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DUALITY OF MULTI-PARAMETER TRIEBEL-LIZORKIN SPACES ASSOCIATED WITH THE COMPOSITION OF TWO SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, we study the duality theory of the multi-parameter Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ associated with the composition of two singular integral operators on $\mathbb{R}^m$ of different homogeneities. Such composition of two singular operators was considered by Phong and Stein in 1982. For $1 < p < \infty$, we establish the dual spaces of such spaces as $(\dot{F}_p^{\alpha,q}(\mathbb{R}^m))^* = \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^m)$, and for $0 < p \leq 1$ we prove $(\dot{F}_p^{\alpha,q}(\mathbb{R}^m))^* = CMO_p^{-\alpha,q'}(\mathbb{R}^m)$. We then prove the boundedness of the composition of two Calderón-Zygmund singular integral operators with different homogeneities on the spaces $CMO_p^{-\alpha,q'}$. Surprisingly, such dual spaces are substantially different from those for the classical one-parameter Triebel-Lizorkin spaces $\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m)$. Our work requires more complicated analysis associated with the underlying geometry generated by the multi-parameter structures of the composition of two singular integral operators with different homogeneities. Therefore, it is more difficult to deal with than the duality result of the Triebel-Lizorkin spaces in the one-paramter settings. We note that for $0 < p \leq 1$, $q = 2$ and $\alpha = 0$, $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is the Hardy space associated with the composition of two singular operators considered in Rev. Mat. Iberoam. **29** (2013), 1127–1157. Our work appears to be the first effort on duality for Triebel-Lizorkin spaces in the multi-parameter setting.

1. Introduction

The classical theory of one-parameter harmonic analysis may be considered as centering around the Hardy-Littlewood maximal operator and its relationship with certain singular integral operators which commute with the usual one-parameter dilations on $\mathbb{R}^m$, given by $\delta(x) = (\delta x_1, \cdots, \delta x_m), \delta > 0$. If this isotropic dilation is replaced by more general non-isotropic groups of dilations, then many non-isotropic variants of the classical theories can be produced, such as the strong maximal functions, multi-parameter singular integral operators, corresponding to the multi-parameter dilations $\delta : x \to (\delta_1 x_1, \delta_2 x_2), x = (x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^m, \delta = (\delta_1, \delta_2), \delta_1, \delta_1 > 0$. Such a multi-parameter theory has been developed extensively over the past decades. We refer the reader to the work in [2–4, 8, 15, 17–21, 27, 32,

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33, 37–39, 42, 52, 54, 56]. Multi-parameter flag singular integrals and their boundedness on L^p and H^p spaces have been studied in [16, 31, 34, 45–48, 50, 55], multi-parameter and multi-linear Coifman-Meyer Fourier multipliers have been investigated in [5, 9, 10, 35, 43, 44], and a theory of multi-parameter singular Radon transforms have been developed in [58–60].

Recently, the authors of [30] developed a theory of new multi-parameter Hardy space associated with the composition of two singular integral operators with different homogeneities and established the boundedness of the composition of such singular integrals on this space. To be more precise, for $\mathbb{R}^m = \mathbb{R}^{m-1} \times \mathbb{R}$ with $x = (x', x_m)$ where $x' \in \mathbb{R}^{m-1}$ and $x_m \in \mathbb{R}$, they consider two kinds of homogeneities:

$$\delta : (x', x_m) \to (\delta x', \delta x_m), \delta > 0,$$

and

$$\delta : (x', x_m) \to (\delta x', \delta^2 x_m), \delta > 0.$$

The first is the classical isotropic dilations occurring in the classical Calderón-Zygmund singular integrals, while the second is non-isotropic and related to the heat equations (also Heisenberg groups). For $x = (x', x_m) \in \mathbb{R}^{m-1} \times \mathbb{R}$, denote $|x|_e = (|x'|^2 + |x_m|^2)^{\frac{1}{2}}$ and $|x|_h = (|x'|^2 + |x_m|)^{\frac{1}{2}}$. The singular integrals considered in [30] are defined in the following.

Definition 1.1. A locally integrable function $\mathcal{K}_1$ on $\mathbb{R}^m \backslash \{0\}$ is said to be a Calderón-Zygmund kernel associated with the isotropic homogeneity if

$$\left| \frac{\partial^\alpha}{\partial x^\alpha} \mathcal{K}_1(x) \right| \leq A |x|_e^{-m-|\alpha|} \qquad \text{for all } |\alpha| \geq 0, \tag{1.1}$$

$$\int_{r_1 < |x|_e < r_2} \mathcal{K}_1(x)\, dx = 0 \tag{1.2}$$

for all $0 < r_1 < r_2 < \infty$.

An operator T_1 is said to be a Calderón-Zygmund singular integral operator associated with the isotropic homogeneity if $T_1(f)(x) = p.v.(\mathcal{K}_1 * f)(x)$, where $\mathcal{K}_1$ satisfies conditions in (1.1) and (1.2).

Definition 1.2. Suppose $\mathcal{K}_2 \in L^1_{loc}(\mathbb{R}^m \backslash \{0\})$. $\mathcal{K}_2$ is said to be a Calderón-Zygmund kernel associated with the non-isotropic homogeneity if

$$\left| \frac{\partial^\alpha}{\partial (x')^\alpha} \frac{\partial^\beta}{\partial (x_m)^\beta} \mathcal{K}_2(x', x_m) \right| \leq B |x|_h^{-m-1-|\alpha|-2\beta} \qquad \forall \ |\alpha| \geq 0, \ \beta \geq 0, \tag{1.3}$$

$$\int_{r_1 < |x|_h < r_2} \mathcal{K}_2(x)\, dx = 0 \tag{1.4}$$

for all $0 < r_1 < r_2 < \infty$.

An operator T_2 is said to be a Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity if $T_2(f)(x) = p.v.(\mathcal{K}_2 * f)(x)$, where $\mathcal{K}_2$ satisfies the conditions in (1.3) and (1.4).

Both the classical Calderón-Zygmund theory and theory of singular integral operators associated with the non-isotropic dilations indicate that both the operators T_1 and T_2 are bounded on L^p for $1 < p < \infty$ and of weak-type $(1, 1)$. Nevertheless, it is shown by Phong and Stein in [51] that in general the composition operator $T_1 \circ T_2$ is not of weak-type (1,1). Moreover, the authors of [51] gave a necessary and

sufficient condition such that the composition operator $T_1 \circ T_2$ is of weak-type (1,1). This answers the question raised by Rivieré in [64]. In fact, the operators studied in [51] are compositions with different homogeneities, and such a composition operator arises naturally in the study of the $\bar{\partial}$-Neumann problem.

It is also well-known that any Calderón-Zygmund singular integral operator associated with the isotropic homogeneity is bounded on the classical Hardy space $H^p(\mathbb{R}^m)$ with $0 < p \leq 1$. A Calderón-Zygmund singular integral operator associated with the non-isotropic homogeneity is not bounded on the classical Hardy space but bounded on the non-isotropic Hardy space (see e.g. [23]). However, the composition operator $T_1 \circ T_2$ is bounded on neither the classical Hardy space nor the non-isotropic Hardy space. Thus, the natural question is to ask on what Hardy space can the composition operator $T_1 \circ T_2$ be bounded? To this end, the authors of [30] introduced a new Hardy space $H^p_{hom}(\mathbb{R}^n)$ associated with the composition of these two different homogeneities and proved that $T_1 \circ T_2$ is indeed bounded on such spaces. Recently, the first author developed in [11] the theory of the Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_p(\mathbb{R}^m)$ associated with the composition of these different homogeneities. Such Triebel-Lizorkin spaces for $0 < p \leq 1$, $\alpha_1 = \alpha_2 = 0$ and $q = 2$ are the Hardy spaces $H^p_{hom}(\mathbb{R}^n)$ considered in [30]. Triebel-Lizorkin spaces form a unifying class of function spaces encompassing many well studied classical function spaces such as Lebesgue spaces, Hardy spaces, the Lipschitz spaces, and the space BMO [22, 62]. Boundedness of singular integrals and pseudo-differential operators on the Triebel-Lizorkin spaces have also been extensively studied; see, for example, Frazier and Jawerth [22] and Torres [61].

The main goals of this paper are to identify the dual spaces $CMO_p^{-\alpha,q'}$ of the new Triebel-Lizorkin spaces $\dot{F}^{\alpha,q}_p(\mathbb{R}^m)$.

We now introduce the new Triebel-Lizorkin spaces associated with different homogeneities. Denote $\mathcal{S}_0(\mathbb{R}^m) = \{f \in \mathcal{S}(\mathbb{R}^m) : \int_{\mathbb{R}^m} f(x)x^\alpha dx = 0, \ \forall \ |\alpha| \geq 0\}$. Let $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\operatorname{supp}\widehat{\psi^{(1)}} \subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_e \leq 2\} \tag{1.5}$$

and

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}\xi', 2^{-j}\xi_m)|^2 = 1 \ \text{ for all } (\xi', \xi_m) \in \mathbb{R}^m \setminus \{0\}. \tag{1.6}$$

Let $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\operatorname{supp}\widehat{\psi^{(2)}} \subseteq \{(\xi', \xi_m) \in \mathbb{R}^{m-1} \times \mathbb{R} : \frac{1}{2} \leq |\xi|_h \leq 2^{1/2}\} \tag{1.7}$$

and

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}\xi', 2^{-2k}\xi_m)|^2 = 1 \ \text{ for all } (\xi', \xi_m) \in \mathbb{R}^m \setminus \{0\}. \tag{1.8}$$

Denote $\psi_{j,k}(x) = \psi^{(1)}_j * \psi^{(2)}_k(x)$, where $\psi^{(1)}_j(x', x_m) = 2^{jm}\psi^{(1)}(2^j x', 2^j x_m)$, $\psi^{(2)}_k(x', x_m) = 2^{k(m+1)}\psi^{(2)}(2^k x', 2^{2k} x_m)$, and $j \wedge k = \min\{j, k\}$, $j \vee k = \max\{j, k\}$. The following discrete Calderón reproducing formula is from [30].

Theorem A. *Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in* (1.5)-(1.6) *and* (1.7)-(1.8)*, respectively. Then*

(1.9)
$$\begin{aligned} f(x', x_m) &= \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} 2^{-(m-1)(j\wedge k)} 2^{-(j\wedge 2k)} (\psi_{j,k} * f)(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m) \\ &\qquad \times \psi_{j,k}(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m), \end{aligned}$$

where the series converges in $L^2(\mathbb{R}^m)$, $\mathcal{S}_0(\mathbb{R}^m)$ and $\mathcal{S}_0'(\mathbb{R}^m)$.

Remark 1.3. Actually, in the proof of Theorem A, the authors of [30] have used the additional assumptions that $\psi^{(1)}$ and $\psi^{(2)}$ are real and radial Schwartz functions. Dispensing with these assumptions, (1.9) should be

$$\begin{aligned} f(x', x_m) &= \sum_{j,k\in\mathbb{Z}} \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} 2^{-(m-1)(j\wedge k)} 2^{-(j\wedge 2k)} (\tilde{\psi}_{j,k} * f)(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m) \\ &\qquad \times \psi_{j,k}(x' - 2^{-(j\wedge k)}\ell', x_m - 2^{-(j\wedge 2k)}\ell_m), \end{aligned}$$

where $\tilde{\psi}_{j,k}(x) = \overline{\psi_{j,k}(-x)}$, which is a generalization of Lemma 2.1 in [22].

For $j, k \in \mathbb{Z}$, we denote $\Pi_{j,k} = \{R = I \times J : I$ are dyadic cubes in $\mathbb{R}^{m-1}$, J are dyadic intervals in $\mathbb{R}$, with the side lengths $l(I) = 2^{-(j\wedge k)}$ and $l(J) = 2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $x_I = 2^{-(j\wedge k)}\ell'$ and $x_J = 2^{-(j\wedge 2k)}\ell_m$, respectively, $(\ell', \ell_m) \in \mathbb{Z}^{m-1} \times \mathbb{Z}\}$. We also set $\mathcal{D} = \bigcup_{j,k} \Pi_{j,k}$.

One should note that, for any $\mu \in \mathbb{Z}$, there exist $j, k \in \mathbb{Z}$ and $j', k' \in \mathbb{Z}$ such that $2^{-(j\wedge k)} = 2^{-\mu}$, $2^{-(j'\wedge k')} = 2^{-\mu}$ respectively. But, for some $(\mu, \nu) \in \mathbb{Z}^2$, there may not exist j, k such that $2^{-(j\wedge k)} = 2^{-\mu}, 2^{-(j\wedge 2k)} = 2^{-\nu}$ since $j \wedge k \le j \wedge 2k$ if $j, k \ge 0$. So $\mathcal{D} \subsetneqq \{R = I \times J : I$ are dyadic cubes in $\mathbb{R}^{m-1}$, J are dyadic intervals in $\mathbb{R}\}$.

With the discrete Calderón reproducing formula, the multi-parameter Triebel-Lizorkin spaces with different homogeneities were introduced in [11] as follows.

Definition 1.4. Let $0 < p, q < \infty, \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. The multi-parameter Triebel-Lizorkin type space with different homogeneities $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is defined by

$$\dot{F}_p^{\alpha,q}(\mathbb{R}^m) = \{f \in \mathcal{S}_0'(\mathbb{R}^m) : \ \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} < \infty\},$$

where

$$\begin{aligned} &\|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)} \\ &= \Big\| \Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} \\ &\qquad \times \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)|^q \chi_I(x')\chi_J(x_m) \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^m)}, \end{aligned}$$

where I are dyadic cubes in $\mathbb{R}^{m-1}$ and J are dyadic intervals in $\mathbb{R}$ with the side lengths $l(I) = 2^{-(j\wedge k)}$ and $l(J) = 2^{-(j\wedge 2k)}$, and the left lower corners of I and the left end points of J are $2^{-(j\wedge k)}\ell'$ and $2^{-(j\wedge 2k)}\ell_m$, respectively.

This multi-parameter Triebel-Lizorkin space is well defined, since it has been proved in [11] that $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is independent of the choice of the functions ψ^1 and ψ^2. This space can also be characterized by its continuous form, that is,

$$\begin{aligned}
&\|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \\
&\times \sum_{(\ell',\ell_m)\in\mathbb{Z}^{m-1}\times\mathbb{Z}} |\psi_{j,k} * f(2^{-(j\wedge k)}\ell', 2^{-(j\wedge 2k)}\ell_m)|^q \chi_I(x')\chi_J(x_m)\Big)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^m)} \\
&\approx \|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q}|\psi_{j,k} * f|^q\Big)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^m)}; \qquad (1.10)
\end{aligned}$$

for a rigorous proof, see [14].

In Definition 1.4, setting $\alpha_1 = \alpha_2 = 0$, $q = 2, 0 < p \le 1$, one obtains Hardy spaces associated with different homogeneities $H^p_{com}(\mathbb{R}^m)$, which was introduced in [30] to study the boundedness of composition operators with different homogeneities.

Note that the multi-parameter structure with different homogeneities is involved in (1.10). If $\psi_{j,k}(x,y)$ in (1.10) is the form $\psi^1_j(x)\cdot\psi^2_k(y)$, then we obtain the Triebel-Lizorkin space of multi-parameter pure product $\dot{\mathtt{F}}_p^{\alpha,q}(\mathbb{R}^n\times\mathbb{R}^m)$ with the norm

$$\|\Big(\sum_{j,k\in\mathbb{Z}} 2^{-(j\alpha_1+k\alpha_2)q}|\psi_{j,k} * f|^q\Big)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^n\times\mathbb{R}^m)}$$

for $f \in \dot{\mathtt{F}}_p^{\alpha,q}(\mathbb{R}^n\times\mathbb{R}^m)$, $0 < p, q < \infty, \alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2$. It has been introduced in [42].

Let q' denote the conjugate of q, so that $1/q + 1/q' = 1$ when $1 \le q \le \infty$. If $0 < q < 1$, it is also convenient to let $q' = \infty$. The first main theorem of this paper concerns the duality of the spaces $\dot{F}_p^{\alpha,q}$ when $p > 1$.

Theorem 1.1. *Suppose $1 < p < \infty$, $0 < q < \infty$, $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2$; then*

$$(\dot{F}_p^{\alpha,q})^* = \dot{F}_{p'}^{-\alpha,q'}.$$

Namely, if $g \in \dot{F}_{p'}^{-\alpha,q'}$, then the map l_g, given by $l_g(f) = \langle f, g\rangle$, defined initially for $f \in \mathcal{S}_0$, extends to a continuous linear functional on $\dot{F}_p^{\alpha,q}$ with $\|l_g\| \lesssim \|g\|_{\dot{F}_{p'}^{-\alpha,q'}}$. Conversely, every $l \in (\dot{F}_p^{\alpha,q})^*$ satisfies $l = l_g$ for some $g \in \dot{F}_{p'}^{-\alpha,q'}$ with $\|l_g\| \approx \|g\|_{\dot{F}_{p'}^{-\alpha,q'}}$.

Though there have been extensive works on dual spaces of multi-parameter Hardy spaces (see [2], [21], [31], [28], [29], [30], etc.), the duality of Triebel-Lizorkin spaces has only been studied in the one-parameter settings started in [22, 62]; see also [1] for anisotropic Triebel-Lizorkin spaces, [36] for weighted anisotropic Triebel-Lizorkin spaces. For $0 < p, q < \infty, \alpha \in \mathbb{R}$, the Triebel-Lizorkin space of one-parameter $\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m)$ with the norm

$$\|\Big(\sum_{j\in\mathbb{Z}} 2^{-jq}|\psi_j * f|^q\Big)^{\frac{1}{q}}\|_{L^p(\mathbb{R}^m)}$$

was investigated in [22, 62]. There it was shown that the dual space of $\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m)$ is

$$(\dot{\mathcal{F}}_p^{\alpha,q}(\mathbb{R}^m))^* = \begin{cases} \dot{\mathcal{F}}_{p'}^{-\alpha,q'}(\mathbb{R}^m), & 1 \le p < \infty; \\ \dot{\mathcal{F}}_\infty^{-\alpha+m(1/p-1),\infty}(\mathbb{R}^m), & 0 < p < 1, \end{cases} \tag{1.11}$$

where $\dot{\mathcal{F}}_\infty^{\alpha,q}(\mathbb{R}^m)$ is defined to be the set of all $f \in \mathcal{S}_0'(\mathbb{R}^m)$ such that

$$\|f\|_{\dot{\mathcal{F}}_\infty^{\alpha,q}(\mathbb{R}^m)} = \sup_{Q \text{ dyadic cubes}} \Big(\frac{1}{|Q|} \int_Q \sum_{j=-\log_2 l(Q)}^{\infty} 2^{-j\alpha q} |\psi_j * f|^q dx \Big)^{1/q} < \infty.$$

It is well known that $\dot{\mathcal{F}}_p^{0,2}(\mathbb{R}^m)$ is the classical Hardy space $\mathcal{H}_p, 0 < p \le 1$. From (1.11), one has

$$(\mathcal{H}_p)^* = \dot{\mathcal{F}}_\infty^{m(1/p-1),\infty}(\mathbb{R}^m).$$

The method to obtain (1.11) no longer works in multi-parameter cases when $0 < p \le 1$. By using techniques of discrete Littlewood-Paley theory developed in [31, 34] for flag Hardy spaces, the authors established the dual spaces for flag Hardy spaces. Using similar ideas of discrete Littlewood-Paley theory, the dual spaces for Hardy spaces on product spaces of homogeneous type and on weighted multi-parameter Hardy spaces have been obtained in [28, 29] and [41]. To give an idea of such dual spaces in the simplest form, we state the dual space of multi-parameter pure product Hardy space $\mathtt{H}_p = \dot{\mathtt{F}}_p^{0,2}(\mathbb{R}^n \times \mathbb{R}^m)$ by another form, for $0 < p \le 1$,

$$(\mathtt{H}_p)^* = CMO_p,$$

where $f \in CMO_p$ is defined by

$$\|f\|_{CMO_p} = \sup_\Omega \Big(\frac{1}{|\Omega|^{\frac{2}{p}-1}} \int_\Omega \sum_{j,k\in\mathbb{Z}} \sum_{R\in\Pi_{j,k}, R\subseteq\Omega} (|\psi_{j,k} * f(x_I, x_J)|^2 \chi_R(x)) dx \Big)^{1/2}$$

for all open sets $\Omega \subseteq R^n \times \mathbb{R}^m$ with finite measure (see [41]). Combining the techniques developed in [22, 62] for one-parameter Triebel-Lizorkin spaces and [34] for multi-parameter Hardy spaces, we investigate the dual spaces of the multi-parameter Triebel-Lizorkin spaces associated with different homogeneities when $0 < p \le 1$. Before we state the duality result, we first give the definition of $CMO_p^{\alpha,q}(\mathbb{R}^m)$.

Definition 1.5. For $0 < p \le 1$, $1 \le q \le \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ and with I, J and x_I, x_J being the same as before, the space $CMO_p^{\alpha,q}(\mathbb{R}^m)$ is defined by

$$CMO_p^{\alpha,q}(\mathbb{R}^m) = \{f \in \mathcal{S}_0'(\mathbb{R}^m) : \ \|f\|_{CMO_p^{\alpha,q}(\mathbb{R}^m)} < \infty\},$$

where

$$\begin{aligned} \|f\|_{CMO_p^{\alpha,q}(\mathbb{R}^m)} &= \sup_\Omega \Big(\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \int_\Omega \sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} \\ &\quad \times \sum_{R\in\Pi_{j,k}, R\subseteq\Omega} (|\psi_{j,k} * f(x_I, x_J)|^q \chi_R(x)) dx \Big)^{1/q}. \end{aligned} \tag{1.12}$$

Remark 1.6. If $0 < p \le 1$, $1 \le q < \infty$, (1.12) is

$$\|f\|_{CMO_p^{\alpha,q}} = \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)} |J|^{\alpha_2} |\psi_{j,k} * f(x_I, x_J)|)^q |R| \Big)^{1/q},$$

and naturally, if $q=\infty$, (1.12) is interpreted as

$$\|f\|_{CMO_p^{\alpha,\infty}} = \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R\subseteq\Omega, R\in\mathcal{D}} |I|^{\alpha_1/(m-1)}|J|^{\alpha_2}||\psi_{j,k} * f(x_I,x_J)|.$$

To see that the space $CMO_p^{\alpha,q}$ is well defined, one needs to show the following theorem, which is actually also one of the main theorems of the paper.

Theorem 1.2. *Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions in* (1.5)-(1.8). *Then if $0<p\le 1$, $1\le q\le\infty$, $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$, one has*

$$\begin{aligned}
&\sup_{\Omega}\Big(\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\int_\Omega \sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \\
&\qquad\qquad \times \sum_{R\in\Pi_{j,k},R\subseteq\Omega} (|\psi_{j,k} * f(x_I,x_J)|^q\chi_R(x))dx\Big)^{1/q} \\
&\approx \sup_{\Omega}\Big(\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\int_\Omega \sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1+(j\wedge 2k)\alpha_2]q} \\
&\qquad\qquad \times \sum_{R\in\Pi_{j,k},R\subseteq\Omega} (|\varphi_{j,k} * f(x_I,x_J)|^q\chi_R(x))dx\Big)^{1/q}
\end{aligned}$$

for $f\in\mathcal{S}_0'$.

The proof of this theorem can follow from the ψ-transforms that correspond between the multi-parameter Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ and the discrete multi-parameter Triebel-Lizorkin sequence spaces $\dot{f}_p^{\alpha,q}$ indexed by the multi-parameter dyadic rectangles in $\mathbb{R}^m$ associated with the underlying structures of the composition of two singular integrals. Since the definition of $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ is independent of the choice of the functions ψ^1 and ψ^2, this theorem is immediate once we prove the following duality theorem (Theorem 1.3). Nevertheless, we offer another proof following the proof of Theorem 2.3.

The dual spaces for $\dot{F}_p^{\alpha,q}$ when $0<p\le 1$ are considerably different from those for $1<p<\infty$ and more difficult to get, in particular in the multi-parameter settings. Therefore, the following duality result is the third main theorem of this paper.

Theorem 1.3. *Suppose $0<p\le 1$, $0<q<\infty$, $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$. Then*

$$(\dot{F}_p^{\alpha,q})^* = CMO_p^{-\alpha,q'}$$

where q' is defined to be ∞ when $0<q\le 1$.

By duality, one can obtain the boundedness of $T_1\circ T_2$ on $CMO_p^{-\alpha,q'}$, which is the last main theorem in this paper.

Theorem 1.4. *Suppose $0<p\le 1$, $1<q\le\infty$, $\alpha=(\alpha_1,\alpha_2)\in\mathbb{R}^2$. Then the composition operator $T=T_1\circ T_2$ is bounded on $CMO_p^{\alpha,q}$.*

Remark 1.7. In this paper, we only carry out the duality theory in the multi-parameter setting associated with the dilations of the composition of two singular integral operators. This is essentially within the framework concerning the translation-invariant environment in the Euclidean spaces. The more general case involving more families of dilations and more parameters has been outlined in Section 6, Appendix. An interested reader can carry out the details following the same

scheme in this paper without essential difficulty but with more technicalities. The perhaps more interesting but more complicated case of considering the translation non-invariant dilations which are more related to the $\overline{\partial}$-problem studied by, e.g., Greiner and Stein [26], Nagel and Stein [49, 50] will be carefully investigated in a future project. We thank the referee for pointing out this more relevant situation to us. In fact, multi-parameter local Hardy space theory parallel to the one-parameter local Hardy space theory of Goldberg [24] has been recently developed by the authors [12] in which the atomic decomposition and duality theory has been established. Applications of proving boundedness of certain classes of operators on such spaces are also given in [12]. With similar ideas, one can also establish that the composition of two translation non-invariant singular integral operators of different homogeneities are bounded on such multi-parameter local Hardy spaces [13]. Furthermore, we have proved the boundedness of multi-parameter pseudo-differential operators and Fourier integral operators on such spaces. Multi-parameter local Hardy space theory can be extended to the setting of multi-parameter local Triebel-Lizorkin and Besov spaces as done in the classical multi-parameter Hardy spaces [11, 14, 16, 42].

The organization of this paper is as follows. Section 2 introduces the multi-parameter ψ-transform S_ψ and its inverse ψ-transform T_ψ. These transforms correspond between the multi-parameter Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ and the discrete multi-parameter Triebel-Lizorkin sequence spaces $\dot{f}_p^{\alpha,q}$ indexed by the multi-parameter dyadic rectangles in $\mathbb{R}^m$ associated with the underlying structures of the composition of two singular integrals. We also introduce the discrete sequence form $C_p^{\alpha,q}$ of the space $CMO_p^{\alpha,q}$. Thus, we prove in Theorem 2.1 that the operators $S_\psi : \dot{F}_p^{\alpha,q} \to \dot{f}_p^{\alpha,q}$ and $T_\psi : \dot{f}_p^{\alpha,q} \to \dot{F}_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $\dot{F}_p^{\alpha,q}$. Then we establish in Theorem 2.3 that the operators $S_\psi : CMO_p^{\alpha,q} \to C_p^{\alpha,q}$ and $T_\psi : C_p^{\alpha,q} \to CMO_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $CMO_p^{\alpha,q}$. The proof of Theorem 2.3 is rather involved, and the underlying geometry of the multi-paramter structures is extensively used. Section 3 concerns the imbedding theorems and gives a characterization of imbedding of ℓ^r spaces into $\dot{f}_p^{\alpha,q}$ and imbedding of $\dot{f}_p^{\alpha,q}$ into ℓ^r spaces. In Section 4, we establish the duality of the sequence space $\dot{f}_p^{\alpha,q}$. Section 5 gives the proof of the duality of the space $\dot{F}_p^{\alpha,q}$ and establishes the boundedness of the composition of two singular integral operators on the dual spaces $CMO_p^{\alpha,q}$.

2. Multi-parameter ψ-transform

In order to prove the duality theorems, following Frazier and Jawerth in the one-parameter case [22] (see also [61]), we should first do these in the corresponding discrete multi-parameter Triebel-Lizorkin sequence spaces. For any $R \in \Pi_{j,k}$, setting $\psi_R(x) = |R|^{1/2}\psi_{j,k}(x' - x_I, x_m - x_J)$, then by (1.9), it's easy to have

$$f(x) = \sum_{R\in\mathcal{D}} \langle f, \psi_R\rangle \psi_R(x). \tag{2.1}$$

Definition 2.1. Suppose that $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in (1.5)-(1.6) and (1.7)-(1.8), respectively. Define the multi-parameter ψ-transform S_ψ as the map taking $f \in \mathcal{S}_0'(\mathbb{R}^m)$ to the sequence $S_\psi f = \{(S_\psi f)_R\}_R$, where

$(S_\psi f)_R = \langle f, \psi_R\rangle$. Define the inverse multi-parameter ψ-transform T_ψ as the map taking a sequence $s = \{s_R\}_R$ to $T_\psi s = \sum_R s_R \psi_R(x)$.

By (2.1), for $f \in \mathcal{S}_0, g \in \mathcal{S}_0'$ one has

$$\langle f, g\rangle = \langle \sum_{R\in\mathcal{D}} (S_\psi f)_R \psi_R(x), g\rangle = \langle S_\psi f, S_\psi g\rangle. \tag{2.2}$$

For a sequence $s = s_R$, one also has the following identity:

$$\langle S_\psi f, s\rangle = \sum_{R\in\mathcal{D}} \langle f, \psi_R\rangle s_R \quad = \quad \langle f, \sum_{R\in\mathcal{D}} s_R \psi_R\rangle = \langle f, T_\psi s\rangle. \tag{2.3}$$

The discrete Triebel-Lizorkin sequence space $\dot{f}_p^{\alpha,q}$ is defined as follows.

Definition 2.2. For $0 < p < \infty$, $0 < q \le \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, define $\dot{f}_p^{\alpha,q}$ to be the collection of all complex-valued sequences $s = \{s_R\}_R$ such that

$$\|s\|_{\dot{f}_p^{\alpha,q}} = \|\Big(\sum_{R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x))^q\Big)^{1/q}\|_{L^p} < \infty \tag{2.4}$$

where $\tilde{\chi}_R(x) = |R|^{-1/2}\chi_R(x)$.

Remark 2.3. If $q = \infty$, (2.4) is interpreted as

$$\|s\|_{\dot{f}_p^{\alpha,\infty}} = \|\sup_{R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R)\|_{L^p} < \infty.$$

We also need the discrete sequence form of $CMO_p^{\alpha,q}$.

Definition 2.4. For $0 < p \le 1$, $1 \le q \le \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, define $C_p^{\alpha,q}$ to be the collection of all complex-valued sequences $t = \{t_R\}_R$ such that

$$\|t\|_{C_p^{\alpha,q}} = \sup_\Omega \Big(\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \int_\Omega \sum_{R\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|t_R|\tilde{\chi}_R(x))^q dx\Big)^{1/q} \tag{2.5}$$

where $\tilde{\chi}_R(x)$ is the same as the form defined in Definition 2.2.

Remark 2.5. Naturally, if $0 < p \le 1$, $1 \le q < \infty$, (2.5) is

$$\|t\|_{C_p^{\alpha,q}} = \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|t_R||R|^{-1/2})^q |R|\Big)^{1/q};$$

and if $0 < p \le 1$, $q = \infty$, (2.5) is interpreted as

$$\|t\|_{C_p^{\alpha,\infty}} = \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R\subseteq\Omega, R\in\mathcal{D}} |I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|t_R||R|^{-1/2}.$$

Then the following generalization of the fundamental result of Theorem 2.2 in [22] holds.

Theorem 2.1. *Suppose $0 < p < \infty$, $0 < q \le \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, and $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in* (1.5)-(1.6) *and* (1.7)-(1.8)*, respectively. The operators $S_\psi : \dot{F}_p^{\alpha,q} \to \dot{f}_p^{\alpha,q}$ and $T_\psi : \dot{f}_p^{\alpha,q} \to \dot{F}_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $\dot{F}_p^{\alpha,q}$.*

Proof. The boundedness of S_ψ is immediate since

$$\|S_\psi(f)\|_{\dot{f}_p^{\alpha,q}} = \|f\|_{\dot{F}_p^{\alpha,q}}$$

from the definition.

We now outline the proof of T_ψ's boundedness. For a sequence $s = \{s_R\}_{R\in\mathcal{D}}$, let $f(x) = T_\psi s = \sum_R s_R \psi_R(x)$. Then by almost orthogonality estimates (e.g. see Lemma 3.1 in [30]), one has

$$\begin{aligned}
&|\psi_{j',k'} * \psi_{j,k}(x_{I'} - x_I, x_{J'} - x_J)| \\
&\quad \lesssim 2^{-|j-j'|L} 2^{-|k-k'|L} \frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1 + 2^{j\wedge j'\wedge k\wedge k'}|x_{I'} - x_I|)^{(M+m-1)}} \\
&\qquad \times \frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1 + 2^{j\wedge j'\wedge 2(k\wedge k')}|x_{J'} - x_J|)^{(M+1)}}.
\end{aligned}$$

Hence for any $v'' \in x_{I'}, v'_m \in x_{J'}$,

$$\begin{aligned}
&|f * \psi_{j',k'}(x_{I'}, x_{J'})| \\
&\quad \lesssim \sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} C_1 \Big\{ \mathcal{M}_s \Big(\sum_{R\in\Pi_{j,k}} |R|^{-1/2} |s_R| \chi_I \chi_J \Big)^\delta (v'', v'_m) \Big\}^{1/\delta}
\end{aligned}$$

for a $\delta > 0$ which can be sufficiently small if one chooses M big enough by Lemma 3.2 in [30]. Summing over j', k' and (ℓ'', ℓ'_m), one has

$$\begin{aligned}
&\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1 + (j'\wedge 2k')\alpha_2]q} \sum_{(\ell'',\ell'_m)} |\psi_{j',k'} * f(x_{I'}, x_{J'})|^q \chi_{I'}(x'') \chi_{J'}(x'_m) \Big)^{\frac{1}{q}} \\
\le\ & C \Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1 + (j'\wedge 2k')\alpha_2]q} \Big[\sum_{j,k} 2^{-|j-j'|L} 2^{-|k-k'|L} \\
&\qquad \times C_1 \Big\{ \mathcal{M}_s \Big(\sum_{R\in\Pi_{j,k}} |R|^{-1/2} |s_R| \chi_I \chi_J \Big)^\delta (v'', v'_m) \Big\}^{1/\delta} \Big]^q \Big)^{\frac{1}{q}}.
\end{aligned}$$

Then by the inequality $(\sum_l a_l)^q \le \sum_l a_l^q$, if $0 < q \le 1$, or Cauchy's inequality with exponents q, q', $\frac{1}{q} + \frac{1}{q'} = 1$, if $q > 1$, we obtain

$$\begin{aligned}
&\Big(\sum_{j',k'\in\mathbb{Z}} 2^{-[(j'\wedge k')\alpha_1 + (j'\wedge 2k')\alpha_2]q} \sum_{(\ell'',\ell'_m)} |\psi_{j',k'} * f(x_{I'}, x_{J'})|^q \chi_{I'}(x'') \chi_{J'}(x'_m) \Big)^{\frac{1}{q}} \\
\lesssim\ & \Big(\sum_{j,k\in\mathbb{Z}} 2^{-[(j\wedge k)\alpha_1 + (j\wedge 2k)\alpha_2]q} \Big\{ \mathcal{M}_s \Big(\sum_{R\in\Pi_{j,k}} |R|^{-1/2} |s_R| \chi_I \chi_J \Big)^\delta (v'', v'_m) \Big\}^{q/\delta} \Big)^{\frac{1}{q}}.
\end{aligned}$$

Applying Fefferman-Stein's vector-valued strong maximal inequality on $L^{p/\delta}(\ell^{q/\delta})$ provided $\delta < \min\{p, q, 1\}$, we complete the proof. □

Next, we will obtain a similar correspondence between $CMO_p^{\alpha,q}$ and $C_p^{\alpha,q}$. Following the proof of Lemma 3.1 in [30], one can obtain the following almost orthogonality estimates.

Lemma 2.2. *Suppose that $\psi_{j,k}$ and $\varphi_{j',k'}$ satisfy the same conditions in* (1.5)-(1.8). *Then for any given integers L_1, L_2 and M, there exists a constant $C = C(L, M) > 0$ such that*

$$|\psi_{j,k} * \varphi_{j',k'}(x', x_m)| \leq C 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} \frac{2^{(j\wedge k)(m-1)} 2^{j\wedge 2k}}{(1+2^{j\wedge k}|x'|)^{(M+m-1)}(1+2^{j\wedge 2k}|x_m|)^{(M+1)}}.$$

Proof. One can write

$$(\psi_{j,k} * \varphi_{j',k'})(x', x_m) = \int_{\mathbb{R}^{m-1}\times\mathbb{R}} (\psi_j^{(1)} * \varphi_{j'}^{(1)})(x'-y', x_m - y_m)(\psi_k^{(2)} * \varphi_{k'}^{(2)})(y', y_m) dy' dy_m.$$

Then by classical almost orthogonality estimates, one has

$$|\psi_j^{(1)} * \varphi_{j'}^{(1)}(u', u_m)| \leq C \frac{2^{(j\wedge j')m}\ 2^{-|j-j'|L_1}}{(1+2^{(j\wedge j')}|u'|)^{(M+m-1)}\ (1+2^{(j\wedge j')}|u_m|)^{(M+1)}} \tag{2.6}$$

and

$$|\psi_k^{(2)} * \varphi_{k'}^{(2)}(y', y_m)| \leq C \frac{2^{(k\wedge k')(m+1)}\ 2^{-|k-k'|L_2}}{(1+2^{(k\wedge k')}|y'|)^{(M+m-1)}\ (1+2^{2(k\wedge k')}|y_m|)^{(M+1)}} \tag{2.7}$$

for any positive integer L_1, L_2 and M. With the same process as in the proof of Lemma 3.1 in [30], we have

$$\begin{aligned} |\psi_{j,k} * \varphi_{j',k'}(x', x_m)| \quad &\leq \quad C 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} \frac{2^{(j\wedge j'\wedge k\wedge k')(m-1)}}{(1+2^{j\wedge j'\wedge k\wedge k'}|x'|)^{(M+m-1)}} \\ &\qquad \times \frac{2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge j'\wedge 2(k\wedge k')}|x_m|)^{(M+1)}}, \end{aligned}$$

which gives

$$\begin{aligned} |\psi_{j,k} * \varphi_{j',k'}(x', x_m)| \quad &\lesssim \quad 2^{-|j-j'|L_1} 2^{-|k-k'|L_2} \frac{2^{(j\wedge k)M-(j\wedge j'\wedge k\wedge k')M} 2^{(j\wedge k)(m-1)}}{(1+2^{j\wedge k}|x'|)^{(M+m-1)}} \\ &\qquad \times \frac{2^{(j\wedge 2k)M-(j\wedge j'\wedge 2(k\wedge k'))M} 2^{j\wedge j'\wedge 2(k\wedge k')}}{(1+2^{j\wedge 2k}|x_m|)^{(M+1)}}. \end{aligned}$$

After observing that

$$j\wedge k - j\wedge j'\wedge k\wedge k' \leq |j-j'| + |k-k'|$$

and

$$j\wedge 2k - j\wedge j'\wedge 2(k\wedge k') \leq |j-j'| + 2|k-k'|,$$

we obtain the desired result. □

The next theorem concerns the actions of the multi-parameter ψ-transform S_ψ and its inverse ψ-transform T_ψ on the space $CMO_p^{\alpha,q}$ and its discrete sequence form $C_p^{\alpha,q}$. We prove that operators $S_\psi : CMO_p^{\alpha,q} \to C_p^{\alpha,q}$ and $T_\psi : C_p^{\alpha,q} \to CMO_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $CMO_p^{\alpha,q}$. The proof of this theorem is rather involved, and the underlying geometry of the multi-parameter structures of the dyadic rectangles associated with the composition of two operators with different homogeneities plays an important role. These sorts of ideas have been initially used in [31] and then [30] for duality of flag Hardy spaces, and similar ideas have been used subsequently for Hardy spaces in different multi-parameter

settings (see [28], [29], [41], etc.). Nevertheless, it is more difficult and complicated to carry out our multi-parameter Triebel-Lizorkin spaces.

Theorem 2.3. *Suppose $0 < p \le 1 \le q \le \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, and $\psi^{(1)}$ and $\psi^{(2)}$ are functions satisfying conditions in* (1.5)-(1.6) *and* (1.7)-(1.8)*, respectively. Then the operators $S_\psi : CMO_p^{\alpha,q} \to C_p^{\alpha,q}$ and $T_\psi : C_p^{\alpha,q} \to CMO_p^{\alpha,q}$ are bounded, and $T_\psi \circ S_\psi$ is the identity on $CMO_p^{\alpha,q}$.*

Proof. We only prove T_ψ is bounded since the rest is obvious. Let $t = \{t_{R'}\}_{R'} \in C_p^{\alpha,q}$ and $f = \sum t_{R'}\psi_{R'}$. When $1 \le q < \infty$, we are going to prove

$$\begin{aligned} &\sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|\varphi_{j,k} * f(x_I, x_J)|)^q |R| \Big)^{1/q} \\ &\lesssim \sup_\Omega \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R'=I'\times J'\subseteq\Omega, R'\in\mathcal{D}} (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q |R'| \Big)^{1/q}. \end{aligned} \tag{2.8}$$

For any $R' \in \Pi_{j'k'}$, by Lemma 2.2, one has

$$\begin{aligned} |\varphi_{j,k} * \psi_{R'}(x_I, x_J)| \quad \le \quad & C|R'|^{1/2} 2^{-|j-j'|(L_1+L_2)} 2^{-|k-k'|(L_1+2L_2)} \\ & \times \frac{2^{(j'\wedge k')(m-1)}}{(1+2^{j'\wedge k'}|x_I - x_{I'}|)^{(M+m-1)}} \\ & \times \frac{2^{j'\wedge 2k'}}{(1+2^{j'\wedge 2k'}|x_J - x_{J'}|)^{(M+1)}}. \end{aligned}$$

Since $|j' \wedge k' - j \wedge k| \le |j - j'| + |k - k'|, |j' \wedge 2k' - j \wedge 2k| \le |j - j'| + 2|k - k'|$, one has

$$\begin{aligned} |\varphi_{j,k} * \psi_{R'}(x_I, x_J)| \quad \le \quad & C|R'|^{-1/2} \frac{2^{-L_1|j'\wedge k' - j\wedge k|}}{(1+2^{j'\wedge k'}|x_I - x_{I'}|)^{(M+m-1)}} \\ & \times \frac{2^{-L_2|j'\wedge 2k' - j\wedge 2k|}}{(1+2^{j'\wedge 2k'}|x_J - x_{J'}|)^{(M+1)}} \end{aligned}$$

for any sufficiently larger L_1, L_1. Using conditions (1.5), (1.7), it is easy to see that

$$\begin{aligned} &(\varphi_{j,k} * \psi_{j',k'}(\cdot - 2^{-(j'\wedge k')}\ell', \cdot - 2^{-(j'\wedge 2k')}\ell_m))^\wedge(\xi', \xi_m) \\ &\quad = \widehat{\varphi_{j,k}}(\xi', \xi_m)\widehat{\psi_{j',k'}}(\xi', \xi_m) \exp(-2\pi i[2^{-(j'\wedge k')}\ell'\xi' + 2^{-(j'\wedge 2k')}\,\ell_m\xi_m]) \\ &\quad = 0 \text{ if } |j' - j| > 1 \text{ or } |k' - k| > 1, \end{aligned}$$

from which follows

$$|\varphi_{j,k} * f(x_I, x_J)|^q \lesssim \sum_{R'} |t_{R'}|^q |\varphi_{j,k} * \psi_{R'}(x_I, x_J)|^q.$$

Hence

$$
\begin{aligned}
&\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|\varphi_{j,k} * f(x_I,x_J)|)^q|R| \\
&\lesssim \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} |I|^{q\alpha_1/(m-1)}|J|^{q\alpha_2}|t_{R'}|^q|R'|^{-q/2}|R| \\
&\times \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{qL_1/(m-1)} \frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{qL_2} \frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M} \\
&= \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} \left(\frac{|I|}{|I'|}\right)^{q\alpha_1/(m-1)} \left(\frac{|J|}{|J'|}\right)^{q\alpha_2} \frac{|R|}{|R'|} \\
&\times \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{qL_1/(m-1)} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{qL_2} \frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M} \frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M} \\
&(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'| \\
&\lesssim \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} \left(\frac{|I|}{|I'|}\vee\frac{|I'|}{|I|}\right)^{q|\alpha_1|/(m-1)+1} \left(\frac{|J|}{|J'|}\vee\frac{|J'|}{|J|}\right)^{q|\alpha_2|+1} \\
&\times \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{qL_1/(m-1)} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{qL_2} \frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M} \frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M} \\
&\times (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'| \\
&= \sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'| \\
&\times \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L} \frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M} \frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M}
\end{aligned}
$$

by setting $qL_1/(m-1) - q|\alpha_1|/(m-1) - 1 = L = qL_2 - q|\alpha_2| - 1$. Thus

$$
\begin{aligned}
&\sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|\psi_{j,k} * f(x_I,x_J)|)^q|R| \Big)^{1/q} \\
&\lesssim \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{R'=I'\times J'\in\mathcal{D}} r(R,R')p(R,R') \\
&\qquad\qquad \times (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'| \Big)^{1/q},
\end{aligned} \tag{2.9}
$$

where

$$
r(R,R') = \left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L} \left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L}
$$

and

$$
p(R,R') = \frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M} \frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M}.
$$

In order to prove inequality (2.8), using (2.9), we only need to prove

$$\sup_{\Omega}\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\Big(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}r(R,R')p(R,R')$$
$$\times(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'|\Big)^{1/q}$$
$$\lesssim\sup_{\Omega}\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\Big(\sum_{R'=I'\times J'\subseteq\Omega,R'\in\mathcal{D}}(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'|\Big)^{1/q}. \tag{2.10}$$

To do this, define

$$\Omega^{0,0}=\bigcup_{R=I\times J\subseteq\Omega}3(I\times J).$$

For any $R\subseteq\Omega$, let $A_{i,l}(R)$ be the collection of dyadic rectangles R' so that

$$A_{0,0}(R)=\{R'=I'\times J'\subseteq\Omega:\operatorname{dist}(I,I')\le\ell(I)\vee\ell(I'),\operatorname{dist}(J,J')\le\ell(J)\vee\ell(J')\},$$

and for $i\ge1$,

$$A_{i,0}(R)=\{R'=I'\times J'\subseteq\Omega:(2^{i-1}\ell(I'))\vee\ell(I)<\operatorname{dist}(I,I')\le(2^i\ell(I'))\vee\ell(I),$$
$$\operatorname{dist}(J,J')\le\ell(J)\vee\ell(J')\},$$

and for $l\ge1$,

$$A_{0,l}(R)=\{R'=I'\times J'\subseteq\Omega:\operatorname{dist}(I,I')\le\ell(I)\vee\ell(I'),$$
$$(2^{l-1}\ell(J'))\vee\ell(J)<\operatorname{dist}(J,J')\le(2^l\ell(J'))\vee\ell(J)\},$$

and for $i,l\ge1$,

$$A_{i,l}(R)=\{R'=I'\times J'\subseteq\Omega:(2^{i-1}\ell(I'))\vee\ell(I)<\operatorname{dist}(I,I')\le(2^i\ell(I'))\vee\ell(I),$$
$$(2^{l-1}\ell(J'))\vee\ell(J)<\operatorname{dist}(J,J')\le(2^l\ell(J'))\vee\ell(J)\},$$

and $i,l\ge0$,

$$A_{i,j}=\{R'=I'\times J'\in\mathcal{D}:\ 3(2^iI'\times2^lJ')\cap\Omega^{0,0}\ne\emptyset\}.$$

It is easy to see that for any $R\subseteq\Omega$, $\bigcup_{i,l\ge0}A_{i,l}(R)=\mathcal{D}$, $A_{i,l}(R)\cap A_{i',l'}(R)=\emptyset$ if $(i,l)\ne(i',l')$ and $A_{i,l}(R)\subseteq A_{i,l}$. Note that for $R'\in A_{i,j}(R)$, $i,l\ge0$,

$$1+\frac{\operatorname{dist}(I,I')}{\ell(I')}\gtrsim2^i,\quad 1+\frac{\operatorname{dist}(J,J')}{\ell(J')}\gtrsim2^l,$$

from which follows

$$p(R,R)\lesssim2^{-(i+l)M}.$$

Hence

$$\begin{aligned}
&\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\Big(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}} r(R,R')p(R,R') \\
&\qquad\qquad\times(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'|\Big)^{1/q} \\
&=\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\Big(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{i,l\geq 0}\sum_{R'\in A_{i,l}(R)} r(R,R')p(R,R') \\
&\qquad\qquad\times(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'|\Big)^{1/q} \\
&=\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\Big(\sum_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{i,l\geq 0}\sum_{R'\in A_{i,l}} \chi_{R'\in A_{i,l}(R)}r(R,R')p(R,R') \\
&\qquad\qquad\times(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'|\Big)^{1/q} \\
&\lesssim\frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}}\Big(\Big(\sum_{R'\in A_{0,0}}+\sum_{i\geq 1}\sum_{R'\in A_{i,0}}+\sum_{l\geq 1}\sum_{R'\in A_{0,l}}+\sum_{i,l\geq 1}\sum_{R'\in A_{i,l}}\Big)2^{-(i+l)M} \\
&\qquad\times\sum_{R\subseteq\Omega,R\in\mathcal{D}}\chi_{R'\in A_{i,l}(R)}r(R,R')(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'|\Big)^{1/q} \\
&=(I_1+I_2+I_3+I_4)^{1/q}.
\end{aligned}$$

We only estimate I_4 since estimates of estimate I_1, I_2 and I_3 can be concluded by applying the same techniques.

For each integer $h\geq 1$, let $\mathcal{F}_h^{i,l}=\{R'=I'\times J'\in A_{i,l}:|3(2^iI'\times 2^lJ')\cap\Omega^{0,0}|\geq\frac{1}{2^h}|2^iI'\times 2^lJ'|\}$. Let

$$\mathcal{D}_h^{i,l}=\mathcal{F}_h^{i,l}\backslash\mathcal{F}_{h-1}^{i,l}\quad\text{and}\quad\Omega_h^{i,l}=\bigcup_{R'\in\mathcal{D}_h^{i,l}}R'.$$

Then

$$\begin{aligned}
I_4 \;=\; &\frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\sum_{i,l\geq 1}\sum_{h\geq 1}\sum_{R'\in\mathcal{D}_h^{i,l}}2^{-(i+l)M}\sum_{R\subseteq\Omega,R\in\mathcal{D}}\chi_{R'\in A_{i,l}(R)}r(R,R') \\
&\qquad\times(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'|.
\end{aligned}$$

To estimate the right-hand side of the above inequality, we only need to estimate

$$\sum_{R\subseteq\Omega,R\in\mathcal{D}}\chi_{R'\in A_{i,l}(R)}r(R,R').$$

Firstly, because $R'\in A_{i,l}(R)$, one has $3R\cap 3(2^iI'\times 2^jJ')\neq\emptyset$. For $R\subseteq\Omega$, there are four cases:

Case 1: $|2^iI'|\geq|I|$, $|2^lJ'|\geq|J|$; Case 2: $|2^iI'|\geq|I|$, $|2^lJ'|\leq|J|$;
Case 3: $|2^iI'|\leq|I|$, $|2^lJ'|\geq|J|$; Case 4: $|2^iI'|\leq|I|$, $|2^lJ'|\leq|J|$.

From the definition of $A_{i,l}(R)$, one can see that if $R'\in$ Case 2, then

$$\ell(J)=(2^{l-1}\ell(J'))\vee\ell(J)<\operatorname{dist}(J,J')\leq(2^l\ell(J'))\vee\ell(J)=\ell(J),$$

which implies Case 2 is an empty set. For the same reason, Case 3 is also an empty set. We split I_4 into two terms:

$$\begin{aligned} I_4 &= \frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}}\sum_{i,l\geq 1}\sum_{h\geq 1}\sum_{R'\in\mathcal{D}_h^{i,l}} 2^{-(i+l)M}\Big(\sum_{R\in\text{Case 1}}+\sum_{R\in\text{Case 4}}\Big)\chi_{R'\in A_{i,l}(R)}r(R,R') \\ &\qquad\qquad \times(|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q|R'| \\ &= I_4^1+I_4^4. \end{aligned}$$

In Case 1, since $R'\in A_{i,l}(R)$ and $R'\in\mathcal{D}_h^{i,l}$, one has

$$|R|\leq|3R\cap 3(2^iI'\times 2^lJ')|\leq|3(2^iI'\times 2^lJ')\cap\Omega^{0,0}|\leq\frac{1}{2^{h-1}}|3(2^iI'\times 2^lJ')|. \tag{2.11}$$

In $r(R,R')$, we should compare the side-length of R with the side-length of R'. We divide $R\subseteq\Omega$ into four categories:

Category 1.1 $|I|\leq|I'|$, $|J|\leq|J'|$; Category 1.2 $|I|\leq|I'|$, $|J|>|J'|$;

Category 1.3 $|I|>|I'|$, $|J|\leq|J'|$; Category 1.4 $|I|>|I'|$, $|J|>|J'|$.

For Category 1.1, (2.11) gives $2^{i(m-1)+l}|R'|=2^{h-1-2m+\eta}|R|$ for some integer $\eta>0$ since I,I' are all dyadic, where 2^{-2m} is used to offset 3^m. For each fixed $\eta>0$, the number of such R's must be less than $7^m2^{h-1-2m+\eta}$ since $R\subseteq 7(2^iI'\times 2^lJ')$. Therefore

$$\begin{aligned} \sum_{R\in\text{Case1},|I|\leq|I'|,|J|\leq|J'|}\chi_{R'\in A_{i,l}(R)}r(R,R') &\leq 7^m\sum_{\eta>0}\frac{(2^{i(m-1)+l})^L}{(2^{h-1-2m+\eta})^{L-1}} \\ &\lesssim 2^{iLm-h(L-1)+lL}. \end{aligned}$$

For Category 1.2, $|I|\leq|I'|$, $|J|>|J'|$. From (2.11), one has

$$|I||J'|\leq|R|\leq|3R\cap 3(2^iI'\times 2^lJ')|\leq|3(2^iI'\times 2^lJ')\cap\Omega^{0,0}|\leq\frac{1}{2^{h-1}}|3(2^iI'\times 2^lJ')|.$$

It follows that

$$|I|\leq\frac{3^m2^{i(m-1)+l}}{2^{h-1}}|I'|;$$

hence $2^{i(m-1)+l}|I'|=2^{h-1-2m+\theta}|I|$ for some integer $\theta>0$. For each fixed $\theta>0$, the number of such I's must be less than $7^{m-1}2^{h-1-2m-l+\theta}$ since $I\subseteq 7(2^iI')$. Moreover from $|J'|<|J|\leq|2^lJ'|$ we have $|2^\beta J'|=|J|$ for some positive integer β with $1\leq\beta\leq l$. For each fixed $\beta>0$, the number of such J's must be less than $72^{(l-\beta)}$ since $J\subseteq 7(2^lJ')$. Hence

$$\begin{aligned} \sum_{R\in\text{Case1},|I|\leq|I'|,|J|>|J'|}\chi_{R'\in A_{i,l}(R)}r(R,R') &\leq 7^m\sum_{\theta>0}\sum_{\beta=1}^{l}\frac{(2^{i(m-1)})^L}{(2^{h-1-2m-l+\theta})^{L-1}}\frac{2^{l-\beta}}{2^{L\beta}} \\ &\lesssim l2^{iLm-h(L-1)+3lL}. \end{aligned}$$

With a similar argument to Category 1.2, for Category 1.3 one can obtain the following estimates:

$$\sum_{R\in\text{Case1},|I|>|I'|,|J|\leq|J'|}\chi_{R'\in A_{i,l}(R)}r(R,R')\lesssim i2^{2iLm-h(L-1)+2lL}.$$

For Category 1.4, from (2.11) one has

$$|R'|\leq\frac{1}{2^{h-1}}|3(2^iI'\times 2^lJ')|,$$

from which follows that $2^{h-1} \le 3^m 2^{i(m-1)+l}$. On the other hand, with $|R'| \le |R| \le |2^i I' \times 2^l J'|$, one has $2^{i(m-1)+l}|R'| = 2^\lambda |R|$ for some integer $0 \le \lambda \le i(m-1)+l$. For each fixed $\lambda \ge 0$, the number of such R's must be less than $7^m 2^\lambda$ since $R \subseteq 7(2^i I' \times 2^l J')$. So

$$\sum_{R\in \text{Case1}, |I|>|I'|, |J|>|J'|} \chi_{R'\in A_{i,l}(R)} r(R,R') = \sum_{\lambda=0}^{i(m-1)+l} 7^m 2^\lambda \Big(\frac{2^\lambda}{2^{i(m-1)+l}}\Big)^L \lesssim 2^{im+l}.$$

Therefore

$$\begin{aligned}
I_4^1 &= \frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{i,l\ge1}\sum_{h\ge1}\sum_{R'\in\mathcal{D}_h^{i,l}} 2^{-(i+l)M} \sum_{R\in\text{Case 1}} \chi_{R'\in A_{i,l}(R)} r(R,R') \\
&\qquad \times (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q |R'| \\
&\lesssim \frac{1}{|\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{i,l\ge1} \Big(\sum_{h\ge1}(2^{iLm-h(L-1)+lL} + l2^{iLm-h(L-1)+3lL} \\
&\quad + i2^{2iLm-h(L-1)+2lL}) + \sum_{h:2^{h-1}\le 3^m 2^{i(m-1)+l}} 2^{im+l}\Big) 2^{-(i+l)M} |\Omega_h^{i,l}|^{\frac{q}{p}-\frac{q}{q'}} \\
&\quad \times \frac{1}{|\Omega_h^{i,l}|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq\Omega_h^{i,l}} (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q |R'| \\
&\lesssim \sum_{i,l\ge1}\Big(\sum_{h\ge1} 2^{4iLm-h(L-1)+4lL} + \sum_{h:2^{h-1}\le 3^m 2^{i(m-1)+l}} 2^{im+l}\Big) \\
&\quad \times 2^{-(i+l)M}(2^{2h})^{\frac{q}{p}-\frac{q}{q'}} \sup_{\bar\Omega} \frac{1}{|\bar\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq\bar\Omega} (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q |R'| \\
&\lesssim \sup_{\bar\Omega} \frac{1}{|\bar\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq\bar\Omega} (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q |R'|
\end{aligned}$$

since $|\Omega_h^{i,l}| \lesssim 2^{2h}|\Omega^{0,0}| \le 2^{2h}|\Omega|$, $0 < p \le 1 \le q \le \infty$ and choosing $M > 4mL$ with L large enough.

In Case 4, firstly, since $R' \in A_{i,l}(R)$ and $R' \in \mathcal{D}_h^{i,l}$, one has

$$|2^i I' \times 2^l J'| \le |3R \cap 3(2^i I' \times 2^l J')| \le |3(2^i I' \times 2^l J') \cap \Omega^{0,0}| \le \frac{1}{2^{h-1}}|3(2^i I' \times 2^l J')|,$$

which follows $h \le 2^{2m+1}$. Moreover, from $|2^i I'| \le |I|$, $|2^l J'| \le |J|$, one has $2^{i(m-1)+l+\sigma}|R'| = |R|$ for some integer $\sigma \ge 0$. For each fixed $\sigma \ge 0$ and any R', the number of such R's must be less than 7^m. In this situation, we have the following estimates:

$$\sum_{R\in\text{Case4}} \chi_{R'\in A_{i,l}(R)} r(R,R') = \sum_{\sigma\ge0} 7^m \Big(\frac{1}{2^{i(m-1)+l+\sigma}}\Big)^L \lesssim 2^{im+l}.$$

Then with the same process, one has

$$I_4^4 \lesssim \sup_{\bar\Omega} \frac{1}{|\bar\Omega|^{\frac{q}{p}-\frac{q}{q'}}} \sum_{R'\subseteq\bar\Omega} (|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2})^q |R'|.$$

We then complete the proof of (2.8).

When $q=\infty$, for $t=\{t_{R'}\}_{R'}\in C_p^{\alpha,q}$ and $f=\sum t_{R'}\psi_{R'}$, we are going to prove

$$\begin{aligned}&\sup_{\Omega}\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|\varphi_{j,k}*f(x_I,x_J)|\\ &\lesssim\sup_{\Omega}\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R'=I'\times J'\subseteq\Omega,R'\in\mathcal{D}}|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\end{aligned}$$

Its proof is similar to the case of $0<q<\infty$; hence we only give an outline.

For convenience, we use the same symbols as above. With the same process, one has

$$\begin{aligned}&\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|\varphi_{j,k}*f(x_I,x_J)|\\ \lesssim\ &\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|t_{R'}||R'|^{-1/2}\\ &\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L_1/(m-1)}\frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L_2}\frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M}\\ =\ &\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}\left(\frac{|I|}{|I'|}\right)^{\alpha_1/(m-1)}\left(\frac{|J|}{|J'|}\right)^{\alpha_2}\\ &\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L_1/(m-1)}\frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L_2}\frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M}\\ &|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\\ \lesssim\ &\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\\ &\times\left(\frac{|I|}{|I'|}\wedge\frac{|I'|}{|I|}\right)^{L}\left(\frac{|J|}{|J'|}\wedge\frac{|J'|}{|J|}\right)^{L}\frac{1}{(1+\frac{\mathrm{dist}(I,I')}{\ell(I')})^M}\frac{1}{(1+\frac{\mathrm{dist}(J,J')}{\ell(J')})^M}\end{aligned}$$

$$\begin{aligned}=\ &\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\sum_{R'=I'\times J'\in\mathcal{D}}r(R,R')p(R,R')|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\\ =\ &\frac{1}{|\Omega|^{\frac{1}{p}-1}}\sup_{R=I\times J\subseteq\Omega,R\in\mathcal{D}}\Big(\sum_{R'\in A_{0,0}(R)}+\sum_{i\geq1}\sum_{R'\in A_{i,0}(R)}+\sum_{l\geq1}\sum_{R'\in A_{0,l}(R)}\\ &\qquad+\sum_{i,l\geq1}\sum_{R'\in A_{i,l}(R)}\Big)2^{-(i+l)M}r(R,R')|I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}\\ =\ &B_1+B_2+B_3+B_4.\end{aligned}$$

We only estimate B_4 since estimates of B_1, B_2 and B_3 can be concluded by applying the same techniques.

For each integer $h\geq1$, let $\mathcal{F}_h^{i,l}=\{R'=I'\times J'\in A_{i,l}(R):|3(2^iI'\times2^lJ')\cap R|\geq\frac{1}{2^h}|2^iI'\times2^lJ'|\}$. Let

$$\mathcal{D}_h^{i,l}=\mathcal{F}_h^{i,l}\backslash\mathcal{F}_{h-1}^{i,l}\quad\text{and}\quad\Omega_h^{i,l}=\bigcup_{R'\in\mathcal{D}_h^{i,l}}R'.$$

Then

$$B_4 = \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{i,l\geq 1}\sum_{h\geq 1}\sum_{R'\in\mathcal{D}_h^{i,l}} 2^{-(i+l)M} r(R,R') \times |I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}.$$

To estimate the right-hand side of the above inequality, we only need to estimate

$$\sum_{R'\in\mathcal{D}_h^{i,l}} r(R,R').$$

Firstly, because $R'\in A_{i,l}(R)$, one has $3R\cap 3(2^iI'\times 2^jJ')\neq\emptyset$. For $R'\in\mathcal{D}_h^{i,l}$, there are also four cases:

Case 1: $|2^iI'|\geq|I|$, $|2^lJ'|\geq|J|$; Case 2: $|2^iI'|\geq|I|$, $|2^lJ'|\leq|J|$;
Case 3: $|2^iI'|\leq|I|$, $|2^lJ'|\geq|J|$; Case 4: $|2^iI'|\leq|I|$, $|2^lJ'|\leq|J|$.

It is easy to see that Case 2, Case 3 are none. Then $B_4 = B_4^1 + B_4^4$.

For $R'\in$ Case 1, one has

$$|R|\leq|3R\cap 3(2^iI'\times 2^lJ')|\leq|3(2^iI'\times 2^lJ')\cap\Omega^{0,0}|\leq\frac{1}{2^{h-1}}|3(2^iI'\times 2^lJ')|.$$

We divide $R'\in$ Case 1 into four categories:

Category 1.1 $|I|\leq|I'|$, $|J|\leq|J'|$; Category 1.2 $|I|\leq|I'|$, $|J|>|J'|$;
Category 1.3 $|I|>|I'|$, $|J|\leq|J'|$; Category 1.4 $|I|>|I'|$, $|J|>|J'|$.

In Category 1.1, we have $2^{i(m-1)+l}|R'| = 2^{h-1-2m+\eta}|R|$ for some integer $\eta > 0$. Moreover, for any fixed $\eta>0$ and R, the number of such R''s is less than $7^m2^{i(m-1)+l}$ since $3R\cap 3(2^iI'\times 2^jJ')\neq\emptyset$ and $|2^iI'|\geq|I|$, $|2^lJ'|\geq|J|$. Hence

$$\sum_{R'\in\text{Category 1.1}} r(R,R') = \sum_{\eta\geq 0} 7^m 2^{i(m-1)+l}\Big(\frac{2^{i(m-1)+l}}{2^{h-1-2m+\eta}}\Big)^L \lesssim 2^{2imL+2lL-hL}.$$

In Category 1.2, one has $2^{i(m-1)+l}|I'| = 2^{h-1-2m+\theta}|I|$ for some integer $\theta>0$. For each fixed $\theta>0$ and R, the number of such I''s must be less than $7^{m-1}2^{i(m-1)}$. Moreover, from $|J'|<|J|\leq|2^lJ'|$, we have $|2^\beta J'| = |J|$ for some positive integer β with $1\leq\beta\leq l$. For each fixed $\beta>0$ and J, the number of such J''s must be less than $72^{(l-\beta)}$. Hence

$$\begin{aligned}\sum_{R\in\text{Category 1.2}} r(R,R') &\leq 7^m\sum_{\eta>0}\sum_{\beta=1}^{l}\frac{2^{i(m-1)}(2^{i(m-1)+l})^L}{(2^{h-1-2m+\theta})^L}\frac{2^{l-\beta}}{2^{L\beta}}\\ &\lesssim l2^{2iLm-hL+2lL}.\end{aligned}$$

With a similar argument, one has

$$\sum_{R\in\text{Category 1.3}} r(R,R')\lesssim i2^{2iLm-hL+2lL}.$$

In Category 1.4, one has $2^{h-1}\leq 3^m2^{i(m-1)+l}$, and with $|R'|\leq|R|\leq|2^iI'\times 2^lJ'|$, one has $2^{i(m-1)+l}|R'| = 2^\lambda|R|$ for some integer $0\leq\lambda\leq i(m-1)+l$. For each fixed $\lambda\geq 0$, the number of such R''s must be less than $7^m2^{i(m-1)+l}$. So

$$\sum_{R\in\text{Category 1.4}} r(R,R') = \sum_{\lambda=0}^{i(m-1)+l} 7^m2^{i(m-1)+l}\Big(\frac{2^\lambda}{2^{i(m-1)+l}}\Big)^L\lesssim 2^{im+l}.$$

Therefore

$$\begin{aligned} B_4^1 &= \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{i,l\geq 1}\sum_{h\geq 1}\sum_{R'\in \text{Case 1}} 2^{-(i+l)M} r(R,R')|I'|^{\alpha_1/(m-1)} \\ &\qquad\qquad \times |J'|^{\alpha_2}|t_{R'}||R'|^{-1/2} \\ &\leq \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} \sum_{i,l\geq 1}\sum_{h\geq 1}\sum_{R'\in \text{Case 1}} 2^{-(i+l)M} r(R,R')|\Omega_h^{i,l}| \\ &\quad \times \frac{1}{|\Omega_h^{i,l}|^{\frac{1}{p}-1}} \sup_{R\in\Omega_h^{i,l}} |I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2} \\ &\lesssim \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R'=I'\times J'\subseteq\Omega, R'\in\mathcal{D}} |I'|^{\alpha_1/(m-1)}|J'|^{\alpha_2}|t_{R'}||R'|^{-1/2}. \end{aligned}$$

With a similar argument for the rest, we can obtain the desired result. We then have completed the proof. □

Proof of Theorem 1.2. Suppose that $\varphi^{(1)}$ and $\varphi^{(2)}$ are functions satisfying conditions in (1.5)-(1.6) and (1.7)-(1.8), respectively. For $f\in CMO_p^{\alpha,q}$, setting $\varphi_R(x) = |R|^{1/2}\varphi_{j,k}(x'-x_I, x_m-x_J)$ with $R\in\Pi_{j,k}$, and $t_R = \langle f,\varphi_R\rangle = \varphi_{j,k} * f(x_I,x_J)$, by Theorem 2.3, we have $f=\sum_R t_R\varphi_R$ and $t=\{t_R\}_R\in C_p^{\alpha,q}$. Then (2.8) gives

$$\begin{aligned} &\sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|\psi_{j,k} * f(x_I,x_J)|)^q |R| \Big)^{1/q} \\ &\lesssim \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-\frac{1}{q'}}} \Big(\sum_{R=I\times J\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|t_R||R|^{-1/2})^q |R| \Big)^{1/q} \end{aligned}$$

The conclusion of Theorem 1.2 follows immediately. □

3. Imbedding theorems

In this section, we give a characterization of imbedding of ℓ^r spaces into $\dot{f}_p^{\alpha,q}$ and imbedding of $\dot{f}_p^{\alpha,q}$ into ℓ^r spaces. This result was first established by Verbitsky [63] in the dyadic cubes with respect to an arbitrary positive locally finite measure on the Euclidean space and was generalized by Bownik [1] to discrete anisotropic Triebel-Lizorkin sequence spaces.

Theorem 3.1. *Assume that* Π *is any subfamily* $\mathcal{D}$ *and* $\{c_R\}_{R\in\Pi}$ *is any positive sequence.*

(i) Suppose $0<p<r\leq q\leq\infty$. *Then the inequality*

$$\Big\| \Big(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R \Big)^{1/q} \Big\|_{L^p} \leq C\|s\|_{\ell^r} \tag{3.1}$$

holds for all scalar sequences $s=\{s_R\}_{R\in\Pi}$ *if and only if*

$$\int \sup_{R\in\Pi} \Big((c_R)^r |R| \Big)^{p/(r-p)} \chi_R(x)dx < \infty. \tag{3.2}$$

(ii) Suppose $0<q\leq r<p<\infty$. *Then the inequality*

$$\Big\| \Big(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R \Big)^{1/q} \Big\|_{L^p} \geq C\|s\|_{\ell^r} \tag{3.3}$$

holds for all scalar sequences $s=\{s_R\}_{R\in\Pi}$ *if and only if* (3.2) *holds.*

To establish Theorem 3.1 we will follow the original approach of Verbitsky [63]. Thus, we recall the following known results.

Lemma 3.2 (Theorem 1 (i)(ii) of [63])**.** *Let $0 < p < r \le q \le \infty$. Then*

$$\Big\| \Big(\sum_{i\in I} (|s_i|^q \varphi_i^q \Big)^{1/q} \Big\|_{L^p} \le C\|s\|_{\ell^r}$$

holds if

$$\int \sup_i \Big[\Big(\phi_i^{r-p}(x) \|\phi_i\|_{L^p}^p \Big) \Big]^{p/(r-p)} dx < \infty.$$

Suppose $0 < q \le r < p < \infty$. Then

$$\Big\| \Big(\sum_{i} (|s_i|^q \varphi_i^q \Big)^{1/q} \Big\|_{L^p} \ge C\|s\|_{\ell^r}$$

holds if

$$\int \sup_i \Big[\Big(\phi_i^{p-r}(x)/\|\phi_i\|_{L^p}^p \Big) \Big]^{p/(p-r)} dx < \infty.$$

Lemma 3.3 (Theorem 1.1 of [53])**.** *Let $0 < p < r < \infty$, I be any index set, and $\{\varphi_i\}_{i\in I}$ be a family in L^p. Then, the inequality*

$$\Big\| \sup_{i\in I} (|s_i|\varphi_i) \Big\|_{L^p} \le C\|s\|_{\ell^r}$$

holds for all scalar sequences $s = \{s_i\}_{i\in I} \in \ell^r$ if and only if there exists a non-negative measurable function $F \ge 0$ with $\int F(x)dx \le 1$, such that

$$\sup_{i\in I} \|F^{-1/p}\varphi_i\|_{L^{r,\infty}(\mu)} < \infty,$$

where $L^{r,\infty}(\mu)$ is a weak-L^r with respect to the measure $du(x) = F(x)dx$ defined by

$$\|f\|_{L^{r,\infty}(\mu)} = \Big(\sup_{t>0} t^r \mu(\{x \in R^m : |f(x)| > t\}) \Big)^{1/r} < \infty$$

for $f \in L^{r,\infty}(\mu)$.

Lemma 3.4 (Remark 3 of [63])**.** *If $0 < q = r < p < \infty$, then*

$$\| \Big(\sum_{i\in I} |s_i|^q \phi_i^q \Big)^{1/q} \|_{L^p} \ge C\|s\|_{\ell^r}$$

holds if and only if there exists $F \ge 0$ such that

$$\int F(x)dx \le 1 \quad \textit{and} \quad \inf_i \|F^{-1/p}\phi_i\|_{L^r(\mu)} > 0,$$

where $d\mu(x) = F(x)dx$.

Proof of Theorem 3.1. We begin with the proof of part (i). Firstly (3.2)$\Rightarrow$ (3.1) is a direct consequence of Lemma 3.2 since $\int (c_R\chi_R(x))^p dx = (c_R)^p|R|$.

Now suppose that (3.1) holds for $p < r$. By imbedding $\ell^q \hookrightarrow \ell^\infty$ and Lemma 3.3, there exists a non-negative measurable function $F \ge 0$ with $\int F(x)dx \le 1$, such that

$$\sup_{R\in\Pi} \|F^{-1/p} c_R\chi_R\|_{L^{r,\infty}(\mu)} = \sup_{R\in\Pi} c_R \|F^{-1/p}\chi_R\|_{L^{r,\infty}(\mu)} < \infty, \tag{3.4}$$

where $d\mu = F dx$. Let $f = F^{-1/p}\chi_R$; then $\|f\|_{L^p(\mu)} = |R|^{1/p}$. Suppose $p < s < r$ and $1/s = t/p + (1-t)/r$ with $0 < t < 1$. Applying the well-known interpolation inequality (e.g. Proposition 1.1.14 of [25])

$$\|f\|_{L^s(\mu)} \le C\|f\|^t_{L^{p,\infty}(\mu)}\|f\|^{1-t}_{L^{r,\infty}(\mu)},$$

one has for any $R \in \Pi$,

$$(\int_R F^{-s/p+1}dx)^{1/s} \le C|R|^{t/p}\|F^{-1/p}\chi_R\|^{1-t}_{L^{r,\infty}(\mu)}.$$

Letting $\delta = s/p - 1$ and combining the above inequality with (3.4), we obtain

$$\Big(c_R|R|^{1/r}\Big)^{pr/(r-p)}\Big(\frac{1}{|R|}\int_R F^{-\delta}dx\Big)^{1/\delta} \le C < \infty.$$

On the other hand, by Hölder's inequality with exponents $\frac{\delta+\varepsilon}{\varepsilon}, \frac{\delta+\varepsilon}{\delta}$ one has

$$\Big(\frac{1}{|R|}\int_R F^{-\delta}dx\Big)^{1/\delta}\Big(\frac{1}{|R|}\int_R F^{\varepsilon}dx\Big)^{1/\varepsilon} \ge 1,$$

for all $\delta, \varepsilon > 0$. Hence

$$\Big(c_R|R|^{1/r}\Big)^{pr/(r-p)} \le C\Big(\frac{1}{|R|}\int_R F^{\varepsilon}dx\Big)^{1/\varepsilon} \le C(M_s(F^{\varepsilon})(x))^{1/\varepsilon}$$

for $x \in R$, where M_s denotes the strong maximal operator. Since M_s is bounded on $L^{1/\varepsilon}$ for $0 < \varepsilon < 1$, we have

$$\int \sup_{R\in\Pi}\Big((c_R)^r|R|\Big)^{p/(r-p)}\chi_R dx \lesssim \int (M_s(F^{\varepsilon})(x))^{1/\varepsilon}dx \lesssim \int F(x)dx < \infty.$$

We then have completed the proof of part (i) of Theorem 3.1.

We now give the proof of part (ii). The second part of Lemma 3.2 gives the proof of (3.2)$\Rightarrow$ (3.3). Now suppose that (3.3) holds. We first prove (3.2) for $q = r$ following the original argument of Verbitsky [63]. By Lemma 3.4, there exists $F \in L^1$, $F \ge 0$, such that

$$\inf_{R\in\Pi}\int F^{1-r/p}(c_R\chi_R)^r dx = \inf_{R\in\Pi}(c_R)^r\int_R F^{1-r/p}dx > 0.$$

It follows from the above inequality that

$$\begin{aligned}\int \sup_{R\in\Pi}\Big((c_R)^r|R|\Big)^{p/(r-p)}\chi_R(x)dx &\le \int \sup_{R\in\Pi}\Big(\frac{1}{|R|}\int_R F^{1-r/p}dy\Big)^{p/(p-r)}\chi_R(x)dx\\ &\le \int \Big(M_s(F^{1-r/p})(x)\Big)^{p/(p-r)}dx\\ &\le C\int F(x)dx < \infty.\end{aligned}$$

When $q < r$, we use the argument of Bownik [1] by taking advantage of the already established duality of $\dot{f}^{\alpha,1}_p$, $p > 1$. Note that by duality

$$\|s\|_{\ell^r} = \sup_{t=\{t_R\}}\frac{(\sum|s_R|^q|t_R|^q)^{1/q}}{\|t\|_{\ell^{rq/(r-q)}}}.$$

Hence (3.3) is equivalent to the inequality

$$(\sum_{R\in\Pi}|s_R|^q|t_R|^q)^{1/q} \le C\|\Big(\sum_{R\in\Pi}|s_R|^q(c_R)^q\chi_R\Big)^{1/q}\|_{L^p}\|t\|_{\ell^{rq/(r-q)}}. \tag{3.5}$$

On the other hand, since $1 < p/q < \infty$, by the already established duality $(\dot{f}^{\alpha,1}_{p/q})^* = \dot{f}^{-\alpha,\infty}_{p/(p-q)}$, one has for $\alpha = (\frac{1}{2}(m-1), \frac{1}{2})$,

$$\sup_{u=\{u_R\}} \frac{|\sum u_R \bar{v}_R|}{\|\sum u_R \chi_R\|_{L^{p/q}}} = \sup_{u=\{u_R\}} \frac{|\langle u, v\rangle|}{\|u\|_{\dot{f}^{\alpha,1}_{p/q}}} = \|v\|_{\dot{f}^{-\alpha,\infty}_{p/(p-q)}} = \|\sup_{R\in\mathcal{D}} |v_R||R|^{-1}\chi_R\|_{L^{p/(p-q)}}. \tag{3.6}$$

Let

$$v_R = \begin{cases} |t_R|^q (c_R)^{-q}, & R \in \Pi; \\ 0, & R \in \mathcal{D}\backslash\Pi, \end{cases}$$

and

$$u_R = \begin{cases} |s_R|^q (c_R)^{q}, & R \in \Pi; \\ 0, & R \in \mathcal{D}\backslash\Pi. \end{cases}$$

Then (3.6) may be rewritten in the following form by taking the qth roots:

$$\sup_{s=\{s_R\}} \frac{\left|\sum_{R\in\Pi} |s_R|^q |t_R|^q\right|^{1/q}}{\left\|(\sum_{R\in\Pi} |s_R|^q (c_R)^q \chi_R)^{1/q}\right\|_{L^p}} = \left\|\sup_{R\in\Pi} |t_R|(c_R)^{-1}|R|^{-1/q}\chi_R\right\|_{L^{pq/(p-q)}}. \tag{3.7}$$

Let $p_1 = pq/(p-q)$, $r_1 = rq/(r-q)$ and $\tilde{c}_R = (c_R)^{-1}|R|^{-1/q}$. Combining (3.5) with (3.7) yields

$$\left\|\sup_{R\in\Pi} |t_R|(\tilde{c}_R)\chi_R\right\|_{L^{p_1}} \le C\|t\|_{\ell^{r_1}}$$

for all $t = \{t_R\}_R$. Using the facts that $p_1 r_1/(r_1 - p_1) = pr/(p-r)$, $p_1 < r_1$, and applying (i) of Theorem 3.1, we get from the preceding inequality

$$\int \sup_{R\in\Pi} \Big((\tilde{c}_R)^{r_1}|R|\Big)^{p_1/(r_1-p_1)} \chi_R(x)dx = \int \sup_{R\in\Pi} \Big((c_R)^{r}|R|\Big)^{p/(r-p)} \chi_R(x)dx < \infty.$$

Hence (3.2) holds for $q < r$. We thus have completed the proof. □

4. Duality of $\dot{f}^{\alpha,q}_p$

Theorem 4.1. *Suppose* $1 < p < \infty$, $0 < q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. *Then*

$$(\dot{f}^{\alpha,q}_p)^* = \dot{f}^{-\alpha,q'}_{p'}.$$

Proof. For any $s \in \dot{f}^{\alpha,q}_p$, $t \in \dot{f}^{\alpha,q'}_{p'}$ we have

$$\begin{aligned}
&|\sum_{R\in\mathcal{D}} s_R \bar{t}_R| \\
\le\ & \int \sum_{R\in\mathcal{D}} |I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x)|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\tilde{\chi}_R(x)dx \\
\le\ & \int \Big(\sum_{R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x))^q\Big)^{1/q} \\
& \qquad \times \Big(\sum_{R\in\mathcal{D}} (|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\tilde{\chi}_R(x))^{q'}\Big)^{1/q'} dx \\
\le\ & \|s\|_{\dot{f}^{\alpha,q}_p}\|t\|_{\dot{f}^{-\alpha,q'}_{p'}},
\end{aligned}$$

by duality if $1 \le q < \infty$ or by imbedding $\ell^q \hookrightarrow \ell^1$ if $0 < q < 1$. This yields that t is a continuous linear functional on $\dot{f}_p^{\alpha,q}$ and

$$\|t\|_{(\dot{f}_p^{\alpha,q})^*} \le \|t\|_{\dot{f}_{p'}^{-\alpha,q'}}.$$

For the converse direction, we split its proof into 2 cases: $(p,q) \in (1,\infty) \times [1,\infty)$ and $(1,\infty) \times (0,1)$.

Case 1: $(p,q) \in (1,\infty) \times [1,\infty)$. This case is elementary. Take any $l \in (\dot{f}_p^{\alpha,q})^*$. Then there exists some sequence $t = t_R$ such that $l(s) = \sum_R s_R \bar{t}_R$ for any $s = \{s_R\}_R \in \dot{f}_p^{\alpha,q}$. Now we need a well-known result that

$$(L^p(l^q))^* = L^{p'}(l^{q'}) \tag{4.1}$$

if $1 < p < \infty$, $1 < q < \infty$, where

$$L^p(l^q) = \Big\{ f = \{f_v\} : \|f\|_{L^p(l^q)} = \|(\sum_v |f_v|^q)^{1/q}\|_{L^p} < \infty \Big\},$$

with the pairing $\langle f, g\rangle = \int \sum_v f_v \bar{g}_v$ for $f \in L^p(l^q)$, $g \in L^{p'}(l^{q'})$ (see e.g. [62]). Let $I : \dot{f}_p^{\alpha,q} \to L^p(l^q)$ be defined by

$$I(s) = \{f_{j,k}\}_{j,k\in\mathbb{Z}}, \qquad \text{where } f_{j,k} = \sum_{R\in\Pi_{j,k}} |I|^{\alpha_1/(m-1)} |J|^{\alpha_2} s_R \tilde{\chi}_R(x).$$

Clearly, the map I is a linear isometry onto a subspace of $L^p(l^q)$. By the Hahn-Banach Theorem, there exists $\tilde{l} \in (L^p(l^q))^*$ such that $\tilde{l} \circ I = l$ and $\|\tilde{l}\| = \|l\|$. By (4.1), $\tilde{l}(f) = \langle f, g\rangle$ for some $g \in L^{p'}(l^{q'})$ with $\|g\|_{L^{p'}(l^{q'})} \le \|l\|$. Hence

$$\begin{aligned} l(s) &= \tilde{l}(I(s)) = \int \sum_{j,k} f_{j,k} \bar{g}_{j,k} \\ &= \int \sum_{j,k} (\sum_{R\in\Pi_{j,k}} |I|^{\alpha_1/(m-1)} |J|^{\alpha_2} s_R \tilde{\chi}_R(x)) \bar{g}_{j,k}(x) dx \\ &= \sum_{j,k} \sum_{R\in\Pi_{j,k}} s_R \Big(|I|^{\alpha_1/(m-1)} |J|^{\alpha_2} |R|^{-1/2} \int_R \bar{g}_{j,k}(x) dx \Big) \\ &= \sum_{j,k} \sum_{R\in\Pi_{j,k}} s_R t_R = \langle t, s\rangle, \end{aligned}$$

for all $s \in \dot{f}_p^{\alpha,q}$, where $t = \{t_R\}_R$ with $t_R = |I|^{\alpha_1/(m-1)} |J|^{\alpha_2} |R|^{-1/2} \int_R \bar{g}_{j,k}(x) dx$ for $R \in \Pi_{j,k}$. Then

$$\begin{aligned} \|t\|_{\dot{f}_{p'}^{-\alpha,q'}} &= \|\sum_{j,k} \sum_{R\in\Pi_{j,k}} (\frac{1}{|R|} \int_R g_{j,k}) \chi_R(x)\|_{L^{p'}} \\ &\le \|\{M_s(g_{j,k})\}\|_{L^{p'}(l^{q'})} \lesssim \|g\|_{L^{p'}(l^{q'})} \le \|l\|. \end{aligned}$$

This completes the proof of Case 1.

Case 2: $(p,q) \in (1,\infty) \times (0,1)$. In this case, $L^p(l^q)$ is not a normed space; hence we can't use the Hahn-Banach theorem.

Take $l \in (\dot{f}_p^{\alpha,q})^*$. Then there exists some sequence $t = t_R$ such that for any $s = \{s_R\}_R \in \dot{f}_p^{\alpha,q}$,

(4.2)
$$|l(s)| = |\sum_R s_R \bar{t}_R| \le C\|s\|_{\dot{f}_p^{\alpha,q}} = C\|\Big(\sum_{R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x))^q\Big)^{1/q}\|_{L^p}.$$

If we prove the estimates

$$\|t\|_{\dot{f}_{p'}^{-\alpha,\infty}} = \|\sup_{R\in\mathcal{D}} (|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R|\tilde{\chi}_R(x)\|_{L^{p'}} < \infty,$$

we then complete the proof.

Define $\Pi = \{R \in \mathcal{D}, t_R \neq 0\}$, and let $u_R = s_R\bar{t}_R, c_R = \frac{|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}}{|R|^{1/2}|t_R|}$ for $R \in \Pi$. We may assume that $s_R\bar{t}_R \ge 0$ for all $R \in \mathcal{D}$ by choosing proper s_R. Moreover we can assume $s_R = 0$ if $R \notin \Pi$. Then (4.2) can be rewritten as

$$\|u\|_{\ell^1} \le c\|\Big(\sum_{R\in\Pi} |u_R|^q (c_R)^q \chi_R\Big)^{1/q}\|_{L^p}$$

for all $u = \{u_R\}_{R\in\Pi}$. Then (ii) of Theorem 3.1 with $0 < q < r = 1 < p < \infty$ yields

$$\int \sup_{R\in\Pi} \Big((c_R\chi_R)|R|\Big)^{p/(1-p)} dx < \infty,$$

that is,

$$\int \sup_{R\in\Pi} \Big((|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}\chi_R)\Big)^{p/(p-1)} dx < \infty.$$

We thus have completed the proof. □

Theorem 4.2. *Suppose* $0 < p \le 1$, $0 < q < \infty$, $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. *Then*

$$(\dot{f}_p^{\alpha,q})^* = C_p^{-\alpha,q'}.$$

Proof. We first assume $1 \le q < \infty$. Suppose $t \in C_p^{-\alpha,q'}$. For any $s \in \dot{f}_p^{\alpha,q}$, set

$$h(x) = \Big(\sum_{R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x))^q\Big)^{1/q},$$

and for $k \in \mathbb{Z}$,

$$\Omega_k = \{x \in \mathbb{R}^m : h(x) > 2^k\},$$

$$\mathcal{B}_k = \{R \in \mathcal{D} : |R \cap \Omega_k| > \frac{1}{2}|R|, |R \cap \Omega_{k+1}| \le \frac{1}{2}|R|\}.$$

One can obtain

$$\begin{aligned} |\sum_{R\in\mathcal{D}} s_R t_R| &= \sum_k \sum_{R\in\mathcal{B}_k} (|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}}) \\ &\qquad \times (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}}|R|^{-\frac{1}{q'}}) \\ &\le \Big\{\sum_k \Big[\sum_{R\in\mathcal{B}_k} (|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}})^{q'}\Big]^{p/q'} \\ &\qquad \times \Big[\sum_{R\in\mathcal{B}_k} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}}|R|^{-\frac{1}{q'}})^q\Big]^{p/q}\Big\}^{1/p}. \end{aligned} \tag{4.3}$$

Let $\tilde{\Omega}_k = \{x \in \mathbb{R}^m, \mathcal{M}_s(\chi_{\Omega_k})(x) > \frac{1}{2}\}$; then $|\tilde{\Omega}_k| \lesssim |\Omega_k|$. One sees that if $R \in \mathcal{B}_k$, then one has $R \subseteq \tilde{\Omega}_k$. So

$$\begin{aligned}
&\Big[\sum_{R\in\mathcal{B}_k}(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}})^{q'}\Big]^{1/q'}\\
&=\frac{1}{|\tilde{\Omega}|^{\frac{1}{p}-\frac{1}{q}}}\Big[\sum_{R\in\mathcal{B}_k}(|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-\frac{1}{2}}|R|^{\frac{1}{q'}})^{q'}\Big]^{1/q'}|\tilde{\Omega}|^{\frac{1}{p}-\frac{1}{q}}\\
&\lesssim \|t\|_{C_p^{-\alpha,q'}}|\Omega_k|^{\frac{1}{p}-\frac{1}{q}}.
\end{aligned}$$

On the other hand, using the fact that if $R \in \mathcal{B}_k$, $R \subseteq \tilde{\Omega}_k$, one also obtains

$$\frac{1}{2}|R| < |R\backslash\Omega_{k+1}| = |R\cap\tilde{\Omega}_k\backslash\Omega_{k+1}|.$$

Hence

$$\begin{aligned}
&\Big[\sum_{R\in\mathcal{B}_k}(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}}|R|^{-\frac{1}{q'}})^{q}\Big]^{1/q}\\
&=\Big[\sum_{R\in\mathcal{B}_k}(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{\frac{1}{2}-\frac{1}{q'}-\frac{1}{q}})^{q}|R|\Big]^{1/q}\\
&\lesssim\Big[\sum_{R\in\mathcal{B}_k}(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R||R|^{-1/2})^{q}|R\cap\tilde{\Omega}_k\backslash\Omega_{k+1}|\Big]^{1/q}\\
&=\Big(\int_{\tilde{\Omega}_k\backslash\Omega_{k+1}}\sum_{R\in\mathcal{B}_k}(|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|s_R|\tilde{\chi}_R(x))^{q}dx\Big)^{1/q}\\
&\le\Big(\int_{\tilde{\Omega}_k\backslash\Omega_{k+1}}h^q(x)dx\Big)^{1/q}\lesssim 2^k|\Omega_k|^{1/q}.
\end{aligned}$$

Combining (4.3) with the above inequality, one obtains

$$|\sum_{R\in\mathcal{D}}s_Rt_R|\lesssim\|t\|_{C_p^{\alpha,q'}}(\sum_k 2^{kp}|\Omega_k|)^{1/p}\lesssim\|t\|_{C_p^{-\alpha,q'}}\|s\|_{\dot{f}_p^{\alpha,q}}.$$

Next, we will prove $C_p^{-\alpha,q'}\supseteq(\dot{f}_p^{\alpha,q})^*$. Let $\ell\in(\dot{f}_p^{\alpha,q})^*$. Then there exists some $t=\{t_R\}_R$ such that for every $s=\{s_R\}_R\in\dot{f}_p^{\alpha,q}$, $\ell(s)=\sum_R s_Rt_R$ and

$$|\sum_R s_Rt_R|\le\|\ell\|_{(\dot{f}_p^{\alpha,q})^*}\|s\|_{\dot{f}_p^{\alpha,q}}.$$

Once having shown $t\in C_p^{-\alpha,q'}$, we will then complete the proof. For any open set $\Omega\subseteq\mathbb{R}^m$ with finite measure, let $X=\{R\in\mathcal{D}:R\subseteq\Omega\}$, and let μ be a measure on X such that the μ-measure of R is $|R|$ if $1\le q<\infty$ or $\mu(R)=1$ if $q=1$. Then by

the above inequality, one has

$$\begin{aligned}
&\Big(\sum_{R\subseteq\Omega, R\in\mathcal{D}} (|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2})^{q'}|R|\Big)^{1/q'}\\
&= \||I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}\|_{l^{q'}(X,d\mu)}\\
&= \sup_{\|s\|_{l^q(X,d\mu)}\le 1}\Big|\sum_{R\subseteq\Omega, R\in\mathcal{D}} |I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2}s_R|R|\Big|\\
&\le \|\ell\|_{(\dot f_p^{\alpha,q})^*}\sup_{\|s\|_{l^q(X,d\mu)}\le 1}\||I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|R|^{-1/2}s_R|R|\|_{\dot f_p^{\alpha,q}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\||I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|R|^{-1/2}s_R|R|\|_{\dot f_p^{\alpha,q}}\\
&= \|\Big(\sum_{R\subseteq\Omega, R\in\mathcal{D}} (|I|^{\alpha_1/(m-1)}|J|^{\alpha_2}|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|R|^{-1/2}|s_R||R|\tilde\chi_R(x))^q\Big)^{1/q}\|_{L^p}\\
&= \|\Big(\sum_{R\subseteq\Omega, R\in\mathcal{D}} (|s_R|\chi_R(x))^q\Big)^{1/q}\|_{L^p}\\
&\le \Big\{\int_\Omega \sum_{R\subseteq\Omega, R\in\mathcal{D}} (|s_R|\chi_R(x))^q dx\Big\}^{1/q}|\Omega|^{\frac1p-\frac1q}
\end{aligned}$$

by Holder's inequality since $0<p\le 1\le q<\infty$. So

$$\begin{aligned}
&\Big(\sum_{R\subseteq\Omega, R\in\mathcal{D}} (|I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2})^{q'}|R|\Big)^{1/q'}\\
&\le \|\ell\|_{(\dot f_p^{\alpha,q})^*}\sup_{\|s\|_{l^q(X,d\mu)}\le 1}\Big\{\int_\Omega \sum_{R\subseteq\Omega, R\in\mathcal{D}} (|s_R|\chi_R(x))^q dx\Big\}^{1/q}|\Omega|^{\frac1p-\frac1q}\\
&\le \|\ell\|_{(\dot f_p^{\alpha,q})^*}|\Omega|^{\frac1p-\frac1q},
\end{aligned}$$

that is, $t\in C_p^{-\alpha,q'}$.

When $0<q<1$, by the trivial imbedding $\dot f_p^{\alpha,q}\to\dot f_p^{\alpha,1}$, one has

$$(\dot f_p^{\alpha,q})^*\supseteq(\dot f_p^{\alpha,1})^* = C_p^{-\alpha,\infty}.$$

To show the other direction, as above, let $\ell\in(\dot f_p^{\alpha,q})^*$. Then there exists some $t=\{t_R\}_R$ such that for every $s=\{s_R\}_R\in\dot f_p^{\alpha,q}$, $\ell(s)=\sum_R s_R t_R$ and

$$|\sum_R s_R t_R|\le\|\ell\|_{(\dot f_p^{\alpha,q})^*}\|s\|_{\dot f_p^{\alpha,q}}.$$

We now prove $\|t\|_{C_p^{-\alpha,\infty}} = \sup_{\Omega} \frac{1}{|\Omega|^{\frac{1}{p}-1}} \sup_{R\subseteq\Omega, R\in\mathcal{D}} |I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2} < \infty$. For any fixed $R = I \times J \in \mathcal{D}$, let $\delta_{Q,R} = 1$ if $Q = R$; otherwise $\delta_{Q,R} = 0$. So

$$\begin{aligned}
&\sup_{R\subseteq\Omega, R\in\mathcal{D}} |I|^{-\alpha_1/(m-1)}|J|^{-\alpha_2}|t_R||R|^{-1/2} \\
&= \sup_{R\subseteq\Omega, R\in\mathcal{D}} \sum_{Q=\tilde{I}\times\tilde{J}\subseteq\Omega} \delta_{Q,R}|\tilde{I}|^{-\alpha_1/(m-1)}|\tilde{J}|^{-\alpha_2}|Q|^{-1/2}|t_Q| \\
&\leq \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{(\dot{f}_p^{\alpha,q})^*}\|\delta_{Q,R}|\tilde{I}|^{-\alpha_1/(m-1)}|\tilde{J}|^{-\alpha_2}|Q|^{-1/2}\|_{\dot{f}_p^{\alpha,q}} \\
&= \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{(\dot{f}_p^{\alpha,q})^*}\||R|^{-1}\chi_R\|_{L^p} \\
&= \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{(\dot{f}_p^{\alpha,q})^*}|R|^{\frac{1}{p}-1} \leq \sup_{R\subseteq\Omega, R\in\mathcal{D}} \|\ell\|_{(\dot{f}_p^{\alpha,q})^*}|\Omega|^{\frac{1}{p}-1}
\end{aligned}$$

since $0 < p \leq 1$, which implies our desired results, and we thus have completed the proof of Theorem 4.2. □

5. Duality of $\dot{F}_p^{\alpha,q}$

In this section we derive the duality of Theorem 1.1 and Theorem 1.3 from Theorem 4.1 and Theorem 4.2, respectively, in the sequence space cases. It is known from Proposition 3.1 in [11] that $\mathcal{S}_0(\mathbb{R}^m)$ is dense in $\dot{F}_p^{\alpha,q}(\mathbb{R}^m)$ for $0 < p, q < \infty$.

Proof of Theorem 1.1. Let $g \in \dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^m)$, $f \in \mathcal{S}_0(\mathbb{R}^m)$ and $1 < p < \infty, 0 < q < \infty$. Then by the identity (2.2) one has $\langle f, g\rangle = \langle S_\psi f, S_\psi g\rangle$. Hence

$$|\langle f, g\rangle| \leq \|S_\psi f\|_{\dot{f}_p^{\alpha,q}(\mathbb{R}^m)}\|S_\psi g\|_{\dot{f}_{p'}^{-\alpha,q'}(\mathbb{R}^m)} \lesssim \|f\|_{\dot{F}_p^{\alpha,q}(\mathbb{R}^m)}\|g\|_{\dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^m)}$$

by Theorem 4.1 and Theorem 2.1. This proves that $\|l_g\| \lesssim \|g\|_{\dot{F}_{p'}^{-\alpha,q'}(\mathbb{R}^m)}$.

Conversely, suppose $l \in (\dot{F}_p^{\alpha,q}(\mathbb{R}^m))^*$. Then $l_1 \equiv l \circ T_\psi \in (\dot{f}_p^{\alpha,q})^*$, so by Theorem 4.1, there exists $t = \{t_R\}_R \in \dot{f}_{p'}^{-\alpha,q'}$ such that

$$l_1(s) = \langle s, t\rangle = \sum_R s_R\bar{t}_R$$

for all $s = \{s_R\}_R \in \dot{f}_p^{\alpha,q}(\mathbb{R}^m)$. Moreover $\|t\|_{\dot{f}_{p'}^{-\alpha,q'}} \approx \|l_1\| \lesssim \|l\|$ for the boundedness of T_ψ. Note that $l_1 \circ S_\psi = l \circ T_\psi \circ S_\psi = l$ since $T_\psi \circ S_\psi$ is an identity by Theorem 2.1. Then letting $g = T_\psi(t)$ and $f \in \mathcal{S}_0(\mathbb{R}^m)$, one has

$$l(f) = l_1(S_\psi(f)) = \langle S_\psi(f), t\rangle = \langle f, T_\psi(t)\rangle = \langle f, g\rangle$$

by (2.3), which implies that $l = l_g$, and by Theorem 2.1 again, one has

$$\|g\|_{\dot{F}_{p'}^{-\alpha,q'}} = \|T_\psi(t)\|_{\dot{F}_{p'}^{-\alpha,q'}} \lesssim \|t\|_{\dot{f}_{p'}^{-\alpha,q'}} \lesssim \|l\|.$$

We have then completed the proof of Theorem 1.1. □

Proof of Theorem 1.3. One can go through the same process as above to finish the proof of Theorem 1.3. We shall omit the details. □

Proof of Theorem 1.4. We assume that $\mathcal{K}_i$ is the kernel of the convolution operator T_i, $i = 1, 2$, and T^* is the conjugate operator of T with the kernel $\mathcal{K}^*$. One may check that

$$\mathcal{K}^* * f(x) = \tilde{\mathcal{K}}_2 * \tilde{\mathcal{K}}_1 * f(x) = \tilde{\mathcal{K}}_1 * \tilde{\mathcal{K}}_2 * f(x)$$

for $f \in C_c^\infty$, where $\tilde{\mathcal{K}}_i(x) = \mathcal{K}_i(-x)$, $i = 1, 2$. Hence T^* is bounded on $\dot{F}_p^{\alpha,q}$ for all $0 < p, q < \infty$, $\alpha \in \mathbb{R}^2$ by Theorem 1.5 in [11] since $\tilde{\mathcal{K}}_i$ satisfy Definition 1.1, Definition 1.2, respectively.

For $\forall\ 1 \le q < \infty$, there exists a $0 < \tilde{q} < \infty$ such that $\tilde{q}' = q$. Then by Theorem 1.3,

$$\begin{aligned}
\|T(f)\|_{CMO_p^{\alpha,q}} &= \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} |\langle T(f), g\rangle| \\
&= \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} |\langle f, T^*(g)\rangle| \\
&\le \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} \|f\|_{CMO_p^{\alpha,q}} \|T^*(g)\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \\
&\lesssim \sup_{\|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \le 1} \|f\|_{CMO_p^{\alpha,q}} \|g\|_{\dot{F}_p^{-\alpha,\tilde{q}'}} \\
&\le \|f\|_{CMO_p^{\alpha,q'}}.
\end{aligned}$$

We thus have completed the proof of Theorem 1.4. □

6. Appendix

The multi-parameter Triebel-Lizorkin spaces we study here are associated with the composition of two singular integral operators with the specific dilations

$$\delta : (x', x_m) \to (\delta x', \delta x_m), \delta > 0$$

and

$$\delta : (x', x_m) \to (\delta x', \delta^2 x_m), \delta > 0.$$

The first is the classical isotropic dilations occurring in the classical Calderón-Zygmund singular integrals, while the second is non-isotropic and related to the heat equations (also Heisenberg groups).

As we explained in the introduction, these two dilations are motivated by the study of weak-$(1, 1)$ boundedness of the composition of two singular integrals by Phong and Stein [51]. This composition of such two singular integral operators is particularly interesting because they essentially arise naturally in the study of the $\bar{\partial}$-Neumann problem (see [26], [49], [50], [51]). This motivates us to study the function spaces associated with the composition of two such dilations and then the boundedness of relevant operators. It is worthwhile to note that the underlying multi-parameter structure we study is intrinsic to the composition of these two dilations. Nevertheless, the multi-parameter structures we consider are still in the framework of the translation-invariant environment. The more general case of translation non-invariant dilations will be studied in a forthcoming project.

Though we restrict our attention to the above two very specific dilations in this paper, all results in this paper can be carried out to the composition with more singular integral operators associated with more general non-isotropic homogeneities. To see this, let

$$\delta_i : (x_1, x_2, \cdots, x_m) \to (\delta_i^{\lambda_{i,1}} x_1, \delta_i^{\lambda_{i,2}} x_2, \cdots, \delta_i^{\lambda_{i,m}} x_m)$$

for $\delta_i > 0, \lambda_{i,t} > 0, 1 \le i \le n$ and $1 \le t \le m$.

For $x \in \mathbb{R}^m$ we denote $|x|_i = \sqrt{|x_1|^{\frac{2}{\lambda_{i,1}}} + |x_2|^{\frac{2}{\lambda_{i,2}}} + \cdots + |x_m|^{\frac{2}{\lambda_{i,m}}}}$. Let $\psi^{(i)} \in \mathcal{S}(\mathbb{R}^m)$ with

$$\operatorname{supp} \widehat{\psi^{(i)}} \subseteq \{(\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^m : \frac{1}{2} \le |\xi|_i \le 2\}$$

and

$$\sum_{j_i \in \mathbb{Z}} |\widehat{\psi^{(i)}}(2^{-j_i\lambda_{i,1}}\xi_1, 2^{-j_i\lambda_{i,2}}\xi_2, \cdots, 2^{-j_i\lambda_{i,m}}\xi_m)|^2 = 1 \quad \forall (\xi_1, \xi_2, \cdots, \xi_m) \in \mathbb{R}^m/\{0\}.$$

Set $\psi_{j_1,j_2,\cdots,j_n}(x) = \psi^{(1)}_{j_1} * \psi^{(2)}_{j_2} * \cdots * \psi^{(n)}_{j_n}(x)$, where

$$\psi^{(i)}_{j_i}(x) = 2^{j_i(\lambda_{i,1}+\lambda_{i,2}+\cdots+\lambda_{i,m})}\psi^{(i)}(2^{j_i\lambda_{i,1}}x_1, 2^{j_i\lambda_{i,2}}x_2, \cdots, 2^{j_i\lambda_{i,m}}x_m).$$

Then we can obtain the following general discrete Calderón reproducing formula:

Theorem B. *Suppose that $\psi^{(i)}, i = 1, \cdots, n$, are functions satisfying the above conditions, respectively. Then*

$$\begin{aligned}
& f(x_1, x_2, \cdots, x_m) \\
= & \sum_{j_1,\cdots,j_n \in \mathbb{Z}} \sum_{(\ell_1,\cdots,\ell_m) \in \mathbb{Z}^m} \prod_{t=1}^{m} 2^{-(j_1\lambda_{1,t} \wedge j_2\lambda_{2,t} \wedge \cdots \wedge j_n\lambda_{n,t})} \\
& \times (\psi_{j_1,j_2,\cdots,j_n} * f)(2^{-(j_1\lambda_{1,1} \wedge j_2\lambda_{2,1} \wedge \cdots \wedge j_n\lambda_{n,1})}\ell_1, \cdots, \\
& \qquad\qquad 2^{-(j_1\lambda_{1,m} \wedge j_2\lambda_{2,m} \wedge \cdots \wedge j_n\lambda_{n,m})}\ell_m) \\
& \times \psi_{j_1,j_2,\cdots,j_n}(x_1 - 2^{-(j_1\lambda_{1,1} \wedge j_2\lambda_{2,1} \wedge \cdots \wedge j_n\lambda_{n,1})}\ell_1, \cdots, \\
& \qquad\qquad x_m - 2^{-(j_1\lambda_{1,m} \wedge j_2\lambda_{2,m} \wedge \cdots \wedge j_n\lambda_{n,m})}\ell_m),
\end{aligned}$$

where the series converges in $L^2(\mathbb{R}^m)$, $\mathcal{S}_0(\mathbb{R}^m)$ and $\mathcal{S}'_0(\mathbb{R}^m)$.

With the above discrete Calderón reproducing formula, the multi-parameter Triebel-Lizorkin spaces with different homogeneities can be introduced as follows:

Definition. Let $0 < p, q < \infty, \alpha = (\alpha_1, \alpha_2, \cdots, \alpha_m) \in \mathbb{R}^m$. The multi-parameter Triebel-Lizorkin type spaces with different homogeneities $\dot{F}^{\alpha,q}_p(\mathbb{R}^m)$ are defined by

$$\dot{F}^{\alpha,q}_p(\mathbb{R}^m) = \{f \in \mathcal{S}'_0(\mathbb{R}^m) : \ \|f\|_{\dot{F}^{\alpha,q}_p(\mathbb{R}^m)} < \infty\},$$

where

$$\begin{aligned}
& \|f\|_{\dot{F}^{\alpha,q}_p(\mathbb{R}^m)} \\
= & \Big\| \Big(\sum_{j_1,\cdots,j_n \in \mathbb{Z}} \prod_{t=1}^{m} 2^{-(j_1\lambda_{1,t} \wedge j_2\lambda_{2,t} \wedge \cdots \wedge j_n\lambda_{n,t})\alpha_t p} \\
& \times \sum_{(\ell_1,\cdots,\ell_m) \in \mathbb{Z}^m} |(\psi_{j_1,j_2,\cdots,j_n} * f)(2^{-(j_1\lambda_{1,1} \wedge j_2\lambda_{2,1} \wedge \cdots \wedge j_n\lambda_{n,1})}\ell_1, \cdots, \\
& \qquad 2^{-(j_1\lambda_{1,m} \wedge \cdots \wedge j_n\lambda_{n,m})}\ell_m)|^q \chi_{I_1}(x_1)\chi_{I_2}(x_2)\cdots\chi_{I_m}(x_m) \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^m)},
\end{aligned}$$

where $\{I_t\}_{t=1,\cdots,m}$ are dyadic intervals in $\mathbb{R}$ with the side length $l(I_t) = 2^{-(j_1\lambda_{1,t} \wedge \cdots \wedge j_n\lambda_{n,t})}$, and the left end points of I_t are $2^{-(j_1\lambda_{1,t} \wedge j_2\lambda_{2,t} \wedge \cdots \wedge j_n\lambda_{n,t})}\ell_t$, respectively.

Applying the same techniques as in this paper, one can establish the duality theory (Theorem 1.1) of the multi-parameter Triebel-Lizorkin spaces associated with these more general non-isotropic dilations. The details of the proofs appear to be very lengthy and complicated to present in the more general situation. Therefore, we shall not discuss these in more detail in this paper.

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