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NÉRON MODELS OF ALGEBRAIC CURVES

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Dedicated to Michel Raynaud on the occasion of his seventy-fifth birthday

Abstract. Let S be a Dedekind scheme with field of functions K. We show that if X_K is a smooth connected proper curve of positive genus over K, then it admits a Néron model over S, *i.e.*, a smooth separated model of finite type satisfying the usual Néron mapping property. It is given by the smooth locus of the minimal proper regular model of X_K over S, as in the case of elliptic curves. When S is excellent, a similar result holds for connected smooth affine curves different from the affine line, with locally finite type Néron models.

1. Introduction

In 1964, A. Néron [11] introduced the notion of Néron models (see Definition 2.2) of abelian varieties over the fraction field of a Dedekind domain, and proved the existence of these models (see the introduction of [1] for a detailed presentation). Since then, this notion has been generalized to smooth commutative algebraic groups and to torsors under these groups (see [1], §6.5). In this work we investigate the case of smooth proper or affine curves.

Let S be a *Dedekind scheme*, that is, a noetherian regular connected scheme of dimension 1. Let $K = K(S)$ be its field of functions. Let X_K be a smooth connected curve over K. When X_K is proper of positive genus, a canonical smooth model is the smooth locus X_{sm} of the minimal proper regular model of X_K over S. When X_K is an elliptic curve, it is well known that X_{sm} is also the Néron model of X_K (see [11] or [8], 10.2.14). The first main result of this work is a generalization of the latter fact to higher genus.

Theorem 1.1 (Theorem 4.1)**.** *Let X_K be a proper smooth connected curve of positive genus over K. Then the smooth locus X_{sm} of the minimal proper regular model of X_K over S is the Néron model of X_K.*

When S is excellent, we actually prove a slightly more general result: for a regular proper connected curve X_K/K of positive arithmetic genus, the smooth locus of X_K admits a Néron model over S, equal to the smooth locus of the minimal proper regular model of X_K over S.

As an immediate consequence of Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let X_K be as in Theorem 1.1. Let Y be a smooth scheme over S and let $f_K\colon Y_K \to X_K$ be a morphism of K-schemes. Then*

(1) *the morphism f_K extends uniquely to a morphism of S-schemes $Y \to X_{\mathrm{sm}}$;*

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(2) (Corollary 4.7) *if Y is proper over S (i.e., Y_K has good reduction over S) and f_K is dominant, then X_{sm} is proper over S and X_K has good reduction over S.*

In the second part of this paper (§5–§7), we consider Néron lft-models (Néron model locally of finite type) of smooth affine curves. The main result of this second part is

Theorem 1.3 (Theorem 7.10)**.** *Suppose S is excellent. Let U_K be an affine smooth and geometrically connected curve over K, not isomorphic to $\mathbb{A}^1_K$. Then U_K admits a Néron lft-model U over S.*

Note that, in general, the scheme U in the above theorem is not of finite type. However, necessary and sufficient conditions (in terms of the points at infinity of U_K) can be found in Proposition 7.11 to ensure that U is of finite type over S.

The paper is organized as follows. Some basic properties of Néron lft-models are assembled in §2. In §3, we prove a crucial technical result (Proposition 3.6) on the image of a morphism $f: Y \to X$ from a smooth S-scheme to a normal relative curve over S. In §4, we prove the main Theorem 4.1 on the existence of Néron models for proper smooth curves of positive genus. In §5, we study the Néron lft-model of open subscheme of a curve having already a Néron lft-model (Theorem 5.1). §6 is devoted to the existence of the Néron model of some special affine open subsets of a smooth conic over local Dedekind schemes S. Finally we prove Theorem 1.3 in §7.

Notation. Throughout this paper, unless explicitly mentioned, the letter S denotes a *Dedekind scheme* (that is, a noetherian regular connected scheme of dimension 1) and K denotes its field of functions $K(S)$. Symbols such as X_K, Y_K, U_K usually denote a scheme over K. On the other hand, for any S-scheme X, X_K also denotes the generic fiber of X. If X_K is a proper smooth and connected curve over K, by *genus of X_K*, we mean the dimension $\dim_K H^1(X_K, \mathcal{O}_{X_K})$.

2. Basic properties

Let S, K be as above.

Definition 2.1. Let X_K be a separated algebraic variety (*i.e.* separated scheme of finite type) over K. A *model of X_K over S* is a **locally finite type**, separated and flat[1] scheme over S endowed with an isomorphism from its generic fiber to X_K.

Definition 2.2 ([1], 10.1/1, 1.2/1)**.** Let X_K be a separated smooth algebraic variety over K. A *Néron lft-model of X_K over S* or an *S-Néron lft-model of X_K* is a smooth model X of X_K over S satisfying the following universal property, called *Néron mapping property*:

for any smooth scheme $Y \to S$, the canonical map (once the generic fiber of X is identified with X_K)

$$\mathrm{Mor}_S(Y, X) \to \mathrm{Mor}_K(Y_K, X_K) \tag{2.1}$$

is a bijection.

A Néron lft-model of finite type is called a *Néron model.*

[1] In this work, we do not require the models to be faithfully flat.

Remark 2.3. (1) The universal property above implies the uniqueness (up to a unique isomorphism) of the Néron lft-model if it exists.

(2) Let X be a model of X_K. As X is separated, the map (2.1) is always injective. So it is enough to check the surjectivity for the Néron mapping property. By the injectivity, when S is local, it is also enough to check the surjectivity with Y smooth of finite type having irreducible fibers over S.

(3) Let X_K be a separated smooth algebraic variety over K. If each connected component of X_K admits an S-Néron lft-model, then X_K also has an S-Néron lft-model given by the disjoint union of the S-Néron lft-models of the connected components. This holds similarly for Néron models.

Proposition 2.4. *Let X be an S-model of X_K. Let $S' \to S$ be a morphism of Dedekind schemes. Denote by K' the function field of S'.*

(1) *Assume that the morphism $S' \to S$ is faithfully flat, and that $X_{S'} := X \times_S S'$ is the Néron lft-model (resp. Néron model) of $X_{K'} := X \times_{\mathrm{Spec}(K)} \mathrm{Spec}(K')$ over S'. Then X is the Néron lft-model (resp. Néron model) of X_K over S.*

(2) *Assume that $S' \to S$ can be written as a filtered inverse limit of affine smooth schemes of finite type over S, and that X is the Néron lft-model (resp. Néron model) of X_K over S. Then the base change $X_{S'}$ is the Néron lft-model (resp. Néron model) of $X_{K'}$ over S'.*

(3) *Assume that $S' \to S$ is an extension of local Dedekind schemes of ramification index* 1[2] *with S* excellent, *and that X is the Néron lft-model over S. Then, $X_{S'}$ is the Néron lft-model over S'.*

Proof. (1) This is a direct application of fpqc descent, so we omit the details here.

(2) Let Y' be a smooth S'-scheme, and let $f_{K'} \colon Y'_{K'} \to X_{K'}$ be a morphism of K'-schemes. Without loss of generality, we may and do assume that the scheme Y' is quasi-compact, hence is of finite presentation over S'. Consequently, the S'-scheme Y' (resp. the morphism $f_{K'} \colon Y'_{K'} \to X_{K'}$) descends to an S_0-scheme Y_0 (resp. to a morphism of $S_{0,K}$-schemes $f_{0,K} \colon Y_0 \times_{S_0} S_{0,K} \to X_{S_0} \times_{S_0} S_{0,K}$) for some (affine) smooth morphism $S_0 \to S$ ([5], IV.8.8.2). Thus, we only need to prove that the morphism $f_{0,K}$ can be extended to a morphism $f_0 \colon Y_0 \to X \times_S S_0$. Consider the composite $Y_0 \to S_0 \to S$, which defines a smooth S-scheme. The morphism $f_{0,K}$ gives a morphism of K-schemes from $Y_0 \times_S \mathrm{Spec}(K)$ to X_K. As X is the S-Néron lft-model of X_K, the Néron mapping property implies existence of a morphism of S-schemes $g \colon Y_0 \to X$ making the following diagram commutative:

$$\begin{tikzcd} Y_0 \arrow[r, dotted, "\exists f_0"'] \arrow[rr, bend left, "g"] \arrow[dr] & X_{S_0} \arrow[r] \arrow[d] & X \arrow[d] \\ & S_0 \arrow[r] & S \end{tikzcd}$$

We obtain a morphism $f_0 \colon Y_0 \to X_{S_0}$ of S_0-schemes extending $f_{0,K}$, as required. If X is the S-Néron model of X_K, then $X_{S'}$ is of finite type over S' and is the S'-Néron model of $X_{K'}$.

(3) Write $S = \mathrm{Spec}(R)$ and $S' = \mathrm{Spec}(R')$. As S is excellent, by [1], 3.6/2, the morphism $R \to R'$ is regular. Consequently, by a result of Néron ([1], 3.6/8;

[2]See [1], 3.6/1, for the definition. The residue extension is required to be (not necessarily algebraic) separable.

see also [12], 2.5, for a more general statement), R' can be written as a filtered inductive limit of smooth R-algebras of finite type. Therefore, $X_{S'}$ is the Néron lft-model over S' by (2). $\square$

Corollary 2.5. *Let S be a Dedekind scheme with field of functions K, and let X be an S-scheme locally of finite type (resp. an S-scheme of finite type) with smooth finite type generic fiber X_K. Then X is the S-Néron lft-model of X_K (resp. is the S-Néron model of X_K) if and only if for any closed point $s \in S$, $X \times_S \operatorname{Spec}(\mathcal{O}_{S,s})$ is the $\operatorname{Spec}(\mathcal{O}_{S,s})$-Néron lft-model of X_K (resp. is the $\operatorname{Spec}(\mathcal{O}_{S,s})$-Néron model of X_K).*

Proof. If X/S is of finite type, then this corollary is proved in [1], 1.2/4. The same proof applies in the locally finite type case. For the convenience of the readers, we reproduce the proof in this case here. The direct implication follows from Proposition 2.4(2). Conversely, if $X \times_S \operatorname{Spec}(\mathcal{O}_{S,s})$ is the Néron lft-model of X_K over $\operatorname{Spec}(\mathcal{O}_{S,s})$ for any closed point $s \in S$, it follows that X/S is smooth and separated ([5], IV.8.10.5(v)). It remains to check the Néron mapping property for X/S (Definition 2.2). Consider Y a smooth S-scheme, and $f_K\colon Y_K \to X_K$ a morphism of K-schemes; we need to extend f_K to a morphism of S-schemes from Y to X. We may assume that Y is quasi-compact, hence is of finite presentation over S. Now, for any closed point $s \in S$, our assumptions imply that one can find an extension $f_{\mathcal{O}_{S,s}}$ of f_K over $\operatorname{Spec}(\mathcal{O}_{S,s})$. As Y/S is of finite presentation and X/S is locally of finite presentation, we may descend $f_{\mathcal{O}_{S,s}}$ to a morphism $f_s\colon Y \times_S V \to X \times_S V$ over some open neighborhood $V \subseteq S$ of s ([5], IV.8.8.2). As X/S is separated, and Y/S is flat, these local extensions of f_K are compatible with each other. As a result, they can be glued together to a morphism $f\colon Y \to S$ extending f_K, as desired. $\square$

The next lemma says that we can restrict ourselves to geometrically connected varieties X_K without loss of generality. Denote by $\overline{K}$ an algebraic closure of K. For any closed point $s \in S$ we denote by K_s^{sh} the fraction field of the strict henselization of the local ring $\mathcal{O}_{S,s}$.

Lemma 2.6. *Let X_K be a separated smooth connected variety over K. Let $K' = K(X_K) \cap \overline{K}$ be the field of constants of $K(X_K)$, let S' be the integral closure of S in K' and let $T \subseteq S'$ be the étale locus of $S' \to S$. Then*

1. *X_K is canonically a separated smooth and geometrically connected variety over K';*
2. *X_K/K admits an S-Néron lft-model (resp. S-Néron model) if and only if $\dim T = 0$, or $\dim T = 1$ and X_K/K' admits a T-Néron lft-model (resp. T-Néron model).*

Proof. (1) As $K' \subseteq \mathcal{O}_{X_K}(X_K) \subseteq K(X_K)$ by the normality of X_K, the latter has a canonical structure of K'-variety. As K' is algebraically closed in $K(X_K)$, X_K/K' is geometrically connected. As $X_K \times_{\operatorname{Spec}(K')} \operatorname{Spec}(\overline{K})$ is a connected component of $X_K \times_{\operatorname{Spec}(K)} \operatorname{Spec}(\overline{K})$, X_K/K' is separated and smooth.

(2) Note that because X_K is smooth over K, K'/K is separable, so $S' \to S$ is finite. If $\dim T = 0$, then S is semi-local and $S' \to S$ is ramified at all closed points. This implies that for all closed points $s \in S$, X_K has no K_s^{sh}-point, so X_K is its own Néron model. Suppose now that $\dim T = 1$ and that X_K/K' has a T-Néron lft-model X. Then the composition with $T \to S$ makes X into a smooth

separated S-scheme, with generic fiber X_K/K. Let us check that it satisfies the Néron mapping property. Let $Y \to S$ be a smooth scheme and let $f_K : Y_K \to X_K$ be a K-morphism. Then $Y_K \to \operatorname{Spec}(K)$ also factors through $Y_K \to \operatorname{Spec}(K')$ via $X_K \to \operatorname{Spec}(K')$. In particular, f_K is a K'-morphism. On the other hand, as Y is normal, $Y \to S$ factors through $Y \to S'$. The latter has image in T by Corollary 3.2[3] and makes Y a smooth T-scheme. So f_K extends to a T-morphism $f : Y \to X$, which is *a fortiori* an S-morphism.

Conversely, if X_K/K has an S-Néron lft-model X, the above arguments show that X is canonically a smooth separated T-scheme, and the generic fiber of X_T is nothing but X_K viewed as a scheme over K'. If $\dim T = 1$, the Néron mapping property of $X \to T$ is immediate to verify.

Finally, as $T \to S$ is of finite type and separated, $X \to T$ is of finite type if and only if $X \to S$ is of finite type. $\square$

For the sake of completeness, we include a discussion about Néron models for zero-dimensional schemes.

Proposition 2.7. *Let X_K be a smooth K-variety of dimension zero. Then X_K admits a Néron model over S.*

Proof. By Remark 2.3(3) and Lemma 2.6, we can suppose X_K/K is geometrically connected, smooth of dimension 0. Then $X_K = \operatorname{Spec}(K)$, and S is clearly the Néron model of X_K over S. $\square$

The following proposition will be used to see when a Néron lft-model is a Néron model (*e.g.* in Proposition 7.11).

Proposition 2.8. *Let $X \to S$ be a separated morphism locally of finite type, such that $X \times_S \operatorname{Spec}(\mathcal{O}_{S,s})$ is of finite type for all s.*

(1) *If X is irreducible and X_K is proper over K, then X is of finite type over S.*
(2) *If X_K is affine, X_s irreducible for all $s \in S$ and S is excellent, then X is of finite type over S.*
(3) *If X is of finite type and if X_K/K is geometrically connected, then X_s is geometrically connected for all s in some dense open subset of S.*

Proof. (1) Let U be a quasi-compact open subset of X such that $X_K \subseteq U$, and let U' be a Nagata compactification of $U \to S$. Then U and U' are of finite type over S sharing the same generic fiber. So there exists a dense open subset V of S such that $U \times_S V \cong U' \times_S V$ is proper over V. The inclusion $U \times_S V \to X \times_S V$ is then open and closed. As $X \times_S V$ is irreducible, $X \times_S V = U \times_S V$, thus is of finite type over V. The missing part $X \times_S (S \setminus V)$ is a finite union of quasi-compact subsets, so X is quasi-compact.

(2) As in (1), to prove X is quasi-compact, we are allowed to shrink S. Let U be a quasi-compact open subset of X containing X_K, and let W be an affine finite type S-scheme such that $W_K = X_K$. As U, W are of finite presentation over S with the same generic fiber, shrinking S if necessary, we can suppose that $U = W$ is affine, and that $U \to S$ is surjective (with S affine). We claim that $X = U$. Let U' be any affine open subset of X not contained in U and $F := U' \setminus (U \cap U')$. Then $F \neq \emptyset$, so it has pure codimension 1 in U' because $U \cap U'$ is affine (see [8], Exercise 4.1.15; the hypothesis S excellent implies that the normalization map of U' is finite). As

[3]We remark that Corollary 3.2 does not make use of what we are proving here.

$F_K = \emptyset$, F is then a finite union of vertical divisors, which is impossible since U_s is dense in X_s. Consequently, $X = U$ is of finite type over S.

(3) This follows from [5], IV.9.7.7, after noticing that, as S is irreducible, a dense locally constructible subset of S contains an open dense subset. $\square$

Remark 2.9. In general, the condition $X \times_S \operatorname{Spec}(\mathcal{O}_{S,s})$ of finite type over $\mathcal{O}_{S,s}$ for all s is not sufficient to conclude that X is of finite type over S, as an example of Oesterlé ([1], 10.1/11) shows. See also Remark 7.12.

3. Image of smooth schemes

Let $f\colon Y \to X$ be a morphism of S-schemes. In this section, we study geometric properties of X at the points of $f(Y)$, when Y is smooth over S. The main result is Proposition 3.6 which states that X is smooth at points of $f(Y)$ under some mild hypothesis. This proposition is the principal ingredient of the proof of Theorem 4.1.

Let Y be a scheme. For any morphism locally of finite type $Z \to Y$, we denote by $\operatorname{sm}(Z/Y) \subseteq Z$ the smooth locus of $Z \to Y$. This is an open subset of Z if Y is locally noetherian.

Lemma 3.1. *Let $Z \to Y$ be a morphism locally of finite type. Suppose that Y, Z are locally noetherian and regular. Then for any section $\sigma\colon Y \to Z$, the image $\sigma(Y)$ is contained in the smooth locus $\operatorname{sm}(Z/Y)$ of $Z \to Y$.*

Proof. See [1], 3.1/2. $\square$

Corollary 3.2. *Let S be a locally noetherian regular scheme. Let $f\colon Y \to X$ be a morphism between two S-schemes locally of finite type. Suppose that Y is smooth over S, and that X is regular. Then $f(Y)$ is contained in the smooth locus $\operatorname{sm}(X/S)$ of $X \to S$.*

Proof. Consider the Y-scheme $Z := X \times_S Y$. Notice that Z is regular as Y is smooth over S. The morphism f induces a section $\sigma\colon Y \to Z$, $y \mapsto (f(y), y)$, of the second projection $Z \to Y$. By Lemma 3.1,

$$\sigma(Y) \subseteq \operatorname{sm}(Z/Y) = \operatorname{sm}(X/S) \times_S Y,$$

hence $f(Y) \subseteq \operatorname{sm}(X/S)$. $\square$

Corollary 3.2 does not hold in general if we remove the regularity hypothesis on X. However, in the situation of relative curves, we can weaken the regularity hypothesis to the normality of X (Proposition 3.6). We first prove some preliminary results.

Lemma 3.3. *Let S be an irreducible locally noetherian scheme, and let X, Y be integral flat S-schemes locally of finite type. Let $f\colon Y \to X$ be a dominant S-morphism. Let $s \in S$. Suppose that f is quasi-finite at some point $y_0 \in Y_s$, and that Y_s is irreducible at y_0. Then X_s is irreducible at $x_0 := f(y_0)$.*

Proof. The property is local on X and Y. In particular, shrinking X and Y if necessary, we can suppose that $f\colon Y \to X$ is quasi-finite and separated. Thus f can be factorized as $Y \to \overline{Y} \to X$ with an open (dense) immersion followed by a finite surjective morphism ([5], IV.8.12.6). Let Z_1, Z_2 be two irreducible components of X_s passing through x_0. By the going-down property of $\overline{Y} \to X$ ([10], 5.E.(v)), there exist irreducible closed subschemes F_1, F_2 of $\overline{Y}$ passing through y_0 such that the induced maps $F_i \to Z_i$ ($i = 1, 2$) are finite and dominant (thus surjective). Let

η be the generic point of S. As X, Y are irreducible and flat over S, both X_s and Y_s are equidimensional of dimension $\dim X_\eta = \dim Y_\eta$ ([5], IV.14.2.3). Consequently,

$$\dim F_i = \dim Z_i = \dim_{x_0} X_s = \dim X_\eta = \dim Y_\eta = \dim_{y_0} Y_s$$

and $F_i \cap Y_s = Y_s$. Therefore $Z_1 = Z_2$ and X_s is irreducible at x_0. □

Lemma 3.4. *Let S be a locally noetherian scheme, let X, Y be two S-schemes locally of finite type, and let $f\colon Y \to X$ be a morphism of S-schemes. Consider $s \in S$, and $y_0 \in Y_s$ a closed point of Y_s. Let $x_0 = f(y_0)$. Suppose that Y_s is regular at y_0 and*

$$\dim_{y_0} Y_s > 1 \quad \textit{and} \quad \operatorname{codim}_{y_0}(f^{-1}(x_0), Y_s) > 0.$$

(The second inequality means that in some irreducible neighborhood of y_0 in Y_s, f is non-constant.) Then there exists a subscheme Z of Y passing through y_0 such that

$$\dim_{y_0} Z_s < \dim_{y_0} Y_s \quad \textit{and} \quad \operatorname{codim}_{y_0}(f^{-1}(x_0) \cap Z, Z_s) > 0,$$

and Z_s is regular at y_0. If furthermore Y is regular (resp. if $Y \to S$ is flat) at y_0, we can assume that the same holds for Z.

Proof. We construct Z locally at y_0 as a hypersurface defined by some $u \in \mathfrak{m}_{y_0}\mathcal{O}_{Y,y_0}$ which must avoid some ideals of $\mathcal{O}_{Y,y_0}$. We notice the following facts:

(i) Let $\Gamma_1, \dots, \Gamma_n$ be the irreducible components of $f^{-1}(x_0)$ of codimension 1 in Y_s passing through y_0. Each Γ_i is defined by a prime principal ideal $\bar{t}_i\mathcal{O}_{Y_s,y_0} \subseteq \mathfrak{m}_{y_0}\mathcal{O}_{Y_s,y_0}$ because Y_s is regular at y_0. As $\dim_{y_0} Y_s > 1$, we have $\mathfrak{m}_{y_0}\mathcal{O}_{Y_s,y_0} \not\subseteq \bar{t}_i\mathcal{O}_{Y_s,y_0}$.

(ii) We have $\mathfrak{m}_{y_0}\mathcal{O}_{Y_s,y_0} \not\subseteq \mathfrak{m}_{y_0}^2\mathcal{O}_{Y_s,y_0}$ because Y_s has positive dimension at y_0.

By the prime avoidance lemma ([10], 1.B), there exists

$$\bar{u} \in \mathfrak{m}_{y_0}\mathcal{O}_{Y_s,y_0} \setminus \left(\mathfrak{m}_{y_0}^2\mathcal{O}_{Y_s,y_0} \cup \big(\bigcup_{i \le n} \bar{t}_i\mathcal{O}_{Y_s,y_0}\big) \right).$$

Lift $\bar{u}$ to some $u \in \mathfrak{m}_{y_0}\mathcal{O}_{Y,y_0}$ and let $Z := V(u)$ be the subscheme of Y defined in some open neighborhood of y_0. Then:

(1) Z_s is regular at y_0; if Y is regular (resp. if $Y \to S$ is flat) at y_0, then the same holds for Z because $u \notin \mathfrak{m}_{y_0}^2\mathcal{O}_{Y,y_0}$ and $\bar{u}$ is not a zero divisor;

(2) $\dim_{y_0} Z_s < \dim_{y_0} Y_s$; and

(3) $\operatorname{codim}_{y_0}(f^{-1}(x_0) \cap Z, Z_s) > 0$, because otherwise $V(\bar{u})$ would be contained in, hence equal to, some irreducible component of $f^{-1}(x_0)$, so $\bar{u} \in \bar{t}_i\mathcal{O}_{Y_s,y_0}$ for some $i \le n$, a contradiction.

Therefore Z satisfies the desired properties. □

Lemma 3.5. *Let S be a Dedekind scheme. Let X be a normal relative curve over S,*[4] *and let Y be a regular scheme which is flat and locally of finite type over S. Suppose Y_s is regular. Let $f\colon Y \to X$ be an S-morphism, and $x_0 = f(y_0)$ for some $y_0 \in Y_s$. Suppose that*

$$\operatorname{codim}_{y_0}(f^{-1}(x_0), Y_s) > 0.$$

Then X_s is irreducible and reduced at x_0. If Y_s is geometrically reduced in a neighborhood of y_0, then X_s is geometrically reduced in a neighborhood of x_0.

[4]By a *relative curve* over S, we mean a flat, locally finite type S-scheme with generic fiber of dimension 1.

Proof. Using Lemma 3.4 repeatedly, we find a subscheme Z of Y containing y_0, flat over S, such that Z_s is regular of dimension 1 and $f|_{Z_s}$ is non-constant. This implies that $f|_{Z_s}$ is quasi-finite and $f|_Z$ is dominant. By Lemma 3.3, X_s is irreducible at x_0. Shrinking X and Y if necessary, we can suppose X_s is irreducible. Hence $Y_s \to X_s$ is dominant (because $f|_{Z_s}$ is non-constant) and X_s is reduced at its generic point. But X is normal, hence X is S_2 and X_s is S_1. This implies that X_s is reduced and $Y_s \to X_s$ is scheme-theoretically dominant. Then the same property holds over $\overline{k(s)}$, which implies that X_s is geometrically reduced if Y_s is geometrically reduced. □

Proposition 3.6. *Let S be a Dedekind scheme, and let X be a normal relative curve over S with smooth generic fiber. Let $f\colon Y \to X$ be a morphism with Y smooth and Y_s irreducible for some closed point $s \in S$. Then either $f(Y_s)$ is one point, or X is smooth at every point of $f(Y_s)$.*

Proof. We may assume that $S = \operatorname{Spec}(R)$ is local, and that $f(Y_s)$ is not one point. Then for all $y_0 \in Y_s$ and $x_0 := f(y_0)$, we have $\operatorname{codim}_{y_0}(f^{-1}(x_0), Y_s) > 0$. Let R' be any discrete valuation ring dominating R. Then $X_{R'} := X \otimes_R R'$ has smooth generic fiber, and its special fiber is reduced at any point x_0' lying over x_0 by Lemma 3.5. So $X_{R'}$ is normal at x_0'. Therefore, to prove X is smooth at x_0, we can enlarge R and suppose it is complete with algebraically closed residue field. Using Lemma 3.4, we can suppose that Y is a relative smooth curve. Then f is quasi-finite, and $\operatorname{Spec} \widehat{\mathcal{O}}_{Y,y_0} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X,x_0}$ is finite. By a result of Raynaud ([13], Appendice, p. 195), X is smooth at x_0. □

Remark 3.7. Let X be an integral relative curve over S with smooth generic fiber. Then X admits a minimal desingularization $X' \to X$. See *e.g.* [8], 8.3.50 and 9.3.32, when X is proper over S. As the construction of X' is local on X, and the minimal desingularization is unique, the same result holds for any integral relative curve X over S with smooth generic fiber.

Corollary 3.8. *Let S be a Dedekind scheme, and let X be an integral relative curve over S. Let $f\colon Y \to X$ be an S-morphism from a smooth S-scheme Y to X. Let $s \in S$ be a closed point.*

(1) *If $f(Y_s)$ is reduced to one point $x_0 \in X_s$, then f factors as $Y \to \widetilde{X} \to X$, where the second morphism is the blowing-up of X along the reduced center x_0.*
(2) *Suppose that X_K is smooth, Y is irreducible and that $f_K\colon Y_K \to X_K$ is dominant. Let $X' \to X$ be the minimal desingularization of X (Remark 3.7). Then $f\colon Y \to X$ factors through $X' \to X$.*

Proof. (1) We have to show that $\mathfrak{m}_{x_0}\mathcal{O}_Y$ is an invertible sheaf of ideals of $\mathcal{O}_Y$. Let $y \in f^{-1}(x_0) = Y_s$, and let π be a generator of $\mathfrak{m}_s\mathcal{O}_{S,s}$. We have

$$\sqrt{\mathfrak{m}_{x_0}\mathcal{O}_{Y,y}} = \sqrt{\pi\mathcal{O}_{Y,y}} = \pi\mathcal{O}_{Y,y}$$

(because Y_s is reduced). As $\pi \in \mathfrak{m}_{x_0}\mathcal{O}_{Y,y} \subseteq \pi\mathcal{O}_{Y,y}$, we find $\mathfrak{m}_{x_0}\mathcal{O}_{Y,y} = \pi\mathcal{O}_{Y,y}$. Therefore $\mathfrak{m}_{x_0}\mathcal{O}_Y$ is an invertible sheaf of ideals, and $Y \to X$ factors through $Y \to \widetilde{X} \to X$.

(2) As the minimal desingularization commutes with restriction to open subsets, and because the property to prove is local at Y, we can suppose Y is quasi-compact. Then $f(Y)$ is contained in a quasi-compact open subset of X. Therefore we can also suppose X is quasi-compact.

As Y is normal and f_K is dominant, f factors through the normalization of X. Furthermore, the normalization map of X is finite (see [8], 8.3.49(d)). So we can suppose X is normal.

Let F be the singular locus of X, which is a finite closed subset of X_s. Let $X_1 \to X$ be the blowing-up along F (with the reduced structure). If $F \cap f(Y_s) = \emptyset$, then f trivially factors as $Y \to X_1 \to X$. In general, for any $x_0 \in F \cap f(Y_s)$, it follows easily from Proposition 3.6 that $f^{-1}(x_0)$ is a union of irreducible components of Y_s and, by (1), f factors through $X_1 \to X$. Similarly as above, the morphism $Y \to X_1$ factors through the normalization map $X_1' \to X_1$. Now we start again with $Y \to X_1'$ and the process will stop at the canonical desingularization $\widetilde{X}$ of X, which is made of a finite sequence of normalizations and blowing-ups of closed singular points. As the minimal desingularization $X' \to X$ is obtained by successive contractions of exceptional divisors contained in the exceptional locus of $\widetilde{X} \to X$ (see the proof of [8], 9.3.32), we deduce that f factors through $X' \to X$. $\square$

4. Néron models of proper smooth curves

Let S be a Dedekind scheme with field of functions K. Let X_K be a proper regular and connected curve over K, of positive arithmetic genus (*i.e.*, $\mathrm{H}^1(X_K, \mathcal{O}_{X_K})$ is non-trivial). When the base S is excellent or X_K is smooth over K, X_K admits a unique minimal proper regular model $X_{\min}$ over S (see [2], Theorem 1.2, or [8], 8.3.45 and 9.3.21). Let X_{sm} denote the smooth locus of $X_{\min}$. The aim of this section is to prove the next theorem. See also Proposition 4.12 for a partial result in higher dimension.

Theorem 4.1. *Let S be a Dedekind scheme with field of functions K. Let X_K be a proper regular connected curve of positive arithmetic genus over K. Assume either S is excellent or X_K/K is smooth. Then X_{sm} is the Néron model of the smooth locus $X_{K,\mathrm{sm}}$ of X_K over S.*

We will deduce Theorem 4.1 from the next proposition.

Proposition 4.2 (see also [1], 7.1/6)**.** *Let J_K be a separated connected smooth K-scheme of finite type, and let $X_K \subseteq J_K$ be a connected smooth closed subscheme of dimension one. Assume that J_K admits a Néron lft-model (resp. Néron model) J over S. Then X_K admits a Néron lft-model (resp. Néron model) over S.*

Proof. Let X_0 denote the scheme-theoretic closure of X_K inside J. Let $p : X \to X_0$ be the minimal desingularization of X_0 (Remark 3.7). We want to prove that the smooth locus X_{sm} of $X \to S$ is the Néron lft-model of X_K over S. As the formation of X_{sm} commutes with localization and strict henselization ([8], Proposition 9.3.28), by Proposition 2.4 we can suppose $S = \mathrm{Spec}(R)$ is strictly local (*i.e.*, R is a strictly henselian discrete valuation ring). Let Y be a smooth scheme over S and let $f_K\colon Y_K \to X_K$ be a morphism of K-schemes. We want to extend f_K to a morphism of S-schemes $Y \to X_{\mathrm{sm}}$. We can suppose Y_s is irreducible (Remark 2.3(2)). If $Y_K(K) = \emptyset$, then $Y = Y_K$, and f_K is a morphism from Y to X.

Suppose $Y_K(K) \neq \emptyset$. If $f_K\colon Y_K \to X_K$ is not dominant, the image $f_K(Y_K)$ consists of a rational point P of X_K. The Zariski closure $\overline{\{P\}}$ of $\{P\}$ in X is contained in X_{sm} (Lemma 3.1) and is the image of a section $\sigma\colon S \to X_{\mathrm{sm}}$. Then f_K extends to $Y \to X_{\mathrm{sm}}$ as composition of the structure morphism $Y \to S$ and the section $\sigma\colon S \to X_{\mathrm{sm}}$.

Now suppose f_K is dominant. The morphism $Y_K \to X_K \to J_K$ extends to a dominant morphism $Y \to X_0$. By Corollary 3.8(2), the latter induces a dominant morphism $Y \to X$. Therefore f_K extends to $Y \to X$, hence to $Y \to X_{\mathrm{sm}}$ (Corollary 3.2). This shows that X_{sm} is the Néron lft-model of X_K over S.

If J is of finite type over S, then X_0 and X above are of finite type over S, thus X_{sm} is of finite type. □

The next proposition is well known.

Proposition 4.3. *Let k be a field, and let C be a projective geometrically integral curve over k of arithmetic genus ≥ 1. Let $U \subseteq C$ be the smooth locus of C/k. Then the canonical morphism*

$$U \to \mathrm{Pic}^1_{C/k}, \quad x \mapsto \mathcal{O}_C(x)$$

(given by the invertible sheaf $\mathcal{O}_{C\times_k U}(D)$, where D is the diagonal of $U \times_k U$) is a closed immersion.

Proof. Let $\mathrm{Div}_{C/k}$ be the scheme of effective Cartier divisors on C (see [1], §8.2). Let $\mathrm{Div}^1_{C/k}$ be the subscheme corresponding to effective Cartier divisors of degree 1. Then the canonical morphism $U \to \mathrm{Div}^1_{C/k}$, $x \mapsto x$, is an isomorphism ([6], Exercise 9.3.8).

Let $f : \mathrm{Div}^1_{C/k} \to \mathrm{Pic}^1_{C/k}$ be the restriction of the canonical morphism $\mathrm{Div}_{C/k} \to \mathrm{Pic}_{C/k}$ (corresponding to $E \mapsto \mathcal{O}_C(E)$). It will be enough to show that f is a closed immersion. It is known that f can be identified to $\mathbb{P}(\mathcal{F}) \to \mathrm{Pic}^1_{C/k}$ for some coherent sheaf $\mathcal{F}$ on $\mathrm{Pic}^1_{C/k}$ ([1], Proposition 8.2/7). In particular, f is proper and its fibers are projective spaces. On the other hand, the map $U \to \mathrm{Pic}^1_{C/k}$ is injective because $p_a(C) > 0$. So the fiber of f at any $y \in \mathrm{Im}(f)$ is a projective space of dimension 0 over $k(y)$, hence isomorphic to $\mathrm{Spec}(k(y))$. This implies that f is a proper (hence closed) immersion. □

Proof of Theorem 4.1. As X_{sm} is a finite type scheme over S, it is enough to show X_{sm} is the Néron lft-model of $X_{K,\mathrm{sm}}$. By Corollary 2.5, to show our theorem we can suppose S is local. Consider the closed immersion $f : X_{K,\mathrm{sm}} \to \mathrm{Pic}^1_{X_K/K}$ defined in Proposition 4.3. On the other hand, $\mathrm{Pic}^1_{X_K/K}$ is a torsor under $J_K := \mathrm{Pic}^0_{X_K/K}$, and J_K has no subgroup isomorphic to $\mathbb{G}_{a,K}$ or $\mathbb{G}_{m,K}$ ([14], Proposition 1.1). Hence J_K has a Néron model over S ([1], Theorem 10.2/1) as well as $\mathrm{Pic}^1_{X_K/K}$ ([1], Corollary 6.5/4). So we can conclude with Proposition 4.2. Indeed let N be the Néron model of $X_{K,\mathrm{sm}}$ over S. Embed it into a proper model, and resolve the singularities without modifying the regular locus (which contains N). Then we get a proper regular model N' containing N as an open subset. The identity on X_K extends to a morphism $N' \to X_{\mathrm{min}}$. By Corollary 3.2, this morphism induces a morphism $N \to X_{\mathrm{sm}}$ which is an isomorphism on the generic fiber. Therefore X_{sm} satisfies the Néron mapping property. □

Remark 4.4. Keep the notation of Theorem 4.1.

(1) The Néron model X_{sm} is not necessarily faithfully flat over S. Indeed, if X_{min} has a multiple fiber above some point $s \in S$, then $(X_{\mathrm{sm}})_s = \emptyset$. However, the faithful flatness holds if $X_K(K) \neq \emptyset$.

(2) Assume $X_K(K) \neq \emptyset$, and embed X_K into its Jacobian J_K by using some rational point of X_K. By the uniqueness of the Néron model, Theorem 4.1 (together with Proposition 4.2) shows that the smooth locus of the minimal desingularization of the scheme-theoretic closure $\overline{X_K} \subset J$ is isomorphic to X_{sm}, where J denotes the Néron model of J_K. Note that, in general, even when $X_{\min}$ is semi-stable, $X_{\mathrm{sm}} \to J$ is not an immersion; see [3], Proposition 9.5.

Remark 4.5. Suppose $S = \mathrm{Spec}(R)$ is local. Let $R \to R'$ be an extension of discrete valuation rings of index 1 (*i.e.*, ramification index 1 and separable residue extension) and let $K' = \mathrm{Frac}(R')$. Then under the condition of Theorem 4.1, $X_{\mathrm{sm}} \times_S \mathrm{Spec}(R')$ is the Néron model of $X_{K',\mathrm{sm}}$ over R'. Indeed, the formation of X_{sm} commutes with completion ([8], 9.3.28). Applying Proposition 2.4(3) to the extension $\widehat{R} \to \widehat{R'}$, we see that $X_{\mathrm{sm}} \times_S \mathrm{Spec}(\widehat{R'})$ is the $\widehat{R'}$-Néron model, hence $X_{\mathrm{sm}} \times_S \mathrm{Spec}(R')$ is the Néron R'-Néron model by Proposition 2.4(1).

Remark 4.6. For any smooth connected curve X_K over K, if it can be embedded as a closed subscheme into a semi-abelian K-variety (or more generally, into a smooth K-group scheme of finite type admitting an S-Néron lft-model), Proposition 4.2 implies immediately that X_K admits also an S-Néron lft-model. For example, this works if X_K is proper of positive genus, or if X_K is affine such that the reduced divisor at infinity of X_K in its regular compactification is separable of degree > 1 over K . But this method does not apply for *all* curves: for instance the complement of a rational point in a proper smooth connected curve cannot be embedded as a closed subscheme into any semi-abelian variety over K. In the second part of this work (§5–§7), we propose a different approach which works for any affine curve. Even better, our method allows a very explicit description of the Néron lft-model with the help of a minimal proper regular model of the regular compactification of the affine curve.

Corollary 4.7. *Let S be a Dedekind scheme with field of functions K, and let X_K be a connected smooth proper curve over K of positive genus. Suppose that there exist a proper smooth variety Y_K having good reduction (i.e., having a proper smooth model Y/S) and a surjective morphism $f_K \colon Y_K \to X_K$ over K. Then X_K has good reduction.*

Proof. Let X_{sm} be the Néron model of X_K over S. Then f_K extends to $f : Y \to X_{\mathrm{sm}}$. As Y is proper over S, the image $f(Y)$ is closed in X_{sm} and is dense because it contains X_K. So $f(Y) = X_{\mathrm{sm}}$. Therefore X_{sm} is proper over S, and X_K has good reduction. □

Remark 4.8. One can give a direct proof of Corollary 4.7 if the smooth S-scheme Y/S is assumed to be *projective*. Indeed, it is enough to show $X_{\mathrm{sm}} = X_{\min}$. So one can suppose S is local. Using a Bertini-type result, we may find a smooth closed subscheme of Y/S such that its generic fiber again dominates X_K. Then we can repeat this argument to lower the relative dimension of the scheme Y/S until we find a smooth relative curve over S whose generic fiber still dominates X_K. Finally we only need to apply [9], Corollary 4.10, to conclude.

Remark 4.9. Corollary 4.7 does not hold in general if $g(X_K) = 0$. Let us consider the following example. Let R be a henselian discrete valuation ring with finite residue field k. Let k'/k be a quadratic extension and let $T^2 + aT + b \in R[T]$ be a lifting of the minimal polynomial of a generator of k'/k. Consider the scheme

$$X = \operatorname{Proj} R[x, y, z]/(x^2 + axy + by^2 + \pi z^2)$$

where π is a uniformizing element of R. Let $R' = R[T]/(T^2 + aT + b)$. This is a finite étale extension of R. Let $K = \operatorname{Frac}(R), K' = \operatorname{Frac}(R')$. Then $X_{K'} \cong \mathbb{P}^1_{K'}$. So $X_{K'}$ has good reduction over R', hence over R when it is viewed as a K-scheme (a smooth projective model over R' is also smooth projective over R).

We have a surjective K-morphism $X_{K'} \to X_K$. However, X_K has no good reduction over S. Indeed, suppose X_K has a proper smooth model $P \to S$. Then P_k is a smooth conic over a finite field, hence has a rational point. As R is henselian, then $X_K = P_K$ has a rational point. The latter then specializes to the (unique) rational point of X_k. But this is a singular point of X_k. As X is regular, we have a contradiction by Lemma 3.1.

Corollary 4.10. *Let X_K be a proper smooth connected curve of positive genus over K. Suppose that for some proper smooth connected variety Y_K over K, the product $X_K \times_K Y_K$ has good reduction over S. Then X_K has good reduction over S.*

Proof. Apply Corollary 4.7 to the projection $X_K \times_K Y_K \to X_K$. □

Let $f_K \colon Y_K \to X_K$ be a finite morphism of proper smooth and connected curves over K. Suppose $g(X_K) \geq 1$. Let $Y_{\min}, X_{\min}$ be the respective minimal proper regular models of Y_K, X_K over S. In general, f_K does not extend to a morphism $Y_{\min} \to X_{\min}$ ([9], Remark 4.5). However, Theorem 4.1 immediately implies the next corollary, answering positively a question raised by A. Pirutka.

Corollary 4.11. *Let $f_K : Y_K \to X_K$ be a finite morphism of proper smooth and connected curves over K, with $g(X_K) \geq 1$. Let Y_{sm} (resp. X_{sm}) be the smooth locus of the minimal proper regular model of Y_K (resp. X_K) over S. Then f_K extends to a morphism $f : Y_{\mathrm{sm}} \to X_{\mathrm{sm}}$.*

For the sake of completeness, let us consider the Néron lft-models of curves of genus 0.

Proposition 4.12. *Let X_K be a smooth projective conic over K.*

(1) *If $X_K = \mathbb{P}^1_K$, then X_K does not have Néron lft-model over S.*
(2) *In general, X_K has a Néron lft-model over S if and only if S is semi-local and if $X_K(K_s^{sh}) = \emptyset$ for any closed point $s \in S$, where K_s^{sh} denotes the fraction field of the strict henselization of $\mathcal{O}_{S,s}$. In this case, X_K is its own Néron model over S.*

Proof. (1) We will argue by contradiction: assume that $\mathbb{P}^1_K$ admits a Néron lft-model P over S. By the Néron mapping property, there exists a morphism of S-schemes $f \colon \mathbb{P}^1_S \to P$ extending the canonical identification between the generic fibers. We claim that f is an isomorphism. As $\mathbb{P}^1_S$ is proper, and P/S is separated, the image $f(\mathbb{P}^1_S)$ is a closed subset of P. Its closed fiber has dimension 1 by Chevalley's semi-continuity theorem ([5], IV.13.1.1). Therefore $\mathbb{P}^1_{k(s)} \to P_s$ is quasi-finite. By Zariski's Main Theorem, f is an open immersion. But f is proper and P is irreducible, thus $f : \mathbb{P}^1_S \to P$ is an isomorphism.

On the other hand, there are many endomorphisms of $\mathbb{P}^1_K$ that cannot extend to an endomorphism of $\mathbb{P}^1_S$, hence $\mathbb{P}^1_S$ is not the Néron lft-model of $\mathbb{P}^1_K$, a contradiction.

(2) Assume first that X_K admits a Néron lft-model over S. By Proposition 2.4(2), $X_{K_s^{sh}}$ admits a Néron lft-model over $\mathrm{Spec}(\mathcal{O}_{S,s}^{sh})$. As a result, $X_{K_s^{sh}} \not\cong \mathbb{P}^1_{K_s^{sh}}$. In other words, $X_K(K_s^{sh}) = \emptyset$ for all closed points $s \in S$. On the other hand, this condition implies that S is semi-local. Conversely, assume $X_K(K_s^{sh}) = \emptyset$ for any closed point $s \in S$. As S is semi-local, X_K is of finite type over S and is its own S-Néron model. $\square$

Higher dimension. Let V be an algebraic variety over an algebraically closed field k. We say that V *contains a rational curve* if V has a subscheme isomorphic to an open dense subscheme of $\mathbb{P}^1_k$. It is easy to see that if V does not contain any rational curve, then for any algebraically closed field extension K/k, V_K does not contain any rational curve.

Proposition 4.13. *Let S be a Dedekind scheme with field of functions K. Let X_K be a smooth proper algebraic variety over K. Suppose X_K has a proper regular model X over S such that no geometric fiber $X_{\bar{s}}$, $s \in S$, contains a rational curve. Then the smooth locus X_{sm} of X is the Néron model of X_K.*

Proof. Let Y be an irreducible smooth scheme of finite type over S and $f_K\colon Y_K \to X_K$ a morphism of K-schemes. Consider the Y-scheme $Z := X \times_S Y \to Y$. Its geometric fibers are $Z_{\bar{y}} = X_s \otimes_{k(s)} \overline{k(y)}$ and they do not contain any rational curve. The morphism f_K induces a section $Y_K \to Z_K$, $y \mapsto (f_K(y), y)$, which extends to a section $Y \to Z$ by [4], Proposition 6.2. Composing this section with the projection $Z \to X$ gives a morphism $f\colon Y \to X$ extending f_K. By Corollary 3.2, $f(Y) \subseteq X_{\mathrm{sm}}$. This proves that X_{sm} is the Néron model of X_K over S. $\square$

Remark 4.14. As an application of Proposition 4.13, we recover the well-known fact that an S-abelian scheme A is the Néron model of its generic fiber A_K ([1], Proposition 1.2/8).

5. Néron lft-models of open curves in the local case

Let X_K be a separated smooth connected curve having a Néron lft-model X over S (*e.g.* X_K is proper of positive genus; see Theorem 4.1). Let U_K be an open dense subscheme of X_K. A natural question is whether U_K has a Néron lft-model U and, if it exists, how it is related to X. In this section, we restrict ourselves to the case where $S = \mathrm{Spec}(R)$ is *local*. Then we will show that the answer is positive under mild hypothesis and we describe explicitly the construction of U.

Let R^{sh} denote the strict henselization of R, $\widehat{R^{sh}}$ the completion of R^{sh}, $K^{sh} = \mathrm{Frac}(R^{sh})$ and $\widehat{K^{sh}} = \mathrm{Frac}(\widehat{R^{sh}})$.

5.1. **Main statement.**

Theorem 5.1. *Let X_K be a separated smooth connected curve over K and let U_K be a dense open subscheme of X_K. Suppose that X_K has a smooth model X over S such that $X_{\widehat{R^{sh}}}$ is the Néron lft-model of $X_{\widehat{K^{sh}}}$. Then U_K has a Néron lft-model U over S. Moreover, denoting by $\Delta_K = X_K \setminus U_K$ the boundary of U_K in X_K, U satisfies the following properties:*

(1) *The scheme U is of finite type over S if and only if X is of finite type and if*

$$\Delta_K \cap X_K(\widehat{K^{sh}}) = \emptyset.$$

If S is excellent, the latter condition is also equivalent to

$$\Delta_K \cap X_K(K^{sh}) = \emptyset.$$

(2) *Let Δ be the Zariski closure of Δ_K in X. Then the identity on the generic fiber U_K extends to an open immersion $X \setminus \Delta \to U$, and the open immersion $U_K \to X_K$ extends to a morphism $U \to X$.*
(3) *Let s be the closed point of S and let $k(s)^{\mathrm{sep}}$ be a separable closure of the residue field $k(s)$ of s. Then $U = X \setminus \Delta$ if and only if $\Delta \cap X_s(k(s)^{\mathrm{sep}}) = \emptyset$.*

Remark 5.2. In the statement of Theorem 5.1, the S-model X is necessarily the Néron lft-model of X_K over S (Proposition 2.4(1)), but the requirement in Theorem 5.1 is slightly stronger than this, except when S is excellent (Proposition 2.4(3)).

5.2. **Construction of U.** Let $k = k(s)$. First we construct a (possibly infinite) sequence of blow-ups of X, then define U as a suitable open subset of the resulting scheme. Note that the following construction can be done for any smooth S-scheme X such that U_K is a dense open subset of X_K, and Lemmas 5.3 and 5.4 hold just under these assumptions.

Put $X_0 := X$ and $\Delta_0 := \Delta$.

(i) Let $\Delta_0' = (\Delta_0)_s \cap \mathrm{sm}(X_0)_s(k^{\mathrm{sep}})$, *i.e.*, the subset of points of $(\Delta_0)_s \cap \mathrm{sm}(X_0)_s$ with separable residue field over k. If $\Delta_0' = \emptyset$, we stop.
(ii) Otherwise, let $X_1 \to X_0$ be the blowing-up of X_0 along Δ_0' (endowed with the reduced structure). Let Δ_1 be the Zariski closure of Δ_K in X_1 and let $\Delta_1' = \Delta_1 \cap \mathrm{sm}(X_1)_s(k^{\mathrm{sep}})$. If $\Delta_1' = \emptyset$, we stop.
(iii) Otherwise blow up $X_2 \to X_1$ along Δ_1' (reduced) and start again with the Zariski closure Δ_2 of Δ_K in X_2. We construct in this way a (possibly infinite) sequence of models locally of finite type X_n of X_K, with $\Delta_n \subset X_n$ the Zariski closure of Δ_K in X_n, and $X_{n+1} \to X_n$ is the blowing-up of $\Delta_n' := \Delta_n \cap \mathrm{sm}(X_n)_s(k^{\mathrm{sep}})$.

Let $U_n = \mathrm{sm}(X_n) \setminus \Delta_n$. The identity map on U_K extends to an open immersion $U_n \to U_{n+1}$. More precisely U_n is U_{n+1} minus the the exceptional divisor of $X_{n+1} \to X_n$. Let $U = \bigcup_{n \geq 0} U_n$. This is a smooth, separated scheme locally of finite type over S, with generic fiber isomorphic to U_K. We will show that U is the Néron lft-model of U_K over S. By construction, we have canonical morphisms

$$X \setminus \Delta \hookrightarrow U \to X, \tag{5.1}$$

and the second morphism has image in $(X \setminus \Delta) \cup (\Delta_0')_s$.

Note that the formation of U commutes with any flat extension of discrete valuation rings $R \to R'$ because taking Zariski closure, blowing-up and taking the smooth locus are all compatible with such an extension. To prove U is the Néron lft-model, we can replace R by $\widehat{R^{sh}}$ and suppose R is strictly henselian and excellent (Proposition 2.4). After this reduction, X is the Néron lft-model of X_K in the situation of Theorem 5.1. But in general the curve U_K might be no longer connected (the curve X_K and its Néron lft-model decompose accordingly). In this situation, if we can deal with each connected component of U_K (which is an open subscheme of a connected component of X_K), the general case follows (Remark 2.3(3)). Hence, we may assume that U_K is still connected.

5.3. **Dilatation** ([1], §3.2). Let R be a discrete valuation ring. Let X be any flat R-scheme of finite type. Let E be a closed subscheme of the special fiber X_s defined by a sheaf of ideals $\mathcal{I} \subset \mathcal{O}_X$. Recall that the *dilatation of* E *on* X is obtained by blowing-up $u : \widetilde{X} \to X$ along E, and then taking the open subset of $\widetilde{X}$ where $u^*\mathcal{I}$ is generated by a uniformizing element of R. Denote by X' the dilatation of E. It satisfies the following universal property: for any flat R-scheme Z, a morphism $Z \to X$ factors through $X' \to X$ if and only if $Z_s \to X_s$ factors through $E \to X_s$ as morphism.

If X/R is smooth and if the center E is smooth over k, by a local computation, one can show that the dilatation X' of E on X is smooth over R, and is equal to the complement in $\widetilde{X}$ of the strict transform of X_s.

5.4. Some technical lemmas.

Lemma 5.3. *Keep the notation of* §5.2 *and assume* R *is strictly henselian and excellent. Suppose that the sequence of blowing-ups* $X_{n+1} \to X_n$ *is infinite. Let* $p_1, p_2 \in X_K$ *be closed points such that, for all* $n \in \mathbb{N}$, *they specialize to a same point* $x_n \in \Delta'_n$. *Then* $p_i \in X_K(K)$ *and* $p_1 = p_2$.

Proof. Let us first prove $p_i \in X_K(K)$. Let $p \in \{p_1, p_2\}$. Let $X'_{n+1} \subseteq X_{n+1} \to X_n$ be the dilatation of Δ'_n on X_n and let $P_n = \{p, x_n\}$ be the reduced Zariski closure of p in X_n. By the construction of X_n, x_{n+1} maps to $x_n \in \Delta'_n$ and it is a smooth point of X_{n+1}. So $x_{n+1} \in X'_{n+1}$ and, for all $n \geq 0$, we have a commutative diagram

$$\begin{tikzcd} & X'_{n+1} \arrow[d, hook] \\ P_{n+1} \arrow[ur] \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ P_n \arrow[r] & X_n. \end{tikzcd}$$

The first vertical arrow is birational and finite. Moreover, P_{n+1} is nothing but the strict transform of P_n in X_{n+1}. By the embedded resolution of singularities (see *e.g.*, [8], 9.2.26; recall that S is excellent), there exists $m \geq 1$ such that P_m is a regular scheme. Then $P_{m+1} \to P_m$ is an isomorphism and we have a factorization

$$\begin{tikzcd} & X'_{m+1} \arrow[d] \\ P_m \arrow[ur] \arrow[r, "\sigma_m"] & X_m. \end{tikzcd}$$

By the universal property of the dilatation (§5.3), the closed immersion $\sigma_{m,s} : P_{m,s} \to X_{m,s}$ factors through $\operatorname{Spec} k(x_m) \subset X_{m,s}$. Hence $P_{m,s} \to \operatorname{Spec} k(x_m) = \operatorname{Spec} k$ is an isomorphism. Therefore $P_m \to S$ is an isomorphism and we have $K(p) = K$.

If $p_1 \neq p_2$, by the embedded resolution of singularities of the Zariski closure of $\{p_1, p_2\}$ in X_0, the Zariski closure becomes a disjoint union of sections in some X_n. Contradiction with the hypothesis in the lemma. $\square$

Lemma 5.4. *Keep the notation of §5.2 and suppose R is strictly henselian and excellent.*

(a) *If $(X_n)_n$ is an infinite sequence, and if $(x_n)_n$ is such that $x_n \in \Delta'_n$ and $x_{n+1} \mapsto x_n$ by $X_{n+1} \to X_n$ for all $n \geq 0$, then there exists $p \in \Delta_K$ such that $x_n \in P_n$ (Zariski closure of p in X_n) for all $n \geq 0$.*

(b) *Let Y be a flat integral S-scheme of finite type with irreducible closed fiber Y_s. Let $f_K : Y_K \to X_K$ be a morphism. Suppose that for all $n \geq 1$, f_K extends to $f_n : Y \to X_n$ and that $f_n(Y_s) \subseteq \Delta'_n$. Then $f_K(Y_K) \cap \Delta_K \neq \emptyset$.*

(c) *The scheme U is of finite type over S if and only if X is of finite type and $\Delta_K \cap X_K(K) = \emptyset$.*

Proof. (a) Let $F_n \subseteq \Delta_K$ be the set of points specializing to x_n. Then $(F_n)_n$ is a decreasing sequence of non-empty finite sets, so $\cap_n F_n \neq \emptyset$ and any point p in the intersection satisfies the required property.

(b) As Y_s is irreducible, $f_n(Y_s)$ is reduced to one point x_n and $x_{n+1} \mapsto x_n$ by $X_{n+1} \to X_n$. Let $p \in \Delta_K$ be such that $x_n \in P_n$ for all $n \geq 0$. Let $q \in Y_K$ be a lifting of some closed point of Y_s. Then $f_K(q)$ and p are closed points of X_K having the same specialization $x_n \in \Delta'_n$ for all n. By Lemma 5.3, $p = f_K(q) \in \Delta_K \cap f_K(Y_K)$.

(c) Suppose there exists $p \in \Delta_K \cap X_K(K)$. For any $n \geq 0$ such that X_n is constructed, p specializes to a point of $\Delta_n \cap \mathrm{sm}(X_n)_s(k)$, so X_{n+1} exists in the construction of §5.2. As $X_{n+1} \to X_n$ consists in blowing-up some smooth points, U_{n+1} contains a dense open subset of the exceptional locus of $X_{n+1} \to X_n$. So $U_n \subsetneq U_{n+1}$ and U is not of finite type. On the other hand, if X is not of finite type, then U_1, thus U, is not of finite type.

Conversely, if Δ_K does not contain rational point of X_K, there exists $m \geq 1$ such that no point of Δ_K specializes to a point of $\mathrm{sm}(X_m)_s(k)$ (Lemma 5.3), and the construction of §5.2 stops at this step. Thus $U = U_m$ is of finite type over S if X is of finite type. □

Lemma 5.5. *Let S be a locally noetherian scheme, and let $f : Z \to T$ be a morphism of locally noetherian flat S-schemes. Let $z_0 \in Z_s$ and $t_0 = f(z_0)$. Suppose $Z_s \to T_s$ is flat at z_0. Then the canonical morphism*

$$f_{z_0} : \operatorname{Spec} \mathcal{O}_{Z,z_0} \to \operatorname{Spec} \mathcal{O}_{T,t_0}$$

is surjective. In particular, for any $P \in T$ such that $t_0 \in \overline{\{P\}}$, there exists $Q \in f^{-1}(P)$ such that $z_0 \in \overline{\{Q\}}$.

Proof. By the fiberwise flatness criterion ([5], IV.11.3.10.1), f is flat at z_0 and f_{z_0} is a flat morphism of local schemes, hence faithfully flat. In particular, f_{z_0} is surjective. The last assertion results from the usual interpretation of the images of $\operatorname{Spec} \mathcal{O}_{T,t_0}$ and $\operatorname{Spec} \mathcal{O}_{Z,z_0}$ in T and Z respectively. □

5.5. **Proof of Theorem 5.1.** Now we prove that U is the Néron lft-model of U_K over S. As noticed previously in §5.2, we can suppose S is strictly local and excellent. In particular $k^{\mathrm{sep}} = k$.

Let Y be a smooth scheme over S and let $f_K : Y_K \to U_K$ be a morphism of K-schemes. We want to extend f_K to a morphism $Y \to U$. We can suppose Y_s is irreducible (Remark 2.3). First f_K extends to a morphism $f_0 : Y \to X$ by the hypothesis on X. Consider the sequence $(X_n)_n$ constructed in §5.2.

(A) Suppose that for some $m \geq 0$, f factors through $f_m\colon Y \to X_m$ and that $f_m(Y_s) \not\subseteq \Delta'_m$. Let us show $f_m(Y) \subseteq U_m$ or, equivalently, that $f_m(Y_s) \cap \Delta_m = \emptyset$ because $f_m(Y) \subseteq \mathrm{sm}(X_m)$. If this is true, then f_K extends to $f_m\colon Y \to U_m \subseteq U$ and we are done. We distinguish two cases:

Case 1: *$f_m(Y_s) = \{x_m\}$ is a singleton.* By hypothesis, $x_m \notin \Delta'_m$. As Y_s is smooth and $Y_s \to \operatorname{Spec} k$ factors through $Y_s \to \operatorname{Spec} k(x_m)$, we have $x_m \in \mathrm{sm}(X_m)_s(k)$. So $x_m \notin \Delta_m$ and $f_m(Y_s) \subseteq U_m$.

Case 2: *$f_m(Y_s)$ is not a singleton.* If there exists $x_m \in f_m(Y_s) \cap \Delta_m$, let $y_m \in f_m^{-1}(x_m)$ and let $p \in \Delta_K$ specializing to x_m. As $\mathrm{sm}(X_m)_s$ is a smooth curve and $(f_m)_s$ is dominant, $(f_m)_s$ is flat at y_m. By Lemma 5.5 applied to $Z = Y$ and $T = \mathrm{sm}(X_m)$, we find a closed point $q \in Y_K$ such that $f_K(q) = p$. This contradicts the hypothesis that $f_K(Y_K) \subseteq U_K$.

(B) Now we show that the condition in (A) is satisfied for some $m \geq 0$. We start with $f_0\colon Y \to X_0$. If $f_0(Y_s) \subseteq \Delta'_0$, then $f_0(Y_s) = \{x_0\}$. As in Corollary 3.8(1), f_0 factors through $f_1 : Y \to X_1$. If $f_1(Y_s) \not\subseteq \Delta'_1$, we are done. Otherwise, we have a $f_2\colon Y \to X_2$. Repeating this construction, we see that if (A) is never satisfied, then for all $n \geq 0$, f_K extends to $f_n\colon Y \to X_n$ with $f_n(Y_s) \subseteq \Delta'_n$. This is impossible by Lemma 5.4(b). Hence U is the Néron lft-model of U_K.

It remains to prove the various properties of U. We first remark that if S is excellent, then K^{sh} is algebraically closed in its completion $\widehat{K^{sh}}$, hence the two conditions in Part (1) are indeed equivalent. Part (1) is a direct consequence of Lemma 5.4(c). Parts (2) and (3) follow from the construction in §5.2: $X \setminus \Delta_0 = U_0$ is open in U and its generic fiber is $X_K \setminus \Delta_K = U_K$; and $U = X \setminus \Delta_0$ is equivalent to $\Delta'_0 = \emptyset$. □

As an application of Theorem 5.1, we have the following result which will be used in Proposition 6.1 in the next section.

Proposition 5.6. *Let S be local. Let P be a regular, proper semi-stable model of $\mathbb{P}^1_K$ over S. Let $\Gamma_1, \cdots, \Gamma_n$ be disjoint sections, $n \geq 2$, in the smooth locus $\mathrm{sm}(P/S)$ such that $\Gamma := \cup_i \Gamma_i$ meets every exceptional divisor of P/S when P_s is not irreducible. Let $V_K = \mathbb{P}^1_K \setminus \Gamma_K$. Let V be the S-model over S obtained by the process described in §5.2 starting with $X := \mathrm{sm}(P/S)$. Then V is the S-Néron lft-model of V_K over S. Moreover, the identity $V_K \to V_K$ and the inclusion $V_K \to P_K$ extend to*

$$\mathrm{sm}(P/S) \setminus \Gamma \to V \to \mathrm{sm}(P/S)$$

and the first morphism is an open immersion.

Proof. As the formation of V commutes with completion and strict henselization, by Proposition 2.4(1), we can suppose S is strictly local and excellent. We prove the result by induction on the number of irreducible components of P_s.

First suppose P_s is irreducible. Start with the case $n = 2$. Then $P \cong \mathbb{P}^1_S$. One can see easily that V, which is isomorphic to the Néron lft-model of $\mathbb{G}_{m,K}$, is obtained by the process described in §5.2 with $X := P$. See [1], 10.1.

If P_s is irreducible and $n \geq 3$, we consider $U_K = \mathbb{P}_K \setminus \{(\Gamma_1)_K, (\Gamma_2)_K\}$ with its Néron lft-model U. Then V can be obtained by the process of §5.2 starting with $X := U$. As $P \setminus (\Gamma_1 \cup \Gamma_2)$ is open in U by the above discussions, and the Zariski closure of $(\Gamma_i)_K$ in P is $\Gamma_i \subset P \setminus (\Gamma_1 \cup \Gamma_2)$ when $i \geq 3$, Theorem 5.1 says that V is the S-Néron lft-model of V_K, $P \setminus \Gamma$ is open in V and the latter maps to U, hence to P.

Now suppose P_s has more than one component. Let E be an exceptional divisor in P. Up to renumbering, we can suppose $\Gamma_1, \dots, \Gamma_r$, $r \leq n-1$, are exactly the sections of P among the Γ_i's not meeting E. Let $\pi : P \to Q$ be the contraction of E and let $q = \pi(E) \in Q_s$. Consider

$$U_K = \mathbb{P}^1_K \setminus \{(\Gamma_1)_K, \dots, (\Gamma_r)_K, (\Gamma_{r+1})_K\}.$$

Then Q is regular, proper and semi-stable and, if we still denote by Γ_i the Zariski closure of $(\Gamma_i)_K$ in Q, $\Gamma_1, \dots, \Gamma_r, \Gamma_{r+1}$ correspond to $r+1$ disjoint sections of Q/S whose union meets every exceptional divisor of Q/S when Q_s is not irreducible. By the induction hypothesis, the Néron lft-model U of U_K is obtained by the process of §5.2 starting from $X := \mathrm{sm}(Q/S)$. As $(\Gamma_{r+1})_s = \{q\}$ and $\pi\colon P \to Q$ is the blow-up of Q along $\{q\}$, by the explicit construction of U, $\mathrm{sm}(P) \setminus (\bigcup_{i \leq r+1} \Gamma_i)$ is open in U. For any $i \geq r+2$, the point of $(\Gamma_i)_K$ specializes to a point of $(\mathrm{sm}(P) \setminus (\bigcup_{i \leq r+1} \Gamma_i))_s$. Then Theorem 5.1 tells us that V_K admits an S-Néron lft-model V, and the latter is obtained from U by blowing-up the closed points $\bigcup_{i \geq r+2} (\Gamma_i)_s$ contained in the open subset $\mathrm{sm}(P) \setminus (\bigcup_{i \leq r+1} \Gamma_i) \subset U$, taking the smooth locus and starting again, etc. In particular, $\mathrm{sm}(P/S) \setminus \Gamma$ is an open subscheme of V and the latter maps to $\mathrm{sm}(P/S)$. □

6. Néron models of open subsets of a smooth conic

In this section, we suppose $S = \mathrm{Spec}(R)$ is local and excellent. We prove the existence of the Néron model for affine open subsets of a smooth projective conic, whose complement is non-empty and consists of ramified points.

Proposition 6.1. *Let S be an excellent local Dedekind scheme with field of functions K. Let C_K be a projective smooth conic over K. Let Δ_K be a non-empty finite closed subset of C_K (endowed with the reduced structure) such that $\Delta_K(K^{sh}) = \emptyset$. Then $U_K := C_K \setminus \Delta_K$ admits a Néron model over S.*

Proof. Assume first $C_K(K) \neq \emptyset$, or equivalently, $C_K \cong \mathbb{P}^1_K$. Consider a smooth proper model isomorphic to $\mathbb{P}^1_S$ of $\mathbb{P}^1_K$. After finitely blowing-ups along smooth separable points of the special fiber of the latter, the construction of §5.2 gives us a regular proper semi-stable model P of $C_K \cong \mathbb{P}^1_K$ such that the intersection $\Delta \cap \mathrm{sm}(P/S)_s(k(s)^{\mathrm{sep}})$ is empty. By successively blowing-down exceptional divisors of P which do not meet the Zariski closure Δ of Δ_K, we can suppose that:

(1) Δ meets every exceptional divisor of P/S if P_s is not irreducible and
(2) Δ meets P_s only at singular points or smooth inseparable points.

Claim. Under the above conditions, $U := \mathrm{sm}(P/S) \setminus \Delta$ is the Néron model of U_K.

To prove the claim we can suppose S is strictly local. In particular, each (reduced) irreducible component of P_s is isomorphic to $\mathbb{P}^1_k$. Let Y be a smooth S-scheme with connected fibers and let $f_K : Y_K \to U_K$ be a morphism of K-schemes. Let $\Gamma_1, \dots, \Gamma_n \subset U(S)$ be disjoint sections such that $n \geq 2$, $\Gamma := \bigcup_i \Gamma_i$ is ample in P and such that $f_K(Y_K) \not\subset \Gamma_K$.

Set $H = \overline{f_K^{-1}(\Gamma_K)} \subset Y$. This is a closed subset of Y, empty or of codimension 1. Let $Y' := Y \setminus H$. We claim that the restriction $f_K|_{Y'_K} : Y'_K \to U_K \setminus \Gamma_K$ extends to a morphism $Y' \to U$. Indeed, let V be the Néron lft-model of $V_K := P_K \setminus \Gamma_K$. Then f_K induces a morphism $f' : Y' \to V$. By Proposition 5.6, we have a morphism $V \to \mathrm{sm}(P/S) \setminus \Gamma = U$. Consequently, f' extends to a morphism $f'' : Y'' \to U$ where $Y'' = Y' \cup Y_K$. As Y_s is connected, the image $f''_s \colon Y''_s \to U_s$ is contained in some connected component. Let U' denote the union of the latter connected component of U_s with U_K. Then f'' factors through $U' \subset U$.

On the other hand, the scheme U' can be obtained by first blowing-down successively all the irreducible components of P_s other than the one containing U'_s, then removing from the resulting proper smooth model of $C_K \cong \mathbb{P}^1_K$ the closure of Δ_K. In particular, the scheme U' is affine. Thus f'' extends to a morphism $Y \to U' \subset U$ since $Y \setminus Y''$ is a closed subset of codimension ≥ 2 of the normal scheme Y ([1], 4.4/2). Therefore U satisfies Néron mapping property, as desired.

For the general case, if $X_K(K^{sh}) = \emptyset$, then X_K is the S-Néron model of itself. Otherwise, there exists a finite unramified extension K' of K such that $C_K(K') \neq \emptyset$. Since K'/K is unramified, the complement of $U_{K'}$ in its smooth compactification $C_{K'}$ consists of closed points which are still ramified of degree > 1 over K'. Therefore, $U_{K'}$ admits a Néron model U' over the semi-local ring S', the normalization of S in K'. Hence U_K admits also an S-Néron model by Proposition 7.4 and Proposition 7.5 (which make no use of what we are proving here). □

Remark 6.2. Keep the notation of Proposition 6.1, and let U be the S-Néron model of U_K. Assume $C_K \cong \mathbb{P}^1_K$ with a proper smooth S-model C such that the Zariski closure Δ of Δ_K (with the reduced structure) is regular. Then the canonical morphism obtained from Néron mapping property $C \setminus \Delta \to U$ is an open immersion. Indeed, as Δ is regular, after blowing up all separable closed points of Δ, we obtain a proper semi-stable model P of C_K such that the conditions (1), (2) in the proof of Proposition 6.1 are satisfied. By the claim in the same proof, $U = \mathrm{sm}(P/S) \setminus \overline{\Delta_K}$. Thereby $C \setminus \Delta$ is an open subset of U.

Remark 6.3. In Proposition 6.1, we cannot drop the assumption that S is excellent. For example, let k be a field of characteristic 2, and $K := k(t, u^2) \subset k[[t]] \subset k((t))$ with $u \in k[[t]]$ transcendental over $k(t)$. The discrete valuation on $k((t))$ induces a discrete valuation on K, and let R be the corresponding (discrete) valuation ring (with $t \in R$ a uniformizer). The completion $\widehat{R}$ of R is $k[[t]] \subset k((t))$, hence $\widehat{K} = \mathrm{Frac}(\widehat{R}) = k((t))$. Note that $u \in \widehat{K}$ is purely inseparable of degree 2 over K (in particular, R is not excellent). Consider $v = u^2 \in K$ and let $U_K = \mathrm{Spec}(K[X_1, X_2]/(X_1^2 - vX_0^2 - X_0))$. This is the underlying scheme of a unipotent group. As $v \notin K^2$, $U_K \not\cong \mathbb{A}^1_K$. On the other hand, $U_{\widehat{K}} \cong \mathbb{A}^1_{\widehat{K}}$. It follows that $U_{\widehat{K}} \cong \mathbb{G}_{a,\widehat{K}}$. Therefore, by [1], 10.2/2, U_K does not admit a Néron lft-model over S.

7. Néron lft-models of affine curves

The aim of this section is to prove the existence of Néron lft-models of affine curves over K different from the affine line (Theorem 7.10). Let S be a Dedekind scheme with field of functions K.

7.1. **Globalize local Néron lft-models.**

Lemma 7.1. *Let S be a Dedekind scheme with K its field of functions. Let U_K be a separated connected smooth curve over K. Suppose that*

(i) *for any closed point $s \in S$, U_K admits a Néron lft-model $U(s)$ over $\mathcal{O}_{S,s}$;*
(ii) *there exists a model of finite type U^0 of U_K over S such that for all $s \in S$, the isomorphism $U^0_K \to U(s)_K$ extends to an open immersion $U^0 \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) \to U(s)$.*

Then U_K admits a Néron lft-model over S.

Proof. The proof is inspired from that of [1], 10.1/7. The Néron lft-model of U_K over S will be obtained by gluing the local Néron lft-models $U(s)$ for all closed points $s \in S$. We will first extend $U(s)$ to a model $V(s)$ which coincides with U^0 above $S \setminus \{s\}$ and then glue the various $V(s)$ in a natural way to obtain the Néron lft-model U of U_K over S.

Fix a closed point s. As $U(s)$ is locally of finite type, any connected component $U(s)_{s,\alpha}$ of $U(s)_s$ is open in $U(s)_s$, hence its union with U_K is a quasi-compact open subset $U_\alpha(s)$ of $U(s)$. We can extend $U_\alpha(s)$ to a separated scheme of finite type U_α over S (use [5], IV.8.10.5). As U_α and U^0 are both of finite type over S and have the same generic fiber, they are S-isomorphic over a dense open subset $S_\alpha \subseteq S \setminus \{s\}$ (and the S-isomorphism is unique once the isomorphisms $U^0_K \to U_K$, $(U_\alpha)_K \to U_K$ are fixed because U_α is separated over S). Now we glue the separated morphisms of finite type

$$U_\alpha \times_S (S_\alpha \cup \{s\}) \to S_\alpha \cup \{s\}, \quad U^0 \times_S (S \setminus \{s\}) \to S \setminus \{s\},$$

above $(S_\alpha \cup \{s\}) \cap (S \setminus \{s\}) = S_\alpha$. The resulting S-scheme V_α is separated and of finite type because these properties are satisfied above $S_\alpha \cup \{s\}$ and $S \setminus \{s\}$. By construction, we have canonically

$$V_\alpha \times_S (S \setminus \{s\}) = U^0 \times_S (S \setminus \{s\}), \quad V_\alpha \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) = U_\alpha(s).$$

Next we glue the various V_α (when $U_\alpha(s)$ runs through the connected components of $U(s)_s$) with the condition $V_\alpha \cap V_{\alpha'} = U^0$ if $\alpha \neq \alpha'$. The resulting S-scheme $V(s)$ satisfies canonically

$$V(s) \times_S (S \setminus \{s\}) = U^0 \times_S (S \setminus \{s\}), \quad V(s) \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) = U(s).$$

Hence $V(s)$ is separated and locally of finite type over S. Moreover, condition (ii) implies that the isomorphism $U^0_K \to V(s)_K = U(s)_K$ extends to an open immersion $U^0 \to V(s)$.

Finally, we glue the various $V(s)$ when s runs through the closed points of S with the condition $V(s) \cap V(s') = U^0$ if $s \neq s'$. The resulting S-scheme U is locally of finite type and $U \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) \cong U(s)$ for all $s \in S$. By Corollary 2.5, U is the Néron lft-model of U_K over$S$, as required. $\square$

Lemma 7.2. *Let S be a Dedekind scheme. Let X_K be a smooth connected separated curve over K and let U_K be an open dense subscheme of X_K. Suppose that X_K has a smooth model X over S such that for all closed points $s \in S$, $X \times_S \operatorname{Spec}(\mathcal{O}_{S,s})$ satisfies the property in Theorem 5.1. Then U_K admits a Néron lft-model over S.*

Proof. Let $\Delta_K := X_K \setminus U_K$, $X^0 \subset X$ any quasi-compact open subset containing X_K, and $\Delta^0 := \overline{\Delta_K} \subset X^0$. By Theorem 5.1, the Néron model of U_K over $\operatorname{Spec}(\mathcal{O}_{S,s})$ exists for all closed point $s \in S$, and by Theorem 5.1(2), $U^0 := X^0 \setminus \Delta^0$

verifies the hypothesis of Lemma 7.1. Thus we can apply Lemma 7.1 to conclude. □

Proposition 7.3. *Let S be a Dedekind scheme. Let X_K be a connected regular proper curve over K of arithmetic genus ≥ 1. Let U_K be a dense open subscheme of X_K contained in the smooth locus of X_K/K. Suppose either S is excellent or the scheme X_K is smooth over K. Then U_K admits a Néron lft-model over S.*

Proof. Let $X = X_{\mathrm{sm}}$ be the smooth locus of the minimal proper regular model of X_K over S. Let $s \in S$ be a closed point. Then $X \times_S \widehat{\mathcal{O}_{S,s}^{sh}}$ is the Néron model of X_K over $\widehat{\mathcal{O}_{S,s}^{sh}}$ by Theorem 4.1 and because in both cases the minimal proper regular as well as its smooth locus commute with strict henselization and completion ([8], 9.3.28). By Lemma 7.2, U_K has a Néron lft-model U over S. □

7.2. **Weil restriction.**

Proposition 7.4 (see also [1], 10.1/4)**.** *Let S be a Dedekind scheme with field of functions K and let X_K be a separated smooth connected curve over K. Let K'/K be a finite extension, and let S' be the normalization of S in K'. Assume that*

(i) *$S' \to S$ is finite (e.g., if S is excellent or K'/K is separable);*
(ii) *$X_{K'}$ admits a Néron lft-model (resp. Néron model) X' over S';*
(iii) *any quasi-compact open subset of X' is quasi-projective over S'.*

Then X_K also admits a Néron lft-model (resp. Néron model) over S.

Proof. Let $s \in S$. Then any finite subset F of $X'_s := X' \times_S \operatorname{Spec} k(s)$ is contained in an affine open subset of X'. Indeed, F is contained in a quasi-compact open subset W of X'. Let V be an affine open neighborhood of s. As $S' \times_S V$ is finite over V, $W \times_S V = W \times_{S'} (S' \times_S V)$ is quasi-projective over V, hence F is contained in an affine open subset of $W \times_S V \subseteq X'$.

The morphism $S' \to S$ is finite and locally free, hence the above property implies that the Weil restriction functor $Y := \mathrm{Res}_{S'/S} X'$ is representable by a smooth S-scheme locally of finite type ([1], 7.6/4). Furthermore, by the functoriality of the Weil restriction functor, one easily checks that Y is the S-Néron lft-model of its generic fiber $Y_K \cong \mathrm{Res}_{K'/K}(X_{K'})$. Finally, remark that as X_K/K is separated, the adjunction map $X_K \to \mathrm{Res}_{K'/K}(X_{K'}) = Y_K$ is a closed immersion, hence it suffices to apply Proposition 4.2 to conclude the existence of the Néron lft-model or the Néron model of X_K. □

The following result is useful when we want to check the condition (iii) of Proposition 7.4.

Proposition 7.5 (Quasi-projectivity)**.** *Let S be a Dedekind scheme with field of functions K, and let U be a connected regular relative curve over S locally of finite type. Suppose that either U_K has a smooth compactification or S is excellent. Then any quasi-compact open subset of U is quasi-projective over S (in the sense of* [5], II.5.3.1*).*

Proof. Let U_0 be a quasi-compact open subset of U. Let $U_0 \subseteq U'_0$ be a Nagata compactification. The hypothesis on U_K or S implies that there exists a desingularization morphism $Z \to U'_0$ which is an isomorphism above U_0. So U_0 is isomorphic to an open subscheme of a regular proper flat S-scheme Z. It is enough to show that $Z \to S$ is projective. This is a theorem of Lichtenbaum when S is affine

([7], Theorem 2.8, or [8], 8.3.16), but the proof works exactly in the same way in the general case: find a positive horizontal Weil divisor H on Z which meets all irreducible components of all fibers of $Z \to S$. As Z is regular, H is defined by an invertible sheaf $\mathcal{L}$ on Z. The hypothesis on H implies that $\mathcal{L}$ is fiberwise ample, hence $\mathcal{L}$ is relatively ample for $Z \to S$. □

Remark 7.6. If the Dedekind scheme S is separated, then any quasi-projective scheme over S is a subscheme of some $\mathbb{P}^N_S$. Indeed, S then has an invertible ample sheaf ([15, Proposition 09NZ]), and one can conclude with [5], II.5.3.3.

7.3. **Affine open subsets of a conic.** In this subsection, we discuss the existence of Néron lft-models of an affine open subset U_K of a smooth projective conic C_K/K. Observe first that $\mathbb{A}^1_K$ does not admit a Néron lft-model over S ([1], 10.1/8). Therefore, we only need to consider the case where U_K is not isomorphic to $\mathbb{A}^1_K$.

Proposition 7.7. *Let S be an excellent Dedekind scheme with field of functions K. Let U_K be an affine open subset of a smooth projective conic C_K over K. Suppose U_K is not isomorphic to $\mathbb{A}^1_K$. Then U_K admits a Néron lft-model U over S.*

Proof. If over an algebraic closure $\overline{K}$ of K, $C_{\overline{K}} \setminus U_{\overline{K}}$ contains at least two points, then there exists a finite extension K'/K such that $U_{K'}$ is isomorphic to an open subscheme of $\mathbb{G}_{m,K'}$. It follows from Proposition 7.4 that we can suppose U_K is an open subscheme of $\mathbb{G}_{m,K}$. The latter has a Néron lft-model G over S, locally on S compatible with any index 1 extension (see the construction of [1], 10.1/5). Consequently, by Lemma 7.2, U_K admits a Néron lft-model over S. For the rest of the proof, we can therefore suppose that $U_{\overline{K}}$ is $C_{\overline{K}}$ minus one point. So $\Delta_K := C_K \setminus U_K$ consists of a single point q_∞ which is purely inseparable of degree > 1 over K because $U_K \not\cong \mathbb{A}^1_K$.

As C_K is smooth over K, there exists a separable extension K'/K such that $C_{K'} \cong \mathbb{P}^1_{K'}$. The point of $C_{K'} \setminus U_{K'}$ is still purely inseparable of degree > 1 over K' because K'/K is separable. Using Proposition 7.4, we can reduce to the case $C_K = \mathbb{P}^1_K$. Let $P \cong \mathbb{P}^1_S$ be a smooth proper model of $\mathbb{P}^1_K$ over S, and let $\Delta = \overline{\{q_\infty\}} \hookrightarrow P$. We know (Proposition 6.1) that for all closed points $s \in S$, U_K admits a Néron model $U(s)$ over $\mathcal{O}_{S,s}$. To find a global Néron lft-model, it is enough to show that for $U^0 := P \setminus \Delta$, the canonical morphism

$$U^0 \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) \to U(s) \tag{7.1}$$

is an open immersion for all s contained in a dense open subset $V \subset S$: the base change $U^0 \times_S V$ then satisfies condition (ii) of Lemma 7.1. As S is excellent, so is Δ. Thus the regular locus of Δ is open in Δ. Shrinking S if necessary, we can assume Δ is regular. Then for any closed point $s \in S$, the morphism (7.1) is an open immersion by Proposition 6.1 and Remark 6.2, and the proposition is proved. □

Corollary 7.8. *Let S be an excellent Dedekind scheme of characteristic $p > 0$, with K its field of functions. Let G_K be a connected smooth K-wound unipotent group of dimension* 1. *Then G_K admits a Néron lft-model over S.*

Remark 7.9. Let S be a Dedekind scheme with field of functions K. One can deduce from Proposition 4.12 and Proposition 4.2 that if X_K is a connected separated smooth K-variety admitting Néron lft-model over S, then X_K does not contain any closed subscheme isomorphic to $\mathbb{P}^1_K$ or $\mathbb{A}^1_K$. Conversely, if X_K is the underlying scheme of a smooth commutative algebraic group over K, the latter condition is also sufficient for the existence of a Néron lft-model when S is local and excellent ([1], 10.2/2).

When S is global and excellent, whether this latter condition is sufficient is still an open question. It is conjectured ([1], 10.3, Conjecture I) that the answer is yes. Some positive examples are known in [1], Chap. 10. Corollary 7.8 provides some evidence in favor of this conjecture. Together with the well-known results for abelian varieties and for tori, we deduce that over any excellent Dedekind scheme S with field of functions K, any smooth connected K-algebraic group G_K of dimension 1 admits a Néron lft-model over S if and only if G_K is not isomorphic to $\mathbb{G}_{a,K}$. In other words, the Conjecture I of 10.3 [1] holds when $\dim(G_K) = 1$. When the unipotent group scheme G_K in Corollary 7.8 admits a regular compactification of genus ≥ 1, or equivalently when $\mathrm{uni}(G_K) = 0$, Corollary 7.8 is a special case of [1], 10.3/5. So the new case provided here is when $\mathrm{uni}(G_K) > 0$, or equivalently when G_K admits a smooth compactification of genus 0. The latter happens only when $\mathrm{char}(K) = 2$ (see the last paragraph of [1], 10.3, p. 316). In this case, G_K is the subgroup of $\mathbb{G}_{a,K}^2 = \mathrm{Spec}(K[X,Y])$ defined by equation $X^2 = Y + aY^2$ for some $a \in K \setminus K^2$, which, as a scheme, is isomorphic to $\mathrm{Proj}(K[T,T']) \setminus V_+(T^2 - aT'^2)$.

7.4. **Néron lft-models for affine curves.** We are now in the position to prove the existence of Néron lft-models for affine curves.

Theorem 7.10. *Let S be an excellent Dedekind scheme with field of functions K. Let U_K be an affine smooth connected curve over K. Then U_K admits a Néron lft-model over S if $U_K \not\cong \mathbb{A}^1_L$ for any finite extension L/K.*

Proof. By Lemma 2.6, we can suppose U_K is geometrically connected. If the regular compactification of U has positive genus, then U_K has a Néron lft-model by Proposition 7.3. Otherwise, U_K is an affine open subset of a smooth projective conic over K, not isomorphic to $\mathbb{A}^1_K$. So U_K admits a Néron lft-model over S by Proposition 7.7. □

Next we examine when the Néron lft-model is of finite type.

Proposition 7.11. *Let S be an excellent Dedekind scheme with field of functions K. Let U_K be an affine smooth geometrically connected curve of K, not isomorphic to $\mathbb{A}^1_K$, and let C_K be its regular compactification. Denote by $\Delta_K := C_K \setminus U_K$. Let C be a relatively minimal regular model of C_K over S and let Δ be the reduced Zariski closure of Δ_K in C. Let U be the Néron lft-model of U_K over S. Then the following properties are true:*

1. *The scheme U/S is of finite type if and only if $\Delta_K(K_s^{sh}) = \emptyset$ for all closed points $s \in S$ and if $\Delta_s \cap C_{\mathrm{sm},s}(k(s)^{\mathrm{sep}}) = \emptyset$ for almost all $s \in S$.*
2. *Assume S is infinite. For each closed point $s \in S$, set $U(s) := U \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$, the local Néron lft-model of U_K over $\mathrm{Spec}(\mathcal{O}_{S,s})$. Let K^{sep} denote a separable closure of K.*
   - (i) *$\Delta_K(K^{\mathrm{sep}}) = \emptyset$ if and only if all the local Néron lft-models $U(s)$ are of finite type.*
   - (ii) *If $\Delta_K(K^{\mathrm{sep}}) \neq \emptyset$, the local Néron lft-models $U(s)$ are not of finite type for all but finitely many closed points $s \in S$. In particular, if K has characteristic 0, then U is never of finite type.*

Proof. (1) The S-scheme U is of finite type if and only if

(a) for all closed points $s \in S$, $U(s)$ is of finite type over $\mathcal{O}_{S,s}$; and if
(b) U is of finite type over some open dense subset of S, or equivalently, U_s is connected for all but finitely many s (Proposition 2.8).

Therefore we only need to show that the conditions of (1) are equivalent to the conditions (a) and (b) above.

First assume that C_K is of genus > 0. Then $U_K \subseteq X_K := \mathrm{sm}(C_K/K)$. Let X be the Néron model of X_K over S, equal to the smooth locus C_{sm} of C/S (Theorem 4.1). The minimal regular model C commutes with strict henselization and completion ([8], 9.3.28), so by Theorem 5.1, for any closed point $s \in S$, $U(s)$ is of finite type over $\mathcal{O}_{S,s}$ if and only if $\Delta_K(K_s^{sh}) = \emptyset$. Now consider the connectedness at a closed point $s \in S$. Shrinking S if necessary, we can suppose $X_s = C_{\mathrm{sm},s}$ is connected. If $\Delta_s \cap C_{\mathrm{sm},s}(k(s)^{\mathrm{sep}}) = \emptyset$, then $U_s = X_s \setminus \Delta_s$ (Theorem 5.1(3)) is connected. Conversely, suppose U_s is connected. By separatedness, $U_s \to X_s$ is an open immersion. Then over $\mathcal{O}_{S,s}$, $U \to X$ is an open immersion, as $X \setminus U$ contains Δ_K, hence $U \subseteq X \setminus \Delta$. By (5.1), we have $U = X \setminus \Delta$ (over $\mathcal{O}_{S,s}$), thus $\Delta_s \cap C_{\mathrm{sm},s}(k(s)^{\mathrm{sep}}) = \emptyset$ (Theorem 5.1(3)).

Now suppose C_K is a smooth conic. Let $s \in S$ be a closed point. If $\Delta_K(K_s^{sh}) = \emptyset$, then $U(s)$ is of finite type over $\mathcal{O}_{S,s}$ by Proposition 6.1. Conversely, suppose $\Delta_K(K_s^{sh}) \neq \emptyset$. It is enough to show $U(s)$ is not of finite type over $\mathcal{O}_{S,s}^{sh}$. Thus we can suppose K is strictly henselian and $\Delta_K(K) \neq \emptyset$. If there are two rational points in Δ_K, then U_K is isomorphic to an open subscheme of $\mathbb{G}_{m,K}$, hence $U(s)$ is not of finite type by Theorem 5.1(1). Otherwise, there exists a non-rational point $q_\infty \in \Delta_K$. We apply again Theorem 5.1(1) to $U_K \subset C_K \setminus \{q_\infty\}$ to conclude that $U(s)$ is not of finite type. So again $U(s)$ is of finite type if and only if $\Delta_K(K_s^{sh}) = \emptyset$.

To see the equivalence between condition (b) above and the second condition of (1), we are allowed to shrink S and suppose that C/S is smooth with connected fibers. The condition $\Delta_s \cap C_{\mathrm{sm},s}(k(s)^{\mathrm{sep}}) = \emptyset$ then implies that $C \setminus \Delta$ is the Néron model of U_K over $\mathcal{O}_{S,s}$ (see Claim in the proof of Proposition 6.1). Thus $U = C \setminus \Delta$ is of finite type with connected fibers over S. Conversely, suppose U/S is of finite type (up to replacing S by some open dense subset). Shrinking S if necessary, the isomorphism $U_K \to (C \setminus \Delta)_K$ extends to an isomorphism $U \cong C \setminus \Delta$. If there exists $x \in \Delta_s \cap C_{\mathrm{sm},s}(k(s)^{\mathrm{sep}})$, then there exists $y \in U_K(K_s^{sh}) = (C \setminus \Delta)_K(K_s^{sh})$ which does not specialize to U_s. Contradiction with the Néron mapping property of U.

(2) As S is infinite, $\Delta_K(K^{\mathrm{sep}}) \neq \emptyset$ if and only if $\Delta_K(K_s^{sh}) \neq \emptyset$ for some (hence for almost all) $s \in S$. Thereby (2) follows directly from (1). □

Remark 7.12. In general it is not true that if $\Delta_K(K^{\mathrm{sep}}) = \emptyset$, then U is of finite type. One can construct examples similar to that of Oesterlé ([1], 10.1/11), showing that the existence of a local Néron models does not imply the existence of a global Néron model. Let S be an excellent Dedekind scheme of characteristic $p > 0$ with infinitely many closed points, such that each closed point of S has perfect residue field (for example, take S a smooth algebraic curve over a perfect field of characteristic p). Let $q_\infty \in \mathbb{P}^1_K$ be a purely inseparable closed point of degree > 1. Let $U_K = \mathbb{P}^1_K \setminus \{q_\infty\}$ and let U be its S-Néron model. Then $U \times_S \mathrm{Spec}(\mathcal{O}_{S,s})$ is of finite type. But as U_s has two connected components for almost all $s \in S$, U is not of finite type over S by Proposition 2.8(3).

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