Schauder system, see page 2883 of [27]; cf. [39].

Other simple examples of nonnegative quasibases can be produced as follows. We use the notation $h^+(t) = \max\{h(t), 0\}$ and $h^-(t) = \max\{-h(t), 0\}$ to denote the positive and negative parts of a real-valued function h.

Theorem 5.2. *Fix $1 \leq p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\varphi_n : \mathbb{R} \to \mathbb{R}$ and $\psi_n : \mathbb{R} \to \mathbb{R}$ are real-valued measurable functions and that $\{\varphi_n\}_{n=1}^\infty$ is a Schauder basis for $L^p[0,1]$ with dual system $\{\psi_n\}_{n=1}^\infty \subset L^q[0,1]$. For each $n \in \mathbb{N}$ define*

$$f_{2n-1}(t) = \varphi_n^+(t), \quad f_{2n}(t) = \varphi_n^-(t), \quad g_{2n-1}(t) = \psi_n(t), \quad g_{2n}(t) = -\psi_n(t).$$

Then $\{f_n\}_{n=1}^\infty$ is a quasibasis for $L^p[0,1]$ with dual system $\{g_n\}_{n=1}^\infty$.

Proof. Consider the partial sum operator $S_N : L^p[0,1] \to L^p[0,1]$ defined by

$$S_N(f) = \sum_{n=1}^N \langle f, g_n\rangle f_n.$$

We proceed with the goal of applying Lemma 2.1.

Step I. Since $\{\varphi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ are biorthogonal, if $n \notin \{2k-1, 2k\}$, then $\langle \varphi_k, g_n\rangle = 0$. So, for all $k, N \in \mathbb{N}$, where $N \geq 2k$,

$$S_N(\varphi_k) = \sum_{n=1}^N \langle \varphi_k, g_n\rangle f_n = \langle \varphi_k, \psi_k\rangle \varphi_k^+ + \langle \varphi_k, -\psi_k\rangle \varphi_k^- = \langle \varphi_k, \psi_k\rangle \varphi_k = \varphi_k. \tag{5.1}$$

Let D denote the finite linear span of $\{\varphi_n\}_{n=1}^\infty$ and note that D is dense in $L^p[0,1]$. It follows immediately from (5.1) that

$$\forall f \in D, \quad \lim_{N\to\infty} \|f - S_N(f)\|_p = 0. \tag{5.2}$$

Step II. This step will show that

$$\sup_{N\in\mathbb{N}} \|S_N\|_{p\to p} < \infty. \tag{5.3}$$

Since

$$\forall n \in \mathbb{N}, \forall f \in L^p[0,1], \quad \langle f, g_{2n-1}\rangle f_{2n-1} + \langle f, g_{2n}\rangle f_{2n} = \langle f, \psi_n\rangle \varphi_n,$$

we have

$$S_{2N}(f) = \sum_{n=1}^N \langle f, \psi_n\rangle \varphi_n. \tag{5.4}$$

Thus since $\{\varphi_n\}_{n=1}^\infty$ is a Schauder basis, (5.4) and Lemma 2.1 yield

$$\sup_{N\in\mathbb{N}} \|S_{2N}\|_{p\to p} < \infty. \tag{5.5}$$

Next, since $\|\varphi_{N+1}^+\|_p \leq \|\varphi_{N+1}\|_p$,

$$\begin{aligned}
\|S_{2N+1}(f)\|_p &= \|S_{2N}(f) + \langle f, g_{2N+1}\rangle f_{2N+1}\|_p \\
&= \|S_{2N}(f) + \langle f, \psi_{N+1}\rangle \varphi_{N+1}^+\|_p \\
&\leq \|S_{2N}(f)\|_p + \|\varphi_{N+1}\|_p |\langle f, \psi_{N+1}\rangle| \\
&\leq \left(\sup_{N\in\mathbb{N}} \|S_{2N}\|_{p\to p}\right) \|f\|_p + \left(\sup_{n\in\mathbb{N}} \|\varphi_n\|_p \|\psi_n\|_q\right) \|f\|_p. \qquad (5.6)
\end{aligned}$$

So, (5.5), (5.6), and Lemma 2.4 yield $\sup_{N\in\mathbb{N}} \|S_{2N+1}\|_{p\to p} < \infty$, and consequently (5.3) holds.

Thus, Lemma 2.1 together with (5.2) and (5.3) completes the proof. □

Example 5.3. In the special case when $\{\varphi_n\}_{n=1}^\infty$ is an orthonormal basis for $L^2[0,1]$ and $\psi_n = \varphi_n$, the quasibasis $\{f_n\}_{n=1}^\infty$ for $L^2[0,1]$ and dual system $\{g_n\}_{n=1}^\infty$ defined in Theorem 5.2 can be shown to satisfy

$$\forall f \in L^2[0,1], \quad (1/2)\|f\|_2^2 \leq \sum_{n=1}^\infty |\langle f, f_n\rangle|^2 \quad \text{and} \quad 2\|f\|_2^2 = \sum_{n=1}^\infty |\langle f, g_n\rangle|^2.$$

In particular, $\{g_n\}_{n=1}^\infty$ is a tight frame for $L^2[0,1]$ and the quasibasis $\{f_n\}_{n=1}^\infty$ satisfies a lower frame inequality. However, by Corollary 4.3, $\{f_n\}_{n=1}^\infty$ is not a frame for $L^2[0,1]$ since each f_n is a.e. nonnegative. See [22] for further discussion of nonframe duals of frames.

Applying Theorem 5.2 to the Haar basis for $L^p[0,1]$ yields the following corollary.

Corollary 5.4. *Fix $1 \leq p < \infty$. The characteristic functions of dyadic intervals form a nonnegative quasibasis for $L^p[0,1]$.*

The techniques related to Example 5.1 and specifically Theorem 2 in [27] can be extended to give the following result which shows that there are in fact no nonnegative unconditional quasibases for $L^p[0,1]$.

Theorem 5.5 (Unconditional quasibases)**.** *Fix $1 \leq p < \infty$. There does not exist an unconditional quasibasis $\{f_n\}_{n=1}^{\infty}$ for $L^p[0,1]$ such that each f_n is a.e. nonnegative.*

Proof. Step I. Proceed by contradiction and suppose that $\{f_n\}_{n=1}^{\infty}$ is an unconditional quasibasis and each f_n is a.e. nonnegative. Using Lemma 2.3 on stability of unconditional quasibases and the density of dyadic step functions in $L^p[0,1]$, we may assume without loss of generality that each f_n is a finite positive linear combination of dyadic step functions. In particular, we will assume that

$$f_n(t) = \sum_{j=1}^{2^{k_n}} d_j^n x_{j,k_n}(t)$$

where

- $x_{j,k_n}(t) = \chi_{I_{j,k_n}}(t)$ is the characteristic function of the dyadic interval $I_{j,k_n} = [\frac{j-1}{2^{k_n}}, \frac{j}{2^{k_n}}]$,
- $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ with $k_n < k_{n+1}$ is a strictly increasing sequence of positive integers,
- for each $n \in \mathbb{N}$ and $1 \leq j \leq 2^{k_n}$, $d_j^n \geq 0$.

Step II. Let $\{g_n\}_{n=1}^{\infty} \subset L^q[0,1]$ be any fixed unconditional dual system for the unconditional quasibasis $\{f_n\}_{n=1}^{\infty}$. If R_m is the mth Rademacher function given by (2.8) and if $c_n^m = \langle R_m, g_n \rangle$, then

$$\forall m \geq 1, \quad R_m = \sum_{n=1}^{\infty} \langle R_m, g_n \rangle f_n = \sum_{n=1}^{\infty} c_n^m f_n,$$

with unconditional convergence in $L^p[0,1]$. Consequently, for each $m \geq 1$, there is an increasing subsequence $\{N_{i,m}\}_{i=1}^{\infty} \subset \mathbb{N}$ such that

$$\forall m \geq 1, \quad R_m(t) = \lim_{i \to \infty} \sum_{n=1}^{N_{i,m}} \langle R_m, g_n \rangle f_n(t) = \lim_{i \to \infty} \sum_{n=1}^{N_{i,m}} c_n^m f_n(t),$$

holds pointwise for a.e. $t \in [0,1]$.

Fix any $1 \leq \alpha \leq 2^{k_n}$. For $j \leq n$, let

$$\Lambda_{\alpha,j,n} = \left\{ 1 \leq l \leq 2^{k_j} : I_{\alpha,k_n} \subset \left[\frac{l-1}{2^{k_j}}, \frac{l}{2^{k_j}} \right] \right\}.$$

For a.e. $t \in I_{\alpha,k_n} = [\frac{\alpha-1}{2^{k_n}}, \frac{\alpha}{2^{k_n}}]$ there holds

$$\begin{aligned} R_{k_n+1}(t) &= \sum_{j=1}^{n} c_j^{k_n+1} f_j(t) + \lim_{i\to\infty} \sum_{j=n+1}^{N_{i,k_n+1}} c_j^{k_n+1} f_j(t) \\ &= \sum_{j=1}^{n} c_j^{k_n+1} \sum_{l=1}^{2^{k_j}} d_l^j x_{l,k_j}(t) + \lim_{i\to\infty} \sum_{j=n+1}^{N_{i,k_n+1}} c_j^{k_n+1} f_j(t) \\ &= \sum_{j=1}^{n} c_j^{k_n+1} \sum_{l\in\Lambda_{\alpha,j,n}} d_l^j + \lim_{i\to\infty} \sum_{j=n+1}^{N_{i,k_n+1}} c_j^{k_n+1} f_j(t). \end{aligned}$$

Since the f_j are a.e. nonnegative, for each $n \in \mathbb{N}$ the pointwise limit $\sum_{j=n+1}^{\infty} |c_j^{k_n+1}| f_j(t)$ exists in $[0,\infty]$ for a.e. $t \in [0,1]$. Moreover, for a.e. $t \in I_{\alpha,k_n}$

(5.7)

$$\sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t) \geq \left| \lim_{i\to\infty} \sum_{j=n+1}^{N_{i,k_n+1}} c_j^{k_n+1} f_j(t) \right| = \left| R_{k_n+1}(t) - \sum_{j=1}^{n} c_j^{k_n+1} \sum_{l\in\Lambda_{\alpha,j,n}} d_l^j \right|.$$

Step III. In this step we show that

$$\int_0^1 \sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt \geq 1. \tag{5.8}$$

For this, we first show that for all $n \in \mathbb{N}$ and all $1 \leq \alpha \leq 2^{k_n}$ there holds

$$\int_{I_{\alpha,k_n}} \sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt \geq \frac{1}{2^{k_n}}. \tag{5.9}$$

Let I'_{α,k_n} and I''_{α,k_n} denote the left and right halves of I_{α,k_n} respectively. Note that $R_{k_n+1}(t) = 1$ for a.e. $t \in I'_{\alpha,k_n}$, and $R_{k_n+1}(t) = -1$ for a.e. $t \in I''_{\alpha,k_n}$. So, by (5.7)

(5.10)

$$\begin{aligned} &\int_{I_{\alpha,k_n}} \sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt \\ &\quad = \int_{I'_{\alpha,k_n}} \sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt + \int_{I''_{\alpha,k_n}} \sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt \\ &\quad \geq \int_{I'_{\alpha,k_n}} \left| 1 - \sum_{j=1}^{n} c_j^{k_n+1} \sum_{l\in\Lambda_{\alpha,j,n}} d_l^j \right| dt + \int_{I''_{\alpha,k_n}} \left| -1 - \sum_{j=1}^{n} c_j^{k_n+1} \sum_{l\in\Lambda_{\alpha,j,n}} d_l^j \right| dt \\ &\quad = \frac{1}{2^{k_n+1}} \left| 1 - \sum_{j=1}^{n} c_j^{k_n+1} \sum_{l\in\Lambda_{\alpha,j,n}} d_l^j \right| + \frac{1}{2^{k_n+1}} \left| 1 + \sum_{j=1}^{n} c_j^{k_n+1} \sum_{l\in\Lambda_{\alpha,j,n}} d_l^j \right|. \end{aligned}$$

Since $|1-z| + |1+z| \geq 2$ holds for all $z \in \mathbb{C}$, (5.10) implies that (5.9) holds.

Now (5.8) follows from (5.9) since

$$\int_0^1 \sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt = \sum_{\alpha=1}^{2^{k_n}} \int_{I_{\alpha,k_n}} \sum_{j=n+1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt \geq \sum_{\alpha=1}^{2^{k_n}} \frac{1}{2^{k_n}} = 1.$$

Step IV. In this step we show that

$$\forall n \in \mathbb{N}, \quad \int_0^1 \sum_{j=1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt < \infty. \tag{5.11}$$

Let $\mathcal{U}_n = \{u_j^n\}_{j=1}^{\infty} \subset \mathbb{K}$ be unimodular scalars $|u_j^n| = 1$ such that $\left|c_j^{k_n+1}\right| = u_j^n c_j^{k_n+1}$ for all $j \geq 1$ and $n \geq 0$. Recalling that $f_n \geq 0$ a.e., the monotone convergence theorem and Hölder's inequality give

$$\begin{aligned}
\int_0^1 \sum_{j=1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt &= \lim_{N\to\infty} \int_0^1 \sum_{j=1}^{N} \left| c_j^{k_n+1} \right| f_j(t)dt \\
&\leq \limsup_{N\to\infty} \left(\int_0^1 \left| \sum_{j=1}^{N} \left| c_j^{k_n+1} \right| f_j(t) \right|^p dt \right)^{1/p} \\
&\leq \limsup_{N\to\infty} \left(\int_0^1 \left| \sum_{j=1}^{N} u_j^n \langle R_{k_n+1}, g_j \rangle f_j(t) \right|^p dt \right)^{1/p} \\
&\leq \limsup_{N\to\infty} \|S_{\mathcal{U}_n,N}(R_{k_n+1})\|_p. \qquad (5.12)
\end{aligned}$$

Using unconditional convergence and Lemma 2.2, (5.12) implies that for each $n \in \mathbb{N}$

$$\int_0^1 \sum_{j=1}^{\infty} \left| c_j^{k_n+1} \right| f_j(t)dt \leq \limsup_{N\to\infty} \|S_{\mathcal{U}_n,N}\|_{p\to p} \|R_{k_n+1}\|_p \leq \sup_{\mathcal{U},N} \|S_{\mathcal{U},N}\|_{p\to p} < \infty.$$

Step V. In this step we select integers

$$1 = r(0) < m(1) < r(1) < m(2) < \cdots < m(i+1) < r(i+1) < \cdots$$

such that for every $i \geq 0$

$$\int_0^1 \sum_{j=r(i)+1}^{m(i+1)} \left| c_j^{k_{r(i)}+1} \right| f_j(t)dt \geq \frac{2}{3}, \tag{5.13}$$

$$\forall 0 \leq l \leq i, \quad \int_0^1 \sum_{j=r(i+1)+1}^{\infty} \left| c_j^{k_{r(l)}+1} \right| f_j(t)dt \leq \frac{1}{2^{i+4}}, \tag{5.14}$$

$$\forall 0 \leq l \leq i, \ \forall j \in A_l, \quad \left| c_j^{k_{r(i+1)}+1} \right| < \frac{1}{2^{i+3}} \left| c_j^{k_{r(l)}+1} \right|, \tag{5.15}$$

where A_l is defined by

$$A_l = \{j \in \mathbb{Z} : r(l)+1 \leq j \leq m(l+1), \ \text{ and } \ \left| c_j^{k_{r(l)}+1} \right| > 0\}. \tag{5.16}$$

To see that (5.13), (5.14), (5.15) are possible, proceed inductively as follows. First note that the base case $m(1) \in \mathbb{N}$ can be selected sufficiently large to satisfy (5.13) because of (5.8). Also, $r(1) \in \mathbb{N}$ can be selected sufficiently large to satisfy (5.14) and (5.15) because of (5.11) and Lemma 2.9.

For the general inductive step, suppose that we have selected integers $1 = r(0) < m(1) < r(1) < \cdots < m(i-1) < r(i-1)$ satisfying (5.13), (5.14), and (5.15). By (5.8) we may select $m(i+1) \in \mathbb{N}$ sufficiently large so that (5.13) holds. By (5.11) and Lemma 2.9, we may select $r(i) \in \mathbb{N}$ sufficiently large so that (5.14) and (5.15) both hold.

Step VI. Define

$$f = \sum_{j=1}^{\infty} \frac{1}{j} R_{k_{r(j)}+1}. \tag{5.17}$$

Khinchine's inequality ensures that the sum (5.17) converges in $L^p[0,1]$ and that $f \in L^p[0,1]$. Moreover,

$$\forall n \geq 1, \quad \langle f, g_n \rangle = \sum_{i=1}^{\infty} \frac{1}{i} \langle R_{k_{r(i)}+1}, g_n \rangle = \sum_{i=1}^{\infty} \frac{1}{i} c_n^{k_{r(i)}+1}. \tag{5.18}$$

Note that if $j \geq 1$ and $n \in A_j$, then by (5.15) and (5.18)

$$\begin{aligned} |\langle f, g_n \rangle| &\geq \frac{1}{j} \left| c_n^{k_{r(j)}+1} \right| - \sum_{i=1}^{j-1} \frac{1}{i} \left| c_n^{k_{r(i)}+1} \right| - \sum_{i=j+1}^{\infty} \frac{1}{i} \left| c_n^{k_{r(i)}+1} \right| \\ &\geq \frac{1}{j} \left| c_n^{k_{r(j)}+1} \right| - \sum_{i=1}^{j-1} \frac{1}{i} \left| c_n^{k_{r(i)}+1} \right| - \sum_{i=j+1}^{\infty} \frac{1}{i} \frac{1}{2^{i+2}} \left| c_n^{k_{r(j)}+1} \right| \\ &\geq \frac{7}{8j} \left| c_n^{k_{r(j)}+1} \right| - \sum_{i=1}^{j-1} \frac{1}{i} \left| c_n^{k_{r(i)}+1} \right|. \end{aligned} \tag{5.19}$$

Step VII. In this step we show that if $j \geq 4$, then

$$\int_0^1 \sum_{n=r(j)+1}^{m(j+1)} |\langle f, g_n \rangle| f_n(t) dt \geq \frac{11}{24j}. \tag{5.20}$$

Using (5.19), (5.16), (5.13), (5.14), and the nonnegativity of the $f_n(t)$, one may compute as follows for every $j \geq 4$:

$$\begin{aligned} \int_0^1 \sum_{n=r(j)+1}^{m(j+1)} |\langle f, g_n \rangle| f_n(t) dt &\geq \int_0^1 \sum_{n \in A_j} |\langle f, g_n \rangle| f_n(t) dt \\ &\geq \int_0^1 \sum_{n \in A_j} \left(\frac{7}{8j} \left| c_n^{k_{r(j)}+1} \right| - \sum_{i=1}^{j-1} \frac{1}{i} \left| c_n^{k_{r(i)}+1} \right| \right) f_n(t) dt \\ &\geq \int_0^1 \sum_{n=r(j)+1}^{m(j+1)} \frac{7}{8j} \left| c_n^{k_{r(j)}+1} \right| f_n(t) dt - \int_0^1 \sum_{n=r(j)+1}^{m(j+1)} \sum_{i=1}^{j-1} \frac{1}{i} \left| c_n^{k_{r(i)}+1} \right| f_n(t) dt \\ &\geq \left(\frac{7}{8j} \right) \left(\frac{2}{3} \right) - \sum_{i=1}^{j-1} \int_0^1 \sum_{n=r(j)+1}^{\infty} \frac{1}{i} \left| c_n^{k_{r(i)}+1} \right| f_n(t) dt \\ &\geq \left(\frac{7}{12j} \right) - \sum_{i=1}^{j-1} \frac{1}{i} \left(\frac{1}{2^{j+3}} \right) \\ &\geq \left(\frac{7}{12j} \right) - \frac{1}{8j^2} \sum_{i=1}^{j-1} \frac{1}{i} \geq \left(\frac{7}{12j} \right) - \frac{j}{8j^2} = \frac{11}{24j}. \end{aligned} \tag{5.21}$$

Step VIII. Define the sequence $\mathcal{E} = \{\epsilon_k\}_{k=1}^{\infty} \subset \mathbb{K}$ with $|\epsilon_k| \leq 1$ so that

$$\epsilon_k \langle f, g_k \rangle = |\langle f, g_k \rangle|$$

if $r(j)+1 \le k \le m(j+1)$ for some $j = 0, 1, 2, \cdots$; and $\epsilon_k = 0$ otherwise.

Define

$$f_{\mathcal{E}}(t) = \sum_{k=0}^{\infty} \epsilon_k \langle f, g_k \rangle f_k = \sum_{j=0}^{\infty} \sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k \rangle| f_k(t).$$

Hölder's inequality, the monotone convergence theorem, and (5.21) imply

$$\begin{aligned} \|f_{\mathcal{E}}\|_p \ge \|f_{\mathcal{E}}\|_1 &= \int_0^1 \sum_{j=0}^{\infty} \sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k \rangle| f_k(t) dt \\ &= \sum_{j=0}^{\infty} \int_0^1 \sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k \rangle| f_k(t) dt \\ &\ge \sum_{j=4}^{\infty} \int_0^1 \sum_{k=r(j)+1}^{m(j+1)} |\langle f, g_k \rangle| f_k(t) dt \\ &= \sum_{j=4}^{\infty} \frac{11}{24j} = \infty. \end{aligned} \tag{5.22}$$

Step IX. The monotone convergence theorem and Lemma 2.2 imply

$$\begin{aligned} \|f_{\mathcal{E}}\|_p &= \left\| \sum_{k=0}^{\infty} \epsilon_k \langle f, g_k \rangle f_k \right\|_p \\ &= \lim_{N \to \infty} \left\| \sum_{k=0}^{N} \epsilon_k \langle f, g_k \rangle f_k \right\|_p \\ &= \lim_{N \to \infty} \|S_{\mathcal{E},N}(f)\|_p \\ &\le \left(\sup_{\mathcal{U},N} \|S_{\mathcal{U},N}\|_{p \to p} \right) \|f\|_p < \infty. \end{aligned}$$

This contradicts (5.22) and completes the proof. □

Since frames and unconditional bases are unconditional quasibases, Theorem 5.5 gives an alternate but less direct proof of Corollaries 4.2 and 4.3. Theorem 5.5 shows that there are no pointwise nonnegative unconditional quasibases for $L^p[0,1]$.

6. Markushevich bases and exact systems

In this section we use the Walsh functions to construct an example of a pointwise nonnegative Markushevich basis for $L^p[0,1], 1 \le p < \infty$.

Let $\{w_n\}_{n=1}^{\infty}$ denote the Walsh functions as defined in Example 2.8. Note that the Walsh functions have the following properties:

- $\forall t \in [0,1], \quad w_1(t) = 1,$
- $\forall n \ge 1, \quad w_n(t) \in \{-1, 1\}$ for a.e. $t \in [0,1]$,
- $\forall n \ge 1, \exists t_n > 0$ such that $w_n(t) = 1$ for a.e. $t \in [0, t_n)$.

Recall from Example 2.8 that the Walsh functions are an orthonormal basis for $L^2[0,1]$, a Schauder basis for $L^p[0,1], 1 < p < \infty$, and a Markushevich basis for $L^p[0,1], 1 \le p < \infty$.

The next result generalizes Theorem 2 in [40] from the Fourier setting to the Walsh setting and from $L^2[0,1]$ to $L^p[0,1]$, cf. [24, 25], and provides examples of nonnegative Markushevich bases.

Theorem 6.1 (Markushevich bases)**.** *Fix $1 \le p < \infty$ and let $\frac{1}{p}+\frac{1}{q}=1$. For $n \ge 2$, let*

$$f_n(t) = \frac{1-w_n(t)}{t} = \frac{w_1(t)-w_n(t)}{t} \quad \text{and} \quad g_n(t) = -tw_n(t). \tag{6.1}$$

The system $\{f_n\}_{n=2}^{\infty} \subset L^p[0,1]$ is a Markushevich basis for $L^p[0,1]$ with biorthogonal system $\{g_n\}_{n=2}^{\infty} \subset L^q[0,1]$. Moreover, for each $n \ge 2$, $f_n \ge 0$ is pointwise nonnegative.

Proof. Step I. A direct computation shows that $\{g_n\}_{n=2}^{\infty}$ is biorthogonal to $\{f_n\}_{n=2}^{\infty}$. This uses the fact that $\{w_n\}_{n=1}^{\infty} \subset L^q[0,1]$ is biorthogonal to $\{w_n\}_{n=1}^{\infty} \subset L^p[0,1]$.

Step II. To see that $\{f_n\}_{n=2}^{\infty}$ is complete in $L^p[0,1]$ suppose $h \in L^q[0,1]$ satisfies $\langle f_n, h\rangle = 0$ for all $n \ge 2$. For each $k \ge 1$ define

$$H_k(t) = h(t)\left(\frac{\chi_{(1/2^k,1]}(t)}{t}\right) = h(t)\left(\frac{1-\chi_{[0,1/2^k]}(t)}{t}\right) \in L^q[0,1].$$

It will be convenient to note that $\chi_{[0,1/2^k]}(t)$ has the following finite Walsh expansion which holds for a.e. $t \in [0,1]$:

$$\chi_{[0,1/2^k]}(t) = \sum_{j=1}^{2^k} \langle \chi_{[0,1/2^k]}, w_j\rangle w_j(t) = 2^{-k}\sum_{j=1}^{2^k} w_j(t). \tag{6.2}$$

Also note that the product of two Walsh functions is a Walsh function, namely $w_n w_k = w_{N(n,k)}$ for some integer $N(n,k) \ge 1$; this can be seen directly from (2.10) and (2.11).

Define $f_1 = 0$. If $k \ge 1$ is fixed, then for all $n \ge 1$ we have

$$\begin{aligned}
\langle w_n, H_k\rangle &= \int_0^1 w_n(t)\left(\frac{w_1(t) - 2^{-k}\sum_{j=1}^{2^k} w_j(t)}{t}\right)\overline{h(t)}dt \\
&= 2^{-k}\sum_{j=1}^{2^k}\int_0^1 w_n(t)\left(\frac{1-w_j(t)}{t}\right)\overline{h(t)}dt \\
&= 2^{-k}\sum_{j=1}^{2^k}\int_0^1 \left(\frac{w_n(t)-1+1-w_{N(j,n)}(t)}{t}\right)\overline{h(t)}dt \\
&= 2^{-k}\sum_{j=1}^{2^k}\big(\langle f_{N(j,n)}, h\rangle - \langle f_n, h\rangle\big) = 2^{-k}\sum_{j=1}^{2^k}(0-0) = 0.
\end{aligned}$$

This implies that $H_k = 0$ for all $k \ge 1$, since the Walsh system is a Markushevich basis for $L^p[0,1]$ and in particular is complete; see Example 2.8. By the definition of H_k this implies that $h(t) = 0$ for a.e. $t \in (1/2^k, 1]$. Since this is true for all $k \ge 1$, it follows that $h = 0$.

Step III. To see that $\{g_n\}_{n=2}^{\infty} \subset L^q[0,1]$ is total, suppose that $f \in L^p[0,1]$ and $\langle f, g_n\rangle = 0$ for all $n \ge 2$. Let

$$F(t) = tf(t) - \langle tf, w_1\rangle w_1(t),$$

and note that $F \in L^1[0,1]$ by Hölder's inequality. By construction, $\langle F, w_1\rangle = 0$. Also, for all $n \geq 2$,

$$\langle F, w_n\rangle = -\langle f, g_n\rangle - \langle tf, w_1\rangle\langle w_1, w_n\rangle = 0.$$

Since the Walsh system is a Markushevich basis for $L^1[0,1]$ it is total in $L^\infty[0,1]$, see Example 2.8, and this implies $F = 0$. Thus

$$tf(t) = \langle tf, w_1\rangle w_1(t) = \langle f, t\rangle. \tag{6.3}$$

If $\langle f, t\rangle \neq 0$, then $f(t) = \langle f, t\rangle/t$ contradicts that $f \in L^p[0,1]$. Therefore, we must have $\langle f, t\rangle = 0$ which by (6.3) implies that $tf(t) = 0$ and hence $f = 0$. □

The next result shows that the system in Theorem 6.1 is an example of a Markushevich basis that is not a Schauder basis; cf. Example 5.9 (c) in [21] for a Fourier analog.

Theorem 6.2. *Fix $1 \leq p < \infty$. The Markushevich basis $\{f_n\}_{n=2}^\infty \subset L^p[0,1]$ defined by (6.1) is not a Schauder basis for $L^p[0,1]$.*

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. For $1 < q < \infty$, the dual system $\{g_n\}_{n=1}^\infty \subset L^q[0,1]$ defined by (6.1) satisfies

$$\|g_n\|_q = \left(\int_0^1 |t|^q dt\right)^{1/q} = \left(\frac{1}{q+1}\right)^{1/q}. \tag{6.4}$$

Also note that $\|g_n\|_\infty = 1$ holds and is consistent with (6.4) when $q = \infty$.

By (2.12) we have $w_{2^{m-1}+1} = R_m$. Thus, for all $1 \leq p < \infty$,

$$\begin{aligned} \|f_{2^{m-1}+1}\|_p^p &\geq \int_0^1 \frac{|1 - w_{2^{m-1}+1}(t)|^p}{t} dt = \int_0^1 \frac{|1 - R_m(t)|^p}{t} dt \\ &= \sum_{k=1}^{2^{m-1}} \int_{(2k-1)2^{-m}}^{(2k)2^{-m}} \frac{2^p}{t} dt = 2^p \sum_{k=1}^{2^{m-1}} \ln\left(1 + \frac{1}{(2k-1)}\right). \end{aligned} \tag{6.5}$$

Since $\ln(1+x) \geq x/(e-1)$ holds whenever $0 \leq x \leq (e-1)$, (6.5) yields

$$\|f_{2^{m-1}+1}\|_p^p \geq \frac{2^p}{(e-1)} \sum_{k=1}^{2^{m-1}} \frac{1}{(2k-1)}. \tag{6.6}$$

It follows from (6.4) and (6.6) that

$$\limsup_{m\to\infty} \|f_{2^{m-1}+1}\|_p \|g_{2^{m-1}+1}\|_q = \infty. \tag{6.7}$$

Thus Lemma 2.4 together with (6.7) show that $\{f_n\}_{n=1}^\infty$ is not a Schauder basis for $L^p[0,1]$. □

7. Extensions to $L^p(\mathbb{R})$

Most of our previous results were proven for $L^p[0,1]$, but they easily extend to $L^p(\mathbb{R})$. The following lemma is an immediate consequence of the definition of quasibasis.

Lemma 7.1. *Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\{f_n\}_{n=1}^\infty$ is an unconditional quasibasis for $L^p(\mathbb{R})$ with dual system $\{g_n\}_{n=1}^\infty \subset L^q(\mathbb{R})$, then $\{f_n\chi_{[0,1]}\}_{n=1}^\infty \subset L^p[0,1]$ is an unconditional quasibasis for $L^p[0,1]$ with dual system $\{g_n\chi_{[0,1]}\}_{n=1}^\infty \subset L^q[0,1]$.*

Lemma 7.1 and Theorem 5.5 give the following.

Corollary 7.2. *Fix $1 \le p < \infty$. There does not exist an unconditional quasibasis $\{f_n\}_{n=1}^{\infty}$ for $L^p(\mathbb{R})$ such that each f_n is a.e. nonnegative.*

The next lemma follows easily from the definitions of Markushevich basis and quasibasis.

Lemma 7.3. *Fix $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $\{f_n\}_{n=1}^{\infty} \subset L^p[0,1]$ and $\{g_n\}_{n=1}^{\infty} \subset L^q[0,1]$. Define*

$$F_{m,n}(t) = \chi_{[0,1]}(t-m) f_n(t-m) \quad \text{and} \quad G_{m,n}(t) = \chi_{[0,1]}(t-m) g_n(t-m). \tag{7.1}$$

The following hold:

(1) *If $\{f_n\}_{n=1}^{\infty}$ is a Markushevich basis for $L^p[0,1]$ with biorthogonal system $\{g_n\}_{n=1}^{\infty} \subset L^q[0,1]$, then the system $\{F_{m,n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is a Markushevich basis for $L^p(\mathbb{R})$ with biorthogonal system $\{G_{m,n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \subset L^q(\mathbb{R})$.*

(2) *If $\{f_n\}_{n=1}^{\infty}$ is a quasibasis for $L^p[0,1]$ with dual system $\{g_n\}_{n=1}^{\infty} \subset L^q[0,1]$, then a suitable enumeration of the system $\{F_{m,n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ is a quasibasis for $L^p(\mathbb{R})$ with a correspondingly enumerated dual system $\{G_{m,n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \subset L^q(\mathbb{R})$.*

Lemma 7.3, Theorem 6.1 and Corollary 5.4 give the following.

Corollary 7.4. *Fix $1 \le p < \infty$. If $\{f_n\}_{n=1}^{\infty} \subset L^p[0,1]$ is the Markushevich basis defined by (6.1), then the system $\{F_{m,n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \subset L^p(\mathbb{R})$ defined by (7.1) is a nonnegative Markushevich basis for $L^p(\mathbb{R})$.*

Corollary 7.5. *Fix $1 \le p < \infty$. If $\{f_n\}_{n=1}^{\infty} \subset L^p[0,1]$ is the conditional quasibasis given by enumerating the dyadic step functions in Corollary 5.4, then the suitably enumerated system $\{F_{m,n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \subset L^p(\mathbb{R})$ defined by (7.1) is a nonnegative conditional quasibasis for $L^p(\mathbb{R})$.*

8. Hamel bases

Let V be a linear space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A set $B \subset V$ is a *Hamel basis* for V if every $v \in V$ can be expressed as a finite linear combination of elements in B. In other words, $B \subset V$ is a Hamel basis if

$$\forall v \in V,\ \exists N \in \mathbb{N},\ b_1, \cdots, b_N \in B,\ c_1, \cdots c_N \in \mathbb{K} \quad \text{such that} \quad v = c_1 b_1 + \cdots + c_N b_N.$$

In contrast to the spanning systems discussed in Section 2, Hamel bases for infinite dimensional Banach spaces are uncountable.

Given a measurable function f let $f_R = \mathrm{Re}(f)$ and $f_I = \mathrm{Im}(f)$ denote the real and imaginary parts of f, let $f_R^+(t) = \max\{f_R(t), 0\}$ and $f_R^-(t) = \max\{-f_R(t), 0\}$ denote the positive and negative parts of f_R, and let f_I^+ and f_I^- denote the positive and negative parts of f_I. In particular, $f = f_R^+ - f_R^- + i f_I^+ - i f_I^-$.

Theorem 8.1. *Let V be a linear space of measurable functions with the property that if $f \in V$, then $f_R^+, f_R^-, f_I^+, f_I^- \in V$. There exists a Hamel basis $B \subset V$ for V such that each $b \in B$ is a.e. nonnegative.*

Proof. The proof follows the usual construction of Hamel bases.

Let $\mathcal{F}$ be the family of all subsets of V with the property that if $F \in \mathcal{F}$, then (i) the elements of F are finitely linearly independent, and (ii) if $f \in F$, then f is

a.e. nonnegative. Consider the partial order on $\mathcal{F}$ given by set inclusion. It can be shown that every chain in $\mathcal{F}$ has an upper bound (take the union of elements in the chain). Therefore, by Zorn's lemma, $\mathcal{F}$ contains a maximal element $B \in \mathcal{F}$.

It follows from the maximality of B that if $g \in V$ is a.e. nonnegative, then g must lie in the finite linear span of B (otherwise maximality would be contradicted by $B \cup \{g\}$). Thus, if $f \in V$, then each of the nonnegative functions $f_R^+, f_R^-, f_I^+, f_I^-$ is in the finite linear span of B. Since V is linear, it follows that B is the desired Hamel basis for V. □

Corollary 8.2. *Fix $0 < p \leq \infty$ and measurable $S \subset \mathbb{R}^d$. There exists a Hamel basis B for the linear space $L^p(S)$ for which each $b \in B$ is a.e. nonnegative.*

9. Questions

Theorem 5.5 shows that nonnegative unconditional quasibases do not exist in $L^p[0,1]$. On the other hand, Example 5.1 and Theorem 6.1 show that it is possible for a nonnegative system in $L^p[0,1]$ to be a quasibasis or Markushevich basis in $L^p[0,1]$. However, it is not clear whether a nonnegative system in $L^p[0,1]$ can be a Schauder basis, i.e., simultaneously a quasibasis and a Markushevich basis.

Question 9.1. Given $1 \leq p < \infty$, does there exist a Schauder basis $\{f_n\}_{n=1}^\infty$ for $L^p[0,1]$ such that each f_n is a.e. nonnegative?

For perspective, note that nonnegative Schauder bases do exist in other function spaces. For example, the Schauder system is a nonnegative Schauder basis for $C[0,1]$, see Section 4.1 in [21], and the system $\{\chi_{[0,1]}(t-n) : n \in \mathbb{Z}\}$ is a nonnegative orthonormal basis for its closed linear span in $L^2(\mathbb{R})$.

Question 9.2. Which function spaces admit pointwise nonnegative Schauder bases?

Example 3.3 shows that the approach used to prove Theorem 3.2 on monotone bases does not extend to $L^1[0,1]$. Moreover, unlike the case $1 < p < \infty$, Corollary 4.2 does not help address Theorem 3.2 for $L^1[0,1]$ since there are no unconditional bases for $L^1[0,1]$.

Question 9.3. Does there exist a monotone basis $\{f_n\}_{n=1}^\infty$ for $L^1[0,1]$ such that each f_n is a.e. nonnegative?

It would be interesting to determine if examples of nonnegative quasibases and Markushevic bases $\{f_n\}_{n=1}^\infty$ for $L^p[0,1]$ (see Theorems 5.2 and 6.1, and Example 5.3) can be endowed with additional control between $\|f\|_p$ and ℓ^r norms of the coefficient sequence $\{\langle f, f_n\rangle\}_{n=1}^\infty$. See [29] for related work involving Bessel (C_q)-systems in the context of Gabor analysis, and see [19,33] for related work on coefficient control in the setting of Schauder bases.

Question 9.4. Given $1 < s \leq 2 \leq t < \infty$, does there exist $A, B > 0$ and a quasibasis or Markushevich basis $\{f_n\}_{n=1}^\infty$ for $L^2[0,1]$ such that each f_n is a.e. nonnegative and

$$\forall f \in L^2[0,1], \quad A\left(\sum_{n=1}^\infty |\langle f, f_n\rangle|^t\right)^{1/t} \leq \|f\|_2 \leq B\left(\sum_{n=1}^\infty |\langle f, f_n\rangle|^s\right)^{1/s} ?$$

Recall that the case $s = t = 2$ is not possible by Corollary 4.3.

Note added in proof. With regard to Question 9.1, Bill Johnson and Gideon Schechtman have recently constructed an example of a nonnegative Schauder basis for L^1, but the question remains open for general L^p. With regard to Question 9.3, it can be shown using results on contractive projections that there are no nonnegative monotone Schauder bases for L^1.

10. Appendix

In this Appendix we briefly sketch proofs of background lemmas on quasibases stated in Section 2.2. As noted earlier, the proofs of Lemma 2.1, Lemma 2.2 and Lemma 2.3 follow familiar methods; we briefly outline the details here for the sake of completeness.

Sketch of the proof of Lemma 2.1. The implication (1) $\implies$ (2) follows from the definition of quasibasis and from the uniform boundedness principle since the convergence of the sequence $\{S_N(f)\}_{N=1}^\infty$ for all $f \in X$ implies that $\sup_N \|S_N\|_{X\to X} < \infty$.

For the implication (2) $\implies$ (1), let $\mathcal{C} = \sup_N \|S_N\|_{X\to X} < \infty$. Given $f \in X$, let $\epsilon > 0$ be arbitrary and take $g \in D$ such that $\|f - g\|_X < \epsilon$. Then

$$\|f - S_N(f)\|_X \le \|f - g\|_X + \|g - S_N(g)\|_X + \|S_N(g - f)\|_X.$$

This implies that $\limsup_{N\to\infty} \|f - S_N(f)\|_X \le \epsilon(1+\mathcal{C})$, and since $\epsilon > 0$ is arbitrary, this completes the proof. □

Sketch of the proof of Lemma 2.2. Since the quasibasis expansion

$$f = \sum_{n=1}^\infty \langle f, g_n\rangle f_n$$

converges unconditionally for each $f \in X$, Theorem 3.15 in [21] implies that for each $f \in X$, $\sup_{\mathcal{U},N} \|S_{\mathcal{U},N}(f)\|_X < \infty$. The proof now follows from the uniform boundedness principle. □

Sketch of the proof of Lemma 2.3. *Step* I. We first show that $\{\varphi_n\}_{n=1}^\infty$ is quasibasis for X. Define the operator $S : X \to X$ by $S(f) = \sum_{n=1}^\infty \langle f, g_n\rangle(f_n - \varphi_n)$. The assumption (2.5) guarantees that the partial sums defining $S(f)$ are Cauchy and hence converge. Moreover, (2.5) implies that for all $f \in X$, $\|S(f)\|_X \le (\epsilon/2)\|f\|_X$, and hence $\|S\|_{X\to X} \le \epsilon/2 < 1$.

Define $T = I - S$, where $I : X \to X$ is the identity operator. Exercise 2.40 in [21] shows that $T : X \to X$ is bounded, one-to-one, onto, and has bounded inverse. It can be verified that for all $f \in X$, $T(f) = \sum_{n=1}^\infty \langle f, g_n\rangle\varphi_n$. Thus, we have

$$\forall f \in X, \quad f = T(T^{-1}(f)) = \sum_{n=1}^\infty \langle T^{-1}(f), g_n\rangle\varphi_n = \sum_{n=1}^\infty \langle f, (T^{-1})^*(g_n)\rangle\varphi_n.$$

In other words, $\{\varphi_n\}_{n=1}^\infty$ is a quasibasis for X with dual system $\{(T^{-1})^*(g_n)\}_{n=1}^\infty$.

Step II. We next show that if $\{f_n\}_{n=1}^\infty$ is an unconditional quasibasis, then the quasibasis $\{\varphi_n\}_{n=1}^\infty$ is unconditional. We need to show that for any permutation σ of $\mathbb{N}$, the following series converges:

$$\lim_{N\to\infty} \sum_{n=1}^N \langle f, (T^{-1})^*(g_{\sigma(n)})\rangle\varphi_{\sigma(n)} = \lim_{N\to\infty} \sum_{n=1}^N \langle T^{-1}(f), g_{\sigma(n)}\rangle\varphi_{\sigma(n)}. \tag{10.1}$$

In particular, it suffices to show that the partial sums in (10.1) are Cauchy.

Since $\{f_n\}_{n=1}^{\infty}$ is an unconditional quasibasis with dual $\{g_n\}_{n=1}^{\infty}$, the following series converges:

$$T^{-1}(f) = \lim_{N\to\infty} \sum_{n=1}^{N} \langle T^{-1}(f), g_{\sigma(n)}\rangle f_{\sigma(n)} = \sum_{n=1}^{\infty} \langle T^{-1}(f), g_{\sigma(n)}\rangle f_{\sigma(n)}, \tag{10.2}$$

and so the partial sums in (10.2) are Cauchy. The assumption (2.5) implies that

$$\left\| \sum_{n=M}^{N} \langle T^{-1}(f), g_{\sigma(n)}\rangle (f_{\sigma(n)} - \varphi_{\sigma(n)}) \right\|_X \leq \epsilon \|T^{-1}(f)\|_X \sum_{n=M}^{N} \frac{1}{2^{\sigma(n)+1}}. \tag{10.3}$$

We may conclude from (10.2) and (10.3) that the sequence of partial sums (10.1) is Cauchy as required since

$$\begin{aligned} \left\| \sum_{n=M}^{N} \langle T^{-1}(f), g_{\sigma(n)}\rangle \varphi_{\sigma(n)} \right\|_X &\leq \left\| \sum_{n=M}^{N} \langle T^{-1}(f), g_{\sigma(n)}\rangle f_{\sigma(n)} \right\|_X \\ &\quad + \epsilon \|T^{-1}(f)\|_X \sum_{n=M}^{N} \frac{1}{2^{\sigma(n)+1}}. \end{aligned}$$

□

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References

[1] Dale Alspach and Edward Odell, *L_p spaces*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 123–159, DOI 10.1016/S1874-5849(01)80005-X. MR1863691 (2003b:46013)

[2] John J. Benedetto, Christopher Heil, and David F. Walnut, *Differentiation and the Balian-Low theorem*, J. Fourier Anal. Appl. **1** (1995), no. 4, 355–402, DOI 10.1007/s00041-001-4016-5. MR1350699 (96f:42002)

[3] John J. Benedetto, Alexander M. Powell, and Özgür Yılmaz, *Sigma-Delta ($\Sigma\Delta$) quantization and finite frames*, IEEE Trans. Inform. Theory **52** (2006), no. 5, 1990–2005, DOI 10.1109/TIT.2006.872849. MR2234460 (2007a:94030)

[4] Bernhard G. Bodmann and Nathaniel Hammen, *Stable phase retrieval with low-redundancy frames*, Adv. Comput. Math. **41** (2015), no. 2, 317–331, DOI 10.1007/s10444-014-9359-y. MR3337494

[5] J. Bourgain, *A remark on the uncertainty principle for Hilbertian basis*, J. Funct. Anal. **79** (1988), no. 1, 136–143, DOI 10.1016/0022-1236(88)90033-X. MR950087 (89f:81025)

[6] J. S. Byrnes, W. Moran, and B. Saffari, *Smooth PONS*, J. Fourier Anal. Appl. **6** (2000), no. 6, 663–674, DOI 10.1007/BF02510701. MR1790249 (2001k:42011)

[7] Jian-Feng Cai, Bin Dong, Stanley Osher, and Zuowei Shen, *Image restoration: total variation, wavelet frames, and beyond*, J. Amer. Math. Soc. **25** (2012), no. 4, 1033–1089, DOI 10.1090/S0894-0347-2012-00740-1. MR2947945

[8] N. L. Carothers, *A short course on Banach space theory*, London Mathematical Society Student Texts, vol. 64, Cambridge University Press, Cambridge, 2005. MR2124948 (2005k:46001)

[9] Peter G. Casazza, *The art of frame theory*, Taiwanese J. Math. **4** (2000), no. 2, 129–201. MR1757401 (2001f:42046)

[10] Peter G. Casazza, Deguang Han, and David R. Larson, *Frames for Banach spaces*, The functional and harmonic analysis of wavelets and frames (San Antonio, TX, 1999), Contemp. Math., vol. 247, Amer. Math. Soc., Providence, RI, 1999, pp. 149–182, DOI 10.1090/conm/247/03801. MR1738089 (2000m:46015)

[11] Wojciech Czaja and Alexander M. Powell, *Recent developments in the Balian-Low theorem*, Harmonic analysis and applications, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2006, pp. 79–100, DOI 10.1007/0-8176-4504-7_5. MR2249306 (2007d:42045)

[12] I. Daubechies and A. J. E. M. Janssen, *Two theorems on lattice expansions*, IEEE Trans. Inform. Theory **39** (1993), no. 1, 3–6, DOI 10.1109/18.179336. MR1211486 (94f:94001)

[13] L. E. Dor and E. Odell, *Monotone bases in L_p*, Pacific J. Math. **60** (1975), no. 2, 51–61. MR0399832 (53 #3674)

[14] D. Freeman, E. Odell, Th. Schlumprecht, and A. Zsák. Unconditional structures of translates for $L_p(\mathbb{R}^d)$, preprint (2012), available at http://arxiv.org/pdf/1209.4619.pdf

[15] Bernard R. Gelbaum, *Notes on Banach spaces and bases*, An. Acad. Brasil. Ci. **30** (1958), 29–36. MR0098974 (20 #5419)

[16] Vivek K. Goyal, Jelena Kovačević, and Jonathan A. Kelner, *Quantized frame expansions with erasures*, Appl. Comput. Harmon. Anal. **10** (2001), no. 3, 203–233, DOI 10.1006/acha.2000.0340. MR1829801 (2002h:94012)

[17] V. K. Goyal, J. Kovačević, and M. Vetterli, *Quantized frame expansions as source-channel codes for erasure channels*, in Proc. IEEE Data Compression Conference, 1999, 325–355.

[18] Karlheinz Gröchenig and Eugenia Malinnikova, *Phase space localization of Riesz bases for $L^2(\mathbb{R}^d)$*, Rev. Mat. Iberoam. **29** (2013), no. 1, 115–134, DOI 10.4171/RMI/715. MR3010124

[19] V. I. Gurariĭ and N. I. Gurariĭ, *Bases in uniformly convex and uniformly smooth Banach spaces* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **35** (1971), 210–215. MR0283549 (44 #780)

[20] Deguang Han and David R. Larson, *Frames, bases and group representations*, Mem. Amer. Math. Soc. **147** (2000), no. 697, x+94, DOI 10.1090/memo/0697. MR1686653 (2001a:47013)

[21] Christopher Heil, *A basis theory primer*, Expanded edition, Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2011. MR2744776 (2012b:46022)

[22] Christopher Heil, Yoo Young Koo, and Jae Kun Lim, *Duals of frame sequences*, Acta Appl. Math. **107** (2009), no. 1-3, 75–90, DOI 10.1007/s10440-008-9410-4. MR2520010 (2010c:42065)

[23] Robert C. James, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292. MR0021241 (9,42c)

[24] K. S. Kazarjan, *The multiplicative completion of certain systems* (Russian, with English and Armenian summaries), Izv. Akad. Nauk Armyan. SSR Ser. Mat. **13** (1978), no. 4, 315–351, 353. MR526648 (80j:42038)

[25] K. S. Kazarian, *On bases and unconditional bases in the spaces $L^p(\mu)$, $1 \le p < \infty$*, Studia Math. **71** (1981/82), no. 3, 227–249. MR667313 (84d:42037)

[26] K. S. Kazarian and Robert E. Zink, *Some ramifications of a theorem of Boas and Pollard concerning the completion of a set of functions in L^2*, Trans. Amer. Math. Soc. **349** (1997), no. 11, 4367–4383, DOI 10.1090/S0002-9947-97-02034-5. MR1443881 (99d:42050)

[27] K. S. Kazarian and Robert E. Zink, *Subsystems of the Schauder system that are quasibases for $L^p[0,1]$, $1 \le p < +\infty$*, Proc. Amer. Math. Soc. **126** (1998), no. 10, 2883–2893, DOI 10.1090/S0002-9939-98-04388-3. MR1452807 (99a:46050)

[28] Rui Liu and Bentuo Zheng, *A characterization of Schauder frames which are near-Schauder bases*, J. Fourier Anal. Appl. **16** (2010), no. 5, 791–803, DOI 10.1007/s00041-010-9126-5. MR2673710 (2011h:46023)

[29] Shahaf Nitzan and Jan-Fredrik Olsen, *From exact systems to Riesz bases in the Balian-Low theorem*, J. Fourier Anal. Appl. **17** (2011), no. 4, 567–603, DOI 10.1007/s00041-010-9150-5. MR2819167 (2012e:42065)

[30] E. Odell, B. Sari, Th. Schlumprecht, and B. Zheng, *Systems formed by translates of one element in $L_p(\mathbb{R})$*, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6505–6529, DOI 10.1090/S0002-9947-2011-05305-5. MR2833566 (2012g:42063)

[31] R. E. A. C. Paley, *A remarkable series of orthogonal functions I*, Proceedings of the London Mathematical Society, 34 (1931), no. 1, 241–264.

[32] Alexander M. Powell, Rayan Saab, and Özgür Yılmaz, *Quantization and finite frames*, Finite frames, Appl. Numer. Harmon. Anal., Birkhäuser/Springer, New York, 2013, pp. 267–302, DOI 10.1007/978-0-8176-8373-3_8. MR2964013

[33] William H. Ruckle, *The extent of the sequence space associated with a basis*, Canad. J. Math. **24** (1972), 636–641. MR0300057 (45 #9105)

[34] F. Schipp, W. R. Wade, and P. Simon, *Walsh series*, Adam Hilger, Ltd., Bristol, 1990. An introduction to dyadic harmonic analysis; With the collaboration of J. Pál. MR1117682 (92g:42001)

[35] Ivan Singer, *Bases in Banach spaces. I*, Springer-Verlag, New York-Berlin, 1970. Die Grundlehren der mathematischen Wissenschaften, Band 154. MR0298399 (45 #7451)

[36] Ivan Singer, *On biorthogonal systems and total sequences of functionals*, Math. Ann. **193** (1971), 183–188. MR0350387 (50 #2880)

[37] A. H. Spaeth, *A determination of the existence of various types of positive systems in L^p*, Ph.D. Thesis, Vanderbilt University, Department of Mathematics, 2013.

[38] A. N. Slepchenko, *On some generalizations of bases in Banach spaces*, Mathematics of the USSR Sbornik, 49 (1984), no. 1, 269–281.

[39] P. L. Ul′janov, *Representation of functions by series, and the classes $\phi(L)$* (Russian), Uspehi Mat. Nauk **27** (1972), no. 2(164), 3–52. MR0399749 (53 #3591)

[40] Gang Joon Yoon and Christopher Heil, *Duals of weighted exponential systems*, Acta Appl. Math. **119** (2012), 97–112, DOI 10.1007/s10440-011-9663-1. MR2915572

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