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# HARMONIC ANALYSIS MEETS CRITICAL KNOTS. CRITICAL POINTS OF THE MÖBIUS ENERGY ARE SMOOTH

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Abstract. Motivated by the Coulomb potential of an equidistributed charge on a curve, Jun O'Hara introduced and investigated the first geometric knot energy, the Möbius energy. We prove that every critical curve of this Möbius energy is of class $C^\infty$ and thus extend the corresponding result due to Freedman, He, and Wang for minimizers of the Möbius energy.

In contrast to the techniques used by Freedman, He, and Wang, our methods do to not use the Möbius invariance of the energy, but rely on purely analytic methods motivated from a formal similarity of the Euler-Langrange equation to the half harmonic map equation for the unit tangent.

## Contents



## Introduction

To find nice geometric representatives within a given knot class, several new energies have been invented in the last two decades. After Shinji Fukuhara investigated the Coulomb energy for polygonal knots [Fuk88], Jun O'Hara looked at several potential energies for a charged curve [O'H91]. Still the first Ansatz of his was an indefinite integral, and he made sense of the energies using the physical method of renormalization. This led to the earliest knot energy for absolutely continuous

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curves $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$, the so-called *Möbius energy*:

$$E^{(2)}(\gamma) := \iint\limits_{(\mathbb{R}/\mathbb{Z})^2} \left( \frac{1}{|\gamma(u) - \gamma(v)|^2} - \frac{1}{d_\gamma(u,v)^2} \right) |\gamma'(u)||\gamma'(v)| \, \mathrm{d}u \, \mathrm{d}v.$$

Here $d_\gamma(u,v)$ denotes the intrinsic distance between $\gamma(u)$ and $\gamma(v)$ on the curve $\gamma$. More precisely,

$$d_\gamma(u,v) := \min \left( \mathscr{L}(\gamma|_{[u,v]}), \mathscr{L}(\gamma) - \mathscr{L}(\gamma|_{[u,v]}) \right) \tag{0.1}$$

provided $|u - v| \leq \frac{1}{2}$, where $\mathscr{L}(\gamma) := \int_0^1 |\gamma'(\theta)| \, \mathrm{d}\theta$ is the length of $\gamma$. The factors $|\gamma'(u)||\gamma'(v)|$ guarantee invariance under reparametrization. Any finite-energy curve is bi-Lipschitz continuous with a constant monotonically depending only on its energy. Therefore, a sequence of embedded curves that (pointwise) converges to a curve with a self-intersection leads to an energy blow-up. However, $E^{(2)}$ does not penalize the pulling tight of small knotted arcs.

Shortly after, Michael Freedman, Zheng-Xu He, and Zhenghan Wang [FHW94] made the seminal discovery that $E^{(2)}$ is not merely invariant under translations, dilations, and orthogonal transformations, but under the full group of Möbius transformations, especially inversions on spheres.

Crucially using the Möbius invariance, they were able to show that there are minimizers of this knot energy within every prime knot class and that these are in fact of class $C^{1,1}$. More precisely, they could show that if $\gamma$ is a local minimizer with respect to the $L^\infty$-topology, and if $\gamma$ is parametrized by arc length, then $\gamma$ is $C^{1,1}$. Together with a bootstrapping argument due to He [He00], one then obtains that local minimizers of the Möbius energy are smooth; see also [Rt10].

The restriction to prime knot classes (for the existence proof in [FHW94]) is due to the fact that the Möbius energy does not prevent the pulling tight of small knotted arcs. Considering an arbitrary minimal sequence of curves, one has to prevent knotted arcs from pulling tight—which is not guaranteed by a uniform bound on the Möbius energy as pointed out above. This can be achieved by applying suitable Möbius transformations which unfortunately does not simultaneously work for more than one knotted component and thereby restricts the argument to prime knot classes. However, motivated by numerical evidence, Rob Kusner and John Sullivan were led to a still open conjecture that there are in fact no minimizers within composite knot classes [KS97].

The original regularity argument in [FHW94] crucially relies on both Möbius invariance of the energy and the local minimality of the respective curve. In this context, one may ask whether these two assumptions are conceptually essential for proving regularity.

We negatively answer this question by providing a new regularity argument in this article, based on the theory of fractional harmonic maps. More precisely, we will prove that even only critical points of the Möbius energy are of class $C^\infty$ under the mildest condition one can think of: that $E^{(2)}(\gamma)$ is finite. This is an assumption which, as shown in a recent work of the first author [Bla12], is equivalent to assuming that $\gamma$ is an injective curve of class $H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, that is,

$$\|\gamma\|_{H^{3/2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)} \equiv \left( \|\gamma\|^2_{L^2(\mathbb{R}/\mathbb{Z})} + \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{|w|^{1+2\frac{1}{2}}} \, \mathrm{d}w \, \mathrm{d}u \right)^{\frac{1}{2}} < \infty.$$

This is actually a crucial observation. Let us fix the parametrization of $\gamma$ by arc-length, i.e. $|\gamma'| \equiv 1$ or, equivalently, $\gamma' \in \mathbb{S}^{n-1}$ for the $(n-1)$-sphere $\mathbb{S}^{n-1}$. Thus the energies $E^{(2)}(\gamma)$ and the energy $F(\gamma')$ where

$$F(f) = \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f(u+w) - f(u)|^2}{|w|^{1+2\frac{1}{2}}} \, \mathrm{d}w \, \mathrm{d}u, \quad f : \mathbb{R}/\mathbb{Z} \to \mathbb{S}^{n-1}$$

may not share geometric features, but seem to share analytic features. Critical points of $F$ are called half-harmonic maps. Those were first studied by Tristan Rivière and Francesca Da Lio in [DLR11a], and for them there is now a wide range of techniques available [DLR11a, DLR11b, DL11, Sch12, Sch11]. A good understanding of those arguments and several extensions thereof will lead to our main result:

**Theorem I** (Stationary points are smooth)**.** *Any critical point $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ of $E^{(2)}$, i.e., any curve $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ which satisfies the Euler-Lagrange equations*

$$\delta E^{(2)}(\gamma; h) = 0 \quad \textit{for all } h \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n),$$

*belongs to $C^\infty$ when parametrized by arc-length.*

Let us briefly comment on the consequences of our result. Firstly, in contrast to [FHW94], it does not merely apply to local minimizers but to all critical points. Albeit the existence of saddle points is still an open question, this extension is conceptually a stronger result than smoothness of local minimizers alone. Saddle points of the Möbius energy are of interest for several reasons. Among other things, the existence of a saddle point within the unknot class would imply that the gradient flow of the Möbius energy *cannot* be used to define a retraction of the class of all unknots onto the circles. Such a retraction exists due to the Smale conjecture proven by Hatcher; see [FHW94, Section 6] for details.

Secondly, by not using the Möbius invariance in our arguments, we demonstrate that this is not the essential tool for proving regularity of minimizers. Thus there is the chance to study other, possibly not Möbius invariant, critical knot energies, using the techniques developed in this article. Candidates for this reasoning are corresponding "non-degenerate critical" members of two-parameter energy families such as generalized versions of the tangent-point energy and integral Menger curvature. The general idea is that the non-linearity of the corresponding Euler-Lagrange equation is *critical* and the term of highest order is related to the fractional Laplacian and not some fractional variant of the *degenerate* $p$-Laplacian.

Additionally, our arguments are not restricted to the situation of one-dimensional domains but can be applied to arbitrary dimensions.

Of course, the price we pay is that, instead of the very appealing geometric argument in [FHW94], we have to adapt the sophisticated techniques originally developed by Tristan Rivière and Francesca Da Lio [DLR11a, DLR11b, DL11] and the third author [Sch12, Sch11] to deal with $\frac{n}{2}$-harmonic maps into manifolds.

The first task in order to prove Theorem I is to derive the Euler-Lagrange equation for such critical points.

Recall that a closed (absolutely continuous) curve $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^n$ is *regular* if there is a positive constant $c = c(\gamma)$ with $|\gamma'(x)| \geq c$ for all $x \in \mathbb{R}/\mathbb{Z}$. It is *simple* if it is injective or, equivalently, embedded.

In [FHW94] it was shown that for simple regular closed curves $\gamma \in C^{1,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and $h \in C^{1,1}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ we have

$$\begin{aligned}
&\delta E^{(2)}(\gamma;h)\\
&:= \lim_{\tau\searrow 0} \frac{E^{(2)}(\gamma+\tau h)-E^{(2)}(\gamma)}{\tau}\\
&= 2\lim_{\varepsilon\searrow 0} \iint\limits_{|u-v|\geq\varepsilon} \left(\frac{\langle\gamma'(u),h'(u)\rangle}{|\gamma'(u)|^2} - \frac{\langle\gamma(u)-\gamma(v),h(u)-h(v)\rangle}{|\gamma(u)-\gamma(v)|^2}\right)\frac{|\gamma'(v)|\,|\gamma'(u)|}{|\gamma(u)-\gamma(v)|^2}\,\mathrm{d}u\,\mathrm{d}v.
\end{aligned}$$

We will show that this formula still holds under weaker assumptions, which require the notion of *Bessel potential spaces.* For $f \in H^{s,\varrho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $s>0$, $s\notin\mathbb{N}$, $\varrho\in(1,\infty)$, let

$$\|f\|_{H^{s,\varrho}} := \left\|\sum_{\ell\in\mathbb{Z}} \left(1+\ell^2\right)^{s/2} \hat{f}_\ell e^{2\pi i\ell\bullet}\right\|_{L^\varrho}. \tag{0.2}$$

Here, $\hat{\cdot}_\ell$ denotes the $\ell$-th Fourier coefficient. The space $f \in H^{s,\varrho}$ consists of all $f \in L^\varrho(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ for which $\|f\|_{H^{s,\varrho}} < \infty$. As usual, we let $H^s := H^{s,2}$. Note that $\|\cdot\|_{H^{3/2}}$ is equivalent to the norm defined above.

**Theorem II** ($E^{(2)} \in C^1(H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty})$)**.** *The energy $E^{(2)}$ is continuously differentiable on the space of injective and regular curves belonging to $H^{3/2}\cap H^{1,\infty}$. Furthermore, if $\gamma\in H^{3/2}$ is injective and parametrized by arc-length and $\varphi\in H^{3/2}\cap H^{1,\infty}$, the first variation*

$$\delta E^{(2)}(\gamma;\varphi) := \lim_{\tau\to 0}\frac{E^{(2)}(\gamma+\tau\varphi)-E^{(2)}(\gamma)}{\tau}$$

*exists and equals*

$$2\lim_{\varepsilon\searrow 0}\iint\limits_{|u-v|\geq\varepsilon}\left(\langle\gamma'(u),\eth(u)\rangle - \frac{\langle\gamma(u)-\gamma(v),\varphi(u)-\varphi(v)\rangle}{|\gamma(u)-\gamma(v)|^2}\right)\frac{\mathrm{d}u\,\mathrm{d}v}{|\gamma(u)-\gamma(v)|^2}.$$

The requirements on $\gamma$ are quite moderate—for arc-length parametrized simple curves we only assume that the Möbius energy is finite (which already implies $H^{3/2}$ regularity as indicated in the subsequent section). As the Möbius energy is invariant under reparametrization, there is no loss of generality in assuming arc-length parametrization.

Though the space $H^{3/2}\cap H^{1,\infty}$ seems somewhat artificial at first sight, it just guarantees that we do not use bad parametrizations of our curves. The proof of this result is similar to [BR12].

In [BR12], improving a previous result [Rt12], the smoothness of critical points was already shown for the case of $E^{(\alpha)} := E^{(\alpha,1)}$, $\alpha\in(2,3)$, where

$$E^{(\alpha,p)}(\gamma) := \iint\limits_{(\mathbb{R}/\mathbb{Z})^2}\left(\frac{1}{|\gamma(u)-\gamma(v)|^\alpha} - \frac{1}{d_\gamma(u,v)^\alpha}\right)^p |\gamma'(u)||\gamma'(v)|\ \mathrm{d}u\,\mathrm{d}v$$

for $\alpha,p\in[1,\infty)$; see [O'H94]. Opposed to the Möbius energy $E^{(2)}=E^{(2,1)}$, there are minimizers of the energies $E^{(\alpha,p)}$ in the case $\alpha p>2$ as shown in [O'H94].

It is worth noting that those energies $E^{(\alpha)}$, $\alpha\in(2,3)$, lead to a *subcritical* Euler-Langrage equation and that in some sense the regularity theory can be based on

Sobolev embeddings for fractional Sobolev and Besov spaces. Using the terminology introduced above, this reflects the "non-degenerate subcritical" case, while, in contrast to this, the Euler-Lagrange equation of the Möbius energy is *critical.* As for well-known critical geometric equations—like the Euler-Lagrange equation of the Willmore functional (see, e.g., [Sim93, Riv08]) or harmonic maps on $\mathbb{R}^2$ (see, e.g., [Hél91, Riv07])—one has first to find a way to gain an $\varepsilon$ of additional regularity (via gaining a $\delta$ of additional integrability) and then start a bootstrapping argument. That is, in a quite natural way, the proof of Theorem I is an immediate consequence of two technically independent steps:

**Theorem III** (Initial regularity)**.** *Let $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\gamma' \in \mathbb{S}^{n-1}$, be a critical point of the Möbius energy, i.e., satisfying $\delta E^{(2)}(\gamma, \varphi) = 0$ for all $\varphi \in C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. Then $\gamma \in C^{1,\alpha}$ for some $\alpha > 0$, and $\gamma \in H^{\frac{3}{2},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for some $p > 2$.*

**Theorem IV** (Bootstrapping)**.** *For some $p > 2$, let $\gamma \in H^{\frac{3}{2},p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\gamma' \in \mathbb{S}^{n-1}$, be a critical point of the Möbius energy. Then $\gamma$ is smooth.*

Theorem III is proven in Section 3, Theorem IV in Section 4. While Theorem IV relies mainly on bringing together Sobolev embeddings and standard commutator estimates for Bessel potential spaces with techniques developed in [Bla12], some very delicate estimates are needed to get anything more than the critical and initial regularity $H^{\frac{3}{2}}$ for critical points as stated in Theorem III.

We will comment on the structure of the proofs in the following sections.

Let us conclude this introduction by remarking that, in contrast to critical points of $E^{(\alpha,1)}$, in the "degenerate cases" where $p > 1$ we do not expect critical points of $E^{(\alpha,p)}$ to be $C^\infty$-smooth: The resulting Euler-Lagrange equation should be in some sense a non-local degenerate elliptic equation—as it is related to the corresponding equation for the fractional $p$-Laplacian. Keeping in mind the regularity theory for elliptic degenerate equations, one might nevertheless expect that critical points are at least a bit more regular than an arbitrary finite-energy curve alone.

## Notation and function spaces

The aim of this section is to introduce the function spaces and operators which will be used throughout this paper.

**Notation.** In general, constants may change from line to line. Note that if we consider the constant factors to be irrelevant with respect to the mathematical argument, for the sake of simplicity we will omit them in the calculations, writing $\lesssim, \gtrsim$ and $\approx$ instead of $\leq, \geq$ and $=$.

**The fractional Laplacian.** Let $|D|^s \equiv (-\partial_x^2)^{s/2} \equiv (-\Delta)^{\frac{s}{2}}$ be the fractional Laplacian on $\mathbb{R}$. In the case of positive powers $s$ of the Laplacian $|D|^s$, $s \in (0,1)$, we use the corresponding formula

$$|D|^s f(x) = \tilde{c}_s \int_{\mathbb{R}} \frac{f(y) - f(x)}{|x-y|^{1+s}} \, \mathrm{d}y. \tag{0.3}$$

The inverse, $I_s \equiv (-\Delta)^{-\frac{s}{2}}$, $s \in (0,1)$, is the Riesz potential given by

$$I_s f(x) = c_s \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-s}} \, \mathrm{d}y. \tag{0.4}$$

For a detailed introduction to the fractional Laplacian we refer to, e.g., [DNPV11], [Sch10, Section 2.5].

**Function spaces.** Let $H^{k,\varrho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $k\in\mathbb{N}$, $\varrho\in(1,\infty)$, denote the usual Sobolev space; recall $H^{0,\varrho}:=L^\varrho$.

The definitions and facts presented here can be found in textbooks such as, e.g., [Tri83, RS96, Tar07]. We also refer to [DNPV11].

**Fractional Sobolev spaces.** We already introduced Bessel potential spaces $H^{s,\varrho}$ in the Introduction which play a fundamental rôle in all parts of this text. In the Hilbert case $\varrho=2$ they coincide with the Sobolev-Slobodeckiĭ spaces $W^{s,\varrho}$ which are equipped with the seminorm

$$[f]_{W^{s,\varrho}} := \left(\int_{\mathbb{R}/\mathbb{Z}}\int_{-1/2}^{1/2}\frac{\left|f^{(k)}(u+w)-f^{(k)}(u)\right|^\varrho}{|w|^{1+\sigma\varrho}}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/\varrho} \tag{0.5}$$

for some $f\in H^{k,\varrho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $k\in\mathbb{N}$, $\sigma\in(0,1)$, $\varrho\in(1,\infty)$ and $s=k+\sigma$. A function $f\in H^{k,\varrho}$ belongs to $W^{s,\varrho}$ if $\|f\|_{H^{s,\varrho}}:=\|f\|_{H^{k,\varrho}}+[f]_{W^{s,\varrho}}<\infty$. The norms $\|\cdot\|_{H^{s,2}}$ and $\|\cdot\|_{W^{s,2}}$ are equivalent.

The use of Sobolev-Slobodeckiĭ spaces is quite convenient as they naturally appear as energy spaces for O'Hara's energies. We will comment on this fact below.

By $H^{s,\varrho}_{\mathrm{ir}}$ we will denote the set of injective and regular curves in $H^{s,\varrho}$. Accordingly, $H^{s,\varrho}_{\mathrm{ia}}$ contains injective $H^{s,\varrho}$-curves parametrized by arc-length.

Without further notice we will frequently use the embedding

$$H^{k+s,\varrho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)\hookrightarrow C^{k,s-1/\varrho}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n),\qquad \varrho\in(1,\infty),\quad s\in(\varrho^{-1},1). \tag{0.6}$$

*The importance of* $H^{3/2}$. At this point, we would like to clarify the natural use of the Sobolev space $H^{3/2}$. To this end we present the following heuristic argument and refer to [Bla12] for a rigorous proof. The integrand of the Möbius energy reads

$$\frac{1}{|\gamma(u+w)-\gamma(u)|^2}-\frac{1}{|w|^2}=\left|\frac{\gamma(u+w)-\gamma(u)}{w}\right|^{-2}\frac{\left(1-\left|\frac{\gamma(u+w)-\gamma(u)}{w}\right|^2\right)}{|w|^2}. \tag{0.7}$$

The main idea is now that, using arc-length parametrization and $|a-b|^2=2-2\langle a,b\rangle$ for $|a|=|b|=1$,

$$\begin{aligned}1-\left|\frac{\gamma(u+w)-\gamma(u)}{w}\right|^2 &= 1-\iint_{[0,1]^2}\langle\gamma'(u+\vartheta_1 w),\gamma'(u+\vartheta_2 w)\rangle\,\mathrm{d}\vartheta_1\,\mathrm{d}\vartheta_2\\ &=\tfrac12\iint_{[0,1]^2}\left|\gamma'(u+\vartheta_1 w)-\gamma'(u+\vartheta_2 w)\right|^2\mathrm{d}\vartheta_1\,\mathrm{d}\vartheta_2.\end{aligned}$$

Furthermore, $(u,w)\mapsto\left|\frac{\gamma(u+w)-\gamma(u)}{w}\right|$ is continuous, taking values in some compactum in $(0,1]$ on $\mathbb{R}/\mathbb{Z}\times[-\frac12,\frac12]$. By means of the functions $g_2: x\mapsto x^{-2}$ and $h_2: x\mapsto\frac{1-x^2}{1-x^2}$, which are analytic and non-negative on $(0,1]$ provided $a\geq 2$, we may express the right-hand side of (0.7) by

$$\frac{1}{2}g_2\left(\left|\frac{\gamma(u+w)-\gamma(u)}{w}\right|\right)h_2\left(\left|\frac{\gamma(u+w)-\gamma(u)}{w}\right|\right) \cdot \iint_{[0,1]^2}\frac{\left|\gamma'(u+\vartheta_1 w)-\gamma'(u+\vartheta_2 w)\right|^2}{|w|^2}\,\mathrm{d}\vartheta_1\,\mathrm{d}\vartheta_2.$$

The first factors are bounded, while the last one resembles the integrand of the Sobolev-Slobodeckiĭ seminorm $[\cdot]_{W^{3/2,2}}$.

According to [Bla12], any embedded arc-length parametrized curve has finite Möbius energy if and only if it belongs to $H^{3/2}$.

**Besov spaces.** In the proof of Theorem IV we will work with the Besov spaces $B^s_{p,q}$. Given $s\in(0,1)$ and $p,q\in[1,\infty)$, one way to define the norm on these spaces is to set

$$|f|_{B^s_{p,q}} := \left(\int_{-1/2}^{1/2}\frac{\left(\int_{\mathbb{R}/\mathbb{Z}}|f(u+w)-f(u)|^p du\right)^{q/p}}{|w|^{1+sq}}\,\mathrm{d}w\right)^{1/q}$$

and to put

$$\|f\|_{B^s_{p,q}} := \|f\|_{L^p} + |f|_{B^s_{p,q}}.$$

The Besov space $B^s_{p,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ then consists of all functions $f\in L^p$ with $|f|_{B^s_{p,q}} < \infty$.

Apart from this definition we just need the Sobolev embedding

$$H^{\tilde{s},\tilde{p}} \subset B^s_{p,q}$$

if $s<\tilde{s}$ and $s-\frac{1}{p}<\tilde{s}-\frac{1}{\tilde{p}}$.

**Lorentz spaces.** A fundamental tool for the proof of Theorem III is the notion of Lorentz spaces which we shortly recapitulate here together with their main properties that we will use in this article. For a measurable function $f:\Omega\to\mathbb{R}$ and $\Omega\subset\mathbb{R}$ one considers the *decreasing rearrangement*

$$f^*(t) := \inf\left\{s>0 : \mathcal{L}^1(\{|f|>s\})\le t\right\},$$

where $\mathcal{L}^1$ denotes the Lebesgue measure. We define

$$|f|_{(p,q),\Omega} := \begin{cases}\left(\int_0^\infty (t^{1/p}f^*(t))^q\frac{dt}{t}\right)^{1/q} & \text{if } p,q\in[1,\infty),\\ \sup_{t>0} t^{1/p}f^*(t) & \text{if } q=\infty.\end{cases}$$

To prevent technical problems, unless $p\in(1,\infty)$ we will only take the spaces $L^{1,1}=L^1$ and $L^{\infty,\infty}=L^\infty$ into consideration.

Furthermore, $|f|_{(p,p)}\approx\|f\|_{L^p}$ for all $p\in[1,\infty]$. If $\Omega=\mathbb{R}$ we will omit $\Omega$ in the notation. Though $|\cdot|_{(p,q),\Omega}$ is not a norm, as it does not obey the triangle inequality, there is a norm $\|\cdot\|_{(p,q),\Omega}$ on the Lorentz spaces which is equivalent to $|\cdot|_{(p,q),\Omega}$. These norms satisfy a *Hölder inequality*, i.e., for $p_1,p_2,p\in[1,\infty)$ and $q_1,q_2,q\in[1,\infty]$ with $1/p_1+1/p_2=1/p$ and $1/q_1+1/q_2=1/q$ we have

$$\|fg\|_{(p,q),\Omega} \lesssim \|f\|_{(p_1,q_1),\Omega}\,\|g\|_{(p_2,q_2),\Omega}. \tag{0.8}$$

For $p_1,p_2,p\in(1,\infty)$ and $q_1,q_2\in[1,\infty]$ with $1/p_1+1/p_2=1/p+1$ and $1/q_1+1/q_2=1/q$, we have the *Young-O'Neil inequality* [Hun66]

$$\|f*g\|_{(p,q),\Omega} \lesssim \|f\|_{(p_1,q_1),\Omega}\,\|g\|_{(p_2,q_2),\Omega}. \tag{0.9}$$

Furthermore, we have the *Sobolev inequality*

$$\|I_s f\|_{(p^*,q)} \lesssim \|f\|_{(p,q)}$$

for all $s \geq 0$, $p \in [1,\infty), q \in [1,\infty]$ and $p^* := \frac{p}{1-sp} \in [1,\infty)$. Further information and proofs can be found in [Hun66, Gra08, Tar07].

The main reason for using Lorentz spaces in the context of critical equations, i.e., equations to which standard Gagliardo-Nirenberg-Sobolev embeddings cannot be applied to gain regularity, is the following fact. Although for functions $f$ the $L^2$-norm of $|D|^{\frac{1}{2}} f$ does not control the $L^\infty$-norm of $f$, the $L^{2,1}$ norm does; i.e. we have the estimate

$$\|f\|_\infty \lesssim \||D|^{\frac{1}{2}} f\|_{(2,1)}.$$

We will also need this in the more general form

$$\|f\|_\infty \lesssim \||D|^s f\|_{(\frac{1}{s},1)} \quad \text{for all } s \in (0,1).$$

## 1. Outline of the proofs

We will briefly outline the structure of the proofs. Details will be carried out in the subsequent sections.

**The proof of Theorem II.** The basic strategy is to approximate $E^{(2)}$ by a family of functionals $\left(E^{(2)}_\varepsilon\right)_{\varepsilon>0}$ which emerge from $E^{(2)}$ by removing an $\varepsilon$-neighborhood from the diagonal $\{u = v\}$, more precisely by restricting the integration domain $(u,v) \in (\mathbb{R}/\mathbb{Z})^2$ to $|u - v|_{\mathbb{R}/\mathbb{Z}} \geq \varepsilon$.

As perturbations of the form $\gamma + \tau h$ will in general not preserve arc-length parametrization, we have to work with arbitrary (but still regular) parametrizations. We state a more general derivative formula, not requiring arc-length, in Proposition 2.2 from which the proof of Theorem II is easily obtained by simplifying this formula in the case of arc-length parametrization.

It is straightforward that any $E^{(2)}_\varepsilon$ is Fréchet differentiable on $H^{3/2}_{\text{ir}} \cap H^{1,\infty}$; see Lemma 2.3. Of course, in order to use arbitrary test functions we have to ensure that $H^{3/2}_{\text{ir}} \cap H^{1,\infty}$ is open in $H^{3/2} \cap H^{1,\infty}$, which is confirmed in Lemma 2.1.

While the pointwise convergence $E^{(2)}_\varepsilon \to E^{(2)}$ is rather clear as the integrand is non-negative, this argument fails for the derivative (which in fact turns out to be a principle value integral). Moreover, due to the fact that bounded $L^1$-sequences are not uniformly integrable, the approximations $E^{(2)}_\varepsilon$ do not even form a Cauchy sequence in $C^0(H^{3/2}_{\text{ir}} \cap H^{1,\infty})$.

In order to prove Proposition 2.2, we state in Lemma 2.4 that $E^{(2)}_\varepsilon$ is *nearly* a Cauchy sequence in $C^1(X_\delta)$ for those subsets $X_\delta \subset H^{3/2}_{\text{ir}} \cap H^{1,\infty}$, $\delta \geq 0$, which satisfy a substitute of the uniform integrability property. Loosely speaking, for any element in $X_\delta$ the contribution of an infinitesimal neighborhood of the diagonal to the $W^{3/2}$-seminorm is bounded by $\delta$; see (2.7). Now Lemma 2.4 states that for any $\gamma_0 \in H^{3/2}_{\text{ir}} \cap H^{1,\infty}$ there is a neighborhood in $X_\delta$ on which the Lipschitz constant of $E^{(2)}_{\varepsilon_1} - E^{(2)}_{\varepsilon_2}$ is bounded in terms of $\delta$ as $\varepsilon_1, \varepsilon_2 \searrow 0$.

This fact is exploited several times in the proof of Proposition 2.2 for deriving the existence of directional derivatives of $E^{(2)}$, Gâteaux differentiability, and continuity of the functional.

**The Euler-Lagrange equation.** From Theorem II we deduce that a stationary knot $\gamma \in H^{\frac{3}{2}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, parametrized by arc-length, satisfies the Euler-Lagrange equation

$$Q(\gamma,h) := \lim_{\varepsilon\to 0} Q_\varepsilon(\gamma,h) = T_1(\gamma,h) + T_2(\gamma,h) \quad \text{for all } h \in C^\infty(\mathbb{R}/\mathbb{Z}), \tag{1.1}$$

where, for $f,g : \mathbb{R} \to \mathbb{R}^n$, $\varepsilon > 0$,

$$Q_\varepsilon(f,g) := \int_0^1 \int_{[-\frac{1}{2},\frac{1}{2}]\setminus(-\varepsilon,\varepsilon)} \Big(\langle f'(u), g'(u)\rangle\, w^2 - \langle f(u+w)-f(u), g(u+w)-g(u)\rangle\Big) \frac{\mathrm{d}w}{w^4}\, \mathrm{d}u, \tag{1.2}$$

$$T_1(f,g) := -\int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle f'(u),\ g'(u)\rangle \left(\frac{1}{|f(u+w)-f(u)|^2} - \frac{1}{|w|^2}\right) \mathrm{d}w\ \mathrm{d}u, \tag{1.3}$$

and

$$T_2(f,g) := \int_0^1 \int_{-\frac{1}{2}}^{\frac{1}{2}} \langle f(u+w)-f(u),\ g(u+w)-g(u)\rangle \cdot \left(\frac{1}{|f(u+w)-f(u)|^4} - \frac{1}{|w|^4}\right) \mathrm{d}w\ \mathrm{d}u. \tag{1.4}$$

The proofs of Theorem III and Theorem IV both rely on this decomposition (1.1) of the first variation dating back to [He00, Formula (4.5)], which already proved to be helpful in the analysis of the functionals $E^{(\alpha)}$ for $\alpha \in (2,3)$ (cf. [BR12]) and the gradient flow of the energies $E^{(\alpha)}$ for $\alpha \in [2,3)$ [Bla13, Bla11].

**The proof of Theorem III.** Most techniques for dealing with critical partial differential equations of fractional order have been developed for equations on the whole Euclidean space. For that reason, we prefer working on the real line rather than working on the circle. To this end, we interpret functions on $\mathbb{R}/\mathbb{Z}$ as functions on $\mathbb{R}$ which are periodic with period 1. We then choose a cutoff function $\eta \in C_0^\infty(\mathbb{R})$, $\eta \equiv 1$ on $[-3,4]$ and consider

$$g = \eta\,\gamma \tag{1.5}$$

instead of $\gamma$.

We will show that for every $u \in \mathbb{R}/\mathbb{Z}$ we have

$$|D|^{\frac{1}{2}} g' \in L^p((u-1/20, u+1/20))$$

for a $p > 2$. Due to the invariance of the problem under shifting the parametrization, it is enough to show this for $u = 1/2$, i.e.

$$|D|^{\frac{1}{2}} g' \in L^p((9/20, 11/20)). \tag{1.6}$$

In order to prove our regularity result, we will prove a Dirichlet growth theorem for the weak $H^{1/2}$-energy of $\gamma'$ on balls in a manner comparable to [DLR11a, Sch12], which are as well in the setting of sphere-valued mappings. In contrast to these papers, the techniques from [DLR11b, DL11, Sch11] deal with a more general setting, but have to work (as we will here) with estimates of the $L^{2,\infty}$-norm instead of the $L^2$-norm. Note nevertheless that our right-hand side is very different from theirs. In order to obtain the estimates of the norms $\||D|^{\frac{1}{2}} g'\|_{(2,\infty)}$ on small balls, we will have to use new arguments.

To prove the regularity theorem, we will begin with an approach appearing in [DLR11a, Sch12] and divide $|D|^{\frac{1}{2}} g'$ into the part parallel to $g'$ (and thus *normal* to the sphere $\mathbb{S}^{n-1}$) and the term perpendicular to $g'$ (*tangential* to the sphere). More precisely, we use that

$$\||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_r} \lesssim \|\langle g', |D|^{\frac{1}{2}} g'\rangle\|_{(2,\infty),B_r} + \max_{\omega_{ij}} \|g'_j \omega_{ij} |D|^{\frac{1}{2}} g'_i\|_{(2,\infty),B_r}, \tag{1.7}$$

where the maximum is over matrices $\omega_{ij} = -\omega_{ji} \in \{-1,0,1\}$. This follows from a version of Lagrange's identity,

$$|b|^2 = \langle a,b\rangle^2 + |a \wedge b|^2, \quad \text{if } |a| = 1;$$

for a detailed argument the interested reader is referred to the appendix of [DLS12].

By Hölder's inequality, the terms of the right-hand side of (1.7) can be estimated by $\|g'\|_\infty \, \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_r}$, which in turn is controlled $\||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_r}$. This however, is not a sufficiently good estimate.

Indeed, the terms on the right-hand side of (1.7) each can be estimated by

$$(1.7) \lesssim \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{\Lambda r}} + \text{tail}, \tag{1.8}$$

for some $\Lambda > 2$, and with the tail being terms that are essentially of lower order. The crucial point is that the exponent 2 on the right-hand side is larger than one, which is the exponent we got from the first simple application of Hölder inequality. This increased exponent is the result of several integration-by-compensation effects. In the theory of two-dimensional harmonic maps this effect usually is seen by employing the Hardy-BMO-duality, more precisely that for any $f$ in the Hardy-space $\mathcal{H}^1$ and any $g \in BMO$ we have

$$\int f \; g \lesssim \|f\|_{\mathcal{H}^1} \; [g]_{BMO}.$$

Since harmonic maps into spheres $f : \mathbb{R}^2 \to \mathbb{S}^{n-1}$ satisfy [Hél90, Hél02]

$$\|\Delta f\|_{\mathcal{H}^1} \lesssim \|\nabla f\|_2^2,$$

and since in two dimensions $[f]_{BMO} \lesssim \|\nabla f\|_2$, one obtains

$$\|\nabla f\|^2_{2,B_r} \lesssim \int_{B_{\Lambda r}} \Delta f \; f + \text{tail} \lesssim \|\nabla f\|^3_{2,B_{\Lambda r}} + \text{tail}.$$

Again, one sees that the exponent 3 on the right-hand side is larger than the exponent 2 on the left-hand side.

The estimate for the normal part, i.e., the first term of the right-hand side of (1.7), is provided in Lemma 3.1. It purely relies on the assumption that $|\gamma'| \equiv 1$ and is not related to the $\gamma$ solving any equation. The main point is that $|D|^{1/2}|\gamma'|^2 \equiv 0$ since $|g'|$ is a constant. Thus we can write

$$\gamma' |D|^{1/2} \gamma' = \frac{1}{2}(\gamma' |D|^{1/2} \gamma' + \gamma' |D|^{1/2} \gamma' - |D|^{1/2} (\gamma' \gamma')).$$

The right-hand side is a three-term commutator introduced in [DLR11a], and the important feature is that the right-hand side now grows quadratically, as needed for (1.8).

For the second term of (1.7) one should notice that it measures $|D|^{1/2}g'$ into a direction tangential to $\mathbb{S}^{n-1}$ at the point $\gamma'$. And it is a common feature of many geometric Euler-Lagrange equations that the equation is zero when tested with functions which are tangential. For our Euler-Lagrange equation (1.1) the term $T_1$ on the right-hand side is indeed zero whenever the testfunction $h'$ is orthogonal to $\gamma'$, i.e. when $h'(u)$ is tangential to $\mathbb{S}^{n-1}$ at the point $\gamma'(u)$ for a.e. $u \in \mathbb{R}/\mathbb{Z}$. The term $T_2$ however does not vanish.

Thus the estimates for the second term of (1.7) proceed as follows: First, we show in (3.5) that $|D|^{1/2}(g'_j\omega_{ij}|D|^{\frac{1}{2}}g'_i)$ can be estimated by a "critical term", which appeared already in the theory of fractional harmonic maps and is treated in Lemma 3.2, and an "essential" term to which the Euler-Lagrange equation is applied. For the latter, we use a representation formula of the operator $Q$; see (3.6). Then in Lemma 3.3, testing $Q$ with tangential test functions, the term $T_2$ from the Euler-Lagrange equation (1.1) remains. To show that this remaining term also exhibits effects of integration-by-compensation, i.e., sufficiently high exponents, it will be controlled in terms of a function $\Gamma$, which then is covered by Lemma 3.4. The estimate of the tangential part is subsumed in Lemma 3.6.

In fact, Lemmata 3.3 and 3.4 provide the essential new estimates in this article on which our entire reasoning crucially relies. Using these and the above-mentioned improved Sobolev embeddings for Lorentz spaces, we are ready to show the estimate in Lemma 3.5. This will allow us to prove a Dirichlet growth of the $L^{2,\infty}$-norm of $|D|^{\frac{1}{2}}g'$.

By localization and iteration techniques we now infer Hölder regularity due to (3.10). An iteration of this argument, using Lemma 3.1 and (3.10) as well as the boundedness of Riesz potential spaces on Morrey spaces, yields the boundedness of both summands of (1.7).

**The proof of Theorem IV.** The strategy is to establish a bootstrapping process in Lemma 4.5 by exploiting a regularity gap between the left- and right-hand sides of (1.1). It relies on estimates for the operators $T_1$ and $T_2$ in Corollary 4.4. It turns out that they can be brought into a common form, so it is enough to deal with a general case in Lemma 4.3. To this end, we need estimates for products and compositions in Bessel potential spaces stated in Lemmata 4.1 and 4.2.

## 2. Euler-Lagrange equation: Proof of Theorem II

This section is devoted to the proof of Theorem II which especially involves the derivation of a formula for the first variation.

First we will need the following lemma to guarantee that $E^{(2)}$ is well defined on a sufficiently small $H^{3/2} \cap H^{1,\infty}$ neighborhood of the curve $\gamma$:

**Lemma 2.1** ($H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ is open in $H^{3/2} \cap H^{1,\infty}$). *For any $\gamma \in H^{3/2}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n) \cap H^{1,\infty}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ there is some $\tau_0 = \tau_0(\gamma) > 0$ with*

$$Y := \left\{\gamma + \varphi \,\middle|\, \varphi \in H^{1,\infty}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n), \|\varphi'\|_{L^\infty} \le \tau_0 \right\} \subset H^{1,\infty}_{\mathrm{ir}}. \tag{2.1}$$

*Moreover, there is a constant $c = c(\gamma) > 0$ with*

$$\min\{|\tilde\gamma(u+w) - \tilde\gamma(u)|, d_{\tilde\gamma}(u+w,u)\} \ge c|w|, \qquad |\tilde\gamma'(u)| \ge c, \tag{2.2}$$

*for all* $\tilde{\gamma} \in Y$ *and* $(u,w) \in U_0$, *where*

$$U_\varepsilon := \mathbb{R}/\mathbb{Z} \times \left([-\tfrac{1}{2}, -\varepsilon] \cup [\varepsilon, \tfrac{1}{2}]\right), \qquad \varepsilon \in [0, \tfrac{1}{2}]. \tag{2.3}$$

*Proof.* We first show that $\gamma$ is bi-Lipschitz. To this end, choose $\delta \in (0, \frac{1}{2})$ with

$$\left( \int_{B_r(z)} \int_{B_r(0)} \frac{|\gamma'(u+w) - \gamma'(u)|^2}{|w|^2} \,\mathrm{d}w \,\mathrm{d}u \right)^{1/2} \leq \tfrac{1}{2}$$

for all $z \in \mathbb{R}/\mathbb{Z}$ and all $r \in [0, \delta]$ which gives

$$\begin{aligned}
&\frac{1}{2r} \int_{B_r(z)} \left| \gamma'(x) - \frac{1}{2r} \int_{B_r(z)} \gamma'(y) \,\mathrm{d}y \right| \mathrm{d}x \\
&\leq \frac{1}{4r^2} \int_{B_r(z)} \int_{B_r(z)} |\gamma'(x) - \gamma'(y)| \,\mathrm{d}x \,\mathrm{d}y \\
&\leq \left( \frac{1}{4r^2} \int_{B_r(z)} \int_{B_r(z)} |\gamma'(x) - \gamma'(y)|^2 \,\mathrm{d}x \,\mathrm{d}y \right)^{1/2} \\
&\leq \left( \int_{B_r(z)} \int_{B_r(z)} \frac{|\gamma'(x) - \gamma'(y)|^2}{|x-y|^2} \,\mathrm{d}x \,\mathrm{d}y \right)^{1/2} \\
&\leq \tfrac{1}{2}.
\end{aligned}$$

Since $\left| \frac{1}{2r} \int_{B_r(z)} \gamma'(y) \,\mathrm{d}y \right| \leq 1$ we deduce that

$$\inf_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \frac{1}{2r} \int_{B_r(z)} |\gamma'(y) - a| \,\mathrm{d}y \leq \tfrac{1}{2}.$$

For $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x-y| \leq 2\delta$ let $r := \frac{1}{2}|x-y|$ and $z \in \mathbb{R}/\mathbb{Z}$ be the midpoint of the shorter arc between $x$ and $y$. Then

$$\begin{aligned}
|\gamma(x) - \gamma(y)| &= \sup_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \int_{B_r(z)} \langle \gamma'(t), a \rangle \,\mathrm{d}t \\
&= \sup_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \int_{B_r(z)} \langle \gamma'(t), \gamma'(t) + (a - \gamma'(t)) \rangle \,\mathrm{d}t \\
&\geq \left( 1 - \inf_{\substack{a \in \mathbb{R}^n \\ |a| \leq 1}} \frac{1}{2r} \int_{B_r(z)} |\gamma'(t) - a| \,\mathrm{d}t \right) |x-y| \\
&\geq \tfrac{1}{2}|x-y|
\end{aligned}$$

for all $x, y \in \mathbb{R}/\mathbb{Z}$ with $|x-y| \leq 2\delta$. Since $\gamma$ is embedded and

$$(x,y) \mapsto \frac{|\gamma(y) - \gamma(x)|}{|y-x|}$$

defines a continuous positive function on $I_\delta := \{(x,y) \in (\mathbb{R}/\mathbb{Z})^2 : |x-y| \geq 2\delta\}$, we furthermore have

$$|\gamma(x) - \gamma(y)| \geq \underbrace{\min_{(\tilde{x},\tilde{y}) \in I_\delta} \frac{|\gamma(\tilde{y}) - \gamma(\tilde{x})|}{|\tilde{y} - \tilde{x}|}}_{>0} |x-y|$$

for all $(x, y) \in I_\delta$. Hence, there is a $c_0 = c_0(\gamma) > 0$ with

$$|\gamma(x) - \gamma(x+w)| \geq c_0|w|$$

for all $w \in [-1/2, 1/2]$. Lessening $c_0$ if necessary, we can also achieve by regularity

$$|\gamma'| \geq c_0 \qquad \text{on } \mathbb{R}/\mathbb{Z}.$$

Letting $\tau_0 := \frac{1}{2}c_0$ we obtain for arbitrary $\tilde{\gamma} \in Y$

$$\begin{aligned} |\tilde{\gamma}(u+w) - \tilde{\gamma}(u)| &\geq |\gamma(u+w) - \gamma(u)| - |(\tilde{\gamma}-\gamma)(u+w) - (\tilde{\gamma}-\gamma)(u)| \\ &\geq c_0|w| - \|\gamma' - \tilde{\gamma}'\|_{L^\infty} |w| \\ &\geq \tfrac{1}{2}c_0|w| \end{aligned}$$

and

$$|\tilde{\gamma}'| \geq |\gamma'| - |\tilde{\gamma}' - \gamma'| \geq c_0 - \|\gamma' - \tilde{\gamma}'\|_{L^\infty} \geq \tfrac{1}{2}c_0.$$

From the latter estimate we deduce by (0.1) for $u \in \mathbb{R}/\mathbb{Z}$, $w \in [-\frac{1}{2}, \frac{1}{2}]$,

$$d_{\tilde{\gamma}}(u+w, u) \geq d_{\tilde{\gamma}}(u \pm w, u) \geq \tfrac{1}{2}c_0|w|.$$

We have established (2.2), which gives (2.1). □

We will use the last lemma to prove the following proposition, from which Theorem II will follow quite easily.

**Proposition 2.2.** *The energy $E^{(2)}$ is continuously differentiable on*

$$\left(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}\right)(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n).$$

*The derivative of $E^{(2)}$ at $\gamma \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ in direction $\varphi \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ exists and is given by*

(2.4)

$$\begin{aligned} &\delta E^{(2)}(\gamma;\varphi) \\ &= 2\lim_{\varepsilon\searrow 0} \iint_{U_\varepsilon} \Bigg\{ \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2}\right) \left\langle \frac{\gamma'(u)}{|\gamma'(u)|^2}, \varphi'(u)\right\rangle \\ &\qquad - \left( \frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u)\rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)}{d_\gamma(u+w,u)^3}\right)\Bigg\} \\ &\qquad\qquad |\gamma'(u+w)||\gamma'(u)|\,\mathrm{d}w\,\mathrm{d}u. \end{aligned}$$

As $\gamma$ is absolutely continuous and regular, the derivative $\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)$ is well defined for almost all $(u,w) \in U_0$. From (0.1) we deduce

(2.5)
$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u) \\ &\quad = \begin{cases} |w| \int_0^1 \left\langle \frac{\gamma'(u+\sigma w)}{|\gamma'(u+\sigma w)|}, \varphi'(u+\sigma w)\right\rangle \mathrm{d}\sigma & \text{if } \mathscr{L}(\gamma|_{[u,u+w]}) < \frac{1}{2}\mathscr{L}(\gamma), \\ -|w| \int_0^1 \left\langle \frac{\gamma'(u+\sigma w)}{|\gamma'(u+\sigma w)|}, \varphi'(u+\sigma w)\right\rangle \mathrm{d}\sigma & \text{if } \mathscr{L}(\gamma|_{[u,u+w]}) > \frac{1}{2}\mathscr{L}(\gamma). \end{cases} \end{aligned}$$

To prove Proposition 2.2, we will first show that the following approximations of the energy $E^{(2)}$, in which we cut off the singular part, are continuously differentiable and provide a formula for the first variation. For $\varepsilon \in (0, \frac{1}{2})$ we set

$$E^{(2)}_\varepsilon(\gamma) := \iint_{U_\varepsilon} \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2}\right) |\gamma'(u+w)||\gamma'(u)|\,\mathrm{d}w\,\mathrm{d}u.$$

**Lemma 2.3.** *For $\varepsilon \in (0, \frac{1}{2})$ the functional $E_\varepsilon^{(2)}$ is continuously differentiable on the space of all injective regular curves in $H^{3/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. The directional derivative at $\gamma \in H^{3/2}_{\mathrm{ir}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ in direction $\varphi \in H^{3/2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \cap H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is given by*

(2.6)

$$\begin{aligned}
&\delta E_\varepsilon^{(2)}(\gamma; \varphi) \\
&\quad = 2 \iint_{U_\varepsilon} \Bigg\{ \left( \frac{1}{|\gamma(u+w) - \gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2} \right) \left\langle \frac{\gamma'(u)}{|\gamma'(u)|^2}, \varphi'(u) \right\rangle \\
&\qquad\qquad - \left( \frac{\langle \gamma(u+w) - \gamma(u), \varphi(u+w) - \varphi(u) \rangle}{|\gamma(u+w) - \gamma(u)|^4} - \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)}{d_\gamma(u+w,u)^3} \right) \Bigg\} \\
&\qquad\qquad\qquad |\gamma'(u+w)||\gamma'(u)| \, \mathrm{d}w \, \mathrm{d}u.
\end{aligned}$$

*Proof.* Applying Lemma 2.1, we obtain an $H^{3/2} \cap H^{1,\infty}$-neighborhood $Y \subset H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ of $\gamma$ such that (2.2) uniformly holds on $Y$ for any element in $U_\varepsilon$. The integrand in (2.6) is almost everywhere the pointwise derivative of the integrand in $E_\varepsilon^{(2)}$. Using (0.1) and (2.5), one sees furthermore that this pointwise derivative is majorized by some $L^1$-function. So, Lebesgue's Theorem permits us to interchange differentiation and integration which, by a suitable reparametrization, results in (2.6).

As for the continuity of $E_\varepsilon^{(2)}$ and $\delta E_\varepsilon^{(2)}$, the only difficulty is to treat the intrinsic distance. Recalling the continuity of the length functional with respect to absolutely continuous curves we can directly read from (0.1) that the integrand of $E_\varepsilon^{(2)}$ defines a continuous operator $(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty})(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to L^1(U_\varepsilon)$.

We claim that the same applies to the integrand of $\delta E_\varepsilon^{(2)}$. To this end we consider (2.6). We can conclude in the same manner as before except for the derivative of the intrinsic distance. According to (2.5) the latter exists for a.e. $(u,w) \in U_\varepsilon \cap \big\{\mathscr{L}(\gamma|_{[u,u+w]}) \neq \frac{1}{2}\mathscr{L}(\gamma)\big\}$. On the other hand, $\big\{\mathscr{L}(\gamma|_{[u,u+w]}) = \frac{1}{2}\mathscr{L}(\gamma)\big\}$ is a null set, since, due to the fact that $\gamma$ is regular, for any fixed $u \in \mathbb{R}/\mathbb{Z}$ there are at most two points $w \in [-\frac{1}{2}, \frac{1}{2}]$ satisfying $\mathscr{L}(\gamma|_{[u,u+w]}) = \frac{1}{2}\mathscr{L}(\gamma)$. As desired, we see that the integrand of $\delta E_\varepsilon^{(2)}$ gives rise to a continuous mapping $(H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty})(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \times H^{1,\infty}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \to L^1(U_\varepsilon)$. Being linear and bounded in the second component, it can be viewed as a continuous mapping from $H^{1,\infty}$ into the linear bounded operators $H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty} \to L^1(U_\varepsilon)$.

Altogether, the integrand of $E_\varepsilon^{(2)}$ is a continuously differentiable functional $H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty} \to L^1(U_\varepsilon)$. The statement now follows from the chain rule and the fact that the integration operator

$$L^1(U_\varepsilon) \to \mathbb{R}, \qquad g \mapsto \iint_{U_\varepsilon} g(u,w) \, \mathrm{d}u \, \mathrm{d}w,$$

is continuously differentiable as it is a bounded linear operator. □

In order to prove Proposition 2.2, we state in Lemma 2.4 below that $E_\varepsilon^{(2)}$ is *nearly* a Cauchy sequence in $C^1(X_\delta)$ for subsets $X_\delta \subset H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$, $\delta \geq 0$, which satisfy the following substitute of the uniform integrability property:

$$\limsup_{\varepsilon\to 0}\sup_{\gamma\in X_\delta}[\gamma']_{H^{1/2}_\varepsilon}\le\delta, \tag{2.7}$$

where we set for $\varepsilon\in(0,\frac{1}{2})$

$$[f]_{H^{1/2}_\varepsilon}:=\left(\iint\limits_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]}\frac{|f(u+w)-f(u)|^2}{w^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2}.$$

The statement involves the Lipschitz constant

$$\operatorname{lip}_Y E=\sup_{\substack{f,\tilde f\in Y\\ f\neq\tilde f}}\frac{\left|E(f)-E(\tilde f)\right|}{\left\|f-\tilde f\right\|}$$

for some real-valued functional $E$ and a subset $Y$ contained in its domain.

**Lemma 2.4.** *We have*

$$E^{(2)}_\varepsilon(\gamma)\to E^{(2)}(\gamma)$$

*for all* $\gamma\in H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$.

*Furthermore, for any* $\gamma_0\in H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$ *there is an open neighborhood* $Y\subset H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$, $\gamma_0\in Y$, *and a constant* $C=C(\gamma_0)<\infty$ *such that*

$$\limsup_{\varepsilon_1,\varepsilon_2\to 0}\operatorname{lip}_{X_\delta\cap Y}(E^{(2)}_{\varepsilon_1}-E^{(2)}_{\varepsilon_2})\le C\delta \tag{2.8}$$

*for all subsets* $X_\delta\subset H^{3/2}\cap H^{1,\infty}$ *satisfying* (2.7) *with* $\delta\in[0,1]$.

*Proof.* From Lemma 2.1 we get an $H^{3/2}\cap H^{1,\infty}$-neighborhood $Y\subset H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$ of $\gamma$ such that (2.2) holds for all $\gamma\in Y$. Making $Y$ smaller if necessary, we may also assume the existence of an $\varepsilon_0>0$ with

$$d_\gamma(u+w,u)=\mathscr{L}(\gamma|_{[u,u+w]})$$

for all $\gamma\in Y$ and $w\in[-\varepsilon_0,\varepsilon_0]$.

In order to bring the integrand in the definition of $E^{(2)}$ and $E^{(2)}_\varepsilon$ into a more convenient form we introduce the function

$$g(\zeta,\eta,\vartheta,\iota):=\frac{\zeta^{-2}-\eta^{-2}}{\eta^2-\zeta^2}\vartheta\iota=\frac{\vartheta\iota}{\zeta^2\eta^2}$$

which is Lipschitz continuous and positive on $[\tilde c,\tilde C]^4$ for any $0<\tilde c<\tilde C<\infty$. We define for $u\in\mathbb{R}/\mathbb{Z}$, $w\in[-\varepsilon_0,\varepsilon_0]$

$$\mathcal{G}_\gamma:(u,w)\mapsto g\left(\left|\int_0^1\gamma'(u+\theta_1 w)\,\mathrm{d}\theta_1\right|,\int_0^1|\gamma'(u+\theta_2 w)|\,\mathrm{d}\theta_2,|\gamma'(u+w)|,|\gamma'(u)|\right).$$

We have chosen $Y$ in such a way that the arguments in $\mathcal{G}$ are uniformly bounded away from zero. On the other hand, any of these arguments is uniformly bounded by $\operatorname{ess\,sup}|\gamma'|\le\|\gamma\|_{H^{1,\infty}}$. Choosing $Y$ to be bounded and using the Lipschitz continuity of $g$ therefore gives $0\le\mathcal{G}_\gamma(u,w)\le C$ as well as $|\mathcal{G}_{\tilde\gamma}(u,w)-\mathcal{G}_\gamma(u,w)|\le C\|\tilde\gamma'-\gamma'\|_{L^\infty}$ for any $u\in\mathbb{R}/\mathbb{Z}$, $w\in[-\varepsilon_0,\varepsilon_0]$, $\gamma,\tilde\gamma\in Y$, and a constant $C$ depending on $Y$ only.

Then we decompose the integrand in the definition of $E^{(2)}$ for $|w| \le \varepsilon_0$ into

$$\begin{aligned}
&\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{d_\gamma(u+w,u)^2}\right)|\gamma'(u+w)||\gamma'(u)| \\
&= \frac{1}{w^2}\left(\frac{1}{\left|\int_0^1 \gamma'(u+\theta_1 w)\,\mathrm{d}\theta_1\right|^2} - \frac{1}{\left(\int_0^1 |\gamma'(u+\theta_2 w)|\,\mathrm{d}\theta_2\right)^2}\right)|\gamma'(u+w)||\gamma'(u)| \\
&= \mathcal{G}_\gamma(u,w)\frac{\left(\int_0^1 |\gamma'(u+\theta_2 w)|\,\mathrm{d}\theta_2\right)^2 - \left|\int_0^1 \gamma'(u+\theta_1 w)\,\mathrm{d}\theta_1\right|^2}{|w|^2} \\
&= \mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}\left(|\gamma'(u+\theta_1 w)||\gamma'(u+\theta_2 w)| - \langle\gamma'(u+\theta_1 w),\gamma'(u+\theta_2 w)\rangle\right)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{|w|^2}.
\end{aligned}$$

Using $2|a||b| - 2\langle a,b\rangle = |a-b|^2 - ||a|-|b||^2$ for $a,b \in \mathbb{R}^n$, this can be written as

$$\begin{aligned}
&\tfrac{1}{2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2} \\
&\quad - \tfrac{1}{2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}\left(|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|\right)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}.
\end{aligned}$$

We first use this to get

$$\begin{aligned}
&E^{(2)}(\gamma) - E^{(2)}_\varepsilon(\gamma) \\
&\quad\le C\int_{\mathbb{R}/\mathbb{Z}}\int_{-\varepsilon}^{\varepsilon}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\,\mathrm{d}w\,\mathrm{d}u \\
&\quad\le C\int_{\mathbb{R}/\mathbb{Z}}\int_{-\varepsilon}^{\varepsilon}\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\,\mathrm{d}w\,\mathrm{d}u \xrightarrow{\varepsilon\to 0} 0,
\end{aligned}$$

which proves the pointwise convergence stated in the lemma.

Let now $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_0$ and set

$$F := E^{(2)}_{\varepsilon_1} - E^{(2)}_{\varepsilon_2}.$$

Using the decomposition of the integrand above, we get

$$\begin{aligned}
F(\gamma) &= \tfrac{1}{2}\int_{\mathbb{R}/\mathbb{Z}}\int_{\varepsilon_1<|w|<\varepsilon_2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}|\gamma'(u+\theta_1 w)-\gamma'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\,\mathrm{d}w\,\mathrm{d}u \\
&\quad - \tfrac{1}{2}\int_{\mathbb{R}/\mathbb{Z}}\int_{\varepsilon_1<|w|<\varepsilon_2}\mathcal{G}_\gamma(u,w)\frac{\iint_{[0,1]^2}\left(|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|\right)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2} \\
&\qquad\mathrm{d}w\,\mathrm{d}u \\
&=: \tfrac{1}{2}F_1^{(2)}(\gamma) - \tfrac{1}{2}F_2^{(2)}(\gamma).
\end{aligned}$$

To estimate the difference $F(\tilde{\gamma}) - F(\gamma)$ for $\gamma, \tilde{\gamma} \in Y$, we first consider

$$\begin{aligned}
&|\mathcal{G}_{\tilde{\gamma}}(u,w) - \mathcal{G}_{\gamma}(u,w)| \\
&\le C\left|\left|\int_0^1 \tilde{\gamma}'(u+\theta_1 w)\,\mathrm{d}\theta_1\right| - \left|\int_0^1 \gamma'(u+\theta_2 w)\,\mathrm{d}\theta_2\right|\right| \\
&\quad + C\left|\int_0^1 \left(|\tilde{\gamma}'(u+\theta w)| - |\gamma'(u+\theta w)|\right)\,\mathrm{d}\theta\right| \\
&\quad + C\big||\tilde{\gamma}'(u+w)| - |\gamma'(u+w)|\big| + C\big||\tilde{\gamma}'(u)| - |\gamma'(u)|\big| \\
&\le C\int_0^1 |\tilde{\gamma}'(u+\theta w) - \gamma'(u+\theta w)|\,\mathrm{d}\theta + C|\tilde{\gamma}'(u+w) - \gamma'(u+w)| + C|\tilde{\gamma}'(u) - \gamma'(u)| \\
&\le C\,\|\tilde{\gamma}' - \gamma'\|_{L^\infty}.
\end{aligned}$$

We arrive at

$$\begin{aligned}
&\left|F_1^{(2)}(\tilde{\gamma}) - F_1^{(2)}(\gamma)\right| \\
&\le \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} |\mathcal{G}_{\tilde{\gamma}}(u,w) - \mathcal{G}_{\gamma}(u,w)| \frac{\iint_{[0,1]^2} |\tilde{\gamma}'(u+\theta_1 w) - \tilde{\gamma}'(u+\theta_2 w)|^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\,\mathrm{d}w\,\mathrm{d}u \\
&\quad + \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} |\mathcal{G}_{\gamma}(u,w)| \\
&\quad \cdot \frac{\iint_{[0,1]^2} \left||\tilde{\gamma}'(u+\theta_1 w) - \tilde{\gamma}'(u+\theta_2 w)|^2 - |\gamma'(u+\theta_1 w) - \gamma'(u+\theta_2 w)|^2\right|\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2} \\
&\qquad \mathrm{d}w\,\mathrm{d}u \\
&\le C\,\|\tilde{\gamma}' - \gamma'\|_{L^\infty} \iint_{[0,1]^2} \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \frac{|\tilde{\gamma}'(u+\theta_1 w) - \tilde{\gamma}'(u+\theta_2 w)|^2}{w^2}\,\mathrm{d}w\,\mathrm{d}u\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\
&\quad + C\,\|\gamma'\|_{L^\infty} \iint_{[0,1]^2} \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \\
&\quad \cdot \frac{|(\tilde{\gamma}'+\gamma')(u+\theta_1 w) - (\tilde{\gamma}'+\gamma')(u+\theta_2 w)||(\tilde{\gamma}'-\gamma')(u+\theta_1 w) - (\tilde{\gamma}'-\gamma')(u+\theta_2 w)|}{w^2} \\
&\qquad \mathrm{d}w\,\mathrm{d}u\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\
&\le C\,[\tilde{\gamma}']^2_{H^{1/2}_{2\varepsilon_2}} \|\tilde{\gamma}' - \gamma'\|_{L^\infty} + C\,\|\gamma'\|_{L^\infty} [\tilde{\gamma}' + \gamma']_{H^{1/2}_{2\varepsilon_2}} [\tilde{\gamma}' - \gamma']_{H^{1/2}_{2\varepsilon_2}}.
\end{aligned}$$

For the second term we compute

$$\begin{aligned}
&\left|F_2^{(2)}(\tilde{\gamma}) - F_2^{(2)}(\gamma)\right| \\
&\le \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} |\mathcal{G}_{\tilde{\gamma}}(u,w) - \mathcal{G}_{\gamma}(u,w)| \frac{\iint_{[0,1]^2} \left(|\tilde{\gamma}'(u+\theta_1 w)| - |\tilde{\gamma}'(u+\theta_2 w)|\right)^2\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2}{w^2}\,\mathrm{d}w\,\mathrm{d}u \\
&\quad + \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} |\mathcal{G}_{\gamma}(u,w)|
\end{aligned}$$

$$\cdot \frac{\iint_{[0,1]^2} \Big| \big(|\tilde\gamma'(u+\theta_1 w)| - |\tilde\gamma'(u+\theta_2 w)|\big)^2 - \big(|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|\big)^2 \Big| \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2}{w^2} \, \mathrm{d}w \, \mathrm{d}u$$

$$\le C \|\tilde\gamma' - \gamma'\|_{L^\infty} \iint_{[0,1]^2} \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]} \frac{|\tilde\gamma'(u+\theta_1 w) - \tilde\gamma'(u+\theta_2 w)|^2}{w^2} \, \mathrm{d}w \, \mathrm{d}u \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2$$

$$+ C \|\gamma'\|_{L^\infty} \iint_{[0,1]^2} \iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon_2,\varepsilon_2]}$$

$$\cdot \frac{|(|\tilde\gamma'(u+\theta_1 w)| - |\tilde\gamma'(u+\theta_2 w)|) + (|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|)|}{|w|}$$

$$\cdot \frac{|(|\tilde\gamma'(u+\theta_1 w)| - |\tilde\gamma'(u+\theta_2 w)|) - (|\gamma'(u+\theta_1 w)| - |\gamma'(u+\theta_2 w)|)|}{|w|} \, \mathrm{d}w \, \mathrm{d}u \, \mathrm{d}\theta_1 \, \mathrm{d}\theta_2$$

$$\le C \|\tilde\gamma' - \gamma'\|_{L^\infty} [\tilde\gamma']^2_{H^{1/2}_{2\varepsilon_2}} + C \|\gamma'\|_{L^\infty} [|\tilde\gamma'| + |\gamma'|]_{H^{1/2}_{2\varepsilon_2}} [|\tilde\gamma'| - |\gamma'|]_{H^{1/2}_{2\varepsilon_2}} .$$

Using the chain and product rule for Sobolev spaces and the formula

$$|\tilde\gamma'| - |\gamma'| = \frac{\langle \tilde\gamma' + \gamma', \tilde\gamma' - \gamma' \rangle}{|\tilde\gamma'| + |\gamma'|},$$

we obtain, assuming $\varepsilon_0 < \frac{1}{4}$, for $C > 0$ depending on the constant from (2.2)

$$[|\tilde\gamma'| - |\gamma'|]_{H^{1/2}_{2\varepsilon_2}} \le [|\tilde\gamma'| - |\gamma'|]_{H^{1/2}(\mathbb{R}/\mathbb{Z})}$$
$$\le C \|\tilde\gamma' - \gamma'\|_{H^{1/2}\cap L^\infty} \|\tilde\gamma' + \gamma'\|_{H^{1/2}\cap L^\infty} \Big( 1 + [\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} + [\gamma']_{H^{1/2}_{2\varepsilon_2}} \Big),$$

where

$$\|\cdot\|_{H^{1/2}\cap L^\infty} := \|\cdot\|_{H^{1/2}} + \|\cdot\|_{L^\infty} .$$

Hence

$$\Big| F_2^{(2)}(\tilde\gamma) - F_2^{(2)}(\gamma) \Big| \le C \Bigg( [\tilde\gamma']^2_{H^{1/2}_{2\varepsilon_2}} + \|\gamma'\|_{L^\infty} \Big( [\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} + [\gamma']_{H^{1/2}_{2\varepsilon_2}} \Big) \|\tilde\gamma' + \gamma'\|_{H^{1/2}\cap L^\infty}$$
$$\cdot \Big( 1 + [\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} + [\gamma']_{H^{1/2}_{2\varepsilon_2}} \Big) \Bigg) \|\tilde\gamma' - \gamma'\|_{H^{1/2}\cap L^\infty} .$$

The claim follows from

$$\limsup_{\varepsilon_2 \searrow 0} \sup_{\tilde\gamma \in X_\delta \cap Y} [\tilde\gamma']_{H^{1/2}_{2\varepsilon_2}} \le \delta.$$

□

*Proof of Proposition* 2.2. In order to prove that *directional derivatives* exist at $\gamma_0 \in H^{3/2}_{\mathrm{ir}} \cap H^{1,\infty}$ for all directions $\varphi \in H^{3/2} \cap H^{1,\infty}$, let $Y$ be as in the proof of Lemma 2.4 and

$$X_0 := \{ \gamma_0 + \tau\varphi \,|\, \tau \in (-1,1) \} .$$

First we observe that $X_0$ satisfies (2.7), thus being an admissible set for Lemma 2.4. Indeed, for $\gamma_\tau := \gamma_0 + \tau\varphi$, $|\tau| \le 1$,

$$\begin{aligned} &\left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma_\tau'(u+w)-\gamma_\tau'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2} \\ &\le \left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma_0'(u+w)-\gamma_0'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2} \\ &\quad + \left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\varphi'(u+w)-\varphi'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2} \\ &\xrightarrow{\varepsilon\searrow 0} 0. \end{aligned} \tag{2.9}$$

From this we deduce, for

$$f_\varepsilon : \tau \mapsto E_\varepsilon^{(2)}(\gamma_0+\tau\varphi),$$

that

$$\begin{aligned} &\left|f_{\varepsilon_1}'(\tau) - f_{\varepsilon_2}'(\tau)\right| = \left|\delta E_{\varepsilon_1}^{(2)}(\gamma_0+\tau\varphi;\varphi) - \delta E_{\varepsilon_2}^{(2)}(\gamma_0+\tau\varphi;\varphi)\right| \\ &\le \limsup_{\theta\to 0}\left|\frac{E_{\varepsilon_1}^{(2)}(\gamma_0+(\tau+\theta)\varphi) - E_{\varepsilon_1}^{(2)}(\gamma_0+\tau\varphi)}{\theta}\right. \\ &\qquad\qquad \left. - \frac{E_{\varepsilon_2}^{(2)}(\gamma_0+(\tau+\theta)\varphi) - E_{\varepsilon_2}^{(2)}(\gamma_0+\tau\varphi)}{\theta}\right| \\ &\le \operatorname{lip}_{X_0\cap Y}\left(E_{\varepsilon_1}^{(2)} - E_{\varepsilon_2}^{(2)}\right)\|\varphi\|_{H^{3/2}\cap H^{1,\infty}} \\ &\xrightarrow{\varepsilon_1,\varepsilon_2\searrow 0} 0 \qquad \text{by (2.8).} \end{aligned} \tag{2.10}$$

As $E_\varepsilon^{(2)} \to E^{(2)}$ pointwise, this proves $(f_\varepsilon)_{\varepsilon>0}$ is a Cauchy sequence in $C^1((-\tau_0,\tau_0))$ converging to $E^{(2)}(\gamma_0+\tau\varphi) = \lim_{\varepsilon\searrow 0} E_\varepsilon^{(2)}(\gamma_0+\tau\varphi)$ as $\varepsilon\to 0$. Hence, especially all directional derivatives of $E^{(2)}$ exist and

$$\delta E^{(2)}(\gamma_0;\varphi) = \lim_{\varepsilon\searrow 0} \delta E_\varepsilon^{(2)}(\gamma_0;\varphi)$$

for all $\gamma_0 \in H_{\mathrm{ir}}^{3/2}\cap H^{1,\infty}$, $\varphi \in H^{3/2}\cap H^{1,\infty}$.

The next step is to establish *Gâteaux differentiability.* To this end we merely have to show $\delta E^{(2)}(\gamma_0,\cdot) \in \left(H^{3/2}\cap H^{1,\infty}\right)^*$ for $\gamma_0 \in H_{\mathrm{ir}}^{3/2}\cap H^{1,\infty}$. Linearity carries over from $E_\varepsilon^{(2)}$. In order to prove boundedness we introduce

$$X_\delta := \{\gamma \in H^{3/2} : \|\gamma-\gamma_0\|_{H^{3/2}} \le \delta\} \qquad \text{for } \delta\in(0,1], \tag{2.11}$$

which also satisfies (2.7), where for $\gamma\in X_\delta$ we have

$$\begin{aligned} &\left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma'(u+w)-\gamma'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2} \\ &\le \left(\iint_{\mathbb{R}/\mathbb{Z}\times[-\varepsilon,\varepsilon]} \frac{|\gamma_0'(u+w)-\gamma_0'(u)|^2}{|w|^2}\,\mathrm{d}w\,\mathrm{d}u\right)^{1/2} + \delta \xrightarrow{\varepsilon\searrow 0} \delta. \end{aligned} \tag{2.12}$$

Now

$$\begin{aligned}\delta E^{(2)}(\gamma_0;\varphi) &= \delta E^{(2)}_\varepsilon(\gamma_0;\varphi) + \delta E^{(2)}(\gamma_0;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)\\ &= \delta E^{(2)}_\varepsilon(\gamma_0;\varphi) + \lim_{\varepsilon_1\searrow 0}\left(\delta E^{(2)}_{\varepsilon_1}(\gamma_0;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)\right),\end{aligned}$$

and thus, arguing as in (2.10) and recalling $\delta E^{(2)}_\varepsilon(\gamma_0;\cdot) \in \left(H^{3/2}\cap H^{1,\infty}\right)^*$,

$$\begin{aligned}|\delta E^{(2)}(\gamma_0;\varphi)| &\le |\delta E^{(2)}_\varepsilon(\gamma_0;\varphi)| + \underbrace{\limsup_{\varepsilon_1\searrow 0}\operatorname{lip}_{X_\delta\cap Y}\left(E^{(2)}_{\varepsilon_1} - E^{(2)}_\varepsilon\right)}_{<\infty}\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}\\ &\le C\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}\end{aligned}$$

for all $\gamma_0 \in H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$ and $\varphi \in H^{3/2}\cap H^{1,\infty}$. Hence, $E^{(2)}$ is Gâteaux differentiable and the differential is given by

$$\left(E^{(2)}\right)'(\gamma_0) = \delta E^{(2)}(\gamma_0;\cdot)$$

for all $\gamma_0 \in H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$, $\varphi \in H^{3/2}\cap H^{1,\infty}$.

Finally, to see that the differential is *continuous*, let $\sigma > 0$ be given and let us choose $\delta > 0$ and $\varepsilon > 0$ so small that

$$\operatorname{lip}_{X_\delta\cap Y}\left(E^{(2)}_{\varepsilon_1} - E^{(2)}_{\varepsilon_2}\right) \overset{(2.8)}{\le} C\delta \le \tfrac{1}{3}\sigma$$

for all $\varepsilon_1, \varepsilon_2 < \varepsilon$, where $X_\delta$ is as in (2.11). Then we have for $\gamma \in X_\delta \cap Y$ and any $\varphi \in H^{3/2}\cap H^{1,\infty}$,

$$\begin{aligned}&|\delta E^{(2)}(\gamma;\varphi) - \delta E^{(2)}(\gamma_0;\varphi)|\\ &\quad\le |\delta E^{(2)}(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma;\varphi)| + |\delta E^{(2)}_\varepsilon(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)|\\ &\qquad + |\delta E^{(2)}_\varepsilon(\gamma_0;\varphi) - \delta E^{(2)}(\gamma_0;\varphi)|\\ &\quad\overset{(2.10)}{\le} |\delta E^{(2)}_\varepsilon(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)| + \tfrac{2}{3}\sigma\,\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}.\end{aligned}$$

Since $E^{(2)}_\varepsilon$ is $C^1$ we deduce that there is an open neighborhood $V \subset X_\delta$ of $\gamma_0$ such that

$$|\delta E^{(2)}_\varepsilon(\gamma;\varphi) - \delta E^{(2)}_\varepsilon(\gamma_0;\varphi)| \le \tfrac{1}{3}\sigma\,\|\varphi\|_{H^{3/2}\cap H^{1,\infty}},$$

and hence

$$|\delta E^{(2)}(\gamma;\varphi) - \delta E^{(2)}(\gamma_0;\varphi)| \le \sigma\,\|\varphi\|_{H^{3/2}\cap H^{1,\infty}}$$

for all $\gamma \in V$. This proves that $\left(E^{(2)}\right)'$ is continuous from $H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$ into $\left(H^{3/2}\cap H^{1,\infty}\right)^*$, and hence $E^{(2)}$ is $C^1(H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty})$. ☐

*Proof of Theorem* II. Using Proposition 2.2 we merely have to derive the formula of the first variation for a curve $\gamma \in H^{3/2}_{\mathrm{ir}}\cap H^{1,\infty}$ parametrized by arc-length and

$\varphi \in H^{3/2} \cap H^{1,\infty}$. As $|\gamma'| \equiv 1$ a.e. we deduce from Lemma 2.3 and Equation (2.5)

$$\delta E^{(2)}(\gamma;\varphi) \xleftarrow{\varepsilon \searrow 0} 2 \iint_{U_\varepsilon} \Bigg\{ \left( \frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{w^2} \right) \langle \gamma'(u), \varphi'(u) \rangle - 2 \Bigg( \frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\frac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u)}{|w|^3} \Bigg) \Bigg\} \,\mathrm{d}w\,\mathrm{d}u,$$

where now

$$\tfrac{\mathrm{d}}{\mathrm{d}\tau}\big|_{\tau=0} d_{\gamma+\tau\varphi}(u+w,u) = |w| \int_0^1 \langle \gamma'(u+\theta w), \varphi'(u+\theta w) \rangle \,\mathrm{d}\theta$$

for all $(u,w) \in \mathbb{R}/\mathbb{Z} \times (-\frac{1}{2}, \frac{1}{2})$. Hence,

$$\begin{aligned} \delta E^{(2)}(\gamma;\varphi) &\xleftarrow{\varepsilon \searrow 0} 2 \iint_{U_\varepsilon} \Bigg\{ \left( \frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{w^2} \right) \langle \gamma'(u), \varphi'(u) \rangle \\ &\quad - \Bigg( \frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\int_0^1 \langle \gamma'(u+\theta w), \varphi'(u+\theta w) \rangle \, d\theta}{w^2} \Bigg) \Bigg\} \,\mathrm{d}w\,\mathrm{d}u \\ &= 2 \iint_{U_\varepsilon} \Bigg\{ \left( \frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{w^2} \right) \langle \gamma'(u), \varphi'(u) \rangle \\ &\quad - \left( \frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} - \frac{\langle \gamma'(u), \varphi'(u) \rangle}{w^2} \right) \Bigg\} \,\mathrm{d}w\,\mathrm{d}u \\ &= 2 \iint_{U_\varepsilon} \left( \frac{\langle \gamma'(u), \varphi'(u) \rangle}{|\gamma(u+w)-\gamma(u)|^2} - \frac{\langle \gamma(u+w)-\gamma(u), \varphi(u+w)-\varphi(u) \rangle}{|\gamma(u+w)-\gamma(u)|^4} \right) \mathrm{d}w\,\mathrm{d}u. \end{aligned}$$

□

## 3. Initial regularity: Proof of Theorem III

**Estimate of the normal part.** We have

$$\langle g', |D|^{\frac{1}{2}} g' \rangle = -\frac{1}{2} H_{\frac{1}{2}}(g', g') + \frac{1}{2} |D|^{\frac{1}{2}} |g'|^2, \tag{3.1}$$

where

$$H_s(a,b) := |D|^s(ab) - a\,|D|^s b - b\,|D|^s a. \tag{3.2}$$

Note that for any $s \in (0,1)$, we have

$$|D|^s |g'|^2 \overset{(1.5)}{=} |D|^s \eta^2 + 2|D|^s(\eta'\gamma' g) + |D|^s(|\eta'|^2|\gamma|^2) \in L^\infty((0,1)). \tag{3.3}$$

In fact, $|D|^s \eta^2 \in L^\infty$ by interpolation inequalities. For the remaining terms we use the quasi-locality, Lemma A.1, and the support of $\eta$ and $\eta'$.

As in [Sch11] we will use pointwise estimates for $H_s$ and some quantitative version of the quasi-locality to estimate the normal part of $|D|^{\frac{1}{2}}g'$:

**Lemma 3.1** (Normal part). *For any $s \in [0, \frac{1}{2})$ there exists $\theta > 0$ such that for any $B_r \subset [0,1]$, $\Lambda > 4$,*

$$\begin{aligned} &\| |D|^s \langle g', |D|^{\frac{1}{2}} g' \rangle \|_{(\frac{2}{1+2s},\infty),B_r} \\ &\precsim \| |D|^{\frac{1}{2}} g' \|^2_{(2,\infty),B_{\Lambda r}} \\ &\quad + \ \Lambda^{-\theta} \ \| |D|^{\frac{1}{2}} g' \|_{(2,\infty),\mathbb{R}} \ \sum_{k=1}^{\infty} 2^{-\theta k} \ \| |D|^{\frac{1}{2}} g' \|_{(2,\infty),B_{2^k\Lambda r}} + r^{\frac{1}{2}+s}. \end{aligned} \tag{3.4}$$

The proof uses only the fact that $|\gamma'| = 1$, in the form of (3.3), and then uses estimates on commutators of non-local differential operators. It is an adaption of similar arguments which have been used in the theory of fractional harmonic maps [DLR11b, DL11, Sch11]. For the reader's convenience, a proof will be given in the appendix.

**Estimate of the tangential part.** It then remains to estimate the part normal to $g'$ (tangential to the sphere), i.e. for $\omega_{ij} = -\omega_{ji} \in \{-1,0,1\}$, $1 \leq i,j \leq n$, we need to estimate suitable norms on small balls of the term $g'_j \omega_{ij} |D|^{\frac{1}{2}} g'_i$.

This is the part where we will actually make use of the Euler-Lagrange equations. We have

$$\begin{aligned} \int_{\mathbb{R}} g'_j \omega_{ij} |D|^{\frac{1}{2}} g'_i \ |D|^{\frac{1}{2}} \varphi &= \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \ |D|^{\frac{1}{2}} \left(\omega_{ij} g'_j \varphi\right) - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \ \omega_{ij} |D|^{\frac{1}{2}} g'_j \ \varphi \\ &\quad - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \ \omega_{ij} H_{1/2}(g'_j, \varphi) \\ &= \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \ |D|^{\frac{1}{2}} \left(\omega_{ij} g'_j \varphi\right) - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g'_i \ \omega_{ij} H_{1/2}(g'_j, \varphi), \end{aligned} \tag{3.5}$$

where we have used that due to $\omega_{ij} = -\omega_{ji}$, the second term on the right-hand side of the first line vanishes.

The second term can be estimated analogously to similar terms in [DLR11b, DL11, Sch11], again using quasi-locality together with Sobolev embeddings.

**Lemma 3.2.** *There is $\theta > 0$, $s_0 \in (0, \frac{1}{2})$ such that for all $\phi \in C_0^\infty(B_r)$, $\Lambda > 16$, $s \in [0, s_0)$ we have*

$$\begin{aligned} &\int |D|^{\frac{1}{2}} g'_i \omega_{ij} H_{1/2}(g'_j, \phi) \, \mathrm{d}x \\ &\quad \precsim \| |D|^{-s+1/2} \phi \|_{(\frac{2}{1-2s},1)} \Big( \| |D|^{\frac{1}{2}} g' \|^2_{(2,\infty),B_{\Lambda r}} + \Lambda^{-\theta} \| |D|^{\frac{1}{2}} g' \|_{(2,\infty),\mathbb{R}} \\ &\qquad \cdot \sum_{k=1}^{\infty} 2^{-\theta(k-1)} \| |D|^{\frac{1}{2}} g' \|_{(2,\infty),B_{2^k\Lambda r}} \Big). \end{aligned}$$

A proof is provided in the appendix.

It remains to estimate the first term on the right-hand side of equation (3.5) for which we will use the Euler-Langrange equation (1.1).

Combining this equation with the formula

(3.6)
$$\int_{\mathbb{R}} |D|^{\frac{1}{2}} f_1' \ |D|^{\frac{1}{2}} f_2'$$
$$= c \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \left( \langle f_1'(u), f_2'(u) \rangle - \frac{\langle f_1(u+w) - f_1(u), f_2(u+w) - f_2(u) \rangle}{w^2} \right) \frac{\mathrm{d}w}{w^2} \,\mathrm{d}u$$

due to He [He99, Proposition 2], we get the following estimate which contains all the information of the Euler-Langrange equation we need to proceed in the proof:

**Lemma 3.3** (Essential estimate of the Euler-Lagrange equation)**.** *There is a constant $C < \infty$ such that*

$$\int_{\mathbb{R}} \left\langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} \phi \right\rangle \leq C \int_{\mathbb{R}} |\phi(u)| \Gamma(u) \,\mathrm{d}u + C \|\phi\|_{L^2} \tag{3.7}$$

*for any $\phi \in C_0^\infty((4/10, 6/10), \mathbb{R}^n)$ with $\langle \phi, \gamma' \rangle \equiv 0$, where*

$$\Gamma(u) := \int_{(-1,1)^3} \int_{-1/4}^{1/4} \frac{|g'(u) - g'(u+s_2 w)| \ |g'(u+s_3 w) - g'(u+s_4 w)|^2}{|w|^2} \ \mathrm{d}w \ \mathrm{d}s.$$

The heart of the proof of Theorem III is the following pointwise estimate of the most problematic term $\Gamma(u)$, which permits us to localize it and which afterwards will be transformed into a bound of its $L^1$-norm.

**Lemma 3.4** (Estimate of the critical term)**.** *We have*

$$\begin{aligned} \Gamma(u) \lesssim{}& \left| |D|^{-\frac{1}{12}} ||D|^{\frac{1}{2}} g'|(u) \right|^2 \ |D|^{-\frac{1}{3}} ||D|^{\frac{1}{2}} g'|(u) \\ &+ |D|^{-\frac{1}{12}} ||D|^{\frac{1}{2}} g'|(u) \ |D|^{-\frac{1}{4}} \left| |D|^{-\frac{1}{12}} ||D|^{\frac{1}{2}} g'|(u) \right|^2 \end{aligned} \tag{3.8}$$

*almost everywhere.*

**Lemma 3.5.** *There exist $R > 0$, $s_0 \in (0, \frac{1}{2})$, and $\sigma > 0$ such that for any $\Lambda > 2$, $s \in [0, s_0)$, $B_{\Lambda r} \subset (\frac{5}{10}, \frac{6}{10})$, $r \in (0, \Lambda^{-1} R)$ and $\varphi \in C_0^\infty(B_r)$ we have*

$$\begin{aligned} &\int g_i' \omega_{ij} |D|^{\frac{1}{2}} g_j' \ |D|^{\frac{1}{2}} \varphi \\ &\quad\lesssim \ \|I_s |D|^{\frac{1}{2}} \varphi\|_{(\frac{2}{1-2s},1)} \Bigg( r^\sigma + \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{\Lambda r}} \\ &\qquad\qquad + \Lambda^{-\theta} \||D|^{\frac{1}{2}} g'\|_{(2,\infty),\mathbb{R}} \sum_{k=1}^{\infty} 2^{-\theta k} \ \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{2^k \Lambda r}} \Bigg). \end{aligned}$$

Combining Lemma 3.5 with Proposition A.3, we get the following counterpart to Lemma 3.1.

**Lemma 3.6** (Tangential part). *There exist $R > 0$, $s_0 \in (0, \frac{1}{2})$, and $\theta, \sigma > 0$, such that for any $\Lambda > 2$, $s \in [0, s_0)$, $B_{\Lambda r} \subset (\frac{5}{10}, \frac{6}{10})$, $r \in (0, \Lambda^{-1}R)$, we have*

(3.9)

$$\begin{aligned}
&\||D|^s \Big(g_i' \omega_{ij} |D|^{\frac{1}{2}} g_j'\Big)\|_{(\frac{2}{2s+1},\infty),B_r} \\
&\lesssim \left( \big(\Lambda^2 r\big)^\sigma + \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{\Lambda^3 r}} + \||D|^{\frac{1}{2}} g'\|_{(2,\infty),\mathbb{R}} \Lambda^{-\theta} \sum_{k=1}^{\infty} 2^{-\theta k} \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{\Lambda^3 2^k r}} \right) \\
&\quad + \Lambda^{-\theta} r^{-s} \sum_{k=1}^{\infty} 2^{-\theta k}\ \||D|^{\frac{1}{2}} g'\|_{(\frac{2}{1+2s},\infty),B_{\Lambda 2^k r}} \\
&\lesssim \left( \big(\Lambda^2 r\big)^\sigma + \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{\Lambda^3 r}} + \||D|^{\frac{1}{2}} g'\|_{(2,\infty),\mathbb{R}} \Lambda^{-\theta} \sum_{k=1}^{\infty} 2^{-\theta k} \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{\Lambda^3 2^k r}} \right) \\
&\quad + \Lambda^{s-\theta} \sum_{k=1}^{\infty} 2^{(s-\theta)k}\ \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{\Lambda 2^k r}}.
\end{aligned}$$

**Conclusion of the proof of Theorem III.** Let us first use (3.9) for $s = 0$ to obtain in view of (1.7) and Lemma 3.1 that for all $\varepsilon > 0$ we have for sufficiently small $r > 0$, $B_r \subset \big(\frac{4}{10}, \frac{6}{10}\big)$ and big enough $\Lambda$,

$$\begin{aligned}
\||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_r} \leq & C\Big( \big(\Lambda^2 r\big)^\sigma + \varepsilon\ \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{\Lambda^3 r}} \Big) \\
& + C\Big( (1 + \||D|^{\frac{1}{2}} g'\|_{(2,\infty),\mathbb{R}}) \Lambda^{-\theta} \sum_{k=2}^{\infty} 2^{-\theta k}\ \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{2^k \Lambda^3 r}} \Big)
\end{aligned}$$

uniformly in $\Lambda$ and $\varepsilon$.

Let us fix such an $r_0$, and consider the above equation for $\Lambda = 2^{m/3}$, $r = 2^{-m-k} r_0$ (w.l.o.g. $r_0 = 1$). Setting $b_0 := \||D|^{\frac{1}{2}} g'\|_{(2,\infty)}$ and $b_k := \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{2^{-k}}}$ the above estimate gives

$$b_{k+m} \leq C 2^{-\sigma(k+m/3)} + \varepsilon b_k + C 2^{-\theta m/3} \left( \sum_{l=1}^{k} 2^{-\theta l} b_{k-l} + C 2^{-\theta k} \right)$$

for every $k \in \mathbb{N}_0$, where $C < \infty$ does not depend on $k$.

Using the iteration argument, Lemma A.8, leads to

$$\||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_r} \leq C_\gamma\ r^{\tilde{\sigma}} \tag{3.10}$$

for all $B_r \subset \big(\frac{9}{20}, \frac{11}{20}\big)$ and $r$ small enough.

This immediately implies Hölder regularity of the solution $g'$. Instead, let us iterate the above argument to get $|D|^{\frac{1}{2}} g' \in L^p_{loc}((9/20, 11/20))$, for some $p > 2$.

Let us assume without loss of generality that $\tilde{\sigma} < \frac{\theta}{2}$. Then, plugging (3.10) into (3.9) one obtains

$$\||D|^s \Big(g_i' \omega_{ij} |D|^{\frac{1}{2}} g_j'\Big)\|_{(\frac{2}{2s+1},\infty),B_r} \lesssim C_{\Lambda,\gamma,s,R}\ r^{\min\{\sigma,\tilde{\sigma},\tilde{\sigma}-s,\theta/2\}}$$

for small enough $0 < s < (0, \tilde{\sigma})$, $r > 0$, and $B_r \subset [9/20, 11/20]$. Here we have used that $2^k \Lambda^3 r \geq R_0$ for some $R_0 > 0$ leads to $2^{-\theta k} \leq 2^{-\theta k/2} \left(\frac{\Lambda^3}{R_0}\right)^{\theta/2} r^{\theta/2}$. So the series in (3.9) converges and is bounded by some small positive power of $r$.

On the other hand, (3.10) and (3.4) together imply for any small enough $r > 0$, $\Lambda := r^{-\frac{1}{2}}$, and $B_r \subset \left(\frac{9}{20}, \frac{11}{20}\right)$,

$$\begin{aligned} \||D|^{\frac{1}{4}}\langle g', |D|^{\frac{1}{2}} g'\rangle\|_{(\frac{4}{3},\infty),B_r} &\overset{(3.4)}{\lesssim} \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{r^{\frac{1}{2}}}} + r^{\frac{\theta}{2}} \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),\mathbb{R}} + r^{\frac{3}{4}} \\ &\overset{(3.10)}{\leq} C_\gamma \left(r^{\tilde{\sigma}} + r^{\frac{\theta}{2}} + r^{\frac{3}{4}}\right). \end{aligned}$$

That is, $|D|^s\left(g_i'\omega_{ij}|D|^{\frac{1}{2}} g_j'\right)$ and $|D|^{\frac{1}{4}}\langle g', |D|^{\frac{1}{2}} g'\rangle$ both belong locally to a Morrey space $\mathcal{L}^{(p,q),\lambda}_{loc}$ on $\left(\frac{9}{20}, \frac{11}{20}\right)$. More precisely, for some $\lambda \in (0,1)$ we have

$$\begin{aligned} |D|^s\left(g_i'\omega_{ij}|D|^{\frac{1}{2}} g_j'\right) &\in \mathcal{L}^{(\frac{2}{2s+1},\infty),\lambda}_{loc}\left(\frac{9}{20}, \frac{11}{20}\right), \\ |D|^{\frac{1}{4}}\langle g', |D|^{\frac{1}{2}} g'\rangle &\in \mathcal{L}^{(\frac{4}{3},\infty),\lambda}_{loc}\left(\frac{9}{20}, \frac{11}{20}\right). \end{aligned}$$

The boundedness of Riesz potentials on Morrey spaces, as shown in [Ada75], implies that for some $p > 2$,

$$g_i'\omega_{ij}|D|^{\frac{1}{2}} g_j',\ \langle g', |D|^{\frac{1}{2}} g'\rangle \in L^p_{loc}\left(\frac{9}{20}, \frac{11}{20}\right).$$

Together, using (1.7) which of course holds also for $L^p$ instead of $L^{2,\infty}$, we have shown

$$|D|^{\frac{1}{2}} g_j' \in L^p_{loc}\left(\frac{9}{20}, \frac{11}{20}\right),$$

which finishes the proof of Theorem III. □

**Proof of Lemma 3.3.** The main idea is to use $\phi$ as the derivative of a test function for the Euler-Lagrange equation. Of course this is not possible generally, but with some precaution we can actually do it up to a benign error term: For $\phi \in C_0^\infty((4/10, 6/10))$ we set

$$h(u) := \int_0^u \phi(v)dv - au,$$

where $a = \int_0^1 \phi(v)dv$ is chosen such that $h(0) = h(1)$. Hence, if we set

$$\begin{aligned} h_\pi(k+t) &:= h(t), \quad \forall t \in [0,1), k \in \mathbb{Z}, \\ \phi_\pi(k+t) &:= \phi(t), \quad \forall t \in [0,1), k \in \mathbb{Z}, \end{aligned}$$

$h_\pi$ is a smooth one periodic function satisfying

$$h_\pi'(u) = \phi_\pi - a,$$

and, as we assume in Lemma 3.3 that $\langle \phi, \gamma' \rangle \equiv 0$, also

$$\langle \phi_\pi, \gamma' \rangle \equiv 0.$$

Since $\gamma$ is a critical point of the Möbius energy, testing the equation (1.1) with $h_\pi$ (recall (1.3), (1.4), (1.2)),

$$Q(\gamma, h_\pi) := \lim_{\varepsilon \to 0} Q_\varepsilon(\gamma, h_\pi) = T_1(\gamma, h_\pi) + T_2(\gamma, h_\pi). \tag{3.11}$$

As $\phi_\pi$ is perpendicular to $\gamma'$ we can estimate the term $T_1(\gamma, h_\pi)$ by

$$T_1(\gamma, h_\pi) = \int_{-1}^{1}\int_{-1/2}^{1/2} \langle \gamma'(u), \phi_\pi(u) - a\rangle \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^2} - \frac{1}{w^2}\right)$$
$$\leq |a|\ E^{(2)}(\gamma) \leq C\|\phi\|_{L^1} \leq C\|\phi\|_{L^2}. \tag{3.12}$$

As for the remaining terms $Q(\gamma, h_\pi)$ and $T_2(\gamma, h_\pi)$, we will identify them essentially with the left-hand side of (3.7) and the $\Gamma$-term on the right-hand side of (3.7), respectively. A technical detail one has to take into account here is that the domain of (3.11) is the torus $\mathbb{R}/\mathbb{Z}$, whereas the respective domain in (3.7) is the real line $\mathbb{R}$. To estimate the other terms, let us introduce for $f_1, f_2 \in H^{3/2}_{loc}(\mathbb{R}, \mathbb{R}^n)$ the operators

$$\begin{aligned}
&\tilde{Q}_\varepsilon(f_1, f_2)\\
&\quad := \int_0^1 \int_{\tilde{I}_\varepsilon} \left(\langle f_1'(u),\ f_2'(u)\rangle - \frac{\langle f_1(u+w) - f(u), f_2(u+w) - f_2(u)\rangle}{w^2}\right) \frac{\mathrm{d}w}{w^2}\,\mathrm{d}u\\
&\quad = \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \frac{(\langle f_1'(u),\ f_2'(u)\rangle - \langle f_1'(u+s_1w), f_2'(u+s_2w)\rangle)}{w^2}\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}w\ \ \mathrm{d}u,
\end{aligned}$$

where $\tilde{I}_\varepsilon := [-1/4, 1/4] \setminus [-\varepsilon, \varepsilon]$,

$$\tilde{Q} := \lim_{\varepsilon \searrow 0} \tilde{Q}_\varepsilon,$$

and

$$\begin{aligned}
&\tilde{T}_2(f_1, f_2)\\
&\quad := \int_0^1 \int_{-1/4}^{1/4} \langle f_1(u+w) - f_1(u),\ f_2(u+w) - f_2(u)\rangle\\
&\qquad\qquad \cdot \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^4} - \frac{1}{|w|^4}\right) \mathrm{d}w\ \ \mathrm{d}u\\
&\quad = \int_0^1 \int_{-1/4}^{1/4} \int_0^1 \int_0^1 \langle f_1'(u+s_1w),\ f_2'(u+s_2w)\rangle\, w^2\\
&\qquad\qquad \cdot \left(\frac{1}{|\gamma(u+w)-\gamma(u)|^4} - \frac{1}{|w|^4}\right) \mathrm{d}s_1\ \ \mathrm{d}s_2\ \ \mathrm{d}w\ \ \mathrm{d}u.
\end{aligned} \tag{3.13}$$

Recall that $[-\frac{1}{2}, \frac{1}{2}]\setminus(-\varepsilon, \varepsilon)$ is used for the definition of $Q$ and $\tilde{I}_\varepsilon := [-1/4, 1/4] \setminus [-\varepsilon, \varepsilon]$ is used for $\tilde{Q}$, so the difference only contains the set where $|w| > \frac{1}{4}$; thus $|w|^{-2}$ is not singular. Quantitatively, this reads as

$$|\tilde{Q}(\gamma, h) - Q(\gamma, h)| + |\tilde{T}_2(\gamma, h) - T_2(\gamma, h)| \leq C\|\gamma'\|_{L^2}\|h'\|_{L^2} \leq C\|\gamma'\|_{L^2}\|\phi\|_{L^2}, \tag{3.14}$$

where we have used the fact that $\gamma$ is bi-Lipschitz in order to deal with $T_2$.

We now compute

$$\tilde{Q}_\varepsilon(\gamma, h_\pi)$$
$$= \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 (\langle \gamma'(u),\ h'_\pi(u)\rangle - \langle \gamma'(u+s_1 w), h'_\pi(u+s_2 w)\rangle)\ \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\ \mathrm{d}u$$
$$= \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 (\langle \gamma'(u),\ \phi_\pi\rangle - \langle \gamma'(u+s_1 w), \phi_\pi(u+s_2 w)\rangle)\ \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\ \mathrm{d}u$$

(3.15)
$$- \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 (\langle \gamma'(u),\ a\rangle - \langle \gamma'(u+s_1 w), a\rangle)\ \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\ \mathrm{d}u$$

(3.16)
$$= \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 (\langle g'(u),\ \phi(u)\rangle - \langle g'(u+s_1 w), \phi(u+s_2 w)\rangle)\ \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\ \mathrm{d}u.$$

Here we have used that $\int_0^1 \gamma'(u+\tilde{w})du = 0$ for all $\tilde{w}$, i.e., the term (3.15) is constantly zero for any $w$. The term (3.16) essentially is the $L^2$-pairing of $|D|^{\frac{1}{2}} g'$ and $|D|^{\frac{1}{2}}\phi$: More precisely we will show

$$|\tilde{Q}(\gamma, h_\pi) - \int_{\mathbb{R}} |D|^{\frac{1}{2}} g' |D|^{\frac{1}{2}} \phi| \leq C_\gamma \|\phi\|_{L^2}. \tag{3.17}$$

*Proof of* (3.17). Using (3.6), we have for any $f_1, f_2 \in C_0^\infty(\mathbb{R})$ as $\varepsilon \to 0$,

(3.18)
$$\int_{\mathbb{R}} |D|^{\frac{1}{2}} f_1' |D|^{\frac{1}{2}} f_2' du + o(1)$$
$$= c \int_{\mathbb{R}} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \int_0^1 \int_0^1 (\langle f_1'(u), f_2'(u)\rangle - \langle f_1'(u+s_1 w), f_2'(u+s_2 w)\rangle)\ \mathrm{d}s_1\ \mathrm{d}s_2 \frac{\mathrm{d}w}{|w|^2}\,\mathrm{d}u.$$

We now connect (3.18) and (3.16). The technical problem is that the integral of $\phi$ is not a feasible test-function for (3.18). Therefore for $\eta_{[-10,10]} \in C_0^\infty((-11,11))$ and $\eta_{[-10,10]} \equiv 1$ in $[-10,10]$, let

$$\psi := \left(u \mapsto \eta_{[-10,10]} \int_0^u \phi\right) \in C_0^\infty(\mathbb{R}).$$

Thus, $\psi$ is a feasible test-function for (3.18), which $\phi$ is not. Moreover,

$$\psi' = \eta_{[-10,10]}\phi + \eta'_{[-10,10]} \int_0^u \phi \overset{\operatorname{supp}\phi}{\equiv} \phi + \eta'_{[-10,10]} \int_0^u \phi. \tag{3.19}$$

We thus arrive at

$$\begin{aligned}
\tilde{Q}_\varepsilon(\gamma,h) &\overset{(3.16)}{=} \int_0^1 \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle g'(u),\ \phi(u)\rangle - \langle g'(u+s_1 w), \phi(u+s_2 w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\, \mathrm{d}u \\
&\overset{\substack{\operatorname{supp}\phi, |w|,\\ (3.19)}}{=} \int_{\mathbb{R}} \int_{\tilde{I}_\varepsilon} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u)\rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\, \mathrm{d}u \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}\backslash[-\varepsilon,\varepsilon]} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u)\rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\, \mathrm{d}u \\
&\quad - \int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u)\rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\, \mathrm{d}u \\
&\overset{(3.18)}{=} \int \langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} \psi'\rangle \\
&\quad - \int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u)\rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\, \mathrm{d}u \\
&\quad + o(1) \\
&\overset{(3.19)}{=} \int \langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} \phi\rangle + \int \langle |D|^{\frac{1}{2}} g', |D|^{\frac{1}{2}} (\eta'_{[-10,10]} \int_0^{\cdot} \phi)\rangle \\
&\quad - \int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u)\rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\, \mathrm{d}u \\
&\quad + o(1).
\end{aligned}$$

Now, by usual interpolation and/or imbedding of Sobolev spaces (see, e.g. [Tar07, Sch10]),

$$\||D|^{\frac{1}{2}} f\|_{2,\mathbb{R}} \lesssim \|f\|_{2,\mathbb{R}} + \|f'\|_{2,\mathbb{R}},$$

we have

$$\||D|^{\frac{1}{2}} (\eta'_{[-10,10]} \int_0^{\cdot} \phi)\|_{2,\mathbb{R}} \lesssim \|\phi\|_1 + \|\phi\|_2 \overset{\operatorname{supp}\phi}{\lesssim} \|\phi\|_2.$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{|w|>\frac{1}{4}} \int_0^1 \int_0^1 \left(\langle g'(u),\ \psi'(u)\rangle - \langle g'(u+s_1 w), \psi'(u+s_2 w)\rangle\right) \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\mathrm{d}w}{w^2}\ \mathrm{d}u \\
&\qquad \lesssim \int_{|w|>\frac{1}{4}} |w|^{-2}\ \|g'\|_2 \|\psi'\|_2 \\
&\qquad \overset{(3.19)}{\lesssim} \|g'\|_{2,\mathbb{R}} \Big(\|\phi\|_{1,\mathbb{R}} + \|\phi\|_{2,\mathbb{R}}\Big) \overset{\operatorname{supp} g}{\lesssim} C_\gamma\ \|\phi\|_{2,\mathbb{R}}.
\end{aligned}$$

Thus, we have shown that (3.17) holds. □

To estimate $\tilde{T}_2(\gamma,h)$ we calculate

$$\begin{aligned}\tilde{T}_2(\gamma,h) &= \int_0^1 \int_{-1/4}^{1/4} \int_0^1 \int_0^1 \langle \gamma'(u+s_1w),\ \phi_\pi(u+s_2w)+a\rangle\, w^2 \\ &\qquad \cdot \left( \frac{1}{|\gamma(u+w)-\gamma(u)|^4} - \frac{1}{|w|^4} \right) \mathrm{d}s_1\ \mathrm{d}s_2\ \mathrm{d}w\ \mathrm{d}u \\ &= \int_0^1 \int_{-1/4}^{1/4} \int_0^1 \int_0^1 \langle \gamma'(u+s_1w),\ \phi(u+s_2w)-\phi(u+s_1w)+a\rangle\, w^2 \\ &\qquad \cdot \left( \frac{1}{|\gamma(u+w)-\gamma(u)|^4} - \frac{1}{|w|^4} \right) \mathrm{d}s_1\ \mathrm{d}s_2\ \mathrm{d}w\ \mathrm{d}u,\end{aligned}$$

and using

$$\begin{aligned} w^2\left(\frac{1}{|\gamma(u+w)-\gamma(u)|^4} - \frac{1}{|w|^4}\right) &= \frac{w^4}{|\gamma(u+w)-\gamma(u)|^4}\left(\frac{1-\frac{|\gamma(u+w)-\gamma(u)|^4}{w^4}}{w^2}\right) \\ &\leq C\frac{1-\frac{|\gamma(u+w)-\gamma(u)|^2}{w^2}}{w^2} \\ &= \frac{C}{2}\int_0^1\int_0^1 \frac{|\gamma'(u+s_3w)-\gamma'(u+s_4w)|^2}{w^2}\,\mathrm{d}s_3\ \mathrm{d}s_4 \end{aligned}$$

we get

(3.20)

$$\begin{aligned} |\tilde{T}_2(\gamma,h)| &\leq |a|E^{(2)}(\gamma) \\ &\quad + \int_0^1 \int_{-1/4}^{1/4} \int_{(0,1)^4} \\ &\quad \cdot \frac{|\gamma'(u+s_1w)-\gamma'(u+s_2w)|\ |\gamma'(u+s_3w)-\gamma'(u+s_4w)|^2\ |\phi(u+s_2w)|}{w^2}\,\mathrm{d}s\,\mathrm{d}w\ \mathrm{d}u \\ &\leq C\|\phi\|_{L^2} + \int_0^1 \int_{-1/4}^{1/4} \int_{(-1,1)^3} \\ &\quad \cdot \frac{|\gamma'(u+s_1w)-\gamma'(u)|\ |\gamma'(u+s_3w)-\gamma'(u+s_4w)|^2\ |\phi(u)|}{w^2}\,\mathrm{d}s\,\mathrm{d}w\ \mathrm{d}u. \end{aligned}$$

From (3.14), (3.12), (3.17), and (3.20) one gets the claim, since $\gamma' = g'$ on $[-1/4, 5/4]$.

**Proof of Lemma 3.4.** Let

$$F(u) := ||D|^{\frac{1}{2}} g'(u)|.$$

Since $g', |D|^{\frac{1}{2}}g' \in L^2$, we obtain

$$\begin{aligned} g'(x)-g'(y) &= I_{\frac{1}{2}}(|D|^{\frac{1}{2}}g')(x) - I_{\frac{1}{2}}(|D|^{\frac{1}{2}}g')(y) \\ &= c_{\frac{1}{2}}\left(\int |x-\xi|^{-1+\frac{1}{2}}\ |D|^{\frac{1}{2}}g'(\xi)\ \mathrm{d}\xi - \int |y-\xi|^{-1+\frac{1}{2}}\ |D|^{\frac{1}{2}}g'(\xi)\ \mathrm{d}\xi\right). \end{aligned}$$

Then

$$|g'(x)-g'(y)| \lesssim \int\limits_{\mathbb{R}} ||\xi-x|^{-1+\frac{1}{2}} - |\xi-y|^{-1+\frac{1}{2}}|\ F(\xi)\ \mathrm{d}\xi,$$

and hence

$$\Gamma(u) \lesssim \int\limits_{(-1,1)^3} \int\limits_{\mathbb{R}^3} \int\limits_{-1/4}^{1/4} F(\xi_1)\ F(\xi_2)\ F(\xi_3)\ k(\xi,u,s,w)\ \mathrm{d}w\ \mathrm{d}\xi\ \mathrm{d}s,$$

where for almost every $s=(s_1,s_2,s_3)\in(-1,1)^3$, $\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3$, $w\in\mathbb{R}$, $u\in\mathbb{R}$,

$$k(\xi,u,s,w)=\frac{m(\xi_1-u,0,s_1,w)\ m(\xi_2-u,s_2,s_3,w)\ m(\xi_3-u,s_2,s_3,w)}{|w|^2} \tag{3.21}$$

and

$$m(a,s,t,w):=||a+sw|^{-\frac{1}{2}}-|a+tw|^{-\frac{1}{2}}|.$$

The characteristic behavior of $k$ is as follows: The factors $m(\cdot,\cdot,\cdot,w)$ will behave like $|w|^\delta$ in a neighborhood of $w=0$ such that they somewhat absorb the singular behaviour of $|w|^{-2}$, that is, $k$ becomes integrable around $w=0$. This is an effect very similar to the behaviour of $H_s(\cdot,\cdot)$; see Lemma A.5, as developed in [Sch11].

More precisely, we will derive the estimate

$$\begin{aligned}
&\int_{[-1,1]^3} k(\xi,u,s,w)\,\mathrm{d}s\\
&\lesssim |w|^{3\delta-2}|\xi_1-u|^{-1/2-\delta} \sum_{(\sigma(2),s(3))\in\{(2,3),(3,2)\}} |\xi_{\sigma(2)}-u|^{-1/2-\delta}
\end{aligned} \tag{3.22a}$$

$$\cdot\int_{-1}^{1} |\xi_{\sigma(3)}-u+tw|^{-1/2-\delta}\,\mathrm{d}t \tag{3.22b}$$

$$+|w|^{3\delta-2}|\xi_1-u|^{-1/2-\delta}\int_{-1}^{1}|\xi_2-u+tw|^{-1/2-\delta}|\xi_3-u+tw|^{-1/2-\delta}\,\mathrm{d}t.$$

We start with some abstract treatment of $m$.

(3.23) In **case 1**, $$\max\left(|a+sw|,|a+tw|\right)\geq 2|s-t||w|,$$

we obtain

$$|a+sw|\approx|a+tw|. \tag{3.24}$$

Applying the mean value theorem, for any $\delta\in(0,1)$ we arrive at

$$\begin{aligned}
m(a,s,t,w) &\lesssim \max\left(|a+sw|^{-3/2},|a+tw|^{-3/2}\right)|s-t||w|\\
&\lesssim \max\left(|a+sw|^{-1/2-\delta},|a+tw|^{-1/2-\delta}\right)\\
&\qquad\cdot\max\left(|a+sw|^{-1+\delta},|a+tw|^{-1+\delta}\right)|s-t||w|\\
&\overset{(3.23)}{\lesssim} \max\left(|a+sw|^{-1/2-\delta},|a+tw|^{-1/2-\delta}\right)\underbrace{|s-t|^\delta}_{\leq 2^\delta}|w|^\delta\\
&\overset{(3.24)}{\lesssim} \min\left(|a+sw|^{-1/2-\delta},|a+tw|^{-1/2-\delta}\right)|w|^\delta.
\end{aligned} \tag{3.25}$$

(3.26) In **case 2**, $$\max\left(|a+sw|,|a+tw|\right)\le 2|s-t||w|,$$

we immediately obtain

$$\begin{aligned} m(a,s,t,w) &\lesssim \max\left(|a+sw|^{-1/2},|a+tw|^{-1/2}\right)\\ &\lesssim \max\left(|a+sw|^{-1/2-\delta},|a+tw|^{-1/2-\delta}\right)\underbrace{|s-t|^{\delta}}_{\le 2^{\delta}}|w|^{\delta}. \end{aligned} \tag{3.27}$$

We begin with the first factor in (3.21). In case 1 we always have

$$m(\xi_1-u,0,s_1,w)\lesssim |\xi_1-u|^{-1/2-\delta}|w|^{\delta}. \tag{3.28}$$

In case 2 we have either $|\xi_1-u+s_1w|\ge \frac{1}{2}|\xi_1-u|$, which immediately results in (3.28), or the opposite

$$|\xi_1-u+s_1w|\le \tfrac{1}{2}|\xi_1-u|, \tag{3.29}$$

which leads to

$$\begin{aligned} &\int_{-1}^{1}\chi_{|\xi_1-u+s_1w|\le|\xi_1-u|}m(\xi_1-u,0,s_1,w)\,\mathrm{d}s_1\\ &\quad\lesssim |w|^{\delta}\int_{|\xi_1-u+s_1w|\le|\xi_1-u|}|\xi_1-u+s_1w|^{-1/2-\delta}\,\mathrm{d}s_1\\ &\quad\lesssim |w|^{\delta-1}\int_{|\sigma|\le|\xi_1-u|}|\sigma|^{-1/2-\delta}\,\mathrm{d}\sigma\lesssim |w|^{\delta-1}|\xi_1-u|^{1/2-\delta}\\ &\quad\overset{(3.31)}{\lesssim} |\xi_1-u|^{-1/2-\delta}|w|^{\delta},\qquad \text{for } \delta\in(0,\tfrac{1}{2}). \end{aligned} \tag{3.30}$$

Here we made use of the fact that, given case 2 for $a:=\xi_1-u$ and (3.29),

$$\begin{aligned} |a|&\le \min\left(|a+sw|+|s||w|,|a+tw|+|t||w|\right)\le \min\left(|a+sw|,|a+tw|\right)+|w|\\ &\overset{(3.29)}{\le} \tfrac{1}{2}|a|+|w| \end{aligned}$$

implies

$$|a|\le 2|w|. \tag{3.31}$$

Applying (3.25) and (3.27), we arrive at

$$\begin{aligned} &\int_{[-1,1]^3}k(\xi,u,s,w)\,\mathrm{d}s\\ &\lesssim |w|^{\delta-2}|\xi_1-u|^{-1/2-\delta}\iint_{[-1,1]^2}m(\xi_2-u,s_2,s_3,w)m(\xi_3-u,s_2,s_3,w)\,\mathrm{d}s_2\,\mathrm{d}s_3\\ &\lesssim |w|^{3\delta-2}|\xi_1-u|^{-1/2-\delta}\iint_{[-1,1]^2}\mu_2\left(|\xi_2-u+s_2w|^{-1/2-\delta},|\xi_2-u+s_3w|^{-1/2-\delta}\right)\\ &\qquad\cdot\mu_3\left(|\xi_3-u+s_2w|^{-1/2-\delta},|\xi_3-u+s_3w|^{-1/2-\delta}\right)\mathrm{d}s_2\,\mathrm{d}s_3, \end{aligned} \tag{3.32}$$

where

$$\mu_i=\mu_i\left(\xi_i-u,s_2,s_3,w\right)\in\left\{\min,\max\right\},\qquad i=2,3,$$

depending on the respective case. If case 1 holds for at least one of the two factors in the integrand, say the first one, we may choose the argument of

$$\mu_2\left(|\xi_2 - u + s_2 w|^{-1/2-\delta}, |\xi_2 - u + s_3 w|^{-1/2-\delta}\right),$$

which contains the same integration variable as the second one. This results in terms of type (3.22b). If, however, case 2 applies to both factors, the integral in (3.32) is bounded by

$$\iint_{[-1,1]^2} \left(|\xi_2 - u + s_2 w|^{-1/2-\delta} + |\xi_2 - u + s_3 w|^{-1/2-\delta}\right) \cdot \left(|\xi_3 - u + s_2 w|^{-1/2-\delta} + |\xi_3 - u + s_3 w|^{-1/2-\delta}\right) \mathrm{d}s_2 \, \mathrm{d}s_3.$$

Expanding the integrand, the terms $|\xi_2 - u + s_i w|^{-1/2-\delta}|\xi_3 - u + s_i w|^{-1/2-\delta}$, $i = 2, 3$, lead us to (3.22b). For the two remaining terms we may separate the integrals which gives

$$\sum_{(\sigma(2),s(3))\in\{(2,3),(3,2)\}} \int_{-1}^{1} |\xi_{\sigma(2)} - u + s_2 w|^{-1/2-\delta} \mathrm{d}s_2 \int_{-1}^{1} |\xi_{\sigma(3)} - u + s_3 w|^{-1/2-\delta} \mathrm{d}s_3.$$

One integral is kept in order to arrive at (3.22a), and the other one is treated analogously to (3.29) and (3.30). In order to estimate $\Gamma(u)$, we obtain thus for $\delta_1, \delta_2 \in (0, \frac{1}{2})$,

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{(-1,1)^3} F(\xi_1)\ F(\xi_2)\ F(\xi_3)\ k(\xi, u, s, w) \, \mathrm{d}s \, \mathrm{d}\xi \\ \approx &\int_{-1}^{1} |w|^{-1+(3\delta-1)} |I_{\frac{1-2\delta}{2}} F(u)|^2 \ I_{\frac{1-2\delta}{2}} F(u - tw) \, \mathrm{d}t \\ &+ \int_{-1}^{1} |w|^{-1+(3\delta-1)} I_{\frac{1-2\delta}{2}} F(u)\ |I_{\frac{1-2\delta}{2}} F(u - tw)|^2 \, \mathrm{d}t, \end{aligned}$$

which implies

$$\begin{aligned} \Gamma(u) \quad \lesssim \quad & \int_{-1}^{1} \int_{\tilde{w}} t^{2-3\delta}\ |\tilde{w}|^{-1+(3\delta-1)} |I_{\frac{1-2\delta}{2}} F(u)|^2 \ I_{\frac{1-2\delta}{2}} F(u - \tilde{w})\ t^{-1} \, \mathrm{d}\tilde{w} \, \mathrm{d}t \\ & + \int_{-1}^{1} \int_{\tilde{w}} t^{2-3\delta}\ |\tilde{w}|^{-1+(3\delta-1)} I_{\frac{1-2\delta}{2}} F(u)\ |I_{\frac{1-2\delta}{2}} F(u - \tilde{w})|^2\ t^{-1} \, \mathrm{d}\tilde{w} \, \mathrm{d}t \\ \overset{\delta\in(\frac{1}{3},\frac{1}{2})}{\approx} \quad & \int_{w} |w|^{-1+(3\delta-1)} |I_{\frac{1-2\delta}{2}} F(u)|^2 \ I_{\frac{1-2\delta}{2}} F(u - w) \, \mathrm{d}w \\ & + \int_{w} |w|^{-1+(3\delta-1)} I_{\frac{1-2\delta}{2}} F(u)\ |I_{\frac{1-2\delta}{2}} F(u - w)|^2 \, \mathrm{d}w \\ \approx \quad & |I_{\frac{1-2\delta}{2}} F(u)|^2 \ I_{\frac{4\delta-1}{2}} F(u) + I_{\frac{1-2\delta}{2}} F(u)\ I_{3\delta-1} |I_{\frac{1-2\delta}{2}} F(u)|^2. \end{aligned}$$

Setting $\delta := \frac{5}{12}$, this is (3.8).

**Proof of Lemma 3.5.** Plugging together (3.5), Lemma 3.2, and Lemma 3.3 we get for small $s$ and some $\theta > 0$ that

$$\begin{aligned}&\int g_i' \omega_{ij}\ |D|^{\frac{1}{2}} g_j'\ |D|^{\frac{1}{2}}\varphi \\ &\lesssim \|\varphi\|_\infty\ \|\Gamma\|_{1,B_r} + \|I_s |D|^{\frac{1}{2}}\varphi\|_{(\frac{2}{1-2s},1)} \\ &\quad\cdot \left( r^{\frac{1}{2}} + \||D|^{\frac{1}{2}} g'\|^2_{(2,\infty),B_{\Lambda r}} + \left\||D|^{\frac{1}{2}} g'\right\|_{(2,\infty),\mathbb{R}} \sum_{k=2}^{\infty} \Lambda^{-\theta(k-1)} \||D|^{\frac{1}{2}} g'\|_{(2,\infty),B_{\Lambda^k r}}\right).\end{aligned}$$

By the estimate of $\Gamma$ from Lemma 3.4, we infer, for $F := ||D|^{\frac{1}{2}} g'|$,

$$\begin{aligned}\|\Gamma\|_{1,B_r} &\lesssim \left\| |I_{\frac{1}{12}}|F||^2\ I_{\frac{1}{3}}|F| + I_{\frac{1}{12}}|F|\ I_{\frac{1}{4}}\left(|I_{\frac{1}{12}}|F||^2\right)\right\|_{1,B_r} \\ &\lesssim \Theta \left( \left\| I_{\frac{1}{3}}|F|\right\|_{(6,\infty),B_r} + \left\| I_{\frac{1}{12}}|F|\right\|_{(\frac{12}{5},\infty),B_r}\right),\end{aligned}$$

where

$$\Theta := \left\| I_{\frac{1}{12}}|F|\right\|^2_{(\frac{12}{5},2),B_r} + \left\| I_{\frac{1}{4}}\left(|I_{\frac{1}{12}}|F||^2\right)\right\|_{(\frac{12}{7},1),B_r}.$$

Observing

$$\begin{aligned}\|I_{\frac{1}{3}}F\|_{(6,\infty)} &\lesssim \|F\|_{(2,\infty)}, \\ \|I_{\frac{1}{12}}F\|_{(\frac{12}{5},2)} &\lesssim \|F\|_2,\end{aligned}$$

and because $\frac{5}{12} + \frac{5}{12} + \frac{1}{6} = 1$, we arrive at

$$\begin{aligned}\|I_{\frac{1}{12}}F\|_{(\frac{12}{5},q)} &\lesssim \|F\|_{(2,q)}, \\ \|I_{\frac{1}{4}}|I_{\frac{1}{12}}F|^2\|_{(\frac{12}{7},1)} &\lesssim \||I_{\frac{1}{12}}F|^2\|_{(\frac{6}{5},1)} = \||I_{\frac{1}{12}}F|\|^2_{(\frac{12}{5},2)} \lesssim \|F\|_2.\end{aligned}$$

Consequently, $\Theta$ is uniformly small, if $r$ is small enough. In order to conclude, it only remains to apply Lemma A.2 to $f := |F|$. □

## 4. Bootstrapping: Proof of Theorem IV

In Theorem III we have shown that $|D|^{\frac{1}{2}} g' \in L^p$ for some $p > 2$. We now work with Bessel-potential/Sobolev spaces $H^{s,q}$ (cf. [RS96, Tar07, Tri83]) and the fact that $|D|^{\frac{1}{2}} g' \in L^p$ readily implies that $g' \in H^{\frac{3}{2},\tilde{p}}$ for some $\tilde{p} \in (2,p)$. The proof of Theorem IV relies on the decomposition of the first variation

$$\begin{aligned}\delta E^{(2)}(\gamma,h) = 2\lim_{\varepsilon\searrow 0}\iint_{U_\varepsilon} \Bigg( &\frac{\langle \gamma'(u), h'(u)\rangle}{|\gamma(u+w)-\gamma(u)|^2} \\ &- \frac{\langle \gamma(u+w)-\gamma(u), h(u+w)-h(u)\rangle}{|\gamma(u+w)-\gamma(u)|^4}\Bigg)\, \mathrm{d}w\, \mathrm{d}u,\end{aligned}$$

$U_\varepsilon$ as in (2.3), into

$$2\left(Q(\gamma,h) - T_1(\gamma,h) - T_2(\gamma,h)\right).$$

For a critical point of the Möbius energy we have

$$Q(\gamma,h) = T(\gamma,h) := T_1(\gamma,h) + T_2(\gamma,h)$$

for all $h \in H^{3/2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$.

Let us bring these terms in a common form. Using

$$\frac{1}{|\gamma(u+w)-\gamma(u)|^\alpha} - \frac{1}{|w|^\alpha} = \frac{|w|^\alpha}{|\gamma(u+w)-\gamma(u)|^\alpha}\left(\frac{1-\frac{|\gamma(u+w)-\gamma(u)|^\alpha}{|w|^\alpha}}{|w|^\alpha}\right)$$

$$= G^\alpha\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\left(\frac{2-2\frac{|\gamma(u+w)-\gamma(u)|^2}{|w|^2}}{|w|^\alpha}\right)$$

$$= \int_0^1\int_0^1 G^\alpha\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\left(\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{|w|^\alpha}\right)\mathrm{d}\tau_1\,\mathrm{d}\tau_2$$

where

$$G^\alpha(z) := \frac{1}{2|z|^\alpha}\cdot\frac{1-|z|^\alpha}{1-|z|^2}$$

is an analytic function away from the origin for $\alpha \geq 2$. We hence get

$$T_1(\gamma,h) = -\int_0^1\int_0^1 T^2_{0,0,\tau_1,\tau_2}(h)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2,$$

$$T_2(\gamma,h) = \int_0^1\int_0^1\int_0^1\int_0^1 T^4_{s_1,s_2,\tau_1,\tau_2}(h)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2\,\mathrm{d}s_1\,\mathrm{d}s_2,$$

where

$$T^\alpha_{s_1,s_2,\tau_1,\tau_2}(h) := \int_0^1\int_{-1/2}^{1/2} G^\alpha\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{w^2} \cdot \gamma'(u+s_1 w)h'(u+s_2 w)\,\mathrm{d}w\ \mathrm{d}u.$$

In the rest of this section we will derive some estimates for the linear operators $T^\alpha_{s_1,s_2,\tau_1,\tau_2}$ that do not depend on $s_1, s_2, \tau_1$, and $\tau_2$.

The proof relies furthermore on the following rules for Bessel potential spaces.

**Lemma 4.1** (Fractional Leibniz rule [CM78], [RS96, Lem. 5.3.7/1 (i)]). *Let $f \in H^{s,p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $g \in H^{s,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ with $s > 0$, $p,q,r \in (1,\infty)$ and $1/p+1/q = 1/r$. Then $fg \in H^{s,r}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ and*

$$\|fg\|_{H^{s,r}} \leq C\left(\|f\|_{H^{s,p}}\|g\|_{L^q} + \|f\|_{L^p}\|g\|_{H^{s,q}}\right).$$

For the following statement, one mainly has to treat $\left\|(D^k\psi)\circ f\right\|_{H^{\sigma,p}}$ for $k \in \mathbb{N}\cup\{0\}$ and $\sigma \in (0,1)$ which is e.g. covered by [RS96, Thm. 5.3.6/1 (i)].

**Lemma 4.2** (Fractional chain rule). *Let $f \in H^{s,p}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $s > 0$, $p \in (1,\infty)$. If $\psi \in C^\infty(\mathbb{R})$ is globally Lipschitz continuous and $\psi$ and all its derivatives vanish at 0, then $\psi\circ f \in H^{s,p}$ and*

$$\|\psi\circ f\|_{H^{s,p}} \leq C\|\psi\|_{C^k}\|f\|_{H^{s,p}},$$

*where $k$ is the smallest integer greater than or equal to $s$.*

The key to the proof of Theorem IV is the following lemma.

**Lemma 4.3.** *Let $\gamma \in H^{\frac{3}{2}+\beta_0,q}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$, $\alpha \geq 2$, and $\beta_0 \geq \beta \geq 0$, $p,q \in (1,\infty)$ be such that $\beta_0 - 1/q > \beta - \frac{1}{2p}$. Then for all $\tau_1,\tau_2,s_1 \in [0,1]$ the function*

$$g(u) := \int_{-1/2}^{1/2} G^\alpha\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{w^2}\gamma'(u+s_1 w)\,\mathrm{d}w$$

*is in $H^{\beta,p}$. Furthermore, there is a constant $C<\infty$ depending on $\|\gamma\|_{H^{3/2+\beta_0,q}}$ and $\alpha$, but not on $\tau_1,\tau_2$, and $s_1$, such that*

$$\|g\|_{H^{\beta,p}} \leq C.$$

*Proof.* Note that

$$\|g\|_{H^{\beta,p}} \leq \int_{-1/2}^{1/2} \frac{\|g_w\|_{H^{\beta,p}}}{|w|^2}\,\mathrm{d}w,$$

where

$$g_w(u) := G^\alpha\left(\frac{\gamma(u+w)-\gamma(u)}{w}\right)\gamma'(u+s_1 w)|\gamma'(u+\tau_1 w)-\gamma(u+\tau_2 w)|^2.$$

Choosing some $\tilde{p} \in (p,p+1)$ which will be determined later on and letting $\tilde{q} := 2\frac{\tilde{p}p}{\tilde{p}-p}$ leads to

$$\frac{1}{p} = \frac{1}{2\tilde{p}} + \frac{1}{2\tilde{p}} + \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}}.$$

Using that

$$\frac{\gamma(u+w)-\gamma(u)}{w} = \int_0^1 \gamma'(u+\tau w)\,\mathrm{d}\tau,$$

that $\gamma$ is bi-Lipschitz, and that $G^\alpha$ is analytic away from the origin, we get according to the fractional chain rule (Lemma 4.2)

$$\left\|G^\alpha\left(\frac{\gamma(\cdot+w)-\gamma(\cdot)}{w}\right)\right\|_{H^{\beta,\tilde{q}}} \leq C\|\gamma\|_{H^{\beta+1,\tilde{q}}} \leq C.$$

Using the fractional Leibniz rule (Lemma 4.1), we derive

$$\begin{aligned}\|g_w\|_{H^{\beta,p}} &\leq C\left\|G^\alpha\left(\frac{\gamma(\cdot+w)-\gamma(\cdot)}{w}\right)\right\|_{H^{\beta,\tilde{q}}}\|\gamma'\|_{H^{\beta,\tilde{q}}}\|\gamma'(\cdot+\tau_1 w)-\gamma'(\cdot+\tau_2 w)\|^2_{H^{\beta,2\tilde{p}}}\\ &\leq C\left(\|\gamma'(\cdot+\tau_1 w)-\gamma'(\cdot+\tau_2 w)\|^2_{L^{2\tilde{p}}}\right.\\ &\qquad\left.+\||D|^{\beta+1}\gamma(\cdot+\tau_1 w)-|D|^{\beta+1}\gamma(\cdot+\tau_2 w)\|^2_{L^{2\tilde{p}}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\|g\|_{H^{\beta,p}} &\leq C\int_{-1/2}^{1/2}\frac{\|\gamma'(\cdot+\tau_1 w)-\gamma'(\cdot+\tau_2 w)\|^2_{L^{2\tilde p}}}{w^2}\,\mathrm{d}w\\ &\quad + C\int_{-1/2}^{1/2}\frac{\||D|^{\beta+1}\gamma(\cdot+\tau_1 w)-|D|^{\beta+1}\gamma(\cdot+\tau_2 w)\|^2_{L^{2\tilde p}}}{w^2}\,\mathrm{d}w\\ &\leq C\int_{-1/2}^{1/2}\frac{\|\gamma'(\cdot)-\gamma'(\cdot+(\tau_2-\tau_1)w)\|^2_{L^{2\tilde p}}}{w^2}\,\mathrm{d}w\\ &\quad + C\int_{-1/2}^{1/2}\frac{\||D|^{\beta+1}\gamma(\cdot)-|D|^{\beta+1}\gamma(\cdot+(\tau_2-\tau_1)w)\|^2_{L^{2\tilde p}}}{w^2}\,\mathrm{d}w\\ &\leq C|\tau_2-\tau_1|\int_{-1}^{1}\frac{\|\gamma'(\cdot)-\gamma'(\cdot+w)\|^2_{L^{2\tilde p}}}{w^2}\,\mathrm{d}w\\ &\quad + C|\tau_2-\tau_1|\int_{-1}^{1}\frac{\||D|^{\beta+1}\gamma(\cdot)-|D|^{\beta+1}\gamma(\cdot+w)\|^2_{L^{2\tilde p}}}{w^2}\,\mathrm{d}w\\ &\leq C\|\gamma'\|^2_{B^{1/2}_{2\tilde p,2}}+C\||D|^{\beta+1}\gamma\|^2_{B^{1/2}_{2\tilde p,2}}\leq C\end{aligned}$$

if $\tilde p\in(p,p+1)$ is chosen so small that

$$\beta_0-\frac{1}{q}>\beta-\frac{1}{2\tilde p}>\beta-\frac{1}{2p}.$$

This proves Lemma 4.3. □

We use the last lemma to prove

**Corollary 4.4.** *Let $\gamma,\beta_0,\beta$,$p$ and $q$ be as in Lemma 4.3 and $p'$ be such that $1/p+1/p'=1$. Then*

- *for all $\alpha\geq 2$ there is a constant $C$ such that*

$$|T^\alpha_{s_1,s_2,\tau_1,\tau_2}(h)|\leq C\|h\|_{H^{1-\beta,p'}}$$

*for all $s_1,s_2,\tau_1,\tau_2\in[0,1]$ and $h\in C^\infty$,*
- *the operator $T(\gamma,\cdot)=T_1(\gamma,\cdot)+T_2(\gamma,\cdot)\in\left(H^{3/2}(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)\right)^*$ can be extended to a bounded operator on $H^{1-\beta,p'}$.*

*Proof.* For $h\in C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R}^n)$ we have

$$T(\gamma,h)=-\int_0^1\int_0^1 T^2_{0,0,t_1,t_2}(h)\,\mathrm{d}t_1\,\mathrm{d}t_2+\int_0^1\int_0^1\int_0^1\int_0^1 T^4_{s_1,s_2,t_1,t_2}(h)\,\mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}t_1\,\mathrm{d}t_2,$$

and hence the second part is an immediate consequence of the first one.

Let $\Lambda^s:=(\mathrm{id}-\Delta)^{\frac{s}{2}}$. Using that $\Lambda^\beta$ is self-adjoint, we get

$$\begin{aligned}T^\alpha_{s_1,s_2,\tau_1,\tau_2}(h) &= \int_{-1/2}^{1/2}\int_{\mathbb{R}/\mathbb{Z}}G^\alpha\left(\frac{\gamma(\cdot+w)-\gamma(\cdot)}{w}\right)\frac{|\gamma'(u+\tau_1 w)-\gamma'(u+\tau_2 w)|^2}{w^2}\\ &\qquad \gamma'(u+s_1 w)h'(u+s_2 w)\,\mathrm{d}u\ \mathrm{d}w\\ &= \int_{\mathbb{R}/\mathbb{Z}}\int_{-1/2}^{1/2}\frac{\left(\Lambda^\beta g_w\right)(u)}{w^2}\left(\Lambda^{-\beta}h'\right)(u+s_2 w)\,\mathrm{d}w\ \mathrm{d}u,\end{aligned}$$

and hence as in the proof of Lemma 4.3,

$$|T_{s_1,s_2,\tau_1,\tau_2}(h)| \leq C \int_{-1/2}^{1/2} \frac{\|g_w\|_{H^{\beta,p}}}{w^2} \,\mathrm{d}w \, \|h\|_{H^{1-\beta,p'}} \leq C \, \|h\|_{H^{1-\beta,p'}} ,$$

where $C < \infty$ as in Lemma 4.3 does not depend on on $s_1, s_2, \tau_1$,or $\tau_2$. □

Using the two statements above, we are led to the following fact from which Theorem IV immediately follows.

**Lemma 4.5.** *Let $\gamma \in H^{3/2+\beta_0,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $\beta_0 \geq 0$, $q \in [2,\infty)$ and $\beta_0 - \frac{1}{q} > -1/2$ be a critical point of the Möbius energy parametrized by arc length. Then,*

- *if $\beta_0 = 0$, we have $\gamma \in H^s$ for all $s < 3/2 + 2(1/2 - 1/q)$,*
- *if $0 < \beta_0 < 1/2$, we have $\gamma \in H^s$ for all $s < 3/2 + 2(\beta_0 + 1/2 - 1/q)$,*
- *if $\beta_0 \geq 1/2$, we have $\gamma \in H^{3/2+\beta_0+1/4}$.*

*Proof.* We set

$$\begin{aligned}
&\beta = 0, && \frac{1}{p} = \frac{2}{q} + \varepsilon, && \text{if } \beta_0 = 0,\\
&\beta = 0, && \frac{1}{p} = \frac{2}{q} - 2\beta_0 + \varepsilon, && \text{if } 0 < \beta_0 < 1/2,\\
&\beta = \beta_0 - 1/4, && \frac{1}{p} = 2/3, && \text{if } \beta_0 \geq \frac{1}{2}
\end{aligned}$$

and see that in each case the exponents satisfy the assumptions for the preceding Corollary 4.4 for all small enough $\varepsilon > 0$, so, for $p'$ with $\frac{1}{p} + \frac{1}{p'} = 1$, the functional $T(\gamma, \cdot) = T_1(\gamma, \cdot) + T_2(\gamma, \cdot)$ can be extended to an operator in $\left(H^{1-\beta,p'}\right)^* \subset (H^{3/2-1/p'-\beta})^*$.

From the fact that $\gamma$ is a critical point of the Möbius energy we then deduce that

$$Q(\gamma, \cdot) \in (H^{3/2-1/p'-\beta})^*$$

and a comparison of the Fourier coefficients gives

$$\gamma \in H^{3/2+\beta+\frac{1}{p'}}.$$

Since

$$3/2 + \beta + \frac{1}{p'} = 5/2 + \beta - \frac{1}{p} = \begin{cases} 3/2 + 2(1/2 - 1/q) - \varepsilon & \text{if } \beta_0 = 0,\\ 3/2 + 2(\beta_0 + 1/2 - 1/q) - \varepsilon & \text{if } 0 < \beta_0 < 1/2,\\ 3/2 + \beta_0 + 1/4 & \text{else,} \end{cases}$$

this proves Lemma 4.5. □

## A. Appendix

In this section we gather some facts, most of which can already be found in [Sch11] in slightly different versions. The main aim is to prove Lemmata 3.1 and 3.2 which both rely on quasi-locality of the Riesz potential $I_s$. Afterwards, we give an easy proof of the iteration lemma needed to deduce Dirichlet growth.

We will mainly deal with functions belonging to the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing smooth functions $\mathbb{R} \to \mathbb{R}$. The statements carry over to more general situations by suitable approximation arguments.

**Quasi-locality.** The essential tool apart from Sobolev inequalities is the following quantitative version of the *quasi-locality* of the fractional Laplacian and the Riesz potential.

**Lemma A.1** (Quasi-locality)**.** *Let $p_1, p_2, q_1, q_2 \in [1, \infty]$, where we assume that $q_2 = 1$ if $p_2 = 1$, $s \in (-1, 1)$ and $\Omega_1, \Omega_2$ be disjoint domains with $d := \operatorname{dist}(\Omega_1, \Omega_2) > 0$ and with positive and finite Lebesgue measure. Then, for any $f \in \mathcal{S}(\mathbb{R})$,*

$$\|\Delta^{\frac{s}{2}}(f\chi_{\Omega_2})\|_{(p_1,q_1),\Omega_1} \lesssim d^{-1-s}|\Omega_1|^{1/p_1}|\Omega_2|^{1-1/p_2}\|f\|_{(p_2,q_2),\Omega_2},$$

*where we set*

$$\Delta^{\frac{s}{2}} := \begin{cases} |D|^s & \text{if } s > 0, \\ \mathrm{id} & \text{if } s = 0, \\ I_{|s|} & \text{if } s < 0. \end{cases}$$

Note, that $\|\cdot\|_{(1,q_1)}$, $\|\cdot\|_{(\infty,q_\infty)}$ are only considered in the inequalities that follow for $q_1 = 1$ and $q_\infty = \infty$.

*Proof.* For $\tilde{k}_s(z) = \frac{1}{|z|^{1+s}}\chi_{\mathbb{R}\setminus B_d}(z)$ and $\operatorname{supp} f \subset \Omega_2$ we have for all $x \in \Omega_1$

$$\Delta^{\frac{s}{2}} f(x) = c_s(\tilde{k}_s * f)(x),$$

and hence

$$|\Delta^{\frac{s}{2}} f(x)| \lesssim \|f\|_{1,\Omega_2}\|\tilde{k}_s\|_\infty \le d^{-1-s}\|f\|_{1,\Omega_2}.$$

Hence,

$$\begin{aligned} \|\Delta^{\frac{s}{2}} f\|_{(p_1,q_1),\Omega_1} &\le |\Omega_1|^{1/p_1}\|\Delta^{\frac{s}{2}} f\|_{\infty,\Omega_1} \le d^{-1-s}|\Omega_1|^{1/p_1}\|f\|_{1,\Omega_2} \\ &\le d^{-1-s}|\Omega_1|^{1/p_1}|\Omega_2|^{1-1/p_2}\|f\|_{(p_2,q_2),\Omega_2}. \end{aligned}$$

□

A quite immediate consequence of this quasi-locality and Sobolev imbeddings is the following lemma. To state it, let

$$A^k_{\Lambda,r} := B_{2^k\Lambda r} - B_{2^{k-1}\Lambda r}. \tag{A.1}$$

**Lemma A.2.** *Let $p \in (1, \infty)$, $q, \tilde{q} \in [1, \infty]$, $s \in (-1, 1/p)$ and $-\frac{1}{p^*} = s - \frac{1}{p}$.*

(i) *For $f \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp} f \subset B_r$ we have*

$$\|\Delta^{-\frac{s}{2}} f\|_{(p^*,q);\mathbb{R}-B_{\Lambda r}} \lesssim \Lambda^{-1+\frac{1}{p}}\|f\|_{(p,\tilde{q}),B_r}$$

*uniformly for all $\Lambda > 2$.*

(ii) *If $s \in [0, \frac{1}{p})$, we have for all $f \in \mathcal{S}(\mathbb{R})$*

$$\|I_s f\|_{(p^*,q);\mathbb{R}-B_{\Lambda r}} \lesssim \Lambda^{-1+\frac{1}{p}}\|f\|_{(p,\tilde{q});B_r} + \|f\|_{(p,q);\mathbb{R}-B_r}$$

*uniformly for all $\Lambda > 2$.*

(iii) *For $s \in [0, \frac{1}{p})$ and any $f \in \mathcal{S}(\mathbb{R})$, we have*

$$\|I_s f\|_{(p^*,q),B_r} \lesssim \|f\|_{(p,q),B_{\Lambda r}} + \Lambda^{-\frac{1}{p^*}} \sum_{k=1}^\infty \left(2^{-k}\right)^{\frac{1}{p^*}} \|f\|_{(p,\tilde{q}),A^k_{\Lambda,r}}$$

*uniformly for all $\Lambda > 2$.*

*Proof.* For the first inequality we use Lemma A.1 and sum up the estimate

$$\|\Delta^{-\frac{s}{2}} f\|_{(p^*,q),A^k_{\Lambda,r}} \lesssim (\Lambda 2^k)^{-1+\frac{1}{p}} \|f\|_{(p,\tilde{q});B_r}.$$

In the second case, we use

$$\|I_s f\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}} \leq \|I_s(\chi_{B_r} f)\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}} + \|I_s((1-\chi_{B_r})f)\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}}$$

and estimate the first term using (i) and the second term using Sobolev's inequality to get

$$\|I_s f\|_{(p^*,q),\mathbb{R}-B_{\Lambda r}} \lesssim \Lambda^{-1+\frac{1}{p}} \|f\|_{(p,\tilde{q}),B_r} + \|f\|_{(p,q);\mathbb{R}-B_r}.$$

To deduce the last inequality, we decompose $f = \chi_{B_{\Lambda r}} f + \sum_{k=1}^{\infty} \chi_{A^k_{\Lambda,r}} f$ and estimate using Sobolev's inequality

$$\|I_s(\chi_{B_{\Lambda r}} f)\|_{(p^*,q),B_r} \lesssim \|f\|_{(p,q),B_{\Lambda r}}$$

and using Lemma A.1

$$\|I_s(\chi_{A^k_{\Lambda,r}} f)\|_{(p^*,q),B_r} \lesssim (\Lambda 2^k)^{-\frac{1}{p^*}} \|f\|_{(p,\tilde{q}),A^k_{\Lambda,r}}.$$

Summing up, this proves the last inequality. □

Finally, we use the quasi-locality to prove

**Proposition A.3.** *For $p \in (1,\infty)$, $q \in [1,\infty]$, $s,t \geq 0$ with $0 < s+t < 1$ there is a $\theta > 0$ such that we have for any $f \in L^{p,q}(\mathbb{R})$, $\Lambda > 2$, and $r > 0$,*

$$\||D|^s f\|_{(p,q),B_r} \lesssim \sup_{\substack{\varphi \in C_0^\infty(B_{\Lambda^2 r}),\\ \||D|^t \varphi\|_{(p',q')} \leq 1}} \int f |D|^{s+t} \varphi + r^{-s}\Lambda^{-\theta} \, \|f\|_{(p,q);B_{\Lambda r}} + r^{-s}\Lambda^{-\theta} \sum_{l=1}^{\infty} 2^{-\theta l} \|f\|_{(p,q),A^l_{\Lambda,r}}.$$

*Proof.* Assume that

$$\sup_{\varphi \in C_0^\infty(B_{\Lambda^2 r}), \||D|^t \varphi\|_{(p',q')} \leq 1} \int f |D|^{s+t} \varphi \leq K.$$

For $g \in C_0^\infty(B_r)$ we consider

$$\int_{\mathbb{R}} (|D|^s f) g \, \mathrm{d}x = \int_{\mathbb{R}} f |D|^s g \, \mathrm{d}x = \int_{B_{\Lambda r}} f |D|^s g \, \mathrm{d}x + \sum_{l=1}^{\infty} \int_{A^l_{\Lambda,r}} f |D|^s g \, \mathrm{d}x.$$

For the first term we use a smooth partition of unity $(\eta_k)_{k \in \mathbb{N} \cup \{0\}}$ with $\operatorname{supp} \eta_0 \subset B_{\Lambda^2 r}$ and $\operatorname{supp} \eta_k \subset B_{(\Lambda^2 2^k + 1)r} - B_{(\Lambda^2 2^{k-1} - 1)r}$ to get

$$\begin{aligned}\left| \int_{B_{\Lambda r}} f |D|^s g \, \mathrm{d}x \right| &\leq \left| \int_{B_{\Lambda r}} f |D|^{t+s} (\eta_0 I_t g) \, \mathrm{d}x \right| + \sum_{l=1}^{\infty} \left| \int_{B_{\Lambda r}} f |D|^{t+s} (\eta_l I_t g) \, \mathrm{d}x \right| \\ &\lesssim K \|g\|_{(p',q')} + \|f\|_{(p,q);B_{\Lambda r}} \sum_{l=1}^{\infty} \||D|^{t+s} (\eta_l I_t g)\|_{(p',q');B_{\Lambda r}}.\end{aligned}$$

Since by Lemma A.1 we have

$$\begin{aligned}\||D|^{t+s} \eta_l I_t g\|_{(p',q');B_{\Lambda r}} &\lesssim (\Lambda^2 2^l r)^{-1-t-s} (\Lambda r)^{1/p'} (\Lambda^2 2^l r)^{1-1/p'} \|\eta_l I_t g\|_{(p',q')} \\ &\leq (2^l)^{-t-s-1/p'} \Lambda^{-(2s+2t+1/p')} r^{-s-t} \|I_t g\|_{(p',q')},\end{aligned}$$

we can estimate this further by

$$K\|g\|_{(p',q')} + \Lambda^{-\theta_1} r^{-s-t} \|f\|_{(p,q);B_{\Lambda r}} \|I_t g\|_{(p',q')}$$

for $\theta_1 = 2t + 2s + 1/p'$.

Since $\operatorname{supp} g \subset B_r$ we get a scaled Poincaré inequality by applying first the Sobolev and then the Hölder inequality

$$\|I_t g\|_{(p',q')} \lesssim \|g\|_{(\frac{p'}{p't+1},q'),B_r} \lesssim r^t \|g\|_{(p',q')}$$

and using the quasi-locality (Lemma A.1) once more,

$$\begin{aligned} \||D|^s g\|_{(p',q'),A^k_{\Lambda r}} &\lesssim (2^k \Lambda r)^{-1-s} r^{1-1/p'} (2^k \Lambda r)^{1/p'} \|g\|_{(p',q')} \\ &= \Lambda^{-\theta_2} 2^{-k\theta_2} r^{-s} \|g\|_{(p',q')} \end{aligned}$$

for $\theta_2 = 1 + s - 1/p'$.

Hence,

$$\begin{aligned} &\int_{\mathbb{R}} (|D|^s f)\ g \,\mathrm{d}x \\ &\quad \le \left( K + \Lambda^{-\theta} r^{-s} \|f\|_{(p,q);B_{\Lambda r}} + r^{-s} \Lambda^{-\theta} \sum_{l=1}^{\infty} 2^{-\theta l} \|f\|_{(p,q);A^l_{\Lambda,r}} \right) \|g\|_{(p',q')} \end{aligned}$$

for $\theta = \min\{\theta_1, \theta_2\}$, which by duality proves the proposition. □

**Proofs of Lemmata 3.1 and 3.2.** The following lemma is the starting point for the estimates of $H_s$ and essentially follows from the mean value theorem or a first-order Taylor expansion.

**Lemma A.4.** *Let $\delta \in [0,1]$, $\alpha \in (0,1)$. Then for almost all $x, y, \xi \in \mathbb{R}^n$ we have*

$$\left| |x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha} \right| \lesssim |x-y|^\delta \left( |y-\xi|^{-1+\alpha-\delta} + |x-\xi|^{-1+\alpha-\delta} \chi_{|x-y|>2|x-\xi|} \right).$$

*Proof.* If $|x-y| > 2|x-\xi|$ we get

$$|y-\xi| \ge |y-x| - |x-\xi| > |x-\xi|,$$

and hence

$$\left| |x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha} \right| \lesssim |x-\xi|^{-1+\alpha} \lesssim |x-y|^\delta |x-\xi|^{-1+\alpha-\delta}.$$

If $|x-y| \le 2|x-\xi|$ we first observe that the above argument leads to

$$\left| |x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha} \right| \lesssim |x-y|^\delta |y-\xi|^{-1+\alpha-\delta}$$

if $|x-y| > 2|y-\xi|$.

To deal with the case that both $|x-y| \le 2|x-\xi|$ and $|x-y| \le 2|y-\xi|$, we observe that then

$$|y-\xi| \le |x-\xi| + |x-y| \le 3|x-\xi|.$$

Hence, we get using the mean value theorem

$$\begin{aligned} \left| |x-\xi|^{-1+\alpha} - |y-\xi|^{-1+\alpha} \right| &\lesssim |x-y| \max\left\{ |x-\xi|^{-2+\alpha}, |y-\xi|^{-2+\alpha} \right\} \\ &\lesssim |x-y| |y-\xi|^{-2+\alpha} \lesssim |x-y|^\delta |y-\xi|^{-1+\alpha-\delta}. \end{aligned}$$ □

We use the lemma above to derive the following pointwise estimate for $H_{1/2+s}$.

**Lemma A.5** ([Sch11]). *For $s \in [0, \frac{1}{2})$ and functions $a, b \in \mathcal{S}(\mathbb{R})$ the following holds for any $\varepsilon, \varepsilon' \in [0, \frac{1}{6} - \frac{s}{3})$:*

$$\begin{aligned}|H_{1/2+s}(a,b)| \lesssim\ & I_{1/6-s/3}(I_{1/6-s/3-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/6-s/3-\varepsilon'}||D|^{1/2-\varepsilon'}b|)\\ &+ I_{1/3-2s/3-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/6-s/3-\varepsilon'}||D|^{1/2-\varepsilon'}b|\\ &+ I_{1/6-s/3-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/3-2s/3-\varepsilon'}||D|^{1/2-\varepsilon'}b|\\ &+ I_{1/4-s/2-\varepsilon}||D|^{1/2-\varepsilon}a|\ I_{1/4-s/2-\varepsilon'}||D|^{1/2-\varepsilon'}b|.\end{aligned}$$

*Proof.* In order to shorten notation, we restrict to $\varepsilon' = \varepsilon$; the general case is parallel. We use the identities $a = I_{\frac{1}{2}-\varepsilon}|D|^{\frac{1}{2}-\varepsilon}a$ and $b = I_{\frac{1}{2}-\varepsilon}|D|^{\frac{1}{2}-\varepsilon}b$. Then,

$$\begin{aligned}&|H_{1/2+s}(a,b)(x)|\\ &\lesssim \iiint \frac{\left(|y-z|^{-1+1/2-\varepsilon} - |x-z|^{-1+1/2-\varepsilon}\right)\left(|y-w|^{-1+1/2-\varepsilon} - |x-w|^{-1+1/2-\varepsilon}\right)}{|y-x|^{1+(1/2+s)}}\\ &\qquad \cdot \left||D|^{\frac{1-2\varepsilon}{2}}a(z)\right|\left||D|^{\frac{1-2\varepsilon}{2}}b(w)\right| \mathrm{d}w\, \mathrm{d}z\, \mathrm{d}y.\end{aligned}$$

Applying Lemma A.4 we get

$$\begin{aligned}&\frac{\left(|y-z|^{-1+1/2-\varepsilon} - |x-z|^{-1+1/2-\varepsilon}\right)\left(|y-w|^{-1+1/2-\varepsilon} - |x-w|^{-1+1/2-\varepsilon}\right)}{|y-x|^{1+(1/2+s)}}\\ &\quad\lesssim \frac{\left(|y-w|^{-1+1/2-\varepsilon-\delta} + |x-w|^{-1+1/2-\varepsilon-\delta}\chi_{|x-y|>2|x-w|}\right)}{|x-y|^{1+1/2+s-2\delta}}\\ &\qquad \cdot \left(|y-z|^{-1+1/2-\varepsilon-\delta} + |x-z|^{-1+1/2-\varepsilon-\delta}\chi_{|x-y|>2|x-z|}\right)\\ &\quad\lesssim \frac{(|y-w||y-z|)^{-1+1/2-\varepsilon-\delta} + (|x-w||y-z|)^{-1+1/2-\varepsilon-\delta}}{|x-y|^{1+1/2+s-2\delta}}\\ &\qquad + \frac{(|y-w||x-z|)^{-1+1/2-\varepsilon-\delta}}{|x-y|^{1+1/2+s-2\delta}}\\ &\qquad + \frac{(|x-w||x-z|)^{-1+1/2-\varepsilon-\delta}}{|x-y|^{1+1/2+s-2\delta}}\chi_{|x-y|>2|x-w|}\ \chi_{|x-y|>2|x-z|}.\end{aligned}$$

For $\delta = 1/3 + s/3$ we hence get

$$\begin{aligned}|H_{1/2+s}(a,b)| \lesssim\ & I_{1/6-s/3}(I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b|)\\ &+ I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/3-2s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b|\\ &+ I_{1/3-2s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/6-s/3-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b| + A,\end{aligned}$$

where

$$\begin{aligned}A &\lesssim \int\int\left(\int \frac{(|x-w||x-z|)^{-1+1/2-\varepsilon-\delta}}{|x-y|^{1+1/2+s-2\delta}}\chi_{|x-y|>2|x-w|}\ \chi_{|x-y|>2|x-z|}\ \mathrm{d}y\right)\\ &\qquad \cdot ||D|^{\frac{1-2\varepsilon}{2}}a(z)|\ ||D|^{\frac{1-2\varepsilon}{2}}b(w)|\, \mathrm{d}w\, \mathrm{d}z\\ &\lesssim \iint |x-w|^{-1+1/4-\varepsilon-s/2}|x-z|^{-1+1/4-\varepsilon-s/2}||D|^{\frac{1-2\varepsilon}{2}}a(z)|\ ||D|^{\frac{1-2\varepsilon}{2}}b(w)|\, \mathrm{d}w\, \mathrm{d}z\\ &\lesssim I_{1/4-s/2-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}a|\ I_{1/4-s/2-\varepsilon}||D|^{\frac{1-2\varepsilon}{2}}b|.\end{aligned}$$

□

To estimate this further, we will use the following fact about *lower order products* which we get using the quasi-locality.

**Lemma A.6** (Lower order products). *Let for $s > 0$, $0 \leq s_1, s_2, s_3 \leq 1/2$, $s_1 + s_2 + s_3 = s$, and let at least two of these $s_i$, $i = 1, 2, 3$, be non-zero. Let $p \in (1, \infty)$, $p_2 \in (1, \frac{1}{s_2}), p_3 \in (1, \frac{1}{s_3})$ and such that*

$$\frac{1}{p} = \frac{1}{p_2} + \frac{1}{p_3} - s.$$

*Assume moreover that*

$$|G(u, v)| \lesssim I_{s_1} \left( I_{s_2}|u| \; I_{s_3}|v| \right).$$

*Then there is some $\theta > 0$ such that we have for any $\Lambda > 4$ and $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$,*

$$\text{(A.2)} \qquad \|G(u, v)\|_{(p,q),B_r} \lesssim \|u\|_{(p_2,q_1),B_{\Lambda r}} \|v\|_{(p_3,q_2),B_{\Lambda r}} + \|u\|_{(p_2,q_1)} \Lambda^{-\theta} \sum_{k=2}^{\infty} 2^{-\theta k} \|v\|_{(p_3,q_2),B_{\Lambda 2^k r}}.$$

*If* $\operatorname{supp} |D|^t v \subset \overline{B_r}$ *for some $t \in [0, \frac{1}{2}]$, we furthermore get for any $k \geq 2$, $\Lambda > 16$,*

$$\text{(A.3)} \qquad \|G(u, v)\|_{(p,q),A^k_{\Lambda,r}} \lesssim \Lambda^{-\theta} 2^{-\theta k} \|u\|_{(p_2,\infty),\mathbb{R}} \; \|v\|_{(p_3,1),\mathbb{R}}.$$

*Proof.* Let

$$\frac{1}{p_2^*} := \frac{1}{p_2} - s_2,$$
$$\frac{1}{p_3^*} := \frac{1}{p_3} - s_3,$$
$$\frac{1}{p_1} := \frac{1}{p_2^*} + \frac{1}{p_3^*} = \frac{1}{p_2} + \frac{1}{p_3} - s_2 - s_3,$$

and

$$\frac{1}{p_1^*} := \frac{1}{p} \equiv \frac{1}{p_2} + \frac{1}{p_3} - s = \frac{1}{p_1} - s_1.$$

Then by Lemma A.2 (iii), for $\Lambda \hat{=} \sqrt{\Lambda}$, $p^* \hat{=} p_1^* = p$, $p \hat{=} p_1$, $s \hat{=} s_1$,

$$\begin{aligned}
\|G(u, v)\|_{(p_1^*,q),B_r} &\lesssim \|I_{s_2}|u| \; I_{s_3}|v|\|_{(p_1,q),B_{\Lambda^{1/2} r}} \\
&\quad + \Lambda^{-1/2p} \sum_{k=1}^{\infty} 2^{-k/p} \|I_{s_2}|u| \; I_{s_3}|v|\|_{(p_1,q),B_{\Lambda^{1/2} 2^k r}} \\
&\lesssim \|I_{s_2}|u|\|_{(p_2^*,q_1),B_{\Lambda^{1/2} r}} \; \|I_{s_3}|v|\|_{(p_3^*,q_2),B_{\Lambda^{1/2} r}} \\
&\quad + \Lambda^{-1/2p} \sum_{k=1}^{\infty} 2^{-k/p} \|I_{s_2}|u|\|_{(p_2^*,q_1),B_{2^k \Lambda^{1/2} r}} \|I_{s_3}|v|\|_{(p_3^*,q_2),B_{2^k \Lambda^{1/2} r}}.
\end{aligned}$$

Applying yet again Lemma A.2 (iii) to both factors in each sum with $\Lambda \hat{=} \sqrt{\Lambda}$ we arrive at

$$\|G(u, v)\|_{(p,q),B_r} \lesssim \|u\|_{(p_2,q_1),B_{\Lambda r}} \|v\|_{(p_3,q_2),B_{\Lambda r}} + \|u\|_{(p_2,q_1)} \Lambda^{-\theta} \sum_{k=2}^{\infty} 2^{-\theta k} \|v\|_{(p_3,q_2),B_{\Lambda 2^k r}}.$$

As for the second estimate, we apply Lemma A.2 (ii) with $\Lambda \hat{=} \sqrt{\Lambda}$ to get

$$\begin{aligned}\|G(u,v)\|_{(p,q),\mathbb{R}-B_{\Lambda r}} &= \|I_{s_1}\left(I_{s_2}|u|\ I_{s_3}|v|\right)\|_{(p_1^*,q),\mathbb{R}-B_{\Lambda r}} \\ &\lesssim \Lambda^{-\theta_1}\|I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,\infty),B_{\Lambda^{1/2}r}} \\ &\quad + \|I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,q),\mathbb{R}-B_{\Lambda^{1/2}r}}\end{aligned}$$

where $\theta_1 = \frac{1}{2} - \frac{1}{2p_1}$. We estimate the first term using Hölder's inequality and Sobolev imbedding theorem to get

$$\Lambda^{-\theta_1}\|I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,\infty),B_{\Lambda^{1/2}r}} \le \Lambda^{-\theta_1}\|u\|_{(p_2,\infty)}\|v\|_{(p_3,1)}.$$

For the second term we use Hölder's inequality and Sobolev's imbedding theorem to get

$$\begin{aligned}\|I_{s_2}|u|\ I_{s_3}|v|\|_{(p_1,q),\mathbb{R}-B_{\Lambda^{1/2}r}} &\lesssim \|I_{s_2}|u|\|_{(p_2^*,q_1),\mathbb{R}-B_{\Lambda^{1/2}r}}\|I_{s_3}|v|\|_{(p_3^*,q_2),\mathbb{R}-B_{\Lambda^{1/2}r}} \\ &\lesssim \|u\|_{(p_2,\infty),\mathbb{R}}\|I_{s_3}|v|\|_{(p_3^*,q_2),\mathbb{R}-B_{\Lambda^{1/2}r}}.\end{aligned}$$

Since

$$\|I_{s_3}|v|\|_{(p_3^*,q_2),\mathbb{R}-B_{\Lambda^{1/2}r}} \lesssim \Lambda^{-\theta_2}\|v\|_{(p_3,1);B_{\Lambda^{1/4}}} + \|v\|_{(p_3,q);\mathbb{R}-B_{\Lambda^{1/4}}}$$

where $\theta_2 = \frac{1}{4} - \frac{1}{4p_3}$ and, using Lemma A.2 (i),

$$\|v\|_{(p_3,q);\mathbb{R}-B_{\Lambda^{1/4}}} = \||D|^t(I_t v)\|_{(p_3,q);\mathbb{R}-B_{\Lambda^{1/4}}} \le \Lambda^{-\theta_2}\|I_t v\|_{(p_3^{**},1)} \le \Lambda^{-\theta_2}\|I_t v\|_{(p_3,1)},$$

we deduce the statement for $\theta := \min\{\theta_1,\theta_2\}$. □

We then have the following

**Lemma A.7.** *There is a $\varepsilon_0 > 0$ such that for $\delta,\varepsilon \in [0,\varepsilon_0)$, for any $a,b \in \mathcal{S}(\mathbb{R})$, $\Lambda > 4$,*

$$\begin{aligned}\|H_{\frac{1}{2}}(a,b)\|_{(2,q),B_r} &\lesssim \||D|^{1/2-\delta}a\|_{(\frac{2}{1-2\delta},q_1),B_{\Lambda r}}\||D|^{1/2-\varepsilon}b\|_{(\frac{2}{1-2\varepsilon},q_2),B_{\Lambda r}} \\ &\quad + \||D|^{1/2-\delta}a\|_{(\frac{2}{1-2\delta},q_1)}\Lambda^{-\theta}\sum_{k=1}^{\infty}2^{-\theta(k-1)}\||D|^{1/2-\varepsilon}b\|_{(\frac{2}{1-2\varepsilon},q_2),B_{2^k\Lambda r}}.\end{aligned}$$

*If* supp $b \subset B_r$ *we furthermore get for any $k \ge 2$, $\Lambda > 16$, $s \in [0,\frac{1}{2})$,*

$$\|H_{\frac{1}{2}}(a,b)\|_{(\frac{2}{1+2s},q),A^k_{\Lambda,r}} \lesssim (\Lambda 2^k)^{-\theta}\||D|^{\frac{1}{2}}a\|_{(2,\infty),\mathbb{R}}\ \||D|^{1/2-\varepsilon}b\|_{(\frac{2}{1-2\varepsilon},1),\mathbb{R}}.$$

*Proof.* Immediately from Lemma A.5, Lemma A.6 where $s = \frac{1}{2} - \varepsilon - \delta$. □

We use the lemma above to estimate the normal part as stated in Lemma 3.1.

*Proof of Lemma* 3.1. With (3.3) we obtain $\left\||D|^s|g'|^2\right\|_{(\frac{2}{1+2s},\infty),B_r} \lesssim r^{\frac{1}{2}+s}$. In order to show (3.4) by the decomposition (3.1) it remains to treat the $H$-term. We rewrite

$$\begin{aligned}|D|^s H_{\frac{1}{2}}(a,b) = &\underbrace{H_{s+\frac{1}{2}}(a,b)}_{=:I} + \underbrace{a|D|^{\frac{1+2s}{2}}b - |D|^s(a|D|^{\frac{1}{2}}b)}_{=:II} \\ &+ \underbrace{b|D|^{\frac{1+2s}{2}}a - |D|^s(b|D|^{\frac{1}{2}}a)}_{=:III}.\end{aligned}$$

We will show that all three terms satisfy the hypotheses of Lemma A.6. Due to Lemma A.5 this is true for the term $I$.

As for $II$ note that $II = 0$ in case $s = 0$. If $s \in (0,1)$, the potential definition (0.4) of $|D|^s$ gives

$$II(x) = c\left( a(x) \int \frac{|D|^{\frac{1}{2}} b(y) - |D|^{\frac{1}{2}} b(x)}{|x-y|^{1+s}} \ \mathrm{d}y \right.$$
$$\left. - \int \frac{a(y)|D|^{\frac{1}{2}} b(y) - a(x)|D|^{\frac{1}{2}} b(x)}{|x-y|^{1+s}} \ \mathrm{d}y \right)$$
$$= c \int \frac{(a(x) - a(y))\, |D|^{\frac{1}{2}} b(y)}{|x-y|^{1+s}} \ \mathrm{d}y.$$

Using $a = I_{\frac{1}{2}} |D|^{\frac{1}{2}} a$ we arrive at

$$|II(x)| \lesssim \int \frac{|a(y) - a(x)| \left| |D|^{\frac{1}{2}} b(y) \right|}{|x-y|^{1+s}} \ \mathrm{d}y$$
$$\lesssim \int \int \frac{\left| |y-z|^{-1+\frac{1}{2}} - |x-z|^{-1+\frac{1}{2}} \right| \left| |D|^{\frac{1}{2}} b(y) \right| \left| |D|^{\frac{1}{2}} a(z) \right|}{|x-y|^{1+s}} \ \mathrm{d}y \ \mathrm{d}z.$$

For almost all $x, y, z$ we get from Lemma A.4 choosing $\delta := \frac{1}{4} + \frac{s}{2}$,

$$\frac{\left| |y-z|^{-1+\frac{1}{2}} - |x-z|^{-1+\frac{1}{2}} \right|}{|x-y|^{1+s}} \leq (|x-z||x-y|)^{-1+\frac{1}{4}-\frac{s}{2}} + (|y-z||x-y|)^{-1+\frac{1}{4}-\frac{s}{2}},$$

which implies, again using (0.4),

$$|II(x)| \lesssim I_{\frac{1-2s}{4}} \left| |D|^{\frac{1}{2}} a \right| \ I_{\frac{1-2s}{4}} \left| |D|^{\frac{1}{2}} b \right| + I_{\frac{1-2s}{4}} \left( \left| |D|^{\frac{1}{2}} b \right| \ I_{\frac{1-2s}{4}} \left| |D|^{\frac{1}{2}} a \right| \right).$$

By symmetry a respective estimate also holds for the term $III$. Applying Lemma A.6 one concludes the proof. □

*Proof of Lemma* 3.2. Again the proof relies on quasi-locality. First we decompose

$$\left| \int |D|^{\frac{1}{2}} g_i' \omega_{ij} H_{1/2}(g_j', \phi) \right| \lesssim \left| \int_{B_{\Lambda^{1/2} r}} |D|^{\frac{1}{2}} g_i' \omega_{ij} H_{1/2}(g_j', \phi) \right|$$
$$+ \sum_{k=1}^{\infty} \left| \int_{A^k_{\Lambda^{1/2}, r}} |D|^{\frac{1}{2}} g_i' \omega_{ij} H_{1/2}(g_j', \phi) \right|$$
$$\lesssim \| |D|^{\frac{1}{2}} g' \|_{(2,\infty), B_{\Lambda^{1/2} r}} \| H_{1/2}(g_j', \phi) \|_{(2,1), B_{\Lambda^{1/2} r}}$$
$$+ \sum_{k=1}^{\infty} \| |D|^{\frac{1}{2}} g' \|_{(2,\infty), A^k_{\Lambda^{1/2}, r}} \| H_{1/2}(g_j', \phi) \|_{(2,1), A^k_{\Lambda^{1/2}, r}}.$$

Using Lemma A.7, the first summand can be estimated by

$$\|I_s|D|^{\frac{1}{2}}\phi\|_{(\frac{2}{1-2s},1)} \left( \||D|^{\frac{1}{2}}g'\|^2_{(2,\infty),B_{\Lambda r}} \right.$$
$$\left. + \||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{\Lambda r}} \Lambda^{-\theta} \sum_{l=1}^{\infty} 2^{-\theta l} \||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{\Lambda 2^l r}} \right).$$

Applying the second part of Lemma A.7, the infinite sum can be estimated by

$$\Lambda^{-\theta}\|I_s|D|^{\frac{1}{2}}\phi\|_{(\frac{2}{1-2s},1)} \||D|^{\frac{1}{2}}g'\|_{(2,\infty)} \sum_{k=2}^{\infty} 2^{-\theta k} \||D|^{\frac{1}{2}}g'\|_{(2,\infty),B_{2^k\Lambda r}}.$$

□

**Iteration lemma.** In order to prove Dirichlet growth, we need an iteration lemma whose proof is based on the technique presented in [Tao01, notes4, p. 11]. The statement is also similar to the corresponding one appearing in [DLR11a]. One should see this as a generalized version of De Giorgi's Iteration Lemma; cf., e.g., [Gia83].

**Lemma A.8** (Iteration lemma). *Let $C < \infty$ and $\theta > 0$ be given.*
*If $b_k \geq 0$, $k \in \mathbb{N}_0$, satisfy*

$$b_{k+m} \leq \varepsilon b_k + C\left(2^{-\theta(k+m)} + 2^{-\theta m}\sum_{l=1}^{k} 2^{-\theta l} b_{k-l}\right) \tag{A.4}$$

*for all $k \in \mathbb{N}_0$ and $\varepsilon > 0$ is small enough, and $m$ is big enough, then*

$$b_k \lesssim 2^{-\tilde{\theta}k}$$

*for all $k \in \mathbb{N}_0$ with $\tilde{\theta} = \theta/2$.*

*Proof.* We will prove that

$$\sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k \leq C \sum_{l=0}^{m} 2^{\tilde{\theta}l} b_l + C. \tag{A.5}$$

Especially, the infinite sum converges and hence the summands are a null series, which proves the lemma.

Multiplying equation (A.4) with $2^{\tilde{\theta}k}$, and summing over $k$ we get

$$\begin{aligned}
&2^{-\tilde{\theta}m}\sum_{k=0}^{\infty} 2^{\tilde{\theta}(k+m)} b_{k+m} \\
&\quad \leq \varepsilon \sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k + C\sum_{k=0}^{\infty} 2^{-\theta(k+m)} 2^{\tilde{\theta}k} + C2^{-\theta m}\sum_{k=0}^{\infty}\sum_{l=1}^{k} 2^{-\theta l} 2^{\tilde{\theta}k} b_{k-l} \\
&\quad \leq \varepsilon \sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k + C2^{-\theta m}\sum_{k'=0}^{\infty}\sum_{l=1}^{\infty} 2^{-\theta l} 2^{\tilde{\theta}(k'+l)} b_{k'} + C2^{-\theta m} \\
&\quad \leq \varepsilon \sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k + \tilde{C}2^{-\theta m}\sum_{k'=0}^{\infty} 2^{\tilde{\theta}k'} b_{k'} + C2^{-\theta m}.
\end{aligned}$$

Assuming that $m$ is so large that $\tilde{C}2^{-m(\theta-\tilde{\theta})} = \tilde{C}2^{-\frac{m\theta}{2}} < 1/4$ and $0 < \varepsilon < \frac{1}{4}2^{-\theta m}$ we get

$$\sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k \leq \frac{1}{2} \sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k + \sum_{l=0}^{m} 2^{\tilde{\theta}l} b_l + C,$$

and hence

$$\sum_{k=0}^{\infty} 2^{\tilde{\theta}k} b_k \leq C \sum_{l=0}^{m} 2^{\tilde{\theta}l} b_l + C$$

if the infinite series converges. If the sum is not known to converge, we apply the above argument to the cut-off series

$$\begin{cases} \tilde{b}_k = b_k & \text{if } k \leq N, \\ \tilde{b}_k = 0 & \text{else} \end{cases}$$

and get the uniform bound $\sum_{k=0}^{N} 2^{\tilde{\theta}k} b_k \lesssim \sum_{l=0}^{m} 2^{\tilde{\theta}l} b_l + 1$. Letting $N \to \infty$ we get (A.5). □

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