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EXCHANGE RELATION PLANAR ALGEBRAS OF SMALL RANK

ZHENGWEI LIU

Abstract. The main purpose of this paper is to classify exchange relation planar algebras with 4 dimensional 2-boxes. Besides its skein theory, we emphasize the positivity of subfactor planar algebras based on the Schur product theorem. We will discuss the lattice of projections of 2-boxes, specifically the rank of the projections. From this point, several results about biprojections are obtained. The key break of the classification is to show the existence of a biprojection. By this method, we also classify another two families of subfactor planar algebras: subfactor planar algebras generated by 2-boxes with 4 dimensional 2-boxes and at most 23 dimensional 3-boxes; subfactor planar algebras generated by 2-boxes, such that the quotient of 3-boxes by the basic construction ideal is abelian. They extend the classification of singly generated planar algebras obtained by Bisch, Jones and the author.

1. Introduction

In [20], Jones classified the indices of subfactors of type II_1 as follows:

$$\{4\cos^2(\frac{\pi}{n}), n = 3, 4, \cdots\} \cup [4, \infty].$$

One approach to the classification of subfactors is to treat the index. Thus the simplest subfactors are those of index less than 4 and then those of index between 4 and 5. An early result is the classification of subfactors of index at most 4; see [14, 17, 33, 39]. This approach has been extremely successful in the hands of Haagerup [16] and others [1, 3, 7, 17, 41]. Recently the classification has been extended upto index 5, see [18, 23, 29, 31, 34].

Below index 4 a deep theorem of Popa's [37] showed that the *standard invariant* is a complete invariant of subfactors of the hyperfinite factor of type II_1. Subfactor planar algebras were introduced by Jones as a diagrammatic axiomatization of the standard invariant [19]. Other axiomatizations are known as Ocneanu's paragroups [33] and Popa's λ-lattices [38].

From the planar algebra perspective it seems far more natural to say that the simplest subfactors are those whose standard invariants are generated by the fewest elements satisfying the simplest relations. The simplest subfactor planar algebra is the one generated by the sequence of Jones projections, also well known as the Temperley-Lieb algebra, denoted by $TL(\delta)$, and TL for short, where δ is the square root of the index. The next most complicated planar algebras after Temperley-Lieb should be those generated by a single element. See [3, 30, 35, 44] for examples.

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For a planar algebra $\mathcal{S} = \{\mathcal{S}_{n,\pm}\}_{n\in\mathbb{N}_0}$, an element in $\mathcal{S}_{n,\pm}$ is called an n-box. Planar algebras generated by 1-boxes were completely analyzed by Jones in [19]. Subfactor planar algebras generated by a non-trivial 2-box were considered by Bisch and Jones, and classified by them for $\dim(\mathcal{S}_{3,\pm}) \leq 12$ in [11]; for $\dim(\mathcal{S}_{3,\pm}) = 13$ in [12]; by Bisch, Jones and the author for $\dim(\mathcal{S}_{3,\pm}) = 14$ in [8]. They are given by the crossed product group subfactor planar algebra $\mathcal{S}^{\mathbb{Z}_3}$, the *free product* of two TL's, well known as Fuss-Catalan [10]; the crossed product subgroup subfactor planar algebra $\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_5\rtimes\mathbb{Z}_2}$; BMW [4, 32, 44], precisely one family from quantum $Sp(4,\mathbb{R})$ and one from quantum $O(3,\mathbb{R})$, respectively. The classification for $\dim(\mathcal{S}_{3,\pm}) = 15$ is still unclear. In these cases, we always have $\dim(\mathcal{S}_{2,\pm}) = 3$, since $\dim(\mathcal{S}_{2,\pm})^2 \leq \dim(\mathcal{S}_{3,\pm})$.

In this paper, we hope to classify subfactor planar algebras generated by 4 dimensional 2-boxes. Observe that the free product of the index 2 subfactor planar algebra and a subfactor planar algebra generated by a 2-box with 15 dimensional 3-boxes has 4 dimensional 2-boxes and 24 dimensional 3-boxes. So we can only expect a classification for at most 23 dimensional 3-boxes.

Theorem 1.1. *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes, with $\dim(\mathcal{S}_{2,\pm}) = 4$ and $\dim(\mathcal{S}_{3,\pm}) \leq 23$. Then $\mathcal{S}$ is one of the following:*

(1) $\mathcal{S}^{\mathbb{Z}_4}$ *or* $\mathcal{S}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$*;*

(2a) $\mathcal{A}*TL$ *or* $TL*\mathcal{A}$*, where $\mathcal{A}$ is generated by a non-trivial 2-box with* $\dim(\mathcal{A}_{3,\pm}) \leq 13$*;*

(2b) $\mathcal{B}*\mathcal{S}^{\mathbb{Z}_2}$ *or* $\mathcal{S}^{\mathbb{Z}_2}*\mathcal{B}$*, where $\mathcal{B}$ is generated by a non-trivial 2-box with* $\dim(\mathcal{A}_{3,\pm}) \leq 14$*;*

(3) $\mathcal{S}^{\mathbb{Z}_2} \otimes TL$.

Another approach to the classification of planar algebras is to consider the relations of the generators, instead of the boundary of dimensions. Several kinds of relations of 2-boxes appeared naturally in planar algebras generated by a non-trivial 2-box with at most 15 dimensional 3-boxes. If $\mathcal{S}$ is a subfactor planar algebra generated by a non-trivial 2-box with $\dim(\mathcal{S}_{3,\pm}) \leq 12$, then $\mathcal{S}_{3,+}/\mathcal{I}_{3,+}$ is abelian, where $\mathcal{I}_{3,+}$ is the basic construction ideal of $\mathcal{S}_{3,+}$, i.e., the two sided ideal of $\mathcal{S}_{3,+}$ generated by the Jones projection. Motivated by this condition, we have the following classification.

Theorem 1.2. *If $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes and $\mathcal{S}_{3,+}/\mathcal{I}_{3,+}$ is abelian, then $\mathcal{S}$ is either depth-2 or the free product $\mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_n$, such that $\mathcal{A}_1$ is Temperley-Lieb or the dual of $\mathcal{S}^{G_1}$, for a group G_1; $\mathcal{A}_n$ is Temperley-Lieb or $\mathcal{S}^{G_n}$, for a group G_n; $\mathcal{A}_m$, for $1 < m < n$, is Temperley-Lieb or $\mathcal{S}^{G_m}$, for an abelian group G_m. The converse statement is also true.*

If $\mathcal{S}$ is a subfactor planar algebra generated by a non-trivial 2-box with $\dim(\mathcal{S}_{3,\pm}) \leq 13$, then $\mathcal{S}$ is an *exchange relation planar algebra* [28]. Motivated by the exchange relation, we have the following classification.

Theorem 1.3. *Suppose $\mathcal{S}$ is an exchange relation planar algebra with* $\dim(\mathcal{S}_{2,\pm}) = 4$*. Then $\mathcal{S}$ is one of the following:*

(1) $\mathcal{S}^{\mathbb{Z}_4}$ *or* $\mathcal{S}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$*;*

(2) $\mathcal{A}*TL$ *or* $TL*\mathcal{A}$*, where $\mathcal{A}$ is generated by a non-trivial 2-box with* $\dim(\mathcal{A}_{3,\pm}) \leq 13$*;*

(3) $\mathcal{S}^{\mathbb{Z}_2} \otimes TL$*;*

(4) $\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_7\rtimes\mathbb{Z}_2}$.

The three classification results rely on a new approach to the complexity of subfactors, the rank of 2-boxes. We will show that the rank of the *coproduct* of two 2-box minimal projections is bounded by the number of length-2 paths between the two corresponding vertices in the principal graph; see Lemma 4.5. We can show that a subfactor planar algebra is a free product by looking at its principal graph; see Theorem 4.26.

In section 2, we recall some facts and notation about planar algebras. In section 3, a diagrammatic interpretation of the tensor product is discovered based on the construction of a biunitary in the planar algebra. It is related to the *flatness* of a planar algebra with respect to two *biprojections*. If a subfactor planar algebra contains two commuting and co-commuting biprojections, then the planar subalgebra generated by the flat parts with respect to the two biprojections forms a tensor product; see Theorem 3.14.

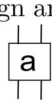
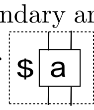
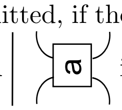
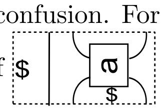
In section 4 , we prove the Schur product theorem for subfactor planar algebras; see Theorem 4.1. Based on it, a new equivalent definition of biprojections in given; see Theorem 4.12. Consequently the support of the *pure depth* 2 *parts* of an irreducible subfactor planar algebra is a biprojection; see Theorem 4.16. From a von Neumann algbera perspective, this tells the existence of an intermediate subfactor which is the crossed product of the smaller factor by a Kac algebra. By the new definition of biprojections, we can talk about the biprojection generated by a 2-box; see Definition 4.13. By the Schur product theorem, we show that the norm of the Fourier transform of a positive 2-box is achieved on the Jones projection; see Lemma 4.18. Then we prove that the Fourier transform of the biprojection generated by a positive 2-box is the spectrum projection of the Fourier transform of the 2-box at its maximal spectrum; see Theorem 4.21. This result generalises a well known result in representation theory; see Proposition 4.22 and the remark following.

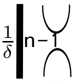
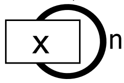
In section 5, we discuss the construction and the decomposition of exchange relation planar algebras under the free product and the tensor product. We obtain two general constructions of exchange relation planar algebras (see Propositions 5.3 and 5.5) and one family of exchange relation planar algebras (see Theorem 5.6). In section 6, we prove the three main classification results.

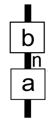
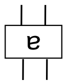
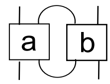
2. Preliminaries

We refer the reader to [22] for the definition of subfactor planar algebras. The dual of a planar algebra is given by switching its shading.

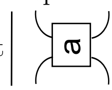
2.1. **Notation.** In a planar tangle, we use a thick string with a number k to indicate k parallel strings. The distinguished intervals of a planar tangle are marked by \$'s (corresponding to $*$'s in [22]).

In this paper, the planar algebra $\mathcal{S} = \{\mathcal{S}_{n,\pm}\}_{n\in\mathbb{N}\cup\{0\}}$ is always the standard invariant of an irreducible subfactor, which is automatically spherical. Equivalently we assume that $\dim(\mathcal{S}_{1,\pm}) = 1$. Since we only work with $\mathcal{S}_{n,+}$, we write $\mathcal{S}_n$ for $\mathcal{S}_{n,+}$, and the dollar sign \$ of a planar tangle is always in an unshaded region. An element in $\mathcal{S}_n$ is written as a rectangle with the dollar sign on the left, called an n-box. The dollar sign and the boundary are omitted, if there is no confusion. For example, we may use  instead of , and  instead of .

The value of a closed circle is δ; id means the identity of $\mathcal{S}_2$; $e_n = \frac{1}{\delta}$  is the Jones projection in $\mathcal{S}_{n+1}$; $e = e_1$. The (unnormalized) Markov trace on $\mathcal{S}_n$ is denoted by $tr_n(x) =$ , $\forall\, x \in \mathcal{S}_n$. When $n = 2$, we write $tr(x)$ for short.

For $a, b \in \mathcal{S}_n$, the product of a and b is defined as $ab =$ . If $a, b \in \mathcal{S}_2$, then we define $a' =$  to be the contragredient of a and $a * b =$  to be the (1-string) coproduct of a and b. Furthermore for $a_i \in \mathcal{S}_2, i = 1, 2, \cdots, k$, we write $*_{i=1}^{k} a_i$ for $a_1 * a_2 * \cdots * a_k$, and a^{*k} for $*_{i=1}^{k} a$.

Note that $\mathcal{S}_2$ is embedded in $\mathcal{S}_3$ by adding one string to the right. Thus a 2-box a can be viewed as an element in $\mathcal{S}_3$, still written as a.

The Fourier transform, i.e., the one click rotation, is an isometry from $\mathcal{S}_2$ to $\mathcal{S}_{2,-}$, and $\mathcal{S}_{2,-}$ is identified as a subspace of $\mathcal{S}_3$ by adding one string to the left. Let us define $1 \boxdot a$ to be the element  of $\mathcal{S}_3$, and $\mathcal{S}_{1,3}$ to be $\{1 \boxdot z | z \in \mathcal{S}_2\}$. Then $\mathcal{S}_{1,3}$ is isomorphic to $\mathcal{S}_{2,-}$ as an algebra. It is easy to check that $(1 \boxdot a)(1 \boxdot b) = 1 \boxdot (a * b)$, $e_2 = \frac{1}{\delta}(1 \boxdot id)$ and $1 \boxdot a'$ is the adjoint of $1 \boxdot a^*$, where a^* is the adjoint of a.

Definition 2.1. For two self-adjoint operators x and y, we say x is weaker (resp. stronger) than y if the support of x (resp.y) is a subprojection of the support of y (resp. x), written as $x \preceq y$ (resp. $y \succeq x$). If $x \preceq y$ and $y \preceq x$, then they have the same support, written as $x \sim y$.

For a self-adjoint operator x and a projection p, $x \preceq p$ is equivalent to $x = pxp$.

Notation 2.2. The support of a two sided ideal of a finite dimensional C*-algebra is the maximal projection in the ideal.

2.2. **Principle graphs, depth-2 subfactors and subgroup subfactors.** We refer the reader to [25] for the definition of the (dual) principal graph of a subfactor. It is also defined for a subfactor planar algebra, since it does not depend on the presumed subfactor [6].

The principal graph and the dual principal graph are parts of the Ocneanu 4-partite principal graph [23, 33].

Suppose $\mathcal{S}$ is the planar algebra of $\mathcal{N} \subset \mathcal{M}$. Let us define $\mathcal{I}_{n+1}$ to be the two sided ideal of $\mathcal{S}_{n+1}$ generated by the Jones projection e_n, called the basic construction ideal; then $\mathcal{I}_{n+1} = \mathcal{S}_{n+1} e_n \mathcal{S}_{n+1} = \mathcal{S}_n e_n \mathcal{S}_n$. Let us define $\mathcal{S}_n / \mathcal{I}_n$ to be the orthogonal complement of $\mathcal{I}_n$ in $\mathcal{S}_n$; there is a bijection between the equivalent classes of minimal projections of $\mathcal{S}_n / \mathcal{I}_n$ and vertices in the principal graph whose distance from the marked point is n.

Definition 2.3. In the principal graph, a vertex is said to be depth-n if its distance from the marked vertex is n. Its multiplicity is the number of length-n paths from the marked vertex to it. The depth of a principal graph is defined to be the maximal depth of its vertices.

Remark 2.4. If a depth-n vertex has multiplicity m, then the vertex corresponds to an $M_m(\mathbb{C})$ component in $\mathcal{S}_n/\mathcal{I}_n$.

Notation 2.5. If the principal graph of a subfactor planar algebra is depth-2, equivalently $\mathcal{S}_3 = \mathcal{I}_3$, then we call it a depth-2 subfactor planar algebra.

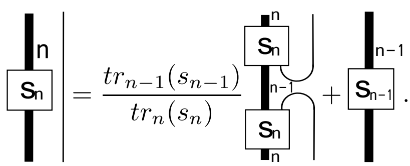
There is a one to one correspondence between depth-2 subfactor planar algebras and finite dimensional Kac algebras, or finite dimensional C* Hopf algebras [26, 40, 42]. Precisely, for any depth-2 subfactor planar algebra $\mathcal{S}$, $\mathcal{S}_2$ forms a Kac algebra. On the other hand for any finite dimensional Kac algebra K there is an outer action of K on the hyperfinite factor $\mathcal{R}$ of type II$_1$. Then $\mathcal{R}' \cap \mathcal{R} \rtimes K = \mathbb{C}$. Thus we obtain an irreducible subfactor planar algebra as the standard invariant of $\mathcal{R} \subset \mathcal{R} \rtimes K$, denoted by $\mathcal{S}^K$. Then $\mathcal{S}^K$ is depth-2, and $(\mathcal{S}^K)_2$ is isomorphic to the dual of K as a Kac algebra. Specially when G is a finite group, we obtain a subfactor planar algebra $\mathcal{S}^G$ of $\mathcal{R} \subset \mathcal{R} \rtimes G$. If H is a subgroup of G, then $\mathcal{R} \rtimes H$ is subfactor of $\mathcal{R} \rtimes G$. Thus we obtain a subfactor planar algebra of $\mathcal{R} \rtimes H \subset \mathcal{R} \rtimes G$.

Definition 2.6. Let us define $\mathcal{S}^G$ to be the planar algebra of the crossed product group subfactor $\mathcal{R} \subset \mathcal{R} \rtimes G$ and $\mathcal{S}^{H \subset G}$ to be the planar algebra of the crossed product subgroup subfactor $\mathcal{R} \rtimes H \subset \mathcal{R} \rtimes G$.

The principal graph of a subgroup subfactor is described in [25, 27].

2.3. **Wenzl's formula.** Suppose $\mathcal{S}$ is a subfactor planar algebra, $\mathcal{I}_{n+1}$ is the basic construction ideal of $\mathcal{S}_{n+1}$, and $\mathcal{S}_{n+1}/\mathcal{I}_{n+1}$ is its orthogonal complement in $\mathcal{S}_{n+1}$. Then $\mathcal{S}_{n+1} = \mathcal{I}_{n+1} \oplus \mathcal{S}_{n+1}/\mathcal{I}_{n+1}$. Let s_{n+1} be the support of $\mathcal{S}_{n+1}/\mathcal{I}_{n+1}$.

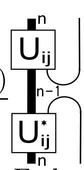
If $\mathcal{S}$ is Temperley-Lieb (with $\delta^2 \geq 4$), then s_n is the n_{th} Jones-Wenzl projection. The following relation is called Wenzl's formula [43]:



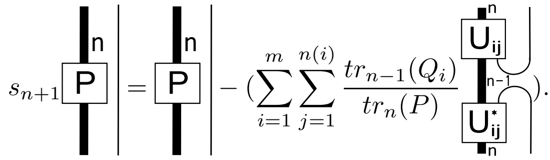
It tells how a minimal projection is decomposed after adding one string to the right.

In general, suppose P is a minimal projection in $\mathcal{S}_n/\mathcal{I}_n$. Note that $\mathcal{S}_{n+1} = \mathcal{I}_{n+1} \oplus \mathcal{S}_{n+1}/\mathcal{I}_{n+1}$. When P is included in $\mathcal{S}_{n+1}$, it is decomposed as two projections $P = P_{old} + P_{new}$, such that $P_{old} \in \mathcal{I}_{n+1}$ and $P_{new} \in \mathcal{S}_{n+1}/\mathcal{I}_{n+1}$. By the definition of s_{n+1}, we have $P_{new} = s_{n+1}P$. Now let us construct P_{old}. Let v be the depth-n vertex in the principal graph corresponding to P, and let V be the central support of P. Suppose v_i, $1 \leq i \leq m$, are the depth-$(n-1)$ vertices adjacent to v, the multiplicity of the edge between v_i and v is $m(i)$, and Q_i is a minimal projection in $\mathcal{S}_{n-1}$ corresponding to v_i. For each i, take partial isometries $\{U_{ij}\}_{j=1}^{m(i)}$ in $\mathcal{S}_n$, such that

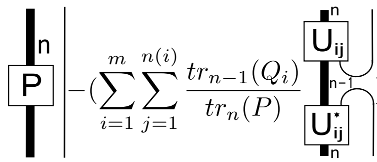
$$U_{ij}^* U_{ij} = P, \ \forall 1 \leq j \leq m(i); \quad \sum_{j=1}^{m(i)} U_{ij} U_{ij}^* = Q_i V.$$

It is easy to check that $\frac{tr_{n-1}(Q_i)}{tr_n(P)}$ 

is a subprojection of P, and they are mutually orthogonal for all i, j. By Frobenius reciprocity, their sum is P_{old}. Then the general Wenzl's formula $P_{new} = P - P_{odd}$ is given as

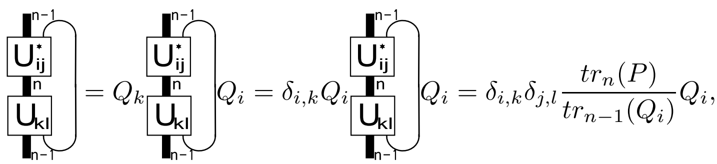


Now we give an alternative proof of the general Wenzl formula without applying Frobenius reciprocity. This proof is useful when the planar algebra is constructed by generators and relations. Based on this proof, we may derive the Bratteli diagram inductively without assuming it is a subfactor planar algebra.

Proposition 2.7. *Take* $P_+ =$ 

. For any $x \in \mathcal{I}_{n+1}$, *we have* $xP_+ = P_+x = 0$.

Proof. For U_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n(i)$, and U_{kl}, $1 \leq k \leq m$, $1 \leq l \leq n(k)$, we have



since $\{Q_i\}$ are mutually inequivalent minimal projections in $\mathcal{S}_{n-1}$, and the last equality follows from computing the trace. So $e_nU_{i,j}P_+ = e_nU_{i,j} - e_nU_{i,j} = 0$.

For any $x \in \mathcal{I}_{n+1}$, we have $xP = \sum_{i=1}^{m}\sum_{j=1}^{n(i)} x_{i,j}e_nU_{i,j}$, for some $x_{i,j} \in \mathcal{S}_n$. So $xP_+ = (xP)P_+ = 0$. Similarly $P_+x = 0$. □

2.4. **Tensor products, free products and biprojections.** Let us recall some facts about tensor products, free products and biprojections due to Bisch and Jones; see [2, 5, 9, 10] for more details.

Notation 2.8 ([19]). Suppose $\mathcal{A}$ and $\mathcal{B}$ are planar algebras and their tensor product is a planar algebra, denoted by $\mathcal{A} \otimes \mathcal{B}$.

The tensor product $\mathcal{A} \otimes \mathcal{B}$ is a planar algebra for which $(\mathcal{A} \otimes \mathcal{B})_n = \mathcal{A}_n \otimes \mathcal{B}_n$, $n \geq 0$, and the action of an unlabeled tangle T from $\bigotimes_{i=1}^{k}(\mathcal{A} \otimes \mathcal{B})_{n_i}$ to $(\mathcal{A} \otimes \mathcal{B})_m$ is defined as a linear extension of the map

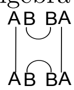
$$T(\bigotimes_{i=1}^{k}(x_i \otimes y_i)) = T(\bigotimes_{i=1}^{k} x_i) \otimes T(\bigotimes_{i=1}^{k} y_i), \ \forall x_i \otimes y_i \in \mathcal{A}_{n_i} \otimes \mathcal{B}_{n_i}, \ 1 \leq i \leq m.$$

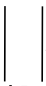
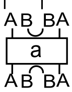
If both $\mathcal{A}$ and $\mathcal{B}$ admit the adjoint operation $*$, then we define $(x \otimes y)^*$ to be $x^* \otimes y^*$.

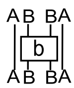
Proposition 2.9 ([19]). *Suppose* $\mathcal{A}$ *and* $\mathcal{B}$ *are subfactor planar algebras. Then* $\mathcal{A} \otimes \mathcal{B}$ *is a subfactor planar algebra.*

Notation 2.10 ([2, 9, 10]). Suppose $\mathcal{A}$ and $\mathcal{B}$ are planar algebras. Their free product is a planar algebra, denoted by $\mathcal{A} * \mathcal{B}$.

Each element in the free product $\mathcal{A} * \mathcal{B}$ is a linear sum of A,B-colour diagrams which consist of non-intersecting A,B-colour strings, labels of $\mathcal{A}$ which only connect with A-coloured strings, and labels of $\mathcal{B}$ which only connect with B-coloured strings. The colour of its boundary points are ordered by $ABBA\ ABBA \cdots ABBA$. For an action of an unlabeled tangle T, we substitute each string of T by a pair of parallel A,B-colour strings and then glue the boundaries.

There is an equivalent definition of the free product. We say an element $x \otimes y \in (\mathcal{A} * \mathcal{B})_n$ is separated by a Temperley-Lieb n-tangle T_n if x can be written as a diagram in unshaded regions of T_n and y can be written a diagram in shaded regions of T_n. Then $\mathcal{A} * \mathcal{B}$ is the planar subalgebra of $\mathcal{A} \otimes \mathcal{B}$ consisting of all separated elements. For example, the diagram 

is separated by the tangle , identified as $\delta_B id \otimes e$, where δ_B is the value of a circle of $\mathcal{B}$. Moreover, 

$= \delta_B a \otimes e$; 

$= id \otimes b$. Consequently we have the following result.

Proposition 2.11 ([2,9])**.** *Suppose $\mathcal{A}$ and $\mathcal{B}$ are subfactor planar algebras. Then $\mathcal{A} * \mathcal{B}$ is a subfactor planar algebra.*

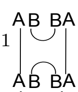
Notation 2.12. The free product is associative but not commutative. The free product of $\mathcal{A}$ and $\mathcal{B}$ may be denoted as either $\mathcal{A} * \mathcal{B}$ or $\mathcal{B} * \mathcal{A}$.

Proposition 2.13. *The dual of $\mathcal{A} * \mathcal{B}$ is the free product of the dual of $\mathcal{B}$ with the dual of $\mathcal{A}$.*

The free product of Temperley-Lieb subfactor planar algebras is called a $Fuss-Catalan$ subfactor planar algebra [10].

Definition 2.14. Suppose Q is a projection in $\mathcal{S}_2$. If $1 \boxdot Q$ is a multiple of a projection in $\mathcal{S}_{1,3}$, then we call Q a biprojection. In this case, $\frac{\delta}{tr(Q)} 1 \boxdot Q$ is the projection.

Notation 2.15. There are two trivial biprojections, e and id, in $\mathcal{S}_2$. A biprojection is said to be non-trivial if it is neither e nor id.

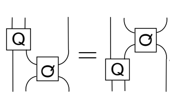
In the free product $\mathcal{A} * \mathcal{B}$, there is a non-trivial biprojection δ_B^{-1} 

.

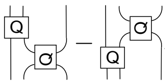
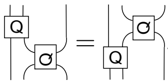
Biprojections were introduced by Bisch while considering the projection onto an intermediate subfactor [5]. Suppose $\mathcal{S}$ is the planar algebra of a subfactor $\mathcal{N} \subset \mathcal{M}$. Then each biprojection Q in $\mathcal{S}_2$ corresponds to an intermediate subfactor $\mathcal{Q}$ of $\mathcal{N} \subset \mathcal{M}$, in the sense that Q is the projection onto $L^2(\mathcal{Q})$ as a subspace of $L^2(\mathcal{M})$.

The following relation of a biprojection was discovered by Bisch in [5].

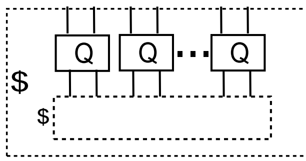
Proposition 2.16 ([5])**.** *Suppose Q is a biprojection in a subfactor planar algebra. Then Q satisfies*



called the exchange relation of the biprojection Q.

Proof. Let x be . Then it is easy to check that $tr_3(x^*x) = 0$. By the positivity of the trace, we have $x = 0$. That means . ☐

It was known to Bisch and Jones and also appeared in [2] that the planar algebra of $\mathcal{N} \subset \mathcal{Q}$ can be realised as a Q cut down on shaded intervals of diagrams in $\mathcal{S}$, denoted by $\mathcal{S}_Q$. That means $(\mathcal{S}_Q)_n = \Psi_Q(\mathcal{S}_n)$, where Ψ_Q is the annular action



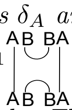
;

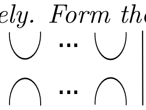
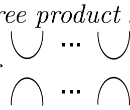
and the action of an unlabeled tangle T on $\mathcal{S}_Q$ is defined to be

$$(\frac{\delta}{\sqrt{tr(Q)}})^{fudge(T)}\Psi_Q \circ T,$$

where $fudge(T) = n - m$, n is number of shaded intervals of the outside boundary of T and m is the number of closed circles after adding a cap at each shaded interval of the (outside and inside) boundary of T. Considering the duality, the planar algebra of $\mathcal{Q} \subset \mathcal{M}$ is realised as a $\frac{\delta}{tr(Q)}Q$ cut down on unshaded intervals of diagrams in $\mathcal{S}$, denoted by $\mathcal{S}^Q$.

The meaning of the fudge factor is explained in the following proposition [9].

Proposition 2.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be subfactor planar algebras with circle parameters δ_A and δ_B respectively. Form the free product $\mathcal{A} * \mathcal{B}$. Let Q be the biprojection* δ_B^{-1} 

and $v_n \in \mathcal{B}$ be  *or* *, when n is odd or even respectively. Then the map*

$$\alpha_Q : \mathcal{A} \Longrightarrow (\mathcal{A} * \mathcal{B})_Q, \ \alpha_Q(x) = x \otimes v_n, \ \forall \ x \in \mathcal{A}_n,$$

is a planar algebra isomorphism.

The following result was first known by Bisch and Jones [9]; see also [2].

Theorem 2.18 ([2, 9])**.** *Let $\mathcal{S}$ be a subfactor planar algebra containing is a biprojection Q. Then the planar subalgebra $\mathcal{S}_Q \vee \mathcal{S}^Q$ of $\mathcal{S}$ generated by the vector spaces $\mathcal{S}_Q$ and $\mathcal{S}^Q$ is naturally the free product $\mathcal{S}_Q * \mathcal{S}^Q$.*

Corollary 2.19. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are subfactor planar algebras generated by 2-boxes. Then $\mathcal{A} * \mathcal{B}$ is generated by 2-boxes.*

Theorem 2.20. *Suppose $\mathcal{S}$ is a subfactor planar algebra. If Q is a biprojection in $\mathcal{S}_2$, and $\mathcal{S}$ is generated by $\{x \in \mathcal{S}_2 | QxQ = x \text{ or } Q * x * Q = (\frac{tr(Q)}{\delta})^2 x\}$ as a planar algebra, then $\mathcal{S} = \mathcal{S}_Q * \mathcal{S}^Q$, and both $\mathcal{S}_Q$ and $\mathcal{S}^Q$ are generated by 2-boxes. In this case, $\mathcal{S}$ is said to be separated by the biprojection Q as a free product.*

Proof. Suppose $\mathcal{A}$ is the planar subalgebra of $\mathcal{S}_Q$ generated by 2-boxes and $\mathcal{B}$ is the planar subalgebra of $\mathcal{S}^Q$ generated by 2-boxes. Then $\mathcal{A} * \mathcal{B} \subseteq \mathcal{S}_Q * \mathcal{S}^Q \subseteq \mathcal{S}$.

On the other hand, if $x \in \mathcal{S}_2$ satisfies $QxQ = Q$, then $x = \alpha_Q^{-1}(x) \otimes \delta_B e$, and if $y \in \mathcal{S}_2$ satisfies $Q * y * Q = (\frac{tr(Q)}{\delta})^2 y$, then $y = id \otimes \beta_Q^{-1}(y)$. If $\mathcal{S}$ is generated by these 2-boxes, then $\mathcal{S} \subseteq \mathcal{A} * \mathcal{B}$. So $\mathcal{A} * \mathcal{B} = \mathcal{S}_Q * \mathcal{S}^Q = \mathcal{S}$. Note that the generating function of a free product is Volculescu's free product of generating functions [9]. By counting the dimension of n-boxes, we obtain $\mathcal{A} = \mathcal{S}_Q$ and $\mathcal{B} = \mathcal{S}^Q$. ☐

Corollary 2.21. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are subfactor planar algebras. If $\mathcal{A} * \mathcal{B}$ is generated by 2-boxes, then both $\mathcal{A}$ and $\mathcal{B}$ are generated by 2-boxes.*

Proof. Suppose Q is the central biprojection $id \otimes e$; then $(\mathcal{A}*\mathcal{B})_2 = \{x \in \mathcal{S}_2 | QxQ = Q \text{ or } Q * x * Q = (\frac{tr(Q)}{\delta})^2 x\}$. The statement follows from Theorem 2.20. ☐

The following result should be known to experts. We give two proofs based on the Schur product theorem; see Theorem 4.1.

Theorem 2.22. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are subfactor planar algebras. Then for any biprojection Q in $(\mathcal{A} * \mathcal{B})_2$, either $Q \geq id \otimes e$ or $Q \leq id \otimes e$.*

Proof. Note that $Q = a \otimes e + id \otimes b$ for some $a \in \mathcal{A}_2, b \in \mathcal{B}_2$. Furthermore we may assume $be = 0$, otherwise a, b are replaced by $\lambda id + a, b - \lambda e$, when $be = \lambda e$. Then this decomposition is unique since $a \otimes e = Q(id \otimes e)$. By assumption Q is a projection, thus $a \otimes e$ and $id \otimes b$ are projections. Then a and b are projections. Moreover $Q = Q'$, thus both $a \otimes e$ and $id \otimes b$ are self-contragredient, and so $a = a'$ and $b = b'$. Furthermore $\frac{tr(Q)}{\delta} Q = Q * Q$, so

$$\begin{aligned}
&a \otimes e + id \otimes b \\
=&a * a \otimes e * e + a * id \otimes e * b + id * a \otimes b * e + id * id \otimes b * b \\
=&\frac{1}{\delta_1} a * a \otimes e + \frac{2tr(a)}{\delta_1 \delta_2} id \otimes b + \delta_1 id \otimes b * b \\
=&\frac{1}{\delta_1} a * a \otimes e + \delta_1 \frac{tr(b)}{\delta_2} id \otimes e + \frac{2tr(a)}{\delta_1 \delta_2} id \otimes b + \delta_1 id \otimes (b * b - \frac{tr(b)}{\delta_2} e).
\end{aligned}$$

Both b and $b * b - \frac{tr(b)}{\delta_2} e$ are orthogonal to e, so

$$\frac{tr(Q)}{\delta} a \otimes e = \frac{1}{\delta_1} a * a \otimes e + \delta_1 \frac{tr(b)}{\delta_2} id \otimes e.$$

If $tr(b) = 0$, then $b = 0$ because b is a projection. Thus $P = a \otimes e \leq id \otimes e$. Otherwise $tr(b) > 0$. By Theorem 4.1, we have that $a * a \otimes e$ is positive. So $id \otimes e \preceq a \otimes e$. Recall that a is a projection, so $a = id$. Then $P \geq id \otimes e$. ☐

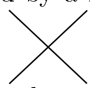
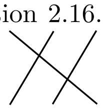
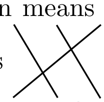
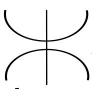
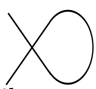
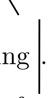
A second proof. Suppose Q is a biprojection. Note that $id \otimes e$ is central. If Q is not a subprojection of $id \otimes e$, then Q has a subprojection $id \otimes b$. By Theorem 4.1, we have

$$id \otimes e \preceq (id * id) \otimes (p' * p) = (id \otimes p)' * (id \otimes p) \preceq Q' * Q \sim Q.$$

So $id \otimes e \leq Q$. ☐

2.5. **Skein theory.** Comparing to group theory, a subfactor planar algebra could be constructed by generators and relations [19]. While trying to construct a subfactor planar algebra $\mathcal{S} = \{\mathcal{S}_{n,\pm}\}_{n\in\mathbb{N}_0}$, we will encounter four problems:

(1) Is $\mathcal{S}$ finite dimensional, i.e., is $\mathcal{S}_{n,\pm}$ finite dimensional for each n?
(2) Is $\mathcal{S}$ evaluable, i.e., is $\mathcal{S}_{0,\pm}$ 1 dimensional?
(3) Is $\mathcal{S}$ the zero planar algebra?
(4) Is the Markov trace positive definite?

If $\mathcal{S}$ is the planar algebra of an irreducible subfactor, then $\dim(\mathcal{S}_{0,\pm}) = \dim(\mathcal{S}_{1,\pm}) = 1$. We shall consider a planar algebra generated by a finite subset of 2-boxes. Then each generator can be viewed as a crossing  with a label at the intersection, and each element in $\mathcal{S}_{n,\pm}$ can be viewed as a linear combination of diagrams with $2n$ boundary points which consist of finitely many crossings and finitely many strings. For example, , as an element in $\mathcal{S}_{3,+}$, is a diagram with 6 boundary points and 2 crossings. What kind of relations should be endowed? One type of relation, termed an exchange relation, is discussed by Landau [28]. It was motivated by the exchange relation of a biprojection discovered by Bisch [5]; see Proposition 2.16. The planar algebra $\mathcal{S}$ has an exchange relation means that the diagram  can be replaced by a finite sum of the diagrams  and , and the diagram  can be replaced by a multiple of a string . Note that a closed string contributes a scalar δ. By these three operations, a face of a diagram can be removed without increasing the number of crossings. Given the number of boundary points, up to isotopy, there are only finitely many diagrams without faces and closed strings. Thus problem (1) is solved. Furthermore if such a diagram has no boundary points, then it has to be the empty diagram. Thus problem (2) is solved. Given generators and an exchange relation, to solve problem (3) is equivalent to checking a finite system of equations. However it is hard to solve theses equations directly. What's worse, it is much harder to solve problem (4). In this paper, we focus on classifying exchange relation planar algebras. The ones that appeared could be constructed by other methods. So we will not deal with problems (3) and (4) directly.

Definition 2.23. Suppose $\mathcal{S}$ is an irreducible subfactor planar algebra. If $\mathcal{S}$ is generated by $\mathcal{S}_2$ with the relations

$$(1 \boxdot a)b = \sum_i c_i(1 \boxdot d_i) + f_i(1 \boxdot id)g_i,$$

for any $a, b \in \mathcal{S}_2$ and for finitely many $c_i, d_i, f_i, g_i \in \mathcal{S}_2$, then $\mathcal{S}$ is called an exchange relation planar algebra.

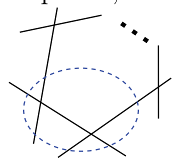
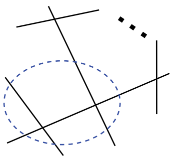
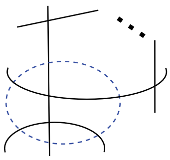
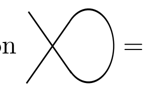
It is easy to check that this definition is equivalent to Landau's definition in [28]. By definition, Temperley-Lieb subfactor planar algebras and depth-2 subfactor planar algebras are exchange relation planar algebras.

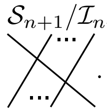
Proposition 2.24. *Suppose $\mathcal{S}$ is an exchange relation planar algebra. Then*

$$\dim(\mathcal{S}_{n+1}) \leq \dim(\mathcal{S}_n)^2 + (\dim(\mathcal{S}_2) - 1)^n$$

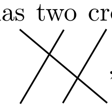
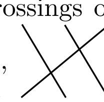
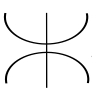
and $\mathcal{P}_3$ is generated by $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ as an algebra. Specifically,

$$\dim(\mathcal{S}_3) \leq \dim(\mathcal{S}_2)^2 + (\dim(\mathcal{S}_2) - 1)^2.$$

Proof. We view 2-boxes as crossings, the labels at the intersection as points, and the strings as edges. Then using exchange relations, we may replace  by  and . By this operation, the number of edges of one face will decrease without adding faces. Combining with the relation , we only need to consider diagrams without faces.

If a diagram with $2n+2$ boundary points, $n+1$ on the top and $n+1$ on the bottom, has no faces, then either it is in the ideal $\mathcal{I}_{n+1}$ generated by the Jones projection e_n, or it has $n+1$ through strings. The dimension of the ideal of $\mathcal{I}_{n+1}$ is at most $\dim(\mathcal{S}_n)^2$. In $\mathcal{S}_{n+1}/\mathcal{I}_{n+1}$, applying the exchange relation, we only need to consider one diagram, . If the label at an intersection is the Jones projection e, then this diagram is in the ideal $\mathcal{I}_{n+1}$. Thus the dimension of $\mathcal{S}_{n+1}/\mathcal{I}_{n+1}$ is at most $(\dim(\mathcal{S}_2) - 1)^n$. Then

$$\dim(\mathcal{S}_{n+1}) \leq \dim(\mathcal{S}_n)^2 + (\dim(\mathcal{S}_2) - 1)^n.$$

If a diagram with 6 boundary points has no faces and it has two crossings or more, then it has to be one of the following three diagrams: , , . Thus $\mathcal{P}_3$ is generated by $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ as an algebra. □

If a closed diagram has no faces, then it is the empty diagram. So each closed diagram is evaluable based on the exchange relation. Furthermore the exchange relation is determined by the algebraic structure of 2-boxes in the following sense.

Definition 2.25. The structure of 2-boxes of a subfactor planar algebra consists of the data of adjoints, contragredients, products and coproducts of 2-boxes.

The following data is also derived from the structure of 2-boxes: the identity *id* is identified as the unique unit of 2-boxes under the product; the value of a closed circle δ is determined by the coproduct of two identities; δe is identified as the unique unit of 2-boxes under the coproduct; the trace of a 2-box is determined by its coproduct with the identity *id*. If the planar algebra is irreducible, then capping a 2-box is also determined.

The following result is known to experts.

Theorem 2.26. *Suppose $\mathcal{S}$ is an exchange relation planar algebra and $\mathcal{A}$ is a subfactor planar algebra generated by* 2*-boxes. If a linear map $\phi : \mathcal{S}_2 \Longrightarrow \mathcal{A}_2$ is surjective and it preserves the structure of* 2*-boxes, i.e., adjoints, contragredients, products and coproducts, then ϕ extends to a planar algebra isomorphism from $\mathcal{S}$ to $\mathcal{A}$.*

Proof. We extend ϕ to the universal planar algebra generated by 2-boxes of $\mathcal{S}$. If

$$(1 \boxdot a)b - (\sum_i c_i(1 \boxdot d_i) + f_i(1 \boxdot id)g_i)$$

is a relation, denoted by y, then $tr(y^*y) = 0$. The computation of $tr(\phi(y)^*\phi(y))$ only depends on the structure of 2-boxes, which are preserved by ϕ, so

$$tr(\phi(y)^*\phi(y)) = tr(y^*y) = 0.$$

Then $\phi(y) = 0$ by the positivity of the trace. So ϕ induces a planar algebra homomorphism from the quotient $\mathcal{S}$ to $\mathcal{A}$. By assumption ϕ is surjective on 2-boxes, and $\mathcal{A}$ is generated by 2-boxes, so ϕ is a planar algebra isomorphism. □

Remark 2.27. A planar algebra homomorphism of subfactor planar algebras induces a homomorphism on the 0-box space $\mathbb{C}$, so it is either zero or injective.

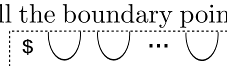
The following classification is given by Bisch and Jones [11, 12].

Theorem 2.28 ([11, 12])**.** *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by a non-trivial* 2*-box with* $\dim(P_3) \leq 13$. *Then $\mathcal{S}$ is one of the following:* (1) $\mathcal{S}^{\mathbb{Z}_3}$; (2) $TL * TL$; (3) $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_5 \rtimes \mathbb{Z}_2}$.

Suppose $\mathcal{S}$ is a subfactor planar algebra generated by $\mathcal{S}_2$ with $\dim(\mathcal{S}_2) = 3$. Assume id, e and r form a basis of $\mathcal{S}_2$. In $\mathcal{S}_3$, there are 5 Temperley-Lieb diagrams, 6 diagrams with one r, the orbits or $r, 1 \boxdot r$ under the rotation, and 3 diagrams with two r, $r(1 \boxdot id)r$, $r(1 \boxdot r)$ and $(1 \boxdot r)r$. If $\dim(\mathcal{S}_3) \leq 13$, then those 14 elements are linear dependent. Without loss of generality, we may assume one diagram with two r is a linear sum of the other 13 diagrams. Up to a 2-click rotation, it can be chosen as $(1 \boxdot r)r$. This implies $\mathcal{S}$ is an exchange relation planar algebra. Conversely if $\mathcal{S}$ is an exchange relation planar algebra with $\dim(\mathcal{S}_2) = 3$, then $\dim(\mathcal{S}_3) \leq 2^2 + 3^2 = 13$. Thus Theorem 2.28 is equivalent to

Theorem 2.29. *Suppose $\mathcal{S}$ is an exchange relation planar algebra with* $\dim(P_2) =$ 3. *Then $\mathcal{S}$ is one of the following:* (1) $\mathcal{S}^{\mathbb{Z}_3}$; (2) $TL * TL$; (3) $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_5 \rtimes \mathbb{Z}_2}$.

3. Tensor products

In this section, sometimes we draw a diagram with all the boundary points on the top. For example, v_n is the Temperley-Lieb n-tangle . Recall that $\mathcal{A} * \mathcal{B} \subset \mathcal{A} \otimes \mathcal{B}$ and it contains a biprojection $id \otimes e$. By Proposition 2.17, there is a planar algebra isomorphism $\alpha_1 : \mathcal{A} \Longrightarrow (\mathcal{A} * \mathcal{B})_{id \otimes e}$, $\alpha(a) = a \otimes v_n$, $\forall\ a \in \mathcal{A}$. By the definition of the tensor product, we have $(\mathcal{A} \otimes \mathcal{B})_{id \otimes e} = \alpha_1(\mathcal{A})$. So $(\mathcal{A} \otimes \mathcal{B})_{id \otimes e} = (\mathcal{A} * \mathcal{B})_{id \otimes e}$. Then we have a planar algebra isomorphism

$$\alpha_1 : \mathcal{A} \Longrightarrow (\mathcal{A} \otimes \mathcal{B})_{id \otimes e},\ \alpha(a) = a \otimes v_n,\ \forall\ a \in \mathcal{A}.$$

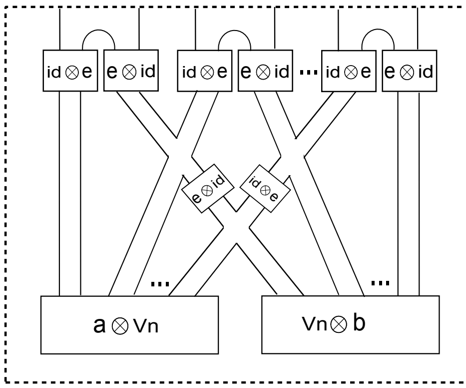
Similarly we have a planar algebra isomorphism

$$\alpha_2 : \mathcal{B} \Longrightarrow (\mathcal{A} \otimes \mathcal{B})_{e\otimes id}, \ \alpha(b) = v_n \otimes b, \ \forall\, b \in \mathcal{B}.$$

The tensor product of planar algebras is defined via simple tensors of vectors. We hope to interpret the simple tensor $a \otimes b$ as a diagram in terms of $\alpha_1(a)$ and $\alpha_2(b)$.

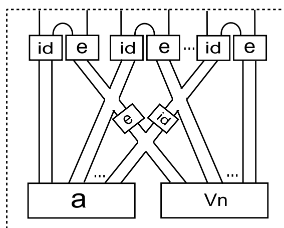
Theorem 3.1. *$\mathcal{A} \otimes \mathcal{B}$ is generated by the two vector spaces $(\mathcal{A} \otimes \mathcal{B})_{id\otimes e}$ and $(\mathcal{A} \otimes \mathcal{B})_{e\otimes id}$ as a planar algebra.*

Proof. For any $a \in \mathcal{A}_n$ and $b \in \mathcal{B}_n$, let us construct an element T in $(\mathcal{A} \otimes \mathcal{B})_n$ as



.

By the definition of the tensor product of planar algebras, the first component of T is

$T_{\mathcal{A}} =$ 

.

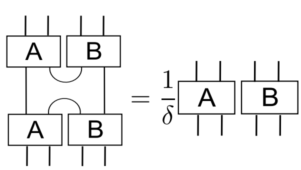
Replacing id, e, v_n in the above tangle by Temperley-Lieb diagrams, we have

$$T_{\mathcal{A}} = \lambda_n a, \text{ for some } \lambda_n > 0.$$

Similarly the second component is $T_{\mathcal{B}} = \lambda_n b$. So $T = \lambda_n a \otimes \lambda_n b$. Note that $a \otimes v_n \in (\mathcal{A} \otimes \mathcal{B})_{id\otimes e}$, $v_n \otimes b \in (\mathcal{A} \otimes \mathcal{B})_{e\otimes id}$, and $\mathcal{A} \otimes \mathcal{B}$ is generated by all $a \otimes b$'s. Thus $\mathcal{A} \otimes \mathcal{B}$ is generated by $(\mathcal{A} \otimes \mathcal{B})_{id\otimes e}$ and $(\mathcal{A} \otimes \mathcal{B})_{e\otimes id}$ as a planar algebra. □

Corollary 3.2. *If both $\mathcal{A}$ and $\mathcal{B}$ are generated by 2-boxes, then $\mathcal{A} \otimes \mathcal{B}$ is generated by 2-boxes.*

Lemma 3.3. *Suppose $\mathcal{S}$ is a subfactor planar algebra and A, B are two biprojections such that $AB = e$ and $A*B$ is a multiple of id. Then $A*B = \frac{1}{\delta}id$ and $tr(A)tr(B) = \delta^2$. Moreover*



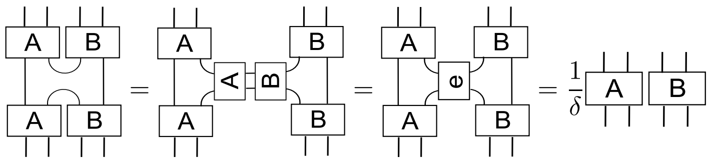
.

Proof. By assumptions, we have

$$AB = (AB)^* = B^*A^* = BA;$$

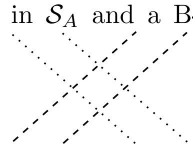
$$A * B = (A * B)' = B' * A' = B * A.$$

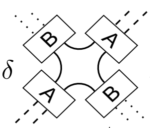
By computing the trace of $A * B$, we have $A * B = \frac{tr(A)tr(B)}{\delta^3}id$. Then $tr((A * B)e) = \frac{tr(A)tr(B)}{\delta^3}$. On the other hand, we view $tr((A * B)e)$ as a diagram; then $tr((A * B)e) = \frac{1}{\delta}tr(AB) = \frac{1}{\delta}$. Thus $tr(A)tr(B) = \delta^2$ and $A * B = \frac{1}{\delta}id$. Using the exchange relation of biprojections (Proposition 2.16), we have

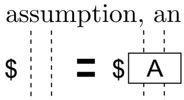
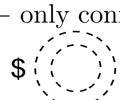
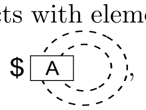
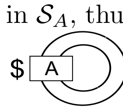
(3.1) 

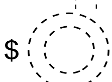
∎

Notation 3.4. Let δ_A, δ_B be $\sqrt{tr(A)}, \sqrt{tr(B)}$; then they are the value of a closed circle in $\mathcal{S}_A, \mathcal{S}_B$ respectively, and $\delta = \delta_A\delta_B$.

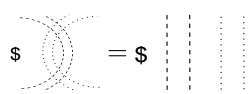
To show $\mathcal{S}_A$ and $\mathcal{S}_B$ are "independent", we use two kinds of coloured strings, an A-colour string $-\,-\,-\,-\,-$ connecting with elements in $\mathcal{S}_A$ and a B-colour string $\cdots\cdots$ connecting with elements in $\mathcal{S}_B$. A crossing  means

δ . Because A and B are biprojections, it does not matter where we put the \$'s. If a non-closed A-colour string does not intersect with a B-colour string, then the A-colour string is just an ordinary Temperley-Lieb string. By our assumption, an A-colour string $-\,-\,-\,-\,-$ only connects with elements in $\mathcal{S}_A$, thus

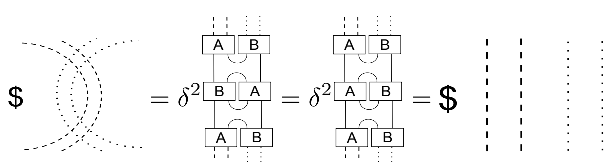
. Moreover we can view  as , i.e., .

Thus  $= tr(A)$. A similar formula holds for B.

Proposition 3.5.

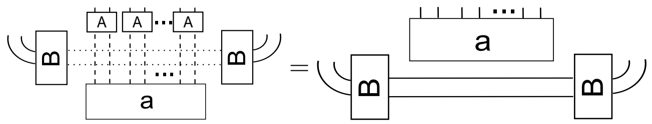


Proof. By Lemma 3.3, we have



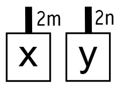
∎

Definition 3.6. For $a \in \mathcal{S}_A$, we call a B-flat if



By definition, the empty diagram $\emptyset$ in $(\mathcal{S}_A)_0$ is B-flat.

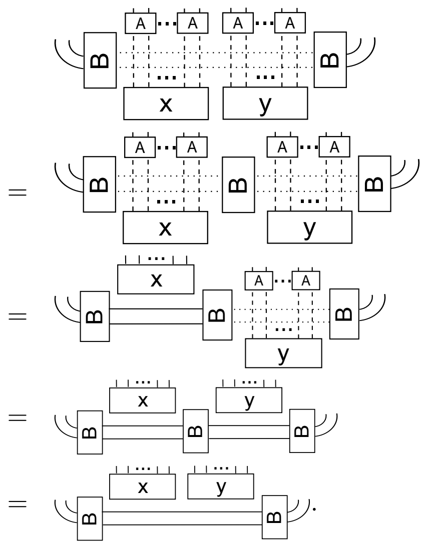
Definition 3.7. Let us define a map $\wedge : \mathcal{S}_m \otimes \mathcal{S}_n \Longrightarrow \mathcal{S}_{m+n}$ as a linear extension of

$x \wedge y =$  $, \ \forall\, x \in \mathcal{S}_m,\ y \in \mathcal{S}_n, m, n \geq 0.$

Its restriction on $\mathcal{S}_A$ induces a map $\wedge : (\mathcal{S}_A)_m \otimes (\mathcal{S}_A)_n \Longrightarrow (\mathcal{S}_A)_{m+n}$.

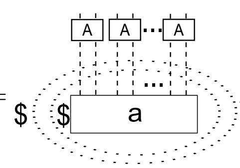
Proposition 3.8. *Suppose $x, y \in \mathcal{S}_A$ are B-flat; then $x \wedge y$ is B-flat.*

Proof.



Thus $x \wedge y$ is B-flat. ☐

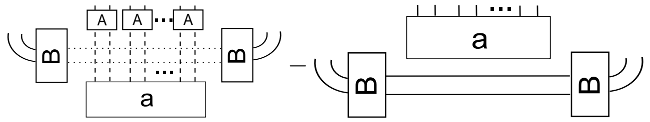
Definition 3.9. Let us define the map Θ_B on $\mathcal{S}_A$ given by the following annular tangle:

$\Theta_B(a) =$  $, \ \forall\, a \in \mathcal{S}_A.$

Proposition 3.10. *For an element $a \in \mathcal{S}_A$, a is B-flat if and only if $\Theta_B(a) = tr(B)a$.*

Proof. If a is B-flat, then $\Theta_B(a) = \Theta_B(\emptyset)a = tr(B)a$.

On the other hand, we assume that $\Theta_B(a) = tr(B)a$; then $\Theta_B(a^*) = (\Theta_B(a))^* = tr(B)a^*$. Take

$x =$ 

Then

$$\begin{aligned} tr(xx^*) &= \Theta_B(\emptyset)tr(aa^*) + \Theta_B(tr(aa^*)\emptyset) - tr(\Theta_B(a)a^*) - tr(a\Theta_B(a^*)) \\ &= tr(B)tr(aa^*) + tr(B)tr(aa^*) - tr(B)tr(aa^*) - tr(B)tr(aa^*) = 0. \end{aligned}$$

Thus $x = 0$. That means a is B-flat. □

Corollary 3.11. *For any $a \in \mathcal{S}_A$, if a is B-flat, then a^* is B-flat.*

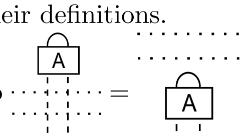
Theorem 3.12. *All Temperley-Lieb elements of $\mathcal{S}_A$ are B-flat. Moreover, all B-flat elements in $\mathcal{S}_A$, denoted by $Flat_B(\mathcal{S}_A)$, form a non-zero planar subalgebra of $\mathcal{S}_A$.*

Proof. We have known that the empty diagram $\emptyset \in (\mathcal{S}_A)_0$ is B-flat and $Flat_B(\mathcal{S}_A)$ is invariant under the adjoint action and the map $\wedge$. Note that the action of a planar tangle can be decomposed as actions of $\wedge$ and annular tangles [19]. Thus it is enough to show that $Flat_B(\mathcal{S}_A)$ is invariant under the annular Temperley-Lieb action. By Proposition 3.10, it is sufficient to prove that the annular Temperley-Lieb action commutes with Θ_B. Observe that, each Temperley-Lieb annular action is a composition of the following five operations:

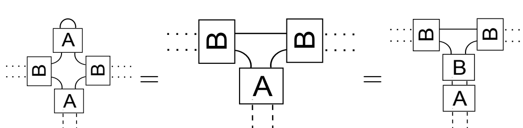
(1) a two-string rotation;
(2) adding a cap on a shaded boundary;
(3) adding a cap on an unshaded boundary;
(4) adding a string in a shaded boundary;
(5) adding a string in an unshaded boundary.

Thus we only need to prove Θ_B commutes with (1)-(5).

Θ_B commutes with (1) follows from their definitions.

Θ_B commutes with (2) is equivalent to 

. That is



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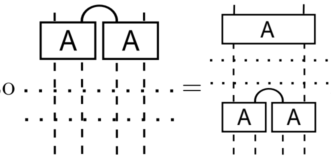
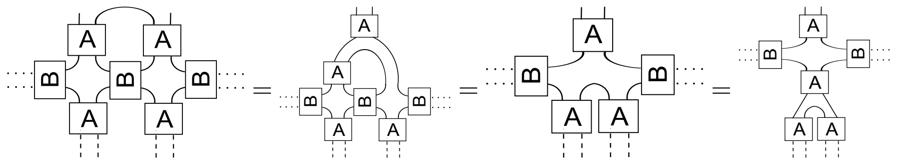
Θ_B commutes with (3) is equivalent to 

That is

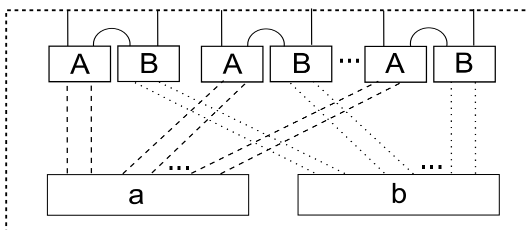


Observe that $\{(\mathcal{S}_A)_n\}$ with the Markov trace forms a sequence of Hilbert spaces, and Θ_B is self-adjoint on them due to the spherical property. Moreover (4) (resp. (5)) is a multiple of the adjoint of (3) (resp. (2)); thus Θ_B commutes with (4) (resp. (5)). □

Notation 3.13. Similarly let us define $Flat_A(\mathcal{S}_B)$ to be the planar subalgebra of $\mathcal{B}$ consisting of all A-flat elements.

Theorem 3.14. *The planar subalgebra of* $\mathcal{S}$ *generated by the two vector spaces* $Flat_B(\mathcal{S}_A)$ *and* $Flat_A(\mathcal{S}_B)$*, denoted by* $Flat_B(\mathcal{S}_A) \vee Flat_A(\mathcal{S}_B)$*, is naturally isomorphic to the tensor product* $Flat_B(\mathcal{S}_A) \otimes Flat_A(\mathcal{S}_B)$*.*

Proof. Let us define a map $\Phi : Flat_B(\mathcal{S}_A) \otimes Flat_A(\mathcal{S}_B) \Longrightarrow \mathcal{S}$ as a linear extension of

$\Phi(a \otimes b) =$ 

,

for any $a \otimes b \in (Flat_B(\mathcal{S}_A) \otimes (Flat_A\mathcal{S}_B))_n, n > 0$, and define $\Phi(\emptyset \otimes \emptyset) = \emptyset$, where $\emptyset$ is the unshaded empty diagram.

To show it is well defined, we prove that Φ preserves the inner product. By equation (3.1) and Proposition 3.5, for any $a_1 \otimes b_1,\ a_2 \otimes b_2 \in (Flat_B(\mathcal{S}_A) \otimes (Flat_A\mathcal{S}_B))_n$, we have

$$\langle \phi(a_1 \otimes b_1), \phi(a_2 \otimes b_2)\rangle_{\mathcal{S}} = tr_n(\phi(a_1 \otimes b_1)^* \phi(a_2 \otimes b_2)) = \delta^{-n} tr_n(a_1^* a_2) tr_n(b_1^* b_2).$$

While computing the inner product in $\mathcal{S}_A$ and $\mathcal{S}_B$, the fudge factor will be involved, and we have

$$\langle a_1, a_2\rangle_{\mathcal{S}_A} = \delta_A^{-n} tr(a_1^* a_2); \quad \langle b_1, b_2\rangle_{\mathcal{S}_B} = \delta_B^{-n} tr(b_1^* b_2).$$

So

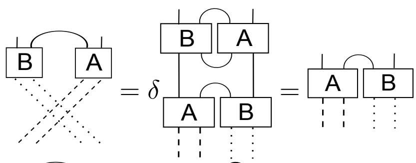
$$\begin{aligned} \langle a_1 \otimes b_1, a_2 \otimes b_2\rangle_{\mathcal{S}_A \otimes \mathcal{S}_B} &= \langle a_1, a_2\rangle_{\mathcal{S}_A} \langle b_1, b_2\rangle_{\mathcal{S}_B} \\ &= \delta_A^{-n} tr(a_1^* a_2) \delta_B^{-n} tr(b_1^* b_2) \\ &= \delta^{-n} tr_n(a_1^* a_2) tr_n(b_1^* b_2) \\ &= \langle \phi(a_1 \otimes b_1), \phi(a_2 \otimes b_2)\rangle_{\mathcal{S}}. \end{aligned}$$

To prove Φ is a planar algebra isomorphism, we need to check that ϕ commutes with the following seven operations:

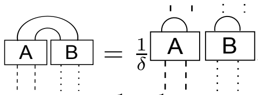
(1) the 2-click rotation;
(2) adding a cap on a shaded boundary;
(3) adding a cap on an unshaded boundary;
(4) adding a string in a shaded boundary;
(5) adding a string in an unshaded boundary;
(6) the adjoint operation;
(7) the $\wedge$ operation.

(1) follows from the flatness of a and b, and the fudge factor is 0 under the 2-click rotation.

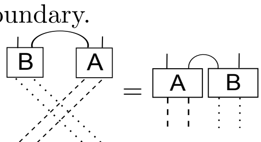
(7) follows from the flatness of a and b, and the fudge factor is 0 under the $\wedge$ operation.

(6) follows from 

, and the flatness of a and b.

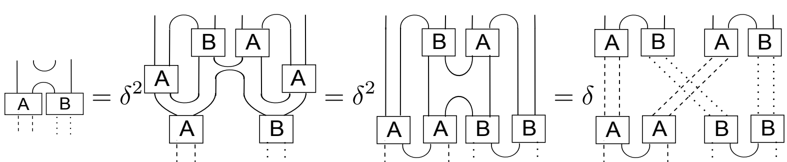
(2) follows from 

, and the flatness of Temperley-Lieb elements. The factor $\delta^{-1} = \delta_A^{-1}\delta_B^{-1}$ contributes -1 to the fudge factor of adding a cap on a shaded boundary.

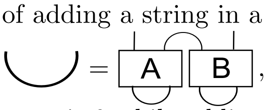
(3) follows from 

, and the flatness of Temperley-Lieb elements. The fudge factor is 0 while adding a cap on an unshaded boundary.

(4) follows from



and the flatness of Temperley-Lieb elements. The factor $\delta = \delta_A\delta_B$ contributes $+1$ to the fudge factor of adding a string in a shaded boundary.

(5) follows from 

, and the flatness of Temperley-Lieb elements. The fudge factor is 0 while adding a string in an unshaded boundary. □

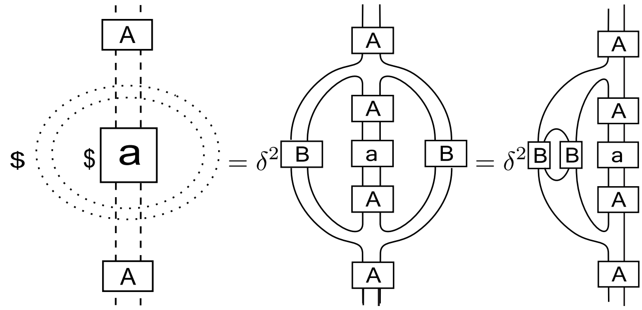
Theorem 3.15. *Suppose $\mathcal{S}$ is a subfactor planar algebra with two biprojections A, B such that*

$$AB = e, \qquad A * B \text{ is a multiple of } id,$$
$$a * B = B * a, \ \forall\ a \preceq A, \qquad A * b = b * A, \ \forall\ b \preceq B,$$

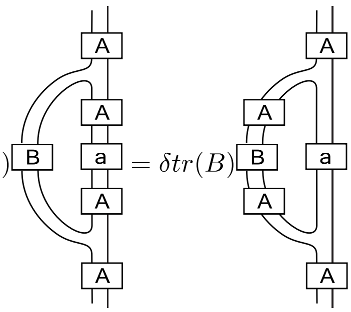
and $\mathcal{S}$ is generated by $\{x \in \mathcal{S}_2 |\ x \preceq A \text{ or } x \preceq B\}$ as a planar algebra. Then the planar subalgebra of $\mathcal{S}$ generated by the two vector spaces $\mathcal{S}_A$ and $\mathcal{S}_B$, denoted by $\mathcal{S}_A \vee \mathcal{S}_B$, is naturally isomorphic to the tensor product $\mathcal{S}_A \otimes \mathcal{S}_B$, $\mathcal{S}_A$ and $\mathcal{S}_B$ are *generated by 2-boxes, and* $\mathcal{S} = \mathcal{S}_A \vee \mathcal{S}_B$*. In this case,* $\mathcal{S}$ *is said to be separated by the biprojections* A *and* B *as a tensor product.*

Proof. For any positive operator $a \in (\mathcal{S}_A)_2$, we have $a \preceq A$, thus $(AaA) * B = a * B = B * a = B * (AaA)$ by assumption. Then

$\Theta_B(a) =$



$= \delta tr(B)$



by the exchange relation of A

$$= \delta tr(B) A(e * a)A = tr(B)AaA = tr(B)a.$$

Thus $a \in Flat_B(\mathcal{S}_A)$ by Proposition 3.10. Any operator in $(\mathcal{S}_A)_2$ is a linear sum of four positive operators, so $(\mathcal{S}_A)_2 \subseteq Flat_B(\mathcal{S}_A)$. Take $\mathcal{A}$ to be the planar subalgebra of $Flat_B(\mathcal{S}_A)$ generated by 2-boxes. Symmetrically $(\mathcal{S}_B)_2 \subseteq Flat_A(\mathcal{S}_B)$. Take $\mathcal{B}$ to be the planar subalgebra of $Flat_A(\mathcal{S}_B)$ generated by 2-boxes. Then $\mathcal{A} \vee \mathcal{B} \subseteq Flat_B(\mathcal{S}_A) \vee Flat_A(\mathcal{S}_B) \subseteq \mathcal{S}$. On the other hand $x \preceq A$ implies $x \in (\mathcal{S}_A)_2$, and $y \preceq B$ implies $y \in (\mathcal{S}_B)_2$. By assumption, we have $\mathcal{S} \subseteq \mathcal{A} \vee \mathcal{B}$. So $\mathcal{A} \vee \mathcal{B} = Flat_B(\mathcal{S}_A) \vee Flat_A(\mathcal{S}_B) = \mathcal{S}$. Then $Flat_B(\mathcal{S}_A) = \mathcal{A}$ and $Flat_A(\mathcal{S}_B) = \mathcal{B}$ by counting the dimensions. So they are generated by 2-boxes. By Theorem 3.14, $\mathcal{S} = Flat_B(\mathcal{S}_A) \vee Flat_A(\mathcal{S}_B)$ is naturally isomorphic to $Flat_B(\mathcal{S}_A) \otimes Flat_A(\mathcal{S}_B)$. $\square$

Corollary 3.16. *Suppose* $\mathcal{A}$, $\mathcal{B}$ *are subfactor planar algebras. If* $\mathcal{A} \otimes \mathcal{B}$ *is generated by 2-boxes, then both* $\mathcal{A}$ *and* $\mathcal{B}$ *are generated by 2-boxes.*

Proof. Considering $A = id \otimes e$, $B = e \otimes id$, and $a \otimes b = \delta^2 (a \otimes e) * (e \otimes b)$, the statement follows from Theorem 3.15. $\square$

4. Biprojections

In this section, we assume that $\mathcal{S}$ is a subfactor planar algebra.

For crossed product group subfactor planar algebra, all 2-box minimal projections are indexed by group elements, and their coproduct behaves like the group multiplication. Motivated by this fact, we will discuss the coproduct of 2-box positive operators of $\mathcal{S}$, specifically the lattice of the supports of those positive operators, based on the Schur product theorem. For a positive operator x, and $e \preceq x$, the support of x^{*k} will be increasing. The limit is a projection P such that

$P * P \sim P$. We shall expect P to be a biprojection, viewed as the biprojection generated by x.

If we assume P is a projection, $e \preceq p$, $P = P'$ and $P * P \sim P$, then by GJS construction [15], we obtain a factor $\mathcal{N} = Gr_0$ consisting of the linear span of diagrams with all boundary points on the left, and a factor $\mathcal{M} = Gr_1$ consisting of the linear span of diagrams with all boundary points on the left except one on the top and one on the bottom. Let us construct $\mathcal{P}$ as a cut down of $\mathcal{M}$ by an action of the projection P on the right. By our assumptions, we can show that $\mathcal{P}$ is an intermediate subfactor of $\mathcal{N} \subset \mathcal{M}$, and P is a biprojection.

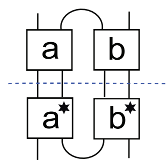
Furthermore we can drop the assumptions $e \preceq p$, $P = P'$, because the support of P^{*k} will contain e and P' when k is large enough. This phenomenon is similar to the fact that any group element g of a finite group generates identity and g^{-1}. We will not expect this phenomenon without the finite-index condition. A well known result of representation theory of finite groups reduces to this phenomenon and the Stone-Weierstrass theorem. See Theorems 4.12 and 4.21, Proposition 4.22 and the remark after that.

Let Q be a projection. Take the maximal projection P subject to the condition $P * Q \preceq Q$; then $P * P \sim P$. By our new result, P is a biprojection. If Q behaves like a normalizer under the coproduct, then the planar algebra $\mathcal{S}$ is a free product; see Theorem 4.26. Moreover we can find such a normalizer Q by looking at the principal graph; see Lemma 4.5. The combination of these two results will be the key break in our main classification results.

Theorem 4.1 (Schur product theorem)**.** *If A, B are positive operators in $\mathcal{S}_2$, then $A * B$ is a positive operator.*

Proof. Set $A = a^*a, B = b^*b$. Then $A * B = \delta\Phi((a(1 \boxdot b))^*(a(1 \boxdot b)))$, where Φ is the conditional expectation from $\mathcal{S}_3$ to $\mathcal{S}_2$. Since $(a(1 \boxdot b))^*(a(1 \boxdot b)) \geq 0$ in $\mathcal{S}_3$, we have $A * B \geq 0$ in $\mathcal{S}_2$. While $tr(A * B) > 0$, it follows that $A * B$ is positive. $\square$

Remark 4.2. Diagrammatically

$A * B =$  $\geq 0.$

Remark 4.3. If we consider the subfactor planar algebra as a planar subalgebra of its graph planar algebra [21, 24], then this Schur product theorem reduces to the classical one.

Definition 4.4. Suppose $X \in \mathcal{S}_2$ is a positive operator, and $X = \sum_{i=1}^{k} C_i P_i$ for some mutually orthogonal minimal projections $P_i \in \mathcal{S}_2$ and $C_i > 0$, for $1 \leq i \leq k$. Let us define the rank of X to be k, denoted by $r(X)$.

It is easy to see that $r(X)$, the rank of X in $\mathcal{S}_2$, is independent of the decomposition. Also, $r(X) = 1$ means X is a multiple of a minimal projection.

Lemma 4.5. *Suppose P and Q are projections in $\mathcal{S}_2$. Then*

$$r(P' * Q) \leq \dim(Q\mathcal{S}_3 P),$$

where $Q\mathcal{S}_3 P = \{QxP | x \in \mathcal{S}_3\}$.

Proof. Suppose $P' * Q = \sum_{i=1}^{k} C_i R_i$, $C_i > 0$, $1 \leq i \leq k$, for some mutually orthogonal minimal projections $R_i \in \mathcal{S}_2$. Take $v_i = Q(1 \boxdot R_i')P$, for $1 \leq i \leq k$. Then $v_i \in Q\mathcal{S}_3 P$. It is easy to check that

$$\begin{aligned} tr_3(v_i^* v_i) &= tr((P' * Q)R_i) > 0, \quad && \forall 1 \leq i \leq k; \\ tr_3(v_i^* v_j) &= 0, && \forall 1 \leq i, j \leq k, \ i \neq j. \end{aligned}$$

So $\{v_i\}$ are mutually orthogonal non-zero vectors. Then

$$r(P' * Q) = k \leq \dim(Q(1 \boxdot R_i')P).$$

□

If P, Q are two minimal projections in $\mathcal{S}_2$, then P, Q correspond to two vertices in the principal graph, and $\dim(Q\mathcal{S}_3 P)$ is the number of length-2 paths between the two vertices. If $\dim(\mathcal{S}_3)$ is small, then for most depth-2 vertices in the principal graph, the number of length-2 paths between them is small. So the rank of their coproduct is small.

Definition 4.6. Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes. Let us define the rank of $\mathcal{S}_2$ to be the maximal number of length-2 paths between a pair of depth-2 vertices in the principal graph, for all pairs of depth-2 vertices.

It was first shown in [36] that a trace-one 2-box minimal projection induces a normalizer of the subfactor. Consequently we have the following result.

Proposition 4.7. *Suppose $\mathcal{S}$ is a subfactor planar algebra and the rank of $\mathcal{S}_2$ is 1; then it is a group subfactor planar algebra $\mathcal{S}^G$, for some group G. The converse statement is also true.*

The following three lemmas are basic facts about positive operators and planar algebras. They will be used frequently.

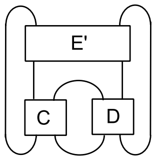
Lemma 4.8. *Suppose A, B, C, D are positive operators in $\mathcal{S}_2$. If $A \preceq C, B \preceq D$, then $A * B \preceq C * D$.*

Proof. Because $\mathcal{S}_2$ is finite dimensional, the spectrum of a positive operator is finite. If $A \preceq C, B \preceq D$, then $A < \lambda C$ and $B < \lambda D$, for some λ large enough. Thus $A * B < \lambda^2 C * D$ by Theorem 4.1. Then $A * B \preceq C * D$. □

Lemma 4.9. *Suppose C and D are two positive operators. If $tr(CD) = 0$, then $CD = 0$. Furthermore if E is a positive operator and $E \preceq C$, then $ED = 0$.*

Proof. If $tr(CD) = 0$, then $tr(C^{\frac{1}{2}} D C^{\frac{1}{2}}) = 0$. Thus $C^{\frac{1}{2}} D^{\frac{1}{2}} = 0$. Then $CD = 0$. Furthermore if $E \preceq C$, then $E < \lambda C$ for some $\lambda > 0$. Thus $0 \leq tr(ED) \leq \lambda tr(CD) = 0$. Thus $ED = 0$. □

Lemma 4.10. *If $C, D, E \in \mathcal{S}_2$, then*



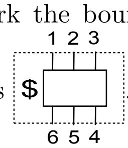
$$=tr((C*D)E') = tr((D*E)C') = tr((E*C)D')$$

$$=tr(E'(C*D)) = tr(C'(D*E)) = tr(D'(E*C))$$
$$=tr(C(E'*D')) = tr(D(C'*E')) = tr(E(D'*C'))$$
$$=tr((E'*D')C) = tr((C'*E')D) = tr((D'*C')E).$$

Recall that a' is the contragredient of a.

Proof. Follows directly from the isotopy and the spherical property of a planar algebra. □

Suppose x is a positive operator in $\mathcal{S}_2$ and $e \preceq x$. Then the support of x^{*k} is an increasing sequence bounded by id, when k approaches to infinity. We will prove that their union is a biprojection.

Before that, let us prove a lemma. For convenience, we mark the boundary points of a 3-box by $1, 2, 3, 4, 5, 6$ from the dollar sign clockwise as 

.

Lemma 4.11. *Suppose P, Q, R are projections in $\mathcal{S}_2$. Then the following are equivalent:*

(1) $P(1 \boxdot Q)R = 0$;
(2) $(P*Q)R = 0$;
(2′) $P(R*Q') = 0$;
(3) $tr((P*Q)R) = 0$.

Proof. (1) $\Longrightarrow$ (2) (resp. (2′)) follows from adding a cap on the 3-box $P(1 \boxdot Q)R$ which connects boundary points $4, 5$ (resp. $2, 3$).

(2),(2′) $\Longrightarrow$ (3) are obvious.

(3) $\Longrightarrow$ (1) follows from the positivity of the trace and

$$tr((P(1 \boxdot Q)R)^*(P(1 \boxdot Q)R)) = tr(P(1 \boxdot Q)R) = 0.$$

□

Theorem 4.12. *Suppose P is a projection in $\mathcal{S}_2$. If $P*P \preceq P$, equivalently $(P*P)P = P*P$, then P is a biprojection.*

Proof. Set $Q = id - P$. The condition $(P*P)P = P*P$ implies $(P*P)Q = 0$. By Lemma 4.11, we have $P(1 \boxdot P)Q = 0$. Then $P(1 \boxdot P) = P(1 \boxdot P)P$. Adding a cap which connects boundary points 2, 3, we obtain

$$\frac{tr(P)}{\delta}P = P(P*P'). \tag{4.1}$$

Computing the trace, we have $\dfrac{tr(P)^2}{\delta} = tr(P(P*P'))$. Note that $P*P'$ is positive by Theorem 4.1. By Hölder's inequality, we have

$$tr(P(P*P')) \leq \|P\| tr(P*P') = \frac{tr(P)^2}{\delta}.$$

The equality of the inequality implies $P * P' \preceq P$. By equation (4.1), we have $\frac{tr(P)}{\delta} P = P * P'$. Note that $(P * P')' = P * P'$, so $P = P'$ and $\frac{tr(P)}{\delta} P = P * P$. Then P is a biprojection. $\square$

In a group, some elements and their inverses will generate a subgroup under group multiplication. Thinking of the coproduct of minimal projections as the group multiplication of group elements, we can define the biprojection generated by an element in $\mathcal{S}_2$.

For an element $y \in \mathcal{S}_2$, take the positive operator $x = e + y^*y + yy^*$. Let P be the union of the support of x^{*k}, for $k = 1, 2, \cdots$. Then $P * P \preceq P$. So P is a biprojection. Moreover $PyP = y$. If Q is a biprojection such that $QyQ = y$, then $QxQ = Q$. So $x^{*k} \preceq Q$. Then $P \leq Q$. That means P is the smallest biprojection "containing" y.

Definition 4.13. Suppose y is an element in $\mathcal{S}_2$ and P is the smallest biprojection satisfying $PyP = y$. Then we call P the biprojection generated by y.

Lemma 4.14. *Suppose $A \in \mathcal{S}_2$ is a positive operator and P is the biprojection generated by A. Then $P \sim \sum_{i=1}^{k} A^{*i}$ for k large enough.*

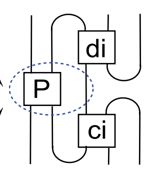
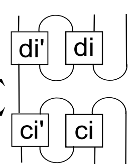
Proof. Let X_k be the support of $\sum_{i=1}^{k} A^{*i}$, for $k = 1, 2, \cdots$. By Theorem 4.1, A^{*i} is positive, so $X_k \leq X_{k+1}$. Since $\mathcal{S}_2$ is finite dimensional, there is an m such that $X_m = X_{m+1}$. Then $X_m * A \preceq X_m$. Thus $X_m * X_m \preceq X_m$. By Theorem 4.12, X_m is a biprojection. Moreover $A \preceq P$ implies $X_m \preceq P$. Thus $X_m = P$. Then $P \sim \sum_{i=1}^{k} A^{*i}$, whenever $k \geq m$. $\square$

Remark 4.15. Note that the generator of the group $\mathbb{Z}$ will not generate the whole group under group multiplication. We can only expect the above result for finite index subfactors.

Let $\iota : \mathcal{S}_2 \to \mathcal{S}_3$ be the inclusion by adding one string to the right, and let $\mathcal{I}_3$ be the basic construction ideal of $\mathcal{S}_3$. Then $\mathcal{S}_2 \cap \mathcal{I}_3 = \{x \in \mathcal{S}_2 | \iota(x) \in \mathcal{I}_3\}$ is a two sided ideal of $\mathcal{S}_2$. Thus the support of $\mathcal{S}_2 \cap \mathcal{I}_3$ is a central projection. A minimal projection of $\mathcal{S}_2$ belongs to $\mathcal{S}_2 \cap \mathcal{I}_3$ if and only if the corresponding vertex in the principal graph is not adjacent to a depth-3 vertex.

Theorem 4.16. *Let P be the support of $\mathcal{S}_2 \cap \mathcal{I}_3$. Then P is a central biprojection and $\mathcal{S}_P$ is depth-2.*

Proof. By definition $\iota(P) \in \mathcal{S}_3$, we have $\iota(P) = \sum_i c_i (1 \boxdot id) d_i$, for finitely many $c_i, d_i \in \mathcal{S}_2$. Then

$\iota(P * P) = \sum_i$  $= \sum_{i,i'}$  .

Thus $P * P \in \mathcal{S}_2 \cap \mathcal{I}_3$. Then $(P * P)P = P * P$. By Theorem 4.12, we have that P is a biprojection. So $P = P'$.

Let $\iota_P : (\mathcal{S}_P)_2 \to (\mathcal{S}_P)_3$ be the inclusion by adding one string to the right. Then for any $x \in (\mathcal{S}_P)_2$, we have $\iota_P(x)$ is a multiple of $P\iota(x)P$. Note that $\iota(x) \in \mathcal{I}_3$, so $\iota_P(x) = \sum_i Pf_i(1 \boxdot id)g_iP$, for finitely many $f_i, g_i \in \mathcal{S}_2$. Note that P is central, so

$$\iota_P(x) = \sum_i f_iP(1 \boxdot id)Pg_i = \sum_i f_iP(1 \boxdot P)Pg_i.$$

The last equation follows from the exchange relation of the biprojection P. Observe that $f_iP, Pg_i \in (\mathcal{S}_P)_2$ and $P(1\boxdot P)P$ is a multiple of the Jones projection in $(\mathcal{S}_P)_3$. Thus $\iota_P(x)$ is in the basic construction ideal of $(\mathcal{S}_P)_3$. Therefore $\mathcal{S}_P$ is depth-2. $\square$

Corollary 4.17. *If $A \in \mathcal{S}_2 \cap \mathcal{I}_3$, then $A' \in \mathcal{S}_2 \cap \mathcal{I}_3$.*

Proof. If $A \in \mathcal{S}_2 \cap \mathcal{I}_3$, then $A = PA$. So $A' = A'P$. Then $A' \in \mathcal{S}_2 \cap \mathcal{I}_3$. $\square$

If $\mathcal{S}$ is the planar algebra of an irreducible subfactor $\mathcal{N} \subset \mathcal{M}$, then the support P of $\mathcal{S}_2 \cap \mathcal{I}_3$ as a central biprojection corresponds to an intermediate subfactor $\mathcal{P}$ of $\mathcal{N} \subset \mathcal{M}$. Note that the planar algebra of $\mathcal{N} \subset \mathcal{P}$ is $\mathcal{S}_P$. So $\mathcal{P} = \mathcal{N} \rtimes K$, where K is the dual of $(\mathcal{S}_P)_2$ as a Kac algebra.

For a positive operator A in $\mathcal{S}_2$, let P be the biprojection generated by A. We will show that $\frac{\delta}{tr(P)} 1 \boxdot P$ is the spectral projection $E_{\{\|1\boxdot(A+A')\|\}}(1 \boxdot (A + A'))$, where $\|\cdot\|$ is the operator norm. To prove this, we need some lemmas.

Lemma 4.18. *Suppose A is a positive operator in $\mathcal{S}_2$. Then*

$$\|1 \boxdot A\| = \frac{tr(A)}{\delta}.$$

Proof. Note that $(1 \boxdot A)e_2 = \frac{tr(A)}{\delta} e_2$, so $\|1 \boxdot A\| \geq \frac{tr(A)}{\delta}$. Recall that $(1 \boxdot A)^* = 1 \boxdot A'$, so $(1 \boxdot A)(1 \boxdot A')$ is positive. Then

$$\|(1 \boxdot A)(1 \boxdot A')\| = \lim_{k\to\infty} tr(((1 \boxdot A)(1 \boxdot A'))^k)^{\frac{1}{k}}.$$

By an isotopy, we have

$$\begin{aligned} tr(((1 \boxdot A)(1 \boxdot A'))^{k+1}) &= \delta tr((A * A')^{*k}(A * A')) \\ &\leq \delta\|A * A'\|tr((A * A')^{*k}) \\ &= \delta^2\|A * A'\|(\frac{tr(A)}{\delta})^{2k}. \end{aligned}$$

Thus $\|(1\boxdot A)(1\boxdot A')\| \leq (\frac{tr(A)}{\delta})^2$. Then $\|(1\boxdot A)\| \leq \frac{tr(A)}{\delta}$. Therefore $\|(1\boxdot A)\| = \frac{tr(A)}{\delta}$. $\square$

Lemma 4.19. *Suppose A is a positive operator in $\mathcal{S}_2$ and $1 \boxdot Q$ is a minimal projection in $\mathcal{S}_{1,3}$. If $Q * A * Q = \frac{tr(A)}{\delta} Q$, then for any self-adjoint operator $B \in \mathcal{S}_2$, $B \preceq A$, we have $Q * B * Q = \frac{tr(B)}{\delta} Q$.*

Proof. For a positive operator $C < A$, by Lemma 4.18, we have

$$\|1 \boxdot C\| = \frac{tr(C)}{\delta}; \quad \|1 \boxdot (A - C)\| = \frac{tr(A - C)}{\delta}.$$

By assumption $1 \boxdot Q$ is minimal; thus

$$\begin{aligned}(1\boxdot Q)(1\boxdot C)(1\boxdot Q) &= \lambda 1\boxdot Q, \quad \text{for some } |\lambda| \leq \frac{tr(C)}{\delta};\\ (1\boxdot Q)(1\boxdot (A-C))(1\boxdot Q) &= \mu 1\boxdot Q, \quad \text{for some } |\mu| \leq \frac{tr(A-C)}{\delta}.\end{aligned}$$

If $Q * A * Q = \frac{tr(A)}{\delta}Q$, then $(1\boxdot Q)(1\boxdot A)(1\boxdot Q) = \frac{tr(A)}{\delta}1\boxdot Q$. Therefore $\frac{tr(A)}{\delta} = \lambda+\mu$. Then $\lambda = \frac{tr(C)}{\delta}$ and $\mu = \frac{tr(A-C)}{\delta}$. Thus $Q*C*Q = \frac{tr(C)}{\delta}Q$. In general, if B is a self-adjoint operator and $B \preceq A$, then B is a linear sum of the C_i's, such that $0 < C_i < A$. So $Q * B * Q = \frac{tr(B)}{\delta}Q$. □

Lemma 4.20. *Suppose X is an operator and P is a projection acting on a finite dimensional Hilbert space. If X has an eigenvalue $\|X\|$ and $E_{\|X\|}(X)$ is the corresponding spectral projection, then $P \leq E_{\|X\|}(X)$ if and only if $PXP = \|X\|P$.*

Proof. If $P \leq E_{\|X\|}(X)$, then obviously $PXP = \|X\|P$.

If $PXP = \|X\|P$, then for any vector v in the Hilbert space such that $Pv = v$, we have $PXv = \|X\|v$. Then

$$\|X\|^2\|v\|_2^2 = \|PXv\|_2^2 \leq \|PXv\|_2^2 + \|(I-P)Xv\|_2^2 = \|Xv\|_2^2 \leq \|X\|^2\|v\|_2^2.$$

So $(I-P)Xv = 0$. Then $Xv = PXv = \|X\|v$. Therefore $P \leq E_{\|X\|}(X)$. □

Theorem 4.21. *Suppose A is a positive operator in $\mathcal{S}_2$, P is the biprojection generated by A and $1\boxdot Q$ is a minimal projection in $\mathcal{S}_{1,3}$. Then the following are equivalent:*

(1) $Q * P * Q = \frac{tr(P)}{\delta}Q$;
(2) $Q * A * Q = \frac{tr(A)}{\delta}Q$;
(3) $Q * A = A * Q = \frac{tr(A)}{\delta}Q$;
(4) $Q * (A + A') * Q = \frac{tr(A+A')}{\delta}Q$.

Consequently $\frac{\delta}{tr(P)}1\boxdot P$ is the spectral projection $E_{\{\|1\boxdot(A+A')\|\}}(1\boxdot(A+A'))$.

Proof. (1) $\Longrightarrow$ (4) $\Longrightarrow$ (2) follows from Lemma 4.19 and the fact $A \preceq A + A' \preceq P$.

(2) $\Longrightarrow$ (3): If $Q*A*Q = \frac{tr(A)}{\delta}Q$, then $(1\boxdot Q)(1\boxdot A)(1\boxdot Q) = \frac{tr(A)}{\delta}(1\boxdot Q)$. By Lemma 4.18, we have $\|1\boxdot A\| = \frac{tr(A)}{\delta}$. By Lemma 4.20, we have $1\boxdot Q \leq E_{\{\|1\boxdot A\|\}}(1\boxdot A)$. Thus $(1\boxdot Q)(1\boxdot A) = (1\boxdot A)(1\boxdot Q) = \frac{tr(A)}{\delta}(1\boxdot Q)$. Then $Q*A = A*Q = \frac{tr(A)}{\delta}Q$.

(3) $\Longrightarrow$ (1): By assumption $1 \boxdot Q$ is a projection, thus $Q * Q = Q$. If $Q * A = A * Q = \frac{tr(A)}{\delta} Q$, then $Q * (\sum_{i=1}^{k} A^{*i}) * Q = \delta^{-1} tr(\sum_{i=1}^{k} A^{*i}) Q$. By Lemma 4.14, we have $P \preceq \sum_{i=1}^{k} A^{*i}$, for k large enough. So $Q * P * Q = \frac{tr(P)}{\delta} Q$ by Lemma 4.19.

Note that $\frac{\delta}{tr(P)} 1 \boxdot P$ is a projection. Moreover

$$\begin{aligned} 1 \boxdot Q \leq \frac{\delta}{tr(P)} 1 \boxdot P \quad &\Longleftrightarrow \quad \frac{\delta}{tr(P)} (1 \boxdot Q)(1 \boxdot P)(1 \boxdot Q) = 1 \boxdot Q \\ &\Longleftrightarrow \quad Q * P * Q = \frac{tr(P)}{\delta} Q. \end{aligned}$$

By Lemma 4.20, we have

$$\begin{aligned} & 1 \boxdot Q \leq E_{\{\|1 \boxdot (A+A')\|\}} (1 \boxdot (A + A')) \\ \Longleftrightarrow \ & (1 \boxdot Q)(1 \boxdot (A + A'))(1 \boxdot Q) = \frac{tr(A + A')}{\delta} 1 \boxdot Q \\ \Longleftrightarrow \ & Q * (A + A') * Q = \frac{tr(A + A')}{\delta} Q. \end{aligned}$$

Thus (1) $\Longleftrightarrow$ (4) implies

$$1 \boxdot Q \leq \frac{\delta}{tr(P)} 1 \boxdot P \quad \Longleftrightarrow \quad 1 \boxdot Q \leq E_{\{\|1 \boxdot (A+A')\|\}} (1 \boxdot (A + A')).$$

So $\frac{\delta}{tr(P)} 1 \boxdot P = E_{\{\|1 \boxdot (A+A')\|\}} (1 \boxdot (A + A'))$. □

Proposition 4.22. *For a finite group G, take $\mathcal{S} = \mathcal{S}^G$ to be a group subfactor planar algebra and A to be the (minimal) central projection corresponding to an (irreducible) representation V of G. Then the following are equivalent:*

(1) *the representation V is faithful;*
(2) $E_{\|1 \boxdot A\|} (1 \boxdot A) = e_2$;
(2′) $E_{\{\|1 \boxdot (A+A')\|\}} (1 \boxdot (A + A')) = e_2$;
(3) *the biprojection generated by A is id;*
(4) *every irreducible representation of G is contained in some tensor power of V.*

Proof. (1) $\Longleftrightarrow$ (2) and (3) $\Longleftrightarrow$ (4) are analogies between subfactors and representation theory. (2) $\Longleftrightarrow$ (2′) follows from Lemmas 4.18 and 4.20. (2′) $\Longleftrightarrow$ (3) follows from Theorem 4.21. □

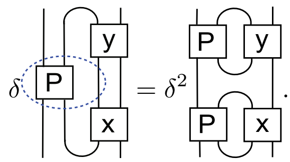
Remark 4.23. Note that (1) $\Longleftrightarrow$ (4) is a well known result in representation theory [13]. Observe that V is faithful is equivalent to the characteristic function of V separates the identity and other group elements. The characteristic function of the regular representation of the group is a multiple of the Dirac function of the identity. Moreover, the characteristic function of the contragradient of V is the complex conjugate of the characteristic function of V. Therefore if the condition of (4) is replaced by the tensor power of V and its contragredient, then (1) $\Longrightarrow$ (4) reduces to the Stone-Weierstrass theorem and the Peter-Weyl theorem. (4) $\Longrightarrow$ (1) is because the regular representation is faithful.

Recall that a trace-one minimal projection in $\mathcal{S}_2$ induces a normalizer [36]. Consequently we have the following proposition.

Proposition 4.24. *Suppose $P \in \mathcal{S}_2$ is a minimal projection and $tr(P) = 1$. Then $\delta P * (\cdot)$ is a *-isomorphism of $\mathcal{S}_2$ as a C^*-algebra. Consequently if $Q \in \mathcal{S}_2$ is a minimal projection, then $\delta P * Q$ is a minimal projection.*

We give a direct proof here.

Proof. Suppose $P, x, y \in \mathcal{S}_2$ and P is a trace-one minimal projection. By Wenzl's formula, we have



Therefore $\delta P * (\cdot)$ is a *-homomorphism of $\mathcal{S}_2$ as a C^*-algebra. Note that $\delta e * (\cdot)$ is the identity map and $\delta P' * \delta P = \delta e$. So $\delta P' * (\cdot)$ is the inverse of $\delta P * (\cdot)$. Thus $\delta P * (\cdot)$ is a *-isomorphism. Consequently if $Q \in \mathcal{S}_2$ is a minimal projection, then $\delta P * Q$ is a minimal projection. ☐

Definition 4.25. If P is a central minimal projection in $\mathcal{S}_2$, such that $tr(P) > 1$ and $r(P*Q) = 1$ (resp. $r(Q*P) = 1$), for any minimal projection Q in $\mathcal{S}_2$, $Q \neq P'$, then we call P a left (resp. right) virtual normalizer. If P is a left and right virtual normalizer, then we call it a virtual normalizer.

Obviously P is a left virtual normalizer if and only if P' is a right virtual normalizer. If the coproduct is commutative, then a left virtual normalizer is a virtual normalizer.

Theorem 4.26. *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by $\mathcal{S}_2$. If $\mathcal{S}_2$ contains a left (or right) virtual normalizer, then either $\mathcal{S}$ is Temperley-Lieb or $\mathcal{S}$ is separated by a non-trivial biprojection as a free product.*

Proof. When $\dim(\mathcal{S}_2) = 2$, we have that $\mathcal{S}$ is Temperley-Lieb.

When $\dim(\mathcal{S}_2) \geq 3$, we assume that P is a left virtual normalizer, so P' is central. By Lemma 4.10, we have $tr(e(P' * P)) = tr((P * e)P) = \dfrac{tr(P)}{\delta}$. So $e \preceq P' * P$, and the coefficient of e in $P' * P$ is $\dfrac{tr(P)}{\delta}$.

Case 1. If $P' * P \preceq e + P'$, then $P' * P \sim e + P'$; otherwise $P' * P \sim e$ implies $tr(P) = 1$, which contradicts the assumption $tr(P) > 1$. Note that $P' * P$ is self-contragredient, so is $e + P'$. Then $P = P'$ and $e + P$ is a biprojection by Theorem 4.12. Computing the trace, we have $P*P = \dfrac{tr(P)}{\delta}e + \dfrac{tr(P)-1}{\delta}P$. For any minimal projection Q orthogonal to $(e + P)$, we have $r(P * Q) = 1$ since P is a left virtual normalizer. Moreover $P * Q \nsim P$, because $tr((P * Q)P) = tr(Q(P * P)) = 0$ by Lemma 4.10. Then $r(P * (P * Q)) = 1$ since P is a left virtual normalizer. Note that

$$P * (P * Q) = (P * P) * Q = \frac{tr(P)}{\delta} e * Q + \frac{tr(P) - 1}{\delta} P * Q;$$

thus $P * Q \sim Q$. Then $(e + P) * Q \sim Q$. Recall that $P = P'$, so $Q' * (e + P) \sim Q'$. By Theorem 2.20, $\mathcal{S}$ is separated by $e + P$ as a free product.

Case 2. Otherwise $(id - e - P')(P' * P) \neq 0$, since e and P' are central. Moreover $(id - e - P')(P' * P)$ is positive by Theorem 4.1. Let S be the support of $(id - P')(P' * P)$; we have $S < id$. Recall that $e \preceq P' * P$ and $P' \neq e$, so $e < S$. We will show that S is a central biprojection which separates B_2 as a free product.

Note that $\mathcal{S}_2$ is a direct sum of matrix algebras. For each $M_n(\mathbb{C})$ summand of $\mathcal{P}_2$, the space $\mathbb{C}P^n$ of minimal projections is path connected in the norm topology. To show the projection S is central, it is enough to show that $SQ_1 = 0 \iff SQ_2 = 0$ when two minimal projections Q_1 and Q_2 are close enough. For a minimal projection Q, $Q \neq P'$, we have

$$\begin{aligned}
& tr((P' * P)Q) > 0 & \\
\iff & tr((P * Q)P) > 0 & \text{by Lemma 4.10} \\
\iff & P * Q = \frac{tr(Q)}{\delta} P & \text{since } P \text{ is a left virtual normailizer} \\
\iff & tr((P * Q)P) = \frac{tr(P)tr(Q)}{\delta} & \\
\iff & tr((P' * P)Q) = \frac{tr(P)tr(Q)}{\delta} & \text{by Lemma 4.10.}
\end{aligned}$$

If Q_1, Q_2 are two minimal projections such that $\|Q_1 - Q_2\| < \dfrac{tr(P)tr(Q)}{\delta \|P' * P\|}$ and $Q_i \neq P'$, for $i = 1, 2$, then

$$\begin{aligned}
tr((P' * P)Q_1) > 0 \quad & \Longrightarrow \quad tr((P' * P)Q_1) = \frac{tr(P)tr(Q_1)}{\delta} \\
& \Longrightarrow \quad tr((P' * P)Q_2) > 0.
\end{aligned}$$

By Theorem 4.1, we have $tr((P' * P)Q_i) \geq 0$, for $i = 1, 2$. Thus

$$tr((P' * P)Q_2) = 0 \Longrightarrow tr((P' * P)Q_1) = 0.$$

Combining with Lemma 4.9, we have

$$\begin{aligned}
SQ_2 = 0 \quad & \Longrightarrow \quad (S + P')Q_2 = 0 \\
& \Longrightarrow \quad (P' * P)Q_2 = 0 \\
& \Longrightarrow \quad tr((P' * P)Q_2) = 0 \\
& \Longrightarrow \quad tr((P' * P)Q_1) = 0 \\
& \Longrightarrow \quad SQ_1 = 0.
\end{aligned}$$

Thus S is central.

Recall that P' is central and S is the support of $(id - P')(P' * P)$, so for any minimal projection R, $R \leq S$, we have $R \neq P'$ and $R \preceq P' * P$. By Lemma 4.10, we have

$$tr((P * R)P) = tr(R(P' * P)) > 0.$$

Recall that P is a left virtual normalizer, so $P * R \sim P$. Then $P * S \sim P$. Therefore

$$P * (S * S) = (P * S) * S \sim P.$$

Then for any minimal projection U, $U \preceq S * S$, we have $P * U \sim P$ by Lemma 4.8. By Lemma 4.10, we have

$$tr(U(P' * P)) = tr((P * U)P) > 0.$$

If $U = P'$, then $e \preceq P * P' = P * U \sim P$. This is a contradiction. Otherwise $U \neq P'$; then $UP' = 0$ since P' is central. Then

$$tr(U(id - P')(P' * P)) = tr(U(P' * P)) > 0.$$

Note that $S \sim (id - P')(P' * P)$; by Lemma 4.9, we have $tr(US) > 0$. Recall that S is central, so $U \leq S$. Thus $S * S \preceq S$. By Theorem 4.12, S is a biprojection.

To show $\mathcal{S}$ is separated by S as a free product, we need the following lemma.

Lemma 4.27. *For any $y \in \mathcal{P}_2$, $yS = 0$, we have $S * y = y$.*

For any element $x \in \mathcal{P}_2$, we have $x = x_1 + x_2$ such that $x_1 S = x_1$, $x_2 S = 0$. By Lemma 4.27, $S * x_2 = x_2$. Note that S is central and $S = S'$, so $x_2' S = 0$. Then $S * x_2' = x_2'$ by Lemma 4.27. So $x_2 * S = x_2$. Then $S x_1 S = x_1$ and $S * x_2 * S = x_2$. By Theorem 2.20, $\mathcal{S}_2$ is separated by S as a free product.

To prove Lemma 4.27, we need the following Lemma.

Lemma 4.28. *Suppose R_1,$R_2 \in \mathcal{S}_2$ are two minimal projections orthogonal to $S + P'$. If $P * R_1 \sim P * R_2$, then $R_1 = R_2$.*

Proof of Lemma 4.27. Recall that $P * S \sim P$, so $S * P' \sim P'$. Suppose R_1 is a minimal projection orthogonal to the central projection $S + P'$. By Lemma 4.10, we have

$$tr((S * R_1)(S + P')) = tr(R_1(S * (S + P')) = 0.$$

Thus $S * R_1$ is orthogonal to $S + P'$. Moreover

$$P * (S * R_1) = (P * S) * R_1 \sim P * R_1.$$

Suppose R_2 is a minimal projection such that $R_2 \preceq S * R_1$. Then R_2 is orthogonal to $S + P'$ and

$$P * R_2 \preceq P * (S * R_1) \sim P * R_1.$$

Recall that P is a left virtual normalizer, so $P * R_1 \sim P * R_2$. By Lemma 4.28, we have $R_2 = R_1$. So $S * R_1 \sim R_1$. Note that for any $y \in \mathcal{S}_2$ with $yS = 0$, we have y is a linear sum of such R_1's and P'. So $S * y = y$. □

Finally let us prove Lemma 4.28.

Proof of Lemma 4.28. Suppose R_1,$R_2 \in \mathcal{S}_2$ are two minimal projections orthogonal to $S + P'$. If $P * R_1 \sim P * R_2$, then $P^{*n} * R_1 \sim P^{*n} * R_2$, for $n = 1, 2, \cdots$. Recall that P is a left virtual normalizer. If $P^{*n} * R_1 \nsim P'$ and $r(P^{*n} * R_1) = 1$, then $r(P^{*(n+1)} * R_1) = 1$.

Case 1. If $P^{*n} * R_1 \nsim P'$, $\forall\, n > 0$, then $r(P^{*n} * R_1) = 1$, $\forall\, n > 0$. By Lemma 4.14, $e \preceq P^{*m}$ for some $m > 0$; thus

$$R_1 \sim P^{*m} * R_1 \sim P^{*m} * R_2 \sim R_2.$$

Then $R_1 = R_2$.

Case 2. If $P^{*n} * R_1 \sim P'$, for some $n > 0$, assuming this n is the minimal one, then $r(P^{*j} * R_1) = 1$, $\forall\, 1 \leq j \leq n$.

Subcase 2.1. If $P' \preceq P^{*k}$, for some $1 \leq k \leq n - 1$, then $e \preceq P * P' \preceq P^{*(k+1)}$ by Lemma 4.8. Thus

$$R_1 \sim P^{*(k+1)} * R_1 \sim P^{*(k+1)} * R_2 \sim R_2.$$

Then $R_1 = R_2$.

Subcase 2.2. Otherwise $P' \nsim P^{*j}$ and $r(P^{*(j+1)}) = 1, \forall\, 1 \leq j \leq n-1$. If $P' \sim P^{*n}$, then $P * P' = P' * P$ and $P' * R_1 \sim P^{*n} * R_1 \sim P'$. By Lemma 4.10, we have

$$tr(R_1(P' * P)) = tr(R_1(P * P')) = tr((P' * R_1)P') > 0.$$

On the other hand, we have $R_1(S + P') = 0$ and $P' * P \preceq S + P'$, so $R_1(P' * P) = 0$ by Lemma 4.9. This is a contradiction.

Subcase 2.3. Otherwise $P' \nsim P^{*j}$ and $r(P^{*(j+1)}) = 1, \forall\, 1 \leq j \leq n$. First we will show that $(P')^{*l}$ is central by induction, for $1 \leq l \leq n+1$. The virtually normalizer P is central, so P' is central. For $1 \leq l \leq n$, suppose $(P')^{*l}$ is central. Take a minimal projection V such that $tr(((P')^{*(l+1)})V) > 0$. By Lemma 4.10, we have

$$tr((P')^{*l}(P * V)) = tr(((P')^{*(l+1)})V) > 0.$$

If $V = P'$, then $(P')^{*(l+1)} = P'$, and it is central. If $V \neq P'$, then $r(P * V) = 1$ since P is a left virtual normalizer. Recall that $r((P')^{*l}) = 1$ and $(P')^{*l}$ is central by induction, so $P * V \sim (P')^{*l}$. Computing the trace, we have $P * V = \dfrac{tr(P)tr(V)}{\delta tr((P')^{*l})}(P')^{*l}$. So

$$tr(((P')^{*(l+1)})V) = tr((P')^{*l}(P * V)) = C(P)tr(V),$$

for a positive constant $C(P)$ only depending on P. Now we have

$$tr(((P')^{*(l+1)})V) > 0 \Longrightarrow tr(((P')^{*(l+1)})V) = C(P)tr(V).$$

Note that the space of minimal projections in the central support of V is path connected in norm topology. When the minimal projection V moves continuously, the assumption $tr(((P')^{*(l+1)})V) > 0$ always holds. Recall that $r((P')^{*(l+1)}) = 1$, so $(P')^{*(l+1)}$ is central. By induction, $(P')^{*(n+1)}$ is central. Recall that $P^{*n} * R_1 \sim P'$; by Lemma 4.10, we have

$$tr(R_1((P')^{*(n+1)})) = tr((P^{*n} * R_1)P') > 0.$$

Recall that $r((P')^{*(n+1)}) = 1$ and $(P')^{*(n+1)}$ is central, so $R_1 \sim (P')^{*(n+1)}$. Similarly $R_2 \sim (P')^{*(n+1)}$. So $R_1 \sim R_2$. Then $R_1 = R_2$. □

□

Remark 4.29. By Corollary 2.21, $\mathcal{S}^S$ is generated by 2-boxes. In $\mathcal{S}^S$, either $tr(P) = 1$ or P is a virtual normalizer. In the latter case, $e + P$ is a biprojection and the planar algebra $\mathcal{S}^S$ is separated by $e + P$ as a free product. Furthermore $(\mathcal{S}^S)_{e+P}$ is Temperley-Lieb, and P is the second Jones-Wenzl projection. After decomposing $\mathcal{S}$ as a free product, a virtual normalizer becomes either a trace-one projection or the second Jones-Wenzl projection of a Temperley-Lieb component in the free product. We leave the details to the reader.

Corollary 4.30. *Suppose $\mathcal{S}$ is a subfactor planar algebra with* $\dim(\mathcal{S}_2) \geq 3$ *which contains a left (or right) virtual normalizer. Then $\mathcal{S}_2$ contains a non-trivial biprojection corresponding to an intermediate subfactor.*

Proof. Let $\mathcal{P} \subseteq \mathcal{S}$ be the subfactor planar algebra generated by $\mathcal{S}_2$. Then $\mathcal{S}_2 = \mathcal{P}_2$. By Theorem 4.26, $\mathcal{P}_2$ contains a non-trivial biprojection. □

5. Constructions and decompositions

5.1. **Exchange relation planar algebras.** In general, it is not easy to show that a subfactor planar algebra is an exchange relation planar algebra. In this section we will give two general constructions of exchange relation planar algebras by the free product and the tensor product. Moreover we will show that the subgroup subfactor planar algebra $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_p \rtimes \mathbb{Z}_2}$ is an exchange relation planar algebra, for an odd prime number p. For the classification, we will show how an exchange relation planar algebra decomposes as a free product or a tensor product.

Proposition 5.1. *If $\mathcal{S}$ is an exchange relation planar algebra, then its dual is an exchange relation planar algebra.*

Proof. Recall that the dual of a subfactor planar algebra is given by switching its shading. Thus the dual of an exchange relation planar algebra is still generated by 2-boxes, and its exchange relation is given by the 180° rotation of the adjoint of the original exchange relation. □

Proposition 5.2. *If $\mathcal{A} * \mathcal{B}$ is an exchange relation planar algebra, then both $\mathcal{A}$ and $\mathcal{B}$ are exchange relation planar algebras.*

Proof. By Theorem 2.20, both $\mathcal{A}$ and $\mathcal{B}$ are generated by 2-boxes. Suppose $Q = id \otimes e$, the central biprojection separating $\mathcal{A} * \mathcal{B}$ as a free product. Then $\mathcal{A}$ is isomorphic to $(\mathcal{A} * \mathcal{B})_Q$. For any $x, y \preceq Q$, we have

$$(1 \boxdot x)y = \sum_i c_i(1 \boxdot d_i) + f_i(1 \boxdot id)g_i,$$

for finitely many two boxes c_i, d_i, f_i, g_i. Then

$$Q(1 \boxdot Qx)yQ = \sum_i Qc_i(1 \boxdot Qd_i)Q + Qf_i(1 \boxdot Q)g_iQ.$$

Note that $Qc_iQ, d_iQ, Qf_i, g_iQ \preceq Q$, so that is the exchange relation of $(\mathcal{A} * \mathcal{B})_Q$. Thus $\mathcal{A}$ is an exchange relation planar algebra. Considering the duality of exchange relation planar algebras, we have that $\mathcal{B}$ is an exchange relation planar algebra. □

Proposition 5.3. *If $\mathcal{A}$ and $\mathcal{B}$ are exchange relation planar algebras, then $\mathcal{A} * \mathcal{B}$ is an exchange relation planar algebra.*

Proof. By Corollary 2.19, $\mathcal{A} * \mathcal{B}$ is generated by 2-boxes. Suppose $Q = id \otimes e$ is the central biprojection which separates $\mathcal{A} * \mathcal{B}$ as a free product. Then any 2-box in $\mathcal{A} * \mathcal{B}$ is of the form $x \otimes e + id \otimes y$, for some $x \in \mathcal{A}_2$, $y \in \mathcal{B}_2$. We need to check the exchange relation for four cases. For any $x_1, x_2 \in \mathcal{A}_2$, $y_1, y_2 \in \mathcal{B}_2$:

(1) the exchange relation of $(1 \boxdot (x_1 \otimes e))(x_2 \otimes e)$ follows from the exchange relation of $\mathcal{A}$;
(2) the exchange relation of $(1 \boxdot (id \otimes y_1))(id \otimes y_2)$ follows from the exchange relation of $\mathcal{B}$;
(3) $(1 \boxdot (x_1 \otimes e))(id \otimes y_1) = (id \otimes y_1)(1 \boxdot (x_1 \otimes e))$;
(4) $(1 \boxdot (id \otimes y_1))(x_1 \otimes e) = (id \otimes y_1')(1 \boxdot id)(x_1 \otimes e)$.

Therefore $\mathcal{A} * \mathcal{B}$ is an exchange relation planar algebra. □

Proposition 5.4. *If $\mathcal{A} \otimes \mathcal{B}$ is an exchange relation planar algebra, then both $\mathcal{A}$ and $\mathcal{B}$ are exchange relation planar algebras.*

Proof. By Corollary 3.16, both $\mathcal{A}$ and $\mathcal{B}$ are generated by 2-boxes. The rest is the same as the proof of Proposition 5.2. □

Its converse is not true. The tensor product of two Temperley-Lieb subfactor planar algebras may not be an exchange relation planar algebra. It is a corollary of Theorem 6.5. However a weak version is true.

Proposition 5.5. *If $\mathcal{A}$ is a depth-2 subfactor planar algebra and $\mathcal{B}$ is an exchange relation planar algebra, then $\mathcal{A}\otimes\mathcal{B}$ is an exchange relation planar algebra.*

Proof. By Corollary 3.2, $\mathcal{A}\otimes\mathcal{B}$ is generated by 2-boxes. Take $A = id\otimes e$ and $B = e\otimes id$. Note that any 2-box in $\mathcal{S}_A\otimes\mathcal{S}_B$ is a finite sum of $x*y$'s for which $x\preceq A, y\preceq B$, so we only need to check the exchange relation for $(1\otimes(x_1*y_1))(x_2*y_2)$, for any $x_1,x_2\preceq A, y_1,y_2\preceq B$. Since $\mathcal{B}$ is an exchange relation planar algebra, and $(\mathcal{A}\otimes\mathcal{B})_B$ is isomorphic to $\mathcal{B}$, we have

$$B(1\boxdot By_1)y_2B=\sum_i Bc_i(1\boxdot Bd_i)B+Bf_i(1\boxdot B)g_iB,$$

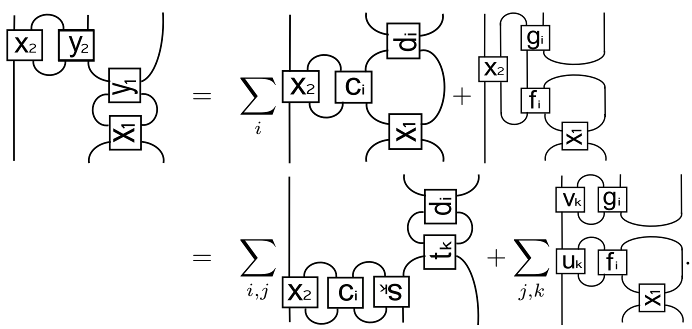
for finitely many $c_i,d_i,f_i,g_i\preceq B$. Then

$$(1\boxdot y_1)y_2=\sum_i c_i(1\boxdot d_i)+f_i(1\boxdot id)g_i,$$

by the exchange relation of the biprojection B. On the other hand,

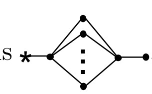
$$x_1=\sum_j s_j(1\boxdot id)t_j;\quad x_2=\sum_k u_k(1\otimes id)v_k,$$

for finitely many $s_j,t_j,u_k,v_k\preceq A$, because $\mathcal{A}$ is a depth-2 subfactor planar algebra. Then



Thus $\mathcal{S}_A\otimes\mathcal{S}_B$ is an exchange relation planar algebra. □

Theorem 5.6. *For an odd prime number p, the subgroup subfactor planar algebra $\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2}$ is an exchange relation planar algebra, where $\mathbb{Z}_p\rtimes\mathbb{Z}_2=\{a,t|a^p=1,t^2=1,tat=a^{-1}\}$.*

Proof. Note that the principal graph of $\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2}$ is , with $\frac{p-1}{2}$ depth-2 vertices. So

$$\dim((\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2})_2)=\frac{p+1}{2};\quad \dim((\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2})_3)=(\frac{p+1}{2})^2+(\frac{p-1}{2})^2.$$

Considering $\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2}$ as a biprojection cutdown of $\mathcal{S}^{\mathbb{Z}_p\rtimes\mathbb{Z}_2}$, it is easy to check that the minimal projections $e, g_1, \cdots, g_{\frac{p-1}{2}}$ of $(\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2})_2$ satisfy the relation

$$\delta g_m * g_n = g_{m+n} + g_{m-n}, \ \forall\ 0 \leq m, n \leq \frac{p-1}{2},$$

where $g_0 = 2e$, $g_{m+n} = g_{p-m-n}$ when $m+n > \frac{p-1}{2}$, and $g_{m-n} = g_{n-m}$ when $m-n < 0$. Note that $(\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2})_{1,3}$ is isomorphic to $(\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2})_{2,-}$ as an algebra. Take

$$\chi_k = \frac{\delta}{p}(1 \boxdot 2e + \sum_{m=1}^{\frac{p-1}{2}} (\omega^{mk} + \omega^{-mk})(1 \boxdot g_m)),$$

where $\omega = e^{\frac{2\pi i}{p}}$, for $k = 0, 1, \cdots, \frac{p-1}{2}$. Then $\chi_0 = 2e_2$ and $\{e_2\} \cup \{\chi_k\}_{k=1}^{\frac{p-1}{2}}$ is the set of minimal projections of $(\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2})_{1,3}$.

Suppose $\mathcal{S}$ is the planar subalgebra of $\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2}$ generated by 2-boxes, and $\mathcal{I}_3$ is the basic construction ideal of $\mathcal{S}_3$. Then $\mathcal{S}_3 = \mathcal{I}_3 \oplus \mathcal{S}_3/\mathcal{I}_3$. Let us define s_3 to be the support of $\mathcal{S}_3/\mathcal{I}_3$; we have Wenzl's formula for g_m,

$$s_3 g_m = g_m - \frac{\delta}{tr(g_m)} g_m (1 \boxdot id) g_m.$$

By a direct computation, we have $tr(s_3 g_m \chi_k) \neq 0$, for $1 \leq m, k \leq \frac{p-1}{2}$, and $\{s_3 g_m \chi_k\}_{1 \leq m,k \leq \frac{p-1}{2}}$ is a set of pairwise orthogonal vectors. By Proposition 2.24, we have

$$\dim(\mathcal{S}_3/\mathcal{I}_3) \leq (\dim(\mathcal{S}_2) - 1)^2 = (\frac{p-1}{2})^2.$$

Thus $\{s_3 g_m \chi_k\}_{1 \leq m,k \leq \frac{p-1}{2}}$ forms a basis of $\mathcal{S}_3/\mathcal{I}_3$, and $\dim(\mathcal{S}_3/\mathcal{I}_3) = (\frac{p-1}{2})^2$. That means $\mathcal{S}$ is an exchange relation planar algebra. Moreover, $\mathcal{S}_3 = \mathcal{S}_3^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2}$. So they have the same principal graph up to depth-3. Then their principal graphs have to be the same by the restriction of the index. So $\mathcal{S} = \mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_p\rtimes\mathbb{Z}_2}$. □

5.2. **Commute relation planar algebras.** To classify subfactor planar algebras generated by 2-boxes subject to the condition that the quotient of 3-boxes by the basic construction ideal is abelian, let us prove two lemmas for the construction and the decomposition of such planar algebras via the free product. For convenience we use following notation.

Notation 5.7. A subfactor planar algebra $\mathcal{S}$ is called a commute relation planar algebra if it is generated by 2-boxes and $\mathcal{S}_3/\mathcal{I}_3$ is abelian. Moreover it is of type AN if $\mathcal{S}_2$ is abelian, of type NA if $\mathcal{S}_{1,3}$ is abelian, and of type AA if both $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ are abelian.

A Temperley-Lieb subfactor planar algebra is a commute relation planar algebra of type AA. A depth-2 subfactor planar algebra is a commute relation planar algebra. Furthermore it is of type AN if and only if it is $\mathcal{S}^G$, for a group G; it is of type NA if and only if it is the dual of $\mathcal{S}^G$, for a group G; and it is of type AA if and only if it is $\mathcal{S}^G$, for an abelian group G.

Proposition 5.8. *Suppose $\mathcal{S}$ is a commute relation planar algebra. Then it is an exchange relation planar algebra. Consequently $\mathcal{S}_3$ is generated by $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ as an algebra.*

Proof. If $\mathcal{S}_3/\mathcal{I}_3$ is abelian, then for any $a, b \in \mathcal{S}_2$, we have $(1 \boxdot a)b - b(1 \boxdot a) \in \mathcal{I}_3$. Thus

$$(1 \boxdot a)b = b(1 \boxdot a) + \sum_i f_i(1 \boxdot id)g_i,$$

for finitely many $f_i, g_i \in \mathcal{S}_2$. □

Lemma 5.9. *Suppose $\mathcal{A}, \mathcal{B}$ are commute relation planar algebras of type NA and AN respectively. Then $\mathcal{S} = \mathcal{A} * \mathcal{B}$ is a commute relation planar algebra. Furthermore if $\mathcal{A}$ is of type AA, then $\mathcal{S}$ is of type AN, and if $\mathcal{B}$ is of type AA, then $\mathcal{S}$ is of type NA.*

Proof. Suppose $\mathcal{A}, \mathcal{B}$ are commute relation planar algebras of types NA and AN respectively. Then by Propositions 5.3 and 5.8, we have that $\mathcal{S} = \mathcal{A} * \mathcal{B}$ is an exchange relation planar algebra. Therefore $\mathcal{S}$ is generated by $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ as an algebra. To prove $\mathcal{S}_3/\mathcal{I}_3$ is abelian, it is enough to prove it for the generating sets $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$. That is, for any $x, y \in \mathcal{S}_2$, we need to show:

(1) $(1 \boxdot x)y - y(1 \boxdot x) \in \mathcal{I}_3$;
(2) $xy - yx \in \mathcal{I}_3$;
(3) $(1 \boxdot x)(1 \boxdot y) - (1 \boxdot y)(1 \boxdot x) \in \mathcal{I}_3$.

Note that $x = a \otimes e + id \otimes b$, $y = c \otimes e + id \otimes d$, for some $a, c \in \mathcal{A}_2$ and $b, d \in \mathcal{B}_2$. Each case splits into four subcases:

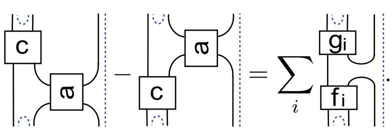
(.1) $x = a \otimes e$, $y = c \otimes e$;
(.2) $x = id \otimes b$, $y = id \otimes d$;
(.3) $x = a \otimes e$, $y = id \otimes d$;
(.4) $x = id \otimes b$, $y = c \otimes e$.

Now we use the A,B-colour diagrams to express elements in the free product. For convenience, we use dotted lines to express B strings. We omit the labels of the boundary of a diagram, which should be ordered as $ABBA\ ABBA \cdots ABBA$.

(1.1) By assumption $\mathcal{A}$ is a commute relation planar algebra, so

$$(1 \boxdot a)c - c(1 \boxdot a) = \sum_i f_i(1 \boxdot id)g_i,$$

for finitely many $f_i, g_i \in \mathcal{A}_2$. Then



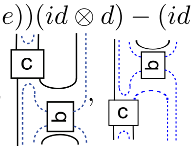
Thus

$$(1 \boxdot (a \otimes e))(c \otimes e) - (c \otimes e)(1 \boxdot (a \otimes e)) \in \mathcal{I}_3.$$

(1.2) By assumption $\mathcal{B}$ is a commute relation planar algebra. Similarly we have

$$(1 \boxdot (id \otimes b))(id \otimes d) - (id \otimes d)(1 \boxdot (id \otimes b)) \in \mathcal{I}_3.$$

(1.3) $(1 \boxdot (a \otimes e))(id \otimes d) - (id \otimes d)(1 \boxdot (a \otimes e)) = 0.$

(1.4) Note that 

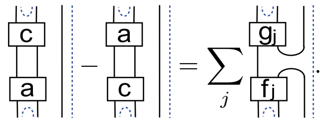
$\in \mathcal{I}_3$; thus

$$(1 \boxdot (id \otimes b))(c \otimes e) - (c \otimes e)(1 \boxdot (id \otimes b)) \in \mathcal{I}_3.$$

(2.1) By assumption $\mathcal{A}$ is a commute relation planar algebra, so

$$ac - ca = \sum_j f_j(1 \boxdot id)g_j,$$

for finitely many $f_j, g_j \in \mathcal{A}_2$. Then



Thus

$$(a \otimes e)(c \otimes e) - (c \otimes e)(a \otimes e) \in \mathcal{I}_3.$$

(2.2) By assumption $\mathcal{B}$ is of type AN, so $bd - db = 0$. Thus $(id \otimes b)(id \otimes d) - (id \otimes b)(id \otimes d) = 0$.

(2.3) $(a \otimes e)(id \otimes d) - (id \otimes d)(a \otimes e) = a \otimes ed - a \otimes de = 0$.

(2.4) $(id \otimes b)(c \otimes e) - (c \otimes e)(id \otimes b) = c \otimes be - c \otimes eb = 0$.

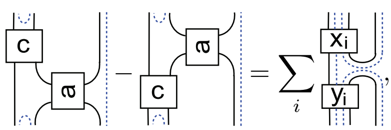
Considering the 180° rotation, the proof of (3) is the same as that of (2) while assuming $\mathcal{A}$ is of type NA.

Therefore $\mathcal{S}$ is a commute relation planar algebra.

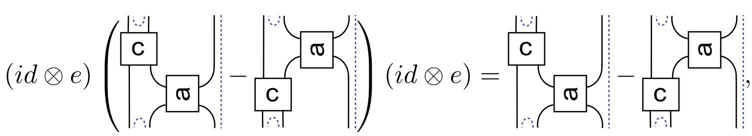
Furthermore if $\mathcal{A}$ is of type AA, then $\mathcal{A}_2$ is abelian. Recall that if $\mathcal{B}$ is of type AN, then $\mathcal{A}_2$ is abelian. So $\mathcal{A} * \mathcal{B}_2$ is abelian and $A * B$ is of type AN. If $\mathcal{B}$ is of type AA, similarly $A * B$ is of type NA by duality. □

Lemma 5.10. *Suppose $\mathcal{A}, \mathcal{B}$ are subfactor planar algebras with circle parameters greater than 1. If $\mathcal{S} = \mathcal{A} * \mathcal{B}$ is a commute relation planar algebra, then $\mathcal{A}, \mathcal{B}$ are commute relation planar algebras of type NA and AN respectively. Furthermore if $\mathcal{S}$ is of type AN, then $\mathcal{A}$ is of type AA, and if $\mathcal{S}$ is of type NA, then $\mathcal{B}$ is of type AA.*

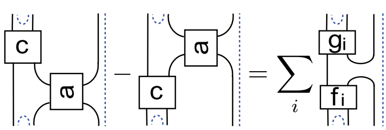
Proof. Because $\mathcal{S}_2$ is a commute relation planar algebra, for any a, c in $\mathcal{A}_2$, we have



for finitely many $x_i, y_i \in \mathcal{S}_2$. Moreover $id \otimes e$ is central and



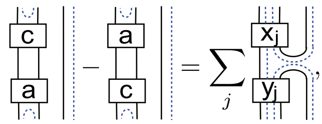
so we may assume that $x_i, y_i \preceq id \otimes e$. Then



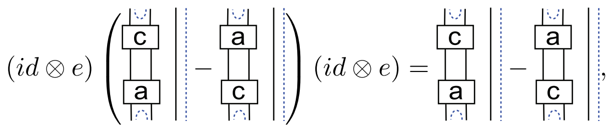
for finitely many $f_i, g_i \in \mathcal{A}_2$. Therefore

$$(1 \boxdot a)c - c(1 \boxdot a) = \sum_i f_i(1 \boxdot id)g_i.$$

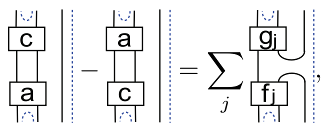
Similarly



for finitely many $x_j, y_j \in \mathcal{S}_2$. Moreover

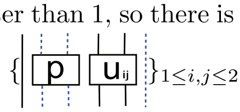


so we may assume that $x_j, y_j \preceq id \otimes e$. Then



for finitely many $f_j, g_j \in \mathcal{A}_2$. Therefore

$$ac - ca = \sum_j f_j (1 \boxdot id) g_j.$$

If $\mathcal{A}_{1,3}$ is not abelian, then there is a system of matrix units $\{u_{11}, u_{12}, u_{21}, u_{22}\}$ in $\mathcal{A}_{1,3}$. By assumption the index of $\mathcal{B}$ is greater than 1, so there is a projection $p \in \mathcal{B}_2$ orthogonal to the Jones projection. Then $\{$  $\}_{1 \leq i,j \leq 2}$ forms a system of matrix units in $\mathcal{S}_3$, and they are in the orthogonal complement of $\mathcal{I}_3$. So $\mathcal{S}_3/\mathcal{I}_3$ is not abelian. This is a contradiction.

Therefore $\mathcal{A}$ is a commute relation planar algebra of type NA. Furthermore if $\mathcal{S}$ is of type AN, then $\mathcal{S}_2$ is abelian. Note that $\mathcal{A}$ is isomorphic to $\mathcal{S}_{id \otimes e}$, so $\mathcal{A}_2$ is abelian. Then $\mathcal{A}$ is of type AA.

Considering the duality, we have that $\mathcal{B}$ is a commute relation planar algebra of type AN. Furthermore if $\mathcal{S}$ is of type NA, then $\mathcal{B}$ is of type AA. □

6. Classifications

Recall that the classification of a subfactor planar algebra generated by a non-trivial 2-box are given by: $\mathcal{S}^{\mathbb{Z}_3}$, $TL * TL$, for at most 12 dimensional 3-boxes [11]; $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_5 \rtimes \mathbb{Z}_2}$, for 13 dimensional 3-boxes [12]; BMW's, precisely one family from quantum $Sp(4, \mathbb{R})$ and one from quantum $O(3, \mathbb{R})$, for 14 dimensional 3-boxes [8]. Now let us prove the main classification results.

Lemma 6.1. *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes with $\dim(\mathcal{S}_2) = 4$. If $\mathcal{S}$ has a biprojection P and $r(P) = 3$, then $\mathcal{S}$ is a free product.*

Proof. If $\mathcal{S}$ has a biprojection P and $r(P) = 3$, then $\dim((\mathcal{S}_P)_2) = 3$. Note that $\dim((\mathcal{S}^P)_2) \geq 2$, so

$$\dim((\mathcal{S}_P * \mathcal{S}^P)_2) = \dim((\mathcal{S}_P)_2) + \dim((\mathcal{S}^P)_2) - 1 \geq 4.$$

By the assumption $\dim(\mathcal{S}_2) = 4$, we have $\mathcal{S}_2 = (\mathcal{S}_P * \mathcal{S}^P)_2$. Note that $\mathcal{S}$ is generated by 2-boxes, so $\mathcal{S} = \mathcal{S}_P * \mathcal{S}^P$. □

Theorem 6.2. *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes with* $\dim(\mathcal{S}_2) = 4$. *If $\mathcal{S}$ has two non-trivial biprojections, then $\mathcal{S}$ is either $TL * TL * TL$ or $TL \otimes TL$.*

Proof. Suppose $\mathcal{S}$ has two non-trivial biprojections P, Q. If $r(P) = 3$, then by Lemma 6.1, $\mathcal{S}$ is separated by P as a free product. By Theorem 2.22, Q is a subprojection of P. So $\mathcal{S} = TL * TL * TL$. If $\mathcal{S} \neq TL * TL * TL$, then $r(P) = 2$. Similarly $r(Q) = 2$. So $PQ = e$. Applying the same argument to the dual of $\mathcal{S}$, we have that $P * Q$ is a multiple of id. Note that the 2-box space of the dual of $\mathcal{S}$ is 4 dimensional, so it is abelian. That means the coproduct of $\mathcal{S}_2$ is commutative. By Theorem 3.15, we have $\mathcal{S} = TL \otimes TL$. □

Lemma 6.3. *Suppose $\mathcal{S}$ is a subfactor planar algebra with* $\dim(\mathcal{S}_2) = 4$. *If $\mathcal{S}_3/\mathcal{I}_3$ is abelian, then $\mathcal{S}$ is either depth-2, or a free product.*

Proof. If $\mathcal{S}_3/\mathcal{I}_3$ is abelian, then in the principal graph, each depth-3 vertex only connects with one depth-2 vertex. Note that $\dim(\mathcal{S}_2) = 4$, so $\dim(\mathcal{S}_2)$ is abelian. That means there is only one edge between each depth-2 vertex and the depth-1 vertex. So there is only one length-2 path between any two different depth-2 vertices. Let e, P_1, P_2, P_3 be the minimal projections of $\mathcal{S}_2$; then $\dim(P_i \mathcal{S}_3 P_j)) = 1$, whenever $1 \leq i, j \leq 3$ and $i \neq j$. By Lemma 4.5, we have $r(P_j' * P_i) = 1$. Thus either $tr(P_i) = 1$, or P_i' is a virtual normalizer. If $tr(P_i) = 1$, for $i = 1, 2, 3$, then $\mathcal{S}_2$ is depth-2. Otherwise one of them is a virtual normailizer. By Theorem 4.26, $\mathcal{S}$ is a free product. □

Lemma 6.4. *Suppose $\mathcal{S}$ is a subfactor planar algebra with* $\dim(\mathcal{S}_2) = 4$. *If $\mathcal{S}_3/\mathcal{I}_3$ is a direct sum of $M_{2\times 2}$ and the $\mathbb{C}$'s, then it is either a free product or $\mathcal{S}^{\mathbb{Z}_2} \otimes TL$.*

Proof. Suppose e, P_1, P_2, P_3 are minimal projections of $\mathcal{S}_2$. Let $a_i, i = 1, 2, 3$, be the depth-2 vertex in the principal graph corresponding to P_i. If $\mathcal{S}_3/\mathcal{I}_3$ is a direct sum of $M_{2\times 2}$ and the $\mathbb{C}$'s, then there is one depth-3 vertex v with multiplicity 2 in the principal graph. Without loss of generality, we assume v is not connected with a_1. Then in the principal graph, there is only one length-2 path between a_1 and a_j, for $j = 2, 3$. That means $\dim(P_1 \mathcal{S}_3 P_j) = 1$. By Lemma 4.5, we have $r(P_1' * P_j) = 1$. Therefore either P_1' is a left virtual normalizer or $tr(P_1) = 1$. If P_1' is a left virtual normalizer, then $\mathcal{S}$ is a free product by Theorem 4.26. Otherwise $tr(P_1) = 1$. By Theorem 4.16, the support T of $\mathcal{S}_2 \cap \mathcal{I}_3$ is a biprojection. If $r(T) = 3$, then $\mathcal{S}$ is a free product by Lemma 6.1. Otherwise $T = e + P_1$.

To sum up, if $\mathcal{S}$ is not a free product, then it has a trace-2 biprojection. Applying the same argument to the dual of $\mathcal{S}$, we also have a trace-2 biprojection in the dual. That gives a biprojection Q in $\mathcal{S}$, and $tr(Q) = \frac{\delta^2}{2}$. If $P = Q$, then $\delta^2 = 4$. This is a contradiction. Otherwise $\mathcal{S} = TL \otimes TL$ by Theorem 6.2. Furthermore $tr(e + P_1) = 2$, so $\mathcal{S} = \mathcal{S}^{\mathbb{Z}_2} \otimes TL$. □

Theorem 6.5. *Suppose $\mathcal{S}$ is an exchange relation planar algebra with* $\dim(\mathcal{S}_2) = 4$. *Then $\mathcal{S}$ is one of the following:*

(1) $\mathcal{S}^{\mathbb{Z}_4}$, *or* $\mathcal{S}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$;
(2) $\mathcal{A} * TL$ *or* $TL * \mathcal{A}$, *for an exchange relation planar algebra $\mathcal{A}$ with* $\dim(\mathcal{A}_2) = 3$;
(3) $\mathcal{S}^{\mathbb{Z}_2} \otimes TL$;
(4) $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_7 \rtimes \mathbb{Z}_2}$.

Recall that an exchange relation planar algebra $\mathcal{A}$ with $\dim(\mathcal{A}_2) = 3$ is the same as a subfactor planar algebra $\mathcal{A}$ generated by a non-trivial 2-box with $\dim(\mathcal{A}_3) \leq 13$; see the arguments at the end of section 2.5.

Proof. Suppose $\mathcal{S}$ is an exchange relation planar algebra with $\dim(\mathcal{S}_2) = 4$.

(1) If $\mathcal{S}$ is depth-2, then it is $\mathcal{S}^{\mathbb{Z}_4}$ or $\mathcal{S}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$. Conversely a depth-2 subfactor planar algebra is an exchange relation planar algebra. Therefore we obtain class (1), $\mathcal{S}^{\mathbb{Z}_4}$, and $\mathcal{S}^{\mathbb{Z}_2\oplus\mathbb{Z}_2}$.

(2) If $\mathcal{S}$ is separated by a non-trivial biprojection Q as a free product, then by Proposition 5.2, both $\mathcal{S}^Q$ and $\mathcal{S}_Q$ are exchange relation planar algebras. Note that

$$\dim{(\mathcal{S}^Q)_2} + \dim{(\mathcal{S}_Q)_2} = \dim(\mathcal{S}_2) + 1 = 5,$$

so one of them is TL and the other is an exchange relation planar algebra $\mathcal{A}$ with $\dim(\mathcal{A}_2) = 3$. Conversely by Proposition 5.3, the free product of an exchange relation planar algebra $\mathcal{A}$ with $\dim(\mathcal{A}_2) = 3$ and TL is an exchange relation planar algebra. Thus we obtain class (2).

(3) By Proposition 5.5, the tensor product of $\mathcal{S}^{\mathbb{Z}_2}$ and TL is an exchange relation planar algebra $\mathcal{S}$ with $\dim(\mathcal{S}_2) = 2 \times 2 = 4$.

(4) By Theorem 5.6, the subgroup subfactor planar algebra $\mathcal{S}^{\mathbb{Z}_2\subset\mathbb{Z}_7\rtimes\mathbb{Z}_2}$ is an exchange relation planar algebra $\mathcal{S}$ with $\dim(\mathcal{S}_2) = \dfrac{7+1}{2} = 4$.

By Proposition 2.24, we have $\dim(\mathcal{S}_3/\mathcal{I}_3) \leq 9$. We need to consider the following four cases:

(a) $\mathcal{S}_3/\mathcal{I}_3$ is abelian;
(b) $\mathcal{S}_3/\mathcal{I}_3$ is a direct sum of $M_{2\times2}$ and the $\mathbb{C}$'s.
(c) $\mathcal{S}_3/\mathcal{I}_3$ contains $M_{2\times2} \oplus M_{2\times2}$;
(d) $\mathcal{S}_3/\mathcal{I}_3 = M_{3\times3}$;

Case (a): If $\mathcal{S}_3/\mathcal{I}_3$ is abelian, then $\mathcal{S}$ is in class (1) or (2) by Lemma 6.3.

Case (b): If $\mathcal{S}_3/\mathcal{I}_3$ is a direct sum of $M_{2\times2}$ and the $\mathbb{C}$'s, then $\mathcal{S}$ is in class (2) or (3) by Lemma 6.4.

To prove cases (c) and (d), let us prove some general results for exchange relation planar algebras. Note that $\dim(\mathcal{S}_2) = 4$, so $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ are abelian. Suppose e, P_1, P_2, P_3 are mutually orthogonal minimal projections of $\mathcal{S}_2$, and $e_2, 1 \boxdot Q_1, 1 \boxdot Q_2$, and $1 \boxdot Q_3$ are mutually orthogonal minimal projections of $\mathcal{S}_{1,3}$. Then

$$(1 \boxdot P_i)(1 \boxdot Q_j) = \lambda_{i,j} 1 \boxdot Q_j, \text{ for some } \lambda_{i,j} \in \mathbb{C}.$$

So $P_i * Q_j = \lambda_{i,j} Q_j$. Let $\mathcal{I}_3$ be the basic construction ideal of $\mathcal{S}_3$; then $\mathcal{S}_3 = \mathcal{I}_3 \oplus \mathcal{S}_3/\mathcal{I}_3$. Let us define s_3 to be the support of $\mathcal{S}_3/\mathcal{I}_3$. Then $\{s_3(1\boxdot Q_j)P_i\}_{1\leq i,j\leq 3}$ are pairwise orthogonal. By assumption $\mathcal{S}$ is an exchange relation planar algebra, so $\mathcal{S}_3/\mathcal{I}_3$ is generated by $\{s_3 P_i(1 \boxdot Q_j)\}_{1\leq i,j\leq 3}$ as a linear space. Then we have $\dim(s_3(1 \boxdot Q_j)\mathcal{S}_3 P_i) \leq 1$ and

$$\dim(s_3(1 \boxdot Q_j)\mathcal{S}_3 P_i)) = 1 \iff s_3 P_i(1 \boxdot Q_j) \neq 0.$$

It is easy to check that

$$\begin{aligned} s_3P_i(1\boxdot Q_j)=0 &\iff tr(s_3P_i(1\boxdot Q_j))=0\\ &\iff tr((P_i-\frac{\delta}{tr(P_i)}P_i(1\boxdot id)P_i)(1\boxdot Q_j))=0\\ &\iff |\lambda_{i,j}|=\frac{tr(P_i)}{\delta}. \end{aligned}$$

Note that P_i corresponds to a depth-2 vertex in the principal graph, denoted by a_i, $1\boxdot Q_j$ corresponds to a depth-2 vertex in the dual principal graph, denoted by b_j, and $\dim(s_3(1\boxdot Q_j)\mathcal{S}_3P_i))$ is the number of length-2 paths from a_i to b_j passing through a depth-3 vertex of the principal graph in the 4-partite principal graph. Thus $\dim(s_3(1\boxdot Q_j)\mathcal{S}_3P_i)\leq 1$ implies that the number of edges connecting a depth-3 vertex of the principle graph with a_i (or b_j) is at most 1.

Case (c): If $\mathcal{S}_3/\mathcal{I}_3$ contains $M_{2\times 2}\oplus M_{2\times 2}$, then there are two depth-3 vertices with multiplicity 2 in the principal graph. Thus there is a vertex a_i which connects with both of them. Moreover there is a vertex b_j which connects with both of them in the 4-partite principal graph. Then there are two length-2 paths from a_i to b_j passing through a depth-3 vertex of the principal graph. This is a contradiction.

Case (d): If $\mathcal{S}_3/\mathcal{I}_3=M_{3\times 3}$, then $\dim(\mathcal{S}_3/\mathcal{I}_3)=9$. Thus $s_3P_i(1\boxdot Q_j)\neq 0$, for any $1\leq i,j\leq 3$. Then $|\lambda_{i,j}|\neq\frac{tr(P_i)}{\delta}$. By Theorem 4.21, the biprojection generated by P_i is id. In the principal graph, there is only one depth-3 vertex, and it connects with each depth-2 vertex with one edge. Thus

$$tr(P_1)=tr(P_2)=tr(P_3)>1;$$
$$\dim(P_i\mathcal{S}_3P_j)=2,\ \forall\ 1\leq i,j\leq 3.$$

Take $c=tr(P_1)$. By Lemma 4.5, we have

$$r(P_i*P_j)\leq 2,\ \forall\ 1\leq i,j\leq 3.$$

Note that among the three projections, at least one is self-contragredient; we assume that $P_1=P_1'$. By Lemma 4.10, we have

$$tr((P_1*P_1)e)=tr((e*P_1)P_1)=\frac{c}{\delta}.$$

So the coefficient of e in P_1*P_1 is $\frac{c}{\delta}$. Recall that $r(P_1*P_1)\leq 2$. Computing the trace, we have

$$P_1*P_1=\frac{c}{\delta}e+\frac{c-1}{\delta}P_k,\ \text{for some } 1\leq k\leq 3.$$

By Theorem 4.12, we have $k\neq 1$; otherwise the biprojection generated by P_1 is $e+P_1$. Without loss of generality, we assume that $k=2$. Then $P_2=P_2'$. By Lemma 4.10,

$$tr((P_1*P_2)P_1)=tr(P_2(P_1*P_1))=\frac{c(c-1)}{\delta}.$$

So the coefficient of P_1 in P_1*P_2 is $\frac{c-1}{\delta}$. Recall that $r(P_1*P_2)\leq 2$. Computing the trace, we have

$$P_1*P_2=\frac{c-1}{\delta}P_1+\frac{1}{\delta}P_l,\ \text{for some } 2\leq l\leq 3.$$

By Theorem 4.12, we have $l \neq 2$; otherwise the biprojection generated by P_1 is $e+P_1+P_2$. So

$$P_1 * P_2 = \frac{c-1}{\delta}P_1 + \frac{1}{\delta}P_3.$$

Applying Lemma 4.10 and computing the trace again, we have

$$P_1 * P_3 = \frac{1}{\delta}P_2 + \frac{c-1}{\delta}P_3.$$

By Lemma 4.10,

$$tr((P_2 * P_2)P_1) = tr((P_1 * P_2)P_2) = 0.$$

So $P_1 \precnsim P_2 * P_2$. Recall that $r(P_2*P_2) \leq 2$ and $e \preceq P_2*P_2$. If $P_2*P_2 \preceq e+P_2$, then $e+P_2$ is a biprojection. This is a contradiction. So $P_2 * P_2 \sim e+P_3$. Computing the trace, we have

$$P_2 * P_2 = \frac{c}{\delta}e + \frac{c-1}{\delta}P_3.$$

By a similar argument, we have

$$P_2 * P_3 = \frac{1}{\delta}P_1 + \frac{c-1}{\delta}P_3;$$

$$P_3 * P_3 = \frac{c}{\delta}e + \frac{c-1}{\delta}P_1.$$

Therefore

$$P_1 * (P_1 * P_2) = \frac{c(c-1)}{\delta^2}e + \frac{(c-1)^2}{\delta^2}P_2 + \frac{1}{\delta^2}P_2 + \frac{c-1}{\delta^2}P_3;$$

$$(P_1 * P_1) * P_2 = \frac{c}{\delta^2}P_2 + \frac{c(c-1)}{\delta^2}e + \frac{(c-1)^2}{\delta^2}P_3.$$

Comparing the coefficient of P_3, we obtain $c=2$. By Theorem 2.26, we have that $\mathcal{S}$ is isomorphic to $\mathcal{S}^{\mathbb{Z}_2 \subset \mathbb{Z}_7 \rtimes \mathbb{Z}_2}$. □

Theorem 6.6. *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes with $\dim(\mathcal{S}_{2,\pm}) = 4$ and $\dim(\mathcal{S}_{3,\pm}) \leq 23$. Then $\mathcal{S}$ is one of the following:*

(1) *$\mathcal{S}^{\mathbb{Z}_4}$ or $\mathcal{S}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$;*
(2a) *$\mathcal{A}*TL$ or $TL*\mathcal{A}$, where $\mathcal{A}$ is generated by a non-trivial 2-box with $\dim(\mathcal{A}_{3,\pm}) \leq 13$;*
(2b) *$\mathcal{B}*\mathcal{S}^{\mathbb{Z}_2}$ or $\mathcal{S}^{\mathbb{Z}_2}*\mathcal{B}$, where $\mathcal{B}$ is generated by a non-trivial 2-box with $\dim(\mathcal{A}_{3,\pm}) \leq 14$;*
(3) *$\mathcal{S}^{\mathbb{Z}_2} \otimes TL$.*

Proof. Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes with $\dim(\mathcal{S}_{2,\pm}) = 4$ and $\dim(\mathcal{S}_{3,\pm}) \leq 23$, and $\mathcal{I}_3$ is the basic construction ideal of $\mathcal{S}_3$. Then $\dim(\mathcal{S}_3/\mathcal{I}_3) \leq 23 - 4^2 = 7$. Thus either $\mathcal{S}_3/\mathcal{I}_3$ is abelian or $\mathcal{S}_3/\mathcal{I}_3$ is a direct sum of $M_{2\times 2}$ and the $\mathbb{C}$'s. By Lemmas 6.3 and 6.4, either $\mathcal{S}$ is one of $\mathcal{S}^{\mathbb{Z}_4}$, $\mathcal{S}^{\mathbb{Z}_2 \oplus \mathbb{Z}_2}$ and $\mathcal{S}^{\mathbb{Z}_2} \otimes TL$, corresponding to classes (1) and (3) in the statement, or $\mathcal{S}$ is separated by a non-trivial biprojection Q as a free product. In the latter case, both $\mathcal{S}^Q$ and $\mathcal{S}_Q$ are generated by 2-boxes. Counting the dimensions of 2-boxes, we have

$$\dim((\mathcal{S}^Q)_2) - 1 + \dim((\mathcal{S}_Q)_2) - 1 = \dim(\mathcal{S}_2) - 1 = 3.$$

Thus one of them is TL, and the other is a subfactor planar algebra generated by 2-boxes with 3 dimensional 2-boxes. Furthermore counting the dimensions of 3-boxes, we have

$$\dim((\mathcal{S}^Q)_3) + \dim((\mathcal{S}_Q)_3) = 3^2 + 2^2 - (3-1)(2-1) + \dim(\mathcal{S}_2) - 4^2 \leq 18.$$

Thus either the Templey-Lieb one is $\mathcal{S}^{\mathbb{Z}_2}$ and the other has at most 14 dimensional 3-boxes, or the other one has at most 13 dimensional 3-boxes. They correspond to classes (2a) and (2b) in the statement. □

Theorem 6.7. *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes such that $\mathcal{S}_3/\mathcal{I}_3$, $\mathcal{S}_2$ and $\mathcal{S}_{1,3}$ are abelian. Then it is a free product of Temperley-Lieb subfactor planar algebras and group subfactor planar algebras for abelian groups. The converse statement is also true.*

Proof. By assumption $\mathcal{S}_3/\mathcal{I}_3$ is abelian, so the multiplicity of a depth-3 vertex in the principal graph is 1. Furthermore $\mathcal{S}_2$ is abelian, so the multiplicity of a depth-2 vertex in the principal graph is 1. Then for two distinct depth-2 vertices, there is only one length-2 path between them. Thus $\dim(P_i\mathcal{S}_3P_j)=1$ in $\mathcal{S}_3$, for any two distinct minimal projections P_i,P_j of $\mathcal{P}_2$. By Lemma 4.5, we have $r(P_j'*P_i)=1$. Thus either $tr(P_i)=1$ or P_i' is a virtual normalizer. If $tr(P_i)=1$, for any minimal projection P_i in $\mathcal{S}_2$, then $\mathcal{S}$ is depth-2. By assumption it is of type AA, so it is a group subfactor planar algebra for some abelian group. Otherwise $\mathcal{S}$ contains a virtual normalizer. By Theorem 4.26, either $\mathcal{S}$ is Temperley-Lieb or $\mathcal{S}$ is separated by a non-trivial biprojection as a free product $\mathcal{A}*\mathcal{B}$. In the latter case, both $\mathcal{A}$ and $\mathcal{B}$ have smaller index. By Lemma 5.10, both $\mathcal{A}$ and $\mathcal{B}$ are commute relation planar algebras of type AA. Then we may decompose them again until they are either depth-2 or Temperley-Lieb. Therefore $\mathcal{S}$ is a free product of Temperley-Lieb subfactor planar algebras and depth-2 subfactor planar algebras, and each of them is of type AA.

Conversely both Temperley-Lieb subfactor planar algebras and group subfactor planar algebras for abelian groups are commute relation planar algebras of type AA. By Lemma 5.9, their free product is a commute relation planar algebra of type AA. □

Theorem 6.8. *Suppose $\mathcal{S}$ is a subfactor planar algebra generated by 2-boxes and $\mathcal{S}_3/\mathcal{I}_3$ is abelian. Then: $\mathcal{S}$ is either depth-2 or the free product $\mathcal{A}_1*\mathcal{A}_2*\cdots*\mathcal{A}_n$ such that $\mathcal{A}_1$ is Temperley-Lieb or the dual of $\mathcal{S}^{G_1}$, for a group G_1; $\mathcal{A}_n$ is Temperley-Lieb or $\mathcal{S}^{G_n}$, for a group G_n; $\mathcal{A}_m$, for $1<m<n$, is Temperley-Lieb or $\mathcal{S}^{G_m}$, for an abelian group G_m. The converse statement is also true.*

In this general case, we still want to show the existence of a virtual normalizer in a commute relation planar algebra, whenever it is not depth-2. Then we may decompose a commute relation planar algebra as a free product of commute relation planar algebras, until they are either Temperley-Lieb or depth-2.

Notation 6.9. Recall that $\mathcal{S}_2\cap\mathcal{I}_3$ is a two sided ideal of $\mathcal{S}_2$. Let us define $\mathcal{S}_2/\mathcal{I}_3$ to be the orthogonal complement of $\mathcal{S}_2\cap\mathcal{I}_3$ in $\mathcal{S}_2$. Then $\mathcal{S}_2=(\mathcal{S}_2\cap\mathcal{I}_3)\oplus\mathcal{S}_2/\mathcal{I}_3$.

Lemma 6.10. *Suppose $\mathcal{S}$ is a commute relation planar algebra. If $\mathcal{S}$ is not depth-2, then each minimal projection P_i in $\mathcal{S}_2/\mathcal{I}_3$ is a virtual normalizer.*

Based on Lemma 6.10, the proof of Theorem 6.8 is similar to the proof of Theorem 6.7.

Proof of Theorem 6.8. Suppose $\mathcal{S}$ is a commute relation planar algebra. If it is not depth-2, then by Lemma 6.10, it contains a virtual normalizer. If $\mathcal{S}$ is not Temperley-Lieb, then by Theorem 4.26 and Lemma 5.10, $\mathcal{S}$ is a free product of two commute relation planar algebras with smaller index. Repeating this process, we have $\mathcal{S} = \mathcal{A}_1 * \mathcal{A}_2 * \cdots * \mathcal{A}_n$ such that each $\mathcal{A}_i$ is either Temperley-Lieb or depth-2. By Lemma 5.10, $\mathcal{A}_1$ is of type NA and $\mathcal{A}_n$ is of type AN; the others are of type AA.

Conversely by Lemma 5.9, their free product is a commute relation planar algebra. □

To prove Lemma 6.10, let us prove some basic results first.

Lemma 6.11. *Suppose $\mathcal{S}$ is a commute relations planar algebra. If P_i, P_j are distinct minimal projections in $\mathcal{S}_2/\mathcal{I}_3$, then $r(P_i' * P_j) = 1$.*

Proof. Suppose P_i, P_j are distinct minimal projections in $\mathcal{S}_2/\mathcal{I}_3$ and v_i, v_j are the corresponding depth-2 vertices in the principal graph. By assumption $\mathcal{S}_3/\mathcal{I}_3$ is abelian, so P_i, P_j are central in $\mathcal{S}_2$. Then the multiplicity of v_i, v_j is 1. Note that the multiplicity of a depth-3 vertex is 1, so there is only one length-2 path between v_i and v_j. By Lemma 4.5, we have $r(P_i' * P_j) = 1$. □

We want to show that $r(P_i * P_j) = 1$ whenever P_i is a minimal projection in $\mathcal{S}_2/\mathcal{I}_3$ and P_j is a minimal projection in $\mathcal{S}_2 \cap \mathcal{I}_3$. If $tr(P_j) = 1$, then $r(P_i * P_j) = 1$ is a minimal. If $tr(P_j) > 1$, we will see $P_i * P_j \sim P_i$.

Lemma 6.12. *Suppose $\mathcal{S}$ is a commute relations planar algebra, P_i, P_j, P_k are minimal projections of $\mathcal{S}_2$, $P_i \in \mathcal{S}_2/\mathcal{I}_3$, and $P_j \in \mathcal{S}_2 \cap \mathcal{I}_3$. If $P_k \preceq P_i * P_j$, then $P_k \in \mathcal{S}_2/\mathcal{I}_3$.*

Proof. Suppose $\mathcal{S}$ is a commute relations planar algebra, $P_k \in \mathcal{S}_2 \cap \mathcal{I}_3$, and $P_k \preceq P_i * P_j$. Then by Lemma 4.10

$$tr(P_i(P_k * P_j')) = tr((P_i * P_j)P_k) > 0.$$

However $P_j, P_k \in \mathcal{S}_2 \cap \mathcal{I}_3$; by Theorem 4.16, we have $P_k * P_j' \in \mathcal{S}_2 \cap \mathcal{I}_3$. So $P_i(P_k * P_j') = 0$. This is a contradiction. □

Lemma 6.13. *Suppose $\mathcal{S}$ is a commute relations planar algebra, P_i, P_j, P_k are minimal projections of $\mathcal{S}_2$, $P_i, P_k \in \mathcal{S}_2/\mathcal{I}_3$, and $P_i \neq P_k$. If $tr(P_j(P_i' * P_k)) > 0$, then $tr(P_j(P_i' * P_k)) = \frac{tr(P_i)tr(P_k)}{\delta}$. Furthermore, $P_j \in \mathcal{S}_2/\mathcal{I}_3$ or $tr(P_j) = 1$.*

Proof. If $tr(P_j(P_i' * P_k)) > 0$, then $tr((P_i * P_j)P_k) > 0$ by Lemma 4.10. Thus $(P_i * P_j)P_k = \lambda P_k$, for some $\lambda > 0$. Note that $\mathcal{I}_3$ is the basic construction ideal of $\mathcal{S}_3$. Let us take s_3 to be the support of $\mathcal{S}_3$. Then $s_3 P_i$ is central in $\mathcal{S}_3$, since $\mathcal{S}_3/\mathcal{I}_3$ is abelian. So

$$(s_3 P_i)(1 \boxdot P_j)P_k = (1 \boxdot P_j)(s_3 P_i)P_k = 0.$$

By Wenzl's formula, we have

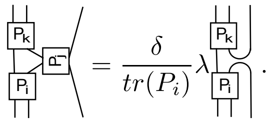
$$s_3 P_i = P_i - \frac{\delta}{tr(P_i)} P_i(1 \boxdot id)P_i.$$

So

$$\begin{aligned} P_i(1 \boxdot P_j)P_k &= \frac{\delta}{tr(P_i)} P_i(1 \boxdot id)P_i(1 \boxdot P_j)P_k \\ &= \frac{\delta}{tr(P_i)} P_i(1 \boxdot id)(P_i * P_j)P_k \quad \text{by an isotopy} \\ &= \frac{\delta}{tr(P_i)} \lambda P_i(1 \boxdot id)P_k, \end{aligned}$$

i.e.,

(6.1)



Then $P_i(1 \boxdot P_j)P_k \neq 0$. Multiplying P_j on the right side (the 3, 4 position), we have

$$P_i(1 \boxdot P_j)P_k = \frac{\delta}{tr(P_i)} \lambda P_i(1 \boxdot P_j)P_k.$$

Thus $\lambda = \frac{tr(P_i)}{\delta}$. Adding a cap on the right bottom (the 4, 5 position), we have

$$tr((P_i * P_j)P_k) = tr(\lambda P_k) = \frac{tr(P_i)tr(P_k)}{\delta}.$$

By Lemma 4.10, $tr(P_j(P_i' * P_k)) = \frac{tr(P_i)tr(P_k)}{\delta}$. Therefore

$$tr(P_j(P_i' * P_k)) > 0 \Longrightarrow tr(P_j(P_i' * P_k)) = \frac{tr(P_i)tr(P_k)}{\delta}.$$

If P_l is a minimal projection in $\mathcal{S}_2$, such that $\|P_l - P_j\| < 1$, then $tr(P_l(P_i' * P_k)) > 0$. Thus $tr(P_l(P_i' * P_k)) = \frac{tr(P_i)tr(P_k)}{\delta}$. Note that the space of projections in the central support of P_j is path connected in norm topology. When P_j moves continuously, $tr(P_j(P_i' * P_k)) > 0$ always holds. On the other hand, $r(P_i' * P_k) = 1$ by Lemma 6.11. So P_j is central. Then $P_j \in \mathcal{S}_2/\mathcal{I}_3$ or $tr(P_j) = 1$. □

Proof of Lemma 6.10. Suppose $\mathcal{S}$ is not depth-2, P_i is a minimal projection in $\mathcal{S}_2/\mathcal{I}_3$, P_j is a minimal projection in $\mathcal{S}_2$, and $P_j \neq P_i'$.

If $P_j \in \mathcal{S}_2/\mathcal{I}_3$, then $r(P_i * P_j) = 1$ by Lemma 6.11.

If $tr(P_j) = 1$, then $r(P_i * P_j) = 1$.

Otherwise $P_j \in \mathcal{S}_2 \cap \mathcal{I}_3$ and $tr(P_j) > 1$. By Lemma 6.12, if $P_k \preceq P_i * P_j$, then $P_k \in \mathcal{S}_2/\mathcal{I}_3$, and $tr(P_j(P_i' * P_k)) = tr((P_i * P_j)P_k) > 0$. Furthermore by Lemma 6.13, if $P_k \neq P_i$, then $P_j \in \mathcal{S}_2/\mathcal{I}_3$ or $tr(P_j) = 1$. This is a contradiction. So $P_k = P_i$. Then $P_i * P_j \sim P_i$.

Therefore P_i is a left virtual normalizer. By Theorem 4.16, P_i' is a minimal projection in $\mathcal{S}_2/\mathcal{I}_3$. So P_i' is a left virtual normalizer. Then P_i is a virtual normalizer. □

Definition 6.14. A subfactor (or a subfactor planar algebra) is said to be k-supertransitive if its principal graph is the Dynkin diagram A_{k+1} up to depth k.

From a subfactor perspective, we have the following weak version of Theorem 6.8.

Corollary 6.15. *Suppose $\mathcal{N} \subset \mathcal{M}$ is a finite index irreducible subfactor such that the quotient of $\mathcal{N}' \cap \mathcal{M}_2$ by the basic construction ideal $(\mathcal{N}' \cap \mathcal{M}_2)e_2(\mathcal{N}' \cap \mathcal{M}_2)$ is abelian, where $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$ is the Jones tower and e_2 is the Jones projection onto $L^2(\mathcal{M})$. Then either $\mathcal{N} \subset \mathcal{M}$ is depth-2 or there exists a sequence of intermediate subfactors $\mathcal{N} \subset \mathcal{R}_1 \subset \cdots \subset \mathcal{R}_n \subset \mathcal{M}$ such that:*

(1) *either $\mathcal{N} \subset \mathcal{R}_1$ is 2-supertransitive or $\mathcal{R}_1 = \mathcal{N} \rtimes G$ for an outer action of a group G;*

(2) *either* $\mathcal{R}_n \subset \mathcal{M}$ *is* 2*-supertransitive or* $\mathcal{R}_n = \mathcal{M}^H$ *for an outer action of a group* H*, where* $\mathcal{M}^H$ *is the fixed point algebra under the action of* H;
(3) *either* $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ *is* 2*-supertransitive or* $\mathcal{R}_{i+1} = \mathcal{R}_i \rtimes A$ *for an outer action of an abelian group* A*, for* $1 \leq i \leq n-1$.

Furthermore any intermediate subfactor of $\mathcal{N} \subset \mathcal{M}$ *is either one of the sequence or an intermediate subfactor of some adjacent pair of the sequence.*

Proof. Suppose $\mathcal{F}$ is the planar algebra of $\mathcal{N} \subset \mathcal{M}$, and $\mathcal{S}$ is the planar subalgebra of $\mathcal{F}_2$ generated by 2-boxes. Since $\mathcal{S}_2 = \mathcal{F}_2$, the basic construction ideal $\mathcal{I}_3$ of $\mathcal{F}_3$ is also the basic construction ideal of $\mathcal{S}_3$. Note that $(\mathcal{N}' \cap \mathcal{M}_2)/((\mathcal{N}' \cap \mathcal{M}_2)e_2(\mathcal{N}' \cap \mathcal{M}_2))$ is abelian means $\mathcal{F}_3/\mathcal{I}_3$ is abelian. Then $\mathcal{S}_3/\mathcal{I}_3$ is abelian. So $\mathcal{S}$ is a commute relation planar algebra. Recall that intermediate subfactors correspond to biprojections in $\mathcal{F}_2 = \mathcal{S}_2$. By Theorem 6.8, $\mathcal{S}$ is either depth-2 or a free product of Temperley-Lieb subfactor planar algebras and depth-2 subfactor planar algebras. Thus either $\mathcal{N} \subset \mathcal{M}$ is depth-2 or there exists a sequence of intermediate subfactors $\mathcal{N} \subset \mathcal{R}_1 \subset \cdots \subset \mathcal{R}_n \subset \mathcal{M}$ corresponding to the sequence of biprojections $P_1, P_2, \cdots, P_n$ which separate $\mathcal{S}$ as a free product such that $\mathcal{S}_{P_{i+1}}^{P_i}$ is either Temperley-Lieb or depth-2, for $1 \leq i \leq n-1$. Note that $\left(\mathcal{S}_{P_{i+1}}^{P_i}\right)_2 = \left(\mathcal{F}_{P_{i+1}}^{P_i}\right)_2$, so $\mathcal{F}_{P_{i+1}}^{P_i}$ is either 2-supertransitive or depth-2. Moreover if $\mathcal{N} \subset \mathcal{R}_1$ is depth-2, then its planar algebra is of type NA; thus $\mathcal{R}_1 = \mathcal{N} \rtimes G$, for an outer action of a group G. If $\mathcal{R}_n \subset \mathcal{M}$ is depth-2, then its planar algebra is of type AN; thus $\mathcal{R}_n = \mathcal{M}^H$, for an outer action of a finite group H. If $\mathcal{R}_i \subset \mathcal{R}_{i+1}$ is depth-2, for some $1 \leq i \leq n-1$, then its planar algebra is of type AA; thus $\mathcal{R}_{i+1} = \mathcal{R}_i \rtimes A$, for an outer action of an abelian group A.

Furthermore, by Theorem 2.22 any intermediate subfactor of $\mathcal{N} \subset \mathcal{M}$ is either one of the sequence or an intermediate subfactor of some adjacent pair of the sequence. □

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Department of Mathematics, Vanderbilt University, Nashville, Tennessee 37240
E-mail address: zhengwei.liu@vanderbilt.edu