

## $n$ -DIMENSIONAL PROJECTIVE VARIETIES WITH THE ACTION OF AN ABELIAN GROUP OF RANK $n - 1$

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ABSTRACT. Let  $X$  be a normal projective variety of dimension  $n \geq 3$  admitting the action of the group  $G := \mathbb{Z}^{\oplus n-1}$  such that every non-trivial element of  $G$  is of positive entropy. We show: ‘ $X$  is not rationally connected’  $\Rightarrow$  ‘ $X$  is  $G$ -equivariant birational to the quotient of a complex torus’  $\Leftrightarrow$  ‘ $K_X + D$  is pseudo-effective for some  $G$ -periodic effective fractional divisor  $D$ ’. To apply, one uses the above and the fact: ‘the Kodaira dimension  $\kappa(X) \geq 0$ ’  $\Rightarrow$  ‘ $X$  is not uniruled’  $\Rightarrow$  ‘ $X$  is not rationally connected’. We may generalize the result to the case of solvable  $G$ .

### 1. INTRODUCTION

We work over the field  $\mathbb{C}$  of complex numbers.

For a normal projective variety  $X$ , we denote by  $\mathrm{NS}(X)$  the Neron-Severi group, i.e., the free abelian group of Cartier divisors modulo algebraic equivalence. Let

$$\mathrm{NS}_{\mathbb{R}}(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

It is a vector space over  $\mathbb{R}$  of finite dimension (called the Picard number of  $X$ ). The *pseudo-effective divisor cone*  $\mathrm{PE}(X)$  is the closure in  $\mathrm{NS}_{\mathbb{R}}(X)$  of effective divisor classes on  $X$ . A divisor is *pseudo-effective* if its class belongs to  $\mathrm{PE}(X)$ .

A projective variety  $X$  is *rationally connected* if some (and hence every) resolution  $X'$  of  $X$  is rationally connected, i.e., any two points of  $X'$  are connected by an irreducible rational curve on  $X'$ . Rational varieties (and more generally unirational varieties) are rationally connected. A variety of Kodaira dimension  $\geq 0$  is not uniruled, so it is not rationally connected (which is condition (i) in Theorem 1.1).

An automorphism  $g$  of a projective variety  $X$  or its representation  $g^*|_{\mathrm{NS}_{\mathbb{R}}(X)}$  is of *positive entropy* if  $g^*L = \lambda L$  for some non-zero nef  $\mathbb{R}$ -Cartier divisor  $L$  and some  $\lambda > 1$ . This is equivalent to saying that the action of  $g$  on the total cohomology group  $H^*(X', \mathbb{R})$  of some (or equivalently every)  $g$ -equivariant resolution  $X'$  of  $X$  has spectral radius  $\rho(g) > 1$ . Indeed,  $g^*|_{\mathrm{NS}_{\mathbb{R}}(X')}$  and  $g^*|_{\mathrm{NS}_{\mathbb{R}}(X)}$  have the same spectral radius since  $g$  permutes the exceptional divisors of  $X' \rightarrow X$ . Further, for smooth  $X$ , we may define the (topological) *entropy* as follows (see [7, Proposition 5.8] or [8, Proposition 3.5], and also [10, §2]):

$$h(g) := \log(\rho(g)).$$

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Let  $X$  be a normal projective variety of dimension  $n$ . In [10], Dinh and Sibony have proved that every commutative subgroup  $G$  of  $\text{Aut}(X)$  of positive entropy has  $\text{rank} \leq n - 1$  (see also [24] for the extension to the solvable groups). In Remarque 4.10 of their arXiv version, they also mention their interest in studying these  $X$  equipped with some  $G$  attaining the maximal rank  $n - 1$ .

In this paper, we consider the maximal rank case. Precisely, we consider the hypothesis:

**Hyp(sA)**  $\text{Aut}(X) \supseteq G := \mathbb{Z}^{\oplus n-1}$ ; every non-trivial element of  $G$  is of positive entropy.

Theorem 2.5 is our main result. For smooth varieties, we have Theorem 1.1; see also key Theorem 2.4 for the general case. The most important assumption in condition (ii) below is the pseudo-effectivity of the adjoint divisor  $K_X + \delta\Delta$ . A Zariski closed subset  $Z \subset X$  is  $G$ -periodic if  $Z$  is stabilized (as a set) by a finite-index subgroup of  $G$ .

Condition (ii) in Theorem 1.1 below is natural and in fact necessary in order for  $X$  to be  $G$ -equivariant birational to a torus quotient. See Main Theorem 2.5. Precisely, condition (i), together with Hyp(sA), implies that  $X$  is non-uniruled and hence  $K_X$  is pseudo-effective, so condition (ii) holds. See the proof of key Theorem 2.4.

**Theorem 1.1.** *Assume that  $X$  and  $G := \mathbb{Z}^{\oplus n-1}$  satisfy Hyp(sA) and  $n = \dim X \geq 3$ . Furthermore, assume any one of the following three conditions holds.*

- (i)  $X$  is not rationally connected.
- (ii)  $X$  is smooth. There exists a reduced simple normal crossing divisor  $\Delta$  which is  $G$ -periodic such that  $K_X + \delta\Delta$  is a pseudo-effective divisor for some  $\delta \in [0, 1)$ ; here  $K_X$  is the canonical divisor of  $X$ .
- (iii)  $X$  is smooth. Every  $G$ -periodic proper subvariety of  $X$  is a point.

Then, replacing  $G$  by a finite-index subgroup, we have the following:

- (1) There is a birational map  $X \dashrightarrow Y$  such that the induced action of  $G$  on  $Y$  is biregular and  $Y = T/F$ , where  $T$  is an abelian variety and  $F$  is a finite group whose action on  $T$  is free outside a finite subset of  $T$ .
- (2) The canonical divisor of  $Y$  is torsion:  $K_Y \sim_{\mathbb{Q}} 0$ . To be precise,  $mK_Y \sim 0$  (linear equivalence) where  $m = |F|$ .
- (3) There is a faithful action of  $G$  on  $T$  such that the quotient map  $T \rightarrow T/F = Y$  is  $G$ -equivariant. Every  $G$ -periodic proper subvariety of  $Y$  or  $T$  is a point.

*Remark 1.2.* (1) Under condition (iii) of Theorem 1.1 (or key Theorem 2.4), we can take  $Y = X$  (cf. Lemma 3.10). See also Remark 2.1 (1).

(2) Hyp(sA) is birational in nature (cf. Lemma 3.1). Conditions (i) and (ii) are also birational in nature. Condition (i) is clear by the definition. For (ii), see the proof in Remark 2.1.

(3) The condition  $n = \dim X \geq 3$  is needed to kill the second Chern class  $c_2$  of  $Y$  by using the perpendicularity of  $c_2$  with the product of common nef eigenvectors of  $G$ .

In summary, Theorem 1.1 (or key Theorem 2.4) and Proposition 2.3 say that if  $G := \mathbb{Z}^{\oplus n-1}$  ( $n \geq 3$ ) acts on a projective  $n$ -fold  $X$  and every non-trivial element of  $G$  is of positive entropy, then:

$$X \text{ is not rationally connected} \Rightarrow X \overset{\text{birational}}{\sim} \text{torus quotient};$$

$$K_X + D \text{ is pseudo-effective for a } G\text{-periodic fractional } D \Leftrightarrow X \overset{\text{birational}}{\sim} \text{torus quot.}$$

For applications, one may combine the above with the following well known fact, where  $\kappa(X)$  is the Kodaira dimension of  $X$ :

$$\kappa(X) \geq 0 \Rightarrow X \text{ is not uniruled} \Rightarrow X \text{ is not rationally connected.}$$

Setting  $\delta = 1$ , the limit case in condition (ii) of Theorem 1.1, we have the following proposition, which is a special case of Proposition 2.6.

**Proposition 1.3.** *Assume that  $X$  and  $G := \mathbb{Z}^{\oplus n-1}$  satisfy Hyp(sa) and  $n = \dim X \geq 2$ . Assume further that  $X$  is smooth,  $\Delta$  is a simple normal crossing reduced divisor which is  $G$ -periodic, and  $K_X + \Delta$  is a pseudo-effective divisor. Then, replacing  $G$  by a finite-index subgroup, there is a birational map  $X \dashrightarrow Y$  such that the following are true.*

- (1)  $Y$  is a normal projective variety. The map  $X \dashrightarrow Y$  is surjective in codimension-1. The induced action of  $G$  on  $Y$  is biregular.
- (2) We have  $K_Y + \Delta_Y \sim_{\mathbb{Q}} 0$ , where  $\Delta_Y$  is the direct image of  $\Delta$  and a reduced divisor.
- (3) Every  $G$ -periodic positive-dimensional proper subvariety of  $Y$  is contained in  $\Delta_Y$ .

Without the pseudo-effectivity of  $K_X + \Delta$  in Proposition 1.3, the following conjecture is well known. The implication '(2)  $\Rightarrow$  (1)' is known in any dimension, while the implication '(1)  $\Rightarrow$  (2)' is known only in dimension  $\leq 2$ . See [15, Lemma 5.11, Theorem 1.1].

**Conjecture 1.4.** *Let  $X$  be a smooth projective variety and let  $\Delta$  be a simple normal crossing reduced divisor. Then the following are equivalent.*

- (1)  $K_X + \Delta$  is not a pseudo-effective divisor.
- (2)  $X \setminus \Delta$  is covered by images of the affine line  $\mathbb{C}$ .

**1.5. Applications of our results** Suppose that  $X$  and  $G$  satisfy Hyp(sa). If every  $G$ -periodic proper subvariety of  $X$  is a point, then  $X = Y$  is a torus quotient (cf. Theorem 1.1 and Lemma 3.10). Otherwise, since the union of all  $G$ -periodic positive-dimensional proper subvarieties of  $X$  is a Zariski-closed proper subset of  $X$  (cf. Lemmas 3.8 and 3.9), replacing  $X$  by a  $G$ -equivariant blowup, we may assume that  $X$  is smooth and this union is a divisor, denoted as  $\Delta$ , with only simple normal crossings.

If  $K_X + \delta\Delta$  is pseudo-effective for some  $\delta \in [0, 1)$  (resp. for  $\delta = 1$ ), we may apply Theorem 1.1 (resp. Proposition 1.3) and say that  $X$  is  $G$ -equivariant birational to a torus quotient (resp. a variety  $Y$  with  $K_Y + \Delta_Y \sim_{\mathbb{Q}} 0$ ).

If  $K_X + \Delta$  is not pseudo-effective, then Conjecture 1.4 (confirmed when  $\dim X \leq 2$ ) asserts that  $X \setminus \Delta$  is covered by images of the affine line  $\mathbb{C}$ ; i.e.,  $X$  is covered by projective rational curves  $\{C_t\}$  with  $C_t$  meeting the  $G$ -periodic locus  $\Delta$  at most once.

**Example 1.6.** In any dimension, there are examples satisfying Hyp(sA) and condition (ii) of Theorem 1.1. Indeed, denote by  $E$  the elliptic curve  $\mathbb{C}/(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$  and by  $T := E \times \cdots \times E$  ( $n$  copies of  $E$ ) an abelian variety of dimension  $n$ . Then  $\text{Aut}(T)$  contains  $G := \mathbb{Z}^{\oplus n-1} \subset \text{GL}_n(\mathbb{Z})$  and  $G$  has a natural action on  $T$  such that  $T$  and  $G$  satisfy Hyp(sA) (cf. [10, Example 4.5]). Consider the automorphism

$$f : T \rightarrow T, (x_1, \dots, x_n) \mapsto (\sqrt{-1}x_1, \dots, \sqrt{-1}x_n).$$

Denote by  $X$  the quotient variety  $T/\langle f \rangle$ . Then  $K_X \sim_{\mathbb{Q}} 0$  and  $X$  has only  $\mathbb{Q}$ -factorial Kawamata log terminal (klt) singularities. Further, when  $n \geq 4$ ,  $X$  is a Calabi-Yau variety in the sense that  $X$  has only canonical singularities,  $K_X \sim_{\mathbb{Q}} 0$ , and the irregularity  $q(X) = 0$ . When  $n \leq 3$ ,  $X$  is a rationally connected variety, in fact, a rational variety; see [22, Theorem 5.9].

Since the diagonal group  $\langle f \rangle$  is normalized by  $G$ , the action of  $G$  on  $T$  descends to a faithful action on  $X$ . This  $X$  and  $G$  satisfy Hyp(sA) and Hyp(B) in §2 with  $D = 0$ . Equivalently, a  $G$ -equivariant resolution  $X'$  of  $X$  and  $G$  satisfy Hyp(sA) and condition (ii) in Theorem 1.1 (cf. Proposition 2.3).

*Remark 1.7.* In dimension two, there are examples satisfying Hyp(sA) and all the other conditions of Proposition 1.3 (resp. the equivalent conditions of Conjecture 1.4). However, we do not know whether there are such examples in dimension  $\geq 3$ .

Indeed, there are surfaces  $X_i$ , for  $i = 1$  (resp.  $i = 2$ ), and an automorphism  $g_i$  of positive entropy such that the union  $\Delta_i$  of all  $g_i$ -periodic curves is a simple normal crossing reduced divisor and  $K_{X_i} + \Delta_i$  is a pseudo-effective (resp. non-pseudo-effective) divisor.

For  $i = 1$ , take  $X'_1$  to be the 12-point blowup of  $\mathbb{P}^2$  as constructed in [6, Example 3.3] with a smooth elliptic curve  $C_1$  being  $g_1$ -periodic, and  $X_1 \rightarrow X'_1$  a  $g_1$ -equivariant blowup such that  $\Delta_1$  is a simple normal crossing divisor. Since  $\Delta_1$  contains the elliptic curve  $C'_1$  lying over  $C_1$ , we have  $K_{X_1} + \Delta_1 \geq K_{X_1} + C' \geq 0$  by the Riemann-Roch theorem.

For  $i = 2$ , take  $X'_2$  to be the 10-point blowup of  $\mathbb{P}^2$  as in [2] or [18, Theorem 1.1] such that  $-K_{X'_2} \sim C_2$  for a cuspidal rational curve  $C_2$ . Take the  $g_2$ -equivariant blowup  $X_2 \rightarrow X'_2$  of the cusp  $P$  of  $C_2$  and two infinitely near points of  $P$  such that the inverse  $\Delta_2$  of  $C_2$  is a simple normal crossing (reduced) divisor. Then the adjoint divisor  $K_{X_2} + \Delta_2$  is not pseudo-effective. Precisely, the adjoint divisor is linearly equivalent to  $-E$  with  $E$  the last  $(-1)$ -curve in the blowup  $X_2 \rightarrow X'_2$ .

**1.8. Related works.** When  $G$  is an abelian subgroup of  $\text{Aut}(X)$ , Dinh-Sibony [10, Theorem 1] proved that the rank of the part of  $G$  with positive entropy is  $\leq \dim X - 1$ . Hence Theorem 1.1 deals with exactly the maximal rank case.

Partial results pertaining to Theorem 1.1 have been obtained in [26] and [27], where one imposed either the assumption that  $X$  and the pair  $(X, G)$  are minimal or the assumption that  $X$  has no  $G$ -periodic positive-dimensional proper subvarieties. These assumptions seem a bit strong, because  $X$  may not have a minimal model  $X'$  when  $X$  is uniruled, and the regular action of  $G$  on  $X$  usually induces only a birational action on  $X'$  (if one exists).

In the key Theorem 2.4 (singular version), we allow  $X$  to have Kawamata log terminal (klt) singularities, which is more natural, from the viewpoint of Log Minimal Model Program (LMMP), than quotient singularities in the previous papers [26] and [27].

See Remark 2.7 for the generalization of Theorem 1.1 (smooth version) and key Theorem 2.4 to the case with  $G$  solvable, like some related partial results in the paper [27].

Our Theorem 1.1 can be compared with Katok-Hertz [13, Corollary 7], where they consider the smooth action of  $\mathbb{Z}^{\oplus n-1}$  on a smooth (real) manifold  $X'$  of dimension  $n$  with the conclusion:  $X'$  is homeomorphic to the connected sum of the compact  $n$ -torus with another manifold.

Theorem 1.1 can also be compared with Cantat-Zeghib [5], where the authors impose the assumption that  $X$  admits the action of a lattice  $\Gamma$  of rank  $n - 1$  in an *almost simple* real Lie group  $H$  (which turns out to be isogenous to  $\mathrm{SL}_n(\mathbb{R})$  or  $\mathrm{SL}_n(\mathbb{C})$ ) and deduce a similar conclusion. They use Margulis' super-rigidity for lattices of higher rank, work out in detail all the actions of  $\Gamma$  on complex  $n$ -tori and also consider the rank  $n$  case (necessarily of non-positive entropy).

Our assumption in Theorem 1.1 is weaker since  $\mathbb{Z}^{\oplus n-1}$  can be thought of as a small part of the lattice  $\mathrm{SL}_n(\mathbb{Z})$  in  $H$ , and hence no super-rigidity is available to induce the action of a big Lie group  $H$ . Instead, we fully utilize the entropy-positivity of  $G$ . Hence our approach is more dynamic in spirit, combined with some algebro-geometric tools.

**1.9. The three ingredients for the proof of Theorems 1.1 and 2.4**

(1) Given a pair  $(X, D)$  of a mildly singular variety  $X$  and a boundary divisor  $D$  supported on  $G$ -periodic divisors, we formally run the log minimal model program (LMMP) and reach a new pair  $(X', D')$  after a divisorial contraction or a flip. See [16, §3.7]. We are able to handle the case when the new pair admits a Fano fibration. The unknown termination of sequence of flips is avoided by running the LMMP directed by an ample divisor. See [4] or [3]. A difficulty occurs: our  $G$  is an infinite group and  $X'$  may have infinitely many extremal rays; how to descend the action of  $G$  on  $X$  to a *biregular* action of  $G$  on  $X'$ ? To overcome this, we have managed to *algebraically* contract only  $G$ -periodic subvarieties, i.e.,  $G$ -periodic extremal rays, and make the LMMP  $G$ -equivariant.

(2) Yau's deep result characterizing étale quotient (or  $Q$ -torus) in terms of the vanishing of the first and second Chern classes has been extended to mildly singular cases, thanks to the recent work of [12].

(3) We use a type of Zariski-decomposition of the adjoint divisor  $K_X + D$  as in [19].

To make this paper easily accessible, we use only the formal process, also sketched in the paper, of the log minimal model program (LMMP), with no detailed technicality involved or required.

The result of this paper explains why manifolds  $Y$  found so far (like [10, Example 4.5]) with the maximal number of commutative automorphisms of positive entropy *all have trivial first Chern class*  $c_1(Y)$ , or  $K_Y \sim_{\mathbb{Q}} 0$ , after a birational change of models.

2. MORE GENERAL RESULTS FOR SINGULAR VARIETIES

We refer to [16] for the conventions and definitions of *Kodaira dimension*, and *Kawamata log terminal (klt)*, *divisorial log terminal (dlt)*, *canonical*, or *log canonical singularities*; see [16, Definition 2.34, 7.73].

We consider the following hypotheses for normal projective varieties  $X$  and  $W$  and a group  $G$  of automorphisms. Denote by  $K_X$  and  $K_W$  their *canonical divisors*.

**Hyp(A)**  $G \leq \text{Aut}(X)$  is a subgroup. The representation  $G^* := G|_{\text{NS}_{\mathbb{R}}(X)}$  is isomorphic to  $\mathbb{Z}^{\oplus n-1}$  where  $n = \dim X$ . Every element of  $G^* \setminus \{\text{id}\}$  is of positive entropy.

**Hyp(B')**  $W$  has Kodaira dimension  $\kappa(W) \geq 0$ .

**Hyp(B'')**  $W$  is non-uniruled; i.e.,  $W$  is not covered by rational curves.

**Hyp(B''')**  $G \leq \text{Aut}(W)$  is a subgroup. For some  $G$ -equivariant resolution  $\eta : X \rightarrow W$  such that the inverse  $\Delta := \eta^{-1}(\text{Sing } W)$  of the singular locus  $\text{Sing } W$  of  $W$  is a simple normal crossing divisor,  $K_X + \delta\Delta$  is a pseudo-effective divisor for some  $\delta \in [0, 1)$ .

**Hyp(B)**  $G \leq \text{Aut}(X)$  is a subgroup. For some effective  $\mathbb{R}$ -divisor  $D$  whose irreducible components are  $G$ -periodic, the pair  $(X, D)$  has at worst  $\mathbb{Q}$ -factorial klt singularities, and  $K_X + D$  is a pseudo-effective divisor.

*Remark 2.1.* (1) If  $X$  and  $G$  satisfy Hyp(A), then the union of all *positive*-dimensional  $G$ -periodic proper subvarieties of  $X$  is a Zariski closed proper subset of  $X$ . See Lemmas 3.8 and 3.9.

(2) Hyp(sA) in the Introduction is stronger than Hyp(A). Hyp(A) is a birational property or more generally a property preserved by generically finite maps. See Lemma 3.1.

(3) Hyp(B') and Hyp(B'') are birational conditions in nature, by the definition.

Hyp(B) is also a birational condition in nature. Indeed, suppose that  $\sigma : X' \rightarrow X$  is a  $G$ -equivariant birational morphism with  $X'$  being  $\mathbb{Q}$ -factorial. We can write

$$K_{X'} + \sigma^*D + E(1) = \sigma^*(K_X + D) + E(2)$$

where  $\sigma^*D$  is the proper transform of  $D$ , where  $E(1)$  and  $E(2)$  are  $\sigma$ -exceptional (and hence  $G$ -periodic) effective divisors with no common components. Now  $D' := \sigma^*D + E(1)$  has components all  $G$ -periodic. The above display shows that the pair  $(X', D')$  is klt since so is  $(X, D)$ . Further,  $K_{X'} + D'$  is pseudo-effective if  $K_X + D$  is pseudo-effective as in Hyp(B); the converse is also true by taking the direct image  $\sigma_*$  of the display.

Similarly, Hyp(B''') is a birational condition in nature by proving as above or using the logarithmic ramification divisor formula.

**Proposition 2.2.** (1) Hyp(B') *implies* Hyp(B'').

(2) Hyp(B'') *implies* Hyp(B''') for any subgroup  $G \leq \text{Aut}(W)$ .

(3) If  $W$  and  $G$  satisfy Hyp(B''') then the  $G$ -equivariant blowup  $X$  in Hyp(B'''), and  $G$  satisfy Hyp(B).

*Proof.* For (1), Hyp(B') says that some (or equivalently every) resolution  $W'$  of  $W$  has Kodaira dimension  $\kappa(W') \geq 0$ . Hence  $mK_{W'} \sim B$  for some integer  $m > 0$  and effective divisor  $B$ . Thus  $K_{W'}$  is pseudo-effective. This means  $W'$  (or equivalently)  $W$  is not uniruled by the uniruled criterion of Miyaoka-Mori and Boucksom-Demailly-Paun-Peternell.

For (2), if  $W$  is non-uniruled, then so is its blowup  $X$ . Thus  $K_X$  is pseudo-effective. So Hyp(B''') holds with  $\delta = 0$ .

For (3) and the  $W$  and  $X$  in  $\text{Hyp}(B''')$ ,  $\text{Sing } W$  and also its inverse  $\Delta$  on  $X$  are stabilized by  $G$ . Set  $D := \delta\Delta$ . Then  $\text{Hyp}(B)$  is satisfied by  $X$  and  $G$ .  $\square$

Proposition 2.3 below, together with 2.2 above, says that under  $\text{Hyp}(A)$ , the  $\text{Hyp}(B)$  in condition (ii) of Theorem 2.4 or 1.1 is not just sufficient but also necessary for  $X$  to be  $G$ -equivariant birational to a torus quotient. See Main Theorem 2.5.

**Proposition 2.3.** *Let  $\sigma : T \rightarrow W$  be a finite morphism from an abelian variety  $T$  onto a normal projective variety  $W$  which is étale in codimension-1 and equivariant under the action of some group  $G$ . Then  $W$  and  $G$  satisfy  $\text{Hyp}(B''')$ .*

*Proof.* Since  $\sigma$  is étale in codimension-1,  $K_T = \sigma^*K_W$ . Since  $K_T \sim 0$ , we have  $K_W \sim_{\mathbb{Q}} 0$ , i.e.,  $mK_W \sim 0$  with  $m = \deg \sigma$ . Since  $T$  is smooth and hence klt, so is  $W$ ; see [16, Proposition 5.20]. Let

$$\eta : X \rightarrow W$$

be a  $G$ -equivariant resolution such that the inverse  $\Delta = \eta^{-1}(\text{Sing } W)$  of the singular locus  $\text{Sing } W$  of  $W$  is a simple normal crossing (reduced) divisor. Write

$$K_X + E(1) = \eta^*K_W + E(2)$$

where  $E(1)$  and  $E(2)$  are  $\eta$ -exceptional effective divisors with no common components. Since  $W$  is klt,  $E(1) = \sum e_i E_i$  is fractional, i.e.,  $e_i \in (0, 1)$ . Choose  $\delta \in (0, 1)$  such that  $E(1) \leq \delta\Delta$ . Then

$$K_X + \delta\Delta \geq K_X + E(1) = \eta^*K_W + E(2) \sim_{\mathbb{Q}} E(2);$$

hence  $K_X + \delta\Delta$  is a pseudo-effective divisor. Therefore,  $W$  and  $G$  satisfy  $\text{Hyp}(B''')$ .  $\square$

Theorem 2.4 below is the key step in proving Main Theorem 2.5.

Regarding the conditions about singularities in (i) and (ii) below, if  $X$  is smooth and  $\Delta$  is a reduced divisor with only simple normal crossings, then both  $X$  and the pair  $(X, \delta\Delta)$ , with  $\delta \in [0, 1)$ , automatically have at worst  $\mathbb{Q}$ -factorial klt singularities. By the proof, Theorem 2.4(4)–(6) hold even when  $n = 2$ .

**Theorem 2.4.** *Assume that  $X$  and  $G$  satisfy  $\text{Hyp}(A)$  and  $n = \dim X \geq 3$ . Furthermore, assume any one of the following three conditions holds.*

- (i)  $X$  is not rationally connected.  $X$  has only  $\mathbb{Q}$ -factorial klt singularities.
- (ii)  $X$  and  $G$  satisfy  $\text{Hyp}(B)$ .
- (iii)  $X$  has only klt singularities. Every  $G$ -periodic proper subvariety of  $X$  is a point.

*Then, replacing  $G$  by a finite-index subgroup, we have the following:*

- (1) *There is a birational map  $X \dashrightarrow Y$  such that the induced action of  $G$  on  $Y$  is biregular and  $Y = T/F$ , where  $T$  is an abelian variety and  $F = \text{Gal}(T/Y)$  acts on  $T$  freely outside a finite subset of  $T$ .*
- (2) *The canonical divisor  $K_Y$  satisfies  $mK_Y \sim 0$  (linear equivalence), where  $m = |F|$ .*
- (3) *The action of  $G$  on  $Y$  lifts to an action of a group  $\tilde{G}$  on  $T$  with  $\tilde{G}/\text{Gal}(T/Y) \cong G$ . Every  $G$ - (resp.  $\tilde{G}$ -) periodic proper subvariety of  $Y$  (resp.  $T$ ) is a point.*

More precisely, we have:

- (4) There is a sequence  $\tau_s \circ \cdots \circ \tau_0$  of birational maps:

$$X = X(0) \xrightarrow{\tau_0} X(1) \xrightarrow{\tau_1} \cdots \xrightarrow{\tau_s} X(s+1) = Y$$

such that each  $X(j) \dashrightarrow X(j+1)$  ( $0 \leq j < s$ ) is either a birational morphism or an isomorphism in codimension-1.  $\tau_s$  is a birational morphism.

- (5) Each pair  $(X(i), D(i))$  ( $0 \leq i \leq s+1$ ) has at worst klt singularities, where  $D(i) \subset X(i)$  is the direct image of  $D$ . The  $X(j)$  ( $0 \leq j \leq s$ ) is  $\mathbb{Q}$ -factorial.  
 (6) The induced action of  $G$  on each  $X(i)$  ( $0 \leq i \leq s+1$ ) is biregular,  
 (7)  $K_{X(s)} = \tau_s^* K_Y \sim_{\mathbb{Q}} 0$  (i.e.,  $m' K_{X(s)} \sim 0$  for some  $m' > 0$ ), and  $D(s) = 0$ .

Hyp(B) in Theorem 2.4 is actually necessary in order for  $X$  to be  $G$ -equivariant birational to a torus quotient:

**Main Theorem 2.5.** *Assume that  $X$  and  $G$  satisfy Hyp(A) and  $n = \dim X \geq 3$ . Then the following are equivalent.*

- (1) Replacing  $G$  by a finite-index subgroup and  $X$  by a  $G$ -equivariant birational model, (the new)  $X$  and  $G$  satisfy Hyp(B).
- (2) Replacing  $G$  by a finite-index subgroup, there exist a  $G$ -equivariant birational map  $X \dashrightarrow W$  and a finite morphism  $T \rightarrow W$  from an abelian variety  $T$  which is Galois and étale in codimension-1, such that the action of  $G$  on  $W$  lifts to an action of a group  $\tilde{G}$  on  $T$  with  $\tilde{G}/\text{Gal}(T/W) \cong G$ . (We can take  $\tilde{G} = G$  if Hyp(sA) is satisfied.)

If we weaken the klt condition for the pair  $(X, D)$  in Hyp(B) for Theorem 2.4 to being dlt, we have the following result which contains Proposition 1.3 as a special case since the pair  $(X, \Delta)$  there is automatically  $\mathbb{Q}$ -factorial dlt.

**Proposition 2.6.** *Assume that  $X$  and  $G$  satisfy Hyp(A) and  $n = \dim X \geq 2$ . Assume further that for some effective  $\mathbb{R}$ -divisor  $D$  whose irreducible components are  $G$ -periodic, the pair  $(X, D)$  has at worst  $\mathbb{Q}$ -factorial divisorial log terminal (dlt) singularities, and  $K_X + D$  is a pseudo-effective divisor. Then, replacing  $G$  by a finite-index subgroup, there is a birational map  $X \dashrightarrow Y$  such that:*

- (1)  $Y$  is a normal projective variety. The map  $X \dashrightarrow Y$  is surjective in codimension-1. The induced action of  $G$  on  $Y$  is biregular.
- (2) The pair  $(Y, D_Y)$  has only log canonical singularities and  $K_Y + D_Y \sim_{\mathbb{Q}} 0$ , where  $D_Y$  is the direct image of  $D$ .
- (3) Every  $G$ -periodic positive-dimensional proper subvariety of  $Y$  is contained in the support of  $D_Y$ .

We may generalize Theorem 2.4 to the case of solvable  $G$ :

*Remark 2.7.* Assume that  $X$  is a smooth projective variety of dimension  $n \geq 3$ . Let  $G \leq \text{Aut}(X)$  and let  $N(G) \subseteq G$  be the set of elements  $g \in G$  of null entropy:  $h(g) = 0$ ; i.e., every eigenvalue of  $g^*|_{\text{NS}_{\mathbb{C}}(X)}$  has modulus 1. Assume that the image  $G \rightarrow \text{GL}(\text{NS}_{\mathbb{C}}(X))$  is solvable and has connected Zariski-closure, and that  $G/N(G) \cong \mathbb{Z}^{\oplus n-1}$  is embedded in  $(\mathbb{R}^{\oplus n-1}, +)$  as a standard lattice by every quasi-nerf sequence. This assumption is weaker than Hyp(A). By [27, Theorem 2.2] and its arXiv version or [9], with  $H^{1,1}(X)$  in its proof replaced by  $\text{NS}_{\mathbb{C}}(X)$  and  $G$  replaced by a finite-index subgroup, we have  $G = N(G)H$  where  $H$  is a subgroup of  $G$  such that  $H|_{\text{NS}_{\mathbb{C}}(X)} \cong \mathbb{Z}^{\oplus n-1}$  and  $N(G)|_{\text{NS}_{\mathbb{C}}(X)}$  is unipotent. Now we can apply Theorem 2.4 to our  $X$  and  $H$  here.



In the joint work with Hu and Dinh [9], we'll show that  $G/N(G) \cong \mathbb{Z}^{\oplus n-1}$  implies  $X$  is a complex torus, or  $N(G)$  is finite so that we can apply Theorem 1.1 (cf. [26, Lemma 2.4]).

**2.8. Theorem 2.4 implies Theorem 1.1**

Note that Hyp(sA) implies Hyp(A). Condition (iii) in Theorem 1.1 clearly implies condition (iii) in Theorem 2.4. Condition (ii) in Theorem 1.1 implies Hyp(B) with  $D := \delta\Delta$ . Thus we can apply Theorem 2.4 with condition (ii). Assume condition (i) in Theorem 1.1. Replacing  $X$  by a  $G$ -equivariant resolution, the (new)  $X$  is still not rationally connected and satisfies Hyp(A), after replacing  $G$  by a finite-index subgroup; see Lemma 3.1. Thus we can apply Theorem 2.4 with condition (i).

Now under condition (i), (ii) or (iii), Theorem 1.1 follows from Theorem 2.4. Indeed, when  $G = \mathbb{Z}^{\oplus n-1}$ , replacing  $G$  by a finite-index subgroup, the faithful action of  $G$  on  $Y$  lifts to a faithful action of  $G$  on  $T$  (cf. [26, Lemma 2.4, §2.15]). This proves Theorem 1.1.

3. PRELIMINARY RESULTS

**Lemma 3.1.** *Assume that a group  $G$  acts biregularly on normal projective varieties  $X_1$  and  $X_2$  of the same dimension  $n$ . Let  $\sigma : X_1 \dashrightarrow X_2$  be a  $G$ -equivariant generically finite map.*

- (1) *Suppose that  $\sigma$  is a morphism or  $\sigma$  is an isomorphism in codimension-1 with both  $X_i$  being  $\mathbb{Q}$ -factorial. If  $X_1$  and  $G$  satisfy Hyp(A), then so do  $X_2$  and  $G$ .*
- (2) *Suppose that  $X_a$  and  $G$ , for some  $a$  in  $\{1, 2\}$ , satisfy Hyp(A). Then, replacing  $G$  by a finite-index subgroup,  $X_i$  and  $G$  satisfy Hyp(A) for both  $i$  in  $\{1, 2\}$ .*

*Proof.* Suppose that  $\sigma$  is a morphism. Identify the representation  $G$  on  $\text{NS}_{\mathbb{R}}(X_2)$  with that on the subspace  $\sigma^* \text{NS}_{\mathbb{R}}(X_2) \subseteq \text{NS}_{\mathbb{R}}(X_1)$ . Let  $K \leq G$  be the subgroup such that the following natural sequence of homomorphisms is exact:

$$1 \rightarrow K|_{\text{NS}_{\mathbb{R}}(X_1)} \rightarrow G|_{\text{NS}_{\mathbb{R}}(X_1)} \xrightarrow{r} G|_{\sigma^* \text{NS}_{\mathbb{R}}(X_2)} \rightarrow 1$$

where  $r$  is the restriction homomorphism and necessarily surjective. If  $g$  is in  $K$ , then  $g^*$  fixes the class  $[H'] = [\sigma^*H]$  with  $H$  any ample divisor on  $X_2$ . Here  $H' = \sigma^*H$  is a nef and big divisor on  $X_1$ , since  $\sigma$  is generically finite. Thus

$$K \leq \text{Aut}_{[H']}(X_1) := \{g \in \text{Aut}(X_1) \mid g^*[H'] = [H']\}$$

where the latter group is a finite extension of the identity connected component  $\text{Aut}_0(X_1)$  of  $\text{Aut}(X_1)$ , by Lieberman [17, Proposition 2.2] or [25, Lemma 2.23]. Since the continuous group  $\text{Aut}_0(X_1)$  acts on the integral lattice  $\text{NS}(X_1)/(\text{torsion})$  as identity, the representation  $K|_{\text{NS}_{\mathbb{R}}(X_1)}$  is a finite subgroup of  $G|_{\text{NS}_{\mathbb{R}}(X_1)}$ .

If  $X_1$  and  $G$  satisfy Hyp(A), then  $G|_{\text{NS}_{\mathbb{R}}(X_1)}$  is isomorphic to  $\mathbb{Z}^{\oplus n-1}$  and its only finite subgroup is  $\{\text{id}\}$ . Thus  $K|_{\text{NS}_{\mathbb{R}}(X_1)} = \{\text{id}\}$ . So we have the isomorphisms

$$\mathbb{Z}^{\oplus n-1} \cong G|_{\text{NS}_{\mathbb{R}}(X_1)} \xrightarrow{r} G|_{\sigma^* \text{NS}_{\mathbb{R}}(X_2)} \cong G|_{\text{NS}_{\mathbb{R}}(X_2)}.$$

Hence  $G$  and  $X_2$  satisfy Hyp(A), and we have proved (1) for the first situation, noting that  $g \in G$  acts on  $X_1$  with positive entropy if and only if the same holds on  $X_2$ . See [25, Lemma 2.6]. In the second situation of (1), we can identify  $\text{NS}_{\mathbb{R}}(X_1)$  and  $\text{NS}_{\mathbb{R}}(X_2)$  so the result is clear.

For (2), let  $W \subset X_1 \times X_2$  be the graph of  $\sigma$ , so  $G$  acts on  $W$  biregularly. Now the two natural projections  $p_i : W \rightarrow X_i$  are both  $G$ -equivariant generically finite morphisms. By (1), it suffices to prove the assertion that  $W$  and  $G$  satisfy Hyp(A), after replacing  $G$  by a finite-index subgroup. By the argument in (1), we have an exact sequence

$$1 \rightarrow K|_{\text{NS}_{\mathbb{R}}(W)} \rightarrow G|_{\text{NS}_{\mathbb{R}}(W)} \xrightarrow{r} G|_{p_a^* \text{NS}_{\mathbb{R}}(X_a)} \rightarrow 1.$$

Thus we have isomorphisms

$$G|_{\text{NS}_{\mathbb{R}}(W)}/K|_{\text{NS}_{\mathbb{R}}(W)} \cong G|_{p_a^* \text{NS}_{\mathbb{R}}(X_a)} \cong G|_{\text{NS}_{\mathbb{R}}(X_a)} \cong \mathbb{Z}^{\oplus n-1}.$$

This and the finiteness of  $K|_{\text{NS}_{\mathbb{R}}(W)}$  as shown in (1) imply that  $G|_{\text{NS}_{\mathbb{R}}(W)} \cong \mathbb{Z}^{\oplus n-1}$ , after replacing  $G$  by a finite-index subgroup; see [26, Lemma 2.4]. This proves the required assertion and also (2). □

The following clean proof of the Hodge index theorem for singular varieties is due to N. Nakayama. Another proof in [26, Lemma 2.5] works only when  $M$  is nef.

**Lemma 3.2.** *Let  $X$  be a projective variety of dimension  $n \geq 2$ . Let  $H_1, \dots, H_{n-1}$  be ample  $\mathbb{R}$ -divisors and  $M$  an  $\mathbb{R}$ -Cartier divisor. Suppose that  $H_1 \cdots H_{n-1} \cdot M = 0 = H_1 \cdots H_{n-2} \cdot M^2$ . Then  $M \equiv 0$  (numerical equivalence).*

*Proof.* Let  $\Sigma := \{L \in \text{NS}_{\mathbb{R}}(X) \mid H_1 \cdots H_{n-1} \cdot L = 0\}$ . Lemma 3.2 follows from property (HI): the higher-dimensional Hodge index theorem holds for  $\mathbb{R}$ -Cartier divisors; i.e., the quadratic form  $I(L_1, L_2) := (H_1 \cdots H_{n-2} \cdot L_1 \cdot L_2)$  is negative definite on  $\Sigma$ . Indeed, property (HI) holds when  $X$  is smooth or when the  $H_i$  are all  $\mathbb{Q}$ -divisors by cutting  $X$  by general hypersurfaces in multiples of  $H_i$  and reducing to the surface case so that the usual Hodge index theorem can be applied.

Now we consider the general case where the  $H_i$  are  $\mathbb{R}$ -divisors.

*Claim 3.3.* Let  $P_1, \dots, P_{n-1}$  be ample  $\mathbb{R}$ -divisors and  $N$  an  $\mathbb{R}$ -Cartier divisor such that  $P_1 \cdots P_{n-1} \cdot N = 0$ . Then  $P_1 \cdots P_{n-2} \cdot N^2 \leq 0$ .

We prove Claim 3.3. Take ample  $\mathbb{Q}$ -divisors  $P_{i,m}$  such that  $P_i = \lim_{m \rightarrow \infty} P_{i,m}$ . There is a unique real number  $r(m)$  such that

$$P_{1,m} \cdots P_{n-1,m} \cdot (N + r(m)P_{n-1}) = 0$$

since  $P_{1,m} \cdots P_{n-1,m} \cdot P_{n-1} > 0$ . By the assumption on  $N$ ,  $\lim_{m \rightarrow \infty} r(m) = 0$ . Since the Hodge index theorem holds for  $\mathbb{Q}$ -divisors  $P_{i,m}$  as mentioned early on, we have

$$P_{1,m} \cdots P_{n-2,m} (N + r(m)P_{n-1})^2 \leq 0.$$

Letting  $m \rightarrow \infty$ , we have  $P_1 \cdots P_{n-2} \cdot N^2 \leq 0$ . This proves Claim 3.3.

We continue the proof of Lemma 3.2. For  $\mathbb{R}$ -Cartier divisors  $L_1, L_2$ , let  $I(L_1, L_2)$  be the intersection number  $(H_1 \cdots H_{n-2} \cdot L_1 \cdot L_2)$ . Then,  $I(\cdot, \cdot)$  defines a bilinear form on the real vector space  $\text{NS}_{\mathbb{R}}(X)$ .

We claim that the symmetric matrix corresponding to  $I(\cdot, \cdot)$  has exactly one positive eigenvalue. Indeed, it has a positive eigenvalue, since  $I(P, P) > 0$  for an ample divisor  $P$ . If  $L$  is an  $\mathbb{R}$ -Cartier divisor such that  $I(P, L) = 0$ , then,  $I(L, L) \leq 0$  by Claim 3.3. Hence, the other eigenvalues are non-positive, and the claim is proved.

We now prove property (HI) by induction on  $n$ . We may assume that  $n \geq 3$ . By the arguments so far, it remains to show the assertion that  $I(\cdot, \cdot)$  is non-degenerate. Assume that property (HI) holds in dimension  $n - 1$ , especially that the assertion

holds for  $n - 1$ . Let  $L$  be an  $\mathbb{R}$ -Cartier divisor on  $X$  such that  $I(L, N) = 0$  for every  $\mathbb{R}$ -Cartier divisor  $N$ . We need to prove that  $L \equiv 0$  (numerical equivalence).

Let  $P$  be a very ample prime divisor. Then,

$$H_{1|P} \cdots H_{n-2|P} \cdot L|P = I(L, P) = 0.$$

By induction, property (HI) holds in dimension  $n - 1$ . Hence, either

- (a)  $L|P \equiv 0$ , or
- (b)  $H_{1|P} \cdots H_{n-3|P} \cdot (L|P)^2 < 0$ , i.e.,  $H_1 \cdots H_{n-3} \cdot P \cdot L^2 < 0$ .

Suppose that (b) holds for every very ample prime divisor  $P$ . Since  $H_{n-2}$  is an ample divisor, we have numerical equivalence  $H_{n-2} \equiv \sum p_i P_i$  for some  $p_i > 0$  and very ample prime divisors  $P_i$ . Since (b) holds for every  $P_i$ , we have  $H_1 \cdots H_{n-2} \cdot L^2 < 0$ . This contradicts  $I(L, L) = 0$ . Thus some very ample prime divisor  $P'$  does not satisfy (b).

Let  $C$  be an arbitrary curve on  $X$ . Then, we can find a very ample prime divisor  $P''$  in  $|mP'|$  for some  $m > 0$  such that  $P''$  contains  $C$ . Since  $P'' (\sim mP')$  does not satisfy (b), it satisfies (a), i.e.,  $L|P'' \equiv 0$ . Thus,  $L \cdot C = L|P'' \cdot C = 0$ . So  $L \equiv 0$ . Therefore,  $I(\cdot, \cdot)$  is non-degenerate. We have proved the assertion. Hence property (HI) holds for any  $n \geq 2$ . This also proves Lemma 3.2. □

Let  $G \leq \text{Aut}(X)$ . We define the subset of *null-entropy elements* of  $G$  as

$$N(G) := \{g \in G \mid \text{the entropy } h(g) = 0\}.$$

$N(G)$  may not be a subgroup of  $G$ .

We quote the following result of [10].

**Lemma 3.4** ([10]). *Let  $X$  be a normal projective variety and  $G \leq \text{Aut}(X)$ . Suppose that  $G^* := G|_{\text{NS}_{\mathbb{R}}(X)} \cong \mathbb{Z}^{\oplus r}$  and every non-trivial element of  $G^*$  is of positive entropy. Then there are nef  $\mathbb{R}$ -Cartier divisors  $L_i$  ( $1 \leq i \leq r + 1$ ) such that each  $L_i$  is a common eigenvector of  $G$  with*

$$g^* L_i = \chi_i(g) L_i \quad (g \in G)$$

for some  $\chi_i(g) \in \mathbb{R}_{>0}$ , such that

$$L_1 \cdots L_{r+1} \neq 0$$

as an element of  $H^{r+1, r+1}(X)$ , and such that the homomorphism

$$\begin{aligned} \varphi : G &\rightarrow (\mathbb{R}^r, +) \\ g &\mapsto (\log \chi_1(g), \dots, \log \chi_r(g)) \end{aligned}$$

has the kernel equal to  $N(G)$  and hence it induces an isomorphism from  $G/N(G)$  ( $\cong G^*$ ) onto a spanning lattice of  $(\mathbb{R}^r, +)$ .

*Proof.* This is proved in [10, Theorems 4.3 and 4.7] by considering the action of  $G$  on  $\text{NS}_{\mathbb{R}}(X)$  instead of that on  $H^{1,1}(X)$ . See also [24, Theorems 1.1 and 1.2].

For the isomorphism  $G^* \cong G/N(G)$ , we just need to check the assertion that the natural representation

$$G \rightarrow \text{GL}(\text{NS}_{\mathbb{R}}(X)), \quad g \mapsto g^*|_{\text{NS}_{\mathbb{R}}(X)}$$

has kernel  $K$  equal to  $N(G)$  (and image equal to  $G^*$  by definition). Indeed, the kernel  $K \leq \text{Aut}_{[H]}(X)$  for any ample divisor class  $[H]$ , and  $\text{Aut}_{[H]}(X)$  is a finite extension of the identity connected component  $\text{Aut}_0(X)$  of  $\text{Aut}(X)$  by [17, Proposition 2.2]. So  $K$  is virtually contained in  $\text{Aut}_0(X)$ , and the latter continuous group

acts on the integral lattice  $\text{NS}(X)/(\text{torsion})$  as identity. Thus  $K \leq N(G)$ . Conversely, since every  $n \in N(G)$  or its action  $n^*$  on  $\text{NS}_{\mathbb{R}}(X)$  is of null entropy, we have  $n^* = \text{id}$  on  $\text{NS}_{\mathbb{R}}(X)$  by the assumption on  $G^*$ . Hence  $n \in K$ . So  $N(G) \leq K$ . We have proved the required assertion  $K = N(G)$ .  $\square$

**3.5.** Applying Lemma 3.4 to our  $G^* = G|_{\text{NS}_{\mathbb{R}}(X)} \cong \mathbb{Z}^{\oplus n-1}$  in  $\text{Hyp}(A)$ , with  $n = \dim X$ , we get nef  $\mathbb{R}$ -Cartier divisors  $L_i$  ( $1 \leq i \leq n$ ), which are common eigenvectors of  $G$ . Set

$$A := L_1 + \cdots + L_n.$$

Then  $(L_1 + \cdots + L_n)^n \geq L_1 \cdots L_n$  and the latter is non-zero in  $H^{n,n}(X, \mathbb{R}) = \mathbb{R}$  by Lemma 3.4 and indeed positive,  $L_i$  being nef. Thus  $A$  is a nef and big divisor.

Since  $L_1 \cdots L_n$  is a scalar, for every  $g \in G$ , we have

$$L_1 \cdots L_n = g^*(L_1 \cdots L_n) = (\chi_1(g) \cdots \chi_n(g))(L_1 \cdots L_n).$$

Hence

$$\chi_1 \cdots \chi_n = 1.$$

**Lemma 3.6.** *Suppose that  $X$  and  $G$  satisfy  $\text{Hyp}(A)$ . Then we have:*

- (1) *The  $A = L_1 + \cdots + L_n$  in 3.5 is a nef and big  $\mathbb{R}$ -Cartier divisor.*
- (2) *For integer  $k \gg 1$ , we have  $A = A_k + E/k$  such that  $A_k$  is an ample  $\mathbb{Q}$ -Cartier divisor and  $E$  is a fixed effective  $\mathbb{R}$ -Cartier divisor.*
- (3) *Suppose further that  $(X, D)$  is a klt pair with  $D$  an effective  $\mathbb{R}$ -Cartier divisor. Then, replacing  $A_k$  by some  $A'_k$  with  $A'_k \sim_{\mathbb{Q}} A_k$  and  $A$  by  $A' := A'_k + E/k$  with  $A' \equiv A$  (numerical equivalence), we may assume that  $A$  is an effective divisor and the pair  $(X, D + A)$  has at worst klt singularities.*

*Proof.* (1) has been proved in 3.5. (2) is a consequence of (1) and was proved in [16, Proposition 2.61] or [19, II, Theorem 3.18] (for  $\mathbb{R}$ -divisors). (3) is true, because  $(X, D)$  is klt and klt is an open condition. Indeed, we just take  $k \gg 1$  and replace  $A_k$  by  $A'_k := \frac{1}{m}B$  with  $B$  a general member of  $|mA_k|$  for some  $m \gg 1$ . See [16, Corollary 2.35 (2)].  $\square$

**Lemma 3.7.** *Assume that  $X$  and  $G$  satisfy  $\text{Hyp}(A)$ . For the  $A$  in 3.5, we have:*

- (1) *Suppose that  $Z \in H^{n-k, n-k}(X)$  is a  $G$ -periodic class for some  $n - k$  in  $\{1, \dots, n - 1\}$ . Then  $A^k \cdot Z = 0$ .*
- (2) *We have  $A^{n-1} \cdot c_1(X) = 0$ , and  $A^{n-2} \cdot c_1(X)^2 = A^{n-2} \cdot c_2(X) = 0$  when  $n \geq 3$ . Here  $c_i(X)$  denotes the  $i$ -th Chern class, so  $c_1(X) = [-K_X]$ , and  $c_2(X)$  is regarded as a linear form on  $n - 2$  copies of  $\text{NS}_{\mathbb{R}}(X)$  as defined in [23, p. 265].*
- (3) *Suppose that  $Z \subset X$  is a  $G$ -periodic positive-dimensional proper subvariety of  $X$ . Then  $A^{\dim Z} \cdot Z = 0$ .*

*Proof.* (2) and (3) are consequences of (1), noting that  $g^*c_i(X) = c_i(X)$ .

(1) was proved in [26, Lemma 2.6]. To be precise, replacing  $G$  by a finite-index subgroup, we may assume that  $g^*Z = Z$  ( $g \in G$ ). Note that for all  $i_j$ , we have

$$(1) \quad L_{i_1} \cdots L_{i_k} \cdot Z = 0.$$

Indeed, since  $\varphi(G) \subset \mathbb{R}^{n-1}$  is a spanning lattice,  $k \leq n - 1$ , and  $\chi_1 \cdots \chi_n = 1$ , we can choose  $g \in G$  such that  $\chi_{i_j}(g) > 1$  for all  $i_j$ . Acting on the left hand side of the equality (1) (a scalar) with  $g^*$  and noting that  $g^*Z = Z$ , we conclude the equality

(1). This, in turn, implies that  $A^k \cdot Z = 0$ , since  $A = \sum L_i$  and hence  $A^k$  is the sum of  $L_{i_1} \cdots L_{i_k}$ . □

The following result was first proved by Nakamaye for  $\mathbb{Q}$ -divisors, and generalized to the following for  $\mathbb{R}$ -divisors by Ein-Lazarsfeld-Mustata-Nakamaye-Popa (see e.g. [11, Example 1.11]). For a nef  $\mathbb{R}$ -Cartier divisor  $L$  on a projective variety  $X$ , let

$$\text{Null}(L) = \bigcup_{L|_Z \text{ not big}} Z$$

where  $Z$  runs over all positive-dimensional subvarieties of  $X$ . Since  $L|_Z$  is nef, it is not big if and only if  $L^{\dim Z} \cdot Z = 0$ .

**Lemma 3.8** (cf. [11]). *Let  $X$  be a normal projective variety and  $L$  a non-zero nef  $\mathbb{R}$ -Cartier divisor. Then  $\text{Null}(L)$  is a Zariski-closed proper subset of  $X$ . (Indeed,  $\text{Null}(L)$  is equal to the augmented base locus  $B_+(L)$ , which we will not use in the sequel.)*

**Lemma 3.9.** *Suppose that  $X$  and  $G$  satisfy Hyp(A). For the  $A$  in 3.5, we have*

$$\text{Null}(A) = \bigcup_{Y \text{ is } G\text{-periodic}} Y$$

where  $Y$  runs over all positive-dimensional  $G$ -periodic proper subvarieties of  $X$ . In particular,  $A$  is ample if and only if every  $G$ -periodic proper subvariety of  $X$  is a point.

*Proof.* The second assertion follows from the first, Lemma 3.7 and the Campana-Peternell generalization of Nakai-Moishezon ampleness criterion to  $\mathbb{R}$ -divisors.

For the inclusion “ $\supseteq$ ”, we just copy the argument in the proof of Lemma 3.7 (1).

The inclusion “ $\subseteq$ ” is as in [26, Lemma 2.6]. Indeed, assume that  $Y \subseteq \text{Null}(A)$  with  $0 < s := \dim Y$ , so that  $A|_Y$  is not big, i.e.,  $A^s \cdot Y = 0$ . Write  $A = A_k + E/k$  as in Lemma 3.6. Since  $A|_Y$  is not big,  $Y \subseteq \text{Supp } E$ . Since  $A = \sum L_i$  with  $L_i$  nef and  $A^s$  is the sum of  $L_{i_1} \cdots L_{i_s}$ , the condition  $A^s \cdot Y = 0$  means  $L_{i_1} \cdots L_{i_s} \cdot Y = 0$  for all  $i_j$ . Since  $L_{i_j}$  are all  $g^*$ -eigenvectors, reversing the process, we get  $A^s \cdot g(Y) = 0$  and hence  $g(Y) \subset \text{Supp } E$  by the above reasoning. Now  $Y$  is contained in the Zariski-closure  $\overline{\bigcup_{g \in G} g(Y)}$ . This closure is  $G$ -stabilized and contained in  $\text{Supp } E$ , and every irreducible component of it is a positive-dimensional  $G$ -periodic proper subvariety of  $X$ . Hence  $Y$  is contained in the right hand side of the equality in Lemma 3.9. This proves the inclusion “ $\subseteq$ ”. □

**Lemma 3.10.** *Theorem 2.4 holds under condition (iii), which is equivalent to the condition that the  $A$  in 3.5 is an ample divisor (cf. Lemma 3.9). Further, we can take  $Y = X$ .*

*Proof.* By Lemma 3.7,  $K_X^i \cdot A^{n-i} = 0$  ( $i = 1, 2$ ), so  $K_X \equiv 0$  by Lemma 3.2 and the ampleness of  $A$ . The known abundance theorem in the case of zero Kodaira dimension implies that  $K_X \sim_{\mathbb{Q}} 0$ . See [16, V. Corollary 4.9].

Let  $m' > 0$  be the smallest integer such that  $m'K_X \sim 0$ . Let

$$\pi : \hat{X} = \text{Spec} \bigoplus_{i=0}^{m'-1} \mathcal{O}(-iK_X) \rightarrow X$$

be the Galois  $\mathbb{Z}/(m')$ -cover which is called the *global index-1 cover* and is étale outside  $\text{Sing } X$  such that  $K_{\hat{X}} \sim 0$  and  $\hat{X}$  has at worst canonical singularities. Since

each  $-iK_X$  is stabilized by  $G$ , the action of  $G$  on  $X$  lifts to a faithful action of  $G$  on  $\hat{X}$ . By Lemma 3.1,  $\hat{X}$  and  $G$  satisfy Hyp(A), after replacing  $G$  by a finite-index subgroup.

Let  $\hat{A}$  be the  $\pi$ -pullback of  $A$ . It is also a sum of common nef eigenvectors of  $G$ . The ampleness of  $A$  implies that of  $\hat{A}$ , since  $\pi$  is a finite morphism. By Lemma 3.7 and since  $n \geq 3$ , we have  $\hat{A}^{n-2} \cdot c_2(\hat{X}) = 0$ . This and the ampleness of  $\hat{A}$  (and Miyaoka’s pseudo-effectivity of  $c_2$ ) imply that  $c_2(\hat{X})$  is zero, as a linear form on the product of  $(n - 2)$ -copies of  $\text{NS}_{\mathbb{R}}(\hat{X})$ ; see [23, pp. 265-267, Proposition 1.1]. Since  $\pi$  is  $G$ -equivariant, every  $G$ -periodic proper subvariety of  $\hat{X}$  is a point, because the same holds on  $X$  by assumption. In particular, the singular locus  $\text{Sing } \hat{X}$  of  $\hat{X}$  is isolated.

We now apply [12, Theorem 1.17] and deduce that  $\hat{X} = T'/F'$  for some abelian variety  $T'$  where the finite group  $F'$  acts on  $T'$  freely outside a codimension-2 subset. Again  $n \geq 3$  is used so that the condition  $(0 =) \dim \text{Sing } \hat{X} \leq n - 3$  in [12] is satisfied. Now  $X$  is covered by the complex torus  $T'$  via the composition

$$T' \rightarrow T'/F = \hat{X} \rightarrow X$$

which is étale in codimension-1. By [1, Sect. 3, especially Proposition 3] or [26, §2.15], the assertions (1) and (3) in Theorem 2.4 hold, with  $Y = X$ . Indeed, since  $\hat{X}$  has no positive-dimensional proper subvariety which is periodic under the action of  $G$ , so do  $X$  and  $T$  under the action of  $G$  and  $\tilde{G}$ . Since  $\tilde{G}$  normalizes  $F := \text{Gal}(T/X)$ , it stabilizes the subset of  $T$  where  $F$  does not act freely, so the latter subset is finite or empty.

The assertion (2) is true because  $K_T \sim 0$ . The others are void since  $Y = X$ .  $\square$

The following is the key step towards the proof of Theorem 2.4. Hyp(wB) is a weaker form of Hyp(B), i.e., without the pseudo-effectivity of  $K_X + D$ .

**Proposition 3.11.** *Assume that  $X$  and  $G$  satisfy Hyp(A), and Hyp(wB): for some effective  $\mathbb{R}$ -divisor  $D$  whose irreducible components are  $G$ -periodic, the pair  $(X, D)$  has at worst  $\mathbb{Q}$ -factorial klt singularities. Assume further  $n := \dim X \geq 2$ . Let  $A = \sum L_i$  be the sum in 3.5 of nef  $\mathbb{R}$ -divisors  $L_i$  which are also  $G$ -eigenvectors such that  $A$  is a nef and big divisor. Replacing  $G$  by a finite-index subgroup and  $A$  by a large multiple, the following are true.*

- (1) *There is a sequence  $\tau_s \circ \dots \circ \tau_0$  of birational maps:*

$$X = X(0) \xrightarrow{\tau_0} X(1) \xrightarrow{\tau_1} \dots \xrightarrow{\tau_s} X(s+1) = Y$$

*such that each  $X(j) \dashrightarrow X(j+1)$  ( $0 \leq j < s$ ) is either a divisorial contraction (and hence a morphism) of a  $(K_{X(j)} + D(j))$ -negative extremal ray or a  $(K_{X(j)} + D(j))$ -flip (and hence an isomorphism in codimension-1); here  $D(i) \subset X(i)$  ( $0 \leq i \leq s+1$ ) is the direct image of  $D$ . The  $\tau_s$  is a birational morphism. So the composition  $X \dashrightarrow Y$  is surjective in codimension-1.*

- (2) *The pair  $(X(i), D(i) + A(i))$  ( $0 \leq i \leq s+1$ ) and hence the pair  $(X(i), D(i))$  has at worst klt singularities (cf. [16, Proposition 2.41, Corollary 2.39]).  $X(j)$  ( $0 \leq j \leq s$ ) is  $\mathbb{Q}$ -factorial.*
- (3) *The induced action of  $G$  on each  $X(i)$  ( $0 \leq i \leq s+1$ ) is biregular.  $X(i)$  and  $G$  satisfy Hyp(A).*

- (4) The direct image  $L(i)_j$  on  $X(i)$  of  $L_j$  is a nef  $\mathbb{R}$ -Cartier  $G$ -eigenvector. Hence the direct image  $A(i)$  on  $X(i)$  of  $A$  is a nef and big  $\mathbb{R}$ -Cartier divisor. Further,  $L(i)_j = \tau_i^* L(i+1)_j$ , so  $A(i) = \tau_i^* A(i+1)$ .
- (5)  $K_Y + D_Y$  is an  $\mathbb{R}$ -Cartier divisor, and  $K_{X(s)} + D(s) = \tau_s^*(K_Y + D_Y)$ , where  $D_Y := D(s+1) = \tau_{s*} D(s)$ .
- (6)  $K_Y + D_Y + A_Y$  is an ample divisor, where  $A_Y := A(s+1) = \tau_* A(s)$ . Hence  $K_{X(s)} + D(s) + A(s) = \tau_s^*(K_Y + D_Y + A_Y)$  is a nef and big divisor.
- (7) The union of all positive-dimensional  $G$ -periodic proper subvarieties of  $X(i)$  ( $0 \leq i \leq s+1$ ) coincides with  $\text{Null}(A(i))$  and hence a Zariski-closed proper subset of  $X(i)$ . Further,  $A(i)|_Z \equiv 0$  (numerical equivalence) for every positive-dimensional subvariety  $Z \subseteq \text{Null}(A(i))$ ; this is especially true when  $Z$  is a component of  $D(i)$ .

The rest of the section is devoted to the proof of Proposition 3.11.

**3.12.** For the convenience of the reader, we recall the traditional LMMP as in [16, §3.7] before running some *directed* LMMP for the klt pair  $(X, D + A)$  chosen in Lemma 3.6. If  $K_X + D + A$  is already nef, then  $(X, D + A)$  is the end product and we stop the LMMP. Suppose that  $K_X + D + A$  is not nef. By the cone theorem ([16, Theorem 3.7]), the closed cone of effective 1-cycles

$$\overline{\text{NE}}(X)$$

contains a  $(K_X + D + A)$ -negative extremal ray  $R = \mathbb{R}_{>0}[\ell]$  generated by an (extremal) rational curve  $\ell$ . There is a corresponding extremal contraction

$$f := \text{Contr}_R : X \rightarrow Y$$

onto a normal projective variety  $Y$  such that fibres of  $f$  are connected, and a curve  $C \subset X$  is contracted by  $f$  to a point on  $Y$  if and only if the class  $[C] \in R$ . Further, if  $B$  is a Cartier divisor on  $X$  such that the intersection  $B \cdot \ell = \text{deg}(B|_\ell) = 0$ , then  $B$  equals the total transform (or pullback)  $f^*B_Y$  for some Cartier divisor  $B_Y$  on  $Y$ .

There are 4 possible cases.

Case (I) (Fano type)  $\dim Y = 0$ ; i.e.,  $Y$  is a point. Then the Picard number  $\rho(X) = 1$  and  $-(K_X + D + A)$  is an ample divisor on  $X$ . Hence  $X$  is of Fano type.

Case (II) (Fano fibration)  $0 < \dim Y < \dim X$ . Let  $F$  be a general fibre of  $f$ . Then  $-(K_X + D + A)$  is positive on every curve  $C \subset F$  and, in fact,  $-(K_X + D + A)|_F = -(K_F + (D + A)|_F)$  is an ample divisor on  $F$ . So  $F$  is of Fano type. This  $f$  is called a Fano-fibration.

Case (III) (divisorial)  $\dim Y = \dim X$  and the *exceptional locus*  $\text{Exc}(f)$  of  $f$  (the subset of  $X$  of points at which  $f$  is not an isomorphism) is a (necessarily) irreducible divisor. In this case  $f$  is a birational morphism. The Picard numbers satisfy  $\rho(Y) = \rho(X) - 1$ . Set  $X(1) := Y$  and let

$$D(1) := f_* D, \quad A(1) := f_* A$$

be the direct images of  $A, D$ , respectively.

Case (IV) (flip)  $\dim Y = \dim X$  and the exceptional locus  $\text{Exc}(f)$  of  $f$  is a Zariski-closed subset of  $X$  of codimension  $\geq 2$  in  $X$ . This  $f : X \rightarrow Y$  is called a flipping contraction. In this case, the natural map

$$X^+ := \text{Proj} \bigoplus_{m \geq 0} \mathcal{O}_Y(f_* [m(K_X + D + A)]) \rightarrow Y$$

is a birational morphism such that  $K_{X^+} + D^+ + A^+$  is relatively ample over  $Y$ . Here  $\lfloor m(K_X + D + A) \rfloor$  is the integral part (or the round down) of the  $\mathbb{R}$ -divisor  $m(K_X + D + A)$ ;  $D^+ \subset X^+$  and  $A^+ \subset X^+$  are the proper transforms of  $D$  and  $A$ . Both birational maps  $X \rightarrow Y$  and  $X^+ \rightarrow Y$  are isomorphisms in codimension-1. In particular, we have the identification of  $\text{NS}_{\mathbb{R}}(X) = \text{NS}_{\mathbb{R}}(X^+)$ . The map  $X \dashrightarrow X^+$  is called a  $(K_X + D + A)$ -flip. Set

$$X(1) := X^+, \quad D(1) := D^+, \quad A(1) := A^+.$$

Set

$$X(0) := X, \quad D(0) := D, \quad A(0) := A.$$

Since  $(X, D + A)$  has only  $\mathbb{Q}$ -factorial klt singularities so does the new pair  $(X(1), D(1) + A(1))$  in Case (III) or (IV) by the LMMP.

If Case (I) or (II) occurs, we get the end product and stop the LMMP. If Case (III) or (IV) occurs, we apply the LMMP to  $(X(1), D(1) + A(1))$  and get divisorial contraction

$$\tau_1 : (X(1), D(1) + A(1)) \rightarrow (X(2), D(2) + A(2))$$

with  $\rho(X(2)) = \rho(X(1)) - 1$ , or flip

$$\tau_1 : (X(1), D(1) + A(1)) \dashrightarrow (X(2), D(2) + A(2)) = (X(1)^+, D(1)^+ + A(1)^+)$$

with  $\rho(X(2)) = \rho(X(1))$ , or Fano type, or Fano fibration. Since the Picard number  $\rho(X(0))$  is finite, Case (III) can only occur finitely many times. Thus there exist a sequence of birational maps

$$X = X(0) \xrightarrow{\tau_0} X(1) \xrightarrow{\tau_1} \dots \dashrightarrow X(s) \xrightarrow{\tau_s} \dots$$

and an integer  $s \geq 0$  such that every

$$\tau_i : (X(i), D(i) + A(i)) \dashrightarrow (X(i+1), D(i+1) + A(i+1))$$

$(i < s)$  is divisorial or flip, and either there is a contraction of Fano type or Fano fibration on  $(X(s), D(s) + A(s))$ , or every contraction  $\tau_j$  ( $j \geq s$ ) is a flip. Here  $D(i) \subset X(i)$  and  $A(i) \subset X(i)$  are the direct images of  $D = D(0)$  and  $A = A(0)$ .

In dimension  $\leq 3$ , the LMMP terminates. In higher dimensions, the termination is not known, i.e., the termination conjecture below is not proven yet: there is no infinite sequence of flips. However, the so-called directed-flip, or  $(K_X + D + A)$ -MMP with scaling, terminates [4, Corollary 1.4.2]. This is one of the key ingredients of our proof.

We continue the proof of Proposition 3.11. We follow the procedures in 3.12 and some steps of [26], but we need to take care of the  $G$ -equivariance of the LMMP.

*Replacing  $G$  by a finite-index subgroup, we may and will assume that every irreducible component of  $D$  is stabilized by  $G$ .*

**Lemma 3.13.** *In 3.12, we can choose the LMMP sequence of divisorial contractions or flips  $\tau_j : X(j) \dashrightarrow X(j+1)$  ( $j < s$ ) such that the following are true for all  $i \in \{0, 1, \dots, s\}$ , after replacing  $G$  by a finite-index subgroup and  $A$  by a large multiple.*

- (1) *The induced action of  $G$  on  $X(i)$  is biregular. The action of  $G$  on  $X(i)$  satisfies Hyp(A).*



- (2) The direct image  $L(i)_j \subset X(i)$  of each  $L_j \subset X$  is nef and a common eigenvector of  $G$ , so the direct image  $A(i) = \sum_{j=1}^n L(i)_j$  on  $X(i)$  of  $A$  is nef and big.
- (3)  $K_{X(s)} + D(s) + bA(s)$  is a nef and big divisor for some (and hence all)  $b \gg 1$ .

*Proof.* Since  $A$  is big, the bigness of the direct image  $A(i)$  of  $A$  is clear. Also the nefness of  $A(i)$  would follow from that of  $L(i)_j$  for all  $j$ . Write  $A = A_k + E/k$  as in Lemma 3.6 such that  $(X, D + A)$  is klt. Fix an ample Cartier divisor  $M$  such that the pair  $(X, D + A + M)$  is nef and klt. This is doable because klt is an open condition and we can replace  $M$  by  $\frac{1}{c}M'$  with  $M'$  a general member of  $|cM|$  for some large integer  $c$ .

The bigness in (3) is clear for  $b \gg 1$ . So assume the contrary that  $K_X + D + bA$  is not nef for any  $b > 1$ . We now consider  $K_X + D + A$ , but  $A$  may be replaced by  $bA$  for some  $b \gg 1$ . Since our boundary divisor  $A$  is larger than some ample divisor  $A_k$ , there are only finitely many  $(K_X + D + A)$ -negative extremal rays  $\mathbb{R}_{>0}[\ell]$  in  $\overline{NE}(X)$ . See the cone theorem [16, Theorem 3.7] or [4, Corollary 3.8.2]. We may assume that all these  $\ell$  satisfy  $A \cdot \ell = 0$  and  $(K_X + D) \cdot \ell < 0$ ; otherwise, we would have  $(K_X + D + A) \cdot \ell > 0$  for all these finitely many  $\mathbb{R}_{>0}[\ell]$ , and hence  $K_X + D + A$  is nef, after replacing  $A$  by a large multiple. Since  $A = \sum_{i=1}^n L_i$  and each  $L_i$  is nef,  $A \cdot \ell = 0$  means  $L_i \cdot \ell = 0$  for all  $i$ . Since  $L_i \cdot g_*\ell = \chi_i(g)L_i \cdot \ell = 0$  and hence  $A \cdot g_*\ell = 0$ , and

$$(K_X + D) \cdot g_*\ell = g^*(K_X + D) \cdot \ell = (K_X + D) \cdot \ell < 0$$

this  $g_*\ell$  ( $= g(\ell)$  as a set, also an extremal curve) satisfies the same conditions as  $\ell$ . So these finitely many extremal rays  $\mathbb{R}_{>0}[\ell]$  are permuted by  $G$  and may be assumed to be fixed by  $G$ , after replacing  $G$  by a finite-index subgroup.

Now we run the  $M$ -directed LMMP for the pair  $(X, D + A)$  as in [3, Definition 2.4]. Remember that we may assume that  $K_X + D + A$  is not nef, while  $K_X + D + A + M$  is nef by the choice of  $M$ . There is an extremal ray  $R = \mathbb{R}_{>0}[\ell]$  such that for

$$\lambda_0 := \inf\{\alpha \geq 0 \mid K_X + D + A + \alpha M \text{ is nef}\}$$

we have  $K_X + D + A + \lambda_0 M$  is nef,  $(K_X + D + A) \cdot \ell < 0$  and  $(K_X + D + A + \lambda_0 M) \cdot \ell = 0$ . As mentioned above, such  $\ell$  satisfies  $A \cdot \ell = 0$  and hence  $L_i \cdot \ell = 0$ ,  $(K_X + D) \cdot \ell < 0$ , and we may assume that the extremal ray  $R$  is  $G$ -stable. Let

$$f = \text{Contr}_R : X \rightarrow Y$$

be the extremal contraction as in 3.12. Since  $R$  is  $G$ -stable and  $f$  is completely determined by  $R$ , the action of  $G$  on  $X$  descends to a biregular action of  $G$  on  $Y$  such that the morphism  $f : X \rightarrow Y$  is  $G$ -equivariant. We consider the 4 cases in 3.12 separately.

Case (I) (Fano type). Since the Picard number  $\rho(X) = 1$ , we take  $H$  to be an ample generator of  $\text{NS}(X)/(\text{torsion}) \cong \mathbb{Z}$ . Then we have  $\text{Aut}(X) \leq \text{Aut}_{[H]}(X)$ . Hence, using [17] as in the proof of Lemma 3.1,  $\text{Aut}(X)$  is a finite extension of the identity component  $\text{Aut}_0(X)$ , so  $\text{Aut}(X)$  and hence  $G$  are of null entropy. This contradicts the assumption that every element of  $G \setminus \{\text{id}\} = \mathbb{Z}^{\oplus n-1} \setminus \{\text{id}\}$  is of positive entropy and that  $n - 1 \geq 1$ .

Case (II) (Fano fibration). Since  $f : X \rightarrow Y$  is a non-trivial  $G$ -equivariant fibration,  $\text{rank}(G) \leq \dim X - 2 = n - 2$ , by [24, Lemma 2.10]. This contradicts the fact  $\text{rank}(G) = n - 1$ .

Case (III) (divisorial). By Lemma 3.1,  $Y$  and  $G$  satisfy Hyp(A). Since  $L_j \cdot \ell = 0$ , a property of the contraction  $f$  implies that  $L_j = f^*L(1)_j$  for some  $\mathbb{R}$ -Cartier divisor  $L(1)_j$  on  $Y$ . See [14, Lemma 3-2-5]. We have  $L(1)_j = f_*L_j$  by the projection formula. Since  $L_j$  is nef, so is  $L(1)_j$ . Clearly, if  $g^*L_j = \chi(g)L_j$ , then  $g^*L(1)_j = \chi(g)L(1)_j$ . Set

$$(X(1), D(1) + A(1)) := (Y, f_*(D + A)).$$

We can continue the  $G$ -equivariant LMMP.

Case (IV) (flip). Since the flip  $X^+$  is uniquely determined by the extremal ray  $R$  (which is stabilized by  $G$ ), there is a biregular action of  $G$  on  $X^+$ . By Lemma 3.1,  $X^+$  and  $G$  satisfy Hyp(A). As in Case (III), we have  $L_j = f^*f_*L_j$  and  $f_*L_j$  is an  $\mathbb{R}$ -Cartier nef divisor on  $Y$ . Let  $L(1)_j$  be the pullback by the birational morphism  $X^+ \rightarrow Y$ . Then it is an  $\mathbb{R}$ -Cartier nef divisor. Clearly, if  $g^*L_j = \chi(g)L_j$ , then  $g^*L(1)_j = \chi(g)L(1)_j$ . Set

$$(X(1), D(1) + A(1)) := (X^+, D^+ + A^+).$$

We can continue the  $G$ -equivariant LMMP.

Since we have settled the cases of Fano type and Fano fibration and since a divisorial contraction decreases the Picard number, setting

$$(X, D + A) = (X(0), D(0) + A(0))$$

we may assume that we have a sequence

$$(X(0), D(0) + A(0)) \xrightarrow{\tau_0} (X(1), D(1) + A(1)) \xrightarrow{\tau_1} \dots \dashrightarrow (X(s), D(s) + A(s)) \xrightarrow{\tau_s} \dots$$

of flips  $\tau_i$  corresponding to the extremal rays  $\mathbb{R}_{>0}[\ell_i]$  with  $(K_{X(i)} + D(i) + A(i)) \cdot \ell_i < 0$  and  $(K_{X(i)} + D(i) + A(i) + \lambda_i M(i)) \cdot \ell_i = 0$ . Here  $D(i)$ ,  $A(i)$  and  $M(i)$  on  $X(i)$  are the direct images of  $D$ ,  $A$  and  $M$  (which is the fixed ample divisor on  $X$ ). The real number  $\lambda_i$  satisfies:

$$\lambda_i = \inf\{\alpha \geq 0 \mid K_{X(i)} + D(i) + A(i) + \alpha M(i) \text{ is nef}\}.$$

If  $K_{X(s)} + D(s) + A(s)$  is nef for some  $s$ , then we are done. Suppose this is not the case for any  $s \geq 0$ . Then we have an infinite sequence of flips

$$\tau_i : (X(i), D(i) + A(i)) \dashrightarrow (X(i + 1), D(i + 1) + A(i + 1))$$

as above.

Since our pair  $(X, D + A)$  is  $\mathbb{Q}$ -factorial klt and the boundary divisor  $D + A$  is a big divisor and larger than an ample divisor  $A_k$  as in Lemma 3.6, we can apply [4, Corollary 1.4.2] or [3, Theorem 1.9 (i)]. Thus the above LMMP must terminate. Hence  $K_{X(s)} + D(s) + A(s)$  must be nef for some  $t \geq 0$ . This proves Lemma 3.13.  $\square$

We continue the proof of Proposition 3.11. Replacing  $X$  by the  $X(s)$  in Lemma 3.13, we may assume that we already have  $K_X + D + A$  nef and big for the  $A$  in Lemma 3.6. By the base point free theorem ([16, Theorem 3.3] or [4, Theorem 3.9.1]), there exist a birational morphism  $\gamma : X \rightarrow Y$  onto a normal projective variety  $Y$  and an ample  $\mathbb{R}$ -Cartier divisor  $B$  on  $Y$  such that  $K_X + D + A = \gamma^*B$ . By the projection formula,  $B = K_Y + D_Y + A_Y$  where  $D_Y = \gamma_*D$  and  $A_Y = \gamma_*A$  are the direct images. So

$$K_X + D + A = \gamma^*(K_Y + D_Y + A_Y).$$

**Lemma 3.14.** *Replacing  $G$  by a finite-index subgroup and  $A$  by a large multiple, we have:*

- (1) *The action  $G$  on  $X$  descends to a biregular action on  $Y$  so that  $\gamma : X \rightarrow Y$  is  $G$ -equivariant.*
- (2)  *$K_Y + D_Y$  is  $\mathbb{R}$ -Cartier and  $K_X + D = \gamma^*(K_Y + D_Y)$ . The pair  $(Y, D_Y + A_Y)$  and hence  $(Y, D_Y)$  have only klt singularities (cf. [16, Proposition 2.41, Corollary 2.39]).*
- (3)  *$L_j = \gamma^*L(Y)_j$  ( $1 \leq j \leq n$ ) for some  $\mathbb{R}$ -Cartier nef divisor  $L(Y)_j$  on  $Y$  which is also a common eigenvector of  $G$ . Hence  $A_Y = \gamma_*A = \sum_{j=1}^n L(Y)_j$  is a nef and big  $\mathbb{R}$ -Cartier divisor, and  $A = \gamma^*A_Y$ .*
- (4)  *$Y$  and  $G$  satisfy Hyp(A). Our  $Y, G$  and  $A_Y = \sum L(Y)_j$  satisfy all the properties in Lemmas 3.4, 3.6 and 3.7 for  $X, G$  and  $A = \sum L_j$ .*

*Proof.* Since  $A = A_k + E/k$  with  $A_k$  ample, the extremal rays  $\mathbb{R}_{>0}[\ell]$  contracted by  $\gamma$  (i.e., perpendicular to the nef and big divisor  $K_X + D + A = \gamma^*(K_Y + D_Y + A_Y)$ ) satisfy

$$0 = \gamma^*(K_Y + D_Y + A_Y) \cdot \ell = (K_X + D_Y + E/k + \varepsilon A_k) \cdot \ell + (1 - \varepsilon)A_k \cdot \ell > (K_X + D_Y + E/k + \varepsilon A_k) \cdot \ell$$

and hence are all  $(K_X + D_Y + E/k + \varepsilon A_k)$ -negative. Thus there are only finitely many such  $\mathbb{R}_{>0}[\ell]$  by the bigness of the (new) boundary divisor  $D_Y + E/k + \varepsilon A_k$ . See the cone theorem [16, Theorem 3.7]. Note that  $\gamma : X \rightarrow Y$  is also a birational contraction of a  $(K_X + D_Y + E/k + \varepsilon A_k)$ -negative extremal face; see the contraction theorem [14, Theorem 3-2-1]. If  $\gamma : X \rightarrow Y$  is an isomorphism, then the lemma is clear. Otherwise, there are such  $\ell$ ; replacing  $A$  by a large multiple, we may assume that the above finitely many extremal rays  $\mathbb{R}_{>0}[\ell]$  (contracted by  $\gamma$ ) satisfy

$$0 = (K_X + D + A) \cdot \ell = (K_X + D) \cdot \ell = A \cdot \ell.$$

Since  $A = \sum_{j=1}^n L_j$  is a sum of nef divisors, we have  $L_j \cdot \ell = 0$ .

Note that if  $\ell$  satisfies the condition  $(K_X + D) \cdot \ell = L_j \cdot \ell = 0$ , then so do the  $G$ -images of  $\ell$ . Since there are only finitely many such extremal rays  $\mathbb{R}_{>0}[\ell]$ , we may assume that all of them are  $G$ -stable, after replacing  $G$  by a finite-index subgroup. This and the fact that the map  $\gamma$  is the contraction of the extremal face generated by these extremal rays imply that  $G$  descends to an action on  $Y$  such that  $\gamma$  is  $G$ -equivariant. By Lemma 3.1,  $Y$  and  $G$  satisfy Hyp(A).

Now the cone theorem ([16, Theorem 3.7] or [14, Theorem 3-2-1 or Lemma 3-2-5]) and that  $L_j \cdot \ell = 0$  for all the above extremal  $\ell$  imply that  $L_j = \gamma^*L(Y)_j$  for some  $\mathbb{R}$ -Cartier divisor  $L(Y)_j$  on  $Y$  which is a nef common eigenvector of  $G$ , since so is  $L_j$ . Further,  $L(Y)_j = \gamma_*L_j$  by the projection formula. By the same cone theorem,  $(K_X + D) \cdot \ell = 0$  implies that  $K_Y + D_Y = \gamma_*(K_X + D)$  is an  $\mathbb{R}$ -Cartier divisor and  $K_X + D = \gamma^*(K_Y + D_Y)$ .

Since  $(X, D + A)$  is klt, so is  $(Y, D_Y + A_Y)$  (and hence  $(Y, D_Y)$ ) by the display preceding Lemma 3.14. This proves Lemma 3.14. □

We continue the proof of Proposition 3.11. Setting  $\tau_s := \gamma$ , by the arguments so far, it remains to prove Proposition 3.11 (7). Its first part is a consequence of Lemmas 3.8 and 3.9, thanks to (3). For the second part of (7), we have only to prove it on  $Y$ , since the sequence  $X \dashrightarrow Y$  is  $G$ -equivalent. Indeed, if  $\tau_i : X(i) \rightarrow X(i + 1)$  is divisorial, then  $A(i) = \tau_i^*A(i + 1)$ ; hence  $\text{Null}(A(i))$  is just the inverse

of  $\text{Null}(A(i+1))$ , using Lemma 3.9. If  $X(i) \rightarrow Y'$  and  $X(i)^+ = X(i+1) \rightarrow Y'$  are the flipping contractions, then  $A(i)$  and  $A(i+1)$  are the pullbacks of some nef and big divisor  $A_{Y'}$  on  $Y'$ , so  $\text{Null}(A(i))$  is the inverse of  $\text{Null}(A_{Y'})$  by using Lemma 3.9 and noting that  $Y'$  and  $G$  also satisfy Hyp(A) by Lemma 3.1.

For the second part of (7), to prove the vanishing of  $A_{Y|Z}$  on  $Y$ , by Lemma 3.7, we may assume that  $k := \dim Z \geq 2$ . As in [26, Lemma 2.9], we prove first:

$$(2) \quad ((K_Y + D_Y + A_Y)|_Z)^{k-1} \cdot A_{Y|Z} = (K_Y + D_Y + A_Y)^{k-1} \cdot A_Y \cdot Z = 0.$$

Indeed, since  $A_Y = \sum L(Y)_i$ , the above mid-term is the summation of the terms

$$(K_Y + D_Y)^{k-1-t} \cdot L(Y)_{j_1} \cdots L(Y)_{j_t} \cdot L(Y)_i \cdot Z$$

where  $0 \leq t \leq k-1 \leq n-2$ . Now the vanishing of each term above can be verified as in Lemma 3.7, since  $g^*(K_Y + D_Y) = K_Y + D_Y$  for  $g \in G$ . The equality (2) above is proved.

The equality (2) and ampleness of  $K_Y + D_Y + A_Y$ , restricted to  $Z$ , imply that

$$((K_Y + D_Y + A_Y)|_Z)^{k-2} \cdot (A_{Y|Z})^2$$

is a non-positive scalar by Lemma 3.2, and hence is zero since  $K_Y + D_Y + A_Y$  and  $A_Y$  are nef. Thus  $A_{Y|Z} \equiv 0$  by Lemma 3.2. This completes the proof of Proposition 3.11.

#### 4. PROOFS OF THEOREMS 2.4 AND 2.5 AND PROPOSITION 2.6

We first prove Theorem 2.4. Theorem 2.4 is true under condition (iii) by Lemma 3.10. Assume condition (i). If  $X$  is not uniruled, then (so is its resolution and hence)  $K_X$  is a pseudo-effective divisor by the uniruledness criterion due to Miyaoka-Mori and Boucksom-Demailly-Paun-Peternell. Thus Hyp(B) holds by letting  $D = 0$ , so condition (ii) holds. If  $X$  is uniruled, the action of  $G$  on  $X$  descends to a biregular action on the base of the *special* MRC (maximal rationally connected) fibration constructed in [20, Theorem 4.18], with general fibres rationally connected varieties. The maximality of  $\text{rank}(G) = n-1$  and [24, Lemma 2.10] imply that this  $G$ -equivariant MRC must be trivial. Since  $X$  is uniruled and hence the general fibre is not a point, the triviality means that the base is a point. So  $X$  is rationally connected, contradicting condition (i).

From now on, we will prove Theorem 2.4 under condition (ii).

Since  $(X, D)$  is klt, if we set  $D^\varepsilon := (1 + \varepsilon)D$ , then  $(X, D^\varepsilon)$  is still klt for small  $\varepsilon \in (0, 1)$ ; see [16, Corollary 2.35]. Choose  $A$  as in Lemma 3.6 such that  $(X, D^\varepsilon + A)$  is klt. Applying Proposition 3.11 to the pair  $(X, D^\varepsilon + A)$ , we get birational maps

$$\tau_i : X(i) \dashrightarrow X(i+1)$$

where  $\tau_j$  ( $0 \leq j < s$ ) is either a divisorial contraction or a flip, corresponding to a  $(K_{X(j)} + D^\varepsilon(j))$ -negative extremal ray, and

$$\tau_s : X(s) \rightarrow X(s+1) = Y$$

is a birational morphism. Here we let

$$D(i), D^\varepsilon(i), A(i)$$

on  $X(i)$  be the direct images of  $D, D^\varepsilon, A$ , respectively; note that

$$D^\varepsilon(i) = (1 + \varepsilon)D(i).$$

The assertions (4) - (6) follow from Proposition 3.11.

Next we show (7). Since  $K_X + D$  and hence  $K_X + D^\varepsilon$  are pseudo-effective, so are their direct images  $K_{X(s)} + D(s)$  and  $K_{X(s)} + D^\varepsilon(s)$ . To simplify the notation, we use

$$\gamma : X \rightarrow Y$$

to denote the  $\tau_s : X(s) \rightarrow X(s+1) = Y$  in Proposition 3.11, and let

$$D = D(s), \quad D^\varepsilon = D^\varepsilon(s) = (1 + \varepsilon)D, \quad A = A(s).$$

By the  $\sigma$ -decomposition for pseudo-effective divisors in [19, III. §1.12],

$$K_X + D^\varepsilon = P + N$$

where  $P$  is in the closed movable cone (generated by fixed-component free Cartier divisors) and  $N$  is an effective divisor.

Replacing  $G$  by a finite-index subgroup, we may and will assume that every  $G$ -periodic divisor on  $X$  is stabilized by  $G$ . See Proposition 3.11 (7).

*Claim 4.1.* We have  $P \equiv 0$ , so  $K_X + D^\varepsilon \equiv N$ . Every irreducible component of  $N$  is stabilized by  $G$ .

*Proof.* We prove Claim 4.1. The uniqueness of the  $\sigma$ -decomposition implies the assertion that  $g^*P \equiv P$  and  $g^*N = N$  for any  $g \in G$  (so every component of  $N$  is  $G$ -periodic and hence stabilized by  $G$ ). Indeed,

$$K_X + D^\varepsilon = g^*(K_X + D^\varepsilon) = g^*P + g^*N$$

and  $(K_X + D^\varepsilon) - g^*N = g^*P$  is movable, so  $g^*N \geq N$  by the minimality of the ‘negative part’  $N$ ; see [19, III, Proposition 1.14]. Applying the above to  $g^{-1}$  we get  $N \geq g^*N$ . The assertion follows.

Now

$$P \cdot M \cdot M_1 \cdots M_{n-2} = P|_M \cdot M_1|_M \cdots M_{n-1}|_M \geq 0$$

for every irreducible divisor  $M$  and nef  $\mathbb{R}$ -Cartier divisors  $M_i$  because  $P|_M$  is a pseudo-effective divisor on  $M$ ; also  $P \cdot A^{n-1} = 0$  for the nef and big divisor  $A$ , by Lemma 3.7. Thus we may apply [21, Lemma 2.2] (by reducing to the hard Lefschetz theorem) to deduce that  $P \equiv 0$ . This proves Claim 4.1. □

*Claim 4.2.*  $N = 0$ . Hence  $K_X + D^\varepsilon \sim_{\mathbb{Q}} 0$ .

*Proof.* We prove Claim 4.2. By Proposition 3.11,  $K_X + D^\varepsilon + A$  is a nef and big divisor. Further,

$$(K_X + D^\varepsilon + A)|_Z = (K_X + D^\varepsilon)|_Z \equiv N|_Z$$

is also nef for every  $G$ -periodic proper subvariety  $Z$  of  $X$ , especially for  $Z = N_i$ , a component of  $N$ . Thus

$$N \cdot M \cdot M_1 \cdots M_{n-2} = N|_M \cdot M_1|_M \cdots M_{n-1}|_M \geq 0$$

for every irreducible divisor  $M$  and nef  $\mathbb{R}$ -Cartier divisors  $M_i$  because  $N|_M$  is a pseudo-effective divisor on  $M$  (even when  $M = N_i$ ); also  $N \cdot A^{n-1} = 0$  for the nef and big divisor  $A$ , by Lemma 3.7. As in Claim 4.1, applying [21, Lemma 2.2], we get  $N \equiv 0$ , so the effective divisor  $N = 0$ . Thus  $K_X + D^\varepsilon \equiv N = 0$ . Since  $(X, D^\varepsilon)$  is klt by Proposition 3.11, the known abundance theorem in the case of zero log Kodaira dimension implies that  $K_X + D^\varepsilon \sim_{\mathbb{Q}} 0$ . See [19, V. Corollary 4.9]. This proves Claim 4.2. □

We return to the proof of Theorem 2.4. Now  $0 \equiv K_X + D^\varepsilon = (K_X + D) + \varepsilon D$ . This and the pseudo-effectivity of  $K_X + D$  as assumed imply that  $K_X + D \equiv 0$  and  $\varepsilon D = 0$ . Hence the effective divisor  $D = 0$ , so  $D^\varepsilon = (1 + \varepsilon)D = 0$ . Thus  $K_X \sim_{\mathbb{Q}} 0$  by Claim 4.2.

Note that in this proof of Theorem 2.4, we have been applying Proposition 3.11 to the pair  $(X, D^\varepsilon)$  and use  $\gamma : X \rightarrow Y$  to denote  $\tau_s : X(s) \rightarrow X(s + 1) = Y$  in Proposition 3.11. The latter says that  $K_Y + \gamma_* D^\varepsilon$  is an  $\mathbb{R}$ -Cartier divisor (with  $\gamma_* D^\varepsilon = 0$  now) and it has  $K_X + D^\varepsilon (= K_X \text{ now})$  as its pullback by  $\gamma$ . Thus  $0 \sim_{\mathbb{Q}} K_X = \gamma^* K_Y$ . Hence  $K_Y \sim_{\mathbb{Q}} 0$ . This proves the assertion (7).

By Proposition 3.11, we have the ampleness of  $K_Y + \gamma_* D^\varepsilon + A_Y (\equiv A_Y \text{ now})$ . Thus we can apply Lemma 3.10 to  $Y$  and  $G$  (cf. Proposition 3.11 (2) and (3)). In particular, assertions (1) - (3) are true. This proves Theorem 2.4 under condition (ii).

We have completed the proof of Theorem 2.4.

*Next we prove Main Theorem 2.5.* Condition (1) implies condition (2) by Theorem 2.4 with  $W := Y$ ; for the remark about “We can take  $\tilde{G} = G \dots$ ”, see [26, Lemma 2.4, §2.15]. Assume condition (2). By Proposition 2.3, our  $W$  and  $G$  satisfy Hyp(B’’) in which the  $G$ -equivariant birational model of  $W$  (also denoted as  $X$  there) and  $G$  clearly satisfy Hyp(B) with  $D = \delta\Delta$ . This proves Main Theorem 2.5. (See also Lemma 3.1 for the birational nature of Hyp(A).)

*Finally we prove Proposition 2.6.* Replacing  $G$  by a finite-index subgroup, we may assume that every  $G$ -periodic divisor is stabilized by  $G$ . As in the proof of Theorem 2.4, let  $K_X + D = P + N$  be the  $\sigma$ -decomposition, where  $P$  is movable and  $N = \sum n_i N_i$  with  $N_i$  irreducible and  $n_i > 0$ . As in Claim 4.1, the uniqueness of such decomposition implies that both  $P$  and  $N$  are stabilized by  $G$ ,  $P \equiv 0$ , and each  $N_i$  is stabilized by  $G$ .

Since  $(X, D)$  is dlt, if we set  $D_\varepsilon := (1 - \varepsilon)D$ , then  $(X, D_\varepsilon)$  is klt for every  $\varepsilon \in (0, 1]$ ; see [16, Proposition 2.41, Corollary 2.39]. Take  $\varepsilon$  small such that  $\varepsilon < \min\{n_i\}$  if  $N \neq 0$ . Choose  $A$  as in Lemma 3.6 such that  $(X, D_\varepsilon + A)$  is klt. We apply Proposition 3.11 to the pair  $(X, D_\varepsilon + A)$ . So we get birational maps

$$\tau_i : X(i) \dashrightarrow X(i + 1)$$

( $0 \leq i \leq s$ ) where  $\tau_j$  ( $j < s$ ) is either a divisorial contraction or a flip, corresponding to a  $(K_{X(j)} + D_\varepsilon(j))$ -negative extremal ray, and  $\tau_s : X(s) \rightarrow X(s + 1) = Y$  is a birational morphism. Here we let  $D(i), D_\varepsilon(i)$  and  $A(i)$  on  $X(i)$  be the direct images of  $D, D_\varepsilon$  and  $A$ ; note that  $D_\varepsilon(i) = (1 - \varepsilon)D(i)$ . Proposition 2.6 (1) follows from Proposition 3.11.

$K_X + D \equiv P + N \equiv N$  implies that  $K_{X(s)} + D(s) \equiv N(s)$ , where  $N(s)$  is the direct image of  $N$ . Set  $\tau = \tau_{s-1} \circ \dots \circ \tau_0 : X = X_0 \dashrightarrow X(s)$ .

*Claim 4.3.* The pair  $(X(s), D(s))$  has only  $\mathbb{Q}$ -factorial log canonical singularities, and  $K_{X(s)} + \Delta(s) \sim_{\mathbb{Q}} 0$ .

*Proof.* We prove Claim 4.3. Let  $N(s) - \varepsilon D(s) = D' - D''$ , with  $D'$  and  $D''$  effective and having no common components. By Proposition 3.11, we have the (big and) nefness of

$$K_{X(s)} + D_\varepsilon(s) + A(s) \equiv N(s) - \varepsilon D(s) + A(s) = D' - D'' + A(s),$$

and hence the nefness of

$$(D' - D'' + A(s))|_{D'_i} = (D' - D'')|_{D'_i}$$

for every component  $D'_i$  of  $D'$ . Thus  $D'_{|D'_i}$  is pseudo-effective, so  $D' = 0$ , as in Claim 4.2. Now  $N(s) - \varepsilon D(s) = -D'' \leq 0$ . This and the choice of  $\varepsilon$  then imply that the direct image  $N(s)$  of  $N$  is 0. Thus  $K_{X(s)} + D(s) \equiv 0$ . Hence  $K_X + D = \tau^*(K_{X(s)} + D(s)) + N$ . Since the pair  $(X, D)$  has only  $\mathbb{Q}$ -factorial (dlt and hence) log canonical singularities (and by taking its log resolution), so has  $(X(s), D(s))$ . This and the known abundance theorem in the case of numerical Kodaira dimension zero imply that (the numerically trivial divisor)  $K_{X(s)} + D(s) \sim_{\mathbb{Q}} 0$ . This proves Claim 4.3.  $\square$

We return to the proof of Proposition 2.6. By Proposition 3.11,  $\tau_s : X(s) \rightarrow Y$  is the contraction of an extremal face. Since  $K_{X(s)} + D(s) (\sim_{\mathbb{Q}} 0)$  is perpendicular to every extremal ray of the face,  $\tau_{s*}(K_{X(s)} + D(s)) = K_Y + D_Y$  is an  $\mathbb{R}$ -Cartier divisor and  $K_{X(s)} + D(s) = \tau_s^*(K_Y + D_Y)$ . This and Claim 4.3 imply that  $K_Y + D_Y \sim_{\mathbb{Q}} 0$  and the pair  $(Y, D_Y)$  has only log canonical singularities. This proves Proposition 2.6 (2).

By Proposition 3.11, we have the ampleness of  $K_Y + \tau_{s*}D_\varepsilon(s) + A_Y = K_Y + (1 - \varepsilon)D_Y + A_Y \sim_{\mathbb{Q}} -\varepsilon D_Y + A_Y$  and hence the ampleness of

$$(-\varepsilon D_Y + A_Y)|_Z = -\varepsilon D_Y|_Z$$

for every  $G$ -periodic positive-dimensional proper subvariety  $Z$  of  $Y$ . So  $Z$  is contained in the support of  $D_Y$ . This proves Proposition 2.6 (3), hence the whole of Proposition 2.6.

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