TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 1, January 2017, Pages 711–754
http://dx.doi.org/10.1090/tran6688
Article electronically published on April 14, 2016

NON-SPLAT SINGULARITY FOR THE ONE-PHASE MUSKAT PROBLEM

DIEGO CÓRDOBA AND TANIA PERNAS-CASTAÑO

Abstract. For the water wave equations, the existence of splat singularities has been shown, i.e., the interface self-intersects along an arc in finite time. The aim of this paper is to show the absence of splat singularities for the incompressible fluid dynamics in porous media.

1. Introduction

The Muskat problem [19] models the evolution of the interface between two fluids of different characteristics in porous media, where the velocity of the fluid is given by Darcy's law:

$$\frac{\mu}{\kappa}u = -\nabla p - (0, g\rho)$$

where $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, $u = (u_1(x,t), u_2(x,t))$ is the incompressible velocity (i.e. $\nabla \cdot u = 0$), $p = p(x,t)$ is the pressure, $\mu = \mu(x,t)$ is the dynamic viscosity, κ is the permeability of the isotropic medium, $\rho = \rho(x,t)$ is the liquid density and g is the acceleration due to gravity. The free boundary is caused by the discontinuity between the densities and viscosities of the fluids; the quantities (μ, ρ) are defined by

$$(\mu, \rho)(x_1, x_2, t) := \begin{cases} (\mu^1, \rho^1), & x \in \Omega^1(t), \\ (\mu^2, \rho^2), & x \in \Omega^2(t) = \mathbb{R}^2 - \Omega^1(t), \end{cases}$$

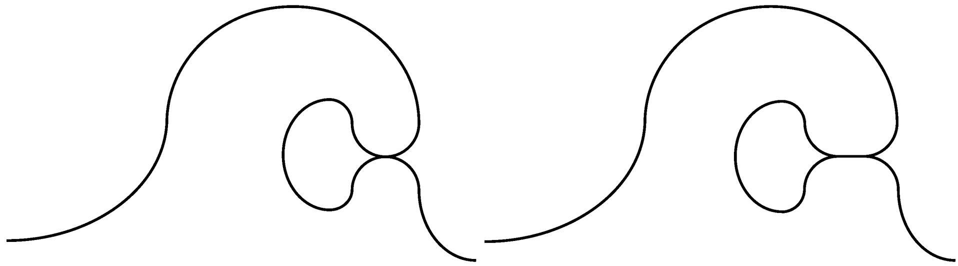
where μ^1, ρ^1, μ^2 and ρ^2 are constants.

We will study only one of the two types of finite time singularities shown for water waves in [3]: the splash and splat singularities. The splash-type singularity (Figure 1(a)) corresponds to the case where the fluid interface self-intersects at a single point. This kind of singularity also occurs for the Muskat problem as proved in [2].

In this paper, we will focus on the splat-type singularity (Figure 1(b)). This singularity is a variation of the former in which the fluid interface self-intersects along an arc. This scenario has been shown to arise for the incompressible Euler equations in the water waves form; see [3], which considers the evolution of the free boundary of a water region in vacuum and irrotational velocity. In [11], these singularities have also been shown to exist for the case with vorticity.

For the Muskat problem, splash singularity cannot be developed in the case in which $\mu^1 = \mu^2$ and $\rho^1 \neq \rho^2$; for more details see [15]. For similar results about two-fluids interfaces see [14], [12]. However, the splash can be achieved with $\mu^1 = \rho^1 = 0$ where $\mathbb{R}^2 - \Omega(t)$ corresponds to the dry region (see [2]).

Received by the editors September 2, 2014 and, in revised form, January 22, 2015.
2010 *Mathematics Subject Classification.* Primary 35Q35.



(a) Splash singularity (b) Splat singularity

FIGURE 1. Finite time singularities

The aim of this work is to show the absence of splat singularities in the case of an interface between an incompressible irrotational fluid and a dry region in porous media. Thus, $\mu^1 = \rho^1 = 0$, i.e.,

$$(\mu, \rho)(x_1, x_2, t) := \begin{cases} (\mu^2, \rho^2), & x \in \Omega(t), \\ (0, 0), & x \in \mathbb{R}^2 - \Omega(t). \end{cases}$$

Let the free boundary be parametrized by

$$\partial\Omega = \{z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) : \alpha \in \mathbb{R}\}$$

so that the periodic condition

$$(z_1(\alpha + 2k\pi, t), z_2(\alpha + 2k\pi, t)) = (z_1(\alpha, t) + 2k\pi, z_2(\alpha, t))$$

holds with initial data $z(\alpha, 0) = z_0(\alpha)$.

From Darcy's law, we deduce that the fluid is irrotational, i.e. $\omega = \nabla \times u = 0$, in the interior of the domain Ω. Therefore, the vorticity is concentrated on the free boundary $z(\alpha, t)$ by a Dirac distribution as follows:

$$\omega(x, t) = \nabla^{\perp} \cdot u(x, t) = \varpi(\alpha, t)\delta(x - z(\alpha, t))$$

where $\varpi(\alpha, t)$ represents the vorticity strength.

The interface $z(\alpha, t)$ evolves with an incompressible velocity field satisfying the Biot-Savart law, which can be explicitly computed and is given by the Birkhoff-Rott integral of the amplitude ϖ along the interface $z(\alpha, t)$:

$$BR(z, \varpi)(\alpha, t) = \frac{1}{2\pi} PV \int_{\mathbb{R}} \frac{(z(\alpha, t) - z(\beta, t))^{\perp}}{|z(\alpha, t) - z(\beta, t)|^2} \varpi(\beta, t) d\beta. \tag{1}$$

We can subtract any term in the tangential direction to the curve in the velocity field without modifying the geometric evolution of the curve

$$z_t(\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t). \tag{2}$$

A wise choice of $c(\alpha, t)$, namely

$$\begin{aligned} c(\alpha, t) = &\frac{\alpha + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\beta z(\beta, t)}{|\partial_\beta z(\beta, t)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) d\beta \\ &- \int_{-\pi}^{\alpha} \frac{\partial_\beta z(\beta, t)}{|\partial_\beta z(\beta, t)|^2} \cdot \partial_\beta BR(z, \varpi)(\beta, t) d\beta, \end{aligned} \tag{3}$$

allows us to remove the dependence on α from the length of the tangent vector to $z(\alpha,t)$ (for more details see [8]):

$$|\partial_\alpha z(\alpha,t)|^2 = A(t).$$

We can close the system using Darcy's law and taking the dot product with $\partial_\alpha z(\alpha,t)$. It is easy to relate ϖ and the free boundary by (see [8]):

$$\varpi(\alpha,t) = -2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t) - 2\kappa g\frac{\rho^2}{\mu^2}\partial_\alpha z_2(\alpha,t). \tag{4}$$

For the stability of the problem we consider the Rayleigh-Taylor condition. Rayleigh [20] and Saffman-Taylor [21] gave a condition that must be satisfied for the linearized model in order to have a solution locally in time, namely that the normal component of the pressure gradient jump at the interface has to have a distinguished sign. This condition can be written as

$$\sigma(\alpha,t) = \frac{\mu^2}{\kappa}BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha^\perp z(\alpha,t) + g\rho^2\partial_\alpha z_1(\alpha,t) > 0.$$

Using Hopf's lemma, the Rayleigh-Taylor condition is satisfied for $\mu^1=\rho^1=0$ (see [2]). For the case of equal viscosities ($\mu^1=\mu^2$), this condition holds when the more dense fluid lies below the interface [4].

This stability has been used to prove local existence in Sobolev spaces, when $\mu^1\neq\mu^2$ and $\rho^1\neq\rho^2$, in [8]. For improvements for local existence results in the case $\mu^1=\rho^1=0$, see [5]. When $\mu^1=\mu^2$ there is local existence and instant analyticity in the stable case; see [4] and [10]. For small data, the fact that $\sigma>0$ has been used to prove global existence as we can check in [6], [22], [13] and [17]. Furthermore, there exists initial data with $\sigma>0$ that in finite time turns to $\sigma<0$ (see [4] and [16]), and later in finite time the interface breaks down [1].

Finally we introduce the function that measures the arc-chord condition

$$\mathcal{F}(z)(\alpha,\beta,t) = \frac{\beta^2}{|z(\alpha)-z(\alpha-\beta)|^2}, \quad \alpha,\beta\in\mathbb{R},$$

with

$$\mathcal{F}(z)(\alpha,0,t) = \frac{1}{|\partial_\alpha z(\alpha,t)|^2}.$$

The main theorem of this paper is the following:

Theorem 1.1. *Let $z_0(\alpha)\in H^k(\mathbb{T})$ for $k\geq 4$ and $\mathcal{F}(z_0)(\alpha,\beta)\in L^\infty$. Then the Muskat problem* (1)-(4) *will not break down in a splat singularity; i.e., there is no time where there exist disjoint intervals $I_1, I_2\in\mathbb{R}$ such that $z(I_1,t)=z(I_2,t)$.*

In order to prove this theorem we have organized the paper as follows.

In sections 2, 3 and 4 we present several a priori estimates that provide instant analyticity for a single curve that initially satisfies the arc-chord and Rayleigh-Taylor conditions. Section 5 is devoted to proving that the region of analyticity does not collapse to the real axis as long as the curve remains smooth and the arc-chord condition remains bounded.

Instant analyticity and exponential decay of the strip of analyticity are shown in [4] for the case where both fluids have equal viscosities ($\mu^1=\mu^2$). In such a case, the formula for the strength of the vorticity is simpler:

$$\varpi(\alpha,t) = -(\rho^2-\rho^1)\partial_\alpha z_2(\alpha,t).$$

In our scenario, the one-phase Muskat problem, the expression (4) of the strength of the vorticity involves the Birkhoff-Rott integral.

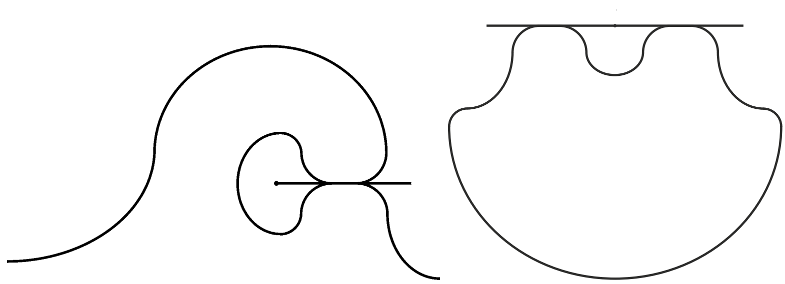
Finally, in section 6, we prove the main theorem using a contradiction argument. The idea of the proof is the following:

Suppose that there exists a splat singularity at time T. If the solution $z(\alpha, t)$ is real-analytic at time T, the formation of a splat singularity would be impossible. This follows from the fact that we would get a real-analytic curve $z(\alpha, t)$ self-intersecting along an arc; therefore $z(\alpha, t)$ should self-intersect at all points.

Since the curve self-intersects, the arc-chord condition fails in our domain Ω, and thus we have no control on the decay of the strip of analyticity. In order to get around this issue it is necessary to apply a transformation defined by $\tilde{z}(\alpha, t)) = P(z(\alpha, t))$ where P is a conformal map (see [3]):

$$P(w) = (\tan(\frac{w}{2}))^{\frac{1}{2}}.$$

This conformal map transforms our domain Ω in $\tilde{\Omega}$, as we can see in Figure 2. The branch of the root will be taken in such a way that it separates the self-intersecting points of the interface.



(a) $\Omega(T)$ domain (b) $\tilde{\Omega}(T)$ domain

FIGURE 2. Finite time singularities

The new contour evolution equation where we handle the splat singularity is (see [2] for more details)

$$\tilde{z}_t(\alpha, t) = Q^2(\alpha, t) BR(\tilde{z}, \tilde{\varpi})(\alpha, t) + \tilde{c}(\alpha)\partial_\alpha \tilde{z}(\alpha, t)$$

where

$$Q^2(\alpha, t) = |\frac{dP}{dw}(z(\alpha, t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha, t)))|^2,$$

$$\tilde{\varpi}(\alpha, t) = -2BR(\tilde{z}, \tilde{\varpi})(\alpha, t) \cdot \partial_\alpha \tilde{z}(\alpha, t) - 2\kappa g \frac{\rho^2}{\mu^2} \partial_\alpha (P_2^{-1}(\tilde{z}(\alpha, t)))$$

and

$$\tilde{c}(\alpha, t) = \frac{\alpha + \pi}{2\pi} \int_{\mathbb{T}} \frac{\partial_\beta \tilde{z}(\beta, t)}{|\partial_\beta \tilde{z}(\beta, t)|^2} \cdot \partial_\beta BR(\tilde{z}, \tilde{\varpi})(\beta, t) d\beta$$
$$- \int_{-\pi}^{\alpha} \frac{\partial_\beta \tilde{z}(\beta, t)}{|\partial_\beta \tilde{z}(\beta, t)|^2} \cdot \partial_\beta BR(\tilde{z}, \tilde{\varpi})(\beta, t) d\beta.$$

Finally we find the Rayleigh-Taylor condition in terms of $\tilde{z}$:

$$\tilde{\sigma}(\alpha, t) = \frac{\mu^2}{\kappa} BR(\tilde{z}, \tilde{\varpi})(\alpha, t) \cdot \partial_\alpha^\perp \tilde{z}(\alpha, t) + \rho^2 g \nabla P_2^{-1}(\tilde{z}(\alpha, t)) \cdot \partial_\alpha^\perp \tilde{z}(\alpha, t). \quad (5)$$

Our final goal in section 6 is to prove instant analyticity and decay of the strip of analyticity for the Muskat problem in the new domain, which allows us to apply our argument of non-splat, i.e., to prove Theorem 1.1.

2. Estimates on $z(\alpha, t)$

Here we show the main estimates that provide instant analyticity of the strip $S(t) = \{\alpha + i\zeta : |\zeta| < \lambda t\}$ for each t. To do that we will need the following estimates from [8]:

$$\|\varpi\|_{H^k} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}), \tag{6}$$

for $k \geq 2$;

$$\|BR(z, \varpi)\|_{H^k} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty} + \|z\|^2_{H^{k+1}}), \tag{7}$$

for $k \geq 2$.

These estimates follow also into the complex strip S, since the time derivative plays no role, and hence any extra term appears in relation with the terms in [8].

Remark 2.1. Inequalities (6) and (7) can also be bounded by a polynomial function; see [9]. In our case, to prove instant analyticity and the decay of the strip, both estimates are valid.

Let λ[1] be given in the definition of $L^2(S)$ and $H^k(S)$:

$$\|z\|^2_{L^2(S)}(t) = \sum_{\pm} \int_{\mathbb{T}} |z(\alpha \pm i\lambda t, t) - (\alpha \pm i\lambda t, 0)|^2 d\alpha,$$

$$\|z\|^2_{H^k(S)}(t) = \|z\|^2_{L^2(S)}(t) + \sum_{\pm} \int_{\mathbb{T}} |\partial^k_\alpha z(\alpha \pm i\lambda t, t)|^2 d\alpha.$$

Remark 2.2. Above $|\cdot|$ is the modulus of a vector in $\mathbb{C}^2$.

2.1. Estimates for the $H^4(S)$ norm.

We shall analyze the evolution of $\|z\|_{H^4(S)}(t)$.

In order to simplify the exposition we write $z(\alpha, t) = z(\alpha)$ for a fixed t, and we denote $\alpha \pm i\lambda t \equiv \gamma$.

It is easy to find that

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}} |z(\alpha \pm i\lambda t) - (\alpha \pm i\lambda t, 0)|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^k(S)}), \tag{8}$$

for $k \geq 3$.

Next, we check that

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}} |\partial^4_\alpha z(\gamma)|^2 d\alpha = \sum_{j=1,2} \frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}} |\partial^4_\alpha z_j(\gamma)|^2 d\alpha,$$

where

$$\frac{1}{2}\frac{d}{dt} \int_{\mathbb{T}} |\partial^4_\alpha z_j(\gamma)|^2 d\alpha = \Re \int_{\mathbb{T}} \overline{\partial^4_\alpha z_j(\gamma)} (\partial_t(\partial^4_\alpha z_j)(\gamma) \pm i\lambda \partial^5_\alpha z_j(\gamma)) d\alpha.$$

[1]At the end of the proof of Theorem 4, we can take any $\lambda < \frac{\min_\alpha(\sigma(\alpha,0))}{2}$.

Then

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\gamma)|^2 d\alpha = \Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z_t(\gamma)d\alpha \pm \Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot i\lambda\partial_\alpha^5 z(\gamma)d\alpha$$
$$\equiv I_1 + I_2.$$

Let us study I_2:

$$I_2 = \Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^5 z(\gamma) i\lambda d\alpha = \int_{\mathbb{T}}\lambda(-\Re(\partial_\alpha^4 z)\Im(\partial_\alpha^5 z)+\Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z))d\alpha$$
$$= 2\lambda\int_{\mathbb{T}}\Im(\partial_\alpha^4 z)\Re(\partial_\alpha^5 z)d\alpha = -2\lambda\int_{\mathbb{T}}\Im(\partial_\alpha^4 z)\Re(\Lambda(H(\partial_\alpha^4 z)))d\alpha$$
$$= -2\lambda\int_{\mathbb{T}}\Lambda^{\frac{1}{2}}(\Im(\partial_\alpha^4 z))\Re(\Lambda^{\frac{1}{2}}H(\partial_\alpha^4 z))d\alpha \leq 2\lambda\|\Lambda^{\frac{1}{2}}\Im(\partial_\alpha^4 z)\|_{L^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}$$
$$\leq 2\lambda\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)},$$

where Λ is defined by the Fourier transform $\widehat{\Lambda f}(\xi) = |\xi|\widehat{f}(\xi)$ and H is the Hilbert transform:

$$\Lambda(f)(x) = \frac{1}{2\pi}PV\int\frac{f(x)-f(y)}{|x-y|^2}dy,$$
$$H(f)(x) = \frac{1}{\pi}PV\int\frac{f(y)}{x-y}dy.$$

Therefore,

$$I_2 \leq 2\lambda\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.$$

Since we have $z_t(\gamma) = BR(z,\varpi)(\gamma) + c(\gamma)\partial_\alpha z(\gamma)$, then

$$I_1 = \Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 BR(z,\varpi)(\gamma)d\alpha + \Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4(c(\gamma)\cdot\partial_\alpha z(\gamma))d\alpha \equiv J_1 + J_2.$$

We will estimate J_1 in the subsections 2.1.1 and 2.1.2 and J_2 in 2.1.3.

2.1.1. *Integrable terms in* $\partial_\alpha^4 BR(z,\varpi)$. We decompose $J_1 = I_3 + I_4 + I_5 + I_6 + I_7$, where

$$I_3 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4\Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big)\varpi(\gamma-\beta)d\alpha d\beta,$$
$$I_4 = \frac{2}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^3\Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big)\partial_\alpha\varpi(\gamma-\beta)d\alpha d\beta,$$
$$I_5 = \frac{3}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2\Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big)\partial_\alpha^2\varpi(\gamma-\beta)d\alpha d\beta,$$
$$I_6 = \frac{2}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha\Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta,$$
$$I_7 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\partial_\alpha^4\varpi(\gamma-\beta)d\alpha d\beta.$$

Below we estimate the highest order term of each I_1. In order to estimate I_j for $j = 4,5,6$, we refer the reader to the paper [8]. We get

$$I_4 + I_5 + I_6 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

The most singular terms for I_3 are those in which four derivatives appear. In order to simplify we write $\Delta\partial_\alpha^k z \equiv \partial_\alpha^k z(\gamma) - \partial_\alpha^k z(\gamma-\beta)$.

One of the two singular terms of I_3 is given by

$$K_1 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(\Delta\partial_\alpha^4 z)^\perp}{|\Delta z|^2}\varpi(\gamma-\beta)d\alpha d\beta,$$

which we decompose into $K_1 = L_1 + L_2$, where

$$L_1 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp\varpi(\gamma-\beta)\Big(\frac{1}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2}\Big)d\alpha d\beta,$$

$$L_2 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp\frac{\varpi(\gamma-\beta)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\alpha d\beta.$$

Let us study L_1. If $\psi = \gamma-\beta+s\beta+t\beta-st\beta$, $\phi=\gamma-\beta+s\beta$ and

(9)

$$\begin{aligned}
B(\gamma,\beta) &\equiv \frac{1}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2}\\
&= \frac{\beta\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(1-s)dtds\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\\
&= \frac{\beta\int_0^1\int_0^1\frac{\partial_\alpha^2 z(\psi)-\partial_\alpha^2 z(\gamma)}{|\psi-\gamma|^\delta}\beta^\delta(-1+s+t-st)^\delta(1-s)dtds\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\\
&+\frac{\beta\partial_\alpha^2 z(\gamma)\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\equiv B_1(\gamma,\beta)+B_2(\gamma,\beta),
\end{aligned}$$

then we have

$$\begin{aligned}
L_1 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp\varpi(\gamma-\beta)B_1(\gamma,\beta)d\alpha d\beta\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp\varpi(\gamma-\beta)B_2(\gamma,\delta)d\alpha d\beta\equiv M_1+M_2.
\end{aligned}$$

It is easy to check that

$$M_1\le C\|\mathcal{F}(z)\|_{L^\infty(S)}^{\frac{3}{2}}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2.$$

Furthermore,

$$\begin{aligned}
B_2(\gamma,\beta) &= \frac{\beta^2\partial_\alpha^2 z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\\
&+\frac{\beta\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}\equiv B_3(\gamma,\beta)+B_4(\gamma,\beta).
\end{aligned}$$

In the same way, we deal with M_2 and we have

$$\begin{aligned}
M_2 &= \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp\varpi(\gamma-\beta)B_3(\gamma,\beta)d\alpha d\beta\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp\varpi(\gamma-\beta)B_4(\gamma,\beta)d\alpha d\beta\equiv N_1+N_2.
\end{aligned}$$

It is clear that

$$N_1\le C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{\mathcal{C}^2}^2\|\varpi\|_{L^\infty}\|\partial_\alpha^4 z\|_{L^2(S)}^2$$

and

$$N_2 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp \varpi(\gamma-\beta)\frac{\beta\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}B(\gamma,\beta)d\alpha d\beta$$
$$+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp \varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4\beta}d\alpha d\beta \equiv N_2^1 + N_2^2.$$

Directly,

$$N_2^1 \le C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{\mathcal{C}^2}^2\|z\|_{\mathcal{C}^1}^2\|\varpi\|_{L^\infty}\|\partial_\alpha^4 z\|_{L^2(S)}^2,$$

and we decompose

$$N_2^2 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\partial_\alpha^4 z(\gamma))^\perp \varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\alpha d\beta$$
$$-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\partial_\alpha^4 z(\gamma-\beta))^\perp \varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\alpha d\beta$$
$$\equiv N_2^{21} + N_2^{22}$$

where

$$N_2^{21} = \frac{1}{2}\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\partial_\alpha^4 z(\gamma))^\perp \frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}H(\varpi)d\alpha$$
$$\le C\|\mathcal{F}(z)\|_{L^\infty}^{\frac{1}{2}}\|z\|_{\mathcal{C}^2}\|\partial_\alpha^4 z\|_{L^2(S)}^2\|\varpi\|_{\mathcal{C}^\delta}$$

and

$$N_2^{22} = -\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\partial_\alpha^4 z(\gamma-\beta))^\perp (\varpi(\gamma-\beta)-\varpi(\gamma))\frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\alpha d\beta$$
$$-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\partial_\alpha^4 z(\gamma-\beta))^\perp \varpi(\gamma)\frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\alpha d\beta$$
$$\le C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|\varpi\|_{\mathcal{C}^1}\|\partial_\alpha^4 z\|_{L^2(S)}^2$$
$$-\frac{1}{2}\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\varpi(\gamma)\frac{\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}H(\partial_\alpha^4 z)d\alpha$$
$$\le C\|\mathcal{F}(z)\|_{L^\infty}^2\|z\|_{\mathcal{C}^2}\|z\|_{\mathcal{C}^1}\|\varpi\|_{\mathcal{C}^1}\|\partial_\alpha^4 z\|_{L^2(S)}^2.$$

Here we have used

$$\|H(f)\|_{L^p} \le C\|f\|_{L^p} \quad \text{for } 1<p<\infty,$$
$$\|H(f)\|_{L^\infty} \le \|f\|_{\mathcal{C}^\delta} \quad \text{for } f\in\mathcal{C}^\delta, \text{ and } 0<\delta<1.$$

Hence, using (6)

$$L_1 \le \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).$$

For L_2 we write $L_2 = M_3 + M_4$, with

$$M_3 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp \frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\alpha d\beta,$$
$$M_4 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\Delta\partial_\alpha^4 z)^\perp \frac{\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\alpha d\beta.$$

Next we write

$$M_3 = \frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\alpha d\beta$$
$$-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma-\beta)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}d\alpha d\beta \equiv N_3+N_4$$

where

$$N_3 = \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma)\frac{\Lambda(\varpi)(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha$$
$$\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\|\Lambda\varpi\|_{L^\infty(S)} \le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\|\varpi\|_{\mathcal{C}^{1,\delta}(S)}$$

and

$$N_4 = -\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma-\beta)\frac{\int_0^1[\partial_\alpha\varpi(\gamma-s\beta)-\partial_\alpha\varpi(\gamma)]ds}{|\partial_\alpha z(\gamma)|^2\beta}d\alpha d\beta$$
$$-\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 z(\gamma-\beta)\frac{\partial_\alpha\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\alpha d\beta$$
$$\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2\|\varpi\|_{\mathcal{C}^2(S)} - \frac{1}{2}\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot H(\partial_\alpha^4 z)(\gamma)\frac{\partial_\alpha\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha$$
$$\le \exp C(\|\mathcal{F}(z)\|_{L^\infty(S)}^2 + \|z\|_{H^4(S)}^2).$$

For M_4,

$$M_4 = \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\Lambda(\partial_\alpha^4 z^\perp)(\gamma)\frac{\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha$$
$$= \int_{\mathbb{T}}\Im(\frac{\varpi}{A(t)})(-\Re(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp)) + \Im(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp)))d\alpha$$
$$+\int_{\mathbb{T}}\Re(\frac{\varpi}{A(t)})(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp)) + \Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp)))d\alpha$$
$$\equiv N_5+N_6.$$

Now we take

$$N_6 = \int_{\mathbb{T}}\Re(\frac{\varpi}{A(t)})(-\Re(\partial_\alpha^4 z_1)\Re(\Lambda(\partial_\alpha^4 z_2)) + \Re(\partial_\alpha^4 z_2)\Re(\Lambda(\partial_\alpha^4 z_1)))d\alpha$$
$$+\int_{\mathbb{T}}\Re(\frac{\varpi}{A(t)})(-\Im(\partial_\alpha^4 z_1)\Im(\Lambda(\partial_\alpha^4 z_2)) + \Im(\partial_\alpha^4 z_2)\Im(\Lambda(\partial_\alpha^4 z_1)))d\alpha$$
$$\equiv N_6^1 + N_6^2$$

where it is easy to find a commutator formula such that, using (see [18])

$$\|\Lambda(fg) - g\Lambda(f)\|_{L^2} \le \|g\|_{\mathcal{C}^{1,\delta}}\|f\|_{L^2}, \tag{10}$$

we get

$$N_6^1 = \int_{\mathbb{T}}(-\Lambda(\Re(\frac{\varpi}{A(t)})\Re(\partial_\alpha^4 z_1)) + \Re(\frac{\varpi}{A(t)})\Re(\Lambda(\partial_\alpha^4 z_1)))\Re(\partial_\alpha^4 z_2)d\alpha$$
$$\le C\|\Re(\frac{\varpi}{A(t)})\|_{\mathcal{C}^{1,\delta}}\|\partial_\alpha^4 z\|_{L^2(S)}^2.$$

In the same way,

$$N_6^2 \le C\|\Re(\frac{\varpi}{A(t)})\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|_{L^2(S)}^2.$$

Thus,

$$N_6 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

For N_5 we have

$$\begin{aligned}
N_5 &= \int_{\mathbb{T}} \Im(\frac{\varpi}{A(t)})(\Re(\partial_\alpha^4 z^\perp)\cdot\Im(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp)))d\alpha \\
&= \int_{\mathbb{T}} (\Lambda(\Im(\frac{\varpi}{A(t)})\Re(\partial_\alpha^4 z^\perp)) - \Im(\frac{\varpi}{A(t)})\Re(\Lambda(\partial_\alpha^4 z^\perp)))\cdot\Im(\partial_\alpha^4 z)d\alpha \\
&+ 2\int_{\mathbb{T}} \Im(\frac{\varpi}{A(t)})\Re(\Lambda(\partial_\alpha^4 z^\perp))\cdot\Im(\partial_\alpha^4 z)d\alpha \\
&\equiv N_5^1 + N_5^2.
\end{aligned}$$

Then,

$$N_5^1 \leq C\|\Im(\frac{\varpi}{A(t)})\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

and

$$\begin{aligned}
N_5^2 &= 2\int_{\mathbb{T}} \Lambda^{\frac{1}{2}}(\Im(\frac{\varpi}{A(t)})\Im(\partial_\alpha^4 z))\cdot\Re(\Lambda^{\frac{1}{2}}(\partial_\alpha^4 z^\perp))d\alpha \\
&\leq 2\|\Lambda^{\frac{1}{2}}(\Im(\frac{\varpi}{A(t)})\Im(\partial_\alpha^4 z))\|_{L^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)} \\
&\leq C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}(\|\partial_\alpha^4 z\|_{L^2(S)} + \|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)} \\
&\leq C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}(\frac{\|\partial_\alpha^4 z\|^2_{L^2(S)}}{2} + \frac{\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}}{2}) \\
&\quad + C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)} \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}$$

Concluding,

$$K_1 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.$$

The other singular term with four derivatives inside I_3 is given by

$$K_2 = -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{(\Delta z)^\perp}{|\Delta z|^4}(\Delta z\cdot\Delta\partial_\alpha^4 z)\varpi(\gamma-\beta)d\alpha d\beta.$$

Here we take $K_2 = L_3 + L_4 + L_5$ where

$$\begin{aligned}
L_3 &= -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\frac{(\Delta z)^\perp}{|\Delta z|^4} - \frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4\beta^3})(\Delta z - \beta\partial_\alpha z(\gamma)) \\
&\qquad\cdot\varpi(\gamma-\beta)\Delta\partial_\alpha^4 z d\alpha d\beta, \\
L_4 &= -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot(\frac{(\Delta z)^\perp}{|\Delta z|^4} - \frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4\beta^3})(\beta\partial_\alpha z(\gamma)\cdot\Delta\partial_\alpha^4 z)\varpi(\gamma-\beta)d\alpha d\beta, \\
L_5 &= -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha z(\gamma)^\perp}{|\partial_\alpha z(\gamma)|^4\beta^3}(\Delta z\cdot\Delta\partial_\alpha^4 z)\varpi(\gamma-\beta)d\alpha d\beta.
\end{aligned}$$

We compute

(11)

$$
\begin{aligned}
C(\gamma,\beta) = \frac{(\Delta z)^{\perp}}{|\Delta z|^4} - \frac{\partial_\alpha z(\gamma)^{\perp}}{|\partial_\alpha z(\gamma)|^4\beta^3} &= \frac{\beta^2\int_0^1\int_0^1 \partial_\alpha^2 z^{\perp}(\eta)(s-1)dsdt}{|\Delta z|^4} \\
&+ \frac{\beta^2\partial_\alpha z^{\perp}(\gamma)\int_0^1\int_0^1 \partial_\alpha^2 z(\psi)(1-s)dsdt}{|\Delta z|^4|\partial_\alpha z(\gamma)|^4} \\
&\cdot \int_0^1 [\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds \int_0^1 [|\partial_\alpha z(\gamma)|^2 + |\partial_\alpha z(\phi)|^2]ds \\
&\equiv C_1(\gamma,\beta) + C_2(\gamma,\beta)
\end{aligned}
$$

and

$$
\Delta z - \beta\partial_\alpha z(\gamma) = \beta^2\int_0^1\int_0^1 \partial_\alpha^2 z(\eta)(s-1)dtds
$$

where $\eta = \gamma - t\beta + st\beta$, allowing us to obtain the desired estimate for the term L_3.

Next we split $L_4 = M_5 + M_6$ since $\Delta\partial_\alpha^4 z = \partial_\alpha^4 z(\gamma) - \partial_\alpha^4 z(\gamma-\beta)$:

$$
\begin{aligned}
M_5 &= -\frac{1}{\pi}\mathfrak{R}\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot C(\gamma,\beta)(\beta\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma))\varpi(\gamma-\beta)d\alpha d\beta, \\
M_6 &= \frac{1}{\pi}\mathfrak{R}\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)} \cdot C(\gamma,\beta)(\beta\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma-\beta))\varpi(\gamma-\beta)d\alpha d\beta.
\end{aligned}
$$

By following the same approach as for L_1 we have

$$
\begin{aligned}
|M_5| &\leq C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \\
&+ |\mathfrak{R}\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2 z^{\perp}(\gamma)(\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma))\frac{H(\varpi)(\gamma)}{|\partial_\alpha z(\gamma)|^4}d\alpha|
\end{aligned}
$$

and

$$
\begin{aligned}
|M_6| &\leq C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \\
&+ |\mathfrak{R}\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2 z^{\perp}(\gamma)(\partial_\alpha z(\gamma)\cdot H(\partial_\alpha^4 z)(\gamma))\frac{\varpi(\gamma)}{|\partial_\alpha z(\gamma)|^4}d\alpha|.
\end{aligned}
$$

Then, the term L_4 is controlled.

To conclude the estimates of K_2, we need to see what happens with the term L_5:

$$
\begin{aligned}
L_5 = &-\frac{1}{\pi}\mathfrak{R}\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^{\perp} z(\gamma)}{|\partial_\alpha z(\gamma)|^4}\Big(\int_0^1 [\partial_\alpha z(\phi)-\partial_\alpha z(\gamma)]ds\cdot\Delta\partial_\alpha^4 z\Big)\frac{\varpi(\gamma-\beta)}{\beta^2}d\alpha d\beta \\
&-\frac{1}{\pi}\mathfrak{R}\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^{\perp} z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\Delta\partial_\alpha^4 z)\frac{\varpi(\gamma-\beta)}{\beta^2}d\alpha d\beta \\
&\equiv M_7 + M_8.
\end{aligned}
$$

For M_7 we proceed in the same way as in L_4 and we get

$$\begin{aligned}M_7 \leq{}& \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\ &- \Re \int_{\mathbb{T}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial^2_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma))H(\varpi)(\gamma)d\alpha\\ &+ \Re \int_{\mathbb{T}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial^2_\alpha z(\gamma)\cdot H(\partial^4_\alpha z)(\gamma))\varpi(\gamma)d\alpha.\end{aligned}$$

Then,

$$M_7 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

To control the term M_8, we decompose it as follows:

$$\begin{aligned}M_8 ={}& -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\Delta\partial^4_\alpha z)\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\alpha d\beta\\ &-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\Delta\partial^4_\alpha z)\frac{\varpi(\gamma)}{\beta^2}d\alpha d\beta\\ \equiv{}& N_7 + N_8.\end{aligned}$$

Since $\Delta\partial^4_\alpha z = \partial^4_\alpha z(\gamma) - \partial^4_\alpha z(\gamma-\beta)$ we have

$$\begin{aligned}N_7 ={}& -\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma))\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\alpha d\beta\\ &+\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma-\beta))\frac{\varpi(\gamma-\beta)-\varpi(\gamma)}{\beta^2}d\alpha d\beta\\ \equiv{}& O_1 + O_2,\end{aligned}$$

where

$$\begin{aligned}O_1 ={}& -2\Re\int_{\mathbb{T}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma))\Lambda(\varpi)(\gamma)d\alpha d\beta\\ \leq{}& C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial^4_\alpha z\|^2_{L^2(S)}\|\Lambda\varpi\|_{L^\infty(S)} \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial^4_\alpha z\|^2_{L^2(S)}\|\varpi\|_{\mathcal{C}^{1,\delta}(S)}\\ \leq{}& \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\end{aligned}$$

and

$$\begin{aligned}O_2 \leq{}& \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\ &+\Re\int_{\mathbb{T}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}(\partial_\alpha z(\gamma)\cdot H(\partial^4_\alpha z)(\gamma))\partial_\alpha\varpi(\gamma)d\alpha\\ \leq{}& \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).\end{aligned}$$

Using integration by parts for Λ,

$$\begin{aligned}N_8 ={}& -2\Re\int_{\mathbb{T}}\Lambda(\overline{\partial^4_\alpha z}\cdot\frac{\partial_\alpha z^\perp}{A^2(t)}\varpi\partial_\alpha z)(\gamma)\cdot\partial^4_\alpha z(\gamma)d\alpha\\ ={}& -2\Re\int_{\mathbb{T}}(\Lambda(\overline{\partial^4_\alpha z}\cdot\frac{\partial_\alpha z^\perp}{A^2(t)}\varpi\partial_\alpha z)(\gamma)-\partial_\alpha z(\gamma)\varpi(\gamma)\frac{\partial^\perp_\alpha z(\gamma)}{A^2(t)}\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma))\cdot\partial^4_\alpha z(\gamma)d\alpha\\ &-2\Re\int_{\mathbb{T}}\frac{\partial^\perp_\alpha z(\gamma)}{A^2(t)}\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma)\varpi(\gamma)d\alpha\\ \equiv{}& O_3 + O_4.\end{aligned}$$

Using the commutator estimate (10),

$$O_3 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

Taking three derivatives of $A(t) = |\partial_\alpha z|^2$ we obtain

$$\partial_\alpha z(\alpha) \cdot \partial^4_\alpha z(\alpha) = -3\partial^2_\alpha z(\alpha) \cdot \partial^3_\alpha z(\alpha).$$

Together with $\Lambda = \partial_\alpha H$ and integrating by parts:

$$\begin{aligned}
O_4 = &-6\Re \int_{\mathbb{T}} H(\overline{\partial^4_\alpha z})(\gamma) \cdot \frac{\partial^2_\alpha z^\perp(\gamma)}{A^2(t)} \partial^2_\alpha z(\gamma) \cdot \partial^3_\alpha z(\gamma) \varpi(\gamma) d\alpha \\
&- 6\Re \int_{\mathbb{T}} H(\overline{\partial^4_\alpha z})(\gamma) \cdot \frac{\partial^\perp_\alpha z(\gamma)}{A^2(t)} \partial^3_\alpha z(\gamma) \cdot \partial^3_\alpha z(\gamma) \varpi(\gamma) d\alpha \\
&- 6\Re \int_{\mathbb{T}} H(\overline{\partial^4_\alpha z})(\gamma) \cdot \frac{\partial^\perp_\alpha z(\gamma)}{A^2(t)} \partial^2_\alpha z(\gamma) \cdot \partial^4_\alpha z(\gamma) \varpi(\gamma) d\alpha \\
&- 6\Re \int_{\mathbb{T}} H(\overline{\partial^4_\alpha z})(\gamma) \cdot \frac{\partial^\perp_\alpha z(\gamma)}{A^2(t)} \partial^2_\alpha z(\gamma) \cdot \partial^3_\alpha z(\gamma) \partial_\alpha \varpi(\gamma) d\alpha \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).
\end{aligned}$$

Then,

$$L_5 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

All previous discussion shows that I_3 satisfies

$$I_3 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)} \|\Lambda^{\frac{1}{2}}(\partial^4_\alpha z)\|^2_{L^2(S)}.$$

2.1.2. *Searching for the Rayleigh-Taylor condition in I_7.* Let us recall the formula for the Rayleigh-Taylor condition:

$$\sigma(\alpha, t) = \frac{\mu^2}{\kappa} BR(z, \varpi)(\alpha, t) \cdot \partial^\perp_\alpha z(\alpha, t) + g\rho^2 \partial_\alpha z_1(\alpha, t).$$

We write I_7 in the form $I_7 = K_8 + K_9$ where

$$\begin{aligned}
K_8 &= \frac{1}{2\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \Big(\frac{(z(\gamma) - z(\gamma - \beta))^\perp}{|z(\gamma) - z(\gamma - \beta)|^2} - \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2 \beta}\Big) \partial^4_\alpha \varpi(\gamma - \beta) d\alpha d\beta, \\
K_9 &= \frac{1}{2\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2 \beta} \partial^4_\alpha \varpi(\gamma - \beta) d\alpha d\beta.
\end{aligned}$$

After an integration by parts we obtain

$$\begin{aligned}
K_8 &= -\frac{1}{2\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \Big(\frac{(\Delta z)^\perp}{|\Delta z|^2} - \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2 \beta}\Big) \partial_\beta(\partial^3_\alpha \varpi(\gamma - \beta)) d\alpha d\beta \\
&= \frac{1}{2\pi} \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\partial^4_\alpha z(\gamma)} \cdot \partial_\beta\Big(\frac{(\Delta z)^\perp}{|\Delta z|^2} - \frac{\partial^\perp_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2 \beta}\Big) \partial^3_\alpha \varpi(\gamma - \beta) d\alpha d\beta.
\end{aligned}$$

We decompose

$$
\begin{aligned}
&\partial_\beta\Big(\frac{(\Delta z)^\perp}{|\Delta z|^2}-\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}\Big)\\
&=\frac{(\Delta\partial_\alpha z)^\perp}{|\Delta z|^2}+\partial_\alpha^\perp z(\gamma)\Big(\frac{1}{|\Delta z|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2}\Big)-2\frac{(\Delta z)^\perp\Delta z\cdot\Delta\partial_\alpha z}{|\Delta z|^4}\\
&-2\frac{(\Delta z)^\perp(\Delta z-\beta\partial_\alpha z(\gamma))\cdot\partial_\alpha z(\gamma)}{|\Delta z|^4}-2\frac{(\Delta z-\beta\partial_\alpha z(\gamma))^\perp\beta|\partial_\alpha z(\gamma)|^2}{|\Delta z|^4}\\
&+\Big(\frac{2\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta^2}-\frac{2\beta^2\partial_\alpha^\perp z(\gamma)|\partial_\alpha z(\gamma)|^2}{|\Delta z|^4}\Big)\\
&\equiv F_1(\gamma,\beta)+F_2(\gamma,\beta)+F_3(\gamma,\beta)+F_4(\gamma,\beta)+F_5(\gamma,\beta)+F_6(\gamma,\beta). \qquad (12)
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
K_8&=\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot F_1(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot F_2(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot F_3(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot F_4(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot F_5(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&+\frac{1}{2\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot F_6(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&\equiv P_1+P_2+P_3+P_4+P_5+P_6.
\end{aligned}
$$

For P_1, P_2, P_3, P_4 and P_5 we can estimate with the same approach as before, and we easily get

$$
P_1+P_2+P_3+P_4+P_5\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
$$

Since

$$
\begin{aligned}
&-\frac{1}{2}F_6(\gamma,\beta)=\partial_\alpha^\perp z(\gamma)\frac{\beta^4|\partial_\alpha z(\gamma)|^4-|\Delta z|^4}{|\Delta z|^4|\partial_\alpha z(\gamma)|^2\beta^2}=U_1(\gamma,\beta)\\
&+\frac{\partial_\alpha^\perp z(\gamma)}{2}\frac{\beta^4\partial_\alpha^2 z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds\int_0^1[|\partial_\alpha z(\gamma)|^2+|\partial_\alpha z(\phi)|^2]ds}{|\Delta z|^4|\partial_\alpha z(\gamma)|^2}\\
&+\partial_\alpha^\perp z(\gamma)\frac{\beta^4\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)\int_0^1\int_0^1\partial_\alpha z(\eta)\cdot\partial_\alpha^2 z(\eta)(s-1)dtds}{|\Delta z|^4|\partial_\alpha z(\gamma)|^2}\\
&+\partial_\alpha^\perp z(\gamma)\frac{\beta^3\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\Delta z|^4}\\
&\equiv U_1(\gamma,\beta)+U_2(\gamma,\beta)+U_3(\gamma,\beta)+U_4(\gamma,\beta)
\end{aligned}
$$

where $U_1(\gamma,\beta)$ is the remainder term that does not cause any trouble, we get

$$\begin{aligned}
P_6 = &-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot U_1(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot U_2(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot U_3(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot U_4(\gamma,\beta)\partial_\alpha^3\varpi(\gamma-\beta)d\alpha d\beta\\
&\equiv Q_1+Q_2+Q_3+Q_4,
\end{aligned}$$

where

$$\begin{aligned}
Q_1 &\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^3(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\|\partial_\alpha^3\varpi\|_{L^2(S)},\\
Q_2 &\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|^2_{\mathcal{C}^2(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\|\partial_\alpha^3\varpi\|_{L^2(S)},\\
Q_3 &\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\|\partial_\alpha^3\varpi\|_{L^2(S)},
\end{aligned}$$

and if we split, then

$$\begin{aligned}
Q_4 = &-\frac{1}{\pi}\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^\perp z(\gamma)\beta^3\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)\partial_\alpha^3\varpi(\gamma-\beta)\\
&\qquad\qquad\times\Big(\frac{1}{|\Delta z|^4}-\frac{1}{|\partial_\alpha z(\gamma)|^4\beta^4}\Big)d\alpha d\beta\\
&-\frac{1}{\pi}\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^4}H(\partial_\alpha^3\varpi)(\gamma)d\alpha.
\end{aligned}$$

It is clear that

$$Q_4 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Thus,

$$P_6 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Therefore,

$$K_8 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

We consider now the K_9 term, which can be written as

$$\begin{aligned}
K_9 &= \frac{1}{2}\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2}H(\partial_\alpha^4\varpi)(\gamma)d\alpha\\
&= \frac{1}{2}\Re\int_{\mathbb{T}}\overline{\partial_\alpha^4 z(\gamma)}\cdot\frac{\partial_\alpha^\perp z(\gamma)}{|\partial_\alpha z(\gamma)|^2}\Lambda(\partial_\alpha^3\varpi)(\gamma)d\alpha\\
&= \frac{1}{2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3\varpi(\gamma)d\alpha.
\end{aligned}$$

Using the formula

$$\begin{aligned}
\varpi(\alpha) &= -2BR(z,\varpi)(\alpha,t)\cdot\partial_\alpha z(\alpha,t)-2\kappa g\frac{\rho^2}{\mu^2}\partial_\alpha z_2(\alpha,t)\\
&= -T(\varpi)(\alpha)-2g\kappa\frac{\rho^2}{\mu^2}\partial_\alpha z_2(\alpha)
\end{aligned}$$

we separate K_9 as a sum of two parts, P_7 and P_8, where

$$P_7 = -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha,$$
$$P_8 = -\frac{1}{2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3 T(\varpi)(\gamma)d\alpha.$$

For P_7 we decompose further: $P_7 = Q_5 + Q_6$ where

$$Q_5 = g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha,$$
$$Q_6 = -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha.$$

Then Q_5 is written as $Q_5 = R_1 + R_2$ with

$$R_1 = g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma) - \partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha,$$
$$R_2 = g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha.$$

Using the commutator estimate (10), we get

$$R_1 \leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

The identity

$$\partial_\alpha z_2(\gamma)\partial_\alpha^4 z_2(\gamma) = \partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma) - \partial_\alpha z_1(\gamma)\partial_\alpha^4 z_1(\gamma)$$

lets us write R_2 as the sum of S_1 and S_2 where

$$S_1 = g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha,$$
$$S_2 = -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha z_1(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha.$$

For S_1 we use an integration by parts and

$$\partial_\alpha z(\gamma)\cdot\partial_\alpha^4 z(\gamma) = -3\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)$$

to get

$$\begin{aligned}
S_1 &= -3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha \\
&= 3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^3 z(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha \\
&\quad + 3g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)}\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).
\end{aligned}$$

Writing Q_6 in the form

$$Q_6 = -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z_2}\cdot\partial_\alpha z_1)(\gamma)-\partial_\alpha z_1(\gamma)\Lambda(\overline{\partial_\alpha^4 z_2})(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)d\alpha$$
$$-g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_1(\gamma)}{A(t)}\Lambda(\overline{\partial_\alpha^4 z_2})(\gamma)\partial_\alpha^4 z_2(\gamma)d\alpha \equiv R_3+R_4,$$

by the commutator estimate, we have

$$R_3 \le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Since

$$S_2+R_4 = -g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_1(\gamma)}{A(t)}\partial_\alpha^4 z(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma)d\alpha$$

we obtain finally

$$P_7 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}) - g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}}\frac{\partial_\alpha z_1(\gamma)}{A(t)}\partial_\alpha^4 z(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z})(\gamma)d\alpha.$$

In the estimate above we can observe how part of $\sigma(\gamma)$ appears in the non-integrable terms.

Let us return to $P_8 = Q_7+Q_8+Q_9+Q_{10}$ where

$$Q_7 = -\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3 BR(z,\varpi)(\gamma)\cdot\partial_\alpha z(\gamma)d\alpha,$$
$$Q_8 = -3\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^2 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^2 z(\gamma)d\alpha,$$
$$Q_9 = -3\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha BR(z,\varpi)(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha,$$
$$Q_{10} = -\Re\int_{\mathbb{T}}\frac{\Lambda(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}BR(z,\varpi)(\gamma)\cdot\partial_\alpha^4 z(\gamma)d\alpha.$$

We will control first the terms Q_8, Q_7 and Q_9 and then we will show how the rest of $\sigma(\gamma)$ appears in Q_{10}.

Using $\Lambda = H\partial_\alpha$ and integrating by parts, we obtain

$$Q_8 = 3\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^3 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^2 z(\gamma)d\alpha$$
$$+3\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A(t)}\partial_\alpha^2 BR(z,\varpi)(\gamma)\cdot\partial_\alpha^3 z(\gamma)d\alpha$$
$$\equiv R_5+R_6.$$

With (7)

$$R_5 \le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\partial_\alpha^3 BR\|_{L^2(S)}\|\partial_\alpha^4 z\cdot\partial_\alpha z\|_{L^2(S)}$$
$$\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$

and

$$R_6 \le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^3(S)}\|\partial_\alpha^2 BR\|_{L^2(S)}\|\partial_\alpha^4 z\cdot\partial_\alpha z\|_{L^2(S)}$$
$$\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

With Q_7 we also integrate by parts to obtain $Q_7 = R_7 + R_8$ where

$$R_7 = \Re \int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \partial_\alpha^4 BR(z,\varpi)(\gamma) \cdot \partial_\alpha z(\gamma) d\alpha,$$

$$R_8 = \Re \int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \partial_\alpha^3 BR(z,\varpi)(\gamma) \cdot \partial_\alpha^2 z(\gamma) d\alpha.$$

Easily we have

$$\begin{aligned} R_8 &\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\partial_\alpha^4 z \cdot \partial_\alpha z\|_{L^2(S)}\|\partial_\alpha^3 BR\|_{L^2(S)} \\ &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}). \end{aligned}$$

In R_7 the application of Leibniz's rule to $\partial_\alpha^3 BR(z,\varpi)$ produces many terms which can be estimated with the same tools used before. For the most singular terms we have the expressions:

$$S_3 = \Re \int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \partial_\alpha BR(z,\partial_\alpha^3 \varpi)(\gamma) \cdot \partial_\alpha z(\gamma) d\alpha,$$

$$S_4 = \Re \int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \int_{\mathbb{R}} \frac{\Delta\partial_\alpha^4 z}{|\Delta z|^2} \cdot \partial_\alpha z(\gamma)\varpi(\gamma-\beta) d\beta d\alpha,$$

$$S_5 = -2\Re \int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \int_{\mathbb{R}} \frac{\Delta z^\perp \cdot \partial_\alpha z(\gamma)}{|\Delta z|^4} (\Delta z \cdot \Delta\partial_\alpha^4 z)\varpi(\gamma-\beta) d\alpha d\beta.$$

Let us consider

$$\begin{aligned} \partial_\alpha BR(z,\varpi)(\gamma) \cdot \partial_\alpha z(\gamma) &= \partial_\alpha (BR(z,\varpi)(\gamma) \cdot \partial_\alpha z(\gamma)) - BR(z,\varpi)(\gamma) \cdot \partial_\alpha^2 z(\gamma) \\ &= \frac{1}{2}\partial_\alpha T(\varpi)(\gamma) - BR(z,\varpi) \cdot \partial_\alpha^2 z(\gamma), \end{aligned}$$

which yields

$$\begin{aligned} S_3 \le C\|\mathcal{F}(z)\|_{L^\infty(S)}&\|\partial_\alpha^4 z\|_{L^2(S)}\|z\|_{\mathcal{C}^1(S)}(\|T(\partial_\alpha^3\varpi)\|_{H^1(S)} \\ &\qquad + \|BR(z,\partial_\alpha^3\varpi)\|_{L^2(S)}\|z\|_{\mathcal{C}^2(S)}) \\ \le \exp C(\|&\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \end{aligned}$$

because $\|T\|_{L^2\to H^1} \le \|\mathcal{F}(z)\|^2_{L^\infty}\|z\|^4_{\mathcal{C}^{2,\delta}}$; for more details see Lemma 3.1 in [8].

Next we write $S_4 = T_1 + T_2$:

$$T_1 = \Re \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta\partial_\alpha^4 z \cdot \partial_\alpha z(\gamma)\varpi(\gamma-\beta)\Big(\frac{1}{|\Delta z|^2} - \frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2}\Big) d\beta d\alpha,$$

$$T_2 = \Re \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta\partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \frac{\varpi(\gamma-\beta)}{A(t)\beta^2} d\beta d\alpha.$$

Using $B_2(\gamma,\beta) = B_3(\gamma,\beta) + B_4(\gamma,\beta)$, we split $T_1 = U_1 + U_2 + U_3$:

$$U_1 = \Re \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta\partial_\alpha^4 z \cdot \partial_\alpha z(\gamma)\varpi(\gamma-\beta) B_1(\gamma,\beta) d\beta d\alpha,$$

$$U_2 = \Re \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta\partial_\alpha^4 z \cdot \partial_\alpha z(\gamma)\varpi(\gamma-\beta) B_3(\gamma,\beta) d\beta d\alpha,$$

$$U_3 = \Re \int_{\mathbb{T}}\int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \Delta\partial_\alpha^4 z \cdot \partial_\alpha z(\gamma)\varpi(\gamma-\beta) B_4(\gamma,\beta) d\beta d\alpha,$$

where

$$B_1(\gamma,\beta)$$
$$=\frac{\beta\int_0^1\int_0^1 \frac{\partial_\alpha^2 z(\psi)-\partial_\alpha^2 z(\gamma)}{|\psi-\gamma|^\delta}\beta^\delta(1+s+t-st)^\delta(1-s)dtds\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2},$$

$$B_3(\gamma,\beta)=\frac{\beta^2\partial_\alpha^2 z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\eta)(s-1)dtds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2},$$

$$B_4(\gamma,\beta)=\frac{\beta\partial_\alpha^2 z(\gamma)2\partial_\alpha z(\gamma)}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2}.$$

Therefore

$$U_1\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^{2,\delta}(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)},$$
$$U_2\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|^2_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}$$

and

$$\begin{aligned}U_3=&2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\beta\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)B(\gamma,\beta)d\beta d\alpha\\&+2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\Delta\partial_\alpha^4 z\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha\\&\equiv V_1+V_2.\end{aligned}$$

Recall that

$$B(\gamma,\beta)\equiv\frac{\beta\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(1-s)dtds\int_0^1\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)ds}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2};$$

then the term V_1 is controlled.

We split $V_2=W_1+W_2$ where

$$W_1=2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma)\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha,$$

$$W_2=-2\Re\int_{\mathbb{T}}\int_{\mathbb{R}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma-\beta)\cdot\partial_\alpha z(\gamma)\varpi(\gamma-\beta)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2\beta}d\beta d\alpha.$$

Easily,

$$\begin{aligned}W_1&=2\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}\partial_\alpha^4 z(\gamma)\cdot\partial_\alpha z(\gamma)H(\varpi)(\gamma)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha\\&\le C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^2(S)}\|H\varpi\|_{L^\infty(S)}\\&\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\end{aligned}$$

and

$$\begin{aligned}W_2\le\;&C\|\mathcal{F}(z)\|^2_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\|z\|_{\mathcal{C}^1(S)}\|z\|_{\mathcal{C}^2(S)}\|\varpi\|_{\mathcal{C}^1(S)}\\&-2\Re\int_{\mathbb{T}}\frac{H(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z^\perp)(\gamma)}{A^2(t)}H(\partial_\alpha^4 z)(\gamma)\cdot\partial_\alpha z(\gamma)\varpi(\gamma)\frac{\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2}d\alpha\\&\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).\end{aligned}$$

Hence,

$$T_1\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

We decompose $T_2 = U_4 + U_5$:

$$U_4 = \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} \Delta \partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \frac{\varpi(\gamma-\beta) - \varpi(\gamma)}{\beta^2} d\beta d\alpha,$$
$$U_5 = \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} \Delta \partial_\alpha^4 z \cdot \partial_\alpha z(\gamma) \frac{\varpi(\gamma)}{\beta^2} d\beta d\alpha.$$

Then we split

$$\begin{aligned}U_4 &= \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} \partial_\alpha^4 z(\gamma) \cdot \partial_\alpha z(\gamma) \frac{\varpi(\gamma-\beta) - \varpi(\gamma)}{\beta^2} d\beta d\alpha \\ &- \Re \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} \partial_\alpha^4 z(\gamma-\beta) \cdot \partial_\alpha z(\gamma) \frac{\varpi(\gamma-\beta) - \varpi(\gamma)}{\beta^2} d\beta d\alpha \\ &\equiv V_3 + V_4\end{aligned}$$

where

$$V_3 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

and

$$\begin{aligned}V_4 &\le C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty(S)} \|\partial_\alpha^4 z\|^2_{L^2(S)} \|z\|_{\mathcal{C}^1(S)} \|\varpi\|_{\mathcal{C}^2(S)} \\ &- \pi \Re \int_{\mathbb{T}} \frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} H(\partial_\alpha^4 z)(\gamma) \cdot \partial_\alpha z(\gamma) \partial_\alpha \varpi(\gamma) d\alpha \\ &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).\end{aligned}$$

Then,

$$U_4 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

For U_5, integrating by parts for Λ we have

$$\begin{aligned}U_5 &= \Re \int_{\mathbb{T}} \Lambda\Big(\frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)}{A^2(t)} \partial_\alpha z \varpi\Big) \cdot \partial_\alpha^4 z(\gamma) d\alpha \\ &= \Re \int_{\mathbb{T}} \Big(\Lambda\Big(\frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)}{A^2(t)} \partial_\alpha z \varpi\Big)(\gamma) - \partial_\alpha z(\gamma) \Lambda\Big(\frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)}{A^2(t)}\Big)(\gamma)\Big) \cdot \partial_\alpha^4 z(\gamma) d\alpha \\ &+ \Re \int_{\mathbb{T}} \partial_\alpha z(\gamma) \Lambda\Big(\frac{H(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)}{A^2(t)}\Big)(\gamma) \cdot \partial_\alpha^4 z(\gamma) d\alpha \\ &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - \Re \int_{\mathbb{T}} \partial_\alpha z(\gamma) \frac{\partial_\alpha(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} \cdot \partial_\alpha^4 z(\gamma) d\alpha.\end{aligned}$$

Now, using $\partial_\alpha z(\alpha) \cdot \partial_\alpha^4 z(\alpha) = -3\partial_\alpha^2 z(\alpha) \cdot \partial_\alpha^3 z(\alpha)$,

$$\begin{aligned}&- \Re \int_{\mathbb{T}} \frac{\partial_\alpha(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 z(\gamma) d\alpha \\ &= 3\Re \int_{\mathbb{T}} \frac{\partial_\alpha(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^2(t)} \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^3 z(\gamma) d\alpha \\ &= -3\Re \int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp(\gamma)}{A^2(t)} \partial_\alpha^3 z(\gamma) \cdot \partial_\alpha^3 z(\gamma) d\alpha \\ &- 3\Re \int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z^\perp(\gamma)}{A^2(t)} \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^4 z(\gamma) d\alpha \\ &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).\end{aligned}$$

Therefore,

$$T_2 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

Thus, S_4 satisfies more identical estimates than T_2.

To conclude with R_7, let us estimate S_5. We split $S_5 = T_3 + T_4$:

$$T_3 = -2\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{H(\overline{\partial^4_\alpha z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} C(\gamma,\beta) \cdot \partial_\alpha z(\gamma)(\Delta z \cdot \Delta \partial^4_\alpha z)\varpi(\gamma-\beta) d\alpha d\beta,$$

$$T_4 = -2\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{H(\overline{\partial^4_\alpha z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \frac{\partial^\perp_\alpha z(\gamma)}{A^2(t)\beta^3} \cdot \partial_\alpha z(\gamma)(\Delta z \cdot \Delta \partial^4_\alpha z)\varpi(\gamma-\beta) d\alpha d\beta.$$

Since $\partial^\perp_\alpha z(\gamma) \cdot \partial_\alpha z(\gamma) = 0$, for (11) we have $C(\gamma,\beta)\cdot\partial_\alpha z(\gamma) = C_1(\gamma,\beta)\cdot\partial_\alpha z(\gamma)$ and $T_4 = 0$.

Recall that

$$C_1(\gamma,\beta) = \frac{\beta^2 \int_0^1 \int_0^1 \partial^2_\alpha z^\perp(\eta)(s-1) ds dt}{|\Delta z|^4}$$

with $\eta = \gamma - t\beta + st\beta$. Using

$$\Delta \partial^k_\alpha z = \beta \int_0^1 \partial^{k+1}_\alpha z(\phi) ds$$

and

$$C_1(\gamma,\beta)\cdot\partial_\alpha z(\gamma) - \frac{\beta^2 \partial^2_\alpha z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)}{|\Delta z|^4}$$
$$= \frac{\beta^2 \int_0^1 \int_0^1 [\partial^2_\alpha z^\perp(\eta) - \partial^2_\alpha z(\gamma)](s-1) dt ds \cdot \partial_\alpha z(\gamma)}{|\Delta z|^4}$$

we get

$$\begin{aligned}
S_5 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&\quad - 2\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \frac{H(\overline{\partial^4_\alpha z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^3(t)} \partial^2_\alpha z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\Delta\partial^4_\alpha z \frac{\varpi(\gamma-\beta)}{\beta} d\alpha d\beta \\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\
&\quad - 2\pi\Re \int_{\mathbb{T}} \frac{H(\overline{\partial^4_\alpha z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^3(t)} \partial^2_\alpha z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\partial^4_\alpha z(\gamma) H(\varpi)(\gamma) d\alpha \\
&\quad - 4\pi\Re \int_{\mathbb{T}} \frac{H(\overline{\partial^4_\alpha z} \cdot \partial_\alpha z^\perp)(\gamma)}{A^3(t)} \partial^2_\alpha z^\perp(\gamma)\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot H(\partial^4_\alpha z)(\gamma)\varpi(\gamma) d\alpha.
\end{aligned}$$

Therefore we can control S_5.

Let us decompose

$$\begin{aligned}
Q_9 =& 3\Re \int_{\mathbb{T}} \frac{H(\overline{\partial^4_\alpha z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \partial^2_\alpha BR(z,\varpi)(\gamma)\cdot\partial^3_\alpha z(\gamma) d\alpha \\
&+ 3\Re \int_{\mathbb{T}} \frac{H(\overline{\partial^4_\alpha z} \cdot \partial_\alpha z^\perp)(\gamma)}{A(t)} \partial_\alpha BR(z,\varpi)(\gamma)\cdot\partial^4_\alpha z(\gamma) d\alpha \equiv R_9 + R_{10},
\end{aligned}$$

using (7):

$$\begin{aligned}
R_9 &\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^3(S)}\|BR\|_{H^2(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha^4 z\|_{L^2(S)}\\
&\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\
R_{10} &\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha BR\|_{L^\infty(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\\
&\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).
\end{aligned}$$

Then $Q_9 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$.

Finally we have to find the rest of $\sigma(\gamma)$ in Q_{10}. To do that let us split $Q_{10} = R_{11}+R_{12}+R_{13}+R_{14}$ where

$$\begin{aligned}
R_{11} &= \Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)}{A(t)} BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha,\\
R_{12} &= \Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)}{A(t)} BR_2(z,\varpi)(\gamma)\partial_\alpha^4 z_2(\gamma)d\alpha,\\
R_{13} &= -\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)} BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha,\\
R_{14} &= -\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial_\alpha^4 z_2}\partial_\alpha z_1)(\gamma)}{A(t)} BR_2(z,\varpi)(\gamma)\partial_\alpha^4 z_2(\gamma)d\alpha.
\end{aligned}$$

Then

$$\begin{aligned}
R_{11} = \Re\int_{\mathbb{T}} &\frac{\Lambda(\overline{\partial_\alpha^4 z_1}\partial_\alpha z_2)(\gamma)-\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)} BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha\\
&+\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)}{A(t)} BR_1(z,\varpi)(\gamma)\partial_\alpha^4 z_1(\gamma)d\alpha,
\end{aligned}$$

and the commutator estimate yields

$$\begin{aligned}
R_{11} &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&\quad+\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma)BR_1(z,\varpi)(\gamma)}{A(t)}\partial_\alpha^4 z_1(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)d\alpha.
\end{aligned}$$

In a similar way we have

$$\begin{aligned}
R_{12} &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&\quad+\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma)BR_2(z,\varpi)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_1})(\gamma)d\alpha,\\
R_{13} &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&\quad-\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)BR_1(z,\varpi)(\gamma)}{A(t)}\partial_\alpha^4 z_1(\gamma)\Lambda(\overline{\partial_\alpha^4 z_2})(\gamma)d\alpha,\\
R_{14} &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\
&\quad-\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)BR_2(z,\varpi)(\gamma)}{A(t)}\partial_\alpha^4 z_2(\gamma)\Lambda(\overline{\partial_\alpha^4 z_2})(\gamma)d\alpha.
\end{aligned}$$

Since

$$\partial_\alpha z_2\partial_\alpha^4 z_2 = \partial_\alpha z\cdot\partial_\alpha^4 z - \partial_\alpha z_1\partial_\alpha^4 z_1 = -3\partial_\alpha^2 z\cdot\partial_\alpha^3 z - \partial_\alpha z_1\partial_\alpha^4 z_1$$

and $H\partial_\alpha = \Lambda$, using integration by parts,

$$\begin{aligned} R_{12} &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &\quad - 3\Re\int_{\mathbb{T}} \partial_\alpha\Big(\frac{BR_2(z,\varpi)(\gamma)}{A(t)}\partial^2_\alpha z \cdot \partial^3_\alpha z\Big) H(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha \\ &\quad - \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) BR_2(z,\varpi)(\gamma)}{A(t)}\partial^4_\alpha z_1(\gamma)\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &\quad - \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma) BR_2(z,\varpi)(\gamma)}{A(t)}\partial^4_\alpha z_1(\gamma)\Lambda(\overline{\partial^4_\alpha z_1})(\gamma)d\alpha. \end{aligned}$$

And in the same way,

$$\begin{aligned} R_{13} &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &\quad + \Re\int_{\mathbb{T}} \frac{\partial_\alpha z_2(\gamma) BR_1(z,\varpi)(\gamma)}{A(t)}\partial^4_\alpha z_2(\gamma)\Lambda(\overline{\partial^4_\alpha z_2})(\gamma)d\alpha. \end{aligned}$$

Therefore,

$$\begin{aligned} &R_{11} + R_{12} + R_{13} + R_{14} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &- \Re\int_{\mathbb{T}} \frac{BR(z,\varpi)(\gamma)\cdot \partial^\perp_\alpha z(\gamma)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha. \end{aligned}$$

Then,

$$\begin{aligned} &P_7 + P_8 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &- \Re\int_{\mathbb{T}} \frac{BR(z,\varpi)(\gamma)\cdot \partial_\alpha z^\perp(\gamma)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha \\ &- g\kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha. \end{aligned}$$

Let us look at these last two terms:

$$\begin{aligned} &- \Re\int_{\mathbb{T}} \frac{BR(z,\varpi)(\gamma)\cdot \partial_\alpha z^\perp(\gamma)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha \\ &- \kappa\frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\partial_\alpha z_1(\gamma)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha \\ &= -\Re\int_{\mathbb{T}} \frac{\sigma(\gamma,t)}{A(t)}\partial^4_\alpha z(\gamma)\cdot\Lambda(\overline{\partial^4_\alpha z})(\gamma)d\alpha \\ &= \int_{\mathbb{T}} \Im\Big(\frac{\sigma}{A(t)}\Big)(-\Re(\partial^4_\alpha z)\cdot\Im(\Lambda(\partial^4_\alpha z)) + \Im(\partial^4_\alpha z)\cdot\Re(\Lambda(\partial^4_\alpha z)))d\alpha \\ &- \int_{\mathbb{T}} \Re\Big(\frac{\sigma}{A(t)}\Big)(\Re(\partial^4_\alpha z)\cdot\Re(\Lambda(\partial^4_\alpha z)) + \Im(\partial^4_\alpha z)\cdot\Im(\Lambda(\partial^4_\alpha z)))d\alpha \equiv Y_1 + Y_2. \end{aligned}$$

We get

$$\begin{aligned} Y_1 &= \int_{\mathbb{T}} (-\Lambda(\Im(\frac{\sigma}{A(t)})\Re(\partial^4_\alpha z)) + \Im(\frac{\sigma}{A(t)})\Re(\Lambda(\partial^4_\alpha z)))\cdot\Im(\partial^4_\alpha z)d\alpha \\ &\leq C\|\frac{\sigma}{A(t)}\|_{\mathcal{C}^{1,\delta}(S)}\|\partial^4_\alpha z\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \end{aligned}$$

and

$$
\begin{aligned}
Y_2 = &-\int_{\mathbb{T}} (\Re(\frac{\sigma}{A(t)}) - m(t))(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\\
&-\int_{\mathbb{T}} m(t)(\Re(\partial_\alpha^4 z)\cdot\Re(\Lambda(\partial_\alpha^4 z)) + \Im(\partial_\alpha^4 z)\cdot\Im(\Lambda(\partial_\alpha^4 z)))d\alpha\\
&\equiv Y_3 + Y_4
\end{aligned}
$$

where

$$m(t) = \min_\gamma \sigma(\gamma, t).$$

Since $\Re(\frac{\sigma}{A(t)}) - m(t) > 0$ using $2g\Lambda(g) - \Lambda(g^2) \geq 0$ (see [7]),

$$
\begin{aligned}
Y_3 &\leq \frac{1}{2}\|\Lambda(\Re(\frac{\sigma}{A(t)}))\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \leq C\|\frac{\sigma}{A(t)}\|_{\mathcal{C}^{1,\delta}(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)}\\
&\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}),\\
Y_4 &= -m(t)\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.
\end{aligned}
$$

Combining all previous estimates, we get

$$I_7 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - m(t)\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.$$

2.1.3. *Estimates on* $\partial_\alpha^4(c(\gamma,t)\cdot\partial_\alpha z(\gamma,t))$ *for* J_2. In the evolution of the norm of $\partial_\alpha^4 z(\gamma)$ it remains to control the term

$$
\begin{aligned}
J_2 = &\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^4 c(\gamma)\partial_\alpha z(\gamma)d\alpha + 4\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^3 c(\gamma)\partial_\alpha^2 z(\gamma)d\alpha\\
&+ 6\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha^2 c(\gamma)\partial_\alpha^3 z(\gamma)d\alpha + 4\Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot\partial_\alpha c(\gamma)\partial_\alpha^4 z(\gamma)d\alpha\\
&+ \Re\int_{\mathbb{T}} \overline{\partial_\alpha^4 z(\gamma)}\cdot c(\gamma)\partial_\alpha^5 z(\gamma)d\alpha \equiv Q_1 + Q_2 + Q_3 + Q_4 + Q_5.
\end{aligned}
$$

Let us recall the formula

$$
\begin{aligned}
c(\alpha,t) = \frac{\alpha+\pi}{2\pi}&\int_{\mathbb{T}} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta\\
&-\int_{-\pi}^{\alpha} \frac{\partial_\beta z(\beta,t)}{|\partial_\beta z(\beta,t)|^2}\cdot\partial_\beta BR(z,\varpi)(\beta,t)d\beta.
\end{aligned}
$$

Then

$$
\begin{aligned}
Q_2 = &4\Re\int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha^2 z(\gamma)\partial_\alpha^3 z(\gamma)\cdot\partial_\alpha BR(z,\varpi)(\gamma)d\alpha\\
&+ 8\Re\int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha^2 z(\gamma)\partial_\alpha^2 z(\gamma)\cdot\partial_\alpha^2 BR(z,\varpi)(\gamma)d\alpha\\
&+ 4\Re\int_{\mathbb{T}} \frac{\overline{\partial_\alpha^4 z(\gamma)}}{A(t)}\cdot\partial_\alpha^2 z(\gamma)\partial_\alpha z(\gamma)\cdot\partial_\alpha^3 BR(z,\varpi)(\gamma)d\alpha \equiv N_1 + N_2 + N_3
\end{aligned}
$$

and

$$
\begin{aligned}
N_1 &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|z\|_{\mathcal{C}^3(S)}\|BR(z,\varpi)\|_{H^1(S)}\|\partial_\alpha^4 z\|_{L^2(S)},\\
N_2 &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|^2_{\mathcal{C}^2(S)}\|\partial_\alpha^2 BR(z,\varpi)\|_{L^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)},\\
N_3 &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial_\alpha^3 BR(z,\varpi)\|_{L^2(S)}\|\partial_\alpha^4 z\|_{L^2(S)}.
\end{aligned}
$$

Thus

$$Q_2 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

In the same way,

$$\begin{aligned} Q_3 = -6\Re \int_{\mathbb{T}} \overline{\partial^4_\alpha z(\gamma)} \cdot \partial^3_\alpha z(\gamma) \frac{\partial^2_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2} \cdot \partial_\alpha BR(z,\varpi)(\gamma) d\alpha \\ - 6\Re \int_{\mathbb{T}} \overline{\partial^4_\alpha z(\gamma)} \cdot \partial^3_\alpha z(\gamma) \frac{\partial_\alpha z(\gamma)}{|\partial_\alpha z(\gamma)|^2} \cdot \partial^2_\alpha BR(z,\varpi)(\gamma) d\alpha \equiv N_4 + N_5 \end{aligned}$$

where

$$\begin{aligned} N_4 &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|z\|_{\mathcal{C}^3(S)}\|\partial^4_\alpha z\|_{L^2(S)}\|\partial_\alpha BR(z,\varpi)\|_{L^2(S)}, \\ N_5 &\leq C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^3(S)}\|z\|_{\mathcal{C}^1(S)}\|\partial^4_\alpha z\|_{L^2(S)}\|\partial^2_\alpha BR(z,\varpi)\|_{L^2(S)}; \end{aligned}$$

thus

$$Q_3 \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

The term Q_4 satisfies

$$\begin{aligned} Q_4 &\leq C\|\partial_\alpha c\|_{L^\infty(S)}\|\partial^4_\alpha z\|^2_{L^2(S)} \leq C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|\partial_\alpha BR\|_{L^\infty(S)}\|\partial^4_\alpha z\|^2_{L^2(S)} \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}), \end{aligned}$$

and for Q_5,

$$\begin{aligned} Q_5 &= \Re \int_{\mathbb{T}} c(\gamma)\overline{\partial^4_\alpha z(\gamma)} \cdot \partial^5_\alpha z(\gamma) d\alpha \\ &= \int_{\mathbb{T}} \Re(c)(\Re(\partial^4_\alpha z)\Re(\partial^5_\alpha z) + \Im(\partial^4_\alpha z)\Im(\partial^5_\alpha z)) d\alpha \\ &+ \int_{\mathbb{T}} \Im(c)(-\Re(\partial^4_\alpha z)\Im(\partial^5_\alpha z) + \Im(\partial^4_\alpha z)\Re(\partial^5_\alpha z)) d\alpha \\ &\equiv Q^1_5 + Q^2_5, \end{aligned}$$

where

$$\begin{aligned} Q^1_5 &= -\frac{1}{2}\int_{\mathbb{T}} \Re(\partial_\alpha c)|\partial^4_\alpha z|^2 d\alpha \leq \|\partial_\alpha z\|_{L^\infty}\|\partial^4_\alpha z\|^2_{L^2(S)} \\ &\leq C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}\|\partial^4_\alpha z\|^2_{L^2(S)} \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \end{aligned}$$

and

$$\begin{aligned} Q^2_5 &= \int_{\mathbb{T}} \Im(\partial_\alpha c)\Re(\partial^4_\alpha z)\Im(\partial^4_\alpha z) d\alpha + 2\int_{\mathbb{T}} \Im(c)\Im(\partial^4_\alpha z)\Re(\partial^5_\alpha z) d\alpha \\ &\leq \|\partial_\alpha c\|_{L^\infty(S)}\|\partial^4_\alpha z\|^2_{L^2(S)} - 2\int_{\mathbb{T}} \Im(c)\Im(\partial^4_\alpha z)\Re(\Lambda(H(\partial^4_\alpha z))) d\alpha \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) - 2\int_{\mathbb{T}} \Lambda^{\frac{1}{2}}(\Im(c)\Im(\partial^4_\alpha z))\Re(\Lambda^{\frac{1}{2}}(H(\partial^4_\alpha z))) d\alpha \\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + K\|\Im(c)\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial^4_\alpha z\|^2_{L^2(S)}. \end{aligned}$$

Finally,

$$\begin{aligned}
Q_1 &= \Re \int_{\mathbb{T}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha^4 z(\gamma) \cdot \partial_\alpha BR(z,\varpi)(\gamma) d\alpha \\
&- 3\Re \int_{\mathbb{T}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha^3 z(\gamma) \cdot \partial_\alpha^2 BR(z,\varpi)(\gamma) d\alpha \\
&- 3\Re \int_{\mathbb{T}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha^2 z(\gamma) \cdot \partial_\alpha^3 BR(z,\varpi)(\gamma) d\alpha \\
&- \Re \int_{\mathbb{T}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 BR(z,\varpi)(\gamma) d\alpha \\
&\equiv N_6 + N_7 + N_8 + N_9
\end{aligned}$$

where

$$\begin{aligned}
N_6 &\le C\|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^1(S)} \|\partial_\alpha BR\|_{L^\infty(S)} \|\partial_\alpha^4 z\|_{L^2(S)}^2, \\
N_7 &\le C\|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^1(S)} \|z\|_{\mathcal{C}^3(S)} \|\partial_\alpha^2 BR\|_{L^2(S)} \|\partial_\alpha^4 z\|_{L^2(S)}, \\
N_8 &\le C\|\mathcal{F}(z)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^1(S)} \|z\|_{\mathcal{C}^2(S)} \|\partial_\alpha^3 BR\|_{L^2(S)} \|\partial_\alpha^4 z\|_{L^2(S)}.
\end{aligned}$$

To estimate N_9, we must proceed in the same way as we did with J_1. We split $N_9 = I_3' + I_4' + I_5' + I_6' + I_7'$:

$$\begin{aligned}
I_3' &= -\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \cdot \partial_\alpha^4 \Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big) \varpi(\gamma-\beta) d\alpha d\beta, \\
I_4' &= -4\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \cdot \partial_\alpha^3 \Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big) \partial_\alpha \varpi(\gamma-\beta) d\alpha d\beta, \\
I_5' &= -6\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \cdot \partial_\alpha^2 \Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big) \partial_\alpha^2 \varpi(\gamma-\beta) d\alpha d\beta, \\
I_6' &= -4\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \cdot \partial_\alpha \Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big) \partial_\alpha^3 \varpi(\gamma-\beta) d\alpha d\beta, \\
I_7' &= -\Re \int_{\mathbb{T}} \int_{\mathbb{R}} \overline{\frac{\partial_\alpha^4 z(\gamma)}{A(t)}} \cdot \partial_\alpha z(\gamma) \partial_\alpha z(\gamma) \cdot \Big(\frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}\Big) \partial_\alpha^4 \varpi(\gamma-\beta) d\alpha d\beta.
\end{aligned}$$

To study these terms we have to repeat all estimates as in section 2.1. We select only the terms with different decompositions and we leave to the reader the remaining easy cases.

If we consider the term corresponding to M_4 in section 2.1 we have

$$\begin{aligned}
\Re(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z) &= \Re(\partial_\alpha^4 z) \cdot \Re(\partial_\alpha z) + \Im(\partial_\alpha^4 z) \cdot \Im(\partial_\alpha z), \\
\Im(\overline{\partial_\alpha^4 z} \cdot \partial_\alpha z) &= -\Re(\partial_\alpha^4 z) \cdot \Im(\partial_\alpha z) + \Im(\partial_\alpha^4 z) \cdot \Re(\partial_\alpha z), \\
\Re(\partial_\alpha^4 z \cdot \partial_\alpha z) &= \Re(\partial_\alpha^4 z) \cdot \Re(\partial_\alpha z) - \Im(\partial_\alpha^4 z) \cdot \Im(\partial_\alpha z), \\
\Im(\partial_\alpha^4 z \cdot \partial_\alpha z) &= \Im(\partial_\alpha^4 z) \cdot \Re(\partial_\alpha z) + \Re(\partial_\alpha^4 z) \cdot \Im(\partial_\alpha z),
\end{aligned}$$

and

$$\partial_\alpha z \cdot \partial_\alpha^4 z = -3\partial_\alpha^2 z \cdot \partial_\alpha^3 z.$$

We can write

$$\begin{aligned}\Re(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z) &= \Re(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z) + 2\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z),\\ \Im(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z) &= \Im(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z) - 2\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z).\end{aligned}$$

Thus

$$\begin{aligned}M_4' &= -2\pi\Re\int_{\mathbb{T}}\frac{\overline{\partial_\alpha^4 z(\gamma)}}{A^2(t)}\cdot\partial_\alpha z(\gamma)\partial_\alpha z(\gamma)\cdot\Lambda(\partial_\alpha^4 z^\perp)(\gamma)\varpi(\gamma)d\alpha\\ &= -2\pi\int_{\mathbb{T}}\Re(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z)\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})\\ &\qquad -\Im(\overline{\partial_\alpha^4 z}\cdot\partial_\alpha z)\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\ &= -2\pi\int_{\mathbb{T}}(\Re(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)+2\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z))\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\ &+2\pi\int_{\mathbb{T}}(\Im(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)-2\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z))\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\ &\equiv M_4^{'1}+M_4^{'2}.\end{aligned}$$

We have

$$\begin{aligned}M_4^{'1} &= -2\pi\int_{\mathbb{T}}\Re(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\ &\quad -4\pi\int_{\mathbb{T}}\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha\\ &\quad\equiv M_4^{'11}+M_4^{'12}.\end{aligned}$$

Clearly,

$$M_4^{'11}\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Since

$$\Re(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)}) = \Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp)) - \Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp))$$

we take

$$\begin{aligned}M_4^{'12} &= -4\pi\int_{\mathbb{T}}\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp))d\alpha\\ &\quad+4\pi\int_{\mathbb{T}}\Im(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp))d\alpha\\ &\quad\le\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\ &\quad+k\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)}\\ &\quad+\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\ &\quad+c\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)}.\end{aligned}$$

For $M_4^{'2}$:

$$\begin{aligned} M_4^{'2} &= 2\pi \int_{\mathbb{T}} \Im(-3\partial_\alpha^2 z\cdot\partial_\alpha^3 z)\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha \\ &- 4\pi \int_{\mathbb{T}} \Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)})d\alpha \\ &\equiv M_4^{'21} + M_4^{'22}. \end{aligned}$$

Clearly,

$$M_4^{'21} \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

Since

$$\Im(\partial_\alpha z\cdot\Lambda(\partial_\alpha^4 z^\perp)\frac{\varpi}{A^2(t)}) = \Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp)) + \Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp))$$

we have

$$\begin{aligned} M_4^{'22} &= -4\pi\int_{\mathbb{T}}\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Im(\Lambda(\partial_\alpha^4 z^\perp))d\alpha \\ &- 4\pi\int_{\mathbb{T}}\Re(\partial_\alpha^4 z)\cdot\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\cdot\Re(\Lambda(\partial_\alpha^4 z^\perp))d\alpha \\ &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &+ k\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)} \\ &+ \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &+ c\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)}. \end{aligned}$$

Using a similar method for the rest of the non-integrable terms we obtain

$$\begin{aligned} J_2 &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C(\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} \\ &\qquad + \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} + \|\Im(c)\|_{H^2(S)})\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)}. \end{aligned}$$

In conclusion,

$$\begin{aligned} I_1 &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) \\ &+ C[\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} \\ &\quad + \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} + \|\Im(c)\|_{H^2(S)} - m(t)]\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)}, \end{aligned}$$

and therefore

(13)

$$\begin{aligned} &\frac12\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\gamma)|^2 d\alpha = I_1 + I_2 \\ &\le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) + C[\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} \\ &\quad + \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} + \|\Im(c)\|_{H^2(S)} - m(t) + 2\lambda]\|\Lambda^{\frac12}\partial_\alpha^4 z\|^2_{L^2(S)}. \end{aligned}$$

3. The evolution of the minimum of $\sigma(\gamma, t)$

Taking the divergence in Darcy's law we obtain

$$\Delta p = 0.$$

Since the pressure is zero on the interface and recalling that the flow is irrotational in the interior of the domain Ω by Hopf's lemma we have

$$\sigma(\alpha, t) = -\frac{\partial p}{\partial \eta}|_{z(\alpha,t)} > 0.$$

In spite of this property, we need to get an a priori estimate for the evolution of the minimum of σ in the strip S in order to absorb the high order terms in (13).

Recall that

$$\sigma(\alpha, t) = \frac{\mu^2}{\kappa} BR(z, \varpi)(\alpha, t) \cdot \partial_\alpha^\perp z(\alpha, t) + g\rho^2 \partial_\alpha z_1(\alpha, t). \tag{14}$$

Lemma 3.1. *Let $z(\gamma, t)$ be a solution of the system with $z(\gamma, t) \in \mathcal{C}([0, T]; H^4(S)) \cap \mathcal{C}^1([0, T]; H^3(S))$, and*

$$m(t) = \min_\gamma \sigma(\gamma, t).$$

Then

$$m(t) \geq m(0) - \int_0^t \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}) ds.$$

Proof. We may consider $\gamma_t \in \mathbb{C}$ such that

$$m(t) = \min_\gamma \sigma(\gamma, t) = \sigma(\gamma_t, t).$$

We may calculate the derivative of $m(t)$ to obtain

$$m'(t) = \sigma_t(\gamma_t, t).$$

The identity (14) yields

$$\begin{aligned}
\sigma_t(\gamma, t) = &\frac{\mu^2}{\kappa} \partial_t BR(z, \varpi)(\gamma, t) \cdot \partial_\alpha^\perp z(\gamma, t) + i\lambda \frac{\mu^2}{\kappa} \partial_\alpha BR(z, \varpi)(\gamma, t) \cdot \partial_\alpha^\perp z(\gamma, t) \\
&+ \frac{\mu^2}{\kappa} BR(z, \varpi)(\gamma, t) \cdot \partial_\alpha^\perp z_t(\gamma, t) + \frac{\mu^2}{\kappa} BR(z, \varpi)(\gamma, t) \cdot i\lambda \partial_\alpha^2 z(\gamma, t) \\
&+ g\rho^2 \partial_\alpha z_{1t}(\gamma, t) + g\rho^2 \partial_\alpha^2 z_1(\gamma, t) \equiv R_1 + R_2 + R_3 + R_4 + R_5 + R_6.
\end{aligned}$$

We can easily estimate

$$\begin{aligned}
|R_2| &\leq \lambda \frac{\mu^2}{\kappa} \|\partial_\alpha BR(z, \varpi)\|_{L^\infty(S)} \|\partial_\alpha z\|_{L^\infty(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}), \\
|R_4| &\leq \lambda \frac{\mu^2}{\kappa} \|BR(z, \varpi)\|_{L^\infty(S)} \|z\|_{\mathcal{C}^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}), \\
|R_6| &\leq g\rho^2 \|z\|_{\mathcal{C}^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}),
\end{aligned}$$

and we have

$$|R_3| + |R_5| \leq C(\|BR(z, \varpi)\|_{L^\infty(S)} + 1) \|\partial_\alpha z_t\|_{L^\infty(S)}.$$

Since $z_t(\gamma) = BR(z,\varpi)(\gamma) + c(\gamma)\partial_\alpha z(\gamma)$,

$$\begin{aligned}\|\partial_\alpha z_t\|_{L^\infty(S)} &\leq \|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}+\|\partial_\alpha c\|_{L^\infty(S)}\|\partial_\alpha z\|_{L^\infty(S)}+\|c\|_{L^\infty(S)}\|\partial_\alpha^2 z\|_{L^\infty(S)}\\ &\leq C\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}(1+\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)})\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).\end{aligned}$$

Then,

$$|R_3+R_5| \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Recall that

$$BR(z,\varpi)(\gamma) = \frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z^\perp}{|\Delta z|^2}\varpi(\gamma-\beta)d\beta;$$

then

$$\begin{aligned}BR_t(z,\varpi)(\gamma) &= \frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z_t^\perp}{|\Delta z|^2}\varpi(\gamma-\beta)d\beta - \frac{1}{\pi}\int_{\mathbb{T}}\frac{\Delta z^\perp(\Delta z\cdot\Delta z_t)}{|\Delta z|^4}\varpi(\gamma-\beta)d\beta\\ &\quad+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z^\perp}{|\Delta z|^2}\varpi_t(\gamma-\beta)d\beta \equiv J_1+J_2+J_3.\end{aligned}$$

We get

$$\begin{aligned}J_1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\Delta z_t^\perp\varpi(\gamma-\beta)(\frac{1}{|\Delta z|^2}-\frac{1}{A(t)\beta^2})d\beta\\ &\quad+\frac{1}{2\pi}\int_{\mathbb{T}}\frac{\Delta z_t^\perp}{A(t)\beta^2}\varpi(\gamma-\beta)d\beta \equiv K_1+K_2.\end{aligned}$$

Using that $\Delta z_t^\perp = \beta\int_0^1\partial_\alpha z_t(\phi)ds$,

$$\begin{aligned}K_1 &= \frac{1}{2\pi}\int_{\mathbb{T}}\varpi(\gamma-\beta)\beta\int_0^1\partial_\alpha z_t^\perp(\phi)dsB(\gamma,\beta)d\beta\\ &\leq C\|\mathcal{F}(z)\|^{\frac{3}{2}}_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\varpi\|_{L^\infty(S)}\|\partial_\alpha z_t\|_{L^\infty(S)}\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).\end{aligned}$$

Since

$$\partial_\alpha^2 z_t = \partial^2 BR(z,\varpi)+\partial_\alpha^2 c\partial_\alpha z+2\partial_\alpha c\partial_\alpha^2 z+c\partial^3 z$$

and

$$\begin{aligned}&\|\partial_\alpha^2 BR(z,\varpi)\|_{L^\infty(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\ &\|\partial_\alpha^2 c\partial_\alpha z\|_{L^\infty(S)} = \|\frac{\partial_\alpha^2 z}{|\partial_\alpha z|^2}\cdot\partial_\alpha BR(z,\varpi)\partial_\alpha z\|_{L^\infty(S)}\\ &\qquad\qquad+\|\frac{\partial_\alpha z}{|\partial_\alpha z|^2}\cdot\partial_\alpha^2 BR(z,\varpi)\partial_\alpha z\|_{L^\infty(S)}\\ &\leq C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}(\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}+\|\partial_\alpha^2 BR(z,\varpi)\|_{L^\infty(S)})\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\\ &2\|\partial_\alpha c\partial_\alpha^2 z\|_{L^\infty(S)} \leq 4\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|\partial_\alpha BR(z,\varpi)\|_{L^\infty(S)}\|\partial_\alpha^2 z\|_{L^\infty(S)}\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}),\end{aligned}$$

then

$$\|\partial_\alpha^2 z_t\|_{L^\infty(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Thus,

$$
\begin{aligned}
K_2 &= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\int_0^1 \partial_\alpha z_t^\perp(\phi)ds}{A(t)\beta}\varpi(\gamma-\beta)d\beta\\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\int_0^1 \partial_\alpha z_t^\perp(\phi)-\partial_\alpha z_t^\perp(\gamma)ds}{A(t)\beta}\varpi(\gamma-\beta)d\beta + \frac{1}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z_t^\perp(\gamma)}{A(t)\beta}\varpi(\gamma-\beta)d\beta\\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \frac{\int_0^1\int_0^1 \partial_\alpha^2 z_t^\perp(\psi)(s-1)dtds}{A(t)}\varpi(\gamma-\beta)d\beta + \frac{1}{2}\frac{\partial_\alpha z_t^\perp(\gamma)}{A(t)}H(\varpi)(\gamma)\\
&\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha^2 z_t\|_{L^\infty(S)}\|\varpi\|_{L^\infty(S)} + K\|\mathcal{F}(z)\|_{L^\infty(S)}\|\partial_\alpha z_t\|_{L^\infty(S)}\|\varpi\|_{\mathcal{C}^\delta(S)}.
\end{aligned}
$$

Therefore,

$$J_1 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

In the same way, it is easy to see that

$$J_2 \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).$$

Finally, since

$$
\begin{aligned}
\frac{\Delta z^\perp}{|\Delta z|^2} - \frac{\partial_\alpha z^\perp(\gamma)}{A(t)\beta} &= \frac{\beta^2\int_0^1\int_0^1 \partial_\alpha^2 z(\psi)(s-1)dtds}{|\Delta z|^2}\\
&+ \frac{\beta^2\partial_\alpha z(\gamma)\int_0^1\int_0^1\partial_\alpha^2 z(\psi)(s-1)dtds\cdot\int_0^1[\partial_\alpha z(\gamma)+\partial_\alpha z(\phi)]ds}{A(t)|\Delta z|^2},
\end{aligned}
$$

$$
\begin{aligned}
J_3 &= \frac{1}{2\pi}\int_{\mathbb{T}}(\frac{\Delta z^\perp}{|\Delta z|^2} - \frac{\partial_\alpha z^\perp(\gamma)}{A(t)\beta})\varpi_t(\gamma-\beta)d\beta + \frac{1}{2\pi}\int_{\mathbb{T}}\frac{\partial_\alpha z^\perp(\gamma)}{A(t)\beta}\varpi_t(\gamma-\beta)d\beta\\
&\equiv K_5 + K_6,
\end{aligned}
$$

where

$$
\begin{aligned}
K_5 &\le C\|\mathcal{F}(z)\|_{L^\infty(S)}\|z\|_{\mathcal{C}^2(S)}\|\varpi_t\|_{L^2(S)},\\
K_6 &= \frac{1}{2}\frac{\partial_\alpha z^\perp(\gamma)}{A(t)}H(\varpi_t)(\gamma) \le C\|\mathcal{F}(z)\|^{\frac{1}{2}}_{L^\infty(S)}\|\varpi_t\|_{\mathcal{C}^\delta(S)}.
\end{aligned}
$$

In order to control $\|\varpi_t\|_{\mathcal{C}^\delta(S)}$ we proceed as in section 9 in [8]. Therefore,

$$|\sigma_t(\gamma,t)| \le \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

give us

$$m'(t) \ge -\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})$$

for almost every t. A further integration yields

$$m(t) \ge m(0) - \int_0^t \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})ds.$$

□

4. Instant analyticity

Theorem 4.1. *Let $z(\alpha,0)=z_0(\alpha)\in H^4$, $\mathcal{F}(z)(z_0)(\alpha,\beta)\in L^\infty$. Then there exists a solution of the Muskat problem $z(\alpha,t)$ defined for $0<t\leq T$ that continues analytically into the strip $S(t)=\{\alpha\pm i\varsigma : |\varsigma|<\lambda t\}$ for each t. Here, λ and T are determined by upper bounds of the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of the $\sigma(\alpha,0)$. Moreover, for $0<t\leq T$, the quantity*

$$\sum_{\pm}\int_{\mathbb{T}}(|z(\alpha\pm i\lambda t)-(\alpha\pm i\lambda t)|^2+|\partial_\alpha^4 z(\alpha\pm i\lambda t)|^2)d\alpha$$

is bounded by a constant determinate by upper bounds for the H^4 norm and the arc-chord constant of the initial data and a positive lower bound of $\sigma(\alpha,0)$.

Proof. For all estimates in above sections we have finally

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\alpha\pm i\lambda t)|^2 \leq{}& \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})\\ &+(2\lambda+C\|f\|(t)-m(t))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}(t)\end{aligned}$$

where

$$\begin{aligned}\|f\|(t)={}&\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}+\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\varpi)\|_{H^2(S)}\\ &+\|\Im(\partial_\alpha z)\Im(\partial_\alpha z\varpi)\|_{H^2(S)}+\|\Im(c)\|_{H^2(S)}.\end{aligned}$$

Note that $\|f\|(0)=0$. If $2\lambda-m(0)<0$ we will show that

$$2\lambda+K\|f\|(t)-m(t)<0$$

for a short time. It yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}}|\partial_\alpha^4 z(\alpha\pm i\lambda t)|^2 d\alpha\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$

as long as $2\lambda+K\|f\|(t)-m(t)<0$. We proceed as in section 8 in [8] to show that

$$\frac{d}{dt}\|\mathcal{F}(z)\|_{L^\infty(S)}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

From the two inequalities above and (8) it is easy to obtain a priori energy estimates that depend upon the negativity of $2\lambda+K\|f\|(t)-m(t)$. We denote

$$\|z\|_{RT}(t)\equiv\|\mathcal{F}(z)\|^2_{L^\infty(S)}(t)+\|z\|^2_{L^2(S)}+\frac{1}{m(t)-2\lambda-C\|f\|}.$$

At this point it is easy to find

$$\|f\|\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})$$

and

$$\frac{d}{dt}(\frac{1}{m(t)-2\lambda-C\|f\|})\leq\frac{\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})}{(m(t)-2\lambda-C\|f\|)^2}.$$

Then,

$$\frac{d}{dt}\|z\|_{RT}(t)\leq \exp C(\|z\|_{RT}(t)),$$

and therefore

$$\|z\|_{RT}\leq -\log(\exp(-C\|z\|_{RT}(0)-C^2t)).$$

Now we approximate the problem as follows:

$$\begin{cases} z_t^\epsilon(\alpha,t) = BR(z^\epsilon,\varpi^\epsilon)(\alpha,t) + c^\epsilon(\alpha,t)\partial_\alpha z^\epsilon(\alpha,t), \\ z^\epsilon(\alpha,0) = \phi_\epsilon * z_0(\alpha) \end{cases}$$

where

$$\begin{aligned} c^\epsilon(\alpha,t) = \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\alpha z^\epsilon(\alpha,t)}{|\partial_\alpha z^\epsilon(\alpha,t)|^2}\cdot \partial_\alpha BR(z^\epsilon,\varpi^\epsilon)(\alpha,t)d\alpha \\ - \int_{-\pi}^{\alpha} \frac{\partial_\alpha z^\epsilon(\beta,t)}{|\partial_\alpha z^\epsilon(\beta,t)|^2}\cdot \partial_\alpha BR(z^\epsilon,\varpi^\epsilon)(\beta,t)d\beta, \end{aligned}$$

$$\varpi^\epsilon(\alpha,t) = -\phi_\epsilon * \phi_\epsilon * (2BR(z^\epsilon,\varpi^\epsilon)\cdot\partial_\alpha z^\epsilon)(\alpha) - 2\kappa\frac{\rho^2}{\mu^2}\phi_\epsilon*\phi_\epsilon*(\partial_\alpha {z_2}^\epsilon)(\alpha)$$

where $\phi_\epsilon(\alpha) = \phi(\frac{\alpha}{\epsilon})/\epsilon$ for $\epsilon > 0$ and ϕ the heat kernel.

Picard's Theorem yields the existence of a solution $z^\epsilon(\alpha)$ in $\mathcal{C}([0,T^\epsilon);H^4)$ which is analytic in the whole space for z_0 satisfying the arc-chord condition and ϵ small enough. Using the same techniques we have denoted above we obtain a bound for $z^\epsilon(\alpha,t)$ in H^4 in the strip $S(t)$ for a small enough T which is independent of ϵ. We need arc-chord condition $z_0 \in H^4$ and $2\lambda - m(0) < 0$. Then we pass to the limit. □

5. Decay estimates on the strip of analyticity

Theorem 5.1. *Let $z(\alpha,0) = z^0(\alpha)$ be an analytic curve in the strip*

$$S = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h(0)\},$$

with $h(0) > 0$ and satisfying:

- *The arc-chord condition, $\mathcal{F}(z^0)(\alpha+i\varsigma,\beta) \in L^\infty(S\times\mathbb{R})$*
- *The curve $z^0(\alpha)$ is real for real α*
- *The functions $z_1^0(\alpha) - \alpha$ and ${z_2}^0(\alpha)$ are periodic with period 2π*
- *The functions ${z_1}^0(\alpha) - \alpha$ and ${z_2}^0(\alpha)$ belong to $H^4(S)$*

Then there exist a time T and a solution of the Muskat problem $z(\alpha,t)$ defined for $0 < t \le T$ that continues analytically into some complex strip for each fixed $t \in [0,T]$. Here T is a small constant depending only on $\exp C(\|\mathcal{F}(z^0)\|^2_{L^\infty(S)} + \|z^0\|^2_{L^2(S)})$.

We will use the following:

Lemma 5.1. *Let $\psi(\alpha \pm i\xi) = \sum_{k=-N}^{N} A_k(t)e^{ik\alpha}e^{\pm k\xi}$ and $h(t) > 0$ be a decreasing function of t. Then*

$$\begin{aligned} \frac{\partial}{\partial t}\sum_{\pm}\int_{\mathbb{T}} |\psi(\alpha \pm ih(t))|^2 d\alpha \le \frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}} \Lambda\psi(\alpha\pm ih(t))\overline{\psi(\alpha\pm ih(t))}d\alpha \\ - 10h'(t)\int_{\mathbb{T}}\Lambda\psi(\alpha)\overline{\psi}(\alpha)d\alpha + 2\Re\sum_{\pm}\int_{\mathbb{T}}\psi_t(\alpha\pm ih(t))\overline{\psi(\alpha\pm ih(t))}d\alpha. \end{aligned}$$

This lemma is a corollary of Lemma 4.2 in [4], and it allows us to prove Theorem 5.1.

Proof of Theorem 5.1. The norms $\|z\|_{L^2(S)}$ and $\|z\|_{H^k(S)}$ are defined as before using the new strip $S(t)$ defined by

$$S(t) = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h(t)\}$$

where $h(t)$ is a positive decreasing function of t.

We use the Galerkin approximation of equation (2), i.e.,

$$z_t^{[N]}(\gamma, t) = \Pi_N[J[z^{[N]}]](\gamma, t)$$

where $\gamma \in \overline{S(t)}$, Π_N will be defined below, and

$$J[z](\alpha, t) = BR(z, \varpi)(\alpha, t) + c(\alpha, t)\partial_\alpha z(\alpha, t).$$

We impose the initial condition

$$z^{[N]}(\alpha, 0) = z^{[N]}(\alpha).$$

Here, for a large enough positive integer N, we define $z^{[N]}(\alpha, 0)$ from $z^0(\alpha)$ by using the projection

$$\Pi_N : \sum_{-\infty}^{\infty} A_k e^{ik\alpha} \to \sum_{-N}^{N} A_k e^{ik\alpha}.$$

We define $z^{[N]}(\alpha)$ by stipulating that

$${z_1}^{[N]}(\alpha) - \alpha = \Pi_N[{z_1}^0(\alpha) - \alpha]$$

and

$${z_2}^{[N]}(\alpha) = \Pi_N[{z_2}^0(\alpha)].$$

For N large enough, the functions $z^{[N]}(\alpha, 0)$ satisfy the arc-chord and Rayleigh-Taylor conditions.

We shall consider the evolution of the most singular quantity

$$\sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^4 z^{[N]}(\alpha \pm ih_N(t), t)|^2 d\alpha$$

where $h_N(t)$ is a smooth positive decreasing function on t, with $h_N(0) = h(0)$, which will be given below. Also we denote

$$S_N(t) = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h_N(t)\}.$$

We will drop the dependency on N from $z^{[N]}$ and $h_N(t)$ in our notation. Using the lemma above,

$$\begin{aligned}
\frac{d}{dt} \sum_{\pm} \int_{\mathbb{T}} |\partial_\alpha^4 z_j(\alpha \pm ih(t))|^2 d\alpha \leq{}& \frac{h'(t)}{10} \sum_{\pm} \int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_j)(\alpha \pm ih(t)) \overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))} \\
&- 10h'(t) \int_{\mathbb{T}} \Lambda(\partial_\alpha^4 z_j)(\alpha) \overline{\partial_\alpha^4 z_j(\alpha)} d\alpha \\
&+ 2 \sum_{\pm} \Re \int_{\mathbb{T}} \Pi_N[\partial_\alpha^4 J_j[z]](\alpha \pm ih(t)) \overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))}.
\end{aligned}$$

Since $\partial_\alpha^4 z_j(\alpha \pm ih(t))$ is a trigonometric polynomial in the range of Π_N,

$$\begin{aligned}
&2\sum_{\pm}\Re\int_{\mathbb{T}}\Pi_N[\partial_\alpha^4 J_j[z]](\alpha \pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))}\\
&= 2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4 J_j[z](\alpha \pm ih(t))\Pi_N[\overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))}]\\
&= 2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4 J_j[z](\alpha \pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z_j(\alpha \pm ih(t))|^2 d\alpha \leq \frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha \pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))}\\
&-10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\overline{\partial_\alpha^4 z_j(\alpha)}d\alpha\\
&+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4 BR(z_j,\varpi)(\alpha \pm ih(t))\overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))}\\
&+2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4(c(\alpha \pm ih(t))\partial_\alpha z_j(\alpha \pm ih(t)))\overline{\partial_\alpha^4 z_j(\alpha \pm ih(t))}\\
&\equiv M_1+M_2+M_3+M_4.
\end{aligned}$$

To estimate M_3 and M_4 we have to repeat the arguments in section 2, with the exception of the term $R_{20}+P_7$. Following in the same way, we will get that

$$\begin{aligned}
M_3 \leq &\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)})+C\|\Im(\frac{\varpi}{A(t)})\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}\\
&-2\Re\int_{\mathbb{T}}\frac{\sigma(\gamma)}{A(t)}\partial_\alpha^4 z_j(\gamma)\cdot\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha
\end{aligned}$$

where $\gamma=\alpha \pm ih(t)$.

In order to avoid problems we write

$$\sigma(\gamma)=\sigma(\alpha)+h(t)g_{\pm}(\alpha)$$

where $g_{\pm}=\frac{1}{h(t)}(\sigma(\gamma)-\sigma(\alpha))$.

Since

$$\sigma(\alpha)=\frac{\mu^2}{\kappa}BR(z,\varpi)(\alpha)\cdot\partial_\alpha^\perp z(\alpha)+g\rho^2\partial_\alpha z_1(\alpha),$$

we can write

$$g_{\pm}=\pm\frac{i\mu^2}{\kappa}\int_0^1\partial_\alpha(BR(z,\varpi)\cdot\partial_\alpha^\perp z)(\gamma t+(t-1)\alpha)dt \pm ig\rho^2\int_0^1\partial_\alpha^2 z_1(\gamma t+(t-1)\alpha)dt;$$

then

$$\|g_{\pm}\|_{H^2(S)}\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Thus, we get

$$\begin{aligned}
&-2\Re\int_{\mathbb{T}}\frac{\sigma(\gamma)}{A(t)}\partial_\alpha^4 z_j(\gamma)\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha = -2\Re\int_{\mathbb{T}}\frac{\sigma(\alpha)}{A(t)}\partial_\alpha^4 z_j(\gamma)\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha\\
&-2h(t)\Re\int_{\mathbb{T}}\frac{g_{\pm}(\alpha)}{A(t)}\partial_\alpha^4 z_j(\gamma)\Lambda(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha \equiv M_3^1+M_3^2.
\end{aligned}$$

On the one hand, since $\Re(\frac{\sigma}{A(t)}) > 0$ and $2g\Lambda(g) - \Lambda(g^2) \geq 0$,

$$\begin{aligned}M_3^1 &= -2\int_{\mathbb{T}} \Re(\frac{\sigma}{A(t)})(\Re(\partial_\alpha^4 z_j)\Re(\Lambda(\partial_\alpha^4 z_j)) + \Im(\partial_\alpha^4 z_j)\Im(\Lambda(\partial_\alpha^4 z_j)))d\alpha\\ &\leq \|\Lambda(\frac{\sigma}{A(t)})\|_{L^\infty(S)}\|\partial_\alpha^4 z\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}).\end{aligned}$$

On the other hand, as in the term N_5 in section 2.1,

$$\begin{aligned}M_3^2 &= -2h(t)\Re\int_{\mathbb{T}} \Lambda^{\frac{1}{2}}(\frac{g_\pm(\alpha)}{A(t)}\partial_\alpha^4 z_j(\gamma))\Lambda^{\frac{1}{2}}(\overline{\partial_\alpha^4 z_j})(\gamma)d\alpha\\ &\leq Ch(t)\|\frac{g_\pm}{A(t)}\|_{H^2(S)}(\|\partial_\alpha^4 z\|_{L^2(S)} + \|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|_{L^2(S)}\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\ &\quad + Ch(t)\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.\end{aligned}$$

For M_1,

$$M_1 \leq \frac{h'(t)}{10}\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)},$$

and for M_4,

$$\begin{aligned}M_4 &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\ &\quad + C(\|\Im(\partial_\alpha z)\Re(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha z)\Im(\partial_\alpha z\frac{\varpi}{A^2(t)})\|_{H^2(S)}\\ &\quad + \|\Im(c)\|_{H^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}\\ &\quad \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.\end{aligned}$$

Then,

$$\begin{aligned}&\frac{d}{dt}\sum_\pm\int_{\mathbb{T}}|\partial_\alpha^4 z_j(\alpha \pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})\\ &- 10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\overline{\partial_\alpha^4 z_j(\alpha)}d\alpha\\ &+ (\exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})h(t) + \frac{h'(t)}{10}\\ &\qquad\qquad + \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)}))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 z\|^2_{L^2(S)}.\end{aligned}$$

Choosing

$$h(t) = \exp(-10\int_0^t G(r)dr)[\int_0^t -10G(r)\exp(10\int_0^r G(s)ds)dr + h(0)]$$

where $G(t) = \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)} + \|z\|^2_{H^4(S)})(t)$, we eliminate the most dangerous term. The other term in the expression above is

$$\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\partial_\alpha^4 z_j(\alpha)d\alpha \leq \frac{C}{h(t)}\sum_\pm\int_{\mathbb{T}}|\partial_\alpha^4 z_j|^2 d\alpha$$

as one sees by examining the Fourier expansion of $\partial_\alpha^4 z_j(\alpha,t)$. Thus,

$$\begin{aligned}&|-10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 z_j)(\alpha)\partial_\alpha^4 z_j(\alpha)d\alpha| \leq C\frac{|h'(t)|}{h(t)}(\|z\|^2_{H^4(S)}+\|\mathcal{F}(z)\|^2_{L^\infty(S)})\\ &\leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).\end{aligned}$$

Finally, we obtain

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z(\alpha\pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(z)\|^2_{L^\infty(S)}+\|z\|^2_{H^4(S)}).$$

Recovering the dependency on N in our notation we have that

$$\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 z^{[N]}(\alpha\pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(z^{[N]})\|^2_{L^\infty(S_N)}+\|z^{[N]}\|^2_{L^2(S_N)}). \tag{15}$$

This estimate is true wherever $t\in[0,T_N]$, where T_N is the maximal time of existence of the solution $z^{[N]}$. In addition inequality (15) shows that we can extend these solutions in $H^4(S)$ up to a small enough time T independent of N and dependent on the initial data. □

6. Non-splat singularity

As we said in the introduction, it is necessary to consider a transformed Muskat problem, and we need to prove instant analyticity and decay estimates in $\tilde{\Omega}$. We will prove that the energy estimates of Theorems 4.1 and 5.1 hold in $\tilde{\Omega}$ for solutions $\tilde{z}$ of equations:

$$\tilde{z}_t(\alpha,t) = Q^2(\alpha,t)BR(\tilde{z},\tilde{\varpi})(\alpha,t)+\tilde{c}(\alpha)\partial_\alpha\tilde{z}(\alpha,t) \tag{16}$$

where

$$Q^2(\alpha,t) = |\frac{dP}{dw}(z(\alpha,t))|^2 = |\frac{dP}{dw}(P^{-1}(\tilde{z}(\alpha,t)))|^2, \tag{17}$$

$$\tilde{\varpi}(\alpha,t) = -2BR(\tilde{z},\tilde{\varpi})(\alpha,t)\cdot\partial_\alpha\tilde{z}(\alpha,t) - 2\kappa g\frac{\rho^2}{\mu^2}\partial_\alpha(P_2^{-1}(\tilde{z}(\alpha,t))) \tag{18}$$

and

$$\begin{aligned}\tilde{c}(\alpha,t) &= \frac{\alpha+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta\\ &\quad -\int_{-\pi}^{\alpha}\frac{\partial_\beta\tilde{z}(\beta,t)}{|\partial_\beta\tilde{z}(\beta,t)|^2}\cdot\partial_\beta BR(\tilde{z},\tilde{\varpi})(\beta,t)d\beta\end{aligned} \tag{19}$$

with $\tilde{z}\in\mathcal{C}([0,T],H^k)$ for $k\geq 4$.

6.1. Instant analyticity in $\tilde{\Omega}$ domain.

We define

$$\begin{aligned}&q^0=(0,0),\quad q^1=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),\quad q^2=(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),\\ &q^3=(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}),\quad q^4=(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}),\end{aligned}$$

which are the singular points of the P^{-1} conformal map. We set $z(\alpha,t)$ to hold $\tilde{z}(\alpha,t)\neq q^l$ for $l=0,1,2,3,4$. In order to get this we fix $\overline{\Omega(0)}$ so that $\frac{dP}{dw}(w)\neq 0$ for any $w\in\overline{\Omega(0)}$ without loss of generality.

We define the energy

$$\|\tilde{z}\|_{RT} \equiv \|\tilde{z}\|^2_{H^k(S)} + \|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \frac{1}{m(Q^2\tilde{\sigma})(t) - 2\lambda - \|g\|(t)} + \sum_{l=0}^{4} \frac{1}{m(q^l)(t)}$$

where

$$\begin{aligned}\|g\|(t) =& C(\|\Im(\partial_\alpha \tilde{z})\Re(\partial_\alpha \tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha \tilde{z})\Im(\partial_\alpha \tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)} \\ &+ \|\Im(\frac{\tilde{\varpi}Q^2}{A(t)})\|_{H^2(S)} + \|\Im(\tilde{c})\|_{H^2(S)})\end{aligned}$$

and

$$m(Q^2\tilde{\sigma})(t) = \min_\alpha Q^2(\alpha,t)\sigma(\alpha,t), \quad m(q^l)(t) = \min_\alpha |\tilde{z}(\alpha,t) - q^l|.$$

Theorem 6.1. *Let $\tilde{z}(\alpha,t)$ be a solution of* (16)-(19)*. Then, the following estimate holds:*

$$\frac{d}{dt}\|\tilde{z}\|_{RT} \leq \exp C(\|\tilde{z}\|_{RT})$$

for C constant.

Remark 6.1. We will show the proof for $k = 4$, the rest of the cases being analogous.

Proof. We have to estimate

$$\frac{d}{dt}\|\partial^4_\alpha \tilde{z}\|^2_{L^2(S)}.$$

We quote [3] for dealing with the Q^2 term. This factor does not introduce a high order term

$$\|Q^2\|_{H^k(S)} \leq \exp C(\|\tilde{z}\|_{RT}).$$

Then we have to repeat all estimates in section 2 in which Q^2 is involved. We will show below how to deal with them.

We find that

$$\begin{aligned}&\frac{d}{dt}\|\partial^4_\alpha \tilde{z}\|^2_{L^2(S)} \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + 2\lambda\|\Lambda^{\frac{1}{2}}\partial^4_\alpha \tilde{z}\|^2_{L^2(S)} \\ &+ J_1 + J_2\end{aligned}$$

where

$$\begin{aligned}J_1 &= \Re\int_{\mathbb{T}} \overline{\partial^4_\alpha \tilde{z}(\gamma)} \cdot \partial^4_\alpha(Q^2(\gamma)BR(\tilde{z},\tilde{\varpi})(\gamma))d\alpha, \\ J_2 &= \Re\int_{\mathbb{T}} \overline{\partial^4_\alpha \tilde{z}(\gamma)} \cdot \partial^4_\alpha(\tilde{c}(\gamma)\partial_\alpha \tilde{z}(\gamma))d\alpha.\end{aligned}$$

We get $J_1 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + I_7$ where

$$I_7 = \Re\int_{\mathbb{T}} \overline{\partial^4_\alpha \tilde{z}(\gamma)} \cdot Q^2(\gamma)\partial^4_\alpha BR(\tilde{z},\tilde{\varpi})(\gamma)d\alpha.$$

As in Remark 2.1 we split $I_7 = \tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6 + \tilde{I}_7$ in the same way so we have

$$\begin{aligned}\tilde{I}_3 + \tilde{I}_4 + \tilde{I}_5 + \tilde{I}_6 \leq& \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\ &+ C\|\Im(\frac{\tilde{\varpi}}{A(t)}Q^2)\|_{H^2(S)}\|\Lambda^{\frac{1}{2}}\partial^4_\alpha \tilde{z}\|^2_{L^2(S)}\end{aligned}$$

and $\tilde{I}_7 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + \tilde{K}_9$ where

$$\tilde{K}_9 = \frac{1}{2}\Re\int_{\mathbb{T}} \overline{\partial^4_\alpha \tilde{z}(\gamma)} \cdot \frac{\partial^\perp_\alpha \tilde{z}(\gamma)}{|\partial_\alpha \tilde{z}|^2} H(\partial^4_\alpha \tilde{\varpi})(\gamma) Q^2(\gamma) d\alpha.$$

Identity $H(\partial_\alpha) = \Lambda$ allows us to rewrite $\tilde{K}_9$ as

$$\tilde{K}_8 = \frac{1}{2}\Re\int_{\mathbb{T}} \Lambda(\overline{\partial^4_\alpha \tilde{z}} \cdot \frac{\partial^\perp_\alpha \tilde{z}}{|\partial_\alpha \tilde{z}|^2} Q^2)(\gamma) \partial^3_\alpha \tilde{\varpi}(\gamma) d\alpha.$$

Using the formula (18), we decompose $\tilde{K}_9 = \tilde{P}_7 + \tilde{P}_8$:

$$\begin{aligned}
\tilde{P}_7 &= -\kappa g \frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha \tilde{z}} \cdot Q^2 \partial^\perp_\alpha \tilde{z})(\gamma)}{A(t)} \partial^4_\alpha (P_2^{-1}(\tilde{z}(\gamma))) d\alpha,\\
\tilde{P}_8 &= -\frac{1}{2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha \tilde{z}} \cdot Q^2 \partial^\perp_\alpha \tilde{z})(\gamma)}{A(t)} \partial^3_\alpha \tilde{T}(\tilde{\varpi})(\gamma) d\alpha
\end{aligned}$$

where $\tilde{T}(\tilde{\varpi}) = -2BR(\tilde{z}, \tilde{\varpi}) \cdot \partial_\alpha \tilde{z}$.

The term $\tilde{P}_8$ can be estimated as was the term P_8 in subsection 2.1.2. An analogous approach provides

$$\begin{aligned}
&\tilde{P}_8 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)})\\
&- \Re\int_{\mathbb{T}} \frac{Q^2(\gamma) BR(\tilde{z}, \tilde{\varpi})(\gamma) \cdot \partial^\perp_\alpha \tilde{z}(\gamma)}{A(t)} \partial^4_\alpha \tilde{z}(\gamma) \cdot \Lambda(\overline{\partial^4_\alpha \tilde{z}})(\gamma) d\alpha.
\end{aligned} \tag{20}$$

For $\tilde{P}_7$ we consider the most singular terms: $\tilde{P}_7 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) + \tilde{P}^1_7$ where

$$\tilde{P}^1_7 = -\kappa g \frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha \tilde{z}} \cdot Q^2 \partial^\perp_\alpha \tilde{z})(\gamma)}{A(t)} \nabla P_2^{-1}(\tilde{z}(\gamma)) \cdot \partial^4_\alpha \tilde{z}(\gamma) d\alpha.$$

Then we split $\tilde{P}^1_7 = \tilde{P}^{11}_7 + \tilde{P}^{12}_7 + \tilde{P}^{13}_7 + \tilde{P}^{14}_7$ by writing the component of the curve:

$$\begin{aligned}
\tilde{P}^{11}_7 &= \kappa g \frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha \tilde{z}_1} Q^2 \partial_\alpha \tilde{z}_2)(\gamma)}{A(t)} \partial_{\tilde{x}_1} P_2^{-1}(\tilde{z}(\gamma)) \partial^4_\alpha \tilde{z}_1(\gamma) d\alpha,\\
\tilde{P}^{12}_7 &= \kappa g \frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha \tilde{z}_1} Q^2 \partial_\alpha \tilde{z}_2)(\gamma)}{A(t)} \partial_{\tilde{x}_2} P_2^{-1}(\tilde{z}(\gamma)) \partial^4_\alpha \tilde{z}_2(\gamma) d\alpha,\\
\tilde{P}^{13}_7 &= -\kappa g \frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha \tilde{z}_2} Q^2 \partial_\alpha \tilde{z}_1)(\gamma)}{A(t)} \partial_{\tilde{x}_1} P_2^{-1}(\tilde{z}(\gamma)) \partial^4_\alpha \tilde{z}_1(\gamma) d\alpha,\\
\tilde{P}^{14}_7 &= -\kappa g \frac{\rho^2}{\mu^2}\Re\int_{\mathbb{T}} \frac{\Lambda(\overline{\partial^4_\alpha \tilde{z}_2} Q^2 \partial_\alpha \tilde{z}_1)(\gamma)}{A(t)} \partial_{\tilde{x}_2} P_2^{-1}(\tilde{z}(\gamma)) \partial^4_\alpha \tilde{z}_2(\gamma) d\alpha.
\end{aligned}$$

The commutator estimate yields

$$
\begin{aligned}
\tilde{P}_7^{11} &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&+ \kappa g \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\partial_{\tilde{x}_1} P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_2(\gamma)\partial_\alpha^4 \tilde{z}_1(\gamma)\Lambda(\overline{\partial_\alpha^4 \tilde{z}_1})(\gamma) d\alpha, \qquad (21)\\
\tilde{P}_7^{12} &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&+ \kappa g \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\partial_{\tilde{x}_2} P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_2(\gamma)\partial_\alpha^4 \tilde{z}_2(\gamma)\Lambda(\overline{\partial_\alpha^4 \tilde{z}_1})(\gamma) d\alpha, \\
\tilde{P}_7^{13} &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&- \kappa g \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\partial_{\tilde{x}_1} P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_1(\gamma)\partial_\alpha^4 \tilde{z}_1(\gamma)\Lambda(\overline{\partial_\alpha^4 \tilde{z}_2})(\gamma) d\alpha, \\
\tilde{P}_7^{14} &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&- \kappa g \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\partial_{\tilde{x}_2} P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_1(\gamma)\partial_\alpha^4 \tilde{z}_2(\gamma)\Lambda(\overline{\partial_\alpha^4 \tilde{z}_2})(\gamma) d\alpha. \qquad (22)
\end{aligned}
$$

Using that

$$\partial_\alpha \tilde{z}_2 \partial_\alpha^4 \tilde{z}_2 = -3\partial_\alpha^2 \tilde{z} \cdot \partial_\alpha^3 \tilde{z} - \partial_\alpha \tilde{z}_1 \partial_\alpha^4 \tilde{z}_1,$$

we get

$$
\begin{aligned}
\tilde{P}_7^{12} &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&- \kappa g \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\partial_{\tilde{x}_2} P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_1(\gamma)\partial_\alpha^4 \tilde{z}_1(\gamma)\Lambda(\overline{\partial_\alpha^4 \tilde{z}_1})(\gamma) d\alpha, \qquad (23)\\
\tilde{P}_7^{13} &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&+ \kappa g \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\partial_{\tilde{x}_1} P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \partial_\alpha \tilde{z}_2(\gamma)\partial_\alpha^4 \tilde{z}_2(\gamma)\Lambda(\overline{\partial_\alpha^4 \tilde{z}_2})(\gamma) d\alpha. \qquad (24)
\end{aligned}
$$

Adding the inequalities (21), (23), (24) and (22) it is easy to check that

$$
\begin{aligned}
\tilde{P}_7 &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&- \kappa g \frac{\rho^2}{\mu^2} \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\nabla P_2^{-1}(\tilde{z}(\gamma))}{A(t)} \cdot \partial_\alpha^\perp \tilde{z}(\gamma)\partial_\alpha^4 \tilde{z}(\gamma) \cdot \Lambda(\partial_\alpha^4 \tilde{z})(\gamma) d\alpha.
\end{aligned}
$$

The above inequality together with (20) lets us obtain

$$
\begin{aligned}
\tilde{K}_9 &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\
&- \Re \int_{\mathbb{T}} \frac{Q^2(\gamma)\tilde{\sigma}(\gamma)}{A(t)} \partial_\alpha^4 \tilde{z}(\gamma) \cdot \Lambda(\partial_\alpha^4 \tilde{z})(\gamma) d\alpha
\end{aligned}
$$

with $\tilde{\sigma}$ given in (5).

Considering $m(Q^2\tilde{\sigma})(t)$ and the pointwise inequality $2f\Lambda(f) \geq \Lambda(f^2)$ we check that

$$\tilde{I}_7 \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) - m(Q^2\tilde{\sigma})(t)\|\partial_\alpha^4 \tilde{z}\|^2_{L^2(S)}.$$

For J_2 it is easy to deal with $\partial_\alpha^4 \tilde{c}$ in the same way as in section 2.1.3. The analogous approach provides

$$\begin{aligned} J_2 &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\ &+ C(\|\Im(\partial_\alpha \tilde{z})\Re(\partial_\alpha \tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)} + \|\Im(\partial_\alpha \tilde{z})\Im(\partial_\alpha \tilde{z}\frac{\tilde{\varpi}Q^2}{A^2(t)})\|_{H^2(S)} \\ &+ \|\Im(\tilde{c})\|_{H^2(S)})\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 \tilde{z}\|^2_{L^2(S)}. \end{aligned}$$

Finally we obtain

$$\begin{aligned} \frac{d}{dt}\|\partial_\alpha^4 \tilde{z}\|^2_{L^2(S)} &\leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)} + \|\tilde{z}\|^2_{H^4(S)}) \\ &+ (2\lambda + \|g\| - m(Q^2\tilde{\sigma}))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4 \tilde{z}\|^2_{L^2(S)}. \end{aligned}$$

Bearing in mind the singular points of the P^{-1} together with the estimation for $m(Q^2\tilde{\sigma})(t)$, which we can obtain in an analogous way as in section 3, we have the desired estimate. □

6.2. **Decay of the strip of analyticity in the $\tilde{\Omega}$ domain.**

Theorem 6.2. *Let $\tilde{z}(\alpha, 0) = \tilde{z}^0(\alpha)$ be an analytic curve in the strip*

$$S = \{\alpha + i\varsigma \in \mathbb{C} : |\varsigma| < h(0)\},$$

with $h(0) > 0$ and satisfying:

* *The arc-chord condition, $\mathcal{F}(\tilde{z}^0)(\alpha + i\varsigma, \beta) \in L^\infty(S \times \mathbb{R})$*
* *The curve $\tilde{z}^0(\alpha)$ is real for real α*
* *The functions $\tilde{z}_1^0(\alpha) - \alpha$ and $\tilde{z}_2{}^0(\alpha)$ are periodic with period 2π*
* *The functions $\tilde{z}_1{}^0(\alpha) - \alpha$ and $\tilde{z}_2{}^0(\alpha)$ belong to $H^4(S)$*

Then there exist a time T and a solution of the Muskat problem in $\tilde{\Omega}$, $\tilde{z}(\alpha, t)$ defined for $0 < t \leq T$ that continues analytically into some complex strip for each fixed $t \in [0, T]$. Here T is a small constant depending only on $\exp C(\|\mathcal{F}(\tilde{z}^0)\|^2_{L^\infty(S)} + \|\tilde{z}^0\|^2_{L^2(S)})$.

Proof. Here we proceed in the same way as in the proof of Theorem 5.1.

After we use the Galerkin approximation, by Lemma 5.1 we get

$$\begin{aligned} &\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))|^2 d\alpha \leq \frac{h'(t)}{10}\sum_{\pm}\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 \tilde{z}_j)(\alpha \pm ih(t))\overline{\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))} \\ &- 10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4 \tilde{z}_j)(\alpha)\overline{\partial_\alpha^4 \tilde{z}_j(\alpha)}d\alpha \\ &+ 2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4(Q^2 BR(\tilde{z}_j, \tilde{\varpi})(\alpha \pm ih(t))\overline{\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))} \\ &+ 2\sum_{\pm}\Re\int_{\mathbb{T}}\partial_\alpha^4(\tilde{c}(\alpha \pm ih(t))\partial_\alpha \tilde{z}_j(\alpha \pm ih(t)))\overline{\partial_\alpha^4 \tilde{z}_j(\alpha \pm ih(t))}. \end{aligned}$$

We write

$$Q^2(\gamma)\tilde{\sigma}(\gamma) = Q^2(\alpha)\tilde{\sigma}(\alpha) + h(t)\tilde{g_\pm}(\alpha)$$

and we have

$$\begin{aligned}
&\frac{d}{dt}\sum_{\pm}\int_{\mathbb{T}}|\partial_\alpha^4\tilde{z}_j(\alpha\pm ih(t))|^2 d\alpha \leq \exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})\\
&-10h'(t)\int_{\mathbb{T}}\Lambda(\partial_\alpha^4\tilde{z}_j)(\alpha)\overline{\partial_\alpha^4\tilde{z}_j(\alpha)}d\alpha\\
&+(\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})h(t)+\frac{h'(t)}{10}\\
&+\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)}))\|\Lambda^{\frac{1}{2}}\partial_\alpha^4\tilde{z}\|^2_{L^2(S)}.
\end{aligned}$$

Choosing

$$h(t)=\exp(-10\int_0^t G(r)dr)[\int_0^t -10G(r)\exp(10\int_0^r G(s)ds)dr+h(0)] \tag{25}$$

where $G(t)=\exp C(\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\|\tilde{z}\|^2_{H^4(S)})(t)$ we get the desired estimation. □

6.3. **Proof of Theorem 1.1.** Let $z_0(\alpha)\in H^4$. From Theorem 4.1 there exists a local solution z that becomes real-analytic in the complex strip $S(t)$.

Suppose that there exists a time T where we have a splat singularity; i.e., the smooth interface collapses along an arc at time T.

From Theorem 5.1, our strip of analyticity is non-zero as long as the regularity of the curve and the arc-chord condition do not fail. But at splat time T, the arc-chord condition blows up, and we cannot guarantee analyticity at that time.

At this point, we transform the system to the tilde domain $\tilde{\Omega}$.

As long as the regularity of the curve and the arc-chord condition do not fail, from Theorem 6.1 we have

$$\frac{d}{dt}\|\tilde{z}\|_{RT}\leq \exp C(\|\tilde{z}\|_{RT})$$

where the constant C depends only on the initial data and

$$\|\tilde{z}\|_{RT}\equiv\|\tilde{z}\|^2_{H^k(S)}+\|\mathcal{F}(\tilde{z})\|^2_{L^\infty(S)}+\frac{1}{m(Q^2\tilde{\sigma})(t)-2\lambda-\|g\|(t)}+\sum_{l=0}^{4}\frac{1}{m(q^l)(t)}.$$

Hence, we can conclude that our transformed curve $\tilde{z}$ is real-analytic into the strip $S(t)$. From the proof of Theorem 6.2, this complex strip decays exponentially until a time that depends on the regularity of the curve and the arc-chord condition too (see equation (25)).

Since in $\tilde{\Omega}$ the arc-chord condition and the regularity of the curve are bounded, the strip of analyticity is non-zero, and therefore we can guarantee the analyticity at time T.

Thus, applying P^{-1}, we have that the analytic curve self-intersects along an arc. Therefore we get a contradiction, and hence Theorem 1.1 is proved.

Acknowledgements

The authors were partially supported by the grant MTM2011-26696 (Spain) and ICMAT Severo Ochoa project SEV-2011-0087.

References

[1] Ángel Castro, Diego Córdoba, Charles Fefferman, and Francisco Gancedo, *Breakdown of smoothness for the Muskat problem*, Arch. Ration. Mech. Anal. **208** (2013), no. 3, 805–909, DOI 10.1007/s00205-013-0616-x. MR3048596

[2] Ángel Castro, Diego Córdoba, Charles Fefferman, and Francisco Gancedo, *Splash singularities for the one-phase Muskat problem in stable regimes.* arXiv:1311.7653v1, 2013.

[3] Angel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and Javier Gómez-Serrano, *Finite time singularities for the free boundary incompressible Euler equations*, Ann. of Math. (2) **178** (2013), no. 3, 1061–1134, DOI 10.4007/annals.2013.178.3.6. MR3092476

[4] Ángel Castro, Diego Córdoba, Charles Fefferman, Francisco Gancedo, and María López-Fernández, *Rayleigh-Taylor breakdown for the Muskat problem with applications to water waves*, Ann. of Math. (2) **175** (2012), no. 2, 909–948, DOI 10.4007/annals.2012.175.2.9. MR2993754

[5] C. H. Arthur Cheng, Rafael Granero-Belinchón, and Steve Shkoller, *Well-posedness of the Muskat problem with H^2 initial data.* arXiv:14127737v1, 2014.

[6] Peter Constantin, Diego Córdoba, Francisco Gancedo, and Robert M. Strain, *On the global existence for the Muskat problem*, J. Eur. Math. Soc. (JEMS) **15** (2013), no. 1, 201–227, DOI 10.4171/JEMS/360. MR2998834

[7] Antonio Córdoba and Diego Córdoba, *A pointwise estimate for fractionary derivatives with applications to partial differential equations*, Proc. Natl. Acad. Sci. USA **100** (2003), no. 26, 15316–15317, DOI 10.1073/pnas.2036515100. MR2032097 (2004k:26009)

[8] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo, *Interface evolution: the Hele-Shaw and Muskat problems*, Ann. of Math. (2) **173** (2011), no. 1, 477–542, DOI 10.4007/annals.2011.173.1.10. MR2753607 (2012a:35368)

[9] Antonio Córdoba, Diego Córdoba, and Francisco Gancedo, *Porous media: the Muskat problem in three dimensions*, Anal. PDE **6** (2013), no. 2, 447–497, DOI 10.2140/apde.2013.6.447. MR3071395

[10] Diego Córdoba and Francisco Gancedo, *Contour dynamics of incompressible 3-D fluids in a porous medium with different densities*, Comm. Math. Phys. **273** (2007), no. 2, 445–471, DOI 10.1007/s00220-007-0246-y. MR2318314 (2008e:76056)

[11] Daniel Coutand and Steve Shkoller, *On the finite-time splash and splat singularities for the 3-D free-surface Euler equations*, Comm. Math. Phys. **325** (2014), no. 1, 143–183, DOI 10.1007/s00220-013-1855-2. MR3147437

[12] Daniel Coutand and Steve Shkoller, *On the impossibility of finite-time splash singularities for vortex sheets.* arXiv:1407.1479v1, 2014.

[13] Joachim Escher and Bogdan-Vasile Matioc, *On the parabolicity of the Muskat problem: well-posedness, fingering, and stability results*, Z. Anal. Anwend. **30** (2011), no. 2, 193–218, DOI 10.4171/ZAA/1431. MR2793001 (2012h:35154)

[14] Charles Fefferman, Alexandru D. Ionescu, and Victor Lie, On the absence of “splash” singularities in the case of two-fluids interfaces. *arXiv:1312.2917v2*, 2013.

[15] Francisco Gancedo and Robert M. Strain, *Absence of splash singularities for surface quasi-geostrophic sharp fronts and the Muskat problem*, Proc. Natl. Acad. Sci. USA **111** (2014), no. 2, 635–639, DOI 10.1073/pnas.1320554111. MR3181769

[16] Javier Gómez-Serrano and Rafael Granero-Belinchón, *On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof*, Nonlinearity **27** (2014), no. 6, 1471–1498, DOI 10.1088/0951-7715/27/6/1471. MR3215843

[17] Rafael Granero-Belinchón, *Global existence for the confined Muskat problem*, SIAM J. Math. Anal. **46** (2014), no. 2, 1651–1680, DOI 10.1137/130912529. MR3196945

[18] Tosio Kato and Gustavo Ponce, *Commutator estimates and the Euler and Navier-Stokes equations*, Comm. Pure Appl. Math. **41** (1988), no. 7, 891–907, DOI 10.1002/cpa.3160410704. MR951744 (90f:35162)

[19] Morris Muskat, *Two fluid systems in porous media. The encroachment of water into an oil sand*, J. Appl. Phys. (5):250–264, 1934.

[20] Lord Rayleigh, *On the instability of jets*, Proc. London Math. Soc. **S1-10**, no. 1, 4, DOI 10.1112/plms/s1-10.1.4. MR1576197

[21] P. G. Saffman and Geoffrey Taylor, *The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid*, Proc. Roy. Soc. London. Ser. A **245** (1958), 312–329. (2 plates). MR0097227 (20 #3697)

[22] Michael Siegel, Russel E. Caflisch, and Sam Howison, *Global existence, singular solutions, and ill-posedness for the Muskat problem*, Comm. Pure Appl. Math. **57** (2004), no. 10, 1374–1411, DOI 10.1002/cpa.20040. MR2070208 (2007f:35235)

Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Cientificas, Nicolás Cabrera, 13-15, 28059 Madrid, Spain
E-mail address: dcg@icmat.es

Instituto de Ciencias Matemáticas, Consejo Superior de Investigaciones Cientificas, Nicolás Cabrera, 13-15, 28059 Madrid, Spain
E-mail address: tania.pernas@icmat.es