

THE NONLINEAR SCHRÖDINGER EQUATION ON THE HALF-LINE

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ABSTRACT. The initial-boundary value problem (ibvp) for the cubic nonlinear Schrödinger (NLS) equation on the half-line with data in Sobolev spaces is analysed via the formula obtained through the unified transform method and a contraction mapping approach. First, the linear Schrödinger (LS) ibvp with initial and boundary data in Sobolev spaces is solved and the basic space and time estimates of the solution are derived. Then, the forced LS ibvp is solved for data in Sobolev spaces on the half-line $[0, \infty)$ for the spatial variable and on an interval $[0, T]$, $0 < T < \infty$, for the temporal variable by decomposing it into a free ibvp and a forced ibvp with zero data, and its solution is estimated appropriately. Furthermore, using these estimates, well-posedness of the NLS ibvp with data $(u(x, 0), u(0, t))$ in $H_x^s(0, \infty) \times H_t^{(2s+1)/4}(0, T)$, $s > 1/2$, is established via a contraction mapping argument. In addition, this work places Fokas' unified transform method for evolution equations into the broader Sobolev spaces framework.

1. INTRODUCTION

A novel approach for solving initial-boundary value problems (ibvps) for linear and integrable nonlinear evolution equations was introduced in 1997 [F1]. This approach, which is known as the unified transform method (UTM) or the Fokas transform method, was motivated in the context of integrable nonlinear equations and can be seen as the ibvp analogue of the celebrated Inverse Scattering Transform method of Gardner, Greene, Kruskal and Miura [GGKM], which was introduced in 1967 for initial value problems.

However, it was immediately understood that the UTM has significant implications in the case of linear ibvps. In particular, it can be used to produce novel solution formulae which involve certain variations of the Fourier transform integrated along contours that lie in the complex spectral (Fourier) plane.

For those problems that can be solved via the classical transform method, the UTM formulae can be reduced to the well-known classical solutions via the use of Cauchy's theorem and contour deformations. Nevertheless, due to the occurrence of the complex contours of integration the novel formulae are uniformly convergent at the boundary of the domain. This feature, in addition to various analytical advantages, yields efficient techniques of numerical integration (it should be noted that any formula obtained via the classical transform method will suffer from lack of

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uniform convergence for inhomogeneous boundary conditions). The most important advantage of the UTM, however, is that it can be employed for solving ibvps of arbitrary order and arbitrary boundary conditions for which the classical techniques often fail (e.g. when the problem is non-self-adjoint). In this respect, for linear ibvps the UTM can be regarded as the counterpart of the Fourier transform method, in the sense that it produces the solution in terms of the “natural” transforms that are intrinsic to the ibvp under study. For a comparison of the UTM against the classical techniques, see [FSp] and [DTV].

The first purpose of this work is to explore the validity of the UTM formulae for data in Sobolev spaces. As the derivation of the UTM formulae for linear ibvps is only done at the level of “nice” initial and boundary data (e.g. of Schwartz class), we wish to investigate the validity of these formulae for larger classes of data which belong to appropriate Sobolev spaces. In particular, we carry out this investigation for the linear Schrödinger (LS) equation formulated on the half-line:

$$(1.1a) \quad iu_t + u_{xx} = 0, \quad x \in (0, \infty), t \in (0, T), T < 1,$$

$$(1.1b) \quad u(x, 0) = u_0(x) \in H_x^s(0, \infty), \quad x \in [0, \infty),$$

$$(1.1c) \quad u(0, t) = g_0(t) \in H_t^{\frac{2s+1}{4}}(0, T), \quad t \in [0, T].$$

Here, the spaces $H_x^s(0, \infty)$ and $H_t^{\frac{2s+1}{4}}(0, T)$ are defined as restrictions of the corresponding spaces over \mathbb{R} according to the following definition.

Definition 1.1. For $s \in \mathbb{R}$ and $a, b \in \mathbb{R}$ (possibly infinite) with $b > a$, we define the Sobolev space $H^s(a, b)$ by

$$(1.2) \quad H^s(a, b) = \left\{ f : f = F|_{(a,b)} \text{ where } F \in H^s(\mathbb{R}) \right\}$$

equipped with the norm

$$(1.3) \quad \|f\|_{H^s(a,b)} \doteq \inf_{F \in H_x^s(\mathbb{R})} \|F\|_{H^s(\mathbb{R})},$$

where the Sobolev norm on \mathbb{R} is defined via the Fourier transform of F by

$$(1.4) \quad \|F\|_{H^s(\mathbb{R})} \doteq \left(\int_{\xi \in \mathbb{R}} (1 + \xi^2)^s |\widehat{F}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

For $s > \frac{1}{2}$, ibvp (1.1) should be supplemented with the compatibility condition

$$(1.5) \quad u_0(0) = g_0(0),$$

which follows from the fact that $u_0(0)$ and $g_0(0)$ must both be equal to $u(0, 0)$. Moreover, if $s > \frac{5}{2}$, then LS implies the following additional conditions for the data at $x = t = 0$:

$$(1.6) \quad g_0^{(j)}(0) = i^j u_0^{(2j)}(0), \quad 1 \leq j < \frac{2s-1}{4}.$$

The UTM can be applied to ibvp (1.1) as follows. Assuming that there exists a solution u and rewriting (1.1a) as a one-parameter family of divergence forms yield via Green’s identity a relation between the Fourier transform \hat{u} of the solution and appropriate transforms of boundary values. This relation, called the global relation, can then be inverted to produce an integral representation for u in terms of the initial condition (1.1b), the boundary condition (1.1c), and the unknown boundary value $u_x(0, t)$. What is more, deforming the contours of integration to appropriate

contours in the complex spectral plane and using certain symmetries of the global relation, it is possible to eliminate the term involving the unknown boundary value by invoking Cauchy’s theorem and Jordan’s lemma.

Carrying out the above described computations, which can be found in every detail in [F2], we obtain the following UTM solution formula for the LS ibvp (1.1):

$$(1.7) \quad u(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx-ik^2t} \hat{u}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-ik^2t} [\hat{u}_0(-k) - 2k\tilde{g}_0(k^2, T)] dk,$$

where

$$(1.8) \quad \hat{u}_0(k) = \int_0^\infty e^{-ikx} u_0(x) dx, \quad k \in \overline{\mathbb{C}}_- \doteq \{k \in \mathbb{C} : \text{Im}k \leq 0\},$$

$$(1.9) \quad \tilde{g}_0(k^2, T) = \int_0^T e^{ik^2t'} g_0(t') dt', \quad k \in \mathbb{C},$$

and ∂D^+ denotes the positively oriented boundary of the first quadrant of the k -complex plane, in which $\text{Re}(ik^2) < 0$ (see Figure 1.1).

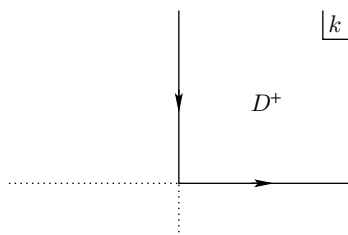


FIGURE 1.1. The region D^+ .

Our result for ibvp (1.1) is summarised as follows.

Theorem 1 (Well-posedness of the LS ibvp (1.1)). *Suppose that $s > \frac{1}{2}$ and $\frac{2s+1}{4} \notin \mathbb{N}_0 + \frac{1}{2}$. Then, the UTM formula (1.7) defines a solution u to the LS ibvp (1.1) together with the compatibility conditions (1.5) and (1.6) in the following sense:*

1. **Space estimate:** *The map $t \mapsto u(t)$ is $C([0, T]; H_x^s(0, \infty))$ with the estimate*

$$(1.10) \quad \sup_{t \in [0, T]} \|u(t)\|_{H_x^s(0, \infty)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right).$$

2. **Time estimates:** *The map $x \mapsto u(x)$ is $C([0, \infty); H_t^{\frac{2s+1}{4}}(0, T))$ with the estimate*

$$(1.11) \quad \sup_{x \in [0, \infty)} \|u(x)\|_{H_t^{\frac{2s+1}{4}}(0, T)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right),$$

and the map $x \mapsto u_x(x)$ is $C([0, \infty); H_t^{\frac{2s-1}{4}}(0, T))$ with the estimate

$$(1.12) \quad \sup_{x \in [0, \infty)} \|u_x(x)\|_{H_t^{\frac{2s-1}{4}}(0, T)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right).$$

In the above estimates, $c_s > 0$ is a constant that depends on s .

3. **Weak solution:** For any $\phi(x) \in C_0^\infty(\mathbb{R})$ such that $\phi(0) = 0$, the function $\langle u(t), \phi \rangle_{L_x^2(0, \infty)}$ is differentiable in $(0, T)$ with

$$(1.13) \quad \frac{d}{dt} \langle u(t), \phi \rangle_{L_x^2(0, \infty)} = -\langle u(t), i\phi_{xx} \rangle_{L_x^2(0, \infty)} + ig_0(t) \overline{\phi_x(0)}.$$

4. **Initial and boundary conditions:** $u(x, 0) = u_0(x)$ for all $x \in [0, \infty)$, and $u(0, t) = g_0(t)$ for all $t \in [0, T]$.

Initial-boundary value problems for linear dispersive evolution equations on the half-line are also analysed in [FSu].

The second purpose of this work is to introduce a new approach for proving well-posedness of nonlinear ibvps. The crucial ingredient of this approach is to use for the linear version of the given nonlinear equation the formula obtained via the UTM. The works of Bona, Sun and Zhang [BSZ1], Colliander and Kenig [CK], and Holmer [H1] show that it is possible to prove well-posedness of nonlinear evolution ibvps with data in Sobolev spaces by using ideas similar to the ones appearing in the initial value problem. In particular, one first obtains a solution formula for the linear ibvp with forcing and uses this formula to derive appropriate linear estimates; then, one replaces the forcing in the linear formula by the nonlinearity and uses the linear estimates together with a contraction mapping argument to infer well-posedness of the nonlinear ibvp.

It is often the case, however, that even the derivation of the linear solution formula is somewhat technical and unintuitive, not to mention the derivation of the relevant linear estimates. The main advantage of the UTM is that it yields explicit formulae for forced linear evolution equations with an arbitrary number of derivatives. Thus, it is not surprising that the “naturally emerging” linear UTM formulae can be used to establish local well-posedness of nonlinear evolution ibvps through a contraction mapping argument.

In order to illustrate our approach, we consider the cubic nonlinear Schrödinger (NLS) equation formulated on the half-line:

$$(1.14a) \quad iu_t + u_{xx} + \lambda u|u|^2 = 0, \quad x \in (0, \infty), \quad t \in (0, T), \quad T < 1, \quad \lambda \in \mathbb{C},$$

$$(1.14b) \quad u(x, 0) = u_0(x) \in H_x^s(0, \infty), \quad x \in [0, \infty),$$

$$(1.14c) \quad u(0, t) = g_0(t) \in H_t^{\frac{2s+1}{4}}(0, T), \quad t \in [0, T].$$

Holmer [H1] has shown local well-posedness of ibvp (1.14) for $0 \leq s < \frac{3}{2}$, $s \neq \frac{1}{2}$. It should be noted that the left end-point of this interval is consistent with the Cauchy problem for NLS, which has critical well-posedness index $s = 0$. The approach of [H1] is based on writing ibvp (1.14) as a superposition of three initial value problems on $\mathbb{R} \times \mathbb{R}$. Since none of these problems admits boundary conditions, the boundary datum (1.14c) enters the analysis via a so-called boundary forcing integral operator, which is defined by means of a Riemann-Liouville fractional integral. As one would expect, several results from the classical theory of the NLS initial value problem are employed in [H1], including Strichartz estimates.

The method of [H1] is inspired by the work of Colliander and Kenig [CK] on ibvps for the generalised Korteweg-de Vries (KdV) equation $u_t + u_{xxx} + \frac{1}{k+1} \partial_x u^{k+1} = 0$, $k \in \mathbb{N}$, where it is shown that the KdV ($k = 1$) is locally well-posed on the half-line for initial and boundary data in $H^s(0, \infty) \times H^{\frac{s+1}{3}}(0, T)$ with $s \geq 0$ (this result was later improved by Holmer to $s \geq -\frac{3}{4}$ [H2]). In this context, we note that

Bona, Sun and Zhang [BSZ1] obtained local well-posedness for the KdV on the half-line for $s > \frac{3}{4}$ based on a different approach, which involves obtaining linear solution formulae via a Laplace transform with respect to the temporal variable. In a preliminary version of an article implementing this approach for NLS (see [BSZ2]), the same authors study local well-posedness of ibvp (1.14) for $s \geq 0$.

As already mentioned, the first step towards showing well-posedness of the NLS ibvp (1.14) is to analyse the corresponding linear ibvp with forcing:

$$\begin{aligned}
 (1.15a) \quad & iu_t + u_{xx} = f(x, t) \in C([0, T]; H_x^s(0, \infty)), \quad x \in (0, \infty), t \in (0, T), T < 1, \\
 (1.15b) \quad & u(x, 0) = u_0(x) \in H_x^s(0, \infty), \quad x \in [0, \infty), \\
 (1.15c) \quad & u(0, t) = g_0(t) \in H_t^{\frac{2s+1}{4}}(0, T), \quad t \in [0, T].
 \end{aligned}$$

The UTM produces the following solution formula for ibvp (1.15):

$$\begin{aligned}
 (1.16) \quad & u(x, t) = S[u_0, g_0, f](x, t) \\
 & \doteq \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx-ik^2t} \hat{u}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx-ik^2t} \hat{u}_0(-k) dk \\
 & \quad - \frac{i}{2\pi} \int_{\mathbb{R}} e^{ikx-ik^2t} \int_0^t e^{ik^2t'} \hat{f}(k, t') dt' dk \\
 & \quad + \frac{i}{2\pi} \int_{\partial D^+} e^{ikx-ik^2t} \int_0^t e^{ik^2t'} \hat{f}(-k, t') dt' dk \\
 (1.17) \quad & + \frac{1}{\pi} \int_{\partial D^+} e^{ikx-ik^2t} k \tilde{g}_0(k^2, T) dk,
 \end{aligned}$$

where $\hat{u}_0(k)$ and $\tilde{g}_0(k^2, T)$ are defined by (1.8) and (1.9),

$$(1.18) \quad \hat{f}(k, t) = \int_0^\infty e^{-ikx} f(x, t) dx,$$

and the contour ∂D^+ is shown in Figure 1.1.

Note that the equation $u = S[u_0, g_0, f]$ introduced by (1.16) denotes the solution u to the LS on the half-line with data (u_0, g_0) and forcing f . With this notation, the solution formula (1.7) to the LS ibvp (1.1) is given by $u = S[u_0, g_0, 0]$.

Further note that, as in the case of the LS ibvp (1.1), if $s > \frac{1}{2}$, then the NLS ibvp (1.14) should be supplemented with the compatibility condition (1.5). What is more, for $s > \frac{5}{2}$ additional *nonlinear* compatibility conditions are introduced through NLS. These extra conditions are not required if we restrict $\frac{1}{2} < s < \frac{5}{2}$, and the situation is similar for the forced LS ibvp (1.15). For this range of s , our well-posedness results for ibvps (1.14) and (1.15) are summarised by the following theorems.

Theorem 2 (Well-posedness of the forced LS ibvp (1.15)). *Suppose that $\frac{1}{2} < s < \frac{5}{2}$. Then the UTM formula (1.17) defines a solution $u \in C([0, T]; H_x^s(0, \infty))$ to ibvp (1.15) with the compatibility condition (1.5) which satisfies the space estimate*

$$(1.19) \quad \sup_{t \in [0, T]} \|u(t)\|_{H_x^s(0, \infty)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} + T \sup_{t \in [0, T]} \|f(t)\|_{H_x^s(0, \infty)} \right),$$

where $c_s > 0$ is a constant depending on s .

Theorem 3 (Well-posedness of the NLS ibvp (1.14)). *Suppose that $\frac{1}{2} < s < \frac{5}{2}$. Then for $T^* > 0$ given by*

$$(1.20) \quad T^* = \min \left\{ T, \frac{1}{32c_s^3 |\lambda| \left(\|u_0\|_{H_x^s(0,\infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} \right)^2} \right\}, \quad c_s = c(s) > 0,$$

there exists a unique solution $u \in C([0, T^]; H_x^s(0, \infty))$ to the NLS ibvp (1.14) with the compatibility condition (1.5) which satisfies the space estimate*

$$(1.21) \quad \|u(t)\|_{H_x^s(0,\infty)} \leq 2c_s \left(\|u_0\|_{H_x^s(0,\infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0,T)} \right), \quad 0 \leq t \leq T^*.$$

Furthermore, the data-to-solution map $\{u_0, g_0\} \mapsto u$ is locally Lipschitz continuous.

Unlike the NLS ibvp, there is extensive literature about the NLS Cauchy problem. Well-posedness in Sobolev spaces H^s for any $s \geq 0$ was proved by Bourgain in [B1] using modern harmonic analysis techniques. This approach was expanded in his monograph [B2] in various directions. Time estimates for the solution of the linear Schrödinger Cauchy problem, like estimate (2.6a), were derived by Kenig, Ponce and Vega in [KPV]. Many more results about the NLS initial value problem can be found in Craig, Kappeler and Strauss [CKS], Cazenave [C], Cazenave and Weissler [CW], Ghidaglia and Saut [GS], Linares and Ponce [LP], Carroll and Bu [CB], Ginibre and Velo [GV], Kato [K], Tsutsumi [Tsu], and the references therein.

The study of the nonlinear Schrödinger equation is strongly motivated by its physical relevance. Indeed, Zakharov [Z] and Hasimoto and Ono [HO] have derived NLS in the context of water waves in infinite and finite depth, respectively; moreover, in the latter case a rigorous justification of the model has been provided by Craig, Sulem and Sulem [CSS]. The equation has also been suggested as a model for rogue waves; see Peregrine [P] and Chabchoub, Hoffmann and Akhmediev [CHA]. Furthermore, in the context of nonlinear optics NLS is the standard model to describe the evolution of almost monochromatic waves in a weakly nonlinear dispersive medium; see for example Boyd [BO] and Newell and Moloney [NM]. Further discussion and references on the physical significance of the equation can be found in the recent book by Lannes [L].

Finally, NLS is completely integrable (see Zakharov and Manakov [ZM]) with the Lax pair

$$\begin{aligned} \mu_x + ik [\sigma_3, \mu] &= Q(x, t)\mu, \\ \mu_t + 2ik^2 [\sigma_3, \mu] &= R(x, t, k)\mu, \quad k \in \mathbb{C}, \end{aligned}$$

where σ_3 is the third Pauli matrix,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and

$$Q(x, t) = \begin{pmatrix} 0 & u(x, t) \\ \lambda \bar{u}(x, t) & 0 \end{pmatrix}, \quad R(x, t, k) = 2kU(x, t) - iU_x \sigma_3 - i\lambda |u|^2 \sigma_3.$$

The Cauchy problem for the cubic NLS equation was solved via the Inverse Scattering Transform method by Zakharov and Shabat [ZS], while the initial-boundary value problem on the half-line and on the interval was analysed by Fokas and collaborators in [FI] and [FIS], respectively. The Cauchy problem for a related generalised NLS equation, which is integrable, has been studied in [FH].

Organisation of the paper. In Section 2, we derive the basic estimates for the LS initial value problem with Sobolev data. In Section 3, using the results of Section 2, we reduce the LS ivvp (1.1) to an ivvp with zero initial datum and a boundary datum which is more convenient for estimating the corresponding solution. We then establish well-posedness of the reduced problem for data in Sobolev spaces and, hence, deduce Theorem 1 for the LS ivvp (1.1). In Section 4, we solve the forced LS ivvp (1.15) (Theorem 2) using Theorem 1. Finally, in Section 5 we establish well-posedness of the NLS ivvp (1.14) (Theorem 3) via a contraction mapping argument.

2. SPACE AND TIME ESTIMATES IN SOBOLEV SPACES FOR THE LS IVP

In order to prove Theorem 1, which establishes well-posedness of the LS ivvp (1.1), we first need to obtain certain estimates for the initial value problem (ivp)

$$(2.1a) \quad iU_t + U_{xx} = 0, \quad x \in \mathbb{R}, \quad t \in (0, T),$$

$$(2.1b) \quad U(x, 0) = U_0(x) \in H_x^s(\mathbb{R}),$$

where U_0 is a Sobolev extension of u_0 from $(0, \infty)$ to \mathbb{R} satisfying

$$(2.2) \quad \|U_0\|_{H_x^s(\mathbb{R})} \leq c \|u_0\|_{H_x^s(0, \infty)}.$$

The solution to ivp (2.1) is given by the formula

$$(2.3) \quad U(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x - i\xi^2 t} \widehat{U}_0(\xi) d\xi, \quad x \in \mathbb{R}, \quad t \in \mathbb{R},$$

where $\widehat{U}_0(\xi)$ denotes the standard Fourier transform with respect to the spatial variable:

$$(2.4) \quad \widehat{U}_0(\xi) = \int_{\mathbb{R}} e^{-i\xi x} U_0(x) dx, \quad \xi \in \mathbb{R}.$$

Using formula (2.3), we establish the following well-posedness theorem for ivp (2.1).

Theorem 4 (LS ivp with Sobolev data). *The function U defined by (2.3) solves the ivp (2.1) in the following sense:*

1. **Space estimate:** *The map $t \mapsto U(t)$ is $C([0, T]; H_x^s(\mathbb{R}))$ with*

$$(2.5) \quad \|U(t)\|_{H_x^s(\mathbb{R})} = \|U_0\|_{H_x^s(\mathbb{R})} \quad \forall t \in [0, T], \quad s \in \mathbb{R}.$$

2. **Time estimates:** *The map $x \mapsto U(x)$ is $C(\mathbb{R}_x; H_t^{\frac{2s+1}{4}}(0, T))$, and the map $x \mapsto U_x(x)$ is $C(\mathbb{R}_x; H^{\frac{2s-1}{4}}(0, T))$ with*

$$(2.6a) \quad \sup_{x \in \mathbb{R}} \|U(x)\|_{H_t^{\frac{2s+1}{4}}(0, T)} \leq c_s (1 + T^{\frac{1}{2}}) \|U_0\|_{H_x^s(\mathbb{R})}, \quad s \geq -\frac{1}{2},$$

$$(2.6b) \quad \sup_{x \in \mathbb{R}} \|U_x(x)\|_{H^{\frac{2s-1}{4}}(0, T)} \leq c_s (1 + T^{\frac{1}{2}}) \|U_0\|_{H_x^s(\mathbb{R})}, \quad s \geq \frac{1}{2}.$$

3. **Weak solution:** *For any given $\phi(x) \in C_0^\infty(\mathbb{R})$ such that $\phi(0) = 0$, the function $\langle U(t), \phi \rangle_{L_x^2(0, \infty)}$ is differentiable in $(0, T)$ with*

$$(2.7) \quad \frac{d}{dt} \langle U(t), \phi \rangle_{L_x^2(0, \infty)} = -\langle U(t), i\phi_{xx} \rangle_{L_x^2(0, \infty)} + iU(0, t) \overline{\phi_x(0)}.$$

4. **Initial condition:** $U(x, 0) = U_0(x)$ for all $x \in \mathbb{R}$.

Proof. 1. Proving the space estimate. Since $\widehat{U}(\xi, t) = e^{-i\xi^2 t} \widehat{U}_0(\xi)$, the estimate follows by the definition of the Sobolev norm and the dominated convergence theorem.

2. Proving the time estimates. We write

$$(2.8) \quad U(x, t) = I_1(x, t) + I_2(x, t),$$

where

$$(2.9) \quad I_1(x, t) \doteq \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x - i\xi^2 t} \theta(\xi) \widehat{U}_0(\xi) d\xi,$$

$$(2.10) \quad I_2(x, t) \doteq \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x - i\xi^2 t} [1 - \theta(\xi)] \widehat{U}_0(\xi) d\xi,$$

and $\theta \in C_0^\infty(\mathbb{R})$ is a smooth cut-off function with compact support such that

$$(2.11) \quad \theta(\xi) = \begin{cases} 1, & |\xi| \leq 1, \\ 0, & |\xi| \geq 2. \end{cases}$$

Estimation of I_1 . By the definition of $\theta(\xi)$,

$$(2.12) \quad I_1(x, t) = \frac{1}{2\pi} \int_{-2}^2 e^{i\xi x - i\xi^2 t} \theta(\xi) \widehat{U}_0(\xi) d\xi, \quad x \in \mathbb{R}, t \in (0, T).$$

Using the Cauchy-Schwarz inequality, for any $j \in \mathbb{N} \setminus \{0\}$, we have

$$\begin{aligned} \left| \partial_t^j I_1(x, t) \right| &= \left| \frac{1}{2\pi} \int_{-2}^2 e^{i\xi x - i\xi^2 t} (-i\xi^2)^j \theta(\xi) \widehat{U}_0(\xi) d\xi \right| \\ &\leq \frac{1}{2\pi} \left(\int_{-2}^2 (1 + \xi^2)^{-s} |\xi^{4j}| |\theta(\xi)|^2 d\xi \right)^{\frac{1}{2}} \cdot \left(\int_{-2}^2 (1 + \xi^2)^s |\widehat{U}_0(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &\leq c_{s,\theta} 2^{2j} \|U_0\|_{H_x^s(\mathbb{R})}, \end{aligned}$$

where $c_{s,\theta} = \|(1 + \xi^2)^{-s/2} \theta(\xi)\|_{L^2(-2,2)} < \infty$. Therefore, for any $j \in \mathbb{N}_0$ we find

$$(2.13) \quad \left\| \partial_t^j I_1(x) \right\|_{L^2(0,T)} \leq c_{s,\theta} 2^{2j} T^{\frac{1}{2}} \|U_0\|_{H_x^s(\mathbb{R})} \quad \forall j \in \mathbb{N}_0, s \in \mathbb{R}, x \in \mathbb{R}.$$

Then, for $m = \mu \in \mathbb{N}_0$, using the definition

$$(2.14) \quad \|I_1(x)\|_{H^\mu(0,T)} = \sum_{j=0}^{\mu} \|\partial_t^j I_1(x)\|_{L^2(0,T)}, \quad \mu \in \mathbb{N}_0,$$

together with estimate (2.13), we get

$$(2.15) \quad \|I_1(x)\|_{H^\mu(0,T)} \leq c_{s,\theta} \frac{2^{2(\mu+1)} - 1}{2^2 - 1} T^{\frac{1}{2}} \|U_0\|_{H_x^s(\mathbb{R})} \quad \forall \mu \in \mathbb{N}_0, s \in \mathbb{R}, x \in \mathbb{R}.$$

We will deal with the non-integer case by employing the following interpolation lemma, which can be proved via Hölder’s inequality.

Lemma 2.1 (Sobolev interpolation). *For $m_1 < m < m_2$, we have*

$$(2.16) \quad \|f\|_{H^m} \leq \|f\|_{H^{m_1}}^{\frac{m_2 - m}{m_2 - m_1}} \|f\|_{H^{m_2}}^{\frac{m - m_1}{m_2 - m_1}}.$$

Therefore, given any $m \geq 0$, Lemma 2.1 with $f = I_1(x)$, $m_1 = [m]$ and $m_2 = [m] + 1$ and estimate (2.15) we have

$$(2.17) \quad \|I_1(x)\|_{H_t^m(0,T)} \leq c_{s,\theta} \frac{2^{2([m]+2)} - 1}{3} T^{\frac{1}{2}} \|U_0\|_{H_x^s(\mathbb{R})}, \quad m \geq 0, \quad s \in \mathbb{R}, \quad x \in \mathbb{R}.$$

Estimation of I_2 . Regarding $I_2(x, t)$, by the definition of $\theta(\xi)$ it can be written as

$$(2.18) \quad I_2(x, t) = \frac{1}{2\pi} \int_{-\infty}^{-1} e^{i\xi x - i\xi^2 t} [1 - \theta(\xi)] \widehat{U}_0(\xi) d\xi + \frac{1}{2\pi} \int_1^{\infty} e^{i\xi x - i\xi^2 t} [1 - \theta(\xi)] \widehat{U}_0(\xi) d\xi.$$

Let

$$(2.19) \quad \sigma(\tau) \doteq \tau^{\frac{1}{2}} = \sqrt{|\tau|} e^{i \arg(\tau)/2}, \quad \arg(\tau) \in [0, 2\pi).$$

The change of variables

$$(2.20) \quad \xi = \xi_1(\tau) = -\sigma(-\tau) = -(-\tau)^{\frac{1}{2}}$$

in the first integral of (2.18) implies that $\tau = -\xi^2$ and maps the interval $(-\infty, -1]$ to the interval $(-\infty, -1]$ and vice-versa. For the second integral of (2.18), we make the change of variables

$$(2.21) \quad \xi = \xi_2(\tau) = \sigma(-\tau) = (-\tau)^{\frac{1}{2}},$$

which, as before, implies that $\tau = -\xi^2$ and maps the interval $[1, \infty)$ to the interval $(-\infty, -1]$ and vice-versa (note, however, the change in direction). Overall, (2.18) becomes

$$(2.22) \quad \begin{aligned} I_2(x, t) = & \frac{1}{2\pi} \int_{-\infty}^{-1} e^{i\xi_1(\tau)x + i\tau t} [1 - \theta(\xi_1(\tau))] \widehat{U}_0(\xi_1(\tau)) \frac{d\tau}{[-2\xi_1(\tau)]} \\ & + \frac{1}{2\pi} \int_{-\infty}^{-1} e^{i\xi_2(\tau)x + i\tau t} [1 - \theta(\xi_2(\tau))] \widehat{U}_0(\xi_2(\tau)) \frac{d\tau}{2\xi_2(\tau)}. \end{aligned}$$

Let

$$(2.23) \quad I_{2,1}(x, t) \doteq \frac{1}{2\pi} \int_{-\infty}^{-1} e^{i\xi_1(\tau)x + i\tau t} [1 - \theta(\xi_1(\tau))] \widehat{U}_0(\xi_1(\tau)) \frac{d\tau}{[-2\xi_1(\tau)]}.$$

Then, by the inverse Fourier transform formula it follows that $\widehat{I}_{2,1}(x, \tau) \equiv 0$ for $\tau \in [-1, \infty)$, while for $\tau \in (-\infty, -1]$ we have

$$(2.24) \quad \widehat{I}_{2,1}(x, \tau) = e^{i\xi_1(\tau)x} [1 - \theta(\xi_1(\tau))] \frac{\widehat{U}_0(\xi_1(\tau))}{[-2\xi_1(\tau)]}.$$

For convenience of notation, we let

$$(2.25) \quad m = \frac{2s + 1}{4}.$$

Taking into consideration that $|e^{i\xi_1(\tau)x}| = 1$ and $|1 - \theta(\xi_1(\tau))| \leq 1$, we then have

$$\begin{aligned}
 \|I_{2,1}(x)\|_{H_t^m(\mathbb{R})}^2 &\leq \int_{-\infty}^{-1} (1 + \tau^2)^m \left| \frac{\widehat{U}_0(\xi_1(\tau))}{[-2\xi_1(\tau)]} \right|^2 d\tau \\
 &= \int_{-\infty}^{-1} (1 + |\xi|^4)^m \frac{|\widehat{U}_0(\xi)|^2}{4|\xi|(-\xi)} (-2\xi) d\xi \quad (\text{since } |\xi| = -\xi) \\
 (2.26) \quad &\simeq \int_{-\infty}^{-1} (1 + \xi^2)^{2m-\frac{1}{2}} |\widehat{U}_0(\xi)|^2 d\xi.
 \end{aligned}$$

Since by (2.25) we have $2m - \frac{1}{2} = s$, from inequality (2.26) we get

$$(2.27) \quad \|I_{2,1}(x)\|_{H_t^m(\mathbb{R})} \lesssim \|U_0\|_{H_x^s(\mathbb{R})}, \quad x \in \mathbb{R}, \quad s \in \mathbb{R},$$

which is the desired estimate for $I_{2,1}(x)$. In a similar way we find that for

$$(2.28) \quad I_{2,2}(x, t) \doteq \frac{1}{2\pi} \int_{-\infty}^{-1} e^{i\xi_2(\tau)x+i\tau t} [1 - \theta(\xi_2(\tau))] \widehat{U}_0(\xi_2(\tau)) \frac{d\tau}{2\xi_2(\tau)}$$

we have the estimate

$$(2.29) \quad \|I_{2,2}(x)\|_{H_t^m(\mathbb{R})} \lesssim \|U_0\|_{H_x^s(\mathbb{R})}, \quad x \in \mathbb{R}, \quad s \in \mathbb{R}.$$

Combining estimates (2.17), (2.27) and (2.29) we conclude that

$$\|U(x)\|_{H_t^m(0,T)} \leq c_{s,\theta} (1 + T^{\frac{1}{2}}) \|U_0\|_{H_x^s(\mathbb{R})}, \quad x \in \mathbb{R}, \quad s \geq -\frac{1}{2},$$

which for $T < 1$ reads

$$\|U(x)\|_{H_t^m(0,T)} \leq c_{s,\theta} \|U_0\|_{H_x^s(\mathbb{R})}, \quad x \in \mathbb{R}, \quad s \geq -\frac{1}{2}.$$

For continuity, we need to show that for any sequence $\{x_n\} \subset \mathbb{R}$ converging to $x \in \mathbb{R}$, we have

$$(2.30) \quad \|U(x_n) - U(x)\|_{H_t^m(0,T)} \xrightarrow{n \rightarrow \infty} 0.$$

By computations similar to those leading to (2.17), we find

$$\begin{aligned}
 &\|I_1(x_n) - I_1(x)\|_{H_t^m(0,T)} \\
 &\leq \left\{ \left(\sum_{j=0}^{\lfloor m \rfloor} c_{s,\theta,j}(x_n, x) \right)^{\lfloor m \rfloor + 1 - m} + \left(\sum_{j=0}^{\lfloor m \rfloor + 1} c_{s,\theta,j}(x_n, x) \right)^{m - \lfloor m \rfloor} \right\} \|U_0\|_{H_x^s(\mathbb{R})},
 \end{aligned}$$

where

$$c_{s,\theta,j}(x_n, x) = \left(\frac{1}{2\pi} \int_{-1}^1 |e^{i\xi x_n} - e^{i\xi x}| (1 + \xi^2)^{-s} \xi^{4j} |\theta(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Note, however, that for all $j \in \mathbb{N}_0$,

$$\int_{-1}^1 |e^{i\xi x_n} - e^{i\xi x}| (1 + \xi^2)^{-s} \xi^{4j} |\theta(\xi)|^2 d\xi \leq 2 \int_{-1}^1 (1 + \xi^2)^{-s} \xi^{4j} |\theta(\xi)|^2 d\xi < \infty;$$

thus by the dominated convergence theorem we have $\lim_{n \rightarrow \infty} c_{s,\theta,j}(x_n, x) = 0$ and hence,

$$\|I_1(x_n) - I_1(x)\|_{H_t^m(0,T)} \xrightarrow{n \rightarrow \infty} 0.$$

Furthermore, recalling the expression (2.24) for the Fourier transform of $I_{2,1}$ with respect to t , we have

$$\begin{aligned} \|I_{2,1}(x_n) - I_{2,1}(x)\|_{H_t^m(\mathbb{R})}^2 &= \int_{\mathbb{R}} (1 + \tau^2)^m |\widehat{I_{2,1}}(x_n, \tau) - \widehat{I_{2,1}}(x, \tau)|^2 d\tau \\ &= \int_{-\infty}^{-1} (1 + \tau^2)^m \left| \left(e^{i\xi_1(\tau)x_n} - e^{i\xi_1(\tau)x} \right) [1 - \theta(\xi_1(\tau))] \frac{\widehat{U}_0(\xi_1(\tau))}{[-2\xi_1(\tau)]} \right|^2 d\tau \\ &\leq \int_{-\infty}^{-1} (1 + \tau^2)^m \left| \left(e^{i\xi_1(\tau)x_n} - e^{i\xi_1(\tau)x} \right) \frac{\widehat{U}_0(\xi_1(\tau))}{[-2\xi_1(\tau)]} \right|^2 d\tau, \end{aligned}$$

since $|1 - \theta(\xi_1(\tau))| \leq 1$ for $\tau \in (-\infty, -1]$. Also, it follows from definitions (2.19) and (2.20) that if $\tau \in (-\infty, -1]$, then $\xi_1(\tau) \in (-\infty, -1]$; hence recalling (2.26) and (2.27), we get

$$\begin{aligned} &\int_{-\infty}^{-1} (1 + \tau^2)^m \left| \left(e^{i\xi_1(\tau)x_n} - e^{i\xi_1(\tau)x} \right) \frac{\widehat{U}_0(\xi_1(\tau))}{[-2\xi_1(\tau)]} \right|^2 d\tau \\ &\leq \int_{-\infty}^{-1} (1 + \tau^2)^m \left| \frac{\widehat{U}_0(\xi_1(\tau))}{[-2\xi_1(\tau)]} \right|^2 d\tau \\ &\lesssim \|U_0\|_{H_x^s(\mathbb{R})} < \infty. \end{aligned}$$

Thus, by the dominated convergence theorem we obtain $\|I_{2,1}(x_n) - I_{2,1}(x)\|_{H_t^m(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$, which implies

$$\|I_{2,1}(x_n) - I_{2,1}(x)\|_{H_t^m(0,T)} \xrightarrow{n \rightarrow \infty} 0.$$

The same can be shown for $I_{2,2}(x, t)$. Overall, we conclude that the map $x \mapsto U(x)$ is $C(\mathbb{R}_x; H_t^m(0, T))$. The argument for the map $x \mapsto U_x(x)$ is analogous.

3. Proving that U is a weak solution. The space $\mathcal{S}(\mathbb{R})$ is dense in $H_x^s(\mathbb{R})$; hence there exists a sequence $\{U_{0,n}(x)\} \subset \mathcal{S}(\mathbb{R})$ converging to the initial data $U_0(x)$ in $H_x^s(\mathbb{R})$. Furthermore, the LS ivp with initial datum $U_{0,n}$ is well-posed with solution $U_n \in C^\infty([0, T]; \mathcal{S}(\mathbb{R}))$ given by formula (2.3) and satisfying the identity

$$(2.31) \quad \frac{d}{dt} \langle U_n(t), \phi \rangle_{L_x^2(0,\infty)} = - \langle U_n(t), i\phi_{xx} \rangle_{L_x^2(0,\infty)} + iU_n(0, t) \overline{\phi_x(0)}$$

for any given test function $\phi(x) \in C_0^\infty(\mathbb{R})$ such that $\phi(0) = 0$. Taking the limit $n \rightarrow \infty$ in the above identity, we conclude that the function U defined by equation (2.3) is a weak solution to the LS ivp (2.1) in the sense of equation (2.7).

4. Proving that the initial condition is satisfied. It follows from the density of $\mathcal{S}(\mathbb{R})$ in $H_x^s(\mathbb{R})$ that $U_n(0) \xrightarrow{n \rightarrow \infty} U(0)$ in $H_x^s(\mathbb{R})$. We know, however, that $U_n(0) = U_{0,n}$, and since $U_{0,n} \xrightarrow{n \rightarrow \infty} U_0$ in $H_x^s(\mathbb{R})$, we conclude that $U(0) = U_0$ in $H_x^s(\mathbb{R})$. □

3. SOLVING THE LS IBVP IN SOBOLEV SPACES

In this section, we prove Theorem 1 for the LS ibvp (1.1). Our strategy is as follows. First, we reduce ibvp (1.1) to a problem with zero initial datum. Then, by constructing a suitable extension of the boundary datum we further reduce this

problem to an ibvp whose UTM solution formula is more convenient to estimate. Finally, we prove well-posedness of the reduced ibvp and thus obtain Theorem 1 for the original problem (1.1). Crucial to this proof will be the boundedness of the Laplace transform in $L_x^2(0, \infty)$.

Preliminary reduction of ibvp (1.1). Define the function v as the difference

$$(3.1) \quad v(x, t) = u(x, t) - U(x, t), \quad x \in [0, \infty), t \in [0, T],$$

where u is the solution to the LS ibvp (1.1) and U satisfies the LS ivp (2.1). By linearity, subtracting these two problems yields the following “zero initial condition” ibvp for v :

$$(3.2a) \quad iv_t + v_{xx} = 0, \quad x \in (0, \infty), t \in (0, T),$$

$$(3.2b) \quad v(x, 0) = 0, \quad x \in [0, \infty),$$

$$(3.2c) \quad v(0, t) = G_0(t) \in H_t^m(0, T), \quad m \doteq \frac{2s + 1}{4},$$

where G_0 is defined by

$$(3.3) \quad G_0(t) \doteq g_0(t) - U(0, t).$$

Note that $G_0 \in H_t^m(0, T)$ since the datum g_0 is prescribed as a function in $H_t^m(0, T)$ and also, by Theorem 4, we have that $U(0, t) \in H_t^m(0, T)$ for $s \geq -\frac{1}{2}$.

The UTM solution formula (1.7) corresponding to the reduced ibvp (3.2) is

$$(3.4) \quad v(x, t) = \frac{1}{\pi} \int_{\partial D^+} e^{ikx - ik^2t} k \tilde{G}_0(k^2, T) dk,$$

where

$$(3.5) \quad \tilde{G}_0(k^2, T) \doteq \int_0^T e^{ik^2t} G_0(t) dt.$$

Further reduction of ibvp (1.1). The space $H_0^m(0, T)$, which is defined as the closure of $C_0^\infty(0, T)$ in $H^m(0, T)$, can be characterised as follows: if $m > \frac{1}{2}$, then $f \in H_0^m(0, T)$ if and only if $f \in H^m(0, T)$ and $f^{(j)}(0) = f^{(j)}(T) = 0$ for all $0 \leq j < m - \frac{1}{2}$. Moreover, given a function $f \in H_0^m(0, T)$ and denoting by F its extension by zero outside $[0, T]$,

$$F(t) = \begin{cases} f(t), & t \in [0, T], \\ 0, & t \in [0, T]^c, \end{cases}$$

we have the following classical result: if $m > \frac{1}{2}$ and $m \notin \mathbb{N} + \frac{1}{2}$, then the mapping $f \mapsto F$ is continuous from $H_0^m(0, T)$ to $H^m(\mathbb{R})$ with the estimate

$$\|F\|_{H^m(\mathbb{R})} \leq c_m \|f\|_{H_0^m(0, T)}.$$

For a proof, see for example Lions and Magenes [LM], §11.3, Theorem 11.4.

The effect of this result on the estimation of the right-hand side of (3.4) is significant: extending the function G_0 by zero outside the interval $[0, T]$, we observe that the term \tilde{G}_0 defined by (3.5) turns into a Fourier transform over \mathbb{R} ; hence the relevant integral can be easily bounded by the standard Sobolev norm of the initial and boundary data.

In this connection, we note that the compatibility condition (1.5), which supplements ibvp (1.1) for $s > \frac{1}{2}$, implies that $G_0(0) = 0$. Also, the additional compatibility conditions (1.6), which hold when $s > \frac{5}{2}$, imply that $G_0^{(j)}(0) = 0$ for all $1 \leq j < m - \frac{1}{2}$. At $t = T$, however, the functions $G_0^{(j)}$ do not necessarily vanish. Thus, it cannot be guaranteed that the extension of $G_0 \in H_t^m(0, T)$ by zero outside $[0, T]$ is a function in $H_t^m(\mathbb{R})$. Instead, choose an extension $G \in H_t^m(\mathbb{R})$ of $G_0 \in H_t^m(0, T)$ such that

$$(3.6) \quad \|G\|_{H_t^m(\mathbb{R})} \leq c \|G_0\|_{H_t^m(0, T)}$$

and consider the function

$$g(t) \doteq \theta(t) \cdot G(t) \in H_t^m(\mathbb{R})$$

with $\theta \in C_0^\infty(\mathbb{R})$ defined by (2.11). Observe that $g(t) \equiv G_0(t)$ on the interval $[0, T]$, $T < 1$. Also, $\text{supp}(g) \subset [-2, 2]$ and, in particular,

$$g^{(j)}(2) = 0, \quad j \in \mathbb{N}_0.$$

Moreover, from the compatibility conditions (1.5) and (1.6), we have

$$g^{(j)}(0) = 0, \quad 0 \leq j < m - \frac{1}{2}.$$

Thus, we infer that $g(t) \in H_0^m(0, 2)$ and its extension h by zero outside $[0, 2]$, given by

$$(3.7) \quad h(t) = \begin{cases} g(t), & t \in [0, 2], \\ 0, & t \in [0, 2]^c, \end{cases}$$

admits the estimate

$$\|h\|_{H_t^m(\mathbb{R})} \leq c_s \|g\|_{H_0^m(0, 2)}, \quad s > \frac{1}{2}.$$

Then, since $\|g\|_{H_0^m(0, 2)} = \|g\|_{H_t^m(0, 2)}$, using the algebra property in $H_t^m(\mathbb{R})$ and the definition of $H_t^m(0, 2)$ as a restriction of $H_t^m(\mathbb{R})$ we find

$$\|h\|_{H_t^m(\mathbb{R})} \leq c_s \|g\|_{H_t^m(\mathbb{R})} \leq c_{s, \theta} \|G\|_{H_t^m(\mathbb{R})} \leq c_{s, \theta} \|G_0\|_{H_t^m(0, T)}, \quad s > \frac{1}{2}.$$

Finally, by definition (3.3), the triangle inequality and estimates (2.6a) and (2.2), we obtain

$$(3.8) \quad \|h\|_{H_t^m(\mathbb{R})} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^m(0, T)} \right), \quad s > \frac{1}{2}.$$

Consider now the following reduced version of ibvp (1.1):

$$(3.9a) \quad iv_t + v_{xx} = 0, \quad x \in (0, \infty), \quad t \in (0, 2),$$

$$(3.9b) \quad v(x, 0) = 0, \quad x \in [0, \infty),$$

$$(3.9c) \quad v(0, t) = h(t) \in H_t^m(\mathbb{R}), \quad t \in [0, 2],$$

where $\text{supp}(h) \subset [0, 2]$. The UTM solution formula (1.7) now reads

$$(3.10) \quad v(x, t) = \frac{1}{\pi} \int_{\partial D^+} e^{ikx - ik^2t} k \hat{h}(-k^2) dk, \quad x \in [0, \infty), \quad t \in [0, 2],$$

where the contour ∂D^+ is depicted in Figure 1.1 and

$$(3.11) \quad \hat{h}(-k^2) \doteq \int_{\mathbb{R}} e^{ik^2t} h(t) dt.$$

Theorem 5 (Reduced LS ibvp with Sobolev data). *Suppose that $s > \frac{1}{2}$. Then, the function v defined by the UTM formula (3.10) solves the reduced ibvp (3.9) in the following sense:*

1. **Space estimate:** *The map $t \mapsto v(t)$ is $C([0, 2]; H_x^s(0, \infty))$ with*

$$(3.12) \quad \sup_{t \in [0, 2]} \|v(t)\|_{H_x^s(0, \infty)} \leq c_s \|h\|_{H_t^m(\mathbb{R})}.$$

2. **Time estimates:** *The map $x \mapsto v(x)$ is $C([0, \infty); H_t^m(0, 2))$ with the estimate*

$$(3.13a) \quad \sup_{x \in [0, \infty)} \|v(x)\|_{H_t^m(0, 2)} \leq c_s \|h\|_{H_t^m(\mathbb{R})},$$

and the map $x \mapsto v_x(x)$ is $C([0, \infty); H_t^{m-\frac{1}{2}}(0, 2))$ with the estimate

$$(3.13b) \quad \sup_{x \in [0, \infty)} \|v_x(x)\|_{H_t^{m-\frac{1}{2}}(0, 2)} \leq c_s \|h\|_{H_t^m(\mathbb{R})}.$$

In all of the above estimates, $c_s > 0$ is a constant depending on s .

3. **Weak solution:** *For any $\phi(x) \in C_0^\infty(\mathbb{R})$ such that $\phi(0) = 0$, the function $\langle v(t), \phi \rangle_{L_x^2(0, \infty)}$ is differentiable in $(0, 2)$ with*

$$(3.14) \quad \frac{d}{dt} \langle v(t), \phi \rangle_{L_x^2(0, \infty)} = -\langle v(t), i\phi_{xx} \rangle_{L_x^2(0, \infty)} + ih(t) \overline{\phi_x(0)}.$$

4. **Initial and boundary conditions:** *$v(x, 0) = 0$ for all $x \in [0, \infty)$ and $v(0, t) = h(t)$ for all $t \in [0, 2]$.*

Proof. **1. Proving the space estimate (3.12).** Starting from formula (3.10), we have

$$\begin{aligned} v(x, t) &= \frac{1}{\pi} \int_{\partial D^+} e^{ikx - ik^2t} k \hat{h}(-k^2) dk \\ &= \frac{1}{\pi} \int_{-\infty}^0 e^{-kx + ik^2t} (ik) \hat{h}(k^2) idk + \frac{1}{\pi} \int_0^\infty e^{ikx - ik^2t} k \hat{h}(-k^2) dk \\ &= v_1(x, t) + v_2(x, t), \end{aligned}$$

where the functions v_1 and v_2 are defined by

$$(3.15a) \quad v_1(x, t) \doteq \frac{1}{\pi} \int_0^\infty e^{-kx + ik^2t} k \hat{h}(k^2) dk,$$

$$(3.15b) \quad v_2(x, t) \doteq \frac{1}{\pi} \int_0^\infty e^{ikx - ik^2t} k \hat{h}(-k^2) dk.$$

Estimate for v_2 . Let

$$(3.16) \quad \widehat{V}_2(k, t) = \begin{cases} 2e^{-ik^2t} k \hat{h}(-k^2), & k \geq 0, \\ 0, & k < 0, \end{cases}$$

and define

$$(3.17) \quad V_2(x, t) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{ikx} \widehat{V}_2(k, t) dk, \quad x \in \mathbb{R}.$$

Note that $V_2|_{x \in \mathbb{R}^+} = v_2$; thus

$$(3.18) \quad \|v_2(t)\|_{H^s(\mathbb{R}^+)}^2 \leq \|V_2\|_{H^s(\mathbb{R})}^2 = 4 \int_0^\infty (1+k^2)^s k^2 |\hat{h}(-k^2)|^2 dk.$$

Now, let $\tau = -k^2 \Rightarrow d\tau = -2kdk$ and $k = \sqrt{-\tau}$. Then,

$$(3.19) \quad \begin{aligned} \|v_2(t)\|_{H^s(\mathbb{R}^+)}^2 &= 2 \int_{-\infty}^0 (1+|\tau|)^s |\tau|^{\frac{1}{2}} |\hat{h}(-\tau)|^2 d\tau \\ &\leq 2 \int_{-\infty}^0 (1+|\tau|^2)^{\frac{s}{2}} (1+\tau^2)^{\frac{1}{4}} |\hat{h}(\tau)|^2 d\tau \leq 2 \|h\|_{H_t^{\frac{2s+1}{4}}(\mathbb{R})}^2. \end{aligned}$$

Estimate for v_1 . Letting

$$(3.20) \quad G_h(k) = k \hat{h}(k^2),$$

we have

$$(3.21) \quad v_1(x, t) = \frac{1}{\pi} \int_0^\infty e^{-kx+ik^2t} G_h(k) dk.$$

We shall prove estimate (3.12) for v_1 by letting $s = [s] + \beta$, $\beta \in [0, 1)$, and considering the three cases: $[s] = 0$; $\beta = 0$; $[s] \neq 0$ and $\beta \neq 0$.

The case $s = \beta \in (0, 1)$. By the definition of the equivalent fractional norm for $H_x^\beta(0, \infty)$, we have

$$(3.22) \quad \begin{aligned} \|v_1(t)\|_\beta^2 &= \int_0^\infty \int_0^\infty \frac{1}{|x-y|^{1+2\beta}} \left| \int_0^\infty (e^{-kx} - e^{-ky}) e^{ik^2t} G_h(k) dk \right|^2 dy dx \\ &= \int_0^\infty \int_0^\infty \frac{1}{|x-y|^{1+2\beta}} \left| \int_0^\infty (e^{-kx} - e^{-ky}) e^{ik^2t} G_h(k) dk \right| \\ &\quad \times \left| \int_0^\infty (e^{-lx} - e^{-ly}) e^{il^2t} G_h(l) dl \right| dy dx \\ &\leq \int_0^\infty \int_0^\infty \frac{1}{|x-y|^{1+2\beta}} \left(\int_0^\infty |e^{-kx} - e^{-ky}| |G_h(k)| dk \right) \\ &\quad \times \left(\int_0^\infty |e^{-lx} - e^{-ly}| |G_h(l)| dl \right) dy dx \\ &= \int_0^\infty \int_0^\infty |G_h(k)| |G_h(l)| \left(\int_0^\infty \int_0^\infty \frac{(e^{-kx} - e^{-ky})(e^{-lx} - e^{-ly})}{|x-y|^{1+2\beta}} dy dx \right) dl dk. \end{aligned}$$

Letting

$$(3.23) \quad \alpha = \left(\frac{k}{l} \right)^{\frac{1}{2}}$$

and making the change of variables

$$x' = (kl)^{\frac{1}{2}} x, \quad y' = (kl)^{\frac{1}{2}} y, \quad \frac{\partial(x', y')}{\partial(x, y)} = \begin{vmatrix} (kl)^{\frac{1}{2}} & 0 \\ 0 & (kl)^{\frac{1}{2}} \end{vmatrix} = kl,$$

we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{(e^{-kx} - e^{-ky})(e^{-lx} - e^{-ly})}{|x - y|^{1+2\beta}} dy dx \\ &= (kl)^{-\frac{1}{2}+\beta} \int_0^\infty \int_0^\infty \frac{(e^{-\alpha x'} - e^{-\alpha y'}) (e^{-\frac{x'}{\alpha}} - e^{-\frac{y'}{\alpha}})}{|x' - y'|^{1+2\beta}} dx' dy'. \end{aligned}$$

Therefore, after dropping the primes estimate (3.22) becomes

$$(3.24) \quad \|v_1(t)\|_\beta^2 \leq \int_0^\infty \int_0^\infty |G_h(k)| |G_h(l)| (kl)^{-\frac{1}{2}+\beta} \Delta(\alpha, \frac{1}{\alpha}, \beta) dl dk,$$

where

$$\begin{aligned} \Delta(\alpha, \frac{1}{\alpha}, \beta) &\doteq \int_0^\infty \int_0^\infty \frac{(e^{-\alpha x} - e^{-\alpha y})(e^{-\frac{x}{\alpha}} - e^{-\frac{y}{\alpha}})}{|x - y|^{1+2\beta}} dy dx \\ (3.25) \quad &= 2 \int_0^\infty \int_0^\infty \frac{e^{-(\alpha+\frac{1}{\alpha})x} - e^{-(\alpha y+\frac{x}{\alpha})}}{|x - y|^{1+2\beta}} dy dx. \end{aligned}$$

The following lemma computes the quantity $\Delta(\alpha, \frac{1}{\alpha}, \beta)$ appearing in (3.24).

Lemma 3.1. *If $\alpha \in [0, \infty]$ and $\beta \in (0, 1)$, then*

$$(3.26) \quad \Delta(\alpha, \frac{1}{\alpha}, \beta) = C_\beta \left(\alpha + \frac{1}{\alpha}\right)^{-1} \left[\alpha^{2\beta} + \left(\frac{1}{\alpha}\right)^{2\beta} - \left(\alpha + \frac{1}{\alpha}\right)^{2\beta} \right]$$

where

$$(3.27) \quad C_\beta = \begin{cases} \frac{\Gamma(1-2\beta)}{\beta}, & \beta \in (0, \frac{1}{2}), \\ \frac{\pi}{\sin(2\pi\beta)\Gamma(2\beta)\beta}, & \beta \in [\frac{1}{2}, 1), \end{cases}$$

and we have the estimate

$$(3.28) \quad \Delta(\alpha, \frac{1}{\alpha}, \beta) \lesssim c_\beta \left(\alpha + \frac{1}{\alpha}\right)^{-1}, \quad c_\beta > 0, \quad \beta \in (0, 1).$$

The proof of Lemma 3.1 is given after the current proof. Combining (3.24) and (3.28) and recalling the definition (3.23) of α , we find

$$\begin{aligned} \|v_1(t)\|_\beta^2 &\lesssim \int_0^\infty \int_0^\infty |G_h(k)| |G_h(l)| (kl)^{-\frac{1}{2}+\beta} \left(\alpha + \frac{1}{\alpha}\right)^{-1} dl dk \\ &= \int_0^\infty \int_0^\infty |G_h(k)| |G_h(l)| \frac{(kl)^\beta}{k+l} dl dk \\ &= \int_0^\infty \int_0^\infty |k|^\beta |l|^\beta |G_h(k)| |G_h(l)| \left(\int_0^\infty e^{-(k+l)x} dx\right) dl dk \\ &\leq \int_0^\infty \int_0^\infty (1+k^2)^{\frac{\beta}{2}} (1+l^2)^{\frac{\beta}{2}} |G_h(k)| |G_h(l)| \left(\int_0^\infty e^{-(k+l)x} dx\right) dl dk \\ (3.29) \quad &= \int_0^\infty \left(\int_0^\infty e^{-kx} (1+k^2)^{\frac{\beta}{2}} |G_h(k)| dk\right) \left(\int_0^\infty e^{-lx} (1+l^2)^{\frac{\beta}{2}} |G_h(l)| dl\right) dx, \end{aligned}$$

which gives the estimate

$$(3.30) \quad \|v_1(t)\|_\beta^2 \lesssim \int_0^\infty \left| \int_0^\infty e^{-kx} (1+k^2)^{\frac{\beta}{2}} |G_h(k)| dk \right|^2 dx.$$

Now, we need the following classical result, which states that the Laplace transform is bounded in $L^2(0, \infty)$.

Lemma 3.2 (L^2 boundedness of the Laplace transform). *Suppose that $Q(k) \in L_k^2(0, \infty)$. Then, the map*

$$Q(k) \mapsto \int_0^\infty e^{-kx} Q(k) dk$$

is bounded from $L_k^2(0, \infty)$ into $L_x^2(0, \infty)$ with

$$(3.31) \quad \left\| \int_0^\infty e^{-kx} Q(k) dk \right\|_{L_x^2(0, \infty)} \leq \sqrt{\pi} \|Q(k)\|_{L_k^2(0, \infty)}.$$

For the sake of completeness we shall include a proof of this lemma here. Also, we mention that there are many proofs of this lemma in the literature. The oldest might be the one found in Titchmarsh [Ti], §11.7, (11.7.4). A more recent one can be found in Epstein and Schotland [ES]. The one given below was suggested by a communication with Loukas Grafakos [G].

Proof. Observing that

$$\int_0^\infty e^{-kx} Q(k) dk = \int_0^\infty e^{-kx/2} k^{-1/4} \cdot e^{-kx/2} k^{1/4} Q(k) dk$$

and applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left| \int_0^\infty e^{-kx} Q(k) dk \right|^2 &\leq \int_0^\infty e^{-kx} k^{-1/2} dk \cdot \int_0^\infty e^{-kx} k^{1/2} |Q(k)|^2 dk \\ &= \sqrt{\pi} x^{-1/2} \int_0^\infty e^{-kx} k^{1/2} |Q(k)|^2 dk. \end{aligned}$$

Combining the last inequality and Fubini’s theorem gives

$$\begin{aligned} \left\| \int_0^\infty e^{-kx} Q(k) dk \right\|_{L_x^2(0, \infty)}^2 &\leq \sqrt{\pi} \int_0^\infty \int_0^\infty x^{-1/2} e^{-kx} k^{1/2} |Q(k)|^2 dk dx \\ &= \sqrt{\pi} \int_0^\infty \left(\int_0^\infty x^{-1/2} e^{-kx} k^{1/2} dx \right) |Q(k)|^2 dk \\ &= \sqrt{\pi} \int_0^\infty \sqrt{\pi} |Q(k)|^2 dk, \end{aligned}$$

which is the desired inequality (3.31). □

Next, using Lemma 3.2 with $Q(k) = (1+k^2)^{\frac{\beta}{2}} |k\hat{h}(k^2)|$ in inequality (3.30), and then making the change of variables $\tau = k^2 \Leftrightarrow k = \tau^{\frac{1}{2}} = \sigma(\tau)$, where $\sigma(\tau)$ is

defined by (2.19), we obtain

$$\begin{aligned}
 \|v_1(t)\|_\beta^2 &\lesssim \int_0^\infty (1+k^2)^\beta |k\hat{h}(k^2)|^2 dk \\
 &= \int_0^\infty (1+|\tau|)^\beta |\tau|^{\frac{1}{2}} |\hat{h}(\tau)|^2 \frac{d\tau}{2} \\
 (3.32) \qquad &\leq \frac{1}{2} \int_{-\infty}^\infty (1+|\tau|)^{\beta+\frac{1}{2}} |\hat{h}(\tau)|^2 d\tau.
 \end{aligned}$$

Note, however, that for $s > -\frac{1}{2}$ we have

$$(1+|\tau|)^2 \leq 2(1+\tau^2) \Leftrightarrow (1+|\tau|)^{\frac{2s+1}{2}} \leq 2^{\frac{2s+1}{4}} (1+\tau^2)^{\frac{2s+1}{4}}.$$

Thus, recalling that $m = \frac{2\beta+1}{4}$ we find

$$(3.33) \qquad \|v_1(t)\|_\beta^2 \leq 2^{\frac{2\beta-3}{4}} \int_{-\infty}^\infty (1+\tau^2)^m |\hat{h}(\tau)|^2 d\tau = 2^{\frac{2\beta-3}{4}} \|h\|_{H_t^m(\mathbb{R})}^2.$$

The case $s = \lfloor s \rfloor \in \mathbb{N}$. Whenever $s = \lfloor s \rfloor = n \in \mathbb{N}$, then

$$(3.34) \qquad \|v_1(t)\|_{H_x^s(0,\infty)}^2 = \sum_{j=0}^n \|\partial_x^j v_1(t)\|_{L_x^2(0,\infty)}^2.$$

Differentiating formula (3.21) for $v_1(t)$ with respect to x , we get

$$\partial_x^j v_1(x,t) = \int_0^\infty e^{-kx} [e^{ik^2t} (-k)^j G_h(k)] dk.$$

Then, applying Lemma 3.2 with $Q(k) = e^{ik^2t} (-k)^j G_h(k)$ we obtain the estimate

$$\|\partial_x^j v_1(t)\|_{L_x^2(0,\infty)}^2 \leq \pi \| |k|^j G_h(k) \|_{L_k^2(0,\infty)}^2 = \pi \| |k|^j k \hat{h}(k^2) \|_{L_k^2(0,\infty)}^2.$$

Furthermore, making the change of variables $\tau = k^2$ and estimating as before, we get

$$(3.35) \qquad \|\partial_x^j v_1(t)\|_{L_x^2(0,\infty)}^2 \lesssim \|h\|_{H_t^{\frac{2j+1}{4}}(\mathbb{R})}^2, \quad j = 0, 1, \dots, n.$$

Combining (3) and (3.34) gives the space estimate (3.12) for v_1 in the case of $s = \lfloor s \rfloor$.

The case $s = \lfloor s \rfloor + \beta$ with $\lfloor s \rfloor > 0$ and $0 < \beta < 1$. In this case, it suffices to estimate $\|\partial_x^{\lfloor s \rfloor} v_1(t)\|_\beta^2$. Working in a way similar to the first case, we obtain the analog of estimate (3.24),

$$\|\partial_x^{\lfloor s \rfloor} v_1(t)\|_\beta^2 \leq \int_0^\infty \int_0^\infty |G_h(k)| |G_h(l)| (kl)^{-\frac{1}{2} + \lfloor s \rfloor + \beta} \Delta(\alpha, \frac{1}{\alpha}, \beta) dl dk,$$

and the same computations as for the case $0 < s < 1$ lead to estimate (3.12) for v_1 .

Proof of Lemma 3.1. Letting $x = r \cos \theta$, $y = r \sin \theta$ in the definition (3.25) of Δ yields

$$\begin{aligned} \Delta &= 2 \int_0^\infty \int_0^{\frac{\pi}{2}} \frac{e^{-(\alpha + \frac{1}{\alpha})r \cos \theta} - e^{-(\alpha r \sin \theta + \frac{r \cos \theta}{\alpha})}}{r^{1+2\beta} |\cos \theta - \sin \theta|^{1+2\beta}} r d\theta dr \\ &= 2 \int_0^{\frac{\pi}{2}} \frac{1}{|\cos \theta - \sin \theta|^{1+2\beta}} \int_0^\infty \left[e^{-(\alpha + \frac{1}{\alpha})r \cos \theta} - e^{-(\alpha \sin \theta + \frac{\cos \theta}{\alpha})r} \right] r^{-2\beta} dr d\theta. \end{aligned}$$

Furthermore, changing variables in the integrals with respect to r and rearranging gives

$$\Delta = 2 \left(\int_0^\infty e^{-\rho} \rho^{-2\beta} d\rho \right) \int_0^{\frac{\pi}{2}} \frac{[(\alpha + \frac{1}{\alpha}) \cos \theta]^{-1+2\beta} - (\alpha \sin \theta + \frac{1}{\alpha} \cos \theta)^{-1+2\beta}}{|\cos \theta - \sin \theta|^{1+2\beta}} d\theta.$$

Next, recall the integral representation of the Gamma function,

$$(3.36) \quad \Gamma(z) \doteq \int_0^\infty e^{-\rho} \rho^{z-1} d\rho, \quad \operatorname{Re}(z) > 0,$$

which can be analytically continued for $\operatorname{Re}(z) < 0$ via the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.$$

Using (3.36) and letting

$$(3.37) \quad D(\alpha, \frac{1}{\alpha}, \beta) \doteq \int_0^{\frac{\pi}{2}} \frac{[(\alpha + \frac{1}{\alpha}) \cos \theta]^{-1+2\beta} - (\alpha \sin \theta + \frac{1}{\alpha} \cos \theta)^{-1+2\beta}}{|\cos \theta - \sin \theta|^{1+2\beta}} d\theta,$$

we find

$$(3.38) \quad \Delta = \begin{cases} 2\Gamma(1-2\beta)D(\alpha, \frac{1}{\alpha}, \beta), & 0 < \beta < \frac{1}{2}, \\ \frac{2\pi}{\sin(2\pi\beta)\Gamma(2\beta)} D(\alpha, \frac{1}{\alpha}, \beta), & \frac{1}{2} < \beta < 1. \end{cases}$$

We will now compute D . Letting $\nu = \tan \theta$, we have

$$D = \int_0^\infty \frac{(\alpha + \frac{1}{\alpha})^{-1+2\beta} - (\alpha\nu + \frac{1}{\alpha})^{-1+2\beta}}{|1-\nu|^{1+2\beta}} d\nu,$$

which can be written as

$$D = \int_0^1 \frac{(\alpha + \frac{1}{\alpha})^{-1+2\beta} (1 + \nu^{-1+2\beta}) - (\alpha + \frac{1}{\alpha} \nu)^{-1+2\beta} - (\alpha\nu + \frac{1}{\alpha})^{-1+2\beta}}{(1-\nu)^{1+2\beta}} d\nu.$$

Integrating, we obtain

$$\begin{aligned} D &= \frac{1}{\beta} \left(\alpha + \frac{1}{\alpha} \right)^{-1} \left\{ \lim_{\nu \rightarrow 1} \frac{1}{(1-\nu)^{2\beta}} \left[\left(\alpha + \frac{1}{\alpha} \right)^{2\beta} - \left(\alpha + \frac{1}{\alpha} \nu \right)^{2\beta} - \left(\frac{1}{\alpha} + \alpha\nu \right)^{2\beta} \right. \right. \\ &\quad \left. \left. + \left[\left(\alpha + \frac{1}{\alpha} \right) \nu \right]^{2\beta} \right] - \left[\left(\alpha + \frac{1}{\alpha} \right)^{2\beta} - \alpha^{2\beta} - \left(\frac{1}{\alpha} \right)^{2\beta} \right] \right\}. \end{aligned}$$

By l'Hôpital's rule, the limit above is equal to zero. Thus,

$$(3.39) \quad D = \frac{1}{\beta} \left(\alpha + \frac{1}{\alpha} \right)^{-1} \left[\alpha^{2\beta} + \left(\frac{1}{\alpha} \right)^{2\beta} - \left(\alpha + \frac{1}{\alpha} \right)^{2\beta} \right].$$

When $\alpha = O(1)$ the square bracket in (3.39) is also $O(1)$, while when α is large the square bracket is approximately equal to $-(1/\alpha)^{2\beta}$, which is small for $\beta \in (0, 1)$. The situation is analogous when α is small, because then $\frac{1}{\alpha}$ is large and the expression for D is symmetric in α and $\frac{1}{\alpha}$. Therefore, we have the estimate

$$(3.40) \quad |D| \lesssim \frac{1}{\beta} \left(\alpha + \frac{1}{\alpha} \right)^{-1}.$$

Inserting (3.39) into the expression (3.38) for Δ , we get

$$(3.41) \quad \Delta = \begin{cases} \frac{\Gamma(1-2\beta)}{\beta} \left(\alpha + \frac{1}{\alpha} \right)^{-1} \left[\alpha^{2\beta} + \left(\frac{1}{\alpha} \right)^{2\beta} - \left(\alpha + \frac{1}{\alpha} \right)^{2\beta} \right], & 0 < \beta < \frac{1}{2}, \\ \frac{\pi}{\sin(2\pi\beta)\Gamma(2\beta)\beta} \left(\alpha + \frac{1}{\alpha} \right)^{-1} \left[\alpha^{2\beta} + \left(\frac{1}{\alpha} \right)^{2\beta} - \left(\alpha + \frac{1}{\alpha} \right)^{2\beta} \right], & \frac{1}{2} < \beta < 1. \end{cases}$$

Also, note that the expression valid for $\frac{1}{2} < \beta < 1$ makes sense even for $\beta = \frac{1}{2}$, since the square bracket vanishes then. Thus, using estimate (3.40) we conclude that

$$\Delta \lesssim c_\beta \left(\alpha + \frac{1}{\alpha} \right)^{-1} \quad \forall \beta \in (0, 1). \quad \square$$

Continuity of the map $t \mapsto v(t)$. To complete the proof of Theorem 5, part 1, we need to show continuity of the map $t \mapsto v(t)$ in $H_x^s(0, \infty)$. For any sequence $\{t_n\} \subset [0, 2]$ converging to $t \in [0, 2]$, we have

$$v(x, t_n) - v(x, t) = [v_1(x, t_n) - v_1(x, t)] + [v_2(x, t_n) - v_2(x, t)]$$

with v_1 and v_2 defined by (3.15), hence

$$(3.42) \quad \|v(t_n) - v(t)\|_{H_x^s(0, \infty)} \leq \|v_1(t_n) - v_1(t)\|_{H_x^s(0, \infty)} + \|v_2(t_n) - v_2(t)\|_{H_x^s(0, \infty)}.$$

The second term in (3.42) can be estimated by using the Fourier transform definition of the Sobolev norm. We have

$$\begin{aligned} \|v_2(t_n) - v_2(t)\|_{H_x^s(0, \infty)}^2 &\leq \int_{-\infty}^{\infty} (1+k^2)^s \left| (e^{-ik^2 t_n} - e^{-ik^2 t}) k \hat{h}(-k^2) \right|^2 dk \\ &\leq 4 \int_{-\infty}^{\infty} (1+k^2)^s \left| k \hat{h}(-k^2) \right|^2 dk \\ &\leq 4 \|h\|_{H_t^m(\mathbb{R})}^2 < \infty; \end{aligned}$$

thus, by the dominated convergence theorem we find $v_2(t_n) \xrightarrow{n \rightarrow \infty} v_2(t)$ in $H_x^s(0, \infty)$.

Regarding the first term in (3.42), we use the L^2 definition of the Sobolev norm. Computations similar to those used for estimating v_1 in (3.15) yield

$$\begin{aligned} \|v_1(t_n) - v_1(t)\|_\beta^2 &\lesssim \int_0^\infty \int_0^\infty |G_h(k)| |G_h(l)| \left| e^{ik^2 t_n} - e^{ik^2 t} \right| \left| e^{il^2 t_n} - e^{il^2 t} \right| (kl)^{-\frac{1}{2}+\beta} \Delta\left(\alpha, \frac{1}{\alpha}, \beta\right) dl dk \\ &\leq c_\beta \int_0^\infty \int_0^\infty |G_h(k)| |G_h(l)| \frac{(kl)^\beta}{k+l} dl dk \lesssim c_\beta \|h\|_{H_t^m(\mathbb{R})}^2 < \infty, \end{aligned}$$

where we have used estimates (3.29)-(3.33). Hence, by the dominated convergence theorem we find $\|v_1(t_n) - v_1(t)\|_\beta \xrightarrow{n \rightarrow \infty} 0$. The proof for $s = [s] + \beta$ is similar. \square

2. Proving the time estimates (3.13). First, we will prove that v and v_x belong to $H_t^m(0, 2)$ and $H_t^{m-\frac{1}{2}}(0, 2)$, respectively. Starting from the UTM formula (3.10), we have

$$\begin{aligned} v(x, t) &= \frac{1}{\pi} \int_{\partial D^+} e^{ikx-ik^2t} k \hat{h}(-k^2) dk \\ &= \frac{1}{\pi} \int_{i\infty}^0 e^{ikx-ik^2t} k \hat{h}(-k^2) dk + \frac{1}{\pi} \int_0^\infty e^{ikx-ik^2t} k \hat{h}(-k^2) dk. \end{aligned}$$

Moreover, the transformation $\tau = k^2 \Leftrightarrow k = \tau^{\frac{1}{2}} = \sigma(\tau)$, with $\sigma(\tau)$ defined by (2.19), gives

$$\begin{aligned} (3.43) \quad v(x, t) &= \frac{1}{\pi} \int_{-\infty}^0 e^{i\sigma(\tau)x-i\tau t} \hat{h}(-\tau) \frac{d\tau}{2} + \frac{1}{\pi} \int_0^\infty e^{i\sigma(\tau)x-i\tau t} \hat{h}(-\tau) \frac{d\tau}{2} \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\tau t} e^{i\sigma(-\tau)x} \hat{h}(\tau) d\tau. \end{aligned}$$

Therefore,

$$\|v(x)\|_{H_t^m(\mathbb{R})}^2 = \int_{\mathbb{R}} (1 + \tau^2)^m \left| e^{i\sigma(-\tau)x} \hat{h}(\tau) \right|^2 d\tau.$$

Noting that for $x \in (0, \infty)$ and $\tau \in \mathbb{R}$,

$$\left| e^{i\sigma(-\tau)x} \right| = e^{-x \operatorname{Im}(\sigma(-\tau))} = e^{-x \sqrt{|\tau|} \sin(\arg(-\tau)/2)} \leq 1,$$

we obtain the estimate

$$\|v(x)\|_{H_t^m(\mathbb{R})}^2 \leq \int_{\mathbb{R}} (1 + \tau^2)^m |\hat{h}(\tau)|^2 d\tau = \|h\|_{H_t^m(\mathbb{R})}^2.$$

Also, differentiating formula (3.43) we have

$$v_x(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\sigma(\tau)x-i\tau t} i\tau^{\frac{1}{2}} \hat{h}(-\tau) d\tau,$$

which implies

$$\begin{aligned} \|v_x(x)\|_{H_t^{m-\frac{1}{2}}(\mathbb{R})}^2 &= \int_{\mathbb{R}} (1 + \tau^2)^{m-\frac{1}{2}} |\tau| |\hat{h}(\tau)|^2 d\tau \\ &\leq \int_{\mathbb{R}} (1 + \tau^2)^m |\hat{h}(\tau)|^2 d\tau = \|h\|_{H_t^m(\mathbb{R})}^2. \end{aligned}$$

Continuity of the maps $x \mapsto v(x)$ and $x \mapsto v_x(x)$. We need to prove that for any sequence $\{x_n\} \subset [0, \infty)$ converging to $x \in [0, \infty)$, we have

$$(3.44) \quad \|v(x_n) - v(x)\|_{H_t^m(0,2)} \xrightarrow{n \rightarrow \infty} 0.$$

Since

$$\begin{aligned} \|v(x_n) - v(x)\|_{H_t^m(\mathbb{R})}^2 &= \left\| \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\tau t} \left[e^{i\sigma(\tau)x_n} - e^{i\sigma(\tau)x} \right] \hat{h}(-\tau) d\tau \right\|_{H_t^m(\mathbb{R})}^2 \\ &= \int_{\mathbb{R}} (1 + \tau^2)^m \left| e^{i\sigma(-\tau)x_n} - e^{i\sigma(-\tau)x} \right|^2 |\hat{h}(\tau)|^2 d\tau \end{aligned}$$

and

$$\int_{\mathbb{R}} (1 + \tau^2)^m \left| e^{i\sigma(-\tau)x_n} - e^{i\sigma(-\tau)x} \right|^2 |\hat{h}(\tau)|^2 d\tau \leq 4 \|h\|_{H_t^m(\mathbb{R})}^2 < \infty,$$

the limit (3.44) follows by the dominated convergence theorem. The continuity argument for $v_x(x) \in H_t^{m-\frac{1}{2}}(0, 2)$ is analogous.

3. Proving that v is a weak solution. The space $C_0^\infty(\mathbb{R})$ is dense in the space $H^m(\mathbb{R})$; hence there exists a sequence $\{h_n(t)\} \subset C_0^\infty(\mathbb{R})$ such that $\|h_n - h\|_{H^m(\mathbb{R})} \xrightarrow{n \rightarrow \infty} 0$ with $h(t) \in H^m(\mathbb{R})$. What is more, the LS ibvp (3.9) with datum h_n is well-posed and has solution $v_n \in C^\infty([0, T]; \mathcal{S}(0, \infty))$ given by the formula

$$v_n(x, t) = \frac{1}{\pi} \int_{\partial D^+} e^{ikx - ik^2t} k \hat{h}_n(-k^2) dk, \quad x \in [0, \infty), \quad t \in [0, 2].$$

Thus, for all $\phi(x) \in C_0^\infty(\mathbb{R})$ such that $\phi(0) = 0$ the sequence $\{v_n(x, t)\}$ satisfies the identity

$$\frac{d}{dt} \langle v_n(t), \phi \rangle_{L_x^2(0, \infty)} = - \langle v_n(t), i\phi_{xx} \rangle_{L_x^2(0, \infty)} + ih_n(t) \overline{\phi_x(0)}.$$

Taking the limit $n \rightarrow \infty$ in the above identity, we conclude that the function v defined by equation (3.10) is a weak solution to the LS ibvp (3.9) in the sense of (3.14).

4. Proving that the initial and boundary conditions are satisfied. From the well-posedness of the LS ibvp in Schwartz spaces it follows that $v_n(x, 0) = 0$ and $v_n(0, t) = h_n(t)$. Moreover, by the density of $C_0^\infty(0, \infty)$ in $H^s(0, \infty)$ we have $\|v_n(x, 0) - v(x, 0)\|_{H_x^s(0, \infty)} \xrightarrow{n \rightarrow \infty} 0$, thus $\|v(0, t)\|_{H_x^s(0, \infty)} \xrightarrow{n \rightarrow \infty} 0$. Also, the density of $C_0^\infty(0, 2)$ in $H^m(0, 2)$ implies $\|v_n(0, t) - v(0, t)\|_{H_t^m(0, 2)} \xrightarrow{n \rightarrow \infty} 0$ so that $\|h_n(t) - v(0, t)\|_{H_t^m(0, 2)} \xrightarrow{n \rightarrow \infty} 0$. However, we have $\|h_n(t) - h(t)\|_{H_t^m(0, 2)} \xrightarrow{n \rightarrow \infty} 0$ and hence $\|v(0, t) - h(t)\|_{H_t^m(0, 2)} \xrightarrow{n \rightarrow \infty} 0$. \square

Deducing Theorem 1 from Theorem 5. As noted earlier, when restricted in $[0, \infty) \times [0, T]$, $T < 1$, the function v , which is the solution of ibvp (3.9) in the sense of Theorem 5, satisfies ibvp (3.2). Moreover, using estimate (3.8) we infer from estimates (3.12) and (3.13) the space estimate

$$(3.45) \quad \sup_{t \in [0, T]} \|v(t)\|_{H_x^s(0, \infty)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right), \quad s > \frac{1}{2},$$

and the time estimates

$$(3.46a) \quad \sup_{x \in [0, \infty)} \|v(x)\|_{H_t^{\frac{2s+1}{4}}(0, T)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right), \quad s > \frac{1}{2},$$

(3.46b)

$$\sup_{x \in [0, \infty)} \|v_x(x)\|_{H^{\frac{2s-1}{4}}(0, T)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right), \quad s > \frac{1}{2}.$$

Further, recall that by linearity the solution to the LS ibvp (1.1) is given by

$$(3.47) \quad u(x, t) = U(x, t) + v(x, t), \quad x \in [0, \infty), \quad t \in [0, T],$$

where U satisfies the LS ivp (2.1) and v satisfies the reduced LS ibvp (3.9). Therefore, combining Theorem 4 and Theorem 5 with estimates (3.12) and (3.13) replaced by estimates (3.45) and (3.46), we obtain Theorem 1.

4. SOLVING THE FORCED LS IBVP IN SOBOLEV SPACES

Having proved Theorem 1 for the LS ibvp (1.1), we now prove Theorem 2 for the forced LS ibvp (1.15) with the same initial and boundary conditions, which establishes well-posedness of this problem in Sobolev spaces. As noted in the Introduction, if $s > \frac{5}{2}$, then additional compatibility conditions that involve the forcing should hold at $x = t = 0$. These conditions are not the same with the corresponding conditions (1.6) for the LS ibvp; in fact, combining the two families of conditions confines the forcing to a very special class of functions in $C([0, T]; H_x^s(0, \infty))$. Therefore, we restrict $\frac{1}{2} < s \leq \frac{5}{2}$ so that the only compatibility condition enforced in both problems is condition (1.5).

Decomposition of ibvp (1.15). By linearity, the forced LS ibvp (1.15) can be decomposed into the following two problems.

Problem 1: LS ibvp on $[0, \infty) \times [0, T]$. This problem reads

$$(4.1a) \quad iu_t + u_{xx} = 0, \quad x \in (0, \infty), \quad t \in (0, T), \quad T < 1,$$

$$(4.1b) \quad u(x, 0) = u_0(x) \in H_x^s(0, \infty), \quad x \in [0, \infty),$$

$$(4.1c) \quad u(0, t) = g_0(t) \in H_t^{\frac{2s+1}{4}}(0, T), \quad t \in [0, T],$$

and its solution is given by the UTM formula (1.17) as

$$(4.2) \quad \begin{aligned} u(x, t) &= S[u_0, g_0, 0](x, t) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx - ik^2t} \hat{u}_0(k) dk - \frac{1}{2\pi} \int_{\partial D^+} e^{ikx - ik^2t} \hat{u}_0(-k) dk \\ &\quad + \frac{1}{\pi} \int_{\partial D^+} e^{ikx - ik^2t} k \tilde{g}_0(k^2, T) dk. \end{aligned}$$

Also, thanks to Theorem 1 we have the space estimate

$$(4.3) \quad \sup_{t \in [0, T]} \|S[u_0, g_0, 0](t)\|_{H_x^s(0, \infty)} \leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right).$$

Problem 2: Forced LS ibvp on $[0, \infty) \times [0, T]$ with zero data. The precise description of this problem is given by

$$(4.4a) \quad iu_t + u_{xx} = f(x, t) \in C([0, T]; H_x^s(0, \infty)), \quad x \in (0, \infty), \quad t \in (0, T), \quad T < 1,$$

$$(4.4b) \quad u(x, 0) = 0, \quad x \in [0, \infty),$$

$$(4.4c) \quad u(0, t) = 0, \quad t \in [0, T],$$

and its solution can be expressed in terms of the solution to ibvp (4.1). Indeed, starting from the UTM solution formula (1.17), we are able to write the solution to ibvp (4.4) as

$$(4.5) \quad \begin{aligned} u(x, t) &= S[0, 0, f](x, t) \\ &= -i \left\{ \int_0^t \left[\frac{1}{2\pi} \int_{\mathbb{R}} e^{ikx - ik^2(t-t')} \hat{f}(k, t') dk \right] dt' \right. \\ &\quad \left. + \int_0^t \left[-\frac{1}{2\pi} \int_{\partial D^+} e^{ikx - ik^2(t-t')} \hat{f}(-k, t') dk \right] dt' \right\} \\ &= -i \int_0^t S[f, 0, 0](x, t - t') dt'. \end{aligned}$$

Combining the solution formula (4.5) with estimate (4.3), we obtain the following space estimate for ibvp (4.4):

$$\begin{aligned}
 \sup_{t \in [0, T]} \|S[0, 0, f](t)\|_{H_x^s(0, \infty)} &= \sup_{t \in [0, T]} \left\| \int_0^t S[f, 0, 0](\cdot, t - t') dt' \right\|_{H_x^s(0, \infty)} \\
 &\leq \sup_{t \in [0, T]} \int_0^t \|S[f, 0, 0](\cdot, t - t')\|_{H_x^s(0, \infty)} dt' \\
 &\leq c_s \sup_{t \in [0, T]} \int_0^t \|f(t')\|_{H_x^s(0, \infty)} dt' \\
 (4.6) \qquad \qquad \qquad &\leq c_s T \sup_{t \in [0, T]} \|f(t)\|_{H_x^s(0, \infty)}.
 \end{aligned}$$

The solution $u(x, t)$ to ibvp (1.15). For $x \in [0, \infty)$, $t \in [0, T]$, $T < 1$, the solution to the forced LS ibvp (1.15) is given by the formula

$$(4.7) \qquad u(x, t) = \Phi f(x, t) = S[u_0, g_0, 0](x, t) + S[0, 0, f](x, t).$$

Indeed, the function Φf satisfies the following.

(a) Forced LS equation:

$$\begin{aligned}
 i(\Phi f)_t + (\Phi f)_{xx} &= i\left(S[u_0, g_0, 0]\right)_t + \left(S[u_0, g_0, 0]\right)_{xx} \\
 &\quad + i\left(S[0, 0, f]\right)_t + \left(S[0, 0, f]\right)_{xx} \\
 &= 0 + f(x, t) = f(x, t).
 \end{aligned}$$

(b) Initial condition (1.15b):

$$\Phi f(x, 0) = S[u_0, g_0, 0](x, 0) + S[0, 0, f](x, 0) = u_0(x) + 0 = u_0(x).$$

(c) Boundary condition (1.15c):

$$\Phi f(0, t) = S[u_0, g_0, 0](0, t) + S[0, 0, f](0, t) = g_0(t) + 0 = g_0(t).$$

Therefore, the function Φf defined by (4.7) satisfies the forced LS ibvp (1.15). What is more, combining the space estimates (4.3) and (4.6) we conclude that $\Phi f(x) \in H_x^s(0, \infty)$ with the space estimate

$$\begin{aligned}
 \sup_{t \in [0, T]} \|\Phi f(t)\|_{H_x^s(0, \infty)} \\
 (4.8) \qquad \qquad \qquad &\leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} + T \sup_{t \in [0, T]} \|f(t)\|_{H_x^s(0, \infty)} \right).
 \end{aligned}$$

The continuity of the map $t \mapsto \Phi f(t)$ follows from Theorem 1, which takes care of $S[u_0, g_0, 0]$, and a similar argument for $S[0, 0, f]$. Thus,

$$u = \Phi f \in C([0, T]; H_x^s(0, \infty)),$$

and the proof of Theorem 2 is complete.

5. WELL-POSEDNESS OF THE NLS IBVP IN SOBOLEV SPACES

We have now developed the tools needed for presenting the proof of Theorem 3, which establishes well-posedness of the NLS ibvp (1.14). Using the UTM formula (4.7), we define an iteration map for the solution, which we then show to be onto and a contraction in the space $C([0, T^*]; H_x^s(0, \infty))$ with T^* defined by equation

(1.20). Having established existence and uniqueness, we then complete the proof by showing that the data-to-solution map is locally Lipschitz continuous.

A. Existence and uniqueness. Setting $f(x, t) = -\lambda u|u|^2(x, t)$ in the solution (4.7) of the forced LS ibvp, we obtain the **iteration map**

$$(5.1) \quad u \mapsto \Phi u = \Phi_{u_0, g_0} u \doteq S[u_0, g_0, 0] + S[0, 0, -\lambda u|u|^2].$$

It now suffices to prove that the map $u \mapsto \Phi u$ is a contraction in the space

$$(5.2) \quad X = C([0, T^*]; H_x^s(0, \infty))$$

equipped with the norm

$$(5.3) \quad \|u\|_X = \sup_{t \in [0, T^*]} \|u(t)\|_{H_x^s(0, \infty)}$$

for an appropriate T^* with $0 < T^* \leq T$.

1. Showing that the map $u \mapsto \Phi u$ is onto X . From the triangle inequality and estimates (4.3) and (4.6), we have

$$(5.4) \quad \begin{aligned} \|\Phi u\|_X &= \sup_{t \in [0, T^*]} \|S[u_0, g_0, 0](t) + S[0, 0, -\lambda u|u|^2](t)\|_{H_x^s(0, \infty)} \\ &\leq \sup_{t \in [0, T^*]} \|S[u_0, g_0, 0](t)\|_{H_x^s(0, \infty)} + \sup_{t \in [0, T^*]} \|S[0, 0, -\lambda u|u|^2](t)\|_{H_x^s(0, \infty)} \\ &\leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} \right) + c_s T^* \sup_{t \in [0, T^*]} \|-\lambda u|u|^2(t)\|_{H_x^s(0, \infty)} \\ &\leq c_s \left(\|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)} + |\lambda| T^* \|u\|_X^3 \right), \end{aligned}$$

with the last inequality due to the algebra property in $H_x^s(0, \infty)$ for $s > \frac{1}{2}$.

Consider the ball $B(0, r) = \{u \in X : \|u\|_X \leq r\}$ with radius

$$(5.5) \quad r = 2c_s \|(u_0, g_0)\|_D,$$

where the norm $\|\cdot\|_D$ in the data space is defined by

$$(5.6) \quad \|(u_0, g_0)\|_D \doteq \|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{\frac{2s+1}{4}}(0, T)}.$$

It then follows from estimate (5.4) that in order to have $\|\Phi u\|_X \leq r$ for $\|u\|_X \leq r$ it suffices to have some $0 < T^* \leq T$ satisfying the **onto condition**

$$(5.7) \quad \frac{r}{2} + c_s |\lambda| T^* r^3 \leq r \iff T^* \leq \frac{1}{8c_s^3 |\lambda| \|(u_0, g_0)\|_D^2}.$$

2. Showing that the map $u \mapsto \Phi u$ is a contraction in X . We shall show that for any $u_1, u_2 \in X$ we have

$$(5.8) \quad \|\Phi u_1 - \Phi u_2\|_X \leq \frac{1}{2} \|u_1 - u_2\|_X.$$

We compute

$$\begin{aligned} \Phi u_1 - \Phi u_2 &= \left(S[u_0, g_0, 0] + S[0, 0, -\lambda u_1|u_1|^2] \right) \\ &\quad - \left(S[u_0, g_0, 0] + S[0, 0, -\lambda u_2|u_2|^2] \right) \\ &= S[0, 0, -\lambda (u_1|u_1|^2 - u_2|u_2|^2)]. \end{aligned}$$

Thus, using estimate (4.6) we find

$$\begin{aligned} \|\Phi u_1 - \Phi u_2\|_X &= \sup_{t \in [0, T^*]} \|S[0, 0, -\lambda(u_1|u_1|^2 - u_2|u_2|^2)](t)\|_{H_x^s(0, \infty)} \\ &\leq c_s |\lambda| T^* \sup_{t \in [0, T^*]} \|(u_1|u_1|^2 - u_2|u_2|^2)(t)\|_{H_x^s(0, \infty)}. \end{aligned}$$

Next, note that

$$u_1|u_1|^2 - u_2|u_2|^2 = (|u_1|^2 + |u_2|^2)(u_1 - u_2) + u_2 u_1 (\overline{u_1 - u_2});$$

hence, by the algebra property we get

$$(5.9) \quad \|\Phi u_1 - \Phi u_2\|_X \leq c_s |\lambda| T^* (\|u_1\|_X + \|u_2\|_X)^2 \|u_1 - u_2\|_X.$$

To show that $u \mapsto \Phi u$ is a contraction on $B(0, r) \subset X$, with r defined by equation (5.5), it suffices to show that there exists $0 < T^* \leq T$ satisfying the **contraction condition** $c_s |\lambda| T^* (r + r)^2 \leq \frac{1}{2}$, which after substituting for r becomes

$$(5.10) \quad T^* \leq \frac{1}{32c_s^3 |\lambda| \|(u_0, g_0)\|_D^2}.$$

Indeed, estimate (5.9) together with condition (5.10) implies the desired contraction inequality (5.8) on $B(0, r)$.

The onto condition (5.7) and the contraction condition (5.10) are both satisfied if we choose

$$(5.11) \quad T^* = \min \left\{ T, \frac{1}{32c_s^3 |\lambda| \|(u_0, g_0)\|_D^2} \right\}.$$

Then, for $s > \frac{1}{2}$ the map $u \mapsto \Phi u$ with Φu defined by (5.1) is both onto and a contraction in $B(0, r) \subset X$. Hence, by the contraction mapping theorem it has a unique fixed point in $B(0, r)$; i.e. the equation $u = \Phi u$, which is an integral equation for the NLS ibvp (1.14), has a unique solution $u \in B(0, r) \subset X$ with r defined by equation (5.5) and lifespan T^* given by equation (5.11).

B. Continuity of the data-to-solution map. To complete the proof of well-posedness of the NLS on the half-line, it remains to show that the data-to-solution map

$$H_x^s(0, \infty) \times H_t^{\frac{2s+1}{4}}(0, T) \ni (u_0, g_0) \mapsto u \in X = C([0, T^*]; H_x^s(0, \infty))$$

with lifespan T^* given by equation (5.11) is continuous. In fact, it turns out that this map is locally Lipschitz.

Let (u_0, g_0) and (v_0, h_0) be two data in a ball $B_\rho \subset D$ of radius $\rho > 0$, whose centre is at a distance r from the origin, and denote by $u = \Phi_{u_0, g_0} u$ and $v = \Phi_{v_0, h_0} v$ the corresponding solutions to the NLS ibvp. As we have seen above, the lifespan of u , which we denote by T_u , is equal to

$$T_u = \min \left\{ T, \frac{1}{32c_s^3 |\lambda| \|(u_0, g_0)\|_D^2} \right\},$$

and the lifespan of v is equal to

$$T_v = \min \left\{ T, \frac{1}{32c_s^3 |\lambda| \|(v_0, h_0)\|_D^2} \right\}.$$

Since $\max \left\{ \|(u_0, g_0)\|_D, \|(v_0, h_0)\|_D \right\} \leq r + \rho$, it follows that

$$(5.12) \quad \min \{T_u, T_v\} \geq \min \left\{ T, \frac{1}{32c_s^3 |\lambda| (r + \rho)^2} \right\} \doteq T_c.$$

Since both T_u and T_v are bigger than T_c , the solutions u and v both exist for $0 \leq t \leq T_c$. For this common lifespan T_c , we define

$$X_c = C([0, T_c]; H_x^s(0, \infty)).$$

Clearly, $X_c \subset X_u$ and $X_c \subset X_v$, where X_u and X_v are defined by (5.2) with T^* replaced by T_u and T_v respectively. We now determine a ball $B(0, r_c) \subset X_c$ such that for any $u, v \in B(0, r_c)$ with data in the ball B_ρ it follows that

$$(5.13) \quad \|u - v\|_{X_c} \leq 2c_s \|(u_0, g_0) - (v_0, h_0)\|_D.$$

Since u and v are obtained as fixed points of the maps $u \mapsto \Phi u$ and $v \mapsto \Phi v$ in X_u and X_v respectively, we have

$$\begin{aligned} \|u - v\|_{X_c} &= \|\Phi u - \Phi v\|_{X_c} \\ &= \left\| \left(S[u_0, g_0, 0] + S[0, 0, -\lambda u|u|^2] \right) - \left(S[v_0, h_0, 0] + S[0, 0, -\lambda v|v|^2] \right) \right\|_{X_c} \\ &\leq \|S[u_0 - v_0, g_0 - h_0, 0]\|_{X_c} + \|S[0, 0, \lambda(u|u|^2 - v|v|^2)]\|_{X_c}, \end{aligned}$$

from which we get the estimate

$$\|u - v\|_{X_c} \leq c_s \|(u_0, g_0) - (v_0, h_0)\|_D + c_s |\lambda| T_c (\|u\|_{X_c} + \|v\|_{X_c})^2 \|u - v\|_{X_c}.$$

Since $u, v \in B(0, r_c) \subset X_c$, the above implies

$$\|u - v\|_{X_c} \leq c_s \|(u_0, g_0) - (v_0, h_0)\|_D + c_s |\lambda| T_c \cdot 4r_c^2 \cdot \|u - v\|_{X_c}.$$

Rearranging, we get

$$\|u - v\|_{X_c} \leq \frac{c_s}{1 - 4c_s |\lambda| T_c r_c^2} \|(u_0, g_0) - (v_0, h_0)\|_D$$

provided that $4c_s |\lambda| T_c r_c^2 < 1$ with T_c defined by equation (5.12). Choosing

$$(5.14) \quad r_c = \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{c_s |\lambda| T_c}} = \max \left\{ \frac{1}{\sqrt{2}} \cdot \frac{1}{2\sqrt{c_s |\lambda| T_c}}, 2c_s (r + \rho) \right\}$$

gives the desired inequality (5.13). Thus, local Lipschitz continuity of the data-to-solution map has been established. □

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