

FINITE GENERATING SETS OF RELATIVELY HYPERBOLIC GROUPS AND APPLICATIONS TO GEODESIC LANGUAGES

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ABSTRACT. Given a finitely generated relatively hyperbolic group G , we construct a finite generating set X of G such that (G, X) has the ‘falsification by fellow traveler property’ provided that the parabolic subgroups $\{H_\omega\}_{\omega \in \Omega}$ have this property with respect to the generating sets $\{X \cap H_\omega\}_{\omega \in \Omega}$. This implies that groups hyperbolic relative to virtually abelian subgroups, which include all limit groups and groups acting freely on \mathbb{R}^n -trees, or geometrically finite hyperbolic groups have generating sets for which the language of geodesics is regular and the complete growth series and complete geodesic series are rational. As an application of our techniques, we prove that if each H_ω admits a geodesic biautomatic structure over $X \cap H_\omega$, then G has a geodesic biautomatic structure.

Similarly, we construct a finite generating set X of G such that (G, X) has the ‘bounded conjugacy diagrams’ property or the ‘neighboring shorter conjugate’ property if the parabolic subgroups $\{H_\omega\}_{\omega \in \Omega}$ have this property with respect to the generating sets $\{X \cap H_\omega\}_{\omega \in \Omega}$. This implies that a group hyperbolic relative to abelian subgroups has a generating set for which its Cayley graph has bounded conjugacy diagrams, a fact we use to give a cubic time algorithm to solve the conjugacy problem. Another corollary of our results is that groups hyperbolic relative to virtually abelian subgroups have a regular language of conjugacy geodesics.

1. INTRODUCTION

In this paper all generating sets generate the group as a monoid and are not necessarily assumed to be closed under taking inverses.

Let (P) be a property of (ordered) generating sets of groups. We say that a group G is (P) -*completable* if every finite (ordered) generating set can be enlarged to a finite (ordered) generating set X of G that has the property (P) .

The results of the paper can be summarized in the following statement.

Theorem 1.1. *Let $I \subseteq \{1, 2, 3, 4, 5\}$. A generating set satisfies (P_I) if it simultaneously satisfies (P_i) , $i \in I$, where*

- (P1) *is “the falsification by fellow traveler property”,*
- (P2) *is “ShortLex is a biautomatic structure”,*
- (P3) *is “admitting a geodesic biautomatic structure”,*
- (P4) *is “the bounded conjugacy diagrams property”, and*

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(P5) is “the neighboring shorter conjugate property”.

Let G be a finitely generated group, hyperbolic relative to a family of (P_I) -completable groups. Then G is (P_I) -completable.

Relatively hyperbolic groups vastly generalize hyperbolic groups and have been intensively studied in the 20 years since their initial suggestion by Gromov [18] and subsequent development by Farb [16], Bowditch [3], Osin [34] and others. One line of research has been to show that they inherit important properties of their parabolic subgroups, and numerous results recording this behavior have been produced (the inheritance of the Rapid Decay property [12], finiteness of the asymptotic dimension [33], decidability of the existential theory with parameters [11] or other algorithmic properties, to name just a few). While lifting characteristics of the parabolic subgroups naturally relies on and extends hyperbolic group machinery, the proofs often have to surmount substantial technical obstacles. In this paper we prove that several geometric, language theoretic and algorithmic properties are inherited from the parabolic subgroups, and in doing so, we produce appropriate generating sets, which are both essential for the properties we discuss and require the development of several technical tools.

Our main theorem, Theorem 1.1, collects the five properties that we show can be lifted from parabolic subgroups. Most importantly, Theorem 1.1 applies to groups that are hyperbolic with respect to abelian subgroups, since finitely generated abelian groups satisfy (P_1) – (P_5) for all finite generating sets. Some of the prominent examples in this class are limit groups in the sense of Sela (see [9, Theorem 0.3]), which coincide with the finitely presented fully residually free groups, as well as groups acting freely on \mathbb{R}^n -trees (see [19, Theorem 7.1]). More generally, it was shown in [23, Theorem 63] that finitely presented Λ -free groups, where Λ is an ordered abelian group, are also hyperbolic relative to non-cyclic abelian subgroups.

Theorem 1.1 also applies to groups that are hyperbolic with respect to virtually abelian groups, since finitely generated virtually abelian groups are $(P_{\{1,3,5\}})$ -completable. Geometrically finite hyperbolic groups are hyperbolic relative to their parabolic subgroups which are virtually abelian (see [36, Theorem 6.1] or [3, Proposition 7.9]). Note that Theorem 1.1 recovers and generalizes the result of Neumann and Shapiro [27, Theorem 4.3], which states that geometrically finite hyperbolic groups have the falsification by fellow traveler property and admit a geodesic biautomatic structure [27, Theorem 5.6].

We now discuss each of these properties and its applications. Formal definitions will be given in Section 2.

1.1. The falsification by fellow traveler property (FFTP). The ‘falsification by fellow traveler’ property is a property of graphs (see Definition 2.1) and was introduced in [27] by W. Neumann and M. Shapiro inspired by ideas of J. Cannon. Informally, a graph has this property if for each non-geodesic path there is a shorter path with the same end-points such that the two paths fellow travel; i.e. there is a global constant that bounds the distance between the two paths when ‘traveling’ along them from one end-point to the other.

Suppose that the Cayley graph $\Gamma(G, X)$ of a finitely generated group G with respect to a finite generating set X has FFTP. Then $\Gamma(G, X)$ is almost convex ([13, Proposition 1]), G has a finite presentation with a Dehn function that is at most quadratic ([13, Proposition 2]) and is of type F_3 ([13, Theorem 1]). Moreover,

the barycentric subdivision of the injective hull of G is a model for the classifying space for proper actions ([25, Theorem 1.4]).

Recall that *the complete growth series* $\widehat{\mathcal{G}}(G, X)(z)$ is the formal power series with coefficients in the group ring $\mathbb{Z}G$ given by

$$\widehat{\mathcal{G}}(G, X)(x) = \sum_{g \in G} gz^{|g|_X} \in \mathbb{Z}G[[z]].$$

There are standard definitions determining when this series is a rational or algebraic function (see [38]). The natural map $\mathbb{Z}G \rightarrow \mathbb{Z}$ sending each g to 1 shows that rationality of $\widehat{\mathcal{G}}(G, X)(z)$ implies the rationality of the usual growth series $\mathcal{G}(G, X)(z) = \sum_{g \in G} z^{|g|_X} \in \mathbb{Z}[[z]]$. In general, if \mathcal{L} is a language over X , one can define the associated complete growth series

$$\widehat{\mathcal{G}}(\mathcal{L})(z) = \sum_{W \in \mathcal{L}} Wz^{\ell_X(W)} \in \mathbb{Z}[X^*][[z]].$$

If the language \mathcal{L} is regular, then the complete growth series is rational, as shown in [38].

If (G, X) has FFTP, then the language of geodesic words over X is regular ([27, Proposition 4.1]), which implies, for example, that dead-end elements in G have bounded depth ([39]); $\Gamma(G, X)$ has finitely many cone types ([28]), so the complete growth series as well as the complete geodesic growth series of G with respect to X are rational (see [17]).

Among all the applications listed above, the one that interests us most is the regularity of the language of geodesics. For a word $W \in X^*$, $\ell(W)$ denotes the length of the word W . For $g \in G$, $|g|_X$ denotes the minimal length of a word $W \in X^*$ representing g . Let $\pi: X^* \rightarrow G$ denote the natural surjection. A word W is *geodesic* if $\ell(W) = |\pi(W)|_X$. The language of geodesics of (G, X) , denoted by $\text{Geo}(G, X)$, is the set

$$\text{Geo}(G, X) := \{W \in X^* \mid \ell(W) = |\pi(W)|_X\}.$$

Proposition 1.2 ([27, Proposition 4.1]). *If (G, X) has the falsification by fellow traveler property, then $\text{Geo}(G, X)$ is regular.*

The only classes of groups that are known to have the falsification by fellow traveler property with respect to any generating set are the hyperbolic and abelian ones. In general, this property is sensitive to changing the generating set, as was shown in [27], where there is an example, due to Cannon, of a virtually abelian group G and two different generating sets X_1 and X_2 , such that $\Gamma(G, X_1)$ has the falsification by fellow traveler property and $\Gamma(G, X_2)$ does not.

Families of groups that have the falsification by fellow traveler property with respect to some generating set include virtually abelian groups [27], geometrically finite hyperbolic groups [27], Coxeter groups and groups acting simply transitively on the chambers of locally finite buildings [31], groups acting cellularly on locally finite CAT(0) cube complexes where the action is simply transitive on the vertices [30], Garside groups [21] and Artin groups of large type [22].

In Section 3 we prove that if a group G has FFTP with respect to some generating set, then G is FFTP-completable. With this fact in mind, it is reasonable to say that a group G has the *falsification by fellow traveler property* if there exists a finite generating set X of G such that $\Gamma(G, X)$ has the falsification by fellow traveler property.

The paper's first main result, which is an immediate consequence of the more technical Theorem 7.1, is the following (see Section 2.2 for a definition of relative hyperbolicity).

Theorem 1.3. *Let G be a finitely generated group, hyperbolic relative to a collection of subgroups with the falsification by fellow traveler property. Then G has the falsification by fellow traveler property.*

The fundamental group of a finite graph of groups with finite edge stabilizers is hyperbolic relative to its vertex groups by [3, Definition 2] and Bass-Serre theory. In particular, if $G = A *_C B$, where C is finite, then G is hyperbolic relative to A and B . Similarly, if G is an HNN extension of A and the associated subgroup is finite, then G is hyperbolic relative to A . We thus obtain the following, which seems to have been previously unknown.

Corollary 1.4. *Suppose that G is the fundamental group of a finite graph of groups where the vertex groups have the falsification by fellow traveler property and the edge groups are finite. Then G has the falsification by fellow traveler property.*

1.2. Geodesically biautomatic groups. Automatic and biautomatic groups feature prominently in geometric group theory, and [14] constitutes an excellent reference for this topic.

We recall here that a geodesic biautomatic structure for G is a regular language \mathcal{L} of geodesic words such that paths that start or end at distance one apart and are labeled by words in \mathcal{L} fellow travel. We note that by an example of Neumann and Shapiro [29], admitting a geodesic biautomatic structure is a generating set dependent property.

The techniques we developed in order to prove Theorem 1.3 lead to:

Theorem 1.5. *Let G be a finitely generated group, hyperbolic relative to a family of subgroups (admitting a geodesic biautomatic structure)-completable. Then G is (admitting a geodesic biautomatic structure)-completable.*

Rebbechi established in his thesis [36] that groups hyperbolic relative to biautomatic groups are biautomatic. However, the geometric properties of his biautomatic structure are not clear. The fact that the biautomatic structure we produce above is geodesic was used in [24, Section 6] to produce Stallings-like graphs for quasi-convex subgroups in toral relatively hyperbolic groups and to compute the intersection of such subgroups.

In terms of proofs, both Rebbechi's and our approach consider what is probably the most natural biautomatic structure for G , namely the set of paths of minimal length which, when traveling inside the parabolic subgroups, follow fixed normal forms. We diverge on how we prove that such a language is regular. One of the technical parts in Rebbechi's thesis is dealing with a variation of the falsification by fellow traveler property in order to obtain that this language is regular. Since we are restricting ourselves to geodesic biautomatic structures, we can use the standard falsification by fellow traveler property to prove that the language is regular. In our case, the difficulties arise when finding the appropriate generating sets.

The extra assumption we have imposed, when compared with Rebbechi's work, also gives a stronger conclusion, namely that the biautomatic structure is geodesic.

1.3. Conjugacy diagram properties. In this paper we also investigate conjugacy diagrams in Cayley graphs of relatively hyperbolic groups. We show that in groups

hyperbolic relative to abelian subgroups conjugacy behaves as in free groups, as explained below.

A *cyclic geodesic* word is a word for which all its cyclic shifts are geodesic. A conjugacy diagram, as defined in Section 8, is a 4-gon in which two opposite sides correspond to conjugate words, and the other two sides to the conjugating element. A conjugacy diagram is *minimal* if the conjugate words U and V are cyclic geodesics, and the conjugator is the shortest possible after cyclically shifting U and V . Minimal conjugacy diagrams have a particularly nice behavior in hyperbolic groups, as shown in [1, III.Г. Lemma 2.9]: the length of two of the opposite sides must be bounded by a universal constant.

More precisely, if $\Gamma(G, X)$ is hyperbolic, then the following holds (see also Lemma 8.2):

(*) there is a constant $K > 0$ such that for any pair of cyclic geodesic words U and V over X representing conjugate elements either $\max\{\ell(U), \ell(V)\} \leq K$ or there is a word C over X such that $\ell(C) \leq K$ and $\pi(CU') = \pi(V'C)$, where U' and V' are cyclic shifts of U and V .

We will say that pairs (G, X) with the property (*) have *bounded conjugacy diagrams* (BCD).

The bounded conjugacy diagrams property is useful for proving that the language of conjugacy geodesics, i.e. the set of minimal length words in a conjugacy class, is regular. Let \sim_c be the equivalence relation defined by conjugation in G . For $g \in G$, we denote by $[g]_c$ the equivalence class of G . A word W is a *conjugacy geodesic* if $\ell(W) = \min_{g \in [\pi(W)]_c} |g|_X$. The language of conjugacy geodesics, denoted by $\text{ConjGeo}(G, X)$, is the set

$$\text{ConjGeo}(G, X) := \{W \in X^* \mid \ell(W) = \min_{g \in [\pi(W)]_c} |g|_X\}.$$

The language of conjugacy geodesics was introduced by the second author and Hermiller in [6], where they proved that its regularity is preserved by graph products and free products with finite amalgamation. In [7], the second author, Hermiller, Holt and Rees showed that hyperbolic groups, virtually abelian groups, extra-large type Artin groups and homogeneous Garside groups have generating sets for which the language of conjugacy geodesics is regular.

In fact, in order to prove the regularity of the language of conjugacy geodesics, one can use a weaker property than BCD, which we call in this paper the *neighboring shorter conjugate* property (NSC). This property appears implicitly in [7, Cor. 3.8], and it says that any cyclic geodesic word that is not a conjugacy geodesic is conjugate (up to cyclic shifts) to a shorter word by a conjugator of a uniformly bounded length (see Definition 8.5).

The following proposition is one of our motivations for studying BCD and NSC.

Proposition 1.6 ([7, Cor. 3.8]). *If (G, X) has FFTP and NSC, then $\text{ConjGeo}(G, X)$ is regular.*

The paper's second main result, which is an immediate consequence of the more technical Theorem 9.17, is the following.

Theorem 1.7. *Let G be a finitely generated group, hyperbolic relative to a family of BCD-completable (resp. NSC-completable) subgroups. Then G is BCD-completable (resp. NSC-completable).*

In contrast with FFTP, we do not know if having BCD (resp. NSC) with respect to some generating set implies BCD-completable (resp. NSC-completable).

A further reason to study the BCD property is the fact that it leads to an efficient solution of the conjugacy problem. The bounded conjugacy diagrams property was used by Bridson and Howie in [2] as part of an algorithm to solve the conjugacy problem for lists in hyperbolic groups. In particular, they used bounded conjugacy diagrams to show that the conjugacy problem in hyperbolic groups can be solved by an algorithm whose time complexity is quadratic in the lengths of the input words. We discuss in Remark 9.18 how to adapt Bridson and Howie's algorithm ([2]) to get a cubic time solution of the conjugacy problem for groups hyperbolic relative to abelian subgroups. As far as we know, this is the first result that bounds the complexity of the conjugacy problem for this family of groups.

After the completion of this paper, the preprint [5] became available. In [5] Inna Bumagina determines the complexity of the conjugacy problem in relatively hyperbolic groups as a function of the complexity of the conjugacy problem in the parabolic subgroups.

1.4. Application to languages.

We now introduce two more languages. Given a total order \prec on the set X , we can extend it to a total order on the monoid X^* by setting for $U, V \in W^*$, $U \prec_{SL} V$ if either $\ell(U) < \ell(V)$ or $\ell(U) = \ell(V)$ and U comes before V in the lexicographic order generated by \prec . We define the set

$\text{ShortLex}(G, X) := \{W \in X^* \mid W \prec_{SL} V \text{ for all } V \in X^* \text{ such that } \pi(V) = \pi(W)\}$ to be the set of shortlex representatives of elements of G , that is, words that are minimal with respect to the length-lexicographic order.

In [7] the following language is also considered:

$$\text{ConjGeoMinSL}(G, X) := \text{ConjGeo}(G, X) \cap \text{ShortLex}(G, X).$$

Theorem 1.1 leads then to a number of results about languages in groups, results which we collect in the following two corollaries.

Corollary 1.8. *Let G be hyperbolic relative to a family of abelian subgroups. Then any finite generating set Y can be completed to a finite generating set X satisfying*

- (1) *(G, X) has the falsification by fellow traveler property,*
- (2) *(G, X) has the bounded conjugacy diagram property,*
- (3) *$\text{ShortLex}(G, X)$ is a biautomatic structure for G .*

In particular, $\text{ConjGeoMinSL}(G, X)$, $\text{ShortLex}(G, X)$, $\text{ConjGeo}(G, X)$ and $\text{Geo}(G, X)$ are regular languages, and their associated complete growth series, as well as $\widehat{\mathcal{G}}(G, X)(z)$, are rational.

In Section 10 we prove that any finite generating set of a virtually abelian group can be enlarged to a finite generating set having the falsification by fellow traveler property, the neighboring shorter conjugate property and a geodesic biautomatic structure simultaneously. Thus by combining these facts with Theorem 1.1 we obtain:

Corollary 1.9. *Let G be hyperbolic relative to a family of virtually abelian subgroups. Then any finite generating set Y can be completed to a finite generating set X satisfying*

- (1) *(G, X) has the falsification by fellow traveler property,*

- (2) (G, X) has the neighboring shorter conjugate property,
- (3) (G, X) admits a geodesic biautomatic structure.

In particular, $\text{Geo}(G, X)$ and $\text{ConjGeo}(G, X)$ are regular languages, and their associated complete growth series, as well as $\widehat{\mathcal{G}}(G, X)(z)$, are rational.

1.5. Construction of generating sets. In Sections 4 and 5 we deal with one of the main difficulties of the paper, which is finding the appropriate generating set, i.e. one that possesses the desired metric properties. Given a group G , hyperbolic relative to a collection of subgroups $\{H_\omega\}_{\omega \in \Omega}$, the Generating Set Lemma (Lemma 5.3) provides a generating set X that relates geodesics in the Cayley graph $\Gamma(G, X)$ to quasi-geodesics in the Cayley graph $\Gamma(G, X \cup \mathcal{H})$, where $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$.

We remark that although our main theorems prove the existence of finite generating sets with various properties and no algorithm has been provided to find these sets, they are in fact computable. By starting with a finite group presentation $\langle X \mid R \rangle$ of a group G and sets of generators, in terms of X , for the finitely many subgroups $\{H_\omega\}_{\omega \in \Omega}$ of G , together with a solution of the word problem for those subgroups on these generators, Dahmani showed in [10] that one can compute an explicit relative presentation for G with an explicit linear relative isoperimetric function. This is enough to produce the many other constants used in our paper (hyperbolicity constant, the constants of the bounded coset penetration property, the constant of Lemma 2.9, etc.). With these data, the sets $\widehat{\Phi}$ and Φ in Theorem 5.2 and the Generating Set Lemma (Lemma 5.3) can be computed following the steps in their corresponding proofs, and with these sets, one can produce the desired generating sets.

2. PRELIMINARIES

We collect notation, definitions and basic results that will be used in the rest of the paper.

2.1. Graphs, words and fellow traveling. Throughout this section Γ will denote a graph, labelled and directed, where loops and multiple edges are allowed. We will use ‘d’ to denote the combinatorial graph distance between vertices.

By $L[0, n]$ we denote the unlabelled graph with vertex set $\{0, 1, 2, \dots, n\}$ and edges joining i to $i+1$ for $i = 0, \dots, n-1$. A *path* p of length n in Γ is a combinatorial graph map $p: L[0, n] \rightarrow \Gamma$. In particular, $p(i)$ is a vertex of Γ , $p(0)$ will be denoted by p_- and $p(n)$ by p_+ . For $i, j \in \mathbb{N}$, $i < j$, we use $[i, j]$ to denote the set $\{i, \dots, j\}$; then $L[i, j]$ represents the subgraph of $L[0, n]$ spanned by the vertices $[i, j]$. A subpath of p is the composition of the graph inclusion $L[i, j]$ into $L[0, n]$ with the map p . We use the notation $p|_{[i, j]}$ for such a subpath. We also adopt the convention that $p(m) = p(n)$ for all $m \geq n$. Let $\ell(p)$ be the length of a path p . A path p in Γ is *geodesic* if $\ell(p)$ is minimal among the lengths of all paths q in Γ with $q_- = p_-$ and $q_+ = p_+$. Let $k > 0$. A path p is a *k-local geodesic* if every subpath of p of length less than or equal to k is geodesic.

Let $\lambda \geq 1$ and $c \geq 0$. A path p is a (λ, c) -quasi-geodesic if for any subpath q of p we have

$$\ell(q) \leq \lambda d(q_-, q_+) + c.$$

Given a set X , we denote by X^* the free monoid generated by X . Elements of X^* are called *words*. Sometimes we write W is a word over X , which means that we view W as an element of X^* . Consider a group G that is generated by X as

monoid. There is a natural monoid surjection $\pi: X^* \rightarrow G$. In order to ease the reading we will make an abuse of notation and identify $W \in X^*$ with $\pi(W) \in G$ throughout the paper. Let $\ell_X(W)$ denote the length of W as a word over X , and let $|W|_X$ be the minimal length of a word over X representing the same element, in G , as W . If the alphabet can be easily understood from the context we will write $\ell(W)$ for $\ell_X(W)$. For $U, W \in X^*$ we write $U \equiv W$ to denote word equality and $U =_G W$ to denote group element equality. Let $\Gamma(G, X)$ be the *Cayley graph* of G with respect to X . We use ‘ d_X ’ for the Cayley graph distance if we need to emphasize the generating set. The labelling of edges in the Cayley graph by elements of X can be extended to paths, so for each path p in $\Gamma(G, X)$ we denote by $\mathbf{Lab}(p) \in X^*$ the word that we obtain reading the labels of the edges along p . Notice that $\ell(p) = \ell_X(\mathbf{Lab}(p))$. A word W is *geodesic* if the path p in $\Gamma(G, X)$ with $\mathbf{Lab}(p) = W$ and $p_- = 1$ is geodesic. A word W is a *cyclic geodesic* if all its cyclic shifts are geodesic.

Let p, q be paths in Γ and $K \geq 0$. We say that p, q *asynchronously K-fellow travel* if there exist non-decreasing functions $\phi: \mathbb{N} \rightarrow \mathbb{N}$ and $\psi: \mathbb{N} \rightarrow \mathbb{N}$ such that $d(p(t), q(\phi(t))) \leq K$ and $d(p(\psi(t)), q(t)) \leq K$ for all $t \in \mathbb{N}$. We say that p, q *synchronously K-fellow travel* if $d(p(t), q(t)) \leq K$ for all $t \in \mathbb{N}$. Let U, V be two words over X , and let p, q be the paths in $\Gamma(G, X)$ with $\mathbf{Lab}(p) \equiv U$, $\mathbf{Lab}(q) \equiv V$ and $p_- = q_- = 1$. We say that U, V asynchronously K -fellow travel (resp. synchronously K -fellow travel) if p and q do.

Definition 2.1. Let Γ be a graph and $K \geq 0$. We say that Γ satisfies the *falsification by K-fellow traveler property* (K -FFTP, for short) if for every non-geodesic path p in Γ there exists a path q in Γ such that $\ell(q) < \ell(p)$, $p_- = q_-$, $p_+ = q_+$ and p and q asynchronously K -fellow travel.

Let G be a group with a finite generating set X . Then (G, X) satisfies K -FFTP if $\Gamma(G, X)$ does. In this case, for any non-geodesic word $W \in X^*$ there exists $U \in X^*$ such that $\ell(U) < \ell(W)$, $U =_G W$, and U and W asynchronously K -fellow travel.

The following is a variation [13, Lemma 1].

Lemma 2.2. *For every $k, K \geq 0$ there exists $M = M(K, k)$ such that the following holds. If p and q are two geodesics in $\Gamma(G, X)$, $d(p_-, q_-) \leq k$ and $d(p_+, q_+) \leq k$, and p and q asynchronously K -fellow travel, then they synchronously M -fellow travel.*

Remark 2.3. Using the previous lemma, Elder showed in [13] that (G, X) has the falsification by fellow traveler property if and only if (G, X) has the *synchronous falsification by fellow traveler property*; i.e. there is a constant $K > 0$ such that for every non-geodesic path p in $\Gamma(G, X)$ there exists a path q in Γ such that $\ell(q) < \ell(p)$, $p_- = q_-$, $p_+ = q_+$ and p and q synchronously K -fellow travel.

The following fact follows easily from the definitions.

Lemma 2.4. *Suppose that p is the composition of paths p_1, p_2, \dots, p_n and q is the composition of paths q_1, q_2, \dots, q_n . If p_i and q_i asynchronously K -fellow travel for all i , then p and q asynchronously K -fellow travel.*

2.2. Relatively hyperbolic groups. We follow [34] for notation and definitions of relatively hyperbolic groups. The definition we use is equivalent to what Farb

calls ‘strong relative hyperbolicity’ in [16], and we will also use his approach when proving Theorem 1.3.

Definition 2.5. Let G be a group, Ω a set, $\{H_\omega\}_{\omega \in \Omega}$ a collection of subgroups of G , and X a subset of G .

The set X is a *generating set relative to $\{H_\omega\}_{\omega \in \Omega}$* if the natural homomorphism from

$$F = (*_{\omega \in \Omega} H_\omega) * F(X)$$

to G is surjective, where $F(X)$ is the free group with basis X .

Assume that X is a generating set relative to $\{H_\omega\}_{\omega \in \Omega}$ and let R be a subset of F whose normal closure is the kernel of the natural map $F \rightarrow G$. In this event, we say that G has relative presentation

$$\left\langle X \bigcup_{\omega \in \Omega} H_\omega \parallel R \right\rangle.$$

If $|X| < \infty$ and $|R| < \infty$, the relative presentation is said to be finite and the group G is said to be *finitely presented relative to the collection of subgroups $\{H_\omega\}_{\omega \in \Omega}$* . Set

$$\mathcal{H} := \bigcup_{\omega \in \Omega} (H_\omega - \{1\}).$$

Given a word W over the alphabet $X \cup \mathcal{H}$ that represents 1 in G there exists an expression for W in F of the form

$$(1) \quad W =_F \prod_{i=1}^n f_i r_i^{\varepsilon_i} f_i^{-1},$$

where $r_i \in R$, $f_i \in F$ and $\varepsilon_i = \pm 1$ for $i = 1, \dots, n$. The smallest possible number n in an expression of type (1) is called the *relative area of W* and is denoted by $\text{Area}_{\text{rel}}(W)$.

A group G is *hyperbolic relative to the collection of subgroups $\{H_\omega\}_{\omega \in \Omega}$* if it is finitely presented relative to $\{H_\omega\}_{\omega \in \Omega}$ and there is a constant $C \geq 0$, called an *isoperimetric constant*, such that

$$\text{Area}_{\text{rel}}(W) \leq C \ell_{X \cup \mathcal{H}}(W)$$

for all words W over $X \cup \mathcal{H}$ that are the identity in G . In particular, $\Gamma(G, X \cup \mathcal{H})$ is Gromov hyperbolic (see [34, Theorem 2.53]). The family of subgroups $\{H_\omega\}_{\omega \in \Omega}$ is called the collection of *peripheral (or parabolic) subgroups* of G . Note that being relatively hyperbolic is a group property independent of the relative presentation [34, Theorem 2.34].

We collect now a series of facts that will be used in the rest of the paper.

Lemma 2.6. Suppose that G is hyperbolic with respect to $\{H_\omega\}_{\omega \in \Omega}$.

- (i) If G is finitely generated (in the ordinary sense), then H_ω is finitely generated for each $\omega \in \Omega$ ([34, Proposition 2.29]).
- (ii) If G is finitely generated (in the ordinary sense), then $|\Omega| < \infty$ ([34, Corollary 2.48]).
- (iii) For $\omega, \mu \in \Omega$, $\omega \neq \mu$ and $g, h \in G$, the following hold: $|H_\omega^g \cap H_\mu^h| < \infty$ and $|H_\omega^g \cap H_\omega| < \infty$ if $g \notin H_\omega$. Here $H^g := g^{-1}Hg$ ([34, Proposition 2.36]).

For the rest of the section, G will be a finitely generated group, $\{H_\omega\}_{\omega \in \Omega}$ a collection of parabolic subgroups of G , and X a finite generating set. As before, $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$.

Definition 2.7. Let p and q be two paths in $\Gamma(G, X \cup \mathcal{H})$.

- (1) An H_ω -component of p is a subpath s such that $\text{Lab}(s) \in (H_\omega)^*$ and s is not contained in any other subpath whose label is a word on H_ω . A subpath s is a *component* if it is an H_ω -component for some $\omega \in \Omega$. For a component s of p and a generating set Y of G , the Y -length of s is $d_Y(s_-, s_+)$.
- (2) Two components s and r (not necessarily in the same path) are *connected* if both are H_ω -components for some $\omega \in \Omega$ and $(s_-)H_\omega = (r_-)H_\omega$.
- (3) A component s of p is *isolated* if it is not connected to any other component r of p .
- (4) The path p does not *backtrack* if all components are isolated.
- (5) The path p does not *vertex backtrack* if for any subpath r of p , $\ell(r) > 1$, $\text{Lab}(r)$ does not represent an element of some H_ω . In particular, if a path does not vertex backtrack, it does not backtrack and all components are edges.
- (6) Let p_1, p_2 be components of p . We write $p_1 < p_2$ if p_1 is traversed before p_2 in the path p . That is, for $p_1 \neq p_2$ and $t, t' \in \mathbb{N}$, if $p(t') \in p_2$ and $p(t) \in p_1$, then $t < t'$.
- (7) We say that p and q are k -similar if

$$\max\{d_X(p_-, q_-), d_X(p_+, q_+)\} \leq k.$$

The following two results are key ingredients in many of the proofs in this paper.

Theorem 2.8 (Bounded Coset Penetration Property [34, Theorem 3.23]). *For any $\lambda \geq 1, c \geq 0, k \geq 0$, there exists a constant $\varepsilon = \varepsilon(\lambda, c, k)$ such that for any two k -similar paths p and q in $\Gamma(G, X \cup \mathcal{H})$ that are (λ, c) -quasi-geodesics and do not backtrack, the following conditions hold.*

- (1) *The sets of vertices of p and q are contained in the closed ε -neighborhoods (with respect to the metric d_X) of each other.*
- (2) *Suppose that s is an H_ω -component of p such that $d_X(s_-, s_+) > \varepsilon$; then there exists an H_ω -component t of q which is connected to s .*
- (3) *Suppose that s and t are connected H_ω -components of p and q respectively. Then s and t are ε -similar.*

Lemma 2.9 ([35, Lemma 2.7]). *Let G be a finitely generated group that is hyperbolic relative to a collection of subgroups $\{H_\omega\}_{\omega \in \Omega}$. Then there exist a finite subset $\Xi \subseteq G$ and constant $L \geq 0$ such that the following condition holds. Let q be a path in $\Gamma(G, X \cup \mathcal{H})$ with $q_- = q_+$ and let p_1, \dots, p_k be the set of isolated components of q . Then the Ξ -lengths of p_1, \dots, p_k satisfy*

$$\sum_{i=1}^k d_\Xi((p_i)_-, (p_i)_+) \leq L\ell(q).$$

Remark 2.10. In [34] all the generating sets are assumed to be symmetric. Also, at the beginning of [34, §3], some additional technical hypotheses are set for the relative generating set, and Theorem 2.8 above is proved under these assumptions. However, it is easy to check that the statements we are using hold if we change a finite relative generating set by another.

3. GENERATING SETS WITH FFTP

In this section we show that if a group has one finite generating set, say X , with FFTP, then it has infinitely many generating sets with this property, because for any positive integer m , the generating set consisting of the ball of radius m over X has FFTP. This is shown in Proposition 3.2.

For convenience in this section we use the synchronous version of FFTP (see Remark 2.3).

Lemma 3.1. *Suppose that (G, X) has M -FFTP. Then for any integer $b \geq 0$ and any $(1, b)$ -quasi-geodesic p in $\Gamma(G, X)$ one can find a geodesic path q in $\Gamma(G, X)$ with $p_- = q_-$, $p_+ = q_+$ such that p and q $(M \cdot b)$ -synchronously fellow travel.*

Proof. Suppose that p is a $(1, b)$ -quasi-geodesic and that $\mathbf{Lab}(p)$ represents the element $g \in G$. Set $p_0 = p$ and define a sequence of paths p_0, p_1, \dots, p_b in $\Gamma(G, X)$ as follows. If p_i is geodesic, then $p_{i+1} = p_i$. If p_i is not geodesic, then there is some path q_i shorter than p_i with the same end-points that synchronously M -fellow travels with p_i , in which case we set $p_{i+1} = q_i$.

Since $\ell(p) \leq |g|_X + b$, if p_i is not geodesic, then $\ell(p_{i+1}) < \ell(p_i)$, and so the path p_b is geodesic. Now, using the fact that p_i and p_{i+1} synchronously M -fellow travel, we get that p_0 and p_b synchronously $(M \cdot b)$ -fellow travel. \square

A word W is *minimal non-geodesic* if it is not geodesic, but all its proper subwords are geodesic.

Proposition 3.2. *Suppose (G, X) has FFTP. Let $m > 0$ and $Z = \{g \in G : |g|_X \leq m\}$. Then (G, Z) has FFTP. In particular, G is FFTP-completable.*

Proof. Let K_X be the falsification by fellow traveler constant for (G, X) . For each $z \in Z$ choose a geodesic word W_z over X that represents z .

Claim 1. Let $U \equiv z_1 \cdots z_n$ be a geodesic word over Z and let $V \equiv W_{z_1} \cdots W_{z_n}$ be the corresponding word over X . We claim that $0 \leq \ell_X(V) - |V|_X \leq m$.

Clearly $\ell_X(V) \geq |V|_X$. Take $a \geq 0$ and $0 \leq b < m$ such that $|V|_X = am + b$. Then $|V|_X > (n-1)m$ because otherwise V could be written as the product of less than n words, each of length $\leq m$, and that would contradict the fact that U is a geodesic over Z . Thus $(n-1)m < |V|_X = am + b$ and therefore $n = \ell_Z(U) \leq a + 1$. Thus, $|V|_X \leq \ell_X(V) \leq m\ell_Z(U) \leq am + m \leq |V|_X + m$, and the claim is proved.

Claim 2. Suppose that $U \equiv z_1 \cdots z_n$ is a minimal non-geodesic word over Z . Then $V \equiv W_{z_1} \cdots W_{z_n}$ is a $(1, 5m)$ -quasi-geodesic word over X .

Take a subword V' of V . Then $V' = AW_{z_i} \cdots W_{z_j}B$ for some $1 < i \leq j < n$, where A is a non-empty suffix of $W_{z_{i-1}}$ and B is a non-empty prefix of $W_{z_{j+1}}$. Since U is a minimal non-geodesic, the subword $z_i \cdots z_j$ is geodesic over Z . Let $C \equiv W_{z_i} \cdots W_{z_j}$. Then by Claim 1, $\ell_X(C) \leq |C|_X + m$. Notice that $|C|_X - 2m \leq |ACB|_X \leq |C|_X + 2m$, and hence

$$\ell_X(ACB) \leq \ell_X(C) + 2m \leq |C|_X + 3m \leq |ACB|_X + 5m,$$

which proves the second claim.

Finally we show that (G, Z) has the synchronous falsification by fellow traveler property with constant $K_Z = 5m \cdot K_X + 6m$ by proving that any minimal non-geodesic in $\Gamma(G, Z)$ synchronously K_Z -fellow travels with a geodesic in $\Gamma(G, Z)$.

Suppose that U is a minimal non-geodesic word over Z , and let V be the corresponding word over X . Let p_X be the path in $\Gamma(G, X)$ with $(p_X)_- = 1$ and such that $\text{Lab}(p_X) \equiv V$, and let p_Z be the path in $\Gamma(G, Z)$ with $(p_Z)_- = 1$, $\text{Lab}(p_Z) \equiv U$.

By Claim 2, p_X is a $(1, 5m)$ -quasi-geodesic, and by Lemma 3.1, there exists a geodesic path q_X in $\Gamma(G, X)$ with $(q_X)_- = (p_X)_-$, $(q_X)_+ = (p_X)_+$ such that p_X and q_X synchronously $(5m \cdot K_X)$ -fellow travel.

Since every subpath of length less than m in q_X represents an element of Z , we can subdivide q_X to obtain a path q_Z in $\Gamma(G, Z)$ such that $q_Z(t) = q_X(tm)$ for $t = 0, 1, \dots, \lfloor \frac{\ell(q_X)}{m} \rfloor$ and $q_Z(t) = (p_Z)_+$ for $t > \lfloor \frac{\ell(q_X)}{m} \rfloor$. Notice that q_Z is a geodesic by construction.

Claim 3. For every $r, s, t \in \mathbb{N}$ such that $p_Z(t) = p_X(s)$, $q_Z(t) = q_X(r)$ we have that $|s - r| \leq 6m$.

Indeed, since p_Z is a minimal non-geodesic, $t \geq d_Z(1, p_Z(t)) \geq t - 1$ and thus

$$(t - 1)m \leq d_X(1, p_Z(t)) = d_X(1, p_X(s)) \leq tm.$$

Since $s = \ell_X(p_X([1, s])) \geq d_X(1, p_X(s))$, we obtain that $s \geq (t - 1)m$. The fact that p_X is a $(1, 5m)$ -quasi-geodesic implies that $s = \ell_X(p_X([0, s])) \leq d_X(1, p_X(s)) + 5m \leq tm + 5m$. In summary,

$$(t - 1)m \leq s \leq tm + 5m.$$

By the construction of q_Z and r , it similarly follows that $(t - 1)m \leq r \leq tm$. Claim 3 now follows.

To conclude the proof, observe that, since q_X is geodesic, we have that for every $t \in \mathbb{N}$ and s, r such that $p_Z(t) = p_X(s)$, $q_Z(t) = q_X(r)$,

$$\begin{aligned} (2) \quad d_Z(p_Z(t), q_Z(t)) &\leq d_X(p_Z(t), q_Z(t)) = d_X(p_X(s), q_X(r)) \\ &\leq d_X(p_X(s), q_X(s)) + d_X(q_X(s), q_X(r)) \leq K_X \cdot 5m + |s - r|. \end{aligned}$$

Using Claim 3 and (2) p_Z and q_Z synchronously $(5m \cdot K_X + 6m)$ -fellow travel. \square

4. CONSTRUCTING PATHS IN $\Gamma(G, X \cup \mathcal{H})$ FROM PATHS IN $\Gamma(G, X)$

Throughout this section let G be hyperbolic relative to $\{H_\omega\}_{\omega \in \Omega}$, let $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$, and let X be a finite generating set.

The main idea of the paper is to transform paths in $\Gamma(G, X)$ into paths in $\Gamma(G, X \cup \mathcal{H})$ while taking advantage of the hyperbolicity of the latter. In this section we establish a canonical way of performing such transformations.

Most of the technicalities in this section are due to the non-trivial intersections of parabolic subgroups. It is worth mentioning that in the case of torsion-free groups the arguments and the proofs in this section can be greatly simplified.

Construction 4.1. The $\{H_\omega\}_{\omega \in \Omega}$ -factorization (or simply factorization) of a word W over X is an expression for W of the form

$$W \equiv A_0 U_1 A_1 \cdots U_n A_n,$$

where A_0, \dots, A_n are words over $X - \mathcal{H}$ and U_1, \dots, U_n are non-empty words over some $X \cap H_\omega$ such that if A_i is empty, then $U_i x$ cannot be a word over any $X \cap H_\omega$, where x is the first letter of U_{i+1} . The number of words U_i is the length of the factorization. We note that the factorization is uniquely determined by W .

We say that a word W over X has *no parabolic shortenings* if each U_i in the factorization of W , U_i a word over $X \cap H_\omega$, is geodesic in $(H_\omega, X \cap H_\omega)$.

For a word W with no parabolic shortenings, we define

$$\widehat{W} \equiv A_0 h_1 A_1 \cdots h_n A_n$$

to be a word over $X \cup \mathcal{H}$, where $U_i =_G h_i \in \mathcal{H}$. Notice that since there are no parabolic shortenings, each U_i is geodesic and non-empty, and hence $h_i \neq 1$. We say that the resulting word \widehat{W} is *derived* from W .

Similarly, if p is a path in $\Gamma(G, X)$ and $\mathbf{Lab}(p)$ has no parabolic shortenings, we denote by \widehat{p} the path in $\Gamma(G, X \cup \mathcal{H})$ with $p_- = \widehat{p}_-$ and $\mathbf{Lab}(\widehat{p}) \equiv \widehat{\mathbf{Lab}(p)}$. This gives a well-defined map

$$\widehat{-}: \{\text{paths in } \Gamma(G, X)\} \rightarrow \{\text{paths in } \Gamma(G, X \cup \mathcal{H})\},$$

$$p \mapsto \widehat{p}.$$

In the next section we will show that for a 2-local geodesic word W that has no parabolic shortenings, we only need to check a finite list of forbidden words to conclude that \widehat{W} is a quasi-geodesic with some fixed parameters. In order to prove this, a first step is Lemma 4.4, where we get sufficient conditions for \widehat{W} to be a 2-local geodesic.

For $t > 0$ and any finite generating set X of G we set $\Theta_X(t) = \{h \in \mathcal{H} : |h|_X \leq t\}$. We will use the notation $\Theta(t)$ instead of $\Theta_X(t)$ when the generating set is clear from the context.

Lemma 4.2. *Let G be hyperbolic relative to $\{H_\omega\}_{\omega \in \Omega}$ and finitely generated by a set X . Let $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$. There exists $m = m(X) > 1$ such that for every $\omega, \mu \in \Omega$, $\mu \neq \omega$,*

$$(H_\omega - \Theta(m)) \cap H_\mu = \emptyset,$$

and for all $\omega \in \Omega$, $f \in H_\omega - \Theta(m)$ and $y \in (X \cup \mathcal{H}) - H_\omega$,

$$|fy|_{X \cup \mathcal{H}} = |yf|_{X \cup \mathcal{H}} = 2.$$

Proof. By Lemma 2.6 (iii), $H_\omega \cap H_\mu$ is finite for all $\omega, \mu \in \Omega$. By Lemma 2.6 (ii), Ω is finite, and hence $I = \bigcup_{\mu \neq \omega} (H_\mu \cap H_\omega)$ is finite and any $m > 0$ such that $I \subseteq \Theta(m)$ satisfies the first claim of the lemma.

In order to prove the second claim, suppose further that $|g|_X > m$ implies $|g|_\Xi > 3L$, for all $g \in \langle \Xi \rangle$, where Ξ and L are the set and the constant of Lemma 2.9.

Let $f \in H_\omega - \Theta(m)$ and $y \in (X \cup \mathcal{H}) - H_\omega$. Since $y \notin H_\omega$ we get $fy \neq_G 1$ and $yf \neq_G 1$. So we only need to consider the case $fy =_G h_1 \in X \cup \mathcal{H}$ and the case $yf =_G h_2 \in X \cup \mathcal{H}$.

Let q_1 (resp. q_2) be the cycle whose label is $\mathbf{Lab}(q_1) \equiv f y h_1^{-1}$ (resp. $\mathbf{Lab}(q_2) \equiv y f h_2^{-1}$). Here f labels an isolated component of q_1 (resp. q_2), since if f were connected to some other component, this should be an H_ω -component because $f \in H_\omega - \Theta(m)$. In this case, we would get that $y \in H_\omega$, which contradicts the hypothesis.

By Lemma 2.9 if $fy =_G h_1$ (resp. $yf =_G h_2$), then $|f|_\Xi \leq L\ell(q_1) = 3L$ (resp. $|f|_\Xi \leq L\ell(q_2) = 3L$), which contradicts $|f|_X > m$. \square

We denote by \mathcal{H}_I the union of all intersections of pairs of parabolic subgroups, that is,

$$(3) \quad \mathcal{H}_I = \left(\bigcup_{\mu \neq \omega} (H_\omega \cap H_\mu) \right) - \{1\}.$$

Notice that by Lemma 2.6, if G is finitely generated, then the set \mathcal{H}_I is finite.

Lemma 4.3. *Suppose that $\mathcal{H}_I \subseteq X$. Let W be a word over X with no parabolic shortenings. Then \widehat{W} does not contain a subword of the form f_1f_2 , where f_1, f_2 in H_ω .*

Proof. Suppose that the factorization of W is $W \equiv A_0U_1A_1 \cdots U_nA_n$. Let $U_i =_G h_i \in \mathcal{H}$ and $\widehat{W} \equiv A_0h_1A_1 \cdots h_nA_n$.

If \widehat{W} contains f_1f_2 as a subword, and f_1, f_2 belong to some H_ω , then $f_1 = h_i$ and $f_2 = h_{i+1}$ and A_{i+1} must be empty. We will show that this leads to a contradiction.

Suppose that h_i and h_{i+1} are elements in H_ω , U_i is a word in $X \cap H_\mu$ and U_{i+1} is a word in $X \cap H_\nu$ with $\mu \neq \nu$. Then $h_i \in H_\omega \cap H_\mu$ and $h_{i+1} \in H_\omega \cap H_\nu$. We claim that U_i is a word in $X \cap H_\omega$. The only case we need to check is $\mu \neq \omega$. Since U_i is a geodesic and represents an element of $H_\mu \cap H_\omega \subseteq \mathcal{H}_I \subseteq X$, we have that $U_i \equiv h_i$, and U_i is a word in $X \cap H_\omega$, as claimed. A similar argument shows that U_{i+1} is a word in $X \cap H_\omega$. Then U_iU_{i+1} is a word in $X \cap H_\omega$ contradicting the maximality of U_i . \square

Lemma 4.4. *Let Y and X be finite generating sets of G . Suppose that*

$$Y \cup \{h \in \mathcal{H} : 0 < |h|_Y \leq 2m\} \subseteq X \subseteq Y \cup \mathcal{H},$$

where $m(Y) > 1$ is the constant of Lemma 4.2. Let W be a 2-local geodesic word over X with no parabolic shortenings. Then \widehat{W} , the word over $X \cup \mathcal{H}$ derived from W , is a 2-local geodesic.

Proof. Fix a generating set X satisfying the hypothesis. It is immediate to see that $X - \mathcal{H} = Y - \mathcal{H}$. Also, by Lemma 4.2, $\mathcal{H}_I \subseteq X$.

Let W be a 2-local geodesic word over X with no parabolic shortenings.

We argue by induction on the length of the factorization of W . If the length of the factorization is zero, then W is a word over $X - \mathcal{H}$ and $\widehat{W} \equiv W$. If \widehat{W} is not a 2-local geodesic, it contains a subword xy , $x, y \in X - \mathcal{H} = Y - \mathcal{H}$ for which $|xy|_Y = 2$ and $|xy|_{X \cup \mathcal{H}} \leq 1$. Then $xy \in X \cup \mathcal{H}$. If $xy \in \mathcal{H}$, since $|xy|_Y = 2$, $xy \in X$. This contradicts that W is a 2-local geodesic over X .

Consider now the case that the factorization is $W \equiv A_0U_1A_1 \cdots U_nA_n$ with $n > 0$, and $\widehat{W} \equiv A_0h_1A_1 \cdots h_nA_n$. The words A_i are words over $X - \mathcal{H}$, and by the previous discussion, all of them are 2-local geodesics over $X \cup \mathcal{H}$.

So if \widehat{W} is not a 2-local geodesic, either some A_i is empty and $|h_ih_{i+1}|_{X \cup \mathcal{H}} \leq 1$, or A_i is non-empty and either $|yh_{i+1}|_{X \cup \mathcal{H}} \leq 1$, where y is the last letter of A_i , or $|h_ix|_{X \cup \mathcal{H}} \leq 1$, where x is the first letter of A_i . In each case we are going to derive a contradiction.

In the first case, if A_i is empty, Lemma 4.3 implies that h_i, h_{i+1} do not belong to the same parabolic subgroup. If $|h_ih_{i+1}|_{X \cup \mathcal{H}} \leq 1$, Lemma 4.2 implies that $|h_i|_Y \leq m$ and $|h_{i+1}|_Y \leq m$ and, in particular, $h_i, h_{i+1} \in X$. Since $h_i, h_{i+1} \in X$ and U_i, U_{i+1} are geodesic, $U_i \equiv h_i$, $U_{i+1} \equiv h_{i+1}$ and h_ih_{i+1} is a subword of W . Since W is a 2-local geodesic, $|h_ih_{i+1}|_X = 2$. If $|h_ih_{i+1}|_{X \cup \mathcal{H}} \leq 1$, then $h_ih_{i+1} \in \mathcal{H}$.

Notice that since $|h_i h_{i+1}|_Y \leq 2m$, if $h_i h_{i+1} \in \mathcal{H}$, then $h_i h_{i+1} \in X$, contradicting $|h_i h_{i+1}|_X = 2$.

Similarly, if A_i is non-empty and $|yh_{i+1}|_{X \cup \mathcal{H}} \leq 1$, where y is the last letter of A_i , then by Lemma 4.2 $|h_{i+1}|_Y \leq m$, so $h_{i+1} \in X$, and since U_{i+1} is geodesic, $U_{i+1} \equiv h_{i+1}$ and yh_{i+1} is a subword of W . Since W is a 2-local geodesic, $|yh_{i+1}|_X = 2$. Thus if $|yh_{i+1}|_{X \cup \mathcal{H}} \leq 1$, then $yh_{i+1} \in \mathcal{H}$, but since $|yh_{i+1}|_Y \leq m + 1$, we obtain $yh_{i+1} \in X$, which is a contradiction.

The last case is analogous. \square

In Section 6 we will show that if for each $\omega \in \Omega$ there is a language $\mathcal{L}_\omega \subseteq \text{Geo}(H_\omega, X \cap H_\omega)$ satisfying a certain fellow traveler property, then we can extend this property to a language of words over X . A first step in this direction is to consider the following language.

Definition 4.5. For each $\omega \in \Omega$ let $\mathcal{L}_\omega \subseteq (X \cap H_\omega)^*$. We define the set $\text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ to be the subset of words W in X^* such that in the factorization $A_0 U_1 A_1 U_2 \cdots U_n A_n$ of W all $U_i \in \bigcup \mathcal{L}_\omega$, $i = 1, \dots, n$.

Let K be a group, Z a generating set, and $\mathcal{L} \subseteq Z^*$. We define the following property for (K, Z, \mathcal{L}) :

- (L1) $\mathcal{L} \subseteq \text{Geo}(K, Z)$ and \mathcal{L} contains at least one representative for each element of K .

We usually require that each $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfy (L1). Notice that in this case, if $W \in \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$, then W has no parabolic shortening and hence \widehat{W} is defined.

Lemma 4.6. Suppose $\mathcal{H}_I \subseteq X$, and suppose that each $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfies (L1).

- (i) If each \mathcal{L}_ω is prefix-closed, then so is $\text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$.
- (ii) If each \mathcal{L}_ω is regular, then so is $\text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$.

Proof. (i) Suppose that $W \in \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ has factorization $A_0 U_1 A_1 U_2 \cdots U_n A_n$. Then a prefix W' of W has factorization $A_0 U_1 A_1 U_2 \cdots U'_i A'_i$, where $i \leq n$, A'_i is a prefix of A_i and $U'_i \equiv U_i$ if A_i is non-empty or U'_i is a non-empty prefix of U_i . Since $\bigcup \mathcal{L}_\omega$ is prefix-closed, it follows that $W' \in \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$.

(ii) Recall that regular languages are closed under concatenation, union, Kleene star and complement. By definition $W \notin \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ if and only if in the factorization $W \equiv A_0 U_1 A_1 U_2 \cdots U_n A_n$ there is some i such that $U_i \notin \bigcup \mathcal{L}_\omega$. Since the languages \mathcal{L}_ω are geodesic and $X_I \subseteq X$, if a geodesic word V represents an element of $H_\omega \cap H_\mu$, $\omega \neq \mu$, then $V \equiv x$, where $x \in X$. Since \mathcal{L}_ω contains at least one representative for $x \in H_\omega$ and x is the unique geodesic word representing x , we have that $x \in \mathcal{L}_\omega$. Thus if a word W belongs to $(X \cap H_\omega)^* \cap (X \cap H_\mu)^*$, then either $W \in \mathcal{L}_\omega \cap \mathcal{L}_\mu$ or $W \notin \mathcal{L}_\omega \cup \mathcal{L}_\mu$.

Let $\mathcal{L}_\omega^c = (X \cap H_\omega)^* - \mathcal{L}_\omega$, $\mathcal{P}_{\mu, \omega} = [(X \cap H_\mu)^* - (X \cap H_\mu \cap H_\omega)^*] \cup (X - \mathcal{H})$ and $\mathcal{S}_\omega = X - H_\omega$. If \mathcal{L}_ω is regular, so is \mathcal{L}_ω^c . Regularity of $\mathcal{P}_{\mu, \omega}$ and \mathcal{S}_ω follows from closure properties of regular languages. The set of words that are not in $\text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ is exactly the set of words that, in their factorization, contain a U_i in some \mathcal{L}_ω^c . Suppose that the factorization of W is $A_0 U_1 A_1 U_2 \cdots U_n A_n$. If $U_i \in (X \cap H_\omega)^*$, then $A_0 U_1 A_1 \cdots A_i$ is either empty or belongs to $\bigcup_{\mu \neq \omega} X^* \mathcal{P}_{\mu, \omega}$ and $A_{i+1} U_{i+1} \cdots A_n$ is either empty or belongs to $\mathcal{S}_\omega X^*$.

Therefore, $X^* - \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ is the set

$$\bigcup_{\omega \in \Omega} \bigcup_{\mu \neq \omega} (X^* \mathcal{P}_{\mu, \omega} \mathcal{L}_\omega^c \mathcal{S}_\omega X^* \cup X^* \mathcal{P}_{\mu, \omega} \mathcal{L}_\omega^c \cup \mathcal{L}_\omega^c \mathcal{S}_\omega X^* \cup \mathcal{L}_\omega^c).$$

It follows that if each \mathcal{L}_ω is regular, $\text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ is regular. \square

At a later point we will need to be able to substitute a subword U_i of W in the factorization of W by another word U'_i in $\bigcup \mathcal{L}_\omega$ representing the same element. Suppose that W' is the word obtained by this substitution. We will need that $\widehat{W} \equiv \widehat{W}'$, but this is not always the case.

Example 4.7. Suppose the factorization of W is $U_1 U_2$, where $U_1 \in (X \cap H_\mu)^*$ and $U_2 \in (X \cap H_\nu)^*$. It might happen that $U_2 =_G U'_2 \in \mathcal{L}_\nu$, but the first letter of U'_2 is in $X \cap H_\mu \cap H_\nu$, so write $U'_2 \equiv x V_2$. Then the factorization of $W' \equiv U_1 U'_2$ is $(U_1 x) V_2$ and $\widehat{W} \not\equiv \widehat{W}'$.

To avoid this problem, we will use geodesic words for which the subwords in $H_\omega \cap X$ are maximal, prioritizing maximality for the left-most subwords, as made precise below.

Definition 4.8. A word W is $\{\mathcal{L}_\omega\}_{\omega \in \Omega}$ -special, or simply special, if the family of languages is clear from the context and if the following hold:

- (1) W is in $\text{Geo}(G, X) \cap \text{Rel}(G, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$, and
- (2i) either the factorization of W has length zero (i.e. W is a word in $X - \mathcal{H}$),
- (2ii) or the factorization of W has $A_0 U_1 A_1 U_2 \cdots U_n A_n$ of length $n > 0$ such that $A_1 U_2 \cdots U_n A_n$ is special and $\ell(U_1) \geq \ell(U'_1)$ for all U'_1 in a factorization $A'_0 U'_1 A'_1 \cdots U'_m A'_m$ of a word W' in $\text{Geo}(G, X) \cap \text{Rel}(G, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ satisfying $W =_G W'$.

Lemma 4.9. Suppose that the collection $\{\mathcal{L}_\omega\}_{\omega \in \Omega}$ satisfies (L1). Then for every $g \in G$ there is a special word representing g .

Proof. Given $g \in G$, we define

$$f(g) = \min\{n \mid W \in \text{Geo}(G, X), W =_G g, \text{ the factorization of } W \text{ has length } n\}.$$

Suppose that $f(g) = 0$. Then there is a geodesic word W over $X - \mathcal{H}$ representing g , and hence W is special.

Let $g \in G$, and assume that $f(g) = n > 0$ and that the lemma holds for all $h \in G$, $f(h) < n$.

Suppose $W \equiv A_0 U_1 A_1 \cdots U_n A_n$ is the factorization of a geodesic word representing g such that $\ell(U_1)$ is maximal; i.e. if $A'_0 U'_1 A'_1 \cdots U'_n A'_n$ is another factorization of a geodesic word representing g , then $\ell(U_1) \geq \ell(U'_1)$.

Let $h =_G A_1 \cdots U_n A_n$. By construction $f(h) < n$. Then there exists a special word B such that $B =_G h$. By (L1) there is $C \in \bigcup \mathcal{L}_\omega$ such that $C =_G U_1$. We claim that $A_0 C B$ is special. First we need to show that $A_0 C B$ is in $\text{Geo}(G, X) \cap \text{Rel}(G, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$. By construction, $A_0 C B =_G W =_G g$, $\ell(C) \leq \ell(U_1)$ and $\ell(B) \leq \ell(A_1 \cdots U_n A_n)$. Therefore, $\ell(A_0 C B) \leq \ell(W)$, and hence $A_0 C B$ is geodesic. It is easy to show that the factorization of $A_0 C B$ is $A_0 C$ followed by the factorization of B' since by maximality of $\ell(U_1) = \ell(C)$, Cx cannot be a word over some $X \cap H_\omega$, where x is the first letter of B . Thus since $C \in \bigcup \mathcal{L}_\omega$ and B is special, $A_0 C B \in \text{Rel}(X, \{\mathcal{L}_\omega\}_\omega)$. The maximality condition of C follows by construction. \square

Now when we substitute a substring $S \in \mathcal{L}_\omega$ inside a special word by another substring $S' \in \mathcal{L}_\omega$, where $S =_G S'$, we obtain again a special word. More precisely:

Lemma 4.10. *Let $W \equiv ACB$ be a special word. Suppose that $\widehat{W} \equiv \widehat{A}h\widehat{B}$, where \widehat{A} and \widehat{B} are derived from A and B , and $h =_G C$. Then $AC'B$ is special for any $C' \in \bigcup \mathcal{L}_\omega$ such that $C' =_G h$.*

Proof. We proceed by induction on the length of the factorization of W . If the length is zero, there is nothing to prove.

So assume that W has factorization $A_0U_1 \cdots U_nA_n$ with $n > 0$ and the result holds for special words with factorization of smaller length.

Since W is special, $W \equiv A_0U_1E$, $U_1 =_G f \in \mathcal{H}$, and E is special. Then $\widehat{W} \equiv A_0f\widehat{E}$.

We have two cases: either $\widehat{A}h \equiv A_0f$ or h is a letter of \widehat{E} .

(i) In the first case $A \equiv A_0$, $B \equiv E$, $h = f$ and $C =_G U_1$. Since both U_1 and C are geodesic, $\ell(U_1) = \ell(C)$. Take any $C' \in \bigcup \mathcal{L}_\omega$, $C' =_G h$. We claim that $A_0C'E$ is special. Since W is special and $\ell(U_1) = \ell(C) = \ell(C')$, we conclude that C' cannot be contained in a longer subword of $A_0C'E$ over some $X \cap H_\omega$. Hence the factorization of $A_0C'E$ is A_0C' followed by the factorization of E . As E is special and $\ell(U_1) = \ell(C')$, we conclude that $A_0C'E$ is special.

(ii) In the second case B is a proper suffix of E . Then $\widehat{E} \equiv \widehat{D}h\widehat{B}$, where \widehat{D} might be the empty word. Let D be a subword of E such that \widehat{D} is derived from D . Since E is special, by the induction hypothesis, $DC'B$ is special for any $C' \in \mathcal{L}_\omega$ such that $C' =_G C =_G h$. Therefore $A_0U_1DC'B$ is special, by definition. \square

5. CONSTRUCTING FINITE GENERATING SETS FOR RELATIVELY HYPERBOLIC GROUPS

Throughout this section let G be hyperbolic relative to $\{H_\omega\}_{\omega \in \Omega}$, $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$, and Y a finite generating set of G .

The objective here is to prove the Generating Set Lemma (Lemma 5.3), a key result of the paper since it provides a finite generating set X (depending on Y) for G which makes it possible to relate geodesics in $\Gamma(G, X)$ to quasi-geodesics in $\Gamma(G, X \cup \mathcal{H})$. The main ingredient of the Generating Set Lemma is Theorem 5.2 below. We recall in Theorem 5.1 the ‘local to global’ quasi-geodesic feature of hyperbolic spaces.

Theorem 5.1 ([8, Section 3, Theorem 1.4]). *Suppose that Γ is a δ -hyperbolic space. For all $\lambda' \geq 1$ and $c' \geq 0$ there exist $k > 0$, $\lambda \geq 1$ and $c \geq 0$ (depending on δ, λ' and c') such that every k -local (λ', c') -quasi-geodesic path is a (λ, c) -quasi-geodesic.*

Theorem 5.2. *Let G be hyperbolic relative to $\{H_\omega\}_{\omega \in \Omega}$, $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$, and Y a finite generating set of G .*

There exists a finite set $\widehat{\Phi}$ of non-geodesic words over $Y \cup \mathcal{H}$ and constants $\lambda \geq 1$ and $c \geq 0$ such that if W is a 2-local geodesic word over $Y \cup \mathcal{H}$ not containing any element of $\widehat{\Phi}$ as a subword, then W is a (λ, c) -quasi-geodesic without vertex backtracking.

Proof. Suppose that $\Gamma(G, Y \cup \mathcal{H})$ is δ -hyperbolic. Take $\lambda' = 4$ and $c' = 0$ and let k, λ, c be the constants provided by Theorem 5.1. Then every k -local $(4, 0)$ -quasi-geodesic is a (λ, c) -quasi-geodesic. Without loss of generality, we can enlarge k to further assume that $\lambda + c \leq k$.

Let Δ be the set of closed paths in $\Gamma(G, Y \cup \mathcal{H})$ of length at most $2k$ in which all the components are isolated or have Y -length less than or equal to m , where m is the constant of Lemma 4.2. Let $L > 0$ and Ξ be the constant and the finite subset of G provided by Lemma 2.9. Then, by Lemma 2.9, if $p \in \Delta$ and p_1, \dots, p_n are the isolated components of p ,

$$\sum d_{\Xi}((p_i)_-, (p_i)_+) \leq L\ell(p) \leq L2k.$$

In particular, since Ξ is finite, the set of labels of paths in Δ is a finite set.

We take $\widehat{\Phi}$ to be the set of labels $\mathbf{Lab}(q)$, where q is a subpath of some $p \in \Delta$, $\ell(q) > \ell(p)/2$. Thus $\widehat{\Phi}$ is a finite set of non-geodesic words over $Y \cup \mathcal{H}$.

Claim 1. If p is a 2-local geodesic path in $\Gamma(G, Y \cup \mathcal{H})$ of length at most k that vertex backtracks, then $\mathbf{Lab}(p)$ contains an element of $\widehat{\Phi}$ as a subword.

Notice first that a 2-local geodesic of length 2 cannot vertex backtrack. Take a 2-local geodesic path p , $2 < \ell(p) \leq k$. We can assume without loss of generality that p vertex backtracks, but no proper subpath of p vertex backtracks. Let r be the edge from p_- to p_+ and suppose that $\mathbf{Lab}(r) \in H_{\omega}$. Since p is 2-local geodesic, all the components of p are single edges. If two components of p were connected, a proper subpath of p would backtrack and hence vertex backtrack. Therefore all components of p are isolated.

If a component p_1 of p were connected to r , the points $p_-, p_+, (p_1)_-$ and $(p_1)_+$ would lie in the same H_{ω} -coset. Since $\ell(p) > 2$, either the subpath from p_- to $(p_1)_-$ or the subpath from $(p_1)_+$ to p_+ would have length greater than 1 and have end-points in the same H_{ω} -coset, which contradicts our minimality assumption. Thus all the components of the path pr^{-1} are isolated. Since $\ell(r) < \ell(p) \leq k$, pr^{-1} is in Δ , and $\mathbf{Lab}(p) \in \widehat{\Phi}$.

This completes the proof of Claim 1.

Claim 2. If p is a 2-local geodesic path in $\Gamma(G, Y \cup \mathcal{H})$ that does not label a k -local $(4, 0)$ -quasi-geodesic, then $\mathbf{Lab}(p)$ contains an element of $\widehat{\Phi}$ as a subword.

We can assume without loss of generality that $\ell(p) \leq k$, and by Claim 1, also assume that p does not backtrack. Let q be a geodesic path with $q_- = p_-$ and $q_+ = p_+$. Since p is not a $(4, 0)$ -quasi-geodesic, we can further assume that $\ell(p) > 4\ell(q)$. Also, p does not backtrack, so all the components of p are isolated. As q is geodesic, all the components of q are isolated, and p and q being 2-local geodesics implies that all the components in p and q are edges.

If all the components on the cycle pq^{-1} are isolated, then $pq^{-1} \in \Delta$ and $\mathbf{Lab}(p) \in \widehat{\Phi}$. So without loss of generality we assume that at least a component of p is connected to a component of q .

We now choose a set of components, i.e. edges, $p_1 < \dots < p_n$, $n \geq 1$, of p such that each p_i is connected to a component q_i of q , $q_1 < q_2 < \dots < q_n$ and no component of the subpath of p from $(p_i)_+$ to $(p_{i+1})_-$ is connected to a component of the subpath of q from $(q_i)_+$ to $(q_{i+1})_-$ for $i = 0, \dots, n$, where we understand that $(p_0)_+ = (q_0)_+ = q_- = p_-$ and $(p_{n+1})_- = (q_{n+1})_- = q_+ = p_+$. See Figure 1.

For $i = 0, \dots, n$, we let r_i denote the subpath of p from $(p_i)_+$ to $(p_{i+1})_-$ and s_i denote the subpath of q from $(q_i)_+$ to $(q_{i+1})_-$. We remark that by hypothesis $n \neq 0$. In general,

$$\ell(p) = n + \sum \ell(r_i) > 4\ell(q) = 4n + 4 \sum \ell(s_i).$$

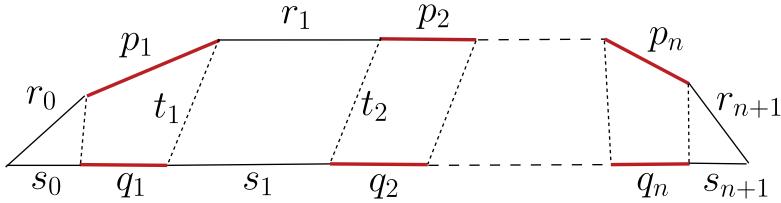


FIGURE 1. Claim 2.

If $\ell(r_i) \leq \ell(s_i) + 2$ for $i = 0, \dots, n$, the inequality

$$4n + 4 \sum \ell(s_i) < n + \sum \ell(r_i) \leq 3n + \sum \ell(s_i)$$

gives a contradiction.

Therefore, there is an i such that $\ell(r_i) > \ell(s_i) + 2$. Let t_i, t_{i+1} be geodesic paths from $(r_i)_-$ to $(s_i)_-$ and $(r_i)_+$ to $(s_i)_+$, respectively. We can assume that $\ell(t_j) = 1$ and $\text{Lab}(t_j) \in H_{\omega_j}$ for $j = i, i+1$. We can view t_i and t_{i+1} as components of the paths r_i and s_i , respectively. If $d_Y((t_i)_-, (t_i)_+) > m$ and t_i is connected to any component of r_i , then the non-vertex backtracking condition implies that this component must be the first edge of r_i and, by Lemma 4.2, this component will have label in H_{ω_i} . Lemma 4.2 also implies that the label of p_i is also in H_{ω_i} , which contradicts the fact that p doesn't vertex backtrack. So if $d_Y((t_i)_-, (t_i)_+) > m$, t_i is not connected to a component of r_i . Similarly, if $d_Y((t_i)_-, (t_i)_+) > m$, t_i is not connected to a component of s_i . If t_i is connected to t_{i+1} , again the non-vertex backtracking condition implies that $\ell(r_i) \leq 1$, contradicting $\ell(r_i) > 2 + \ell(s_i)$. Therefore t_i is either isolated in the closed path $o = r_i t_{i+1} s_i^{-1} t_i^{-1}$ or $d_Y((t_i)_-, (t_i)_+) \leq m$. The same is true for t_{i+1} . By construction, no component of s_i is connected to a component of r_i , so it follows that the closed path $o = r_i t_{i+1} s_i^{-1} t_i^{-1}$ has all components isolated or of Y -length at most m and the length of o is at most $2k$. Therefore $o \in \Delta$, and since $\ell(r_i) > \ell(s_i) + 2$, it follows that $\text{Lab}(r_i) \in \widehat{\Phi}$.

This completes the proof of Claim 2.

Hence, by Claim 2, any 2-local geodesic path p in $\Gamma(G, Y \cup \mathcal{H})$ such that $\text{Lab}(p)$ does not contain any subword in $\widehat{\Phi}$ is a (λ, c) -quasi-geodesic.

Moreover, we remark that such p does not vertex backtrack. Suppose p vertex backtracked. Claim 1 implies that there is a subpath p_1 of p with $\ell(p_1) > k$ and $d((p_1)_-, (p_1)_+) \leq 1$. Since k was chosen such that $\lambda + c \leq k$, Claim 2 implies $d((p_1)_-, (p_1)_+) > 1$, which leads to a contradiction. \square

Lemma 5.3 (Generating Set Lemma). *Let G be a finitely generated group, hyperbolic with respect to a family of subgroups $\{H_\omega\}_{\omega \in \Omega}$, and let Y be a finite generating set.*

There exist $\lambda \geq 1$, $c \geq 0$ and a finite subset \mathcal{H}' of $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$ such that for every finite generating set X of G with

$$Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H},$$

there is a finite subset Φ of non-geodesic words over X satisfying: if a word $W \in X^*$ has no parabolic shortenings and does not contain subwords in Φ , then the word $\widehat{W} \in (X \cup \mathcal{H})^*$ is a 2-local geodesic (λ, c) -quasi-geodesic without vertex backtracking.

In particular, for every $\omega \in \Omega$ and $h \in H_\omega$, $|h|_X = |h|_{X \cap H_\omega}$.

Proof. Let λ, c , and $\widehat{\Phi}$ be the constants and the set given by Theorem 5.2 applied to $(G, Y, \{H_\omega\}_{\omega \in \Omega})$, and let $m = m(G, Y, \{H_\omega\}_{\omega \in \Omega})$ be the constant provided by Lemma 4.2.

For each $\widehat{U} \in \widehat{\Phi}$ let $\widehat{V}(\widehat{U})$ be a geodesic word over $Y \cup \mathcal{H}$ such that $\widehat{U} =_G \widehat{V}(\widehat{U})$. Let

$$\mathcal{H}' = \{h \in \mathcal{H} : h \text{ letter in some } \widehat{V}(\widehat{U}), \widehat{U} \in \widehat{\Phi}\} \cup \{h \in \mathcal{H} : 0 < |h|_Y \leq 2m\}.$$

Let X be a finite subset of G satisfying

$$Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}.$$

Clearly $\langle X \rangle = G$ and $Y \cup \mathcal{H} = X \cup \mathcal{H}$.

Let Φ be the set of words U over X that are either non-geodesics of length 2 or words with no parabolic shortenings such that the derivation \widehat{U} of U lies in $\widehat{\Phi}$. Since $\widehat{\Phi}$ is finite and X is finite, it follows that Φ is finite.

We first show that the words in Φ are not geodesic over X . Pick $U \in \Phi$; we only need to consider the case when the derivation \widehat{U} from U is a word in $\widehat{\Phi}$. By our choice of X , $\widehat{V}(\widehat{U})$ is a word over X and is geodesic over $X \cup \mathcal{H} = Y \cup \mathcal{H}$. Hence $\widehat{V}(\widehat{U})$ must be geodesic when viewed as a word over X . We denote by $V(U)$ the word $\widehat{V}(\widehat{U})$ viewed as a word over X . We have that $\ell_X(U) \geq \ell_{X \cup \mathcal{H}}(\widehat{U}) > \ell_{X \cup \mathcal{H}}(\widehat{V}) = \ell_X(V)$ and then U is non-geodesic.

Let W be a word with no parabolic shortenings that does not contain any subword of Φ , and \widehat{W} the word derived from W . Since W is 2-local geodesic, by Lemma 4.4, \widehat{W} is a 2-local geodesic. By construction of Φ , \widehat{W} does not contain subwords of $\widehat{\Phi}$. Hence, by Theorem 5.2, \widehat{W} is a (λ, c) -quasi-geodesic without vertex backtracking.

Finally, assume moreover that W is a geodesic word over X , $W =_G h \in H_\omega$. Since \widehat{W} does not vertex backtrack, \widehat{W} must have length 1. By the Construction 4.1 W is a word over $X \cap H_\omega$, which means that for $h \in H_\omega$, $|h|_X = |h|_{X \cap H_\omega}$ for all $\omega \in \Omega$. This completes the proof of the lemma. \square

Corollary 5.4. *Let G be a finitely generated group, hyperbolic relative to a family $\{H_\omega\}_{\omega \in \Omega}$. There exists a finite generating set X and constants $k \geq 0$, $\lambda_1 \geq 1$ and $c_1 \geq 0$ such that the following hold:*

- (a) *The inclusion $\Gamma(H_\omega, X \cap H_\omega) \rightarrow \Gamma(G, X)$ is an isometric embedding.*
- (b) *If $W \in X^*$ is a k -local geodesic word with no parabolic shortenings, then W is a (λ_1, c_1) -quasi-geodesic.*

Proof. Take λ, c and X as in Lemma 5.3, and $\Phi \subset X^*$ the associated finite set of non-geodesic words. Claim (a) follows from the last claim of the lemma.

To see (b), take $k = \max\{\ell(U) \mid U \in \Phi\}$ and consider any word W_1 over X that is a k -local geodesic with no parabolic shortenings. Let W_2 be a geodesic word over X representing the same element as W_1 . By Lemma 5.3, \widehat{W}_1 labels a (λ, c) -quasi-geodesic over $X \cup \mathcal{H}$ without vertex backtracking, so

$$(4) \quad \ell(\widehat{W}_1) \leq \lambda |\widehat{W}_1|_{X \cup \mathcal{H}} + c \leq \lambda \ell(\widehat{W}_2) + c \leq \lambda \ell(W_2) + c.$$

By Lemma 5.3, \widehat{W}_2 also labels a (λ, c) -quasi-geodesic over $X \cup \mathcal{H}$. Let \widehat{p}_1 and \widehat{p}_2 be 0-similar paths in $\Gamma(G, X \cup \mathcal{H})$ labelled by \widehat{W}_1 and \widehat{W}_2 , respectively. Let $\varepsilon = \varepsilon(\lambda, c, 0)$ be the constant of the Bounded Coset Penetration property (Theorem 2.8), and $\widehat{r}_1, \dots, \widehat{r}_n$ be the components of \widehat{p}_1 of X -length greater than ε . By the BCP property, there are components $\widehat{s}_1, \dots, \widehat{s}_n$ of \widehat{p}_2 such that \widehat{s}_i is connected to \widehat{r}_i and $d_X((\widehat{r}_i)_-, (\widehat{r}_i)_+) \leq d_X((\widehat{s}_i)_-, (\widehat{s}_i)_+) + 2\varepsilon$.

Then we have that

$$\sum_{i=1}^n d_X((\widehat{r}_i)_-, (\widehat{r}_i)_+) \leq \sum_{i=1}^n (d_X((\widehat{s}_i)_-, (\widehat{s}_i)_+) + 2\varepsilon) \leq \ell(W_2) + 2n\varepsilon \leq (2\varepsilon + 1)\ell(W_2).$$

We can assume that the edges in \widehat{p}_1 that do not belong to $\widehat{r}_1, \dots, \widehat{r}_n$ have X -length less than or equal to ε . By (4), there are at most $\lambda\ell(W_2) + c$ such edges. Finally

$$\begin{aligned} \ell(W_1) &= \sum_{e \text{ edge in } \widehat{p}_1} d_X(e_-, e_+) \leq \varepsilon(\lambda\ell(W_2) + c) + (2\varepsilon + 1)\ell(W_2) \\ &= (\varepsilon\lambda + 2\varepsilon + 1)\ell(W_2) + \varepsilon c. \end{aligned}$$

Since W_1 is arbitrary and $\ell(W_2) = |W_1|_X$, we get the desired result with $\lambda_1 = \varepsilon\lambda + 2\varepsilon + 1$ and $c_1 = c\varepsilon$. \square

6. RELATIVELY HYPERBOLIC GROUPS WITH GEODESIC BICOMBINGS

In this section we suppose that G is hyperbolic relative to $\{H_\omega\}_{\omega \in \Omega}$, X is a finite generating set of G and for each parabolic subgroup H_ω we have a preferred set of geodesic words \mathcal{L}_ω over $X \cap H_\omega$ that fellow travel. The aim of this section is to extend the fellow traveler property to words over X .

More precisely, let K be a group, Z a finite generating set of K and \mathcal{L} a set of words over Z . We define the following two properties for (K, Z, \mathcal{L}) , and we recall the property (L1).

- (L1) $\mathcal{L} \subseteq \text{Geo}(K, Z)$ (i.e. \mathcal{L} is a set of geodesics over Z) and \mathcal{L} contains at least one representative for each element of K .
- (L \forall) There is $M > 0$ so that for every $W \in \mathcal{L}$, $g, h \in K$ and all $U \in \mathcal{L}$ with $U =_G gWh$, p, q asynchronously $M(|g|_Z + |h|_Z)$ -fellow travel, where $\text{Lab}(p) \equiv U$, $\text{Lab}(q) \equiv W$, $p_- = 1$, $q_- = g$.
- (L \exists) There is $M > 0$ so that for every $W \in \mathcal{L}$, $g, h \in K$ there exists $U \in \mathcal{L}$ with $U =_G gWh$ such that p, q asynchronously $M(|g|_Z + |h|_Z)$ -fellow travel, where $\text{Lab}(p) \equiv U$, $\text{Lab}(q) \equiv W$, $p_- = 1$, $q_- = g$.

It would be equivalent to state (L \exists) and (L \forall) for $g, h \in Z^*$ rather than in K , as the following lemma shows. We have omitted its proof since it is exactly the same as that of [27, Lemma 4.5]. There the case $\mathcal{L} = \text{Geo}(K, Z)$, where (K, Z) has M-FFTP, was considered.

Lemma 6.1. *Let K be a group finitely generated by Z . Suppose \mathcal{L} is a set of geodesic words over Z for which there is $M > 0$ such that for all $x, y \in Z$, $W \in \mathcal{L}$, for all (resp. there exist) $U, V \in \mathcal{L}$ with $U =_G x^{-1}W$, $V =_G Wy^{-1}$, the path p asynchronously M -fellow travels with q_U and q_V , where $\text{Lab}(p) \equiv W$, $\text{Lab}(q_U) \equiv U$, $\text{Lab}(q_V) \equiv V$, $p_- = 1 = (q_V)_-$, and $(q_U)_- = x$. Then \mathcal{L} satisfies (L \forall) (resp. (L \exists)).*

Example 6.2. In view of Lemma 6.1, if (K, Z) has FFTP, then $(K, Z, \text{Geo}(K, Z))$ satisfies (L1) and (L \exists).

Similarly, if \mathcal{L} is a geodesic biautomatic structure for (K, Z) , then (K, Z, \mathcal{L}) satisfies (L1) and (L \forall).

Remark 6.3. In view of Lemma 2.2, if (K, Z, \mathcal{L}) satisfies (L1), (K, Z, \mathcal{L}) satisfies (L \forall) if and only if it satisfies a synchronous version of (L \forall). The same holds for (L \exists).

The main result of the section is the following. We refer the reader to Definition 4.5 for the definition of $\text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$.

Proposition 6.4. *Let G be a finitely generated group, hyperbolic with respect to a family of subgroups $\{H_\omega\}_{\omega \in \Omega}$. Let Y be a finite generating set of G and $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$.*

There exists a finite subset \mathcal{H}' of \mathcal{H} such that for every finite generating set X of G satisfying

- (i) $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$ and
- (ii) for all $\omega \in \Omega$, there is a language \mathcal{L}_ω such that $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfies (L1) and (L \forall) (resp. (L1) and (L \exists)),

the triple (G, X, \mathcal{L}) satisfies (L1) and (L \forall) (resp. (L1) and (L \exists)), where $\mathcal{L} = \text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$.

Remark 6.5. A set \mathcal{L} such that (K, Z, \mathcal{L}) satisfies (L1) and (L \forall) and which contains exactly one representative for each element of the group is called a *geodesic bicombing* in [37]. It is easy to adapt the proof of Proposition 6.4 in a way that if we further assume that each \mathcal{L}_ω is a geodesic bicombing, then $\text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ contains a geodesic bicombing.

Lemma 6.6. *Let $\lambda \geq 1$, $c \geq 0$ and $k \geq 0$. Let $\varepsilon = \varepsilon(\lambda, c, k)$ be the constant of the Bounded Coset Penetration property (Theorem 2.8). There exists $K_1 = K_1(\varepsilon, \lambda, c)$ such that the following hold.*

Let p and q be two k -similar (λ, c) -quasi-geodesics. Then for any subpath p_1 of p , $\ell(p_1) > K_1$, and any subpath q_1 of q such that p_1 and q_1 are ε -similar, there is a vertex u in p_1 , $u \notin \{(p_1)_-, (p_1)_+\}$, and a vertex v in q_1 such that $d_X(u, v) \leq \varepsilon$.

Proof. Let K_1 be a constant satisfying

$$K_1/\lambda - c > \lambda(2\varepsilon + 2 + c) + 2\varepsilon.$$

Let p , q , p_1 and q_1 be as in the hypothesis of the lemma.

Suppose that the lemma does not hold; i.e. for all u in p_1 , $u \notin \{(p_1)_-, (p_1)_+\}$, and all vertices v in q_1 , we have $d_X(u, v) > \varepsilon$.

By Theorem 2.8, for each vertex u in p_1 there is a vertex v in q such that $d_X(u, v) \leq \varepsilon$. As shown in Figure 2, let $u_- \neq (p_1)_+$ be the closest vertex in p_1 to $(p_1)_+$ for which there exists v_- in the subpath from q_- to $(q_1)_-$ such that $d_X(u_-, v_-) \leq \varepsilon$. Notice that we are allowing $u_- = (p_1)_-$. Let u_+ be the closest vertex in p_1 to u_- , between u_- and $(p_1)_+$. In this case there is a vertex v_+ in the subpath from $(q_1)_+$ to q_+ such that $d_X(v_+, u_+) \leq \varepsilon$.

Thus $d_{X \cup \mathcal{H}}(v_-, v_+) \leq 1 + 2\varepsilon$.

Since p is a (λ, c) -quasi-geodesic

$$d_{X \cup \mathcal{H}}((p_1)_-, (p_1)_+) \geq K_1/\lambda - c > \lambda(2\varepsilon + 2 + c) + 2\varepsilon.$$

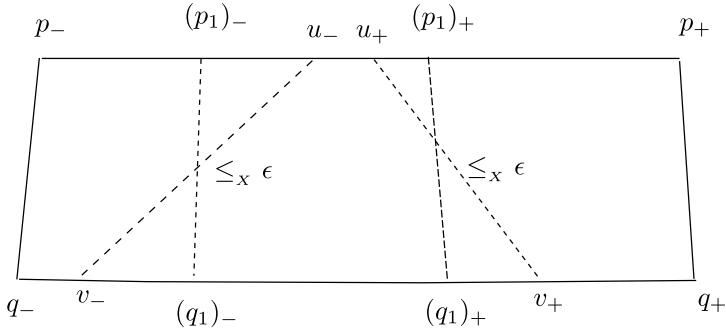


FIGURE 2

Thus

$$d_{X \cup \mathcal{H}}((q_1)_-, (q_1)_+) \geq d_{X \cup \mathcal{H}}((p_1)_-, (p_1)_+) - 2\varepsilon \geq \lambda(2\varepsilon + 2 + c).$$

Let q_2 be the subpath of q from v_- to v_+ . Then q_1 is a subpath of q_2 and hence

$$\ell(q_2) \geq \ell(q_1) \geq d_{X \cup \mathcal{H}}((q_1)_-, (q_1)_+) \geq \lambda(2\varepsilon + 2 + c).$$

Since q is a (λ, c) -quasi-geodesic, we obtain from the previous equation that

$$2\varepsilon + 2 \leq \ell(q_2)/\lambda - c \leq d_{X \cup \mathcal{H}}((q_2)_-, (q_2)_+) = d_{X \cup \mathcal{H}}(v_-, v_+),$$

which contradicts $d_{X \cup \mathcal{H}}(v_-, v_+) \leq 1 + 2\varepsilon$. \square

Lemma 6.7. *Let $\lambda \geq 1$, $c \geq 0$, $k \geq 0$ and $B > 0$. There exists $K_2 = K_2(\lambda, c, k, B)$ such that the following hold. Let p and q be two paths in $\Gamma(G, X)$ with no parabolic shortenings. Suppose that \hat{p} and \hat{q} are ε -similar and are subpaths of two k -similar (λ, c) -quasi-geodesics, and assume that the X -length of the components of \hat{p} and \hat{q} is at most B .*

Then p and q asynchronously K_2 -fellow travel.

Proof. Let K_1 be the constant of Lemma 6.6. We will prove the lemma by induction on $\ell(\hat{p})$. Suppose that $\ell(\hat{p}) \leq K_1$. Then, since all the components of \hat{p} have X -length at most B , p is a path of length at most $B \cdot K_1$. We can bound $\ell(q)$ in the following way. First observe that $d_{X \cup \mathcal{H}}(\hat{q}_-, \hat{q}_+) \leq 2\varepsilon + \ell(\hat{p}) \leq 2\varepsilon + K_1$. Since \hat{q} is a (λ, c) -quasi-geodesic, $\ell(\hat{q}) \leq \lambda(2\varepsilon + K_1) + c$, so $\ell(q) \leq B[\lambda(2\varepsilon + K_1) + c]$. As $d_X(p_-, q_-) \leq \varepsilon$ and $d_X(p_+, q_+) \leq \varepsilon$, p and q asynchronously K_2 -fellow travel, where

$$K_2 := BK_1 + B[\lambda(2\varepsilon + K_1) + c] + 2\varepsilon \geq \ell(p) + \ell(q) + 2\varepsilon.$$

Suppose that $\ell(\hat{p}) > K_1$, and we have proven the result for shorter paths. By Lemma 6.6, there is a vertex u in \hat{p} , $u \notin \{\hat{p}_-, \hat{p}_+\}$, and a vertex v in \hat{q} such that $d_X(u, v) \leq \varepsilon$. Then u and v divide \hat{p} and \hat{q} into paths \hat{p}_1, \hat{p}_2 and \hat{q}_1, \hat{q}_2 , respectively, \hat{p}_i and \hat{q}_i are ε -similar, and $\ell(\hat{p}_i) < \ell(\hat{p})$. For $i = 1, 2$, let p_i and q_i be the subpaths of p and q projecting to \hat{p}_i and \hat{q}_i , respectively, via $\widehat{-}$. By induction, p_i and q_i , $i = 1, 2$, asynchronously K_2 -fellow travel and hence, by Lemma 2.4, p and q asynchronously K_2 -fellow travel. \square

We note that the proof of Lemma 6.8 below follows the same lines as [27, Lemma 4.7], except here we use $(L\forall)/(L\exists)$ instead of the falsification by fellow traveler property. Recall that ‘special words’ were introduced in Definition 4.8.

Lemma 6.8. *Let $\lambda \geq 1$ and $M, c, k \geq 0$. Suppose $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfies $(L1)$ for all $\omega \in \Omega$. There exists a constant $K_3 = K_3(\lambda, c, k, M)$ such that the following holds.*

Let p and q be paths in $\Gamma(G, X)$ such that $\max\{d_X(p_-, q_-), d_X(p_-, q_+)\} \leq k$. Suppose that $\mathbf{Lab}(p), \mathbf{Lab}(q) \in \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$, and \widehat{p} and \widehat{q} are (λ, c) -quasi-geodesics without backtracking.

- (a) *If all $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfy $(L\forall)$ with fellow traveler constant M , then p and q asynchronously K_3 -fellow travel.*
- (b) *If all $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfy $(L\exists)$ with fellow traveler constant M and $\mathbf{Lab}(q)$ is special, then there exists a geodesic path ϱ in $\Gamma(G, X)$ with $\mathbf{Lab}(\varrho)$ special and $\mathbf{Lab}(\widehat{\varrho}) \equiv \mathbf{Lab}(\widehat{q})$, such that ϱ and p asynchronously K_3 -fellow travel.*

Proof. Let p and q be two paths in $\Gamma(G, X)$ satisfying the hypotheses of the lemma.

Let $\varepsilon = \varepsilon(\lambda, c, k)$ be the constant of Theorem 2.8. Without loss of generality we can assume that $\varepsilon \geq k$. By the Bounded Coset Penetration property (Theorem 2.8), for every vertex v of \widehat{p} there is a vertex u in \widehat{q} with $d_X(u, v) \leq \varepsilon$, and if s is a component of \widehat{p} with $d_X(s_-, s_+) > \varepsilon$, then there is a component r of \widehat{q} connected to s such that $d_X(s_-, r_-) \leq \varepsilon$ and $d_X(s_+, r_+) \leq \varepsilon$.

Let $\varepsilon_2 = \varepsilon(\lambda, c, \varepsilon)$ be the constant of Theorem 2.8. Suppose that $\widehat{p}_1 < \widehat{p}_2 < \dots < \widehat{p}_n$ are the (isolated) components of \widehat{p} satisfying $d_X((\widehat{p}_i)_-, (\widehat{p}_i)_+) > \varepsilon_2 + 2\varepsilon$.

Claim. There exist components $\widehat{q}_1 < \dots < \widehat{q}_n$ of \widehat{q} such that \widehat{q}_i is connected to \widehat{p}_i , $d_X((\widehat{p}_i)_-, (\widehat{q}_i)_-) < \varepsilon$ and $d_X((\widehat{p}_i)_+, (\widehat{q}_i)_+) < \varepsilon$ for $i = 1, \dots, n$.

By Theorem 2.8 (2) and (3), there exist components $\widehat{q}_1 < \dots < \widehat{q}_n$ of \widehat{q} and a permutation σ of $\{1, \dots, n\}$ such that for each $i = 1, \dots, n$, \widehat{p}_i and $\widehat{q}_{\sigma(i)}$ are connected, $d_X((\widehat{p}_i)_-, (\widehat{q}_{\sigma(i)})_-) \leq \varepsilon$ and $d_X((\widehat{p}_i)_+, (\widehat{q}_{\sigma(i)})_+) \leq \varepsilon$. The lack of backtracking in both \widehat{p} and \widehat{q} implies that an isolated component \widehat{p}_i cannot be connected to two different isolated components of \widehat{q} . Also notice that $d_X((\widehat{q}_{\sigma(i)})_-, (\widehat{q}_{\sigma(i)})_+) > \varepsilon_2$.

We will show that σ is the identity. Suppose that $i > \sigma(i)$ for some i .

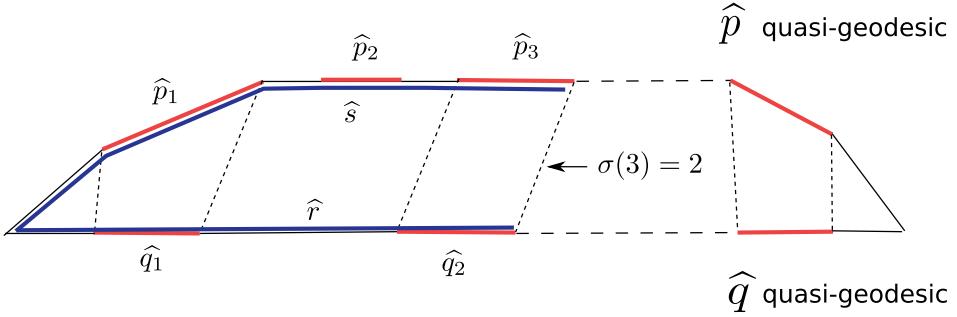
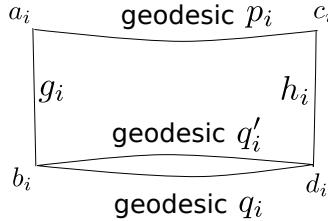
Let \widehat{s} be the subpath of \widehat{p} from \widehat{p}_- to $(\widehat{p}_i)_+$ and \widehat{r} the subpath of \widehat{q} from \widehat{q}_- to $\widehat{q}_{\sigma(i)}_+$. Then \widehat{s} and \widehat{r} are ε -similar (recall that $k \leq \varepsilon$) (λ, c) -quasi-geodesics without backtracking. Hence, by Theorem 2.8(2), the components $\widehat{p}_1, \dots, \widehat{p}_{i-1}$ are connected to some components of \widehat{s} . Since \widehat{q} does not backtrack, for $l = 1, \dots, i-1$, \widehat{p}_l should be connected to $\widehat{q}_{\sigma(l)}$. Thus $\widehat{q}_{\sigma(1)}, \dots, \widehat{q}_{\sigma(i-1)}$ lie in \widehat{s} , contradicting that $i > \sigma(i)$. See Figure 3.

A similar argument holds if $i < \sigma(i)$. This completes the proof of the claim.

Now, for $i = 1, \dots, n$, let p_i be the subpath of p that is sent to the component \widehat{p}_i via Construction 4.1, and similarly for q_i . Note that since we are assuming there are no parabolic shortenings, p_i is a geodesic path in the copy $\Gamma(gH_\omega, X \cap H_\omega)$ of $\Gamma(H_\omega, X \cap H_\omega)$ contained in $\Gamma(G, X)$, where $g = (p_i)_-$.

Let $a_i = (p_i)_-$, $b_i = (q_i)_-$, and view them as elements of G . Set $g_i = a_i^{-1}b_i$. Similarly, let $c_i = (p_i)_+$, $d_i = (q_i)_+$ and $h_i = d_i^{-1}c_i$. See Figure 4.

Let $U_i \equiv \mathbf{Lab}(p_i)$. Since $\mathbf{Lab}(p) \in \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$, by Construction 4.1, U_i is a word in \mathcal{L}_{ω_i} . Let $K_3 = \max\{K_2, M(\varepsilon_2 + 2\varepsilon)\}$. We are going to prove the conditions (a) and (b) separately.

FIGURE 3. Proof of Claim: the paths \hat{r} and \hat{s} (here $k = 0$).FIGURE 4. Geodesic q_i replaced by geodesic q'_i .

(a) Since $\mathbf{Lab}(q) \in \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$, by Construction 4.1, $V_i \equiv \mathbf{Lab}(q_i)$ is a word in \mathcal{L}_{ω_i} . By (L \forall), p_i and q_i asynchronously $(2\varepsilon M)$ -fellow travel.

Let r_0, \dots, r_n (resp. s_0, \dots, s_n) be the subpaths of p (resp. q) such that p is the composition of paths $r_0, p_1, r_1, p_2, \dots, r_n, p_n, r_n$ (q is the composition of paths $s_0, q_1, s_1, \dots, q_n, s_n$). Since the components of each \hat{r}_i have X -length bounded by $B = (\varepsilon_2 + 2\varepsilon)$ and \hat{r}_i and \hat{s}_i are ε -similar, Lemma 6.7 implies that r_i and s_i asynchronously K_2 -fellow travel.

Then by Lemma 2.4, p and q asynchronously K_3 -fellow travel.

(b) By (L \exists) there is a word V_i over $X \cap H_{\omega_i}$ in \mathcal{L}_{ω_i} such that $V_i =_G g_i U_i h_i$ and the paths p_i and q'_i asynchronously $(2\varepsilon M)$ -fellow travel, where $(q'_i)_- = (q_i)_-$ and $\mathbf{Lab}(q'_i) \equiv V_i$. Notice that $(q'_i)_+ = (q_i)_+$.

We replace the subpaths q_i of q by the paths q'_i to obtain ϱ . Notice that, by (L1), q'_i and q_i are geodesic over $X \cap H_{\omega_i}$, and since $\mathbf{Lab}(q'_i) =_G \mathbf{Lab}(q_i)$, $\ell(q'_i) = \ell(q_i)$. Therefore $\ell(q) = \ell(\varrho)$ and $\mathbf{Lab}(\varrho)$ is geodesic. Since $\mathbf{Lab}(q)$ is special, by Lemma 4.10, $\mathbf{Lab}(\varrho)$ is special and $\mathbf{Lab}(\varrho) \equiv \mathbf{Lab}(\hat{q})$.

Using the same argument as in case (a) with ϱ instead of q we get that p and ϱ asynchronously K_3 -fellow travel. \square

We are now ready to prove Proposition 6.4.

Proof of Proposition 6.4. Let \mathcal{H}' , λ and c be the sets and constants provided by the Generating Set Lemma (Lemma 5.3). Fix X , a finite generating set for G satisfying (i) and (ii), and let $\Phi = \Phi(X)$ be the set provided by the Generating Set Lemma.

By Lemma 4.9, $\mathcal{L} = \text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ contains at least one representative for each element of G , and hence (L1) holds for (G, X, \mathcal{L}) . By (ii), all $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfy (L \forall) (resp. (L \exists)). We need to show that (G, X, \mathcal{L}) satisfies (L \forall) (resp. (L \exists)) with fellow traveler constant M .

Let $K_3(\lambda, c, 1, M)$ be the constant of Lemma 6.8. Let U be any word in $\text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$. Then U does not contain subwords of Φ and has no parabolic shortenings. By Lemma 5.3, \widehat{W} labels a (λ, c) -quasi-geodesic path without backtracking. Let $x, y \in X \cup \{1\}$.

Suppose that all $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfy (L \exists).

Let V be any special word in $\text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ such that $V =_G xUy$. By Lemma 4.9 such V exists.

It follows from Lemma 6.8(b) that there is a special V' such that p and q asynchronously K_3 -fellow travel, where $p_- = 1$, $q_- = x$, $\text{Lab}(p) = U$ and $\text{Lab}(q) = V'$. Now, by Lemma 6.1, (G, X, \mathcal{L}) satisfies (L \exists).

Suppose that all $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ satisfy (L \forall). Let V be any word in $\text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ such that $V =_G xUy$. Let p, q be paths in $\Gamma(G, X)$, $\text{Lab}(p) \equiv U$, $\text{Lab}(q) \equiv V$ with $p_- = 1$ and $q_- = x$. It follows from Lemma 6.8(a) that p and q K_3 -asynchronously fellow travel. Now, by Lemma 6.1, $\text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ satisfies (L \forall). \square

7. FFTP AND BIAUTOMATICITY FOR RELATIVELY HYPERBOLIC GROUPS

The primary goal of this section is to prove our first main result, Theorem 1.3 of the Introduction. We also prove the existence of geodesic biautomatic structures in Theorem 7.6.

Theorem 7.1. *Let G be a finitely generated group, hyperbolic relative to a family of subgroups $\{H_\omega\}_{\omega \in \Omega}$. Let Y be a finite generating set of G and let $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$.*

There exists a finite subset \mathcal{H}' of \mathcal{H} such that, for every finite generating set X of G satisfying

- (i) $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$, and
 - (ii) for all $\omega \in \Omega$, $X \cap H_\omega$ generates H_ω and $(H_\omega, X \cap H_\omega)$ has FFTP,
- the pair (G, X) has FFTP.*

Proof. Let \mathcal{H}' , λ and c be the sets and constants provided by the Generating Set Lemma (Lemma 5.3). Fix X , a finite generating set for G satisfying (i) and (ii), and let $\Phi = \Phi(X)$ be the set provided by the Generating Set Lemma.

By Lemma 2.6(ii) Ω is finite, so there is an $M > 0$ such that $(H_\omega, X \cap H_\omega)$ has M -FFTP for every $\omega \in \Omega$. Let $M_1 = \max\{\ell_X(U) : U \in \Phi\}$.

Let W be a non-geodesic word over X . There are several possibilities.

(i) If W is not 2-local geodesic, that is, $W \equiv AxyB$, where $xy =_G z$, $z \in X$, then W asynchronously 1-fellow travels with AzB .

(ii) If W is 2-local geodesic and has parabolic shortenings, then $W \equiv ACB$, where C is a word over $X \cap H_\omega$, for some $\omega \in \Omega$, that is not geodesic. Then there exists a shorter word C' over $X \cap H_\omega$ such that C and C' asynchronously M -fellow travel. Hence W asynchronously M -fellow travels with $AC'B$.

(iii) If W contains some word $U \in \Phi$, then $W \equiv AUB$, and since U is non-geodesic there exists a shorter word V such that $U =_G V$. Then W and AVB M_1 -asynchronously fellow travel.

(iv) If W is a 2-local geodesic with no parabolic shortenings that does not contain any subword of Φ , then by Lemma 5.3, \widehat{W} is a (λ, c) -quasi-geodesic without backtracking. In this case we use the following lemma.

Lemma 7.2. *There exists a constant $M_2 = M_2(\lambda, c, \delta)$ such that the following holds. For every non-geodesic path p in $\Gamma(G, X)$ with no parabolic shortenings and with \widehat{p} a (λ, c) -quasi-geodesic without backtracking, there exists a shorter path q in $\Gamma(G, X)$ with $q_- = p_-$ and $q_+ = p_+$ such that p and q asynchronously M_2 -fellow travel.*

Proof. We will apply Lemma 6.8 with $\mathcal{L}_\omega = \text{Geo}(H_\omega, X \cap H_\omega)$. Notice that $(H_\omega, X \cap H_\omega, \mathcal{L}_\omega)$ trivially satisfies (L1), and (L \exists) follows from Lemma 6.1.

Let p be a non-geodesic path with no parabolic shortenings such that \widehat{p} is a (λ, c) -quasi-geodesic without backtracking. Notice that $\text{Lab}(p)$ is in $\text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$. By Lemma 4.9, there exists a geodesic path q in $\Gamma(G, X)$ with the same end-points as p and such that $\text{Lab}(q)$ is special. Notice that q does not contain subwords of Φ and has no parabolic shortenings. Thus $\text{Lab}(q) \in \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$ and by the Generating Set Lemma (Lemma 5.3), \widehat{q} is a (λ, c) -quasi-geodesic without backtracking.

Let $M_2 = K_3(\lambda, c, 0, M)$ be the constant of Lemma 6.8(b). Then there is a geodesic path ϱ in $\Gamma(G, X)$ with the same end-points as q and such that p and ϱ asynchronously M_2 -fellow travel. \square

Thus Lemma 7.2 implies that there exists a shorter word W' such that W and W' asynchronously M_2 -fellow travel.

So in all cases, for $K = \max\{1, M, M_1, M_2\}$, a non-geodesic word over X asynchronously K -fellow travels with a shorter word, and we thus obtain the falsification by fellow traveler property for the group G with generating set X . \square

We are now ready to prove Theorem 1.3 of the Introduction.

Proof of Theorem 1.3. Suppose that G is hyperbolic relative to $\{H_\omega\}_{\omega \in \Omega}$. For each $\omega \in \Omega$ there is a finite generating set Y_ω of H_ω such that (H_ω, Y_ω) has FFTP. Let Y be a finite generating set of G such that $Y \cap \mathcal{H} = \bigcup_{\omega \in \Omega} Y_\omega$.

According to Theorem 7.1 it is enough to show that for any finite subset \mathcal{H}' of \mathcal{H} there is a finite generating set X of G such that $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$ and $(H_\omega, H_\omega \cap X)$ has FFTP for all $\omega \in \Omega$.

By Lemma 4.2, there is $m > 0$ such that if $g \in H_\omega \cap H_\mu$, $\mu \neq \omega$, then $|g|_Y \leq m$. For each $\omega \in \Omega$, let

$$k_\omega = \max\{|h|_{Y_\omega} : h \in (\mathcal{H}' \cup \Theta_Y(m))^{\pm 1} \cap H_\omega\} + 1$$

and let $k = \max\{k_\omega : \omega \in \Omega\}$. Let

$$X = Y \cup \left(\bigcup_{\omega \in \Omega} \{h \in H_\omega : |h|_{Y_\omega} \leq k\} \right).$$

Then X is a finite generating set of G and $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$. Observe that $H_\omega \cap X = \{h \in H_\omega : |h|_{Y_\omega} \leq k\}$ and then by Proposition 3.2, $(H_\omega, H_\omega \cap X)$ has the falsification by fellow traveler property. \square

7.1. Geodesically biautomatic relatively hyperbolic groups. After presenting a preliminary version of this paper at the Group Theory International Webinar, Olga Kharlampovich asked if it was possible to use our techniques to give another proof of Rebbechi's result about biautomaticity of groups hyperbolic relative to biautomatic groups with a prefix closed normal form [36]. We can indeed recover this result when some additional assumptions are made, assumptions which hold for virtually abelian groups, for example. One of the main technicalities in Rebbechi's thesis is dealing with a variation of the falsification by fellow traveler property to obtain that certain languages are regular. In our approach, however, we use generating sets with the standard falsification by fellow traveler property to obtain the regularity of the language. Moreover, the biautomatic structure we obtain is geodesic.

We use [14, Lemma 2.5.5] as the definition for biautomaticity.

Definition 7.3. Let G be a group, X a finite generating set and \mathcal{L} a regular language over X that maps onto G . Then \mathcal{L} is a *biautomatic structure* if there exists a constant M such that for each $W \in \mathcal{L}$, each pair $x, y \in X \cup \{1\}$, and all $U \in \mathcal{L}$ with $U =_G xWy$, the paths p and q synchronously M -fellow travel, where $\text{Lab}(p) = W$, $\text{Lab}(q) = U$, $p_- = 1$ and $q_- = x$.

We say that \mathcal{L} is a *geodesic biautomatic structure* if all words in \mathcal{L} are geodesic over G . It is clear from the definition that if \mathcal{L} is a geodesic biautomatic structure for G , then \mathcal{L} satisfies (L1) and (L \forall).

The proof of the following lemma is basically the same as that of [14, Theorem 2.5.9].

Lemma 7.4. Let \mathcal{L} be a geodesic biautomatic structure for (G, X) and let \mathcal{L}^{pc} be the prefix closure of \mathcal{L} . Then \mathcal{L}^{pc} is a geodesic biautomatic structure.

The following is based on [27, Proposition 4.1] and can be viewed as the restriction of the FFTP to prefix-closed regular languages over X .

Proposition 7.5. Let X be a finite generating set for G and $\mathcal{L} \subseteq X^*$ be a prefix-closed regular language. Suppose that there is $C > 0$ such that for any non-geodesic word $U \in \mathcal{L}$ there is a word V , $\ell(V) < \ell(U)$, $V =_G U$, such that U and V asynchronously C -fellow travel. Then $\text{Geo}(G, X) \cap \mathcal{L}$ is a regular language.

Proof. Let A be the automaton of [27, Proposition 4.1]; that is, X is the input alphabet, the states are given by the set

$$S = \{\rho\} \cup \{\phi \in \text{Maps}(\Theta(C) \rightarrow \{-C, \dots, -1, 0, 1, \dots, C\}) \mid \phi(1) = 0\},$$

where ρ is the fail state and $\Theta(C) = \{g \in G : |g|_X \leq C\}$. The transition function $T: S \times X \rightarrow S$ is given by $T(\phi, x) = \rho$ if $\phi(x) \neq 1$, and if this is not the case, then for $g \in \Theta(C)$, $T(\phi, x)(g) = \phi(xg) - 1$ if $xg \in \Theta(C)$ and $T(\phi, x)(g) = \min\{\phi(h) : h \in \Theta(C), d(h, xg) \leq 1\}$ if $xg \notin \Theta(C)$. We note that all but ρ are accepting states and therefore $\mathcal{L}(A)$, the language accepted by A , is prefix-closed.

Given a word W accepted by A at state ϕ , we get that for all $g \in \Theta(C)$ there is a word W' of length $\ell(W) + \phi(g)$ such that $Wg =_G W'$ and W and W' asynchronously C -fellow travel.

Claim. $W \in \mathcal{L}$ is accepted by A if and only if W is geodesic.

We prove the claim by induction on $\ell(W)$. If $\ell(W) = 1$ it is easy to see that A accepts W .

So suppose that our claim holds for words U , $\ell(U) \leq n$. Let $W \in \mathcal{L}$, $\ell(W) = n+1$. Since \mathcal{L} is prefix-closed, there is $x \in X$ and $U \in \mathcal{L}$ such that $W \equiv Ux$, $\ell(U) = n$.

If U is not geodesic, W is not geodesic. Since, by induction, U is not accepted by A , neither is W . So assume that U is geodesic and has been accepted at state ϕ . If W is not geodesic, by assumption there is a geodesic word $V =_G W$ such that W and V asynchronously C -fellow travel. Since U is geodesic and W is not, $\ell(V) \leq \ell(U)$. Thus $\phi(x) = \ell(V) - \ell(U) \neq 1$, and W is not accepted.

Conversely, if W is geodesic, then $\phi(x) = 1$, so $T(\phi, x) \neq \rho$. This completes the proof of the claim.

By the claim, $\text{Geo}(G, X) \cap \mathcal{L} = \mathcal{L}(A) \cap \mathcal{L}$. Since the class of regular languages is closed under intersection, $\text{Geo}(G, X) \cap \mathcal{L} = \mathcal{L}(A) \cap \mathcal{L}$ is regular. \square

We can now state a criterion for biautomaticity.

Theorem 7.6. *Let G be a finitely generated group, hyperbolic relative to a family of subgroups $\{H_\omega\}_{\omega \in \Omega}$. Let Y be a finite generating set of G and $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$.*

There exists a finite subset $\mathcal{H}' \subseteq \mathcal{H}$ such that, for every finite generating set X of G satisfying

- (i) $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$, and
- (ii) for all $\omega \in \Omega$, there is a geodesic biautomatic structure \mathcal{L}_ω for $(H_\omega, H_\omega \cap X)$, the set $\text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega^{\text{pc}}\}_{\omega \in \Omega})$ is a geodesic biautomatic structure for (G, X) .

Proof. Let \mathcal{H}'_1 be the set of Proposition 6.4. Let \mathcal{H}'_2 , λ and c be the sets and constants provided by the Generating Set Lemma (Lemma 5.3). Let $\mathcal{H}' = \mathcal{H}_I \cup \mathcal{H}'_1 \cup \mathcal{H}'_2$. Fix X , a finite generating set for G satisfying (i)-(ii), and let $\Phi = \Phi(X)$ be the set provided by the Generating Set Lemma.

By Lemma 7.4, the language $\mathcal{L}_\omega^{\text{pc}}$ is a prefix-closed biautomatic structure, and by Lemma 4.6 $\text{Rel}(X, \{\mathcal{L}_\omega^{\text{pc}}\}_{\omega \in \Omega})$ is regular and prefix-closed.

Let $\mathcal{L}_\Phi = X^* - \bigcup_{W \in \Phi} X^*WX^*$ be the set of words that do not contain subwords in Φ . Since Φ is finite, this language is regular. By definition \mathcal{L}_Φ is prefix-closed. Thus $\mathcal{L} = \text{Rel}(X, \{\mathcal{L}_\omega^{\text{pc}}\}_{\omega \in \Omega}) \cap \mathcal{L}_\Phi$ is regular, prefix-closed and for all $W \in \mathcal{L}$, \widehat{W} is a (λ, c) -quasi-geodesic with no parabolic shortenings. Since for any geodesic U the word \widehat{U} is a (λ, c) -quasi-geodesic we obtain, by Lemma 6.8(a), that if $W \in \mathcal{L}$ is not geodesic, then it asynchronously M -fellow travels with a geodesic word.

By Proposition 7.5, $\text{Geo}(G, X) \cap \mathcal{L} = \text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega^{\text{pc}}\}_{\omega \in \Omega})$ is regular. The synchronous fellow traveler property follows from Proposition 6.4 and Remark 6.3. \square

If $\text{ShortLex}(H_\omega, H_\omega \cap X)$ is a biautomatic structure for H_ω , then it clearly satisfies (ii) of Theorem 7.6. We have the following.

Theorem 7.7. *Suppose the assumptions of Theorem 7.6 hold. Then if X is an ordered generating set satisfying (i) of Theorem 7.6 and*

- (ii') for all $\omega \in \Omega$, $\text{ShortLex}(H_\omega, H_\omega \cap X)$ is a biautomatic structure for H_ω (here the order on $H_\omega \cap X$ is the restriction of the order on X),

then $\text{ShortLex}(G, X)$ is a biautomatic structure for G .

Proof. If (ii') holds, then (ii) also holds, so we have that, by Theorem 7.6,

$$\mathcal{L} = \text{Geo}(G, X) \cap \text{Rel}(X, \{\mathcal{L}_\omega\}_{\omega \in \Omega})$$

is a geodesic biautomatic structure for G , where $\mathcal{L}_\omega = \text{ShortLex}(H_\omega, H_\omega \cap X)$. It is easy to see that $\text{ShortLex}(G, X) \subseteq \mathcal{L}$. Then, by [14, Theorem 2.5.1], $\text{ShortLex}(G, X)$ is a regular language. The biautomaticity follows from the fellow traveler property for words in \mathcal{L} . \square

8. CAYLEY GRAPHS WITH BOUNDED CONJUGACY DIAGRAMS

Let G be a group and X a generating set. An n -gon $p_1 \cdots p_n$ in $\Gamma(G, X)$ is a sequence of paths p_1, p_2, \dots, p_n in $\Gamma(G, X)$ such that $(p_i)_+ = (p_{i+1})_-$ for $i = 1, \dots, n-1$, and $(p_n)_+ = (p_1)_-$. A *conjugacy diagram* over (G, X) is a quadruple (p, q, r, s) where p and q are paths, r, s are geodesic paths with the same label, and $psq^{-1}r^{-1}$ is a 4-gon in $\Gamma(G, X)$. See Figure 5.

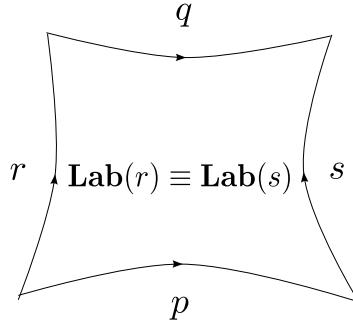


FIGURE 5. A conjugacy diagram over (G, X) .

Suppose that (p, q, r, s) and (p', q', r', s') are conjugacy diagrams. We write $(p, q, r, s) \sim (p', q', r', s')$ if $\text{Lab}(p')$ and $\text{Lab}(q')$ are cyclic permutations of $\text{Lab}(p)$ and $\text{Lab}(q)$, respectively. Let $\lambda \geq 1$ and $c \geq 0$. A conjugacy diagram (p, q, r, s) is a *minimal conjugacy* (λ, c) -diagram if all cyclic permutations of $\text{Lab}(p), \text{Lab}(q)$ are (λ, c) -quasi-geodesic and $\ell(r) \leq \ell(r')$ for all $(p', q', r', s') \sim (p, q, r, s)$.

Let $k \geq 0$. We will say that (G, X) has k -bounded minimal conjugacy (λ, c) -diagrams if for every minimal conjugacy (λ, c) -diagram (p, q, r, s) ,

$$(5) \quad \min\{\max\{\ell(p), \ell(q)\}, \ell(r)\} \leq k.$$

Example 8.1. Let G be a finitely generated abelian group and X any finite generating set. Then (G, X) has 0-bounded minimal conjugacy $(1, 0)$ -diagrams.

More generally, if G is a finitely generated finite-by-abelian group, then every conjugacy class is finite, and two elements in the same conjugacy class are conjugated by an element of the finite subgroup. This is exactly the class of finitely generated BFC-groups (groups with bounded finite conjugacy classes). BFC-groups were studied by B. H. Neumann in [26].

The idea behind bounded minimal conjugacy $(1, 0)$ -diagrams comes from solving the conjugacy problem in free groups when working over free generating sets. There, two words are conjugate if after cyclic reduction one word is a cyclic permutation of the other. So in free groups minimal conjugacy $(1, 0)$ -diagrams have the bound

$k = 0$ for free generating sets. This can be generalized to hyperbolic groups. The following lemma is a well-known result, which can be found in a slightly different version in [1, III.Γ.Lemma 2.9] and [2, Lemma 7.3]. We notice that its proof only depends on the δ -hyperbolicity of the Cayley graph, and it remains valid if one relaxes geodesics to quasi-geodesics. We leave the details of the proof to the reader.

Lemma 8.2. *Let G be a group and Z a (possibly infinite) generating set such that $\Gamma(G, Z)$ is hyperbolic. Given $\lambda \geq 1$, $c \geq 0$, there exists $K = K(G, Z, \lambda, c)$ such that (G, Z) has K -bounded minimal conjugacy (λ, c) -diagrams.*

The previous lemma motivates the following definition.

Definition 8.3. Let G be a group and X a generating set. We say that (G, X) has *bounded conjugacy diagrams* (BCD) if there is some $k \geq 0$ such that (G, X) has k -bounded minimal conjugacy $(1, 0)$ -diagrams.

Example 8.4. As already mentioned, abelian and hyperbolic groups have BCD with respect to any generating set. It is not hard to see that right-angled Artin groups have BCD with respect to the standard generating set.

The BCD property turns out to be useful for efficiently solving the conjugacy problem (see [2, Section 7] or Remark 9.18). However, in order to prove the regularity of the language of conjugacy geodesics, a weaker condition suffices (see Proposition 1.6). This condition requires that any long enough cyclic geodesic word in X^* that is not a conjugacy geodesic has a cyclic permutation that can be shortened via conjugation by an element of bounded length, as the following definition explains.

Definition 8.5. Let $B \geq 0$. We say that a group has the property B -NSC, which stands for the *neighboring shorter conjugate* property, if for any cyclic geodesic U such that $\ell(U) \geq B$ and U is conjugate to some element of length less than $\ell(U)$, there is a cyclic permutation U' of U and $g \in G$ with $|g|_X \leq B$ such that $|gU'g^{-1}|_X < \ell(U)$.

We say then that (G, X) satisfies the NSC property if there is some $B \geq 0$ such that (G, X) is B -NSC.

Remark 8.6. If (G, X) has A -BCD, then it also has A -NSC, where $A \geq 0$.

9. CONJUGACY DIAGRAMS IN RELATIVELY HYPERBOLIC GROUPS

In this section we assume that G is hyperbolic with respect to $\{H_\omega\}_{\omega \in \Omega}$, X is a finite generating set of G and $\mathcal{H} = \bigcup(H_\omega - \{1\})$. Also, $\lambda \geq 1$ and $c > 0$ are fixed constants.

We first need the following result, which is a version of [35, Proposition 3.2].

Lemma 9.1. *There exists $D = D(G, X, \lambda, c) > 0$ such that the following hold. Let $\mathcal{P} = p_1 p_2 \cdots p_n$ be an n -gon in $\Gamma(G, X \cup \mathcal{H})$ and I a distinguished subset of sides of \mathcal{P} such that if $p_i \in I$, p_i is an isolated component in \mathcal{P} , and if $p_i \notin I$, p_i is a (λ, c) -quasi-geodesic. Then*

$$\sum_{i \in I} d_X((p_i)_-, (p_i)_+) \leq Dn.$$

For the rest of the section D will denote the constant of Lemma 9.1, and we assume $D > 1$.

The following corollary is an immediate application of Lemma 9.1.

Corollary 9.2. *Let $p_1p_2p_3p_4$ be a 4-gon in $\Gamma(G, X \cup \mathcal{H})$, where all components of p_1 are edges and are isolated in the closed path $p_1p_2p_3p_4$. Suppose that p_1, p_2, p_3, p_4 are (λ, c) -quasi-geodesics. Then $d_X((p_1)_-, (p_1)_+) \leq (2\ell(p_1) + 3)D$.*

Proof. Consider the polygon with sides p_2, p_3, p_4 and the edges of p_1 . Let I be the (isolated) components of p_1 . By Lemma 9.1, the total sum of the X -lengths of the components of p_1 is less than or equal to $(\ell(p_1) + 3)D$. Then $d_X((p_1)_-, (p_1)_+) \leq \ell(p_1) + (\ell(p_1) + 3)D \leq (2\ell(p_1) + 3)D$. \square

We need some extra terminology. A minimal conjugacy (λ, c) -diagram (p, q, r, s) in $\Gamma(G, X \cup \mathcal{H})$ is *without vertex backtracking* if none of the cyclic permutations of $\text{Lab}(p)$ and $\text{Lab}(q)$ vertex backtrack. In particular, all components of p and q are edges. Also, since r and s are geodesic, all their components are edges. We say that a component is *isolated in (p, q, r, s)* if it is isolated in the closed path $rqs^{-1}p^{-1}$. Finally, we say that p (resp. q) is *parabolic* if $\text{Lab}(p)$ (resp. $\text{Lab}(q)$) is a letter from \mathcal{H} .

From now on (p, q, r, s) will be a minimal conjugacy (λ, c) -diagram without vertex backtracking and $K > 0$ the constant of Lemma 8.2 for $(G, X \cup \mathcal{H})$, that is,

$$(6) \quad \min\{\max\{\ell(p), \ell(q)\}, \ell(r)\} \leq K.$$

The rest of this section is concerned with analyzing such minimal conjugacy diagrams, first arriving at Theorem 9.13, where these diagrams are shown to be bounded (in terms of X -length) if p and q are not parabolic, and culminating with the proof of Theorem 1.7.

The first and easiest case is when all components are isolated in (p, q, r, s) .

Corollary 9.3. *If all components of p, q, r, s are isolated in (p, q, r, s) , then*

$$\min\{\max\{d_X(p_-, p_+), d_X(q_-, q_+)\}, d_X(r_-, r_+)\} \leq (2K + 3)D.$$

Proof. The result follows from (6) and Corollary 9.2. \square

The second case, in which non-isolated components are allowed, is split into a number of lemmas that explain either how the components are positioned in the diagram or how their length with respect to X can be bounded in terms of other known parameters. Each of the lemmas exploits the minimality of (p, q, r, s) .

We say that a vertex v is *adjacent* to an edge e if $v = e_-$ or $v = e_+$.

Lemma 9.4. *Let u be a vertex of p and v a vertex of r . If $d_{X \cup \mathcal{H}}(u, v) \leq 1$, then v is adjacent to the first edge of r .*

Proof. The proof of this lemma is easiest to follow while consulting Figure 6.

Suppose v is not adjacent to the first edge of r , and let r_1 be the subpath of r from v to r_+ . Note that $\ell(r_1) \leq \ell(r) - 2$. Let $z \in X \cup \mathcal{H}$ be the label of the edge from u to v , and let p_1 be the subpath of p from p_- to u and p_2 be the subpath of p from u to p_+ .

Let $U_1 \equiv \text{Lab}(p_1)$, $U_2 \equiv \text{Lab}(p_2)$ and $V_1 \equiv \text{Lab}(r_1)$.

If p' is the path with label U_2U_1 starting at u , and r' and s' are the paths with labels zV_1 starting at p'_- and p'_+ , respectively, it is easy to check that $q' = q$ satisfies $q'_- = r'_+$ and $q'_+ = s'_+$.

Thus (p', q', r', s') is a conjugacy diagram, $\ell(r') = \ell(r_1) + 1 < \ell(r)$ and $\text{Lab}(p')$ and $\text{Lab}(q')$ are cyclic permutations of $\text{Lab}(p)$ and $\text{Lab}(q)$, and hence $(p', q', r', s') \sim (p, q, r, s)$. This contradicts the minimality of (p, q, r, s) . \square

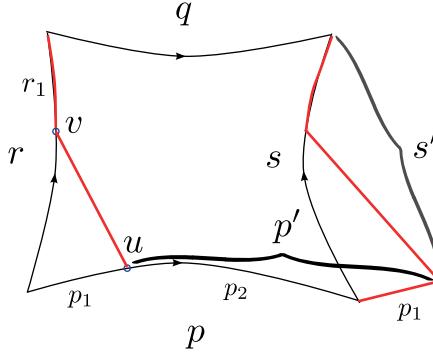


FIGURE 6. Proof of Lemma 9.4.

The next lemma shows that if two consecutive sides of (p, q, r, s) have connected components, these components occur at the same corner.

Lemma 9.5. *If p has a component connected to a component of r , these components must be the first edge of p and the first edge of r , respectively. The same behavior holds for the pairs of sides (q, r) , (p, s) and (q, s) .*

Proof. Suppose that p_1 is a component of p , r_1 a component of r , and p_1 and r_1 are connected H_ω -components. Also suppose that r_1 is not the first edge. Then $(r_1)_+$ is not adjacent to the first edge of r , and $d_{X \cup H}((r_1)_+), (p_1)_+) \leq 1$, which contradicts Lemma 9.4. So r_1 must be the first edge of r .

Suppose now p_1 is not the first edge of p , that is, $(p_1)_- \neq p_-$. Since r_1 is the first edge of r , $p_- = r_- = (r_1)_- \in (p_1)_+ H_\omega$. Thus the subpath of p from p_- to $(p_1)_+$ has length at least 2, and its label represents some element of H_ω , contradicting that (p, q, r, s) is without vertex backtracking. \square

Let m be the constant provided by Lemma 4.2. In particular,

$$(7) \quad \text{if } g \in H_\mu \cap H_\omega, \omega \neq \mu, \text{ then } |g|_X \leq m.$$

Lemma 9.6. *If p is not parabolic, then there exists a minimal conjugacy (λ, c) -diagram (p', q, s', r') such that $(p', q, s', r') \sim (p, q, r, s)$, and if a component p_1 of p' is connected to a component s_1 of s' , then $d_X((s_1)_-, (s_1)_+) \leq m$. Also, $\mathbf{Lab}(s')$ and $\mathbf{Lab}(s)$ share a suffix of length $\ell(s) - 1$.*

Proof. Suppose that (p, q, r, s) does not satisfy the claim of the lemma. That is, there exists a component p_1 of p connected to a component s_1 of s such that $d_X((s_1)_-, (s_1)_+) > m$. By Lemma 9.5, p_1 is the last edge of p and s_1 is the first edge of s .

Thus there is $\omega \in \Omega$ such that $\mathbf{Lab}(p_1) \equiv h_1 \in H_\omega$, and $\mathbf{Lab}(p) \equiv Uh_1$. Since p is not parabolic, U is non-empty. Suppose that $\mathbf{Lab}(s_1) \equiv h_2 \in H_\omega$, $\mathbf{Lab}(s) \equiv h_2V$. Let $h_3 = h_1h_2$. Then $(h_3V)\mathbf{Lab}(q)(h_3V)^{-1} =_G h_1U$, and we have a minimal conjugacy (λ, c) -diagram $(p', q, r', s') \sim (p, q, r, s)$, where $\mathbf{Lab}(p') \equiv h_1U$, $\mathbf{Lab}(r') \equiv \mathbf{Lab}(s') = h_3V$ and $\mathbf{Lab}(s')$ and $\mathbf{Lab}(s)$ share a suffix of length $\ell(s) - 1$.

If (p', q, r', s') does not satisfy the claim of the lemma, then there is a component p'_1 of p' connected to a component s'_1 of s' with $d_X((s'_1)_-, (s'_1)_+) > m$. By Lemma 9.5, p'_1 is the last edge of p' and s'_1 the first edge of s' . Thus there is $\mu \in \Omega$ such

that $\mathbf{Lab}(p'_1) = g_1$ and $\mathbf{Lab}(s'_1) = g_2 \in H_\mu$. Recall that on one hand $\mathbf{Lab}(s'_1) = h_3 \in H_\omega$, and on the other hand $\mathbf{Lab}(s'_1) = g_2 \in H_\mu$. Since $|\mathbf{Lab}(s'_1)|_X > m$, it follows by (7) that $\mu = \omega$. Hence $\mathbf{Lab}(p) \equiv U'g_1h_1$ with $g_1, h_1 \in H_\omega$, contradicting the no vertex backtracking hypothesis. \square

Similarly we obtain the following.

Lemma 9.7. *If q is not parabolic, there exists a minimal conjugacy (λ, c) -diagram (p, q', s', r') such that $(p, q', s', r') \sim (p, q, r, s)$, and if a component q_1 of q' is connected to a component s_1 of s' , then $d_X((s_1)_-, (s_1)_+) \leq m$. Also, $\mathbf{Lab}(s')$ and $\mathbf{Lab}(s)$ share a prefix of length $\ell(s) - 1$.*

Corollary 9.8. *If p and q are not parabolic, then there exists a minimal conjugacy (λ, c) -diagram $(p', q', s'', r'') \sim (p, q, r, s)$ such that if a component s_0 of s' is connected to a component of p' or q' , then $d_X((s_0)_-, (s_0)_+) \leq m$.*

Proof. Suppose that $\ell(s) = 1$. Then the result follows from either Lemma 9.6 or Lemma 9.7. If $\ell(s) > 1$ and a component of s is connected to a component of p , then it has to be the first one by Lemma 9.5. By Lemma 9.6 we obtain a minimal conjugacy (λ, c) -diagram $(p', q, s', r') \sim (p, q, r, s)$ such that if the first component s_1 of s' is connected to a component of p , $d_X((s_1)_-, (s_1)_+) \leq m$ and s' and s share a suffix of length $\ell(s) - 1$. Now, if q has a component connected to s , by Lemma 9.5, it has to be the last component of s which is the same as the last component of s' . We use Lemma 9.7 to obtain a minimal conjugacy (λ, c) -diagram $(p', q', s'', r'') \sim (p', q, r', s')$ such that if the last component s_2 of s'' is connected to a component of q' , then $d_X((s_2)_-, (s_2)_+) \leq m$ and s'' and s' share a suffix of length $\ell(s) - 1$. The only component of s'' that can be connected to p' is the first one and is exactly s_1 . \square

The next two lemmas deal with the case when p or q is long.

Lemma 9.9. *Suppose that $\ell(p) > \lambda(2K + 1 + c)$. Then $\ell(q) > 1$, $\ell(r) = \ell(s) \leq K$ and no component of r is connected to a component of s .*

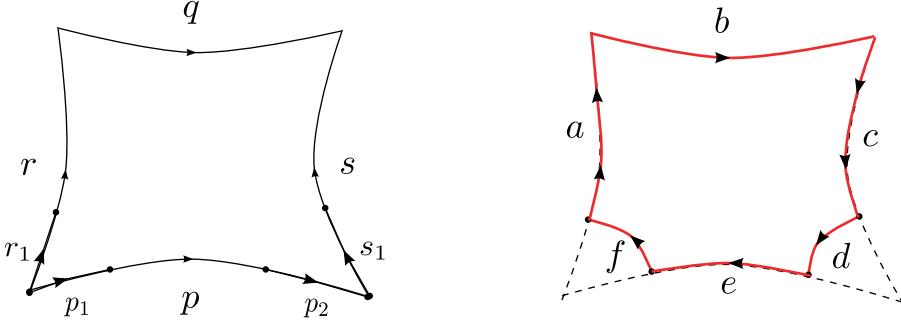
Proof. Let u be a vertex of r and v a vertex of s . Suppose that $d_{X \cup \mathcal{H}}(u, v) \leq 1$. Since p is a (λ, c) -quasi-geodesic, $d_{X \cup \mathcal{H}}(p_-, p_+) \geq \ell(p)/\lambda - c > 2K + 1$. Since $\ell(p) > K$, by (6) one has $\ell(r) = \ell(s) \leq K$. Therefore $d_{X \cup \mathcal{H}}(u, v) \geq d_{X \cup \mathcal{H}}(p_-, p_+) - d_{X \cup \mathcal{H}}(u, p_-) - d_{X \cup \mathcal{H}}(v, p_+) > 2K + 1 - 2K = 1$, which is a contradiction.

In particular $\ell(q) > 1$ and no component of r is connected to a component of s . \square

Lemma 9.10. *If $\max\{\ell(p), \ell(q)\} \geq \lambda(2K + 1 + c)$ there is $(p', q', r', s') \sim (p, q, r, s)$ such that $d_X(r'_-, r'_+) = d_X(s'_-, s'_+) \leq (2K + 3)D + 2m$.*

Proof. By Lemma 9.9 neither p nor q is parabolic. By Corollary 9.8 there is $(p', q', r', s') \sim (p, q, r, s)$ such that if a component s'_1 of s' is connected to a component of p' or of q' , then $d_X((s'_1)_-, (s'_1)_+) \leq m$. By Lemma 9.9 no component of s' is connected to a component of r' , so all the components of s' are either isolated in (p', q', r', s') or are connected to p' or q' and have X -length less than m . By Lemma 9.5 there are at most 2 components of s' connected to components of p' or q' .

Consider the polygon with sides p' , q' , r' and the edges of s' . This polygon has at most $K + 3$ sides, and we set I to be the set of isolated components of s' . Applying Lemma 9.1 we obtain that the sum of the X -lengths of the isolated components of s' is at most $(K + 3)D$. By the previous discussion, the sum of

FIGURE 7. Polygon $abcdef$ for Case 1 of the proof of Lemma 9.12.

the X -length of the non-isolated components of s' is at most $2m$. Since $D > 1$, $d_X(s'_-, s'_+) \leq (K+3)D + 2m + KD \leq (2K+3)D + 2m$. \square

The following is immediate from the fact that (p, q, r, s) is minimal.

Lemma 9.11. *If $\ell(r) = \ell(s) > 1$, then no component of p is connected to a component of q .*

The next lemma gives an upper bound for $d_X(p_-, p_+)$ in terms of the $(X \cup \mathcal{H})$ -length of p . This will help us deal with the case when p and q are short.

Lemma 9.12. *Suppose that $\ell(r) = \ell(s) > 1$ and p is not parabolic. Then*

$$d_X(p_-, p_+) \leq 4(\ell(p) + 4)D + 4m.$$

Proof. Let I be the set of components of p . Then

$$d_X(p_-, p_+) \leq \ell(p) + \sum_{t \in I} d_X(t_-, t_+).$$

Our strategy is to bound $\sum_{t \in I} d_X(t_-, t_+)$. Notice that $\ell(p) = \ell(p')$, $\ell(q) = \ell(q')$ and the lengths of the components of p' are the same as the lengths of the components of p for all $(p', q', r', s') \sim (p, q, r, s)$. So without loss of generality, we can change (p, q, r, s) by a minimal conjugacy diagram $(p', q', r', s') \sim (p, q, r, s)$, and by Corollary 9.8 we can assume that if a component s_1 of s is connected to a component of p or of q , then $d_X((s_1)_-, (s_1)_+) \leq m$.

By Lemma 9.11, no component of p is connected to a component of q . If no component of p is connected to a component of r or s , then the lemma follows from Corollary 9.2.

Case 1. We consider first the case when p has components connected to r and s . In this case, by Lemma 9.5 there are only two components of p that are non-isolated in (p, q, r, s) : the first edge p_1 of p connected to the first edge r_1 of r and the last edge p_2 of p connected to first edge s_1 of s . Since p is not parabolic, $p_1 \neq p_2$. Consider the 6-gon $abcdef$ obtained from (p, q, r, s) by “cutting the bottom corners”. See Figure 7. That is, a is the subpath of r from $(r_1)_+$ to r_+ , $b = q$, c is the subpath of s^{-1} from s_+ to $(s_1)_+$, d is an edge from $(s_1)_+$ to $(p_2)_-$ labeled by $\text{Lab}(s_1)^{-1}\text{Lab}(p_2)^{-1} \in \mathcal{H}$, e is the (possibly empty) subpath of p^{-1} from $(p_2)_-$ to $(p_1)_+$, and finally f is the edge from $(p_1)_+$ to $(r_1)_+$ labeled by $\text{Lab}(p_1)^{-1}\text{Lab}(r_1) \in \mathcal{H}$. Then all the sides of $abcdef$ are (λ, c) -quasi-geodesics and all the components of e are isolated in $abcdef$ by construction. We note that f and d might be paths of length 0 (i.e. vertices).

Claim 1. f is either an isolated component in $abcdef$ or $d_X(f_-, f_+) \leq m$.

To prove the claim, we assume that f is not isolated and $d_X(f_-, f_+) > m$. Suppose that r_1 and p_1 are H_ω -components. Then f is an H_ω -component, and since $d_X(f_-, f_+) > m$, it is not an H_ν -component, for any $\nu \neq \omega$. Thus, if f is connected to another component t of $abcdef$ it must be an H_ω -component. Since r is geodesic t cannot be a component of a . By Lemma 9.11, p_1 is not connected to a component of q , and thus t cannot be a component of b . By Lemma 9.5, p_1 is not connected to a component of s , and thus t cannot be a component of c . Since p has no vertex backtracking, t cannot be a component of e . Thus $t = d$. Then t is an H_ω -component, and since $\mathbf{Lab}(s_1) \equiv \mathbf{Lab}(r_1)$, s_1 is also an H_ω -component. This implies that p_2 is an H_ω -component and is connected to p_1 , contradicting the fact that p has no vertex backtracking. The claim is proved.

Similarly we get:

Claim 2. d is either an isolated component in $abcdef$ or $d_X(d_-, d_+) \leq m$.

We consider the polygon with sides a, b, c, d, f and the edges of e . This is a polygon with at most $\ell(p) + 3$ sides, and all its sides are (λ, c) -quasi-geodesics. Recall that I is the set of components of p . Let I' be the set of isolated components of e in $abcdef$ together with d and f in case they are isolated. Then by Lemma 9.1 the total sum of the X -lengths of the components of I' is $(\ell(p) + 3)D$. By Claim 1 and Claim 2, $\max\{d_X(d_-, d_+), d_X(f_-, f_+)\} \leq \max\{m, (\ell(p) + 3)D\}$. By hypothesis, $d_X((r_1)_-, (r_1)_+) \leq m$, and thus we obtain that $d_X((p_1)_-, (p_1)_+) \leq \max\{m, (\ell(p) + 3)D\} + m$, and similarly for $d_X((p_2)_-, (p_2)_+)$. Hence

$$\begin{aligned} \sum_{t \in I} d_X(t_-, t_+) &\leq \sum_{t \in I'} d_X(t_-, t_+) + d_X((p_1)_-, (p_1)_+) + d_X((p_2)_-, (p_2)_+) \\ &\leq (\ell(p) + 3)D + 2 \max\{m, (\ell(p) + 3)D\} + 2m \\ &\leq 3(\ell(p) + 3)D + 4m. \end{aligned}$$

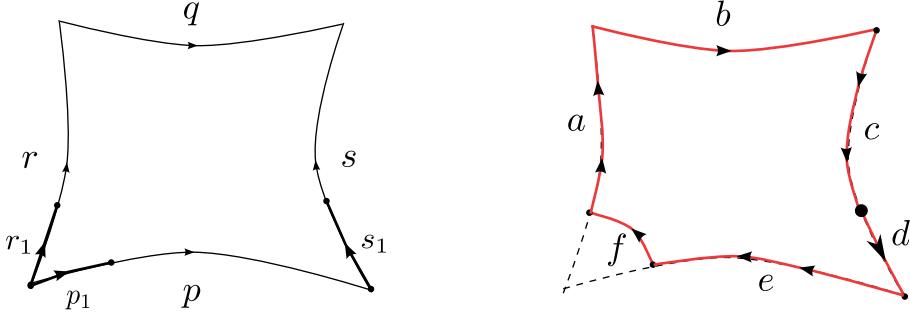
The lemma now follows easily.

Case 2. We consider now the case when p has no component connected to s . In this case, by Lemma 9.5, the only non-isolated component is the first edge p_1 of p connected to the first edge r_1 of r .

Let s_1 be the first edge of s . If r_1 is an H_ω -component, so is s_1 . By hypothesis, s_1 is not connected to a component of p . Since $\ell(s) > 1$, it follows from Lemma 9.5 that s_1 is not connected to a component of q . Since s is geodesic, s_1 is not connected to a component of s . Suppose s_1 is connected to a component r_2 of r . Then there is a vertex $v = (r_2)_+$ of r (the furthest away from r_-) such that $d_{X \cup \mathcal{H}}(u, v) \leq 1$, where $u = p_+ = (s_1)_-$, and hence, by Lemma 9.4, u is a vertex of r_1 and thus $r_2 = r_1$. If r_1 and s_1 are connected, we obtain a contradiction to the non-parabolicity of p . Therefore s_1 is isolated.

We consider the 6-gon $abcdef$ obtained from (p, q, r, s) by “cutting the bottom left corner”. See Figure 8.

That is, a is the subpath of r from $(r_1)_+$ to r_+ , $b = q$, c is the subpath of s^{-1} from s_+ to $(s_1)_+$, $d = s_1^{-1}$, e is the subpath of p^{-1} from p_+ to $(p_1)_-$, and finally f is the edge from $(p_1)_+$ to $(r_1)_+$ labelled by $\mathbf{Lab}(p_1)^{-1}\mathbf{Lab}(r_1) \in \mathcal{H}$. Then all the sides of $abcdef$ are (λ, c) -quasi-geodesics and all the components of e are isolated in $abcdef$ by construction, and d is an isolated component in $abcdef$.

FIGURE 8. Polygon $abcdef$ for Case 2 of the proof of Lemma 9.12.

Note that we allow the degenerate case of f being a vertex. Arguing in a similar way as in Claim 1 we obtain:

Claim 3. f is either an isolated component in $abcdef$ or $d_X(f_-, f_+) \leq m$.

We consider the polygon with sides a, b, c, d, f and the edges of e . This is a polygon with at most $\ell(p) + 4$ sides and all its sides are (λ, c) -quasi-geodesics. Let I' be the set of isolated components of e in $abcdef$ together with f if it is isolated in $abcdef$. Then by Lemma 9.1 the total sum of the X -length of the components of I' is $(\ell(p) + 4)D$. Thus, from Claim 3, $d_X(f_-, f_+) \leq \max\{m, (\ell(p) + 4)D\}$. Since $s_1 = d$ is isolated, $d_X((r_1)_-, (r_1)_+) = d_X((s_1)_-, (s_1)_+) \leq (\ell(p) + 4)D$. Combining this with $d_X(f_-, f_+) \leq \max\{m, (\ell(p) + 4)D\}$, we obtain

$$d_X((p_1)_-, (p_1)_+) \leq \max\{m, (\ell(p) + 4)D\} + (\ell(p) + 4)D.$$

Hence

$$\begin{aligned} \sum_{t \in I} d_X(t_-, t_+) &\leq \sum_{t \in I'} d_X(t_-, t_+) + d_X((p_1)_-, (p_1)_+) \\ &\leq (\ell(p) + 4)D + \max\{m, (\ell(p) + 4)D\} + (\ell(p) + 4)D \\ &\leq 3(\ell(p) + 4)D + m. \end{aligned}$$

This case now easily follows.

Case 3. p has no component connected to r . In this case we can argue as in Case 2.

Recall that the case when p has no component connected to r or s has already been considered, and hence we have proved the lemma in all possible cases. \square

Collecting all the results up to now produces the following general statement about conjugacy diagrams, independent of the Cayley graphs of the parabolic subgroups.

Theorem 9.13. *Let G be a finitely generated group, hyperbolic relative to a collection of subgroups $\{H_\omega\}_{\omega \in \Omega}$. Let X be a finite generating set of G , $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$, $\lambda \geq 1$ and $c \geq 0$.*

There exists a constant $k \geq 0$ such that for every minimal conjugacy (λ, c) -diagram without vertex backtracking (p, q, r, s) , where p and q are not parabolic, one can find $(p', q', r', s') \sim (p, q, r, s)$ such that

$$\min\{\max\{d_X(p'_-, p'_+), d_X(q'_-, q'_+)\}, d_X(r'_-, r'_+)\} \leq k.$$

Proof. We set $k = \max\{(2K+3)D + 2m, 4(\lambda(2K+1+c) + 4)D + 4m\}$.

If $\ell(r) = \ell(s) = 1$, by Corollary 9.8, there exists a conjugacy diagram $(p', q', s', r') \sim (p, q, r, s)$ such that either $d_X(r'_-, r'_+) \leq m$ or r' is isolated in (p', q', r', s') . In the latter case, we use Corollary 9.2 to conclude that $d_X(r'_-, r'_+) \leq 4D$.

Suppose that $\max\{\ell(p), \ell(q)\} \geq \lambda(2K+1+c)$. Then by Lemma 9.10 there is $(p', q', r', s') \sim (p, q, r, s)$ such that $d_X(r'_-, r'_+) \leq (2K+3)D + 2m \leq k$.

Now suppose $\max\{\ell(p), \ell(q)\} < \lambda(2K+1+c)$ and $\ell(r) > 1$. Since p and q are not parabolic, Lemma 9.12 implies

$$\max\{d_X(p_-, p_+), d_X(q_-, q_+)\} \leq 4(\lambda(2K+1+c) + 4)D + 4m \leq k.$$

□

Lemma 9.14. *Suppose that p is parabolic (and hence p is a component) and is connected to a component of q . Then $\mathbf{Lab}(p), \mathbf{Lab}(q), \mathbf{Lab}(r), \mathbf{Lab}(s)$ are letters of some H_ω .*

Proof. If p is an H_ω -component connected to an H_ω -component of q , after a cyclic permutation of q , we can assume that this component of q is the first edge of q and $\mathbf{Lab}(r) \equiv \mathbf{Lab}(s) \in H_\omega$. Since no cyclic permutation of $\mathbf{Lab}(q)$ is allowed to have vertex backtracking due to the fact that (p, q, r, s) is minimal, we can conclude that q is of length one and hence parabolic. □

We now include the property BCD of the parabolic subgroups in the discussion.

Lemma 9.15. *Assume that for every $\omega \in \Omega$ and every $h \in H_\omega$, $|h|_X = |h|_{X \cap H_\omega}$ and $(H_\omega, X \cap H_\omega)$ has A-BCD for some $A \geq 0$. Suppose that there is a cyclic geodesic word over $X \cap H_\omega$ representing $h \in H_\omega$. Then for any $h', g \in H_\omega$ such that $h = gh'g^{-1}$, we have $|h|_X \leq 2A + |h'|_X$.*

Proof. If $|h|_{X \cap H_\omega} = |h|_X \leq A$ there is nothing to prove. So assume $|h|_{X \cap H_\omega} \geq A$. Since h is conjugate to h' , there exists a cyclic geodesic word U over $X \cap H_\omega$ such that $\ell(U) \leq |h'|_{X \cap H_\omega}$ and U represents some conjugate of h . Let V be a cyclic geodesic over $X \cap H_\omega$ representing h . Then there is an A -bounded minimal $(1, 0)$ -diagram (p', q', r', s') in $(H_\omega, H_\omega \cap X)$ with p' and q' labelling cyclic permutations of V and U , respectively. Since $|h|_{X \cap H_\omega} \geq A$, it follows that $\ell(s') = \ell(r') \leq A$. Then

$$\ell(p') = \ell(V) = |h|_{X \cap H_\omega} \leq \ell(s') + \ell(r') + \ell(q') \leq 2A + |h'|_{X \cup H_\omega} = 2A + |h'|_X.$$

□

Lemma 9.16. *Assume that for every $\omega \in \Omega$ and every $h \in H_\omega$, $|h|_X = |h|_{X \cap H_\omega}$. Suppose that there is a cyclic geodesic word U over $X \cap H_\omega$ representing $\mathbf{Lab}(p)$ and p is not connected to a component of q .*

- (a) *If $(H_\omega, X \cap H_\omega)$ has A-BCD, then $d_X(p_-, p_+) \leq 2A + 4D + m$.*
- (b) *If $(H_\omega, X \cap H_\omega)$ has A-NSC, then either $d_X(p_-, p_+) \leq 4D + m$ or there is a cyclic permutation U' of U and words V and C over $X \cap H_\omega$, $\ell(V) < \ell(U)$, $\ell(C) \leq A$ such that $CU'C^{-1} =_G V$.*

Proof. If p is not connected to a component of r or s , then p is isolated and by Corollary 9.2, $d_X(p_-, p_+) \leq 4D$.

If p is connected to a component of r , by Lemma 9.5, it has to be the first edge of r , and hence p is also connected to the first edge of s . Suppose that r_1 is the first edge of r , s_1 is the first edge of s and p' is a path of length one from $(r_1)_+$ to $(s_1)_+$. Let r' be the subpath of r from $(r_1)_+$ to r_+ and s' the subpath of s from $(s_1)_+$ to s_+ .

We claim that $d_X(p'_-, p'_+) \leq 4D + m$. Suppose the converse, i.e. $d_X(p'_-, p'_+) > 4D + m$. Then, by Corollary 9.2, p' can't be isolated in the 4-gon $p's'q^{-1}r'^{-1}$. Also, p' can't be connected to a component q_1 of q , since in this case, $d_X(p'_-, p'_+) > m$ implies that q_1 is also an H_ω -component and has to be connected to p . A similar argument shows that p' can't be connected to a component of r' or s' . This leads to a contradiction, and hence the claim is proved.

(a) Since $d_X(p'_-, p'_+) \leq 4D + m$, by Lemma 9.15 we get $d_X(p_-, p_+) \leq 2A + 4D + m$.

(b) If $d_X(p_-, p_+) > 4D + m$, then, since $d_X(p'_-, p'_+) \leq 4D + m$, the A -NSC property gives the existence of U' , V and C with the desired properties. \square

The following statement implies Theorem 1.7 in the Introduction.

Theorem 9.17. *Let G be a finitely generated group, hyperbolic relative to a family of subgroups $\{H_\omega\}_{\omega \in \Omega}$. Let Y be a finite generating set and $\mathcal{H} = \bigcup_{\omega \in \Omega} (H_\omega - \{1\})$. There exists a finite subset \mathcal{H}' of \mathcal{H} such that the following hold. For every finite set X satisfying*

$$Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$$

there is a finite subset Φ of non-geodesic words over X such that if W is a word over X with no parabolic shortenings and without subwords in Φ , then W represents the trivial element if and only if W is empty. Moreover,

(a) *If $(H_\omega, X \cap H_\omega)$ has BCD for all $\omega \in \Omega$, then there is a constant B such that for every pair of words U and V over X representing conjugate elements in G and such that none of their cyclic shifts have parabolic shortenings or contain subwords in Φ , there is an element $g \in G$ and cyclic shifts U' of U and V' of V such that $gU'g^{-1} =_G V'$ and*

$$\min\{\max\{\ell(U), \ell(V)\}, |g|_X\} \leq B.$$

(b) *If $(H_\omega, X \cap H_\omega)$ have NSC for all $\omega \in \Omega$, then (G, X) has NSC.*

Proof. Let λ, c, \mathcal{H}' be the constants and set provided by the Generating Set Lemma (Lemma 5.3). Let X be any generating set of G satisfying $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$, and let $\Phi = \Phi(X)$ be the set of non-geodesic words produced by the Generating Set Lemma.

For any word W over X such that none of its cyclic shifts have parabolic shortenings or contain a subword in Φ , the Generating Set Lemma implies that \widehat{W} is a cyclic (λ, c) -quasi-geodesic without vertex backtracking. Also, we have that $|h|_X = |h|_{X \cap H_\omega}$ for all $h \in H_\omega$ and $\omega \in \Omega$.

(a) Assume that $(H_\omega, H_\omega \cap X)$ have A -BCD for all $\omega \in \Omega$, where $A \geq 0$.

Let U, V be two words over X representing conjugate elements such that no cyclic shifts have parabolic shortenings or contain subwords in Φ . Let (p, q, r, s) be a minimal conjugacy (λ, c) -diagram in $\Gamma(G, X \cup \mathcal{H})$, where $\text{Lab}(p)$ and $\text{Lab}(q)$ are some cyclic permutations of \widehat{U} and \widehat{V} , respectively. If U and V do not represent an element of a parabolic subgroup, we obtain by Theorem 9.13 that there exist a cyclic permutation U' of U and a cyclic permutation V' of V and $g \in G$ such

that $gU'g^{-1} = V'$ and $\min\{\max\{\ell(U'), \ell(V')\}, |g|_X\} \leq k$, where k is the constant provided by Theorem 9.13.

So we only need to consider the case when U or V labels elements of the parabolic subgroups.

Suppose that p is parabolic. If p is connected to a component of q , then by Lemma 9.14, U and V are words in $X \cap H_\omega$ for some $\omega \in \Omega$ and conjugate in H_ω . Then there exists a cyclic permutation U' of U and a cyclic permutation V' of V and $h \in H_\omega$ such that $hU'h^{-1} = V'$ and $\min\{\max\{\ell(U'), \ell(V')\}, |h|_X\} \leq A$.

If p is not connected to a component of q , then by Lemma 9.16(a), $d_X(p_-, p_+) \leq 2A + 4D + m$. In this case, q is not connected to a component of p , and there are two cases. The first case is when q is also parabolic. Here again, using Lemma 9.16 we obtain that $d_X(q_-, q_+) \leq 2A + 4D + m$. The second case is when q is not parabolic. In this case, by Lemma 9.9, $\ell(q) \leq \lambda(2K + 1 + c)$. Then by Lemma 9.12, $d_X(q_-, q_+) \leq 4(\lambda(2K + 1 + c) + 4)D + 4m$.

Therefore $\max\{\ell(U), \ell(V)\} \leq B := \max\{4(\lambda(2K + 1 + c) + 4)D + 4m, 2A + 4D + m\}$ and then (G, X) has B -BCD.

(b) Assume that $(H_\omega, H_\omega \cap X)$ has A -NSC for all $\omega \in \Omega$, where $A \geq 0$.

We are going to show that (G, X) has B -NSC with

$$B = A + 5D + 4m + k + 4\lambda(2K + 1 + c).$$

Let U, V be two cyclic geodesic words over X representing conjugate elements. Suppose that $\ell(V) < \ell(U)$ and $\ell(U) \geq B$.

Let (p, q, r, s) be a minimal conjugacy (λ, c) -diagram in $\Gamma(G, X \cup \mathcal{H})$, where **Lab**(p) and **Lab**(q) are some cyclic permutations of \widehat{U} and \widehat{V} , respectively. If U and V do not represent an element of a parabolic subgroup, we obtain by Theorem 9.13 that there exist a cyclic permutation U' of U and a cyclic permutation V' of V , and $g \in G$, such that $gU'g^{-1} = V'$ and $\min\{\max\{\ell(U'), \ell(V')\}, |g|_X\} \leq k$, where k is the constant provided by Theorem 9.13.

So we only need to consider the cases when p or q is parabolic.

Assume first that p is parabolic.

If p is connected to a component of q , then by Lemma 9.14, q is parabolic and U and V are conjugate in H_ω . The result in this case follows from the A -NSC of $(H_\omega, X \cap H_\omega)$ and the fact that $|h|_X = |h|_{X \cap H_\omega}$ for $h \in H_\omega$. If p is parabolic and is not connected to a component of q , then the result follows from Lemma 9.16(b).

Assume now that q is parabolic.

If q is connected to a component of p , the result follows by arguing as in the case p parabolic connected to a component of q . If q is not connected to a component of p , we consider two cases separately. In both we will conclude that $d_X(r_-, r_+) \leq B$.

Suppose first that $\ell(r) = 1$. If **Lab**(r) $\in X$, then $d_X(r_-, r_+) = 1 \leq B$. If r is just a component, it can't be connected to q , since this will imply that p vertex backtracks or p is a component, and neither of the two situations holds by hypothesis. Then, by Lemma 9.6, we can assume that either $d_X(r_-, r_+) \leq m \leq B$ or r is isolated. If r is isolated, then by Corollary 9.2, $d_X(r_-, r_+) \leq 5D \leq B$.

Suppose now that $\ell(r) > 1$. Now by Lemma 9.11, $B \leq (4\ell(r) + 4)D + 4m$. Since $B > (4\lambda(2K + 1 + c) + 4)D + 4m$, $\ell(r) > \lambda(2K + 1 + c)$ and then by Lemma 9.9, $d_X(r_-, r_+) \leq (2K + 3)D + 2m \leq B$. \square

Remark 9.18. We observe here that Theorem 9.17 provides a cubic-time algorithm for solving the conjugacy problem in groups hyperbolic relative to abelian subgroups. This algorithm is in the same spirit as [2, Algorithm 7.A].

The conjugacy problem in groups hyperbolic relative to parabolic subgroups with solvable conjugacy problem has been shown to be solvable by Bumagin in [4]. Recently and independently, bounds of the complexity have been obtained by Bumagin [5]. Rebbechi [36] or our Corollary 1.8 proves that groups hyperbolic with respect to abelian subgroups are biautomatic, so this also implies the solvability of the conjugacy problem in the main class of groups considered in this paper.

Suppose that G is a finitely generated group, hyperbolic relative to a family $\{H_\omega\}_{\omega \in \Omega}$ of abelian subgroups. Using Theorem 9.17 we can find a finite generating set X and a finite set Φ of non-geodesic words over X . Let B be the constant provided by Theorem 9.17(a). We assume that the sets X and Φ have been provided to us and do not include the complexity of obtaining these sets in our discussion below. This approach is often employed when considering algorithmic problems in hyperbolic groups, where a Dehn presentation is assumed, and the complexity of obtaining such a presentation from an arbitrary one is incorporated into a constant (produced by a computable function that took into account various parameters of the presentation).

Given a word U over $X \cap H_\omega$, since H_ω is abelian, one can find in $O(\ell(U))$ steps a geodesic word U' over $X \cap H_\omega$ such that $U' =_G U$. Therefore, given a word W over X , one can produce in a linear number of steps a word W' , $\ell(W') \leq \ell(W)$, that cyclically has no parabolic shortenings and represents a conjugate of W . Using the same argument as that in [2, Algorithm DA], from a word W that cyclically has no parabolic shortenings, when replacing subwords in Φ by geodesic words representing the same elements, we can produce a word W' that cyclically has no subwords in Φ and such that W' represents a conjugate of W with $\ell(W') \leq \ell(W)$. Moreover, if W represents the trivial element, W' is empty. We call this algorithm “the reducing algorithm”.

Now given two words U and V , we use the reducing algorithm to produce in a linear number of steps words U' and V' that cyclically have no parabolic shortenings and do not contain subwords in Φ . By Theorem 9.17(a) either $\max\{\ell(U'), \ell(V')\} \leq B$, and we decide if U' and V' are conjugate in a constant number of steps (depending on B), or $\max\{\ell(U'), \ell(V')\} > B$, and we know that if U and V are conjugate, then there are cyclic shifts U'' of U' and V'' of V' and a conjugator of length at most B . Then we can decide if U and V are conjugate by applying the reducing algorithm to all words of the form $CU''C^{-1}(V'')^{-1}$, where C is a word of length less than B and U'' and V'' are cyclic shifts of U' and V' , respectively. The number of such words is $O(\ell(U') \cdot \ell(V'))$. Thus the complexity of the algorithm is $O((\ell(U) + \ell(V))\ell(U)\ell(V))$.

We remark that [2] argues that in the hyperbolic case it is only necessary to check whether cyclic shifts of U' are conjugate to V' by a word of a bounded length. This is the reason why the algorithm there is quadratic. A similar situation might occur in our context. An algorithm with even lower complexity for the conjugacy problem for hyperbolic groups was obtained by Epstein and Holt in [15].

Also notice that the algorithm provides a conjugator for U' and V' in case U and V are conjugate. By keeping track of the cyclic reductions necessary to obtain U' from U and V' from V one can produce a conjugator for U and V . We remark

that an upper bound for the length of the conjugator was obtained by O'Connor in [32].

10. PROOF OF THEOREM 1.1 AND APPLICATION TO VIRTUALLY ABELIAN PARABOLICS

In this section we prove all the remaining main results stated in the Introduction. We start with the proof of Theorem 1.1, which uses a combination of Theorems 7.1, 9.17, 7.6 and 7.7.

Proof of Theorem 1.1. Let (P) be a property of (ordered) generating sets as in Theorem 1.1. Assume that G is hyperbolic with respect to a family $\{H_\omega\}_{\omega \in \Omega}$ of groups that are (P)-completable. Let Y be any finite (ordered) generating set of G .

Let \mathcal{H}'_1 be the subset of \mathcal{H} provided by Theorem 7.1, \mathcal{H}'_2 the subset of \mathcal{H} provided by Theorem 7.6 and \mathcal{H}'_3 the subset of \mathcal{H} provided by Theorem 9.17. Set $\mathcal{H}' = \mathcal{H}'_1 \cup \mathcal{H}'_2 \cup \mathcal{H}'_3$.

Since $\{H_\omega\}_{\omega \in \Omega}$ are (P)-completable, there is a finite (ordered) generating set X of G satisfying $Y \cup \mathcal{H}' \subseteq X \subseteq Y \cup \mathcal{H}$ and such that for all $\omega \in \Omega$, $(H_\omega, H_\omega \cap X)$ has (P). Now the result follows from Theorems 7.1, 7.6, 7.7 and 9.17. \square

We now turn to proving Corollaries 1.8 and 1.9.

It will be convenient to set the following notation for the set of cyclic geodesics:

$$\text{CycGeo}(G, X) := \{W \in X^* \mid \text{for all cyclic shifts } W' \text{ of } W, W' \in \text{Geo}(G, X)\}.$$

We remark that $\text{ConjGeo}(G, X) \subseteq \text{CycGeo}(G, X) \subseteq \text{Geo}(G, X)$.

Proposition 10.1. *Let G be a finitely generated virtually abelian group. Any finite generating set of G is contained in a finite generating set Z such that (G, Z) has NSC and FFTP and (G, Z) admits a geodesic prefix-closed biautomatic structure.*

Proof. We will refer a number of times to the proof of [7, Proposition 3.3], and in particular, the generating set we work with here is the one used in that proof, as well as in [20, Prop. 6.3]. We also need to introduce some terminology. For a set A and $x_1, \dots, x_n \in A$ we say that $V \equiv x_1 \cdots x_n$ is a *piecewise subword* of a word W over A if W belongs to the set $x_1 A^* x_2 A^* \cdots x_n A^*$.

Let Z_0 be a generating set for G . Since G is residually finite, there is $N \triangleleft G$ torsion-free abelian of finite index in G such that $Z_0 \cap N = \emptyset$. Enlarge Z_0 to Z_1 , if necessary, to assume that $\bigcup_{z \in Z_1} zN = G$.

We build the generating set Z for G as follows. Let $Y := (Z_1 - N)^{\pm 1}$. Note that $Z_0 \subseteq Y$. Let X' be the set of all $x \in N$ such that $x =_G W \neq_G 1$ for some $W \in Y^*$ with $\ell(W) \leq 4$. Let X'' be the closure of the set $(Z_0 \cap N) \cup X'$ in G under inversion and conjugation. We now identify N with a lattice in $\mathbb{R} \otimes N$ as in [14, Theorem 4.2.1], and we take the convex hull \mathcal{K} in $\mathbb{R} \otimes N$ of the union of the images of X'' . It follows from [14, Theorem 4.2.1] that there is some $n \in \mathbb{N}$, $n > 0$, such that $X = N \cap n\mathcal{K}$ is a generating set of N that admits a geodesic prefix-closed biautomatic structure \mathcal{L} invariant under the G/N -action.

Let $Z := X \cup Y$. By [14, Corollary 4.2.4], $\mathcal{L} \cup \mathcal{L}Y$ is a geodesic prefix-closed biautomatic structure for (G, Z) .

Moreover,

- (i) $X \subset N$, $Y \subset G - N$,

- (ii) both X and Y are closed under inversion,
- (iii) X is closed under conjugation by elements of G ,
- (iv) Y contains at least one representative of each non-trivial coset of N in G , and
- (v) if $W =_G xy$ with $W \in Y^*$, $\ell(W) \leq 3$, $x \in N$ and $y \in Y \cup \{\epsilon\}$, then $x \in X$.

Write the finite set X as $X = \{x_1, \dots, x_m\}$. For each $x \in X$ and $y \in Y$, let x^y denote the generator in X that represents the group element $y^{-1}xy$. Similarly, if $v = x_{i_1} \cdots x_{i_k} \in X^*$, the symbol v^y denotes the word $x_{i_1}^y \cdots x_{i_k}^y$. An immediate consequence of properties (i)-(v) above is that the language $L := \text{Geo}(G, Z)$ of geodesics of G over Z satisfies the property that $L \subseteq X^* \cup X^*YX^* \cup X^*YX^*YX^*$.

In order to see that (G, Z) has FFTP, observe that Z is in fact one of the generating sets produced by Neumann and Shapiro in [27, Proposition 4.4] to show that (G, Z) has FFTP.

We need to show that (G, Z) also has NSC.

Let $\tilde{L} := \text{ConjGeo}(G, Z)$ be the set of conjugacy geodesics of G over Z . Then $\tilde{L} := \text{ConjGeo}(G, Z) \subseteq L$, and so \tilde{L} can be partitioned as the union of the subsets $\tilde{L}_0 := \tilde{L} \cap X^*$, $\tilde{L}_1 := \tilde{L} \cap X^*YX^*$, and $\tilde{L}_2 := \tilde{L} \cap X^*YX^*YX^*$. The set $C := \text{CycGeo}(G, Z)$ can also be partitioned as the union of the subsets $C_0 = C \cap X^*$, $C_1 := C \cap X^*YX^*$, and $C_2 := C \cap X^*YX^*YX^*$.

It was shown in the proof of [7, Proposition 3.3] that $X^* \cap L = \tilde{L}_0$, so $C_0 = \tilde{L}_0$ since $L_0 \subseteq C_0 \subseteq X^* \cap L = \tilde{L}_0$. Thus if $U \in X^*$ is a cyclic geodesic, there is no word strictly shorter than U that is conjugate to it, and the NSC property is vacuous in this case.

We now turn to $C_1 - \tilde{L}_1$. One can further partition the set $\tilde{L}_1 = \bigcup_{r \in Y} \tilde{L}_{1,r}$, where $\tilde{L}_{1,r} := \{v_1 rv_2 \in \tilde{L} \mid v_1, v_2 \in X^*\}$ for each r in Y . It was shown in the proof of [7, Proposition 3.3] that the set $\tilde{L}_{1,r}$ is exactly the set of all words that do not contain a piecewise subword lying in a particular finite set, denoted there by $\tilde{S}'_{1,r} \subset X^*rX^*$. The set $\tilde{S}'_{1,r}$ is by construction closed under Y -shuffles and shuffles, so in particular under cyclic permutations. (An operation on words over Z given by replacement $ayxb \rightarrow ax^{y^{-1}}yb$ with $a, b \in Z^*$, $x \in X$, and $y \in Y$ is called a Y -shuffle. An operation on words over X given by a replacement $ux_ix_jv \rightarrow ux_jx_iv$ is a shuffle.) The set $\tilde{S}'_{1,r}$ can thus be seen as the set of minimal non-conjugacy geodesic words in X^*rX^* , and so for each word W in $\tilde{S}'_{1,r}$ there is a $g_W \in G$ such that $|g_W^{-1}Wg_W|_Z < |W|_Z$. Take $k_r := \max\{|g_w|_Z \mid w \in \tilde{S}'_{1,r}\}$ and $k_1 := \max\{k_r \mid r \in Y\}$.

Let $V \in C_1 - \tilde{L}_1$. Then it contains a piecewise subword in $S := \bigcup_{r \in Y} \tilde{S}'_{1,r}$. So $V \equiv V_0a_1V_1a_2V_2 \cdots a_nV_n$, where $a_1a_2 \cdots a_n \in S$, $V_0, \dots, V_n \in X^*$. After Y -shuffles and shuffles we can obtain a word V' such that $V' =_G V$ and $V' \equiv UW$, where $U \in X^*$, $W \in S$, and $\ell(U) + \ell(W) = \ell(V') = \ell(V)$. Take the element g_W and notice that

$$g_W^{-1}Vg_W =_G g_W^{-1}V'g_W =_G (g_W^{-1}Ug_W)(g_W^{-1}Wg_W),$$

by construction $|g_W^{-1}Wg_W|_Z < |W|_Z \leq \ell(W)$, and by (iii) $|(g_W^{-1}Ug_W)|_Z \leq |U|_Z \leq \ell(U)$. Thus $|g_W^{-1}Vg_W|_Z < \ell(U) + \ell(W) = \ell(V) = |V|_Z$.

A similar argument works for $C_2 - \tilde{L}_2$. There is a constant, call it k_2 , such that for every $V \in C_2 - \tilde{L}_2$ there is a $g \in G$, $|g| < k_2$, for which V satisfies $|g^{-1}Vg|_Z < |V|_Z$. Thus (G, Z) has k -NSC, where $k = \max\{k_1, k_2\}$. \square

We do not know if virtually abelian groups have generating sets that satisfy both FFTP and BCD. We remark here that Derek Holt has produced an example of a virtually abelian group that does not have BCD with the generating set used in the proof of Proposition 10.1.

Proof of Corollary 1.8. The result follows from Theorem 1.1 and the fact that for any finite generating set Z of an abelian group A , (A, Z) has FFTP and BCD, and for any order of Z , $\text{ShortLex}(A, Z)$ is a biautomatic structure (by [14, Theorem 4.3.1] and [14, Definition 2.5.4]). The regularity of the languages is a consequence of Proposition 1.6, Proposition 1.2 and the closure of regular languages under intersection. \square

Proof of Corollary 1.9. The result follows from Theorem 1.1 and Proposition 10.1. The regularity of the languages is a consequence of Proposition 1.6, Proposition 1.2 and the closure of regular languages under intersection. \square

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