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ON POISSON OPERATORS AND DIRICHLET-NEUMANN MAPS IN H^s FOR DIVERGENCE FORM ELLIPTIC OPERATORS WITH LIPSCHITZ COEFFICIENTS

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ABSTRACT. We consider second order uniformly elliptic operators of divergence form in $\mathbb{R}^{d+1}$ whose coefficients are independent of one variable. Under the Lipschitz condition on the coefficients we characterize the domain of the Poisson operators and the Dirichlet-Neumann maps in the Sobolev space $H^s(\mathbb{R}^d)$ for each $s \in [0,1]$. Moreover, we also show a factorization formula for the elliptic operator in terms of the Poisson operator.

1. INTRODUCTION

In this paper we consider the second order elliptic operator of divergence form in $\mathbb{R}^{d+1} = \{(x,t) \in \mathbb{R}^d \times \mathbb{R}\}$,

$$\mathcal{A} = -\nabla \cdot A\nabla, \qquad A = A(x) = \big(a_{i,j}(x)\big)_{1\le i,j\le d+1}\ . \tag{1.1}$$

Here $d \in \mathbb{N}$, $\nabla = (\nabla_x, \partial_t)^\top$ with $\nabla_x = (\partial_1, \cdots, \partial_d)^\top$, and each $a_{i,j}$ is complex-valued and assumed to be t-independent. The adjoint matrix of A will be denoted by A^*. We assume the uniform ellipticity condition

$$\mathrm{Re}\langle A(x)\eta, \eta\rangle \ge \nu_1|\eta|^2, \qquad |\langle A(x)\eta, \zeta\rangle| \le \nu_2|\eta||\zeta| \tag{1.2}$$

for all $\eta, \zeta \in \mathbb{C}^{d+1}$ with positive constants ν_1, ν_2. Here $\langle\cdot,\cdot\rangle$ denotes the inner product of $\mathbb{C}^{d+1}$, i.e., $\langle \eta, \zeta\rangle = \sum_{j=1}^{d+1} \eta_j \bar{\zeta}_j$ for $\eta, \zeta \in \mathbb{C}^{d+1}$. For later use we set

$$A' = (a_{i,j})_{1\le i,j\le d}, \qquad b = a_{d+1,d+1},$$
$$\mathbf{r_1} = (a_{1,d+1}, \cdots, a_{d,d+1})^\top, \qquad \mathbf{r_2} = (a_{d+1,1}, \cdots, a_{d+1,d})^\top.$$

We will also use the notation $\mathcal{A}' = -\nabla_x \cdot A'\nabla_x$. In this paper, we are concerned with the Poisson operator and the Dirichlet-Neumann map associated with $\mathcal{A}$, which play fundamental roles in the boundary value problems for the elliptic operators. They are defined through $\mathcal{A}$-extension of the boundary data on $\mathbb{R}^d = \partial\mathbb{R}^{d+1}_+$ to the upper half space.

Definition 1.1. (i) For a given $h \in \mathcal{S}'(\mathbb{R}^d)$ we denote by $M_h : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ the multiplication $M_h u = hu$.

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(ii) We denote by $E_{\mathcal{A}} : \dot{H}^{1/2}(\mathbb{R}^d) \to \dot{H}^1(\mathbb{R}^{d+1}_+)$ the $\mathcal{A}$-extension operator; i.e., $w = E_{\mathcal{A}} f$ is the solution to the Dirichlet problem

$$\begin{cases} \mathcal{A}u = 0 \text{ in } \mathbb{R}^{d+1}_+, \\ \quad u = f \text{ on } \partial\mathbb{R}^{d+1}_+ = \mathbb{R}^d. \end{cases} \tag{1.3}$$

The one parameter family of linear operators $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$, defined by $E_{\mathcal{A}}(t)f = (E_{\mathcal{A}}f)(\cdot,t)$ for $f \in \dot{H}^{1/2}(\mathbb{R}^d)$, is called the Poisson semigroup associated with $\mathcal{A}$.

(iii) We denote by $\Lambda_{\mathcal{A}} : D_{L^2}(\Lambda_{\mathcal{A}}) \subset \dot{H}^{1/2}(\mathbb{R}^d) \to \dot{H}^{-1/2}(\mathbb{R}^d)$ the Dirichlet-Neumann map associated with $\mathcal{A}$, which is defined through the sesquilinear form

$$\langle \Lambda_{\mathcal{A}} f, g\rangle_{\dot{H}^{-\frac{1}{2}},\dot{H}^{\frac{1}{2}}} = \langle A\nabla E_{\mathcal{A}} f, \nabla E_{\mathcal{A}} g\rangle_{L^2(\mathbb{R}^{d+1}_+)}, \qquad f,g \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^d). \tag{1.4}$$

Here $\langle\cdot,\cdot\rangle_{\dot{H}^{-1/2},\dot{H}^{1/2}}$ denotes the duality coupling of $\dot{H}^{-1/2}(\mathbb{R}^d)$ and $\dot{H}^{1/2}(\mathbb{R}^d)$.

Here $\dot{H}^s(\mathbb{R}^d)$ is the homogeneous Sobolev space of order $s \in \mathbb{R}$, and $D_H(T)$ denotes the domain of a linear operator T in a Banach space H. Since the ellipticity condition (1.2) ensures that $E_{\mathcal{A}}$ is well-defined in $\dot{H}^{1/2}(\mathbb{R}^d)$ via the Lax-Milgram theorem, it is not difficult to see that $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$ is realized as a strongly continuous and analytic semigroup in $\dot{H}^{1/2}(\mathbb{R}^d)$ and in $H^{1/2}(\mathbb{R}^d)$ (see, e.g., [28, Proposition 2.4]). Then the generator of the Poisson semigroup will be denoted by $-\mathcal{P}_{\mathcal{A}}$, and $\mathcal{P}_{\mathcal{A}}$ is called the *Poisson operator* (associated with $\mathcal{A}$). As for the Dirichlet-Neumann map $\Lambda_{\mathcal{A}}$, it is well known from the theory of sesquilinear forms that (1.2) guarantees the generation of a strongly continuous and analytic semigroup in $L^2(\mathbb{R}^d)$; see [21]. On the other hand, the realization of the Poisson semigroup in $L^2(\mathbb{R}^d)$ is nothing but the solvability of the elliptic boundary value problem (1.3) for L^2 boundary data (see [28] for details); there have been many works on this subject by now. Moreover, as is explained in the book [22], the characterizations of $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ and $D_{L^2}(\Lambda_{\mathcal{A}})$ essentially correspond to the regularity problem and the Neumann problem for the t-independent elliptic operator $\mathcal{A}$ in $\mathbb{R}^{d+1}_+$. There is also a lot of literature on this elliptic boundary value problem. Before stating our results, let us recall literature on this problem in terms of the class of the matrix A.

As far as the authors know, these problems are affirmatively settled at least for the following classes of A.

(I) A is a constant matrix, i.e., $A(x) = A$;

(II) A is Hermite, i.e., $A^* = A$;

(III) A is block type, i.e., $\mathbf{r_1} = \mathbf{r_2} = 0$;

(IV) A is a small L^∞ perturbation of B satisfying one of (I)-(III) above.

Case (I) is easy since one can directly derive the solution formula for (1.3) with the aid of the Fourier transform. Case (II) is a classical problem, for it is closely related to the Laplace equation in Lipschitz domains. The Dirichlet problem (1.3) is solved by [9] when A is in a special class called the Jacobian type, and the Neumann problem is solved by [19] in this case. The reader is also referred to [37], where the Dirichlet and Neumann problems in Lipschitz domains are studied by using the layer potential, and to [10] for the characterization of $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ when A is Jacobian type. The case of general real symmetric matrices is solved by [4, 20] for the Dirichlet problem and by [4, 23] for the Neumann problem. The results of [5, 16] include the case of general Hermite matrices. Case (III) is considered in [4, 6]. In this case, the Poisson operator essentially coincides with the Dirichlet-Neumann map, and the characterization $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ is known as the Kato square root problem for divergence form elliptic operators, which is settled by [6]; see also [8]. Recently, in [7] it is generalized for the block triangular cases, i.e., the cases when $\mathbf{r_1} = 0$ or $\mathbf{r_2} = 0$, where the regularity problem or the Neumann problem is solved respectively. Case (IV) is solved in [13] when B is a constant matrix and in [2, 4, 5, 18] when B is a Hermite, or block matrix. In [28], the authors of the present paper showed the L^2 solvablity of (1.3) and verified the characterization $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$, when $\mathbf{r_1} + \mathbf{r_2}$ and b are real, and each $\nabla_x \cdot \mathbf{r_i}$ ($\mathbf{i} = 1, 2$) belongs to $L^d(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$. We note that in the cases (II)-(IV) the coefficients of A are not discontinuous in general. However, it is shown in [24] that if one imposes only (1.2) and the coefficients are discontinuous, the Dirichlet problem (1.3) is not always solvable for boundary data in $L^2(\mathbb{R}^d)$. This means that some additional conditions on A such as (I)-(IV) are required in order to extend the Poisson semigroup in $H^{1/2}(\mathbb{R}^d)$ as a semigroup in $L^2(\mathbb{R}^d)$.

Most of the works discussed above concern the solvability of the boundary value problems in $L^2(\mathbb{R}^d)$ and the uniform (in the t variable) $H^1(\mathbb{R}^d)$ estimate of the solution for the boundary value problems in $\mathbb{R}^{d+1}_+$. On the other hand, even at the cost of strong regularity conditions on the matrix A, we often need the uniform estimates in higher order Sobolev spaces as well as a robust solvability result without any structural condition on A. For example, these results will be useful when one first wants to consider the approximation of the equations with singular coefficients by smooth ones.

With this motivation we assume in this paper the Lipschitz regularity for the matrix A:

$$\mathrm{Lip}(A) = \sum_{i,j} \sup_{x,y\in\mathbb{R}^d} \frac{|a_{i,j}(x) - a_{i,j}(y)|}{|x-y|} < \infty. \tag{1.5}$$

Note that the Lipschitz regularity is a natural condition so that $D_{L^2(\mathbb{R}^{d+1})}(\mathcal{A})$ is $H^2(\mathbb{R}^{d+1})$ for any $d \geq 1$. Our aim is to give the realization of the Poisson semigroup and the characterization of the domain of its generator in $H^s(\mathbb{R}^d)$ for all $s \in [0, 1]$, not only in $L^2(\mathbb{R}^d)$. The precise statement of the result is given as follows:

Theorem 1.2. *Let $A = A(x)$ be a t-independent complex coefficient matrix satisfying* (1.2) *and* (1.5)*. Then the following statements hold:*

(i) *The Poisson semigroup in $H^{1/2}(\mathbb{R}^d)$ is extended as a strongly continuous and analytic semigroup in $L^2(\mathbb{R}^d)$. Moreover, its generator $-\mathcal{P}_{\mathcal{A}}$ satisfies $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ with equivalent norms, and $I + \mathcal{P}_{\mathcal{A}}$ admits a bounded $H^\infty(\Sigma_\varphi)$-calculus in $L^2(\mathbb{R}^d)$ for some $\varphi \in (0, \frac{\pi}{2})$.*

(ii) *Let $s \in [0, 1]$. Then $H^s(\mathbb{R}^d)$ is invariant under the action of the Poisson semigroup $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ in $L^2(\mathbb{R}^d)$, and its restriction on $H^s(\mathbb{R}^d)$ defines a strongly continuous and analytic semigroup in $H^s(\mathbb{R}^d)$. Moreover, its generator, denoted again by $-\mathcal{P}_{\mathcal{A}}$, satisfies $D_{H^s}(\mathcal{P}_{\mathcal{A}}) = H^{1+s}(\mathbb{R}^d)$ with equivalent norms.*

Here $\Sigma_\varphi = \{z \in \mathbb{C} \mid z \neq 0, |\arg z| < \varphi\}$, $\varphi \in (0, \pi)$, is an open sector in $\mathbb{C}$ with angle φ. For the reader's convenience, we state the definition of a bounded $H^\infty(\Sigma_\varphi)$-calculus for sectorial operators in Appendix A.3. For more details, see, e.g., [15, Chapter 5].

Remark 1.3. In Theorem 1.2 we do not need any structural conditions such as (I)-(IV). As for the literature in this direction, we refer to [11], where the Dirichlet problem is solved for the L^2 boundary data in a bounded Lipschitz domain with a small Lipschitz constant under the smallness condition on the Carleson norm of $\sup\{\delta(X)|\nabla A(X)|^2 : |X-Z| < \delta(Z)/2\}$. Here X is a point of the domain and $\delta(X)$ is its distance from the boundary. (In fact, they also obtained L^p solvability when the Carleson norm is just finite. See also related results in the references therein.) Furthermore, in [32] the solution of the Dirichlet problem in a bounded Lipschitz domain is constructed for the boundary value in the Sobolev and Besov spaces, under the Carleson-type conditions on the domain and the matrix A. In view of local regularity, the Lipschitz condition (1.5) assumed in our theorem is rather strong. However, we would mention two differences between these results and ours. Firstly our solution obtained here has stronger uniform (or boundary) estimates in higher order Sobolev spaces than theirs. Such uniform estimates are represented in terms of the mapping property of $\mathcal{P}_{\mathcal{A}}$ in $H^s(\mathbb{R}^d)$, $s \in [0,1]$. In particular, the result with $s = 1$ and the expansion (1.7) below are crucial in obtaining a standard elliptic estimate for the Stokes problem in a domain with noncompact boundary [30].

Secondly we should point out the lack of the compactness of the boundary in our case. In fact, the authors in [11, 32] apply a localization argument which enables them to approximate $A(x)$ by a constant matrix in each localized domain. Then the boundary value problem is reduced to a finite sum of the problems for small VMO perturbation of the constant matrices. However, in our case one cannot use such a localization procedure, since $A(x)$ is not necessarily close to a constant matrix as $|x| \to \infty$. This difficulty is overcome with the aid of the calculus of the symbol (1.6).

In contrast to approaches taken in the aforementioned results, we analyze $\mathcal{P}_{\mathcal{A}}$ by looking at its principal symbol, which is explicitly calculated as

$$\mu_{\mathcal{A}}(x,\xi) = -\frac{\mathbf{v}(x)\cdot\xi}{2} + i\Big\{\frac{1}{b(x)}\langle A'(x)\xi,\xi\rangle - \frac{1}{4}(\mathbf{v}(x)\cdot\xi)^2\Big\}^{\frac{1}{2}}, \ \mathbf{v} = \frac{\mathbf{r_1}+\mathbf{r_2}}{b}. \tag{1.6}$$

Here $x, \xi \in \mathbb{R}^d$. As expected, the associated pseudo-differential operator $-i\mu_{\mathcal{A}}(\cdot, D_x)$ is shown to be an approximation of $\mathcal{P}_{\mathcal{A}}$. More precisely, we will establish the expansion

$$-\mathcal{P}_{\mathcal{A}} = i\mu_{\mathcal{A}}(\cdot, D_x) + S_{\mathcal{A},1}, \tag{1.7}$$

where $S_{\mathcal{A},1}$ is a bounded operator in $H^s(\mathbb{R}^d)$ for $s \in (0,1)$ and is also bounded from $H^{j+\epsilon}(\mathbb{R}^d)$ to $H^j(\mathbb{R}^d)$ for $j = 0, 1$ and $\epsilon > 0$; see Remark 3.7, Proposition 3.11, and Theorem A.2 for details. Since $\mu_{\mathcal{A}}$ is Lipschitz in x and homogeneous of degree 1 in ξ, one may apply the general theory of pseudo-differential operators with nonsmooth symbols to $\mu_{\mathcal{A}}(\cdot, D_x)$ to show Theorem 1.2 at least for $s < 1$; see [1, 12, 25, 31, 35]. Here we provide another approach to the analysis of $\mathcal{P}_{\mathcal{A}}$ which does not rely on the detailed properties of $\mu_{\mathcal{A}}(\cdot, D_x)$ obtained from this general theory; see Remark 3.7. For example, we do not need a symbol smoothing technique used in the literature above. Indeed, the key ingredient underlying the proof of Theorem 1.2 in our argument is the factorizations of operators $\mathcal{A}'$ and $\mathcal{A}$ in terms of $\mathcal{P}_{\mathcal{A}}$, which we will state in the next theorem. We note here that the assertion (ii) includes the critical case $s = 1$, which seems to be out of reach of the general theory of the pseudo-differential operators in the works cited above.

In order to state the next result, let us recall the realization of $\mathcal{A}$ in $L^2(\mathbb{R}^{d+1})$:

$$\begin{aligned} D_{L^2}(\mathcal{A}) = \big\{ u \in H^1(\mathbb{R}^{d+1}) \mid &\text{ there is } F \in L^2(\mathbb{R}^{d+1}) \text{ such that} \\ &\langle A\nabla u, \nabla v\rangle_{L^2(\mathbb{R}^{d+1})} = \langle F, v\rangle_{L^2(\mathbb{R}^{d+1})} \text{ for all } v \in H^1(\mathbb{R}^{d+1})\big\}, \\ \mathcal{A}u = F \quad \text{for } u \in D_{L^2}(\mathcal{A}). \end{aligned} \tag{1.8}$$

Note that $D_{L^2}(\mathcal{A}) = H^2(\mathbb{R}^{d+1})$ holds with equivalent norms because of (1.5). The realization of $\mathcal{A}' = -\nabla_x \cdot A'\nabla_x$ in $L^2(\mathbb{R}^d)$ is defined in a similar manner, and we have $D_{L^2}(\mathcal{A}') = H^2(\mathbb{R}^d)$ with equivalent norms since A' is Lipschitz continuous. The following theorem shows the factorization of operators $\mathcal{A}$ and $\mathcal{A}'$, and it clarifies the relation of the Poisson operator and the Dirichlet-Neumann map:

Theorem 1.4. *Under the same assumption as in Theorem* 1.2*, the following statements hold:*

(i) *The realization of $\mathcal{A}'$ in $L^2(\mathbb{R}^d)$ and the realization of $\mathcal{A}$ in $L^2(\mathbb{R}^{d+1})$ are respectively factorized as*

$$\mathcal{A}' = M_b \mathcal{Q}_{\mathcal{A}} \mathcal{P}_{\mathcal{A}}, \qquad \mathcal{Q}_{\mathcal{A}} = M_{1/b}(M_{\bar{b}} \mathcal{P}_{\mathcal{A}^*})^*, \tag{1.9}$$

$$\mathcal{A} = -M_b(\partial_t - \mathcal{Q}_{\mathcal{A}})(\partial_t + \mathcal{P}_{\mathcal{A}}). \tag{1.10}$$

Here $(M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*$ is the adjoint operator of $M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*}$ in $L^2(\mathbb{R}^d)$, while $\mathcal{P}_{\mathcal{A}^*}$ is the Poisson operator in $L^2(\mathbb{R}^d)$ associated with $\mathcal{A}^* = -\nabla \cdot A^*\nabla$.

(ii) *It follows that $D_{L^2}(\Lambda_{\mathcal{A}}) = D_{L^2}(\mathcal{Q}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ with equivalent norms and that*

$$\mathcal{P}_{\mathcal{A}} = M_{1/b}\Lambda_{\mathcal{A}} + M_{\mathbf{r_2}/b} \cdot \nabla_x, \tag{1.11}$$

$$\mathcal{Q}_{\mathcal{A}} = M_{1/b}\Lambda_{\mathcal{A}} - M_{\mathbf{r_1}/b} \cdot \nabla_x - M_{(\nabla_x \cdot \mathbf{r_1})/b}, \tag{1.12}$$

as the operators in $L^2(\mathbb{R}^d)$.

Remark 1.5. The operators $\partial_t + \mathcal{P}_{\mathcal{A}}$ and $\partial_t - \mathcal{Q}_{\mathcal{A}}$ are respectively defined as sum operators in $L^2(\mathbb{R}^{d+1})$. In particular, we have

$$D_{H^1}(\partial_t + \mathcal{P}_{\mathcal{A}}) = \{u \in H^1(\mathbb{R}^{d+1}) \mid \partial_t u,\ \mathcal{P}_{\mathcal{A}} u \in H^1(\mathbb{R}^{d+1})\} = H^2(\mathbb{R}^{d+1}),$$

by Theorem 1.2 (ii) with $s = 1$, and hence, the identities (1.9)-(1.10) are valid including the relation among the domains of the operators. The factorization (1.9) is considered as an operator-theoretical description of the Rellich identity; see Section 2. The factorization (1.10) is essentially a consequence of (1.9), but it plays an important role for the inhomogeneous boundary value problem. For example, consider the following Dirichlet problem:

$$\begin{cases} \mathcal{A}u = F & \text{in } \mathbb{R}^{d+1}_+, \\ u = g & \text{on } \partial\mathbb{R}^{d+1}_+. \end{cases}$$

Then our factorization formula leads to the formal integral representation

$$u(t) = e^{-t\mathcal{P}_{\mathcal{A}}} g + \int_0^t e^{-(t-s)\mathcal{P}_{\mathcal{A}}} \int_s^\infty e^{-(\tau - s)\mathcal{Q}_{\mathcal{A}}} M_{1/b} F(\tau)\, \mathrm{d}\tau\, \mathrm{d}s,$$

which directly leads to a solvability result under suitable conditions on the inhomogeneous term; see [28] for the details. Such a representation formula is available also for the Neumann problem, and it is applied to establish the Helmholtz decomposition in a domain above a Lipschitz graph in some anisotropic Lebesgue spaces [27].

Finally we state the counterpart of Theorem 1.2 for the Dirichlet-Neumann map in $H^s(\mathbb{R}^d)$.

Theorem 1.6. *Let $s \in [0,1]$. Under the same assumption as in Theorem 1.2, $H^s(\mathbb{R}^d)$ is invariant under the action of the Dirichlet-Neumann semigroup $\{e^{-t\Lambda_{\mathcal{A}}}\}_{t\geq 0}$ in $L^2(\mathbb{R}^d)$, and its restriction on $H^s(\mathbb{R}^d)$ defines a strongly continuous and analytic semigroup in $H^s(\mathbb{R}^d)$. Moreover, its generator, denoted again by $-\Lambda_{\mathcal{A}}$, satisfies that $D_{H^s}(\Lambda_{\mathcal{A}}) = H^{1+s}(\mathbb{R}^d)$ holds with equivalent norms.*

We note that $\Lambda_{\mathcal{A}}$ admits a bounded $H^\infty(\Sigma_\varphi)$-calculus in $L^2(\mathbb{R}^d)$ for some $\varphi \in (0, \frac{\pi}{2})$, for $\Lambda_{\mathcal{A}}$ is an injective m-φ-accretive operator in $L^2(\mathbb{R}^d)$ for some $\varphi \in (0, \frac{\pi}{2})$ by its definition, and therefore, we can apply [15, Section 7.1.1, Corollary 7.1.17]. We also refer to [12], where the authors studied the Dirichlet-Neumann map for the Laplace operator in a bounded domain with $C^{1+\alpha}$ boundary in the L^p framework. See also [36] for general properties of the Dirichlet-Neumann map and the relation with the layer potentials.

This paper is organized as follows. In Section 2 we state some general results on Poisson operators from [28], which plays a central role in our argument. Section 3 is the core of this paper. In Section 3.1 we study the Poisson semigroup and its generator in $L^2(\mathbb{R}^d)$ with the aid of the calculus of the symbol $\mu_{\mathcal{A}}$, while the Dirichlet-Neumann map in $L^2(\mathbb{R}^d)$ is studied in Section 3.2. The analysis of these operators in $H^s(\mathbb{R}^d)$ is performed in Sections 3.3 - 3.5. As stated in Remark 3.7, our approach recovers some properties of the pseudo-differential operator $\mu_{\mathcal{A}}(\cdot, D_x)$ in $H^s(\mathbb{R}^d)$, which is stated in the appendix.

2. Preliminaries

In this section we recall some results in [28]. As stated in the introduction, the Poisson semigroup $\{E_{\mathcal{A}}(t)\}_{t\geq 0}$ defines a strongly continuous and analytic semigroup in $H^{1/2}(\mathbb{R}^d)$, and thus we have the representation $E_{\mathcal{A}}(t) = e^{-t\mathcal{P}_{\mathcal{A}}}$ with its generator $-\mathcal{P}_{\mathcal{A}}$. The next proposition gives the condition so that $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ is extended as a semigroup in $L^2(\mathbb{R}^d)$.

Proposition 2.1 ([28, Proposition 3.3])**.** *The following two statements are equivalent.*

(i) *$D_{H^{1/2}}(\mathcal{P}_{\mathcal{A}}) \subset D_{L^2}(\Lambda_{\mathcal{A}^*})$ and $\|\Lambda_{\mathcal{A}^*} f\|_{L^2(\mathbb{R}^d)} \leq C\|f\|_{H^1(\mathbb{R}^d)}$ holds for $f \in D_{H^{1/2}}(\mathcal{P}_{\mathcal{A}})$.*

(ii) *$\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ is extended as a strongly continuous semigroup in $L^2(\mathbb{R}^d)$ and $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ is continuously embedded in $H^1(\mathbb{R}^d)$.*

Moreover, if condition (ii) *(and hence,* (i)*) holds, then $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ is continuously embedded in $D_{L^2}(\Lambda_{\mathcal{A}})$, $H^1(\mathbb{R}^d)$ is continuously embedded in $D_{L^2}(\Lambda_{\mathcal{A}^*})$, and it follows that*

$$\mathcal{P}_{\mathcal{A}} f = M_{1/b}\Lambda_{\mathcal{A}} f + M_{\mathbf{r_2}/b} \cdot \nabla_x f, \tag{2.1}$$

$$\langle A' \nabla_x f, \nabla_x g\rangle_{L^2(\mathbb{R}^d)} = \langle \mathcal{P}_{\mathcal{A}} f, \Lambda_{\mathcal{A}^*} g + M_{\bar{\mathbf{r}}_\mathbf{1}} \cdot \nabla_x g\rangle_{L^2(\mathbb{R}^d)} \tag{2.2}$$

for $f \in D_{L^2}(\mathcal{P}_{\mathcal{A}})$ and $g \in H^1(\mathbb{R}^d)$.

In order to show $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ we will use

Proposition 2.2 ([28, Corollary 3.5, Proposition 3.6])**.** *Assume that $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ and $\{e^{-t\mathcal{P}_{\mathcal{A}^*}}\}_{t\geq 0}$ are extended as strongly continuous semigroups in $L^2(\mathbb{R}^d)$ and* *that $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ and $D_{L^2}(\mathcal{P}_{\mathcal{A}^*})$ are continuously embedded in $H^1(\mathbb{R}^d)$. Then we have*

$$\langle A'\nabla_x f, \nabla_x g\rangle_{L^2(\mathbb{R}^d)} = \langle \mathcal{P}_{\mathcal{A}} f, M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*} g\rangle_{L^2(\mathbb{R}^d)}, \qquad f \in D_{L^2}(\mathcal{P}_{\mathcal{A}}),\ g \in D_{L^2}(\mathcal{P}_{\mathcal{A}^*}), \tag{2.3}$$

$$C'\|f\|_{H^1(\mathbb{R}^d)} \le \|\mathcal{P}_{\mathcal{A}} f\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{H^1(\mathbb{R}^d)}, \qquad f \in D_{L^2}(\mathcal{P}_{\mathcal{A}}). \tag{2.4}$$

If in addition $\liminf_{t\to 0} \|\,\mathrm{d}/\,\mathrm{d}t\ e^{-t\mathcal{P}_{\mathcal{A}}} f\|_{L^2(\mathbb{R}^d)} < \infty$ holds for all $f \in C_0^\infty(\mathbb{R}^d)$, then $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ with equivalent norms.

We note that a similar sufficient condition for the characterization $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ with equivalent norms is given in [3, Theorem 4.1]. The identities (2.2) and (2.3) are variants of the classical Rellich identity [34], but when the matrix A possesses a limited smoothness it is important to specify for which functions these identities are verified. The Rellich-type identity is verified and used by [19] when A is real symmetric and by [7] when $\mathbf{r_2} = 0$ without an extra regularity condition on A. See also [17, 20, 33] for the study of the elliptic boundary value problem in relation to the Rellich identity.

As for the factorizations of $\mathcal{A}'$ and $\mathcal{A}$, we have

Proposition 2.3 ([28, Lemma 3.8]). *Assume that the semigroups $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\ge 0}$ and $\{e^{-t\mathcal{P}_{\mathcal{A}^*}}\}_{t\ge 0}$ in $H^{1/2}(\mathbb{R}^d)$ are extended as strongly continuous semigroups in $L^2(\mathbb{R}^d)$ and that $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = D_{L^2}(\mathcal{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$ holds with equivalent norms. Then $H^1(\mathbb{R}^d)$ is continuously embedded in $D_{L^2}(\Lambda_{\mathcal{A}}) \cap D_{L^2}(\Lambda_{\mathcal{A}^*})$ and*

$$\mathcal{P}_{\mathcal{A}} f = M_{1/b}\Lambda_{\mathcal{A}} f + M_{\mathbf{r_2}/b}\cdot\nabla_x f, \qquad f \in H^1(\mathbb{R}^d), \tag{2.5}$$

$$\mathcal{P}_{\mathcal{A}^*} g = M_{1/\bar{b}}\Lambda_{\mathcal{A}^*} g + M_{\bar{\mathbf{r}}_{\mathbf{1}}/\bar{b}}\cdot\nabla_x g, \qquad g \in H^1(\mathbb{R}^d). \tag{2.6}$$

Moreover, the realizations of $\mathcal{A}'$ in $L^2(\mathbb{R}^d)$ and of $\mathcal{A}$ in $L^2(\mathbb{R}^{d+1})$ are respectively factorized as

$$\mathcal{A}' = M_b\mathcal{Q}_{\mathcal{A}}\mathcal{P}_{\mathcal{A}}, \qquad \mathcal{Q}_{\mathcal{A}} = M_{1/b}(M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*, \tag{2.7}$$

$$\mathcal{A} = -M_b(\partial_t - \mathcal{Q}_{\mathcal{A}})(\partial_t + \mathcal{P}_{\mathcal{A}}). \tag{2.8}$$

Here $(M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*$ is the adjoint of $M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*}$ in $L^2(\mathbb{R}^d)$.

The identity (2.7) follows from (2.3), and in this sense, it is considered as an operator-theoretical description of the Rellich identity. The exact factorizations for elliptic operators such as (2.7)-(2.8) have an important application, e.g., to the mathematical analysis of the fluid mechanics. Indeed, it plays a crucial role in revealing the structure of solenoidal vector fields in a domain with graph boundary [29] and also in the analysis of the Stokes operator [30].

3. Analysis of Poisson operator in $H^s(\mathbb{R}^d)$

To study the Poisson operator we consider the boundary value problem

$$\begin{cases} \mathcal{A}u = F & \text{in } \mathbb{R}^{d+1}_+, \\ u = g & \text{on } \partial\mathbb{R}^{d+1}_+. \end{cases} \tag{3.1}$$

Let $x, \xi \in \mathbb{R}^d$ and let $\mu_{\mathcal{A}} = \mu_{\mathcal{A}}(x,\xi) \in \{\mu \in \mathbb{C} \mid \operatorname{Im}\mu > 0\}$ be the root of

$$b(x)\mu^2 + \big(\mathbf{r_1}(x) + \mathbf{r_2}(x)\big)\cdot\xi\mu + \langle A'(x)\xi, \xi\rangle = 0. \tag{3.2}$$

Then we have

$$\mu_{\mathcal{A}}(x,\xi) = -\frac{\mathbf{v}(x)\cdot\xi}{2} + i\Big\{\frac{1}{b(x)}\langle A'(x)\xi,\xi\rangle - \frac{1}{4}(\mathbf{v}(x)\cdot\xi)^2\Big\}^{\frac{1}{2}},\ \mathbf{v} = \frac{\mathbf{r_1}+\mathbf{r_2}}{b}. \tag{3.3}$$

Here the square root in (3.3) is taken as the principal branch. From (1.2) one can check the estimates

$$|\mu_{\mathcal{A}}(x,\xi)| \le C|\xi|, \qquad \operatorname{Im}\mu_{\mathcal{A}}(x,\xi) \ge C'|\xi|, \tag{3.4}$$

$$\operatorname{Re}\big(\langle A'\xi,\xi\rangle - \frac{b}{4}(\mathbf{v}\cdot\xi)^2\big) \ge \nu_1\big(|\xi|^2 + \frac{|\mathbf{v}\cdot\xi|^2}{4}\big), \tag{3.5}$$

where C, C' are positive constants depending only on ν_1, ν_2. As is well known, $\mu_{\mathcal{A}}$ describes the principal symbol of the Poisson operator.

3.1. **Domain of Poisson operator in $L^2(\mathbb{R}^d)$.** The aim of this section is to prove that the domain of the Poisson operator in $L^2(\mathbb{R}^d)$ is $H^1(\mathbb{R}^d)$. For a given $h \in \mathcal{S}(\mathbb{R}^d)$ we set

$$\big(U_{\mathcal{A},0}(t)h\big)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{it\mu_{\mathcal{A}}(x,\xi)+ix\cdot\xi}\hat{h}(\xi)\,\mathrm{d}\xi, \tag{3.6}$$

where $\hat{h}$ is the Fourier transform of h. The operator $U_{\mathcal{A},0}(t)$ represents the principal part of the Poisson semigroup, and we first give some estimates of $U_{\mathcal{A},0}(t)$. To this end let us introduce the operator

$$\big(G_p(t)h\big)(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} p(x,\xi,t)e^{it\mu_{\mathcal{A}}(x,\xi)+ix\cdot\xi}\hat{h}(\xi)\,\mathrm{d}\xi \tag{3.7}$$

for a given measurable function $p = p(x,\xi,t)$ on $\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+$.

Lemma 3.1. *Let $T\in(0,\infty]$. Assume that $p = p(x,\xi,t)$ satisfies*

$$\sup_{x\in\mathbb{R}^d,\xi\neq0}\ \sup_{0<t<T}\ \Big(\sum_{k=0}^{d+1}(1+t|\xi|)^{-l_k}|\xi|^k|\nabla_\xi^k p(x,\xi,t)| + (t|\xi|)^{-l_{j_0}}|\xi|^{j_0}|\nabla_\xi^{j_0}p(x,\xi,t)|\Big) \le L < \infty \tag{3.8}$$

for some $l_k \ge 0$ and for some $l_{j_0} > 0$, $j_0 \in \{0,\cdots,d\}$. Then we have

$$\sup_{0<t<T}\|G_p(t)h\|_{L^2(\mathbb{R}^d)} \le C\|h\|_{L^2(\mathbb{R}^d)}, \tag{3.9}$$

where C depends only on d, ν_1, ν_2, l_k, l_{j_0}, and L. Furthermore, if p satisfies

$$\sup_{x\in\mathbb{R}^d,\xi\neq0}\ \sup_{t>0}\ \sum_{k=0}^{d+1}(t|\xi|)^{-l_k}|\xi|^k|\nabla_\xi^k p(x,\xi,t)| \le L < \infty \tag{3.10}$$

for some $l_k > 0$, $k = 0,\cdots,d+1$, then

$$\int_0^\infty \|G_p(t)h\|_{L^2(\mathbb{R}^d)}^2\frac{\mathrm{d}t}{t} \le C'\|h\|_{L^2(\mathbb{R}^d)}^2, \tag{3.11}$$

where C' depends only on d, ν_1, ν_2, l_k, and L.

The proof of Lemma 3.1 is rather standard and will be stated in the appendix for the convenience of the reader. Now we have

Lemma 3.2. *Let $k \in \mathbb{N} \cup \{0\}$. For $h \in \mathcal{S}(\mathbb{R}^d)$ and $t > 0$ it follows that*

$$\begin{aligned} &\|t^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} U_{\mathcal{A},0}(t)h\|_{L^2(\mathbb{R}^d)} + \|te^{-t}\nabla_x U_{\mathcal{A},0}(t)h\|_{L^2(\mathbb{R}^d)} + \|tU_{\mathcal{A},0}(t)(-\Delta_x)^{\frac{1}{2}}h\|_{L^2(\mathbb{R}^d)} \\ &\qquad + \|[U_{\mathcal{A},0}(t), \nabla_x]h\|_{L^2(\mathbb{R}^d)} \le C\|h\|_{L^2(\mathbb{R}^d)}, \end{aligned} \tag{3.12}$$

where $[B_1, B_2]$ is the commutator of the operators B_1, B_2, and

$$\int_0^\infty \|e^{-t}U_{\mathcal{A},0}(t)h\|^2_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)}\,\mathrm{d}t \le C\|h\|^2_{L^2(\mathbb{R}^d)}. \tag{3.13}$$

In particular, $\lim_{t\to 0} U_{\mathcal{A},0}(t)h = h$ in $L^2(\mathbb{R}^d)$ for any $h \in L^2(\mathbb{R}^d)$.

Proof. The estimate (3.12) is a direct consequence of Lemma 3.1. For example, we take $p(x,\xi,t) = t|\xi|$ for the estimate of $tU_{\mathcal{A},0}(t)(-\Delta_x)^{1/2}h$, and take $p(x,\xi,t) = it\nabla_x\mu_{\mathcal{A}}(x,\xi)$ for $[U_{\mathcal{A},0}, \nabla_x]h$, and so on. As for (3.13), we use the Schur lemma as in the proof of (3.11). By (3.12) and $\|f\|_{\dot{H}^{1/2}} \le \|f\|_{L^2}^{1/2}\|\nabla_x f\|_{L^2}^{1/2}$ it is easy to see that $t^{1/2}e^{-t}\|U_{\mathcal{A},0}(t)h\|_{\dot{H}^{1/2}(\mathbb{R}^d)} \le C\|h\|_{L^2(\mathbb{R}^d)}$ for all $t > 0$. Let $t \ge s > 0$, and let ψ_s be the function defined in the proof of Lemma 3.1 in the appendix. Then we have from $\psi_s = \Delta_x\tilde{\psi}_s$ and (3.12),

$$\begin{aligned} &t^{\frac{1}{2}}\|e^{-t}U_{\mathcal{A},0}(t)\psi_s * h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} \\ &\qquad \le t^{\frac{1}{2}}\|e^{-t}U_{\mathcal{A},0}(t)\Delta_x\tilde{\psi}_s * h\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)}\|\nabla_x e^{-t}U_{\mathcal{A},0}(t)\psi_s * h\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)} \\ &\qquad \le Ct^{-\frac{1}{2}}\|\nabla_x\tilde{\psi}_s * h\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)}\|h\|^{\frac{1}{2}}_{L^2(\mathbb{R}^d)} \le Ct^{-\frac{1}{2}}s^{\frac{1}{2}}\|h\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Let $s \ge t > 0$. Then the relation $\nabla_x U_{\mathcal{A},0}(t) = t^{1/2}G_p(t)(-\Delta_x)^{1/4} + U_{\mathcal{A},0}(t)\nabla_x$ with $p = it^{1/2}|\xi|^{-1/2}\nabla_x\mu_{\mathcal{A}}$ combined with (3.9) and (3.12) yields

$$\begin{aligned} \|\nabla_x U_{\mathcal{A},0}(t)h\|_{L^2(\mathbb{R}^d)} &\le Ct^{\frac{1}{2}}\|(-\Delta_x)^{\frac{1}{4}}\psi_s * h\|_{L^2(\mathbb{R}^d)} + C\|\nabla_x\psi_s * h\|_{L^2(\mathbb{R}^d)} \\ &\le C(t^{\frac{1}{2}}s^{-\frac{1}{2}} + s^{-1}), \end{aligned}$$

which implies

$$\begin{aligned} t^{\frac{1}{2}}\|e^{-t}U_{\mathcal{A},0}(t)\psi_s * h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} &\le Ct^{\frac{1}{2}}e^{-t}(t^{\frac{1}{4}}s^{-\frac{1}{4}} + s^{-\frac{1}{2}})\|h\|_{L^2(\mathbb{R}^d)} \\ &\le Ct^{\frac{1}{4}}s^{-\frac{1}{4}}\|h\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Collecting the above, we can apply the Schur lemma [14, pp. 643-644] to $\{t^{1/2}e^{-t}(-\Delta_x)^{1/4}U_{\mathcal{A},0}(t)\}_{t>0}$ to obtain (3.13). The last statement of the proposition follows from (3.12) and the density argument. The proof is complete. □

We look for a solution u to (3.1) with $F = 0$ and $g = h$ of the form

$$u = M_\chi U_{\mathcal{A},0}h + U_{\mathcal{A},1}h, \tag{3.14}$$

where $\chi = \chi(t)$ is a smooth cut-off function such that $\chi(t) = 1$ if $t \in [0,1]$ and $\chi(t) = 0$ if $t \ge 2$, and $U_{\mathcal{A},1}h$ is a solution to (3.1) with $F = -\mathcal{A}(M_\chi U_{\mathcal{A},0}h)$ and $g = 0$.

Lemma 3.3. *For any $h \in \mathcal{S}(\mathbb{R}^d)$ there exists a unique solution $U_{\mathcal{A},1}h \in \dot{H}^1_0(\mathbb{R}^{d+1}_+)$ to (3.1) with $F = -\mathcal{A}(M_\chi U_{\mathcal{A},0}h)$ and $g = 0$, which satisfies*

$$\|\nabla U_{\mathcal{A},1}h\|_{L^2(\mathbb{R}^{d+1}_+)} \le C\|(I-\Delta_x)^{-\frac14}h\|_{L^2(\mathbb{R}^d)}. \tag{3.15}$$

Proof. For simplicity we write U_0 and U_1 for $U_{\mathcal{A},0}$ and $U_{\mathcal{A},1}$. We set

$$A' = (a_{i,j})_{1\le i,j\le d}, \qquad \mathbf{a}' = \nabla_x \cdot A' = \Big(\sum_{1\le k\le d} \partial_k a_{k,j}\Big)_{1\le j\le d},$$

and

$$\Pi h = \begin{pmatrix} \Pi' h \\ \Pi_{d+1}h \end{pmatrix} = \begin{pmatrix} G_{itA'\nabla_x\mu_\mathcal{A}}h \\ G_{\nabla_x\cdot \mathbf{r_1}}h + G_{it(\mathbf{r_1}+\mathbf{r_2})\cdot\nabla_x\mu_\mathcal{A}}h \end{pmatrix} \in \mathbb{C}^{d+1}. \tag{3.16}$$

Then a direct computation yields

$$-\mathcal{A}U_0 h = \nabla\cdot \Pi h + G_\zeta h,$$
$$\zeta(x,\xi,t) = i\big(\mathbf{r_1}(x) + \mathbf{r_2}(x) + itA'(x)\xi\big)\cdot\nabla_x\mu_\mathcal{A}(x,\xi) + i\mathbf{a}'(x)\cdot\xi.$$

Hence $U_1 h$ should be constructed as the solution of (3.1) with $g=0$ and

$$\begin{aligned} F &= -\mathcal{A}(M_\chi U_0 h) = \nabla\cdot M_\chi \Pi h + M_\chi G_\zeta h + Rh,\\ Rh &= \nabla_x\cdot M_{(\mathbf{r_1}+\mathbf{r_2})\partial_t\chi}U_0h + \partial_t M_{b\partial_t\chi}U_0 h - M_{\partial_t\chi}\Pi_{d+1}h - M_{b\partial_t^2\chi + \nabla_x\cdot\mathbf{r_2}\partial_t\chi}U_0 h. \end{aligned} \tag{3.17}$$

To obtain (3.15), let us estimate each term of F in $\dot{H}^{-1}(\mathbb{R}^{d+1}_+)$. Note that Rh is supported in $\{(x,t)\in\mathbb{R}^{d+1}_+ \mid 1\le t\le 2\}$ by the definition of χ. In particular, it is not difficult to show

$$|\langle Rh,\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \le C\|(I-\Delta_x)^{-\frac14}h\|_{L^2(\mathbb{R}^d)}\|\nabla\varphi\|_{L^2(\mathbb{R}^{d+1}_+)}, \qquad \varphi\in H^1_0(\mathbb{R}^{d+1}_+). \tag{3.18}$$

Thus we focus on the leading terms $\nabla\cdot M_\chi\Pi h$ and $M_\chi G_\zeta h$. By using Lemma 3.1 one can easily check the estimates

$$\begin{aligned} |\langle \nabla\cdot M_\chi \Pi h,\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| &\le \|M_\chi\Pi h\|_{L^2(\mathbb{R}^{d+1}_+)}\|\nabla\varphi\|_{L^2(\mathbb{R}^{d+1}_+)}\\ &\le C\|h\|_{L^2(\mathbb{R}^d)}\|\nabla\varphi\|_{L^2(\mathbb{R}^{d+1}_+)},\\ |\langle M_\chi G_\zeta h,\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| &= \Big|\int_0^\infty \langle M_\chi G_\zeta h, \int_0^t \partial_s\varphi\,\mathrm{d}s\rangle_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\Big|\\ &\le \int_0^\infty t^{\frac12}\|M_\chi G_\zeta h\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t\|\nabla\varphi\|_{L^2(\mathbb{R}^{d+1}_+)}\\ &\le C\|h\|_{L^2(\mathbb{R}^d)}\|\nabla\varphi\|_{L^2(\mathbb{R}^{d+1}_+)}, \end{aligned}$$

for $\varphi\in H^1_0(\mathbb{R}^{d+1}_+)$. Next we consider the estimate of $M_\chi G_\zeta(-\Delta_x)^{\frac14}h$. By using the following general relation for G_p that

$$G_p(t)h = \partial_t G_{p/(i\mu_\mathcal{A})}(t)h - G_{\partial_t p/(i\mu_\mathcal{A})}(t)h, \qquad G_p(t)(-\Delta_x)^{\frac14}h = G_{p|\xi|^{1/2}}(t)h, \tag{3.19}$$

and by using $\partial_t^2\zeta = 0$ and $\partial_t\mu_\mathcal{A} = 0$, we observe that it suffices to estimate

$$\partial_t M_\chi G_{\zeta|\xi|^{1/2}/(i\mu_\mathcal{A})}h + \partial_t M_\chi G_{\partial_t\zeta|\xi|^{1/2}/\mu_\mathcal{A}^2}h,$$

for the other terms are of lower order. We see that

$$\begin{aligned}&|\langle \partial_t M_\chi G_{\zeta|\xi|^{1/2}/(i\mu_{\mathcal{A}})} h + \partial_t M_\chi G_{\partial_t \zeta|\xi|^{1/2}/\mu_{\mathcal{A}}^2} h, \varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \\ &\quad \le C\|M_\chi G_{\tilde{\zeta}} h\|_{L^2(\mathbb{R}^{d+1}_+)} \|\nabla\varphi\|_{L^2(\mathbb{R}^{d+1}_+)}\end{aligned}$$

for $\varphi \in H^1_0(\mathbb{R}^{d+1}_+)$, where $\tilde{\zeta} = \zeta|\xi|^{1/2}/(i\mu_{\mathcal{A}}) + \partial_t\zeta|\xi|^{1/2}/\mu_{\mathcal{A}}^2$. Then the bound of $\|M_\chi G_{\tilde{\zeta}} h\|_{L^2(\mathbb{R}^{d+1}_+)}$ is reduced to that of $\int_{\mathbb{R}_+} \|G_p(t)h\|^2_{L^2(\mathbb{R}^d)} t^{-1}\,\mathrm{d}t$ with p satisfying (3.10), and hence, $\|M_\chi G_{\tilde{\zeta}} h\|_{L^2(\mathbb{R}^{d+1}_+)} \le C\|h\|_{L^2(\mathbb{R}^d)}$ follows from (3.11), as desired. Similarly, we have

$$\|M_\chi \Pi(-\Delta_x)^{\frac{1}{4}} h\|_{L^2(\mathbb{R}^{d+1}_+)} \le C\|h\|_{L^2(\mathbb{R}^d)}.$$

Collecting the above, we arrive at

(3.20)
$$|\langle -\mathcal{A}(M_\chi U_0 h), \varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \le C\|(I-\Delta_x)^{-\frac{1}{4}} h\|_{L^2(\mathbb{R}^d)} \|\nabla\varphi\|_{L^2(\mathbb{R}^{d+1}_+)}, \ \varphi \in H^1_0(\mathbb{R}^{d+1}_+).$$

Thus, by the Lax-Milgram theorem there is a unique solution $U_1 h \in \dot{H}^1_0(\mathbb{R}^{d+1}_+)$ to (3.1) with $F = -\mathcal{A}(M_\chi U_0 h)$ and $g = 0$ which satisfies (3.15). The proof is complete. □

Lemmas 3.2 and 3.3 imply the estimate $\|e^{-t\mathcal{P}_{\mathcal{A}}} h\|_{L^2(\mathbb{R}^d)} \le C\|h\|_{L^2(\mathbb{R}^d)}$ for $t \in (0,1]$ and $h \in H^1(\mathbb{R}^d)$. Since the same argument can be applied for $\{e^{-t\mathcal{P}_{\mathcal{A}^*}}\}_{t\ge 0}$, we have

Corollary 3.4. *The semigroups $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\ge0}$ and $\{e^{-t\mathcal{P}_{\mathcal{A}^*}}\}_{t\ge0}$ in $H^{1/2}(\mathbb{R}^d)$ are extended as strongly continuous analytic semigroups in $L^2(\mathbb{R}^d)$. Moreover, $H^1(\mathbb{R}^d)$ is continuously embedded in $D_{L^2}(\Lambda_{\mathcal{A}}) \cap D_{L^2}(\Lambda_{\mathcal{A}^*})$, and $D_{L^2}(\mathcal{P}_{\mathcal{A}})$ and $D_{L^2}(\mathcal{P}_{\mathcal{A}^*})$ are continuously embedded in $H^1(\mathbb{R}^d)$. We also have the estimate*

$$\|e^{-t\mathcal{P}_{\mathcal{A}}} f\|_{L^2(\mathbb{R}^d)} \le Ct^{-\frac{1}{2}} \|f\|_{H^{-\frac{1}{2}}(\mathbb{R}^d)}, \qquad 0 < t < 1. \tag{3.21}$$

Proof. By the variational characterization of $\dot{H}^{1/2}(\mathbb{R}^d)$, the estimate (3.15) implies that

$$\|U_{\mathcal{A},1}(t)h\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^d)} \le \|\nabla U_{\mathcal{A},1} h\|_{L^2(\mathbb{R}^{d+1}_+)} \le C\|h\|_{H^{-\frac{1}{2}}(\mathbb{R}^d)}, \qquad t > 0. \tag{3.22}$$

Thus (3.13) and (3.22) verify the condition (i) of [28, Proposition 4.3], which gives $D_{L^2}(\mathcal{P}_{\mathcal{A}}) \hookrightarrow H^1(\mathbb{R}^d)$. The same is true for $D_{L^2}(\mathcal{P}_{\mathcal{A}^*})$, and then we also obtain the embedding $H^1(\mathbb{R}^d) \hookrightarrow D_{L^2}(\Lambda_{\mathcal{A}}) \cap D_{L^2}(\Lambda_{\mathcal{A}^*})$ by Proposition 2.1. Now it remains to show (3.21). Let us recall the representation $e^{-t\mathcal{P}_{\mathcal{A}}} f = M_\chi U_{\mathcal{A},0}(t) f + U_{\mathcal{A},1}(t) f$. By the definition of G_p in (3.7) we have $U_{\mathcal{A},0}(t)f = t^{-1/2} G_{t^{1/2}\langle\xi\rangle^{1/2}} (I-\Delta_x)^{-1/4} f$, where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$. Then it is easy to see that $p(x,\xi,t) = t^{1/2}\langle\xi\rangle^{1/2}$ satisfies (3.8) with $T = 1$, and therefore,

$$\begin{aligned}\|U_{\mathcal{A},0}(t)f\|_{L^2(\mathbb{R}^d)} &= t^{-\frac{1}{2}} \|G_{t^{1/2}\langle\xi\rangle^{1/2}} (I-\Delta_x)^{-1/4} f\|_{L^2(\mathbb{R}^d)} \\ &\le Ct^{-\frac{1}{2}} \|(I-\Delta_x)^{-1/4} f\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

On the other hand, we have already proved the desired estimate for $U_{\mathcal{A},1}(t)f$ by (3.15). The proof is complete. □

In order to establish the characterization of $D_{L^2}(\mathcal{P}_\mathcal{A})$ and $D_{L^2}(\mathcal{P}_{\mathcal{A}^*})$, we need further estimates of $U_{\mathcal{A},1}$ as follows.

Lemma 3.5. *For any $h \in H^{1/2}(\mathbb{R}^d)$ we have* $\mathrm{d}/\,\mathrm{d}t\, U_{\mathcal{A},1}(t)h \in C([0,\infty); H^{1/2}(\mathbb{R}^d))$ *and*

$$\sup_{t>0} \|\frac{\mathrm{d}}{\mathrm{d}t} U_{\mathcal{A},1}(t)h\|_{H^{\frac12}(\mathbb{R}^d)} \le C\|h\|_{H^{\frac12}(\mathbb{R}^d)}. \tag{3.23}$$

Proof. As in the proof of Lemma 3.3 we write U_0 and U_1 for $U_{\mathcal{A},0}$ and $U_{\mathcal{A},1}$. First we assume that $h \in D_{L^2}(\mathcal{P}_\mathcal{A})$. Let us recall that $U_1(t)h \in \dot H^1_0(\mathbb{R}^{d+1}_+)$ solves (3.1) with $F = -\mathcal{A}(M_\chi U_0 h) = \nabla \cdot M_\chi \Pi h + M_\chi G_\zeta h + Rh$ and $g = 0$. We first see $\lim_{\delta\to0} d/dt U_1(\delta)h$ exists in $H^{-1}(\mathbb{R}^d)$. Set $\mathbb{R}^{d+1}_{\delta,+} = \{(x,t) \in \mathbb{R}^{d+1} \mid t > \delta\}$ for $\delta > 0$. Since $\mathrm{d}/\,\mathrm{d}t\, U_1 h \in \bigcap_{\delta>0} H^1(\mathbb{R}^{d+1}_{\delta,+})$, we have

$$\begin{aligned}\langle \frac{\mathrm{d}}{\mathrm{d}t} U_1(\delta)h &+ M_{\mathbf{r_2}/b} \cdot \nabla_x U_1(\delta)h, \phi\rangle_{L^2(\mathbb{R}^d)}\\ &= -\langle A\nabla U_1 h, \nabla\varphi\rangle_{L^2(\mathbb{R}^{d+1}_{\delta,+})} - \langle M_\chi \Pi' h, \nabla_x\varphi\rangle_{L^2(\mathbb{R}^{d+1}_{\delta,+})}\\ &\quad + \langle \partial_t M_\chi \Pi_{d+1} h + M_\chi G_\zeta h + Rh, \varphi\rangle_{L^2(\mathbb{R}^{d+1}_{\delta,+})}\end{aligned} \tag{3.24}$$

for any $\delta \in (0,1/2)$ and $\varphi \in H^1(\mathbb{R}^{d+1}_{\delta,+})$ with $\varphi|_{t=\delta} = M_{1/\bar b}\phi$, $\phi \in H^{1/2}(\mathbb{R}^d)$. By taking $\varphi = e^{-(t-\delta)\mathcal{P}_{\mathcal{A}^*}} M_{1/\bar b}\phi$, a similar calculation as in the proof of Lemma 3.3 yields the estimate for the R.H.S. of (3.24) from above by $C\|h\|_{H^{1/2}(\mathbb{R}^d)}\|M_{1/\bar b}\phi\|_{H^{1/2}(\mathbb{R}^d)}$, and hence by $C\|h\|_{H^{1/2}(\mathbb{R}^d)}\|\phi\|_{H^{1/2}(\mathbb{R}^d)}$. On the other hand, we have

$$\begin{aligned}|\langle M_{\mathbf{r_2}/b} &\cdot \nabla_x U_1(\delta)h, \phi\rangle_{L^2(\mathbb{R}^d)}|\\ &\le C\|\nabla_x U_1(\delta)h\|_{L^2(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)}\\ &= C\|\nabla_x\big(e^{-\delta\mathcal{P}_\mathcal{A}} - U_0(\delta)\big)h\|_{L^2(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)}\\ &\le C\big(\|[\nabla_x, e^{-\delta\mathcal{P}_\mathcal{A}}]h\|_{L^2(\mathbb{R}^d)} + \|[\nabla_x, U_0(\delta)]h\|_{L^2(\mathbb{R}^d)}\big)\|\phi\|_{L^2(\mathbb{R}^d)}\\ &\quad + C\|U_1(\delta)\nabla_x h\|_{L^2(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)}.\end{aligned} \tag{3.25}$$

In particular, we see $\|M_{\mathbf{r_2}/b} \cdot \nabla_x U_1(\delta)h\|_{L^2(\mathbb{R}^d)} \to 0$ as $\delta \to 0$, for the facts $h \in D_{L^2}(\mathcal{P}_\mathcal{A})$ and $D_{L^2}(\mathcal{P}_\mathcal{A}) \hookrightarrow H^1(\mathbb{R}^d)$ by Corollary 3.4 imply $\lim_{\delta\to0} \|[\nabla_x, e^{-\delta\mathcal{P}_\mathcal{A}}]h\|_{L^2(\mathbb{R}^d)} = 0$, while the convergence $\lim_{\delta\to0} \|[\nabla_x, U_0(\delta)]h\|_{L^2(\mathbb{R}^d)} = 0$ follows from (3.12) and the density argument. The limit $\lim_{\delta\to0} \|U_1(\delta)\nabla_x h\|_{L^2(\mathbb{R}^d)} = 0$ follows from Lemma 3.3 with h replaced by $\nabla_x h$. By applying (3.24) to another $\delta' \in (0,1/2)$ and then estimating the difference $\langle\, \mathrm{d}/\,\mathrm{d}t\, U_1(\delta)h - \mathrm{d}/\,\mathrm{d}t\, U_1(\delta')h, \phi\rangle_{L^2(\mathbb{R}^d)}$ we conclude that $\mathrm{d}/\,\mathrm{d}t\, U_1(\delta)h$ converges to some limit, denoted by $S_{\mathcal{A},1}h$, in $H^{-1/2}(\mathbb{R}^d)$ as $\delta \to 0$. We claim that

$$\|S_{\mathcal{A},1}h\|_{H^s(\mathbb{R}^d)} \le C_s\|h\|_{H^s(\mathbb{R}^d)}, \qquad h \in D_{L^2}(\mathcal{P}_\mathcal{A}),\ s \in (0, \frac12]. \tag{3.26}$$

To this end, first note that the following equality with the choice of $\varphi(t) = e^{-t\mathcal{P}_{\mathcal{A}^*}}\phi$, $\phi \in H^{1/2}(\mathbb{R}^d)$ holds:

$$
\begin{aligned}
\langle S_{\mathcal{A},1}h, M_{\bar{b}}\phi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} &= \lim_{\delta\to 0}\langle \frac{\mathrm{d}}{\mathrm{d}t}U_1(\delta)h, M_{\bar{b}}\phi\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} \\
&= \lim_{\delta\to 0}\langle \frac{\mathrm{d}}{\mathrm{d}t}U_1(\delta)h, M_{\bar{b}}\varphi(\delta)\rangle_{H^{-\frac{1}{2}},H^{\frac{1}{2}}} \\
&= \lim_{\delta\to 0}\langle M_b\frac{\mathrm{d}}{\mathrm{d}t}U_1(\delta)h + M_{\mathbf{r_2}}\cdot\nabla_x U_1(\delta)h, \varphi(\delta)\rangle_{L^2(\mathbb{R}^d)} \\
&= -\langle A\nabla U_1 h, \nabla\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)} - \langle M_\chi \Pi' h, \nabla_x\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)} \\
&\quad + \langle \partial_t M_\chi \Pi_{d+1}h + M_\chi G_\zeta h + Rh, \varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)} \\
&= -\langle M_\chi \Pi' h, \nabla_x\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)} \\
&\quad + \langle \partial_t M_\chi \Pi_{d+1}h + M_\chi G_\zeta h + Rh, \varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}.
\end{aligned}
\tag{3.27}
$$

Here for each line we have respectively used $\varphi(\delta) = e^{-\delta\mathcal{P}_{\mathcal{A}^*}}\phi \to \phi$ in $H^{1/2}(\mathbb{R}^d)$ for $\phi \in H^{1/2}(\mathbb{R}^d)$, $M_{\mathbf{r_2}/b}\cdot\nabla_x U_1(\delta)h \to 0$ in $L^2(\mathbb{R}^d)$, (3.24), and then the fact that $\langle A\nabla U_1 h, \nabla\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)} = 0$, since $U_1 h \in \dot{H}^1_0(\mathbb{R}^{d+1}_+)$ and $\varphi(t) = e^{-t\mathcal{P}^*_{\mathcal{A}}}\phi$. Hence our next task is to estimate the right-hand side of (3.27). Let $s \in (0, 1/2]$. We will show that

$$
\begin{aligned}
&|\langle M_\chi \Pi' h, \nabla_x\varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| + |\langle \partial_t M_\chi G_{it(\mathbf{r_1}+\mathbf{r_2})\cdot\nabla_x\mu_{\mathcal{A}}}h, \varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \\
&\quad + |\langle M_\chi \partial_t G_{\nabla_x\cdot\mathbf{r_2}}h + M_\chi G_\zeta h, \varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \le C\|h\|_{H^s(\mathbb{R}^d)}\|(I-\Delta_x)^{-\frac{s}{2}}\phi\|_{L^2(\mathbb{R}^d)},
\end{aligned}
\tag{3.28}
$$

which gives (3.26), since the other terms in (3.27) are of lower order and easy to handle. We observe that $\varphi(t) = e^{-t\mathcal{P}_{\mathcal{A}^*}}\phi = M_\chi U_{\mathcal{A}^*,0}(t)\phi + U_{\mathcal{A}^*,1}(t)\phi$, and thus it suffices to check (3.27) with φ replaced by $U_{\mathcal{A}^*,0}(t)\phi$, for $U_{\mathcal{A}^*,1}(t)\phi$ is of lower order if s is less than or equal to $1/2$ thanks to Lemma 3.3. The argument below is based on the quadratic estimate as in (3.11). Note that the counterpart of Lemma 3.1 is valid for $U_{\mathcal{A}^*,0}(t)$. Firstly we see from the definition of Π' in (3.16) that

$$
\begin{aligned}
&|\langle M_\chi \Pi' h, \nabla_x U_{\mathcal{A}^*,0}(t)(-\Delta_x)^{\frac{s}{2}}\phi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \\
&\le \int_0^\infty M_\chi \|G_{itA'\nabla_x\mu_{\mathcal{A}}}(t)h\|_{L^2(\mathbb{R}^d)}\|\nabla_x U_{\mathcal{A}^*,0}(t)(-\Delta_x)^{\frac{s}{2}}\phi\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \\
&\le \Big(\int_0^\infty M_\chi \|t^{1-s}G_{iA'\nabla_x\mu_{\mathcal{A}}}(t)h\|^2_{L^2(\mathbb{R}^d)}\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}} \\
&\quad \times \Big(\int_0^\infty M_\chi \|t^{1+s}\nabla_x U_{\mathcal{A}^*,0}(t)(-\Delta_x)^{\frac{s}{2}}\phi\|^2_{L^2(\mathbb{R}^d)}\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{2}} \\
&\le C\|h\|_{\dot{H}^s(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)}, \qquad s \in [0, \frac{1}{2}],
\end{aligned}
$$

by applying Lemma 3.1. Similarly, we have

$$
\begin{aligned}
&|\langle \partial_t M_\chi G_{it(\mathbf{r_1}+\mathbf{r_2})\cdot\nabla_x\mu_{\mathcal{A}}}h, \varphi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \\
&= |\langle M_\chi G_{it(\mathbf{r_1}+\mathbf{r_2})\cdot\nabla_x\mu_{\mathcal{A}}}h, \frac{\mathrm{d}}{\mathrm{d}t}U_{\mathcal{A}^*,0}(t)(-\Delta_x)^{\frac{s}{2}}\phi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \\
&\le C\|h\|_{\dot{H}^s(\mathbb{R}^d)}\|\phi\|_{L^2(\mathbb{R}^d)}, \qquad s \in [0, \frac{1}{2}].
\end{aligned}
$$

Finally we consider the term $M_\chi \partial_t G_{\nabla_x\cdot \mathbf{r_2}} h + M_\chi G_\zeta h = M_\chi G_\eta h$, $\eta = \mu_{\mathcal{A}} \nabla_x \cdot \mathbf{r_2} + \zeta$. Lemma 3.1 yields

$$B_s(h) := \Big(\int_0^\infty M_\chi \|t^{1-s} G_\eta(t) h\|^2_{L^2(\mathbb{R}^d)} \frac{\mathrm{d}t}{t}\Big)^{\frac12} \le C\|h\|_{\dot H^s(\mathbb{R}^d)}, \qquad s \in [0, \frac12].$$

Thus it follows that

$$\begin{aligned}
|\langle M_\chi G_\eta h, U_{\mathcal{A}^*,0}(t)\phi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| &\le \int_0^\infty M_\chi \|G_\eta(t)h\|_{L^2(\mathbb{R}^d)} \|U_{\mathcal{A}^*,0}(t)\phi\|_{L^2(\mathbb{R}^d)} \,\mathrm{d}t \\
&\le \sup_{t>0} \|U_{\mathcal{A}^*,0}(t)(I-\Delta_x)^{-1}\phi\|_{L^2(\mathbb{R}^d)} \int_0^\infty M_\chi \|G_\eta(t)h\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t \\
&+ \int_0^\infty M_\chi \|G_\eta(t)h\|_{L^2(\mathbb{R}^d)} \|U_{\mathcal{A}^*,0}(t)(-\Delta_x)(I-\Delta_x)^{-1}\phi\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}t,
\end{aligned}$$

and the first term of the right-hand side is bounded from above by

$$C\|(I-\Delta_x)^{-1}\phi\|_{L^2(\mathbb{R}^d)} \Big(\int_0^\infty \chi(t) t^{2s} \frac{\mathrm{d}t}{t}\Big)^{\frac12} B_s(h) \le C_s B_s(h) \|(I-\Delta_x)^{-1}\phi\|_{L^2(\mathbb{R}^d)},$$

while the second term is estimated by

$$\begin{aligned}
&B_s(h)\Big(\int_0^\infty M_\chi \|t^s U_{\mathcal{A}^*,0}(t)(-\Delta_x)^{\frac{s}{2}}(-\Delta_x)^{1-\frac{s}{2}}(I-\Delta_x)^{-1}\phi\|^2_{L^2(\mathbb{R}^d)} \frac{\mathrm{d}t}{t}\Big)^{\frac12} \\
&\le C B_s(h) \|(-\Delta_x)^{1-\frac{s}{2}}(I-\Delta_x)^{-1}\phi\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thus we have

$$|\langle M_\chi G_\eta h, U_{\mathcal{A}^*,0}(t)\phi\rangle_{L^2(\mathbb{R}^{d+1}_+)}| \le C_s \|(I-\Delta_x)^{-\frac{s}{2}}\phi\|_{L^2(\mathbb{R}^d)} \|h\|_{\dot H^s(\mathbb{R}^d)},$$

where C_s is a constant which tends to ∞ as $s \to 0$. Collecting the above, we arrive at (3.26).

Now let $V_1(t)h \in C([0,\infty); H^{1/2}(\mathbb{R}^d))$ be the unique weak solution to (3.1) with $F = -\partial_t \mathcal{A}(M_\chi U_0 h) = \nabla\cdot \partial_t M_\chi \Pi h + \partial_t M_\chi G_\zeta h + \partial_t R h$ and $g = S_{\mathcal{A},1}h$. Hence it has the form

$$V_1(t)h = e^{-t\mathcal{P}_{\mathcal{A}}} S_{\mathcal{A},1}h + W_{\mathcal{A}}(t)h = \chi U_0(t) S_{\mathcal{A},1}h + U_1(t) S_{\mathcal{A},1}h + W_{\mathcal{A}}(t)h, \tag{3.29}$$

where $W_{\mathcal{A}}h \in \dot H^1_0(\mathbb{R}^{d+1}_+)$ is a weak solution to (3.1) with $F = \nabla\cdot\partial_t M_\chi \Pi h + \partial_t M_\chi G_\zeta h + \partial_t R h$ and $g = 0$. Note that

$$\int_0^\infty M_\chi \big(\|t^{\frac12}\partial_t \frac{\mathrm{d}}{\mathrm{d}t}\Pi h\|^2_{L^2(\mathbb{R}^d)} + \|t^{\frac12} G_\zeta(t)h\|^2_{L^2(\mathbb{R}^d)}\big)\frac{\mathrm{d}t}{t} \le C\|h\|^2_{\dot H^{\frac12}(\mathbb{R}^d)}$$

by Lemma 3.1, which implies the estimate $\|\nabla W_{\mathcal{A}}h\|_{L^2(\mathbb{R}^{d+1}_+)} \le C\|h\|_{\dot H^{1/2}(\mathbb{R}^d)}$, and thus

$$\|W_{\mathcal{A}}(t)h\|_{\dot H^{\frac12}(\mathbb{R}^d)} \le \|\nabla w\|_{L^2(\mathbb{R}^{d+1}_{t,+})} \le C\|h\|_{\dot H^{1/2}(\mathbb{R}^d)}, \qquad t>0. \tag{3.30}$$

On the other hand, since $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\ge0}$ defines a strongly continuous semigroup in $H^{1/2}(\mathbb{R}^d)$, we have from (3.26)

$$\|e^{-t\mathcal{P}_{\mathcal{A}}} S_{\mathcal{A},1}h\|_{H^{\frac12}(\mathbb{R}^d)} \le C\|S_{\mathcal{A},1}h\|_{H^{\frac12}(\mathbb{R}^d)} \le C\|h\|_{H^{\frac12}(\mathbb{R}^d)}, \qquad 0<t\le 2. \tag{3.31}$$

Finally let us prove $V_1(t)h = \mathrm{d}/\mathrm{d}t\ U_1(t)h$. To see this, we note that for each $\delta \in (0, 1/2)$, $\mathrm{d}/\mathrm{d}t\ U_1(t+\delta)h$ is the unique weak solution to (3.1) with $F = \tau_\delta\big(\nabla \cdot \partial_t M_\chi \Pi h + \partial_t M_\chi G_\zeta h + \partial_t R h\big)$ and $g = \mathrm{d}/\mathrm{d}t\ U_1(\delta)h$, where $\tau_\delta f(t) = f(t+\delta)$. Hence we can write

$$\frac{\mathrm{d}}{\mathrm{d}t} U_1(t+\delta)h = e^{-t\mathcal{P}_\mathcal{A}} \frac{\mathrm{d}}{\mathrm{d}t} U_1(\delta)h + W_\mathcal{A}^{(\delta)}(t)h, \qquad \delta \in (0, 1/2), \tag{3.32}$$

where $W_\mathcal{A}^{(\delta)}h$ is the solution to (3.1) with $F = \tau_\delta\big(\nabla \cdot \partial_t M_\chi \Pi h + \partial_t M_\chi G_\zeta h + \partial_t R h\big)$ and $g = 0$. Then, arguing as in the estimate of $W_\mathcal{A}h$, we obtain a uniform bound of $W_\mathcal{A}^{(\delta)}h$ in $\dot H^1_0(\mathbb{R}^{d+1}_+)$, and it is not difficult to show $W_\mathcal{A}^{(\delta)}h$ weakly converges to $W_\mathcal{A}h$ in $\dot H^1_0(\mathbb{R}^{d+1}_+)$. On the other hand, we have from (3.21)

$$\|e^{-t\mathcal{P}_\mathcal{A}} S_{\mathcal{A},1}h - e^{-t\mathcal{P}_\mathcal{A}} \frac{\mathrm{d}}{\mathrm{d}t} U_1(\delta)h\|_{L^2(\mathbb{R}^d)} \le C t^{-\frac12} \|S_{\mathcal{A},1}h - \frac{\mathrm{d}}{\mathrm{d}t} U_1(\delta)h\|_{H^{-\frac12}(\mathbb{R}^d)},$$

which converges to zero as $\delta \to 0$ by the definition of $S_{\mathcal{A},1}h$. Hence, the right-hand side of (3.32) converges to $V_1(t)h$ in $L^2_{loc}(\mathbb{R}^{d+1}_+)$. On the other hand, since $\mathrm{d}/\mathrm{d}t\ U_1 h \in \bigcap_{\delta>0} H^1(\mathbb{R}^{d+1}_{\delta,+})$, the left-hand side of (3.32) converges to $\mathrm{d}/\mathrm{d}t\ U_1(t)h$ in $L^2(\mathbb{R}^d)$ as $\delta \to 0$ for each $t > 0$. Thus we have $\mathrm{d}/\mathrm{d}t\ U_1(t)h = V_1(t)h$, and then (3.30) - (3.31) imply $\|\mathrm{d}/\mathrm{d}t\ U_1(t)h\|_{H^{1/2}(\mathbb{R}^d)} \le C\|h\|_{H^{1/2}(\mathbb{R}^d)}$ for $0 < t \le 2$. For $t > 2$ we have from the equality $\mathrm{d}/\mathrm{d}t\ U_1(t)h = \mathrm{d}/\mathrm{d}t\ e^{-t\mathcal{P}_\mathcal{A}}h$ that

$$\|\frac{\mathrm{d}}{\mathrm{d}t}\ U_1(t)h\|_{H^{\frac12}(\mathbb{R}^d)} \le C\|e^{-s\mathcal{P}_\mathcal{A}}h\ |_{s=1}\ \|_{L^2(\mathbb{R}^d)} \le C\|h\|_{L^2(\mathbb{R}^d)}.$$

Hence, (3.23) holds when $h \in D_{L^2}(\mathcal{P}_\mathcal{A})$. Note that $D_{L^2}(\mathcal{P}_\mathcal{A})$ is dense in $H^{1/2}(\mathbb{R}^d)$. Hence the estimate (3.23) for $h \in D_{L^2}(\mathcal{P}_\mathcal{A})$ is extended to all $h \in H^{1/2}(\mathbb{R}^d)$ by the density argument. The proof is complete. □

Corollary 3.6. *It follows that* $D_{L^2}(\mathcal{P}_\mathcal{A}) = D_{L^2}(\mathcal{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$ *with equivalent norms.*

Proof. Note that the same result as in Lemma 3.5 is valid for $\mathrm{d}/\mathrm{d}t\ U_{\mathcal{A}^*,1}(t)$. On the other hand, by Lemma 3.2 and the definition (3.6), it is straightforward to see

$$\sup_{t>0} \|\frac{\mathrm{d}}{\mathrm{d}t} U_{\mathcal{A},0}(t)h\|_{L^2(\mathbb{R}^d)} + \sup_{t>0} \|\frac{\mathrm{d}}{\mathrm{d}t} U_{\mathcal{A}^*,0}(t)h\|_{L^2(\mathbb{R}^d)} < \infty \quad \text{for } h \in C_0^\infty(\mathbb{R}^d)$$

by the definition (3.6). Thus we have

$$\sup_{t>0} \|\frac{\mathrm{d}}{\mathrm{d}t} e^{-t\mathcal{P}_\mathcal{A}}h\|_{L^2(\mathbb{R}^d)} + \sup_{t>0} \|\frac{\mathrm{d}}{\mathrm{d}t} e^{-t\mathcal{P}_{\mathcal{A}^*}}h\|_{L^2(\mathbb{R}^d)} < \infty \quad \text{for } h \in C_0^\infty(\mathbb{R}^d). \tag{3.33}$$

Hence Proposition 2.2, together with Corollary 3.4, shows $D_{L^2}(\mathcal{P}_\mathcal{A}) = D_{L^2}(\mathcal{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$ with equivalent norms. The proof is complete. □

Remark 3.7. Let $\Phi : A \longmapsto \mu_\mathcal{A}$ be the map defined by (A.1). The arguments of the present section essentially rely on the integration by parts technique, and in particular, we did not use the mapping properties of the pseudo-differential operator $\Phi(A)(\cdot, D_x)$ such as the equivalence $\|\Phi(A)(\cdot, D_x)f\|_{L^2(\mathbb{R}^d)} + \|f\|_{L^2(\mathbb{R}^d)} \simeq \|f\|_{H^1(\mathbb{R}^d)}$. Since the above proof implies the identity $-\mathcal{P}_\mathcal{A} = i\Phi(A)(\cdot, D_x) + S_{\mathcal{A},1}$, where $S_{\mathcal{A},1} = \lim_{t\to 0} \mathrm{d}/\mathrm{d}t\ U_{\mathcal{A},1}(t)$ is a lower order operator, our result actually gives an alternative proof (although it is lengthy) of the mapping properties of $\Phi(A)(\cdot, D_x)$ in $H^s(\mathbb{R}^d)$ which are well known in the theory of pseudo-differential operators with nonsmooth coefficients; cf. [1, 12, 25, 31, 35]. Especially, the fact that $i\Phi(A)(\cdot, D_x)$

in $L^2(\mathbb{R}^d)$ with the domain $H^1(\mathbb{R}^d)$ generates a strongly continuous and analytic semigroup in $L^2(\mathbb{R}^d)$ is recovered by regarding $i\Phi(A)(\cdot, D_x)$ as a perturbation from $-\mathcal{P}_{\mathcal{A}}$. More precise statements will be given in Theorem A.2.

3.2. **Domain of Dirichlet-Neumann map in $L^2(\mathbb{R}^d)$.** In this section we consider the domain of the Dirichlet-Neumann map in $L^2(\mathbb{R}^d)$. The result is stated as follows.

Theorem 3.8. *It follows that $D_{L^2}(\Lambda_{\mathcal{A}}) = D_{L^2}(\Lambda_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$ with equivalent norms.*

Proof. It suffices to show $D_{L^2}(\Lambda_{\mathcal{A}}) = H^1(\mathbb{R}^d)$. Let $\mu_{\mathcal{A}}$ be as in (3.3). Let $f \in H^1(\mathbb{R}^d)$. Since we have already shown $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ in the previous section, Proposition 2.3 gives

$$\mathcal{P}_{\mathcal{A}} f = M_{1/b}\Lambda_{\mathcal{A}} f + M_{\mathbf{r_2}/b} \cdot \nabla_x f, \tag{3.34}$$

while, as stated in Remark 3.7, we have

$$\mathcal{P}_{\mathcal{A}} f = -i\mu_{\mathcal{A}}(\cdot, D_x) f - S_{\mathcal{A},1} f = -iM_{1/b}\lambda_{\mathcal{A}}(\cdot, D_x) f - S_{\mathcal{A},1} f + M_{\mathbf{r_2}/b} \cdot \nabla_x f, \tag{3.35}$$

where $S_{\mathcal{A},1}$ is the linear operator given in the proof of Lemma 3.5 and $\lambda_{\mathcal{A}}(x, \xi) = b(x)\mu_{\mathcal{A}}(x, \xi) + \mathbf{r_2}(x) \cdot \xi$. Now let us define the linear operator $\mathcal{J}_{\mathcal{A}}$ in $L^2(\mathbb{R}^d)$ by

$$D_{L^2}(\mathcal{J}_{\mathcal{A}}) = H^1(\mathbb{R}^d), \qquad \mathcal{J}_{\mathcal{A}} f = -i\lambda_{\mathcal{A}}(\cdot, D_x) f - M_b S_{\mathcal{A},1} f, \tag{3.36}$$

which gives $\Lambda_{\mathcal{A}} f = \mathcal{J}_{\mathcal{A}} f$ for $f \in H^1(\mathbb{R}^d)$ by (3.34) - (3.35). Thanks to Lemma A.1 and Theorem A.2 together with Remark 3.7 the operator $i\lambda_{\mathcal{A}}(\cdot, D_x)$ in $L^2(\mathbb{R}^d)$ with $D_{L^2}(\lambda_{\mathcal{A}}(\cdot, D_x)) = H^1(\mathbb{R}^d)$ generates a strongly continuous and analytic semigroup in $L^2(\mathbb{R}^d)$. On the other hand, we have from (3.26)

$$\|M_b S_{\mathcal{A},1} f\|_{L^2(\mathbb{R}^d)} \le C\|S_{\mathcal{A},1} f\|_{H^s(\mathbb{R}^d)} \le C\|f\|_{H^s(\mathbb{R}^d)} \quad \text{for all } s \in (0, \frac{1}{2}].$$

In particular, $M_b S_{\mathcal{A},1}$ is a lower order operator, and hence the standard perturbation theory (cf. [26, Section 2.4]) implies that $-\mathcal{J}_{\mathcal{A}}$ generates a strongly continuous and analytic semigroup $\{e^{-t\mathcal{J}_{\mathcal{A}}}\}_{t\ge 0}$ in $L^2(\mathbb{R}^d)$. Now we observe that $u(t) = e^{-t\mathcal{J}_{\mathcal{A}}} f$, $f \in L^2(\mathbb{R}^d)$, satisfies $0 = \partial_t u + \mathcal{J}_{\mathcal{A}} u = \partial_t u + \Lambda_{\mathcal{A}} u$ for $t > 0$ and $u(t) \to f$ as $t \to 0$ in $L^2(\mathbb{R}^d)$. Thus we also have the representation $u(t) = e^{-t\Lambda_{\mathcal{A}}} f$, that is, $e^{-t\mathcal{J}_{\mathcal{A}}} = e^{-t\Lambda_{\mathcal{A}}}$ for all $t \ge 0$. This proves $\mathcal{J}_{\mathcal{A}} = \Lambda_{\mathcal{A}}$, i.e., $D_{L^2}(\Lambda_{\mathcal{A}}) = D_{L^2}(\mathcal{J}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$. The proof is complete. □

3.3. **Domain of Poisson operator in $H^1(\mathbb{R}^d)$.** In this section we study the Poisson operator in $H^1(\mathbb{R}^d)$. First we consider the operator $\mathcal{Q}_{\mathcal{A}} = M_b(M_{1/\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*$ in $L^2(\mathbb{R}^d)$, which is the key step to proving $D_{H^1}(\mathcal{P}_{\mathcal{A}}) = H^2(\mathbb{R}^d)$.

Theorem 3.9. *Let $\mathcal{Q}_{\mathcal{A}} = M_{1/b}(M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*$. Then $D_{L^2}(\mathcal{Q}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ with equivalent norms and $\mathcal{Q}_{\mathcal{A}} f = M_{1/b}\Lambda_{\mathcal{A}} f - M_{\mathbf{r_1}/b} \cdot \nabla_x f - M_{(\nabla_x \cdot \mathbf{r_1})/b} f$ for $f \in H^1(\mathbb{R}^d)$.*

Proof. Assume that $f \in H^1(\mathbb{R}^d) = D_{L^2}(\Lambda_{\mathcal{A}})$ (by Theorem 3.8). Then for any $g \in D_{L^2}(\mathcal{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$ (by Corollary 3.6) we have from Proposition 2.3

$$\langle f, M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*} g\rangle_{L^2(\mathbb{R}^d)} = \langle f, \Lambda_{\mathcal{A}^*} g + M_{\bar{\mathbf{r}}_1} \cdot \nabla_x g\rangle_{L^2(\mathbb{R}^d)} = \langle \Lambda_{\mathcal{A}} f - \nabla_x \cdot M_{\mathbf{r_1}} f, g\rangle_{L^2(\mathbb{R}^d)}.$$

In particular, we have the estimate $|\langle f, M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*} g\rangle_{L^2(\mathbb{R}^d)}| \le C\|f\|_{H^1(\mathbb{R}^d)}\|g\|_{L^2(\mathbb{R}^d)}$ for all $g \in D_{L^2}(\mathcal{P}_{\mathcal{A}^*})$. This implies $f \in D_{L^2}((M_{\bar{b}}\mathcal{P}_{\mathcal{A}^*})^*) = D_{L^2}(\mathcal{Q}_{\mathcal{A}})$ and

$$\mathcal{Q}_{\mathcal{A}} f = M_{1/b}\Lambda_{\mathcal{A}} f - M_{\mathbf{r_1}/b} \cdot \nabla_x f - M_{(\nabla_x \cdot \mathbf{r_1})/b} f, \quad f \in H^1(\mathbb{R}^d);$$

that is, we have from (3.34)-(3.35)

(3.37)
$$\mathcal{Q}_{\mathcal{A}} f = -i\mu_{\mathcal{A}}(\cdot, D_x) f - M_{(\mathbf{r_1}+\mathbf{r_2})/b} \cdot \nabla_x f - S_{\mathcal{A},1} f - M_{\nabla_x \cdot \mathbf{r_1}} f, \qquad f \in H^1(\mathbb{R}^d).$$

To prove the converse embedding we appeal to the argument of the proof of Theorem 3.8. Set $q_{\mathcal{A}}(x,\xi) = \mu_{\mathcal{A}}(x,\xi) + b(x)^{-1}(\mathbf{r_1}(x) + \mathbf{r_2}(x)) \cdot \xi$ and let $\mathcal{K}_{\mathcal{A}}$ be the linear operator in $L^2(\mathbb{R}^d)$ defined by

$$D_{L^2}(\mathcal{K}_{\mathcal{A}}) = H^1(\mathbb{R}^d), \qquad \mathcal{K}_{\mathcal{A}} f = -iq_{\mathcal{A}}(\cdot, D_x) f - S_{\mathcal{A},1} f - M_{(\nabla_x \cdot \mathbf{r_1})/b} f. \tag{3.38}$$

Then $\mathcal{K}_{\mathcal{A}} f = \mathcal{Q}_{\mathcal{A}} f$ for $f \in H^1(\mathbb{R}^d)$ by (3.37). On the other hand, Lemma A.1 with Remark 3.7 shows that $iq_{\mathcal{A}}(\cdot, D_x)$ generates a strongly continuous and analytic semigroup in $L^2(\mathbb{R}^d)$, and hence so is true for $\mathcal{K}_{\mathcal{A}}$, since the operators $S_{\mathcal{A},1}$ and $M_{(\nabla_x \cdot \mathbf{r_1})/b}$ are of lower order. Then, arguing as in the proof of Theorem 3.8, we conclude that $e^{-t\mathcal{K}_{\mathcal{A}}} = e^{-t\mathcal{Q}_{\mathcal{A}}}$ for all $t \geq 0$. Thus we have $\mathcal{K}_{\mathcal{A}} = \mathcal{Q}_{\mathcal{A}}$, as desired. The proof is complete. ☐

Theorem 3.10. *The restriction of $\{e^{-t\mathcal{P}_{\mathcal{A}}}\}_{t\geq 0}$ in $L^2(\mathbb{R}^d)$ on the invariant subspace $H^1(\mathbb{R}^d)$ defines a strongly continuous and analytic semigroup. Moreover, we have $D_{H^1}(\mathcal{P}_{\mathcal{A}}) = H^2(\mathbb{R}^d)$ with equivalent norms.*

Proof. The first statement is trivial since we have already proved $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$. Thus it suffices to show $D_{H^1}(\mathcal{P}_{\mathcal{A}}) = H^2(\mathbb{R}^d)$. Since $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = D_{L^2}(\mathcal{P}_{\mathcal{A}^*})$ $= H^1(\mathbb{R}^d)$ by Corollary 3.6, we have from (2.7)

$$\begin{aligned} u \in H^2(\mathbb{R}^d) &\Longleftrightarrow u \in D_{L^2}(\mathcal{A}') \Longleftrightarrow u \in D_{L^2}(\mathcal{Q}_{\mathcal{A}}\mathcal{P}_{\mathcal{A}}) \\ &\Longleftrightarrow \mathcal{P}_{\mathcal{A}} u \in D_{L^2}(\mathcal{Q}_{\mathcal{A}}) \Longleftrightarrow \mathcal{P}_{\mathcal{A}} u \in H^1(\mathbb{R}^d), \end{aligned}$$

where we have used $D_{L^2}(\mathcal{Q}_{\mathcal{A}}) = H^1(\mathbb{R}^d)$ by Theorem 3.9. It is also easy to see that $\|u\|_{H^2(\mathbb{R}^d)} \simeq \|\mathcal{P}_{\mathcal{A}} u\|_{H^1(\mathbb{R}^d)} + \|u\|_{H^1(\mathbb{R}^d)}$. The proof is complete. ☐

3.4. Further estimates for remainder part of Poisson operator in $H^s(\mathbb{R}^d)$.

Let $S_{\mathcal{A},1}$ be the bounded linear operator in $H^{1/2}(\mathbb{R}^d)$ defined by

$$S_{\mathcal{A},1} h = \lim_{t\to 0} \mathrm{d}/\,\mathrm{d}t\; U_{\mathcal{A},1}(t) h$$

as in the proof of Lemma 3.5. In this section we study the mapping property of $S_{\mathcal{A},1}$ in $H^s(\mathbb{R}^d)$.

Proposition 3.11. *Let $0 < s, \epsilon < 1$. Then we have*

$$\|S_{\mathcal{A},1} h\|_{H^s(\mathbb{R}^d)} \leq C\|h\|_{H^s(\mathbb{R}^d)}, \qquad \|S_{\mathcal{A},1} h\|_{H^1(\mathbb{R}^d)} \leq C\|h\|_{H^{1+\epsilon}(\mathbb{R}^d)}. \tag{3.39}$$

Proof. Let $h \in \mathcal{S}(\mathbb{R}^d)$. The estimate (3.39) with $s \in (0, 1/2]$ is already proved by (3.26). Next we consider the case $s \in (1/2, 1)$. Let us recall that $U_{\mathcal{A},1}(t)h$ is the solution to (3.1) with F given by (3.17) and $g = 0$. By [28, Theorem 5.1] the characterization of $D_{L^2}(\mathcal{P}_{\mathcal{A}}) = D_{L^2}(\mathcal{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$ provides the integral representation of $U_{\mathcal{A},1}(t)h$ such that

$$U_{\mathcal{A},1}(t)h = \int_0^t e^{-(t-s)\mathcal{P}_{\mathcal{A}}} \int_s^\infty e^{-(\tau-s)\mathcal{Q}_{\mathcal{A}}} M_{1/b}\big(\nabla \cdot M_\chi \Pi h + M_\chi G_\zeta h + Rh\big)\,\mathrm{d}\tau\,\mathrm{d}s,$$

which gives

$$S_{\mathcal{A},1} h = \int_0^\infty e^{-\tau\mathcal{Q}_{\mathcal{A}}} M_{1/b}\big(\nabla \cdot M_\chi \Pi h + M_\chi G_\zeta h + Rh\big)\,\mathrm{d}\tau. \tag{3.40}$$

Here we will only show

$$\| \int_0^\infty e^{-\tau \mathcal{Q}_\mathcal{A}} M_{1/b} \nabla_x \cdot M_\chi \Pi' h \, \mathrm{d}\tau \|_{\dot{H}^s(\mathbb{R}^d)} \le C \|h\|_{H^s(\mathbb{R}^d)}, \tag{3.41}$$

for the other terms are treated in a similar manner. We note that the term $\Pi' h$ is not differentiable in x (see the definition (3.16)), and thus the term $e^{-\tau \mathcal{Q}_\mathcal{A}} M_{1/b} \nabla_x \cdot M_\chi \Pi' h$ in (3.41) has to be interpreted as

(3.42)
$$e^{-\tau \mathcal{Q}_\mathcal{A}} M_{1/b} \nabla_x \cdot M_\chi \Pi' h = (I + \mathcal{Q}_\mathcal{A}) e^{-\tau \mathcal{Q}_\mathcal{A}} \big(- \nabla_x (I + \mathcal{P}_{\mathcal{A}^*})^{-1} M_{1/\bar{b}} \big)^* \cdot M_\chi \Pi' h,$$

where we have used the formal adjoint relation

$$(I + \mathcal{Q}_\mathcal{A})^{-1} M_{1/b} \nabla_x = \big(- \nabla_x (I + \mathcal{P}_{\mathcal{A}^*})^{-1} M_{1/\bar{b}} \big)^*.$$

Since $\big(-\nabla_x (I + \mathcal{P}_{\mathcal{A}^*})^{-1} M_{1/\bar{b}}\big)^*$ is a bounded linear operator in $L^2(\mathbb{R}^d)$ by $D_{L^2}(\mathcal{P}_{\mathcal{A}^*}) = H^1(\mathbb{R}^d)$, the right-hand side of (3.42) is well defined for each $\tau > 0$. Then from $\mathcal{Q}_\mathcal{A} e^{-\tau \mathcal{Q}_\mathcal{A}} = - \mathrm{d}/\mathrm{d}\tau e^{-\tau \mathcal{Q}_\mathcal{A}}$ and from the integration by parts together with $\Pi' h|_{t=0} = 0$ for $h \in \mathcal{S}(\mathbb{R}^d)$ (due to the definition of Π') the estimate (3.41) is essentially reduced to

(3.43)
$$\| \int_0^\infty e^{-\tau \mathcal{Q}_\mathcal{A}} B^* \cdot M_\chi \partial_\tau \Pi' h \, \mathrm{d}\tau \|_{\dot{H}^s(\mathbb{R}^d)} \le C \|h\|_{H^s(\mathbb{R}^d)}, \ B = -\nabla_x (I + \mathcal{P}_{\mathcal{A}^*})^{-1} M_{1/\bar{b}},$$

since the other terms are of lower order. We appeal to the duality argument and consider the integral

$$\begin{aligned} & \langle (-\Delta_x)^{\frac{s}{2}} \int_0^\infty e^{-\tau \mathcal{Q}_\mathcal{A}} B^* \cdot M_\chi \partial_\tau \Pi' h \, \mathrm{d}\tau, \varphi \rangle_{L^2(\mathbb{R}^d)} \\ & = \int_0^\infty \langle B^* \cdot M_\chi \partial_\tau \Pi' h, M_{\bar{b}} e^{-\tau \mathcal{P}_{\mathcal{A}^*}} M_{1/\bar{b}} (-\Delta_x)^{\frac{s}{2}} \varphi \rangle_{L^2(\mathbb{R}^d)} \, \mathrm{d}\tau \end{aligned} \tag{3.44}$$

for $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then we have

(3.45)
$$\begin{aligned} \text{R.H.S. of (3.44)} \le C \Big(& \int_0^\infty M_\chi \tau^{2(1-s)} \|\partial_\tau \Pi' h\|_{L^2(\mathbb{R}^d)}^2 \frac{\mathrm{d}\tau}{\tau} \Big)^{\frac{1}{2}} \\ & \times \Big(\int_0^\infty M_\chi \tau^{2s} \| M_{\bar{b}} e^{-\tau \mathcal{P}_{\mathcal{A}^*}} M_{1/\bar{b}} (-\Delta_x)^{\frac{s}{2}} \varphi \|_{L^2(\mathbb{R}^d)}^2 \frac{\mathrm{d}\tau}{\tau} \Big)^{\frac{1}{2}}. \end{aligned}$$

By (3.16) we see $\partial_\tau \Pi' = G_{i(1+\tau i \mu_\mathcal{A}) A' \nabla_x \mu_\mathcal{A}}$, and it is straightforward to check that

$$p(x, \xi, \tau) = \tau^{1-s} |\xi|^{-s} (1 + \tau i \mu_\mathcal{A}) i A' \nabla_x \mu_\mathcal{A}, \quad s \in (0,1),$$

satisfies the condition (3.10). Hence we have from (3.11)

$$\begin{aligned} \int_0^\infty M_\chi \tau^{2(1-s)} \|\partial_\tau \Pi' h\|_{L^2(\mathbb{R}^d)}^2 \frac{\mathrm{d}\tau}{\tau} &= \int_0^\infty M_\chi \| G_p (-\Delta_x)^{\frac{s}{2}} h \|_{L^2(\mathbb{R}^d)}^2 \frac{\mathrm{d}\tau}{\tau} \\ &\le C \| (-\Delta_x)^{\frac{s}{2}} h \|_{L^2(\mathbb{R}^d)}^2. \end{aligned}$$

Next we estimate the second integral of the right-hand side of (3.45). By the duality argument and $D_{L^2}(\mathcal{Q}_\mathcal{A}) = H^1(\mathbb{R}^d)$ it is easy to see $\| M_{\bar{b}} e^{-\tau \mathcal{P}_{\mathcal{A}^*}} M_{1/\bar{b}} (-\Delta_x)^{\frac{\kappa}{2}} \varphi \|_{L^2(\mathbb{R}^d)} \le C \tau^{-\kappa} \|\varphi\|_{L^2(\mathbb{R}^d)}$ for any $\kappa \in [0,1]$ and $\tau \in (0,2)$. Let $\{\psi_r\}_{r>0}$ be the family of functions introduced in Appendix A.2 (with s replaced by r). Then one can verify the estimates

$$\begin{aligned}
&\tau^s\|M_{\bar b}e^{-\tau\mathcal{P}_{\mathcal{A}^*}}M_{1/\bar b}(-\Delta_x)^{\frac{s}{2}}\psi_r*\varphi\|_{L^2(\mathbb{R}^d)}\\
&\quad=\tau^s\|M_{\bar b}e^{-\tau\mathcal{P}_{\mathcal{A}^*}}M_{1/\bar b}(-\Delta_x)^{\frac12}(-\Delta_x)^{\frac{s-1}{2}}\psi_r*\varphi\|_{L^2(\mathbb{R}^d)}\\
&\quad\le C\tau^{-1+s}r^{1-s}\|\varphi\|_{L^2(\mathbb{R}^d)},\qquad 0<r\le\tau<2,\\
&\tau^s\|M_{\bar b}e^{-\tau\mathcal{P}_{\mathcal{A}^*}}M_{1/\bar b}(-\Delta_x)^{\frac{s}{2}}\psi_r*\varphi\|_{L^2(\mathbb{R}^d)}\\
&\quad\le\tau^s\|(-\Delta_x)^{\frac{s}{2}}\psi_r*\varphi\|_{L^2(\mathbb{R}^d)}\\
&\quad\le C\tau^s r^{-s}\|\varphi\|_{L^2(\mathbb{R}^d)},\qquad 0<\tau\le r,\ 0<\tau<2.
\end{aligned}$$

Hence the Schur lemma [14, pp. 643-644] yields

$$\int_0^\infty M_\chi\tau^{2s}\|M_{\bar b}e^{-\tau\mathcal{P}_{\mathcal{A}^*}}M_{1/\bar b}(-\Delta_x)^{\frac{s}{2}}\varphi\|^2_{L^2(\mathbb{R}^d)}\frac{\mathrm{d}\tau}{\tau}\le C\|\varphi\|^2_{L^2(\mathbb{R}^d)},$$

as desired. This completes the proof of (3.39) with $s\in(0,1)$. To prove the second estimate in (3.39) we go back to the representation (3.40). Here we will only show, instead of (3.43),

$$\|\int_0^\infty e^{-\tau\mathcal{Q}_{\mathcal{A}}}B^*\cdot M_\chi\partial_\tau\Pi'h\,\mathrm{d}\tau\|_{\dot H^1(\mathbb{R}^d)}\le C\|h\|_{H^{1+\epsilon}(\mathbb{R}^d)},\qquad \epsilon\in(0,1).\tag{3.46}$$

We use the identity $\partial_\tau\Pi'h=G_q(I-\Delta_x)^{(1+\epsilon)/2}h$ with

$$q=i(1+|\xi|^2)^{-(1+\epsilon)/2}(1+\tau i\mu_{\mathcal{A}})A'\nabla_x\mu_{\mathcal{A}}.$$

For $h\in\mathcal{S}(\mathbb{R}^d)$ the limit $\lim\limits_{\tau\to0}G_q(I-\Delta_x)^{(1+\epsilon)/2}h$ exists in $L^2(\mathbb{R}^d)$, and we also have

$$\sup_{0<\tau<2}\|G_q(\tau)f\|_{L^2(\mathbb{R}^d)}\le C\|f\|_{L^2(\mathbb{R}^d)},\qquad f\in\mathcal{S}(\mathbb{R}^d).\tag{3.47}$$

The proof of (3.47) is postponed to the appendix. Then from $D_{L^2}(\mathcal{Q}_{\mathcal{A}})=H^1(\mathbb{R}^d)$ we have

(3.48)

$$\begin{aligned}
&\text{L.H.S. of (3.46)}\\
&\quad\le C\|\mathcal{Q}_{\mathcal{A}}\int_0^\infty M_\chi e^{-\tau\mathcal{Q}_{\mathcal{A}}}B^*\cdot G_q(I-\Delta_x)^{\frac{1+\epsilon}{2}}h\,\mathrm{d}\tau\|_{L^2(\mathbb{R}^d)}+\text{ (lower order)}\\
&\quad\le C\lim_{\tau\to0}\|B^*\cdot G_q(I-\Delta_x)^{\frac{1+\epsilon}{2}}h\|_{L^2(\mathbb{R}^d)}\\
&\qquad+C\int_0^\infty M_\chi\|e^{-\tau\mathcal{Q}_{\mathcal{A}}}B^*\cdot\partial_\tau G_q(I-\Delta_x)^{\frac{1+\epsilon}{2}}h\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau+\text{ (lower order)}\\
&\quad\le C\|h\|_{H^{1+\epsilon}(\mathbb{R}^d)}+C\int_0^\infty M_\chi\|\partial_\tau G_q(I-\Delta_x)^{\frac{1+\epsilon}{2}}h\|_{L^2(\mathbb{R}^d)}\,\mathrm{d}\tau+\text{ (lower order)}.
\end{aligned}$$

By the definition of q we see $\partial_\tau G_q=\tau^{-1+\epsilon}G_{\tilde q}$ with $\tilde q=\tau^{1-\epsilon}(iq\mu_{\mathcal{A}}+\partial_\tau q)$ and $\tilde q$ satisfies the condition (3.8) with $T=2$. Thus, (3.9) implies the R.H.S. of (3.48) $\le C\|h\|_{H^{1+\epsilon}(\mathbb{R}^d)}$. The proof is complete. $\square$

3.5. **Proofs of Theorems 1.2, 1.4, and 1.6.**

Proof of Theorem 1.2. The assertion (ii) of Theorem 1.2 with $s = 0$ and $s = 1$ is already proved in Corollary 3.6 and Theorem 3.10. Then the case $s \in (0, 1)$ follows from the interpolation inequality, and the details are omitted. It remains to show the last statement of (i). By Theorem 3.10 we have

$$u \in D_{L^2}(\mathcal{P}_{\mathcal{A}}^2) \Longleftrightarrow \mathcal{P}_{\mathcal{A}} u \in D_{L^2}(\mathcal{P}_{\mathcal{A}}) = H^1(\mathbb{R}^d) \Longleftrightarrow u \in D_{H^1}(\mathcal{P}_{\mathcal{A}}) = H^2(\mathbb{R}^d). \tag{3.49}$$

It is also easy to see the norm equivalence between $H^2(\mathbb{R}^d)$ and $D_{L^2}(\mathcal{P}_{\mathcal{A}}^2)$. Then the sectorial operator $T = I + \mathcal{P}_{\mathcal{A}}$ in $L^2(\mathbb{R}^d)$, which is invertible by [28, Remark 2.6], satisfies

$$(L^2(\mathbb{R}^d), D_{L^2}(T^2))_{\frac{1}{2},2} = (L^2(\mathbb{R}^d), H^2(\mathbb{R}^d))_{\frac{1}{2},2} = H^1(\mathbb{R}^d) = D_{L^2}(T). \tag{3.50}$$

By the Komatsu theorem [15, Theorem 6.6.8] the identity (3.50) implies that the operator T admits a bounded $H^\infty(\Sigma_\varphi)$-calculus in $L^2(\mathbb{R}^d)$ for some $\varphi \in (0, \frac{\pi}{2})$. The proof is complete. □

Proof of Theorem 1.4. The assertions follow from Corollary 3.6 and Theorem 3.9 together with Proposition 2.3. The proof is complete. □

Proof of Theorem 1.6. The case $s = 0$ is already proved by Theorem 3.8. It suffices to consider the endpoint case $s = 1$. Theorem 3.8 implies that $H^1(\mathbb{R}^d)$ is invariant under the action of $\{e^{-t\Lambda_{\mathcal{A}}}\}_{t\ge 0}$ and the restriction of this semigroup in $H^1(\mathbb{R}^d)$ is also analytic and strongly continuous. Hence it suffices to show that the generator of this restriction semigroup satisfies $D_{H^1}(\Lambda_{\mathcal{A}}) = H^2(\mathbb{R}^d)$ with equivalent norms. By the proof of Theorem 3.8 we have

$$\Lambda_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}, \qquad \mathcal{J}_{\mathcal{A}} = -i\lambda_{\mathcal{A}}(\cdot, D_x) - M_b S_{\mathcal{A},1} \quad \text{as an operator in } L^2(\mathbb{R}^d),$$

where $\lambda_{\mathcal{A}}(\cdot, D_x)$ is the pseudo-differential operator with its symbol $\lambda_{\mathcal{A}}(x, \xi) = b(x)\mu_{\mathcal{A}}(x, \xi) + \mathbf{r_2}(x) \cdot \xi$. On the other hand, by Lemma A.1 and Theorem A.2 the operator $i\lambda_{\mathcal{A}}(\cdot, D_x)$ generates a strongly continuous and analytic semigroup in $H^1(\mathbb{R}^d)$, and $D_{H^1}(\lambda_{\mathcal{A}}(\cdot, D_x)) = H^2(\mathbb{R}^d)$ holds with equivalent norms. Then the same is true also for $\mathcal{J}_{\mathcal{A}}$, for b is Lipschitz and $S_{\mathcal{A},1}$ is of lower order by Proposition 3.11. Since it is easy to see that $\Lambda_{\mathcal{A}} = \mathcal{J}_{\mathcal{A}}$ as an operator in $H^1(\mathbb{R}^d)$, we conclude that $D_{H^1}(\Lambda_{\mathcal{A}}) = H^2(\mathbb{R}^d)$ holds with equivalent norms. The proof is complete. □

Remark 3.12. In the proof of Theorem 1.2 we have established the expansion $\mathcal{P}_{\mathcal{A}} = -i\mu_{\mathcal{A}}(\cdot, D_x) + R$, where $\mu_{\mathcal{A}}(\cdot, D_x)$ is the pseudo-differential operator with symbol (1.6) and R is a bounded operator in $H^s(\mathbb{R}^d)$ for $s \in (0, 1)$, while it is a bounded operator from $H^{1+\epsilon}(\mathbb{R}^d)$, $\epsilon > 0$, to $H^1(\mathbb{R}^d)$. Similar expansion is obtained also for $\Lambda_{\mathcal{A}}$. Then one can apply the results of [12, Theorem 4.8] to obtain a stronger statement that $\mathcal{P}_{\mathcal{A}}$ (and $\Lambda_{\mathcal{A}}$) admits a bounded H^∞-calculus in $H^s(\mathbb{R}^d)$, $s \in [0, 1)$. Our proof of bounded H^∞-calculus for $\mathcal{P}_{\mathcal{A}}$ is based on the characterization $D_{H^1}(\mathcal{P}_{\mathcal{A}}) = H^2(\mathbb{R}^d)$ and the Komatsu theorem [15, Proposition 2.7], which is different from the approach in [12].

Appendix A

A.1. **Remark on pseudo-differential operator $\mu_{\mathcal{A}}(\cdot, D_x)$.** In view of the definition (3.3) for $\mu_{\mathcal{A}}(x,\xi)$, which is the root of (3.2) with positive imaginary part, it is natural to introduce the map $\Phi : A \longmapsto \mu_{\mathcal{A}}$, i.e.,

$$\Phi(A) = -\frac{\mathbf{v}(x)\cdot\xi}{2} + i\Big\{\frac{1}{b(x)}\langle A'(x)\xi,\xi\rangle - \frac{1}{4}(\mathbf{v}(x)\cdot\xi)^2\Big\}^{\frac{1}{2}}, \qquad \mathbf{v} = \frac{\mathbf{r_1}+\mathbf{r_2}}{b}, \tag{A.1}$$

where A is a matrix satisfying the ellipticity condition (1.2). We denote by $\mathcal{R}_{Lip}(\Phi)$ the range of Φ for the Lipschitz class of A, that is,

$$\begin{aligned}\mathcal{R}_{Lip}(\Phi) = \{\mu(x,\xi) \in \mathrm{Lip}(\mathbb{R}^d\times\mathbb{R}^d) \mid \text{there is a matrix } A \in (\mathrm{Lip}(\mathbb{R}^d))^{(d+1)\times(d+1)} \\ \text{satisfying (1.2) for some } \nu_1,\nu_2>0 \text{ such that } \mu = \Phi(A)\}.\end{aligned}$$

The next lemma is used in the study of $\Lambda_{\mathcal{A}}$ and $\mathcal{Q}_{\mathcal{A}}$.

Lemma A.1. *Assume that $\mu = \Phi(A) \in \mathcal{R}_{Lip}(\Phi)$ with $A = (a_{i,j})_{1\le i,j\le d+1}$. Set $b = a_{d+1,d+1}$, $\mathbf{r_1} = (a_{j,d+1})_{1\le j\le d}$, and $\mathbf{r_2} = (a_{d+1,j})_{1\le j\le d}$. Then the functions $\lambda(x,\xi)$ and $q(x,\xi)$ defined by*

$$\lambda(x,\xi) = b(x)\mu(x,\xi) + \mathbf{r_2}(x)\cdot\xi, \qquad q(x,\xi) = \mu(x,\xi) + \frac{\mathbf{r_1}(x)+\mathbf{r_2}(x)}{b(x)}\cdot\xi \tag{A.2}$$

belong to $\mathcal{R}_{Lip}(\Phi)$.

Proof. Since μ solves (3.2), λ and q respectively satisfy

$$\begin{aligned}&\lambda^2 + (\mathbf{r_1}(x)-\mathbf{r_2}(x))\cdot\xi\lambda + b(x)\langle A'(x)\xi,\xi\rangle - \mathbf{r_1}(x)\cdot\xi\ \mathbf{r_2}(x)\cdot\xi = 0,\\ &b(x)q^2 - (\mathbf{r_1}(x)+\mathbf{r_2}(x))\cdot\xi q + \langle A'(x)\xi,\xi\rangle = 0.\end{aligned}$$

Set $M' = \big(m_{i,j}\big)_{1\le i,j\le d} = \big(|b|^2 a_{i,j} - \bar{b}a_{d+1,i}a_{j,d+1}\big)_{1\le i,j\le d}$, $\mathbf{s_1} = \bar{b}\mathbf{r_1}$, $\mathbf{s_2} = -\bar{b}\mathbf{r_2}$, and set $N' = A'$, $\mathbf{u_1} = -\mathbf{r_1}$, $\mathbf{u_2} = -\mathbf{r_2}$. Then the matrices

$$M = \begin{pmatrix} M' & \mathbf{s_1} \\ \mathbf{s_2}^{\top} & \bar{b}\end{pmatrix}, \qquad N = \begin{pmatrix} N' & \mathbf{u_1} \\ \mathbf{u_2}^{\top} & b\end{pmatrix} \tag{A.3}$$

satisfy (1.2) and (1.5) (with possibly different ellipticity constants), while we have $\lambda = \Phi(M)$ and $q = \Phi(N)$. The proof is complete. $\square$

As stated in Remark 3.7, in the proof of the characterization $D_{H^s}(\mathcal{P}_{\mathcal{A}}) = H^{1+s}(\mathbb{R}^d)$ we did not use the mapping property of the pseudo-differential operator $\mu_{\mathcal{A}}(\cdot, D_x)$, which has a representation

$$\mu_{\mathcal{A}}(\cdot,D_x)f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d}\mu_{\mathcal{A}}(x,\xi)e^{ix\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi \quad \text{for } f\in\mathcal{S}(\mathbb{R}^d). \tag{A.4}$$

The results of Theorem 1.2 and Proposition 3.11 yield

Theorem A.2. *Let $s \in [0,1]$. Let $\mu = \Phi(A) \in \mathcal{R}_{Lip}(\Phi)$. Then the associated pseudo-differential operator $i\mu(\cdot,D_x)$ generates a strongly continuous and analytic semigroup in $H^s(\mathbb{R}^d)$, and $D_{H^s}(\mu(\cdot,D_x)) = H^{1+s}(\mathbb{R}^d)$ holds with equivalent norms. Moreover, for the Poisson operator $\mathcal{P}_{\mathcal{A}}$ associated with $\mathcal{A} = -\nabla\cdot A\nabla$ we have the identity*

$$\mathcal{P}_{\mathcal{A}}f = -i\mu(\cdot,D_x)f - S_{\mathcal{A},1}f, \qquad f\in H^1(\mathbb{R}^d), \tag{A.5}$$

where $S_{\mathcal{A},1}$ is the linear operator defined in the proof of Lemma 3.5*, which is bounded in $H^s(\mathbb{R}^d)$, $s \in (0,1)$. In particular, there exist $\lambda_0 \geq 0$ such that $\lambda_0 - i\mu(\cdot, D_x)$ admits a bounded $H^\infty(\Sigma_\varphi)$-calculus in $L^2(\mathbb{R}^d)$ for some $\varphi \in (0, \frac{\pi}{2})$.*

Remark A.3. In fact, by applying the general results of [12] for pseudo-differential operators with nonsmooth symbols, it follows that $\lambda_0 - i\mu(\cdot, D_x)$ admits a bounded H^∞-calculus in $H^s(\mathbb{R}^d)$, $s \in [0,1)$. In this sense, the properties of $i\mu(\cdot, D_x)$ stated in Theorem A.2 themselves are not essentially new. As commented in Remark 3.7, the special feature of our proof is that we use the information of $\mathcal{P}_{\mathcal{A}}$ to derive the properties of $-i\mu(\cdot, D_x)$, where the underlying key structure is the factorizations of $\mathcal{A}'$ and $\mathcal{A}$ in Theorem 1.4.

Proof of Theorem A.2. For $f \in H^{1+s}(\mathbb{R}^d)$ we define $\mu(\cdot, D_x) f = \lim_{n\to\infty} \mu(\cdot, D_x) f_n$ in $H^s(\mathbb{R}^d)$, where $\{f_n\}$ is a sequence in $\mathcal{S}(\mathbb{R}^d)$ converging to f in $H^{1+s}(\mathbb{R}^d)$. This is well defined since (A.5) holds for $f \in \mathcal{S}(\mathbb{R}^d)$, and then Theorem 1.2 and Proposition 3.11 imply $\|\mu(\cdot, D_x) f\|_{H^s(\mathbb{R}^d)} \leq \|\mathcal{P}_{\mathcal{A}} f\|_{H^s(\mathbb{R}^d)} + \|S_{\mathcal{A},1} f\|_{H^s(\mathbb{R}^d)} \leq C\|f\|_{H^{1+s}(\mathbb{R}^d)}$ for $f \in \mathcal{S}(\mathbb{R}^d)$. Since $D_{H^s}(\mathcal{P}_{\mathcal{A}}) = H^{1+s}(\mathbb{R}^d)$ and $\mathcal{P}_{\mathcal{A}}$ is closed in $H^s(\mathbb{R}^d)$, we observe also from Proposition 3.11 that the above realization of $i\mu(\cdot, D_x)$ in $H^s(\mathbb{R}^d)$ satisfies (A.5) for any $f \in H^{1+s}(\mathbb{R}^d)$. Hence $i\mu(\cdot, D_x)$ defined above is a perturbation from $-\mathcal{P}_{\mathcal{A}}$ by $S_{\mathcal{A},1}$ which is a lower order operator, and the desired properties of $i\mu(\cdot, D_x)$ then follow from the ones of $-\mathcal{P}_{\mathcal{A}}$ by the perturbation theory of sectorial operators [15, Proposition 5.5.3]. The proof is complete. □

A.2. **Proofs of Lemma 3.1 and** (3.47).

Proof of Lemma 3.1. We set

$$G_p(x, y, t) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} p(x, \xi, t) e^{it\mu_{\mathcal{A}}(x,\xi,t)} e^{iy\cdot\xi} \, \mathrm{d}\xi. \tag{A.6}$$

Then $\big(G_p(t)h\big)(x) = (2\pi)^{d/2} \int_{\mathbb{R}^d} G_p(x, x-y, t) h(y) \, \mathrm{d}y$, and thus it suffices to show

$$|G_p(x, y, t)| \leq C t^{-d} (1 + \frac{|y|}{t})^{-d-\epsilon}, \qquad 0 < t < T, \ x, y \in \mathbb{R}^d, \tag{A.7}$$

for some $\epsilon > 0$. When $|y| \leq t$ we have from (3.4) and (3.8)

$$|G_p(x, y, t)| \leq C \int_{\mathbb{R}^d} e^{-ct|\xi|} \, \mathrm{d}\xi \leq C t^{-d} \leq C t^{-d} (1 + \frac{|y|}{t})^{-d-1}.$$

Next we consider the case $|y| \geq t$. For any multi-index α with its length $|\alpha| = j_0$, integration by parts yields

$$\begin{aligned} y^\alpha G_p(x, y, t) &= \frac{i^{j_0}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (\partial_\xi^\alpha p) e^{it\mu_{\mathcal{A}}} e^{iy\cdot\xi} \, \mathrm{d}\xi \\ &\quad + \frac{i^{j_0}}{(2\pi)^{\frac{d}{2}}} \sum_{\alpha \geq \beta, |\beta| \neq 0} C_{\alpha,\beta} \int_{\mathbb{R}^d} (\partial_\xi^{\alpha-\beta} p)(\partial_\xi^\beta e^{it\mu_{\mathcal{A}}}) e^{iy\cdot\xi} \, \mathrm{d}\xi \\ &=: I + II. \end{aligned}$$

Let $\chi_R(\xi)$ be a smooth function such that $\chi_R = 1$ for $|\xi| \leq R$, $\chi_R = 0$ for $|\xi| \geq 2R$, and $\|\nabla^k \chi_R\|_{L^\infty} \leq C R^{-k}$. We divide I into a low frequency part I_1 and a high frequency part I_2, where

$$I_1 = \frac{i^{j_0}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \chi_R \cdots \, \mathrm{d}\xi, \qquad I_2 = \frac{i^{j_0}}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (1 - \chi_R) \cdots \, \mathrm{d}\xi.$$

Then (3.8) leads to $|I_1| \le Ct^{l_{j_0}} \int_{|\xi|\le 2R} |\xi|^{-j_0+l_{j_0}}\,\mathrm{d}\xi \le Ct^{l_{j_0}} R^{d-j_0+l_{j_0}}$, while integration by parts combined with (3.4) and (3.8) gives $|y^\gamma I_2| \le C\int_{|\xi|\ge R} |\xi|^{-d-1}\,\mathrm{d}\xi \le CR^{-1}$ for any multi-index γ satisfying $|\gamma| = d+1-j_0$. Similarly, we divide II into II_1 and II_2, where

$$II_1 = \frac{i^{j_0}}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} \chi_R \cdots \mathrm{d}\xi, \qquad II_2 = \frac{i^{j_0}}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} (1-\chi_R)\cdots \mathrm{d}\xi.$$

Then we have from $|\beta| \ge 1$ that $|II_1| \le Ct\int_{|\xi|\le 2R} |\xi|^{-j_0+1}\,\mathrm{d}\xi \le CtR^{d-j_0+1}$, while II_2 is estimated as $|y^\gamma II_2| \le CR^{-1}$. Collecting these, we see that

$$\begin{aligned}|G_p(x,y,t)| &\le C|y|^{-j_0}(I_1+I_2+II_1+II_2)\\ &\le C|y|^{-j_0}\big(t^{l_{j_0}}R^{d-j_0+l_{j_0}} + tR^{d-j_0+1} + |y|^{-d-1+j_0}R^{-1}\big).\end{aligned}$$

By taking

$$R = t^{-\frac{l_{j_0}}{d+1-j_0+l_{j_0}}}|y|^{-\frac{d+1-j_0}{d+1-j_0+l_{j_0}}} \text{ if } l_{j_0} \in (0,1], \quad R = t^{-\frac{1}{d+2-j_0}}|y|^{-\frac{d+1-j_0}{d+2-j_0}} \text{ if } l_{j_0} \ge 1,$$

we get the desired estimate (A.7) also for $|y| \ge t$. Now the proof of (3.9) is complete.

To prove (3.11) let $\psi \in C_0^\infty(\mathbb{R}^d)$ be a real-valued function with zero average such that

$$\int_0^\infty \|\psi_s * f\|^2_{L^2(\mathbb{R}^d)}\frac{\mathrm{d}s}{s} = \|f\|^2_{L^2(\mathbb{R}^d)}, \qquad f \in L^2(\mathbb{R}^d).$$

Here $\psi_s(x) = s^{-d}\psi(x/s)$. We may take $\psi = \Delta\tilde{\psi}$ so that $\|s^{-1}\nabla_x\tilde{\psi}_s * f\|_{L^2(\mathbb{R}^d)} \le C\|f\|_{L^2(\mathbb{R}^d)}$ holds. Thanks to (3.9) we have $\|G_p(t)\psi_s * h\|_{L^2(\mathbb{R}^d)} \le C\|h\|_{L^2(\mathbb{R}^d)}$ for all $t, s > 0$. Moreover, when $t \ge s > 0$ we apply (3.9) to p replaced by $p_1 = t\xi p$ and get

$$\begin{aligned}\|G_p(t)\psi_s * h\|_{L^2(\mathbb{R}^d)} &= \|G_p(t)\nabla_x\cdot\nabla_x\tilde{\psi}_s * h\|_{L^2(\mathbb{R}^d)}\\ &= t^{-1}\|G_{p_1}(t)\cdot\nabla_x\tilde{\psi}_s * h\|_{L^2(\mathbb{R}^d)}\\ &\le Ct^{-1}\|\nabla_x\tilde{\psi}_s * h\|_{L^2(\mathbb{R}^d)} \le Ct^{-1}s\|h\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

When $s \ge t > 0$ we take $l = \min\{l_0, \cdots, l_{d+1}\} > 0$ and set $p_2 = (t|\xi|)^{-l/2}p$. Then it is easy to see that p_2 satisfies (3.8). Hence we have

$$\begin{aligned}\|G_p(t)\psi_s * h\|_{L^2(\mathbb{R}^d)} &= t^{\frac{l}{2}}\|G_{p_2}(t)(-\Delta_x)^{\frac{l}{4}}\psi_s * h\|_{L^2(\mathbb{R}^d)}\\ &\le Ct^{\frac{l}{2}}\|(-\Delta_x)^{\frac{l}{4}}\psi_s * h\|_{L^2(\mathbb{R}^d)} \le Ct^{\frac{l}{2}}s^{-\frac{l}{2}}\|h\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

Now we can apply the Schur lemma (cf. see [14, pp. 643-644]) to $\{G_p(t)\}_{t>0}$, which leads to (3.11). The proof of Lemma 3.1 is complete. □

Proof of (3.47). With the notation (A.6) it suffices to show that

$$|G_q(x,y,t)| \le C\min\{|y|^{-d+\delta},\ |y|^{-d-\delta}\}, \qquad x, y \in \mathbb{R}^d,\ 0 < t < 2 \tag{A.8}$$

for some $\delta > 0$, where $q(x,\xi,t) = i(1+|\xi|^2)^{-(1+\epsilon)/2}(1+it\mu_{\mathcal{A}})A'\nabla_x\mu_{\mathcal{A}}$. For any multi-index α with $d-1 \le |\alpha| \le d$ we have

$$\begin{aligned}y^\alpha G_q(x,y,t) &= \frac{i^{|\alpha|}}{(2\pi)^{\frac{d}{2}}}\sum_{\alpha\ge\beta} C_{\alpha,\beta}\int_{\mathbb{R}^d}(\partial_\xi^{\alpha-\beta}q)(\partial_\xi^\beta e^{it\mu_{\mathcal{A}}})e^{iy\cdot\xi}\,\mathrm{d}\xi\\ &= \frac{i^{|\alpha|}}{(2\pi)^{\frac{d}{2}}}\sum_{\alpha\ge\beta} C_{\alpha,\beta}\Big(\int_{\mathbb{R}^d}\chi_R\cdots\mathrm{d}\xi + \int_{\mathbb{R}^d}(1-\chi_R)\cdots\mathrm{d}\xi\Big).\end{aligned}$$

Here χ_R is the cut-off function as in the proof of (A.7). By the definition of q we see that

$$|\int_{\mathbb{R}^d} \chi_R \cdots \,\mathrm{d}\xi| \le C\int_{|\xi|\le 2R} |\xi|^{-|\alpha|+1}(1+|\xi|)^{-1-\epsilon}\,\mathrm{d}\xi \le CR^{d+1-|\alpha|}$$

and

$$|y\int_{\mathbb{R}^d} (1-\chi_R) \cdots \,\mathrm{d}\xi| \le C\int_{|\xi|\ge R} |\xi|^{-|\alpha|}(1+|\xi|)^{-1-\epsilon}\,\mathrm{d}\xi \le CR^{d-|\alpha|-1-\epsilon}.$$

Thus it follows that $|G_q(x,y,t)| \le C|y|^{-|\alpha|}(R^{d+1-|\alpha|} + |y|^{-1}R^{d-|\alpha|-1-\epsilon})$ for $x, y \in \mathbb{R}^d$ and $0 < t < 2$. If $|y| \le 1$, then we take $|\alpha| = d-1$, while if $|y| > 1$, then take $|\alpha| = d$. Then, putting $R = |y|^{-\kappa}$ with sufficiently small $\kappa > 0$, we get (A.8). The proof is complete. □

A.3. **Definition of bounded H^∞-calculus for sectorial operators.** Here, following [15], we briefly recall the definition of a bounded H^∞-calculus for a closed and densely defined sectorial ([15, Section 2.1]) operator $T : D_X(T) \subset X \to X$ in a Banach space X. Assume that the range of T is also dense in X and the spectrum of T is contained in $\overline{\Sigma_{\varphi'}}$ for some $\varphi' \in (0,\varphi)$. Assume also that $\|z(z-T)^{-1}\|_{\mathcal{L}(X)}$ is uniformly bounded in $\mathbb{C}\setminus(\Sigma_\varphi \cup \{0\})$. Let $H^\infty(\Sigma_\varphi)$ denote the space of all bounded and holomorphic functions $f : \Sigma_\varphi \to \mathbb{C}$ equipped with the supremum norm. We also introduce the subspace $H_0^\infty(\Sigma_\varphi)$ of $H^\infty(\Sigma_\varphi)$ as

$$\begin{aligned} H_0^\infty(\Sigma_\varphi) = \big\{ f \in H^\infty(\Sigma_\varphi) \mid & \text{ there are } C > 0 \text{ and } s > 0 \text{ such that} \\ & |f(z)| \le C\min\{|z|^s, |z|^{-s}\} \ \text{ for all } z \in \Sigma_\varphi \big\}. \end{aligned}$$

Then for every $f \in H_0^\infty(\Sigma_\varphi)$ the integral

$$f(T) = \frac{1}{2\pi i}\int_{\partial\Sigma_\varphi} f(z)(z-T)^{-1}\,\mathrm{d}z$$

is well defined and converges absolutely in $\mathcal{L}(X)$, and hence, we have $f(T) \in \mathcal{L}(X)$.

Definition A.4. We say that T admits a bounded $H^\infty(\Sigma_\varphi)$-calculus in X if there exists $C \ge 0$ such that

$$\|f(T)\|_{\mathcal{L}(X)} \le C\|f\|_\infty, \qquad f \in H_0^\infty(\Sigma_\varphi).$$

Under the definition above, by a suitable approximation argument (cf. [15, Proposition 5.3.4]), the operator $f(T)$ is well defined in $\mathcal{L}(X)$ and satisfies the estimate $\|f(T)\|_{\mathcal{L}(X)} \le C\|f\|_\infty$ even for $f \in H^\infty(\Sigma_\varphi)$.

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