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HOCHSTER DUALITY IN DERIVED CATEGORIES AND POINT-FREE RECONSTRUCTION OF SCHEMES

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ABSTRACT. For a commutative ring R, we exploit localization techniques and point-free topology to give an explicit realization of both the Zariski frame of R (the frame of radical ideals in R) and its Hochster dual frame as lattices in the poset of localizing subcategories of the unbounded derived category $D(R)$. This yields new conceptual proofs of the classical theorems of Hopkins-Neeman and Thomason. Next we revisit and simplify Balmer's theory of spectra and supports for tensor triangulated categories from the viewpoint of frames and Hochster duality. Finally we exploit our results to show how a coherent scheme $(X, \mathcal{O}_X)$ can be reconstructed from the tensor triangulated structure of its derived category of perfect complexes.

CONTENTS

INTRODUCTION

One of the remarkable achievements of stable homotopy theory is the classification of thick subcategories in the finite stable homotopy category by Devinatz-Hopkins-Smith [10]. This result migrated to commutative algebra in the work of Hopkins [13] and Neeman [22], was generalized to the category of perfect complexes over a coherent (i.e. quasi-compact quasi-separated) scheme by Thomason [27], and found a version in modular representation theory in the work of Benson-Carlson-Rickard [6]. A theorem of a similar flavor is the classification of radical thick tensor ideals in a tensor triangulated category by Balmer [4]. In each case, the thick subcategories (or radical thick tensor ideals) are classified in terms of unions of closed subsets with quasi-compact complement in a coherent scheme X.

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What apparently was not noticed is that these classifying subsets are precisely the open sets in the Hochster dual topology of the Zariski topology on X, and that Hochster duality, originally a rather puzzling result of Hochster [12], has a very simple description in the setting of point-free topology [15], i.e. working with frames of open sets instead of with points. That Hochster duality is involved in these classification results was first noticed by Buan, Krause and Solberg [8] and independently in [19], where the frame viewpoint was perhaps first exploited; see also the recent [14].

For R a commutative ring, we denote by $D^\omega(R)$ its derived category of perfect complexes. The Zariski frame of R is the frame of radical ideals in R or, equivalently, the frame of Zariski open sets in $\operatorname{Spec} R$. The affine case of Thomason's theorem can be phrased as follows.

Theorem. *The thick subcategories of $D^\omega(R)$ form a coherent frame which is Hochster dual to the Zariski frame of R.*

This formulation is the starting point for our investigations: since it states a clean conceptual relationship between two algebraic structures, with no mention of point sets, there should be a conceptual and point-free explanation of it. We achieve such an explanation as a byproduct of a more general analysis of frames and lattices of thick subcategories, localizing subcategories, and tensor ideals in derived categories of a commutative ring, and more generally of a coherent scheme.

We work in the unbounded derived category $D(R)$. The thick subcategories of $D^\omega(R)$ are in one-to-one correspondence with the *compactly generated* localizing subcategories of $D(R)$. A key ingredient in our proof of the above theorem is the following (Proposition 2.1.13).

Proposition. *Every localizing subcategory of $D(R)$ generated by a* finite *set of compact objects is of the form* $\operatorname{Loc}(R/I)$ *for I a finitely generated ideal of R, and depends only on its radical $\sqrt{I}$.*

Note that R/I is typically *not* a compact object, but it generates the same localizing subcategory as the Koszul complex of I, which *is* compact. The radicals of finitely generated ideals form a distributive lattice called the *Zariski lattice* and we show (Proposition 2.1.17) the following.

Proposition. *These localizing subcategories form a distributive lattice isomorphic to the opposite of the Zariski lattice. The correspondence is given by*

$$\operatorname{Loc}(R/I) \quad \leftrightarrow \quad \sqrt{I}.$$

This result contains the essence of Thomason's affine result quoted above. More precisely, Thomason's result follows by *coherence*, namely the fact that the frame of compactly generated localizing subcategories is determined by its finite part; this is the lattice counterpart of compact generation.

Having described the dual of the Zariski frame explicitly inside $D(R)$, we proceed to show that also the Zariski frame itself can be realized inside $D(R)$; see Theorem 2.2.16.

Theorem. *The localizing subcategories of $D(R)$ generated by modules of the form R_f form a coherent frame isomorphic to the Zariski frame of R.*

Again by coherence, the essence of this correspondence is in the finite part, where it is given by the surprisingly simple correspondence

$$\mathrm{Loc}(R_{f_1}, \ldots, R_{f_n}) \leftrightarrow \sqrt{(f_1, \ldots, f_n)}.$$

The results explained so far make up Section 2, which finishes with a short explanation of the standard procedure of extracting points; this is convenient for comparison with results of Neeman and Thomason.

Before coming to the general case of a coherent scheme in Section 4, we need some abstract theory of radical thick tensor ideals in a tensor triangulated category. We revisit Balmer's theory of spectra and supports and provide a substantial simplification of this theory, using a point-free approach. Large parts of Balmer's paper [4] are subsumed in the following single theorem (3.1.9).

Theorem. *In a tensor triangulated category $\mathcal{T}$, the radical thick tensor ideals form a coherent frame, provided there is only a set of them.*

This coherent frame we call the *Zariski frame* of $\mathcal{T}$ and denote it $\boldsymbol{Zar}(\mathcal{T})$, as it is constructed from the ring-like object $(\mathcal{T}, \otimes, \mathbf{1})$ in the same way as classically a commutative ring R gives rise to the frame of radical ideals in R. The Zariski frame of $\mathcal{T} = D^\omega(R)$ is naturally identified with the frame of compactly general localizing subcategories of $D(R)$ featured in our first main theorem.

Furthermore, just as in the case of rings as observed by Joyal [17] in the early 1970s, the Zariski frame enjoys a universal property; see Theorem 3.2.3:

Theorem. *The support*

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \boldsymbol{Zar}(\mathcal{T}) \\ a & \longmapsto & \sqrt{a} \end{array}$$

is initial among supports.

With these general results in hand, we can finally assemble our precise affine results to establish the following version of Thomason's theorem [27, Theorem 3.15]:

Theorem. *Let X be a coherent scheme. Then the Zariski frame of $D^\omega_{\mathrm{qc}}(X)$ is Hochster dual to the Zariski frame of X.*

Once again the point-free methods give a more elementary and conceptual proof, avoiding for example technical tools such as Absolute Noetherian Approximation.

Finally, our explicit description of the Zariski frame inside $D(R)$ readily endows it with a sheaf of rings, encompassing the local structure necessary to reconstruct also the structure sheaf of X:

Theorem. *A coherent scheme $(X, \mathcal{O}_X)$ can be reconstructed from the tensor triangulated category $D^\omega_{\mathrm{qc}}(X)$ of perfect complexes.*

Balmer [3] had previously obtained such a reconstruction theorem in the special case where X is topologically noetherian. The general case was obtained by Buan-Krause-Solberg in [8, Theorem 9.5]. However both these results rely on point sets and invoke Thomason's classification theorem, whereas our reconstruction is more direct in the sense that it refers directly to the frame of open sets and exploits the reduction to affine schemes to exhibit the sheaf locally as simply $\mathrm{Loc}(R_f) \mapsto R_f$.

Indeed, the main characteristic of our work is that the approach is essentially point-free. Point-free methods have a distinctive constructive flavor contrasted with point-set topology: in general, points (e.g. maximal ideals in rings) are only available through choice principles such as Zorn's lemma. In the present paper we assume the axiom of choice, but exploit the fact that point-free arguments tend to be simpler and more direct.

Another novelty is that we work with unbounded chain complexes, which is the key to understanding Hochster duality: while for compactly generated localizing subcategories this is mostly for convenience, its Hochster dual frame, consisting of localizing subcategories generated by localizations of the ring, is something we think cannot be realized inside $D^\omega(R)$. Having this realization in $D(R)$ is of course the key point in getting at the structure sheaf in so elegant a way.

Finally it is noteworthy that while the classical proofs of the classification theorems relied on the Tensor Nilpotence Theorem (see for instance Rouquier [26] for a discussion of these), we follow Balmer instead, *deducing* the Tensor Nilpotence Theorem and tying it to the fact that the Zariski frame (like any coherent frame) gives rise to a Kolmogorov topological space.

To finish this introduction we comment on the role of Hochster duality in these developments, leading to some questions about duality that are poorly understood. Starting with a commutative ring R, on one hand we can study its "internal" structure, performing the construction of the Zariski frame on $(R, \cdot, 1)$, which gives the spectrum of R with the Zariski topology. On the other hand we can study its "external" structure by performing the Zariski frame construction of the ring-like object $(D^\omega(R), \otimes, \mathbf{1})$. The Zariski frame of this object is the Hochster dual of $\operatorname{Spec}(R)$! This duality puts in correspondence, from the short exact sequence

$$0 \to I \to R \to R/I \to 0$$

(for I a finitely generated ideal), the Zariski open set $D(I)$ with the Hochster open set $\operatorname{Loc}(R/I)$. The Zariski construction is essentially the same in both cases and enjoys similar universal properties. The duality is therefore somehow encoded in taking D^ω. We think this phenomenon deserves further exploration. When passing to the finer data of structure sheaves, the duality points towards a remarkable duality between local rings and domains, which has been only little studied; see Johnstone [15, V.4] for a starting point.

1. Preliminaries

1.1. **Localizing subcategories in triangulated categories.** In this subsection we review some basic properties of localizing categories. Expert readers can skip this subsection. Further details can be found in [23] or in [20]. Let $\mathcal{T}$ be a triangulated category, throughout this subsection assumed to admit arbitrary sums. We denote by $\mathcal{T}^\omega$ the full subcategory of *compact* objects, namely those objects $c \in \mathcal{T}$ for which the functor $\operatorname{Hom}_\mathcal{T}(c, -) : \mathcal{T} \to \boldsymbol{Ab}$ commutes with arbitrary sums or indeed all homotopy colimits. Our main example will be $D(R)$, the derived category of a commutative ring R.

Definition 1.1.1. A *localizing* subcategory is a full triangulated subcategory $\mathcal{L} \subset \mathcal{T}$ stable under arbitrary sums. It is then automatically closed under retracts by the Eilenberg swindle argument. If S is a set of objects in $\mathcal{T}$, then the smallest triangulated category containing S is called the localizing subcategory generated by S and is denoted by $\operatorname{Loc}(S)$. Similarly, using products instead of sums we get the notion of a *colocalizing* subcategory and a colocalizing subcategory generated by S, denoted by $\mathrm{Coloc}(S)$.

Informally, $\mathrm{Loc}(S)$ is the smallest category whose objects can be built out of the objects in S using suspensions, extensions (triangles) and arbitrary sums. For instance, in the derived category of a ring R we have $\mathrm{Loc}(R) = D(R)$. If two objects a and b generate the same localizing subcategory, $\mathrm{Loc}(a) = \mathrm{Loc}(b)$, then, inspired by topology and Bousfield localizations (see below), we will say that they are *cellularly equivalent*. For other notions of cellularity in chain complexes, see [18]. A full, triangulated subcategory $\mathcal{S} \subset \mathcal{T}$ is called *compactly generated* if it is of the form $\mathcal{S} = \mathrm{Loc}(S)$ for some set S of compact objects. In this case there is a uniform way to describe the objects in $\mathrm{Loc}(S)$ which turns the informal description into the following rigorous statement.

Proposition 1.1.2 (Rouquier [25, Thm. 4.22 and Prop. 3.13])**.** *Let $\mathcal{T}$ be a triangulated category, and S a set of compact objects. Then for any object $a \in \mathrm{Loc}(S)$, there exists a sequence of maps*

$$F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} \cdots F_i \xrightarrow{f_i} F_{i+1} \cdots$$

such that F_0 and the cone of each f_i comprise a direct sum of copies of suspensions of objects in S, and

$$a = \operatorname*{hocolim}_i F_i.$$

If moreover a is compact, then there is $n \in \mathbb{N}$ such that a is a direct summand of F_n, and F_0 and the cone of each f_i (for $i < n$) can be taken to be finite sums of objects in S. In particular there exists a finite *subset $K \subset S$ such that $a \in \mathrm{Loc}(K)$.*

Given an object a in a compactly generated localizing subcategory $\mathrm{Loc}(S)$, we will call any of the sequences provided by Proposition 1.1.2 a *recipe* for a.

Corollary 1.1.3. *For S a set of compact objects, we have*

$$\mathrm{Loc}(S)^\omega = \mathrm{Loc}(S) \cap \mathcal{T}^\omega,$$

where the superscript ω stands for the full subcategory of compact objects.

1.1.4. *Bousfield colocalizations.* Recall that a *Bousfield colocalization functor* in $\mathcal{T}$ is a pair (Γ, η), where $\Gamma : \mathcal{T} \to \mathcal{T}$ is an endofunctor, and $\eta : \Gamma \to \mathrm{Id}$ is a natural transformation such that $\Gamma\eta : \Gamma^2 \to \Gamma$ is an isomorphism, so Γ is idempotent, and $\Gamma\eta = \eta\Gamma$. A Bousfield colocalization functor on $\mathcal{T}^{\mathrm{op}}$ is called a *Bousfield localization functor*.

Consider also, for any full subcategory $\mathcal{C} \subset \mathcal{T}$ closed under suspension, its right and left orthogonal categories:

(1) $\mathcal{C}^\perp = \{x \in \mathcal{T} \mid \mathrm{Hom}(c, x) = 0 \text{ for all } c \in \mathcal{C}\}$. This is always a colocalizing subcategory and $\mathcal{C}^\perp = \mathrm{Loc}(\mathcal{C})^\perp$.
(2) ${}^\perp\mathcal{C} = \{x \in \mathcal{T} \mid \mathrm{Hom}(x, c) = 0 \text{ for all } c \in \mathcal{C}\}$. This is always a localizing subcategory and ${}^\perp\mathcal{C} = {}^\perp\,\mathrm{Coloc}(\mathcal{C})$.

In the compactly generated case, which is enough for our purposes, Brown Representability [23, Ch.8] ensures that both Bousfield localization and colocalization functors exist.

Proposition 1.1.5. *Let S be a set of compact objects in $\mathcal{T}$. Then there are both a Bousfield localization L_S and colocalization Γ_S functors associated to S. The* *essential image of* Γ_S *is then* $\mathrm{Loc}(S)$ *and every object* $x \in \mathcal{T}$ *fits into a localization triangle:*

$$\Gamma_S(x) \longrightarrow x \longrightarrow L_S(x) \longrightarrow \Sigma\Gamma_S(x).$$

Moreover in this case $\mathrm{Loc}(S) = {}^{\perp}(\mathrm{Loc}(S)^{\perp})$.

Borrowing from the terminology in algebraic topology, we will call the Bousfield colocalization functor Γ_S, a *cellularization* functor. For more properties of Bousfield (co)localizations in triangulated categories, we refer the interested reader to [23, Chapter 9] or [20].

1.1.6. *Tensor triangulated categories.* Sometimes we will use a *tensor* triangulated category, $(\mathcal{T}, \otimes, \mathbf{1})$, that is a symmetric monoidal structure $\otimes$ on $\mathcal{T}$, compatible with the triangulated structure; i.e. tensoring with an object is an exact, triangulated endofunctor of $\mathcal{T}$. The unit of the tensor product will be denoted by $\mathbf{1}$. For our main example, the derived category $D(R)$, the tensor product is the *derived* tensor product, denoted plainly $\otimes$, and henceforth plainly called the tensor product. Note that the tensor product of two perfect complexes is again perfect, and that the unit object R is perfect; hence also $D^{\omega}(R)$ is a tensor triangulated category.

Definition 1.1.7. A full subcategory I of $(\mathcal{T}, \otimes, \mathbf{1})$, is a *thick tensor ideal* if it is:

(1) a full triangulated subcategory,
(2) closed under finite sums,
(3) thick: if $a \oplus b \in I$, then $a \in I$ and $b \in I$,
(4) absorbing for the tensor product: if $a \in I$ and $b \in \mathcal{T}$, then $a \otimes b \in I$.

The following two observations follow from straightforward arguments.

Lemma 1.1.8. *Let* $(\mathcal{T}, \otimes, \mathbf{1})$ *be a tensor triangulated category such that* $\mathrm{Loc}(\mathbf{1}) = \mathcal{T}$.

(1) Any localizing subcategory is a tensor ideal.
(2) Let x*,* y *and* z *be objects in* $\mathcal{T}$*. If* $y \in \mathrm{Loc}(x)$*, then* $y \otimes z \in \mathrm{Loc}(x \otimes z)$*.*

In the presence of a tensor structure a Bousfield (co)localization functor often has a very peculiar form; for a proof see for instance Balmer and Favi [5]:

Theorem 1.1.9. *Let* $\mathcal{S}$ *be a thick tensor ideal in* $(\mathcal{T}, \otimes, \mathbf{1})$ *for which there are both a Bousfield localization* $L_{\mathcal{S}}$ *and a Bousfield colocalization* $\Gamma_{\mathcal{S}}$ *functors; for instance* S *is generated by compact objects. Let*

$$\Gamma_{\mathcal{S}}(\mathbf{1}) \longrightarrow \mathbf{1} \longrightarrow L_{\mathcal{S}}(\mathbf{1}) \longrightarrow \Sigma\Gamma_{\mathcal{S}}(\mathbf{1})$$

denote the localization triangle for the tensor unit. Then the following are equivalent:

(1) The subcategory $\mathcal{S}^{\perp}$ *is a tensor ideal.*
(2) There is an isomorphism of functors $L_{\mathcal{S}} \simeq L_{\mathcal{S}}(\mathbf{1}) \otimes -$.
(3) There is an isomorphism of functors $\Gamma_{\mathcal{S}} \simeq \Gamma_{\mathcal{S}}(\mathbf{1}) \otimes -$.

1.2. **Frames and Hochster duality.** In this subsection we recall some generalities on frames and Hochster duality. Our main reference for this material is Johnstone [15].

Definition 1.2.1. A *frame* is a complete lattice F in which finite meets distribute over arbitrary joins:

$$\forall a \in F,\ \forall S \subset F, \qquad a \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} (a \wedge s).$$

A *frame map* is a lattice map required to preserve arbitrary joins. Let $\boldsymbol{Frm}$ denote the category of frames and frame maps.

The motivating example is the frame of open sets in a topological space. There the join operation is given by union of open sets, and finite meet is given by intersection. Sending a topological space to its frame of open sets constitutes a functor $\boldsymbol{Top} \to \boldsymbol{Frm}^{\mathrm{op}}$. This functor has a right adjoint, the functor of *points*: a *point* of a frame F is a frame map $x : F \to \{0, 1\}$, and the set of points form a topological space in which the open subsets are those of the form $\{x : F \to \{0, 1\} \mid x(u) = 1\}$ for some $u \in F$. The topological spaces occurring in this way are precisely the *sober spaces* (i.e. every irreducible closed set has a unique generic point); these include any Hausdorff space and the underlying topological space of any scheme. Altogether the adjunction between topological spaces and frames restricts to a contravariant equivalence between sober spaces and *spatial* frames, i.e. having enough points [15, II.1.5].

Topological spaces homeomorphic to the spectrum of a ring were called *spectral spaces* by Hochster [12], now more commonly called *coherent spaces* [15]. Hochster [12] characterized the spectral spaces intrinsically as the sober spaces for which the quasi-compact open sets form a sublattice that is a basis for the topology. A *spectral map* between spectral spaces is a continuous map for which the inverse image of a quasi-compact open is quasi-compact.

The frame-theoretic counterpart of a spectral space is a coherent frame. Recall from [15, II.3.1] that an element c in a frame F is called *finite* if whenever we have $c \leq \bigvee_{s \in S} s$ for some subset $S \subset F$, there exists a finite subset $K \subset S$ such that already $c \leq \bigvee_{s \in K} s$. A frame is *coherent* when every element can be expressed as a join of finite elements and the finite elements form a sublattice. This amounts to requiring that one is finite and that the meet of two finite elements is finite. A coherent frame is spatial [15, Thm. II.3.4]. A frame map is called *coherent* if it takes finite elements to finite elements.

A coherent frame F can be reconstructed from its sublattice F^{ω} of finite elements [15, Proposition II.3.2]: F is canonically isomorphic to the frame of *ideals* in the lattice F^{ω}. Recall that an *ideal* of a lattice is a down-set, closed under finite joins.

The functors 'taking-ideals' (which amounts to join completion) and 'taking-finite-elements' constitute an equivalence of categories between distributive lattices and coherent frames (with coherent maps) [15, Corollary II.3.3]. Altogether the relationships are summarized in the following theorem known as Stone duality.

Theorem 1.2.2 (Stone, 1939; Joyal [16], 1971)**.** *The category of spectral spaces and spectral maps is contravariantly equivalent to the category of coherent frames and coherent maps, which in turn is equivalent to the category of distributive lattices.*

1.2.3. *Hochster duality.* For a spectral space X, Hochster [12] constructed a new topology on X by taking as basic open subsets the closed sets with quasi-compact complements. This space $X^\vee$ is called the *Hochster dual* of X. (When X is a noetherian space, the Hochster open sets can also be characterized as the specialization-closed subsets.) He proved that $X^\vee$ is spectral again, and that $X^{\vee\vee} \simeq X$.

Hochster duality becomes a triviality in the setting of distributive lattices and frames: under the equivalences of Theorem 1.2.2, a spectral space X corresponds to a coherent frame F (the frame of open sets in X) and to a distributive lattice F^ω (the finite elements in F or, equivalently, the lattice of quasi-compact open sets in X). Now the following definitions match the topological ones.

Definition 1.2.4. The Hochster dual of a distributive lattice is simply the opposite lattice. The Hochster dual of a coherent frame F is the ideal lattice of $(F^\omega)^{\mathrm{op}}$, i.e. its join completion (corresponding to the way Hochster defined the dual by generating a topology from the closed sets with quasi-compact complement).

1.2.5. *Points.* The points of a frame F correspond bijectively to *prime ideals* of F, that is, ideals $\mathcal{I}$ for which $1 \in \mathcal{I}^c$ and ($a \wedge b \in \mathcal{I} \Rightarrow a \in \mathcal{I}$ or $b \in \mathcal{I}$). Namely, to a point $x : F \to \{0,1\}$ corresponds the prime ideal $x^{-1}(0)$. In a frame, every prime ideal $\mathcal{P}$ is principal: the *generating element* is $u_\mathcal{P} := \bigvee_{b \in \mathcal{P}} b$; then we have

$$\mathcal{P} = (u_\mathcal{P}) = \{b \in F \mid b \leq u_\mathcal{P}\}.$$

A frame element generating a prime ideal is called a *prime element.*

It should be noted that for F a coherent frame, the points of F are in natural bijection with the points of the Hochster dual frame $F^\vee$.

1.3. **The Zariski frame.** Let R be a commutative ring. The prime spectrum $\mathrm{Spec}_Z(R)$ (with the Zariski topology) is by definition a spectral space. The corresponding coherent frame of open subsets of $\mathrm{Spec}_Z(R)$ is called the *Zariski frame* of R. It can be described directly as the frame of radical ideals in R: the join of a family of radical ideals is the radical of the ideal generated by their union, and the bottom element is $\sqrt{0}$; the meet is intersection in the ring R, and the top element is $\sqrt{1} = R$. We denote this frame by $\boldsymbol{RadId}(R)$. The finite elements in $\boldsymbol{RadId}(R)$ are the radicals of finitely generated ideals. These form the distributive lattice $\boldsymbol{RadId}(R)^\omega = \boldsymbol{RadfgId}(R)$, called the *Zariski lattice.*

Under the above correspondences between notions of point, the points of the Zariski frame coincide with the usual points of $\mathrm{Spec}_Z(R)$, the prime ideals of R. Precisely:

Lemma 1.3.1. *Given a frame-theoretic prime ideal $\mathcal{P} \subset \boldsymbol{RadId}(R)$, its generating element $u_\mathcal{P} = \bigvee_{I \in \mathcal{P}} I \in \boldsymbol{RadId}(R)$ is a prime ideal of the ring R. Conversely, any ring-theoretic prime ideal $\mathfrak{p} \subset R$ defines a frame prime ideal $\{b \in \boldsymbol{RadId}(R) \mid b \subset \mathfrak{p}\}$.*

Definition 1.3.2 (Joyal [17], 1975)**.** A *support* for R (with values in a frame) is a pair (F, d) where F is a frame and d is a map $d : R \to F$ satisfying

$$d(1) = 1, \qquad d(fg) = d(f) \wedge d(g),$$
$$d(0) = 0, \qquad d(f+g) \leq d(f) \vee d(g).$$

A *morphism of supports* is a frame map compatible with the map from R.

The *Zariski support* is the frame of radical ideals, with the map

$$\begin{array}{rcl} R & \longrightarrow & \boldsymbol{RadId}(R), \\ f & \longmapsto & \sqrt{(f)}. \end{array}$$

Theorem 1.3.3 (Joyal [17], 1975)**.** *For any commutative ring the Zariski support is the initial support.*

In other words, for any support $d : R \to F$, there is a unique frame map $\boldsymbol{RadId}(R) \to F$ making the following diagram commute:

$$\begin{tikzcd} R \arrow[rr, "d"] \arrow[dr, "\sqrt{\ }"'] & & F \\ & \boldsymbol{RadId}(R). \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

This map is defined as $J \mapsto \bigvee\{d(f) \mid f \in J\}$.

Equivalently, this result can be formulated in terms of distributive lattices: the initial distributive-lattice-valued support is the Zariski lattice $\boldsymbol{RadfgId}(R)$. In fact Joyal constructed this distributive lattice syntactically by freely generating it by symbols $d(f)$ and dividing out by the relations. The syntactic Zariski lattice, often called the Joyal spectrum, is a cornerstone in constructive commutative algebra (Hilbert's program); see for example Coquand-Lombardi-Schuster [9] and the many references therein.

2. Localizing subcategories in $D(R)$

Throughout this section we fix a commutative ring R, and we work in the derived category $D(R)$. We will write our complexes homologically: differentials lower degree by one.

2.1. **Compactly generated localizing categories and** $\mathrm{Spec}_H R$**.**

Recall that the compact objects in $D(R)$ are precisely the perfect complexes, i.e. quasi-isomorphic to bounded complexes of finitely generated projective modules, a.k.a. strictly perfect complexes (for a precise definition of perfect complexes see [28, Def. 2.2.10] and for the relation to strictly perfect complexes see [28, Thm. 2.4.3]). Since there is only a set of isomorphism classes of finitely projective modules, there is only a set of compact objects in $D(R)$, up to isomorphism. As a consequence there is only a set of compactly generated localizing subcategories in $D(R)$, and they are naturally ordered by inclusion. The bottom element is clearly $\mathrm{Loc}(0) = \{0\}$, and the top element is $\mathrm{Loc}(R) = D(R)$. We wish now to understand this poset.

We first turn to a local description of these localizing subcategories. It happens that in order to find nice parameterizing objects for the localizing subcategories it is important not to stick to perfect complexes but also to consider the non-compact objects in $D(R)$.

2.1.1. *Local structure of compactly generated localizing categories.* The following result due to Dwyer and Greenlees is an important building block in the present work.

Proposition 2.1.2 ([11, Proposition 6.4])**.** *If $I \subset R$ a finitely generated ideal, then $\mathrm{Loc}(R/I) = \mathrm{Loc}(K(I))$, where $K(I)$ is any Koszul complex of the ideal I.*

Recall that the Koszul complex of an element $f \in R$ is the perfect complex

$$K(f) : 0 \longrightarrow R \xrightarrow{f} R \longrightarrow 0$$

where the source of f is in degree 0. Given a finite family of elements $f_1, \ldots, f_n$ in R, the Koszul complex of this family is by definition the complex

$$K(f_1) \otimes \cdots \otimes K(f_n).$$

A Koszul complex $K(I)$ for a finitely generated ideal I is the Koszul complex of any of its finite generating subsets. Complexes of the form R/I in fact abound in localizing subcategories as shown by the next result:

Lemma 2.1.3 ([18]). *Let E be a chain complex and $I \subset R$ an ideal. Assume that there is a chain map $f : E \to R/I$ that induces an epimorphism in homology. Then $R/I \in \operatorname{Loc}(E)$.*

We now build up to Proposition 2.1.13, which generalizes the Dwyer-Greenlees result by showing that in fact any perfect complex C is cellularly equivalent to some quotient R/I for some finitely generated ideal I.

Proposition 2.1.4. *Let C be a complex such that its homology $H_*(C)$ is a finitely generated R-module. Then there exist finitely many ideals $J_1, \ldots, J_m$ in R such that*

$$\operatorname{Loc}(C) = \operatorname{Loc}(R/J_1, \ldots, R/J_m).$$

Proof. We argue by induction on the minimal number d of homogenous generators of H_*C. If $d = 0$, C is acyclic, hence quasi-isomorphic to the 0 complex and $J = R$ is the desired ideal.

If $d = 1$, then C is quasi-isomorphic to a cyclic module. Indeed, without loss of generality we may assume that the only non-zero homology module is in degree 0. We then have a zig-zag of quasi-isomorphisms:

$$\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} \arrow[r] & \cdots \\
\cdots \arrow[r] & C_1 \arrow[r] \arrow[u, equal] \arrow[d] & Z_01 \arrow[r] \arrow[u, hook] \arrow[d, two heads] & 0 \arrow[r] \arrow[u] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & H_0C \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

and since in $D(R)$ we have $C \simeq H_0C \simeq R/J$ for some ideal J, in this case the proposition is trivial.

Assume that the proposition has been proved for all complexes with homology generated by less that $d \geq 1$ homogeneous generators. Let C be a complex with homology generated by $d+1$ homogeneous generators $x_1, \ldots, x_{d+1}$, which we may assume are ordered in decreasing homological degree. Without loss of generality we may also assume that x_{d+1} is in degree 0. As in the case $d = 1$ we may also assume that C is in fact zero in degree < 0. If $x_j, \ldots, x_{d+1}$ are the generators of H_0C, denote by D the submodule generated by $x_j, \ldots, x_d$. Then we have a chain map, which by construction is an epimorphism in homology:

$$\begin{tikzcd}
\cdots \arrow[r] & C_1 \arrow[r] \arrow[d] & C_0 \arrow[r] \arrow[d, two heads] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & H_0C \arrow[r] \arrow[d, two heads] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & H_0C/D \simeq R/J_{d+1} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

Let $C' = \ker(C \to R/J_{m+1})$. By direct inspection, the homology of the complex C' is given by $H_nC' = H_nC$ if $n \neq 0$, and $H_0C' = H_0C/D = R/J_{d+1}$; in particular H_*C' is generated by d elements. The short exact sequence of complexes

$$0 \longrightarrow C' \longrightarrow C \longrightarrow R/J_{d+1} \longrightarrow 0$$

induces a triangle

$$C' \longrightarrow C \longrightarrow R/J_{d+1} \longrightarrow \Sigma C',$$

where the middle arrow is surjective in homology by construction. From Lemma 2.1.3 we conclude that $R/J_{d+1} \in \mathrm{Loc}(C)$, and from the triangle that $C' \in \mathrm{Loc}(C)$. In particular $\mathrm{Loc}(C', R/J_{d+1}) \subset \mathrm{Loc}(C)$. Conversely, the triangle shows that $\mathrm{Loc}(C', R/J_{d+1}) \supset \mathrm{Loc}(C)$ and we conclude by applying the induction hypothesis to the complex C'. $\square$

Another important result by Dwyer and Greenlees provides a characterization of the complexes in $\mathrm{Loc}(R/I)$:

Lemma 2.1.5 ([11, Proposition 6.12])**.** *Let $I \subset R$ be a finitely generated ideal. Then a complex E belongs to $\mathrm{Loc}(R/I)$ if and only if for any $x \in H_*(E)$ there exists an integer $p \in \mathbb{N}$ such that $I^p \cdot x = 0$.*

Corollary 2.1.6. *For any two finitely generated ideals I, J in R we have*

$$\sqrt{I} \subset \sqrt{J} \Longleftrightarrow \mathrm{Loc}(R/J) \subset \mathrm{Loc}(R/I).$$

Proof. According to Lemma 2.1.5, $\mathrm{Loc}(R/J) \subset \mathrm{Loc}(R/I)$ if and only if R/J is an I-torsion complex, and this happens if and only if $\exists n \in \mathbb{N}$ such that $I^n \subset J$ and hence if and only if $\sqrt{I} = \sqrt{I^n} \subset \sqrt{J}$. $\square$

Corollary 2.1.7. *For finitely generated ideals I and J, we have $\sqrt{I} = \sqrt{J}$ if and only if* $\mathrm{Loc}(R/I) = \mathrm{Loc}(R/J)$.

Corollary 2.1.8. *Let I be a finitely generated ideal in R. Then*

$$\mathrm{Loc}(R/J \mid J \supset I, J \textit{ fin. gen.}) = \mathrm{Loc}(R/I).$$

In [11, Proposition 6.11], Dwyer and Greenlees show that the cellularization of a module M with respect to R/I computes the I-local cohomology of M. In particular, Corollary 2.1.7 has the following well-known interpretation in terms of local cohomology. Denote by $H^I_*(M)$ the I-local cohomology of an R-module M.

Proposition 2.1.9. *Let M be an R-module and I, J two finitely generated ideals. If $\sqrt{I} = \sqrt{J}$, then there is a canonical isomorphism $H^I_*(M) \simeq H^J_*(M)$.*

Notice that in the noetherian case this isomorphism is proved by showing that both terms are isomorphic to $H_*^{\sqrt{I}}(M)$, so that these isomorphisms are induced by the inclusions $I \subset \sqrt{I} = \sqrt{J} \supset J$, but this is definitely not true for non-noetherian rings. In the non-noetherian case, the isomorphisms are induced by the inclusions $I \subset I + J \supset J$, for if $\sqrt{I} = \sqrt{J}$, then $\sqrt{I} = \sqrt{I+J} = \sqrt{J}$.

Proposition 2.1.10. *Let I and J be finitely generated ideals in R. Then*

$$\mathrm{Loc}(R/I, R/J) = \mathrm{Loc}(R/I \oplus R/J) = \mathrm{Loc}(R/(I \cdot J)) = \mathrm{Loc}(R/I \cap J).$$

Proof. The first equality is a triviality since localizing subcategories are closed under retracts and direct sums, and the last because $\sqrt{I \cdot J} = \sqrt{I \cap J}$. For the second equality, first notice that both R/I and R/J are $R/(I \cap J)$-modules, so by Lemma 2.1.5, $\mathrm{Loc}(R/I, R/J) \subset \mathrm{Loc}(R/(I \cap J))$. For the reverse inclusion, consider the following commutative diagram of R-modules:

$$\begin{tikzcd}
0 \arrow[r] & I/(I \cap J) \arrow[r] \arrow[d, equal] & R/I \cap J \arrow[r] \arrow[d] & R/J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (I+J)/J \arrow[r] & R/J \arrow[r] & R/(I+J) \arrow[r] & 0,
\end{tikzcd}$$

where the left equality is one of the classical isomorphism theorems. This tells us that $R/(I \cap J)$ is a pullback in R-Mod of R/J, R/I and $R/(I+J)$, and therefore that $R/(I \cap J)$ is in $D(R)$ a homotopy pullback of R/J, R/I and $R/(I+J)$. Now, $R/(I+J)$ is both an R/I and an R/J-module, so $R/(I+J) \in \mathrm{Loc}(R/I, R/J)$, and since localizing subcategories are closed under homotopy pullbacks, $R/(I \cap J) \in \mathrm{Loc}(R/I, R/J)$ as desired. $\square$

This proposition together with Proposition 2.1.4 yields immediately that:

Corollary 2.1.11. *Let S be a noetherian ring and let C be a perfect complex over S. Then there exists a finitely generated ideal $J \subset S$ such that* $\mathrm{Loc}(C) = \mathrm{Loc}(S/J)$.

To get rid of the noetherian assumption, we need the following classical result (see the Appendix of [22]).

Lemma 2.1.12. *Let R be a commutative ring and let C be a perfect complex in $D(R)$. Then there exists a noetherian subring $S \subset R$ and a perfect complex C_S in $D(S)$ such that $C = C_S \otimes_S R$.*

Proposition 2.1.13. *Let R be a commutative ring, and let C be a compact object in $D(R)$. Then there exists a finitely generated ideal $I \subset R$ such that* $\mathrm{Loc}(C) = \mathrm{Loc}(R/I)$.

Proof. By Lemma 2.1.12, we can find a noetherian subring $S \subset R$ and a perfect complex C_S such that $C_S \otimes_S R = C$. For this complex C_S, Corollary 2.1.11 provides us with a finitely generated ideal $J \subset S$ such that $\mathrm{Loc}_S(C_S) = \mathrm{Loc}_S(S/J)$. We claim that $JR \subset R$, the R-ideal generated by the image of J, in R is the finitely generated ideal we are looking for. To see this, note first that by Proposition 2.1.2 we have $\mathrm{Loc}_S(C_S) = \mathrm{Loc}_S(S/J) = \mathrm{Loc}_S(K(J))$. As C_S and $K(J)$ are compact, there exists a finite recipe as in Proposition 1.1.2 in $D(S)$ to build $K(J)$ from C_S, say of length $n+1$,

$$F_0 \xrightarrow{f_0} F_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} F_n.$$

Applying the triangulated functor $-\otimes_S R : D(S) \to D(R)$ to the above sequence of maps shows that $K(J) \otimes_S R$ belongs to $\mathrm{Loc}(C_S \otimes R) = \mathrm{Loc}(C)$. The complex $K(J)$ is strictly perfect, hence flat, so to compute $K(J) \otimes_S R \in D(R)$, we may use the underived tensor product. By direct inspection, for any finite generating set of the ideal $J \subset S$ with associated Koszul complex $K(J)$, the complex $K(J) \otimes_S R$ (underived) is in fact equal to a Koszul complex for the ideal JR, so $\mathrm{Loc}(R/JR) = \mathrm{Loc}(K(JR)) \subset \mathrm{Loc}(C)$. Exchanging the roles of C_S and $K(J)$ in the above argument shows in the same way that $\mathrm{Loc}(C) \subset \mathrm{Loc}(K(JR)) = \mathrm{Loc}(R/JR)$. ☐

2.1.14. *The lattice of compactly generated localizing subcategories.* We turn now to the global structure of the poset of compactly generated localizing subcategories in $D(R)$. This poset has meets and arbitrary joins: if $\{S_\alpha\}_{\alpha\in A}$ is a set of sets of compact objects, the join is given by

$$\bigvee_{\alpha\in A} \mathrm{Loc}(S_\alpha) = \mathrm{Loc}(\bigcup_{\alpha\in A} S_\alpha).$$

The meet is a bit more complicated, since a priori the intersection of $\mathrm{Loc}(S_1)$ and $\mathrm{Loc}(S_2)$ might not be compactly generated. Nevertheless, it contains a largest compactly generated subcategory, namely the localizing subcategory generated by the compact objects it contains:

$$\mathrm{Loc}(S_1) \wedge \mathrm{Loc}(S_2) = \mathrm{Loc}\Big(\big(\mathrm{Loc}(S_1) \cap \mathrm{Loc}(S_2)\big) \cap D^\omega(R)\Big).$$

Denote by $\boldsymbol{CGLoc}(D(R))$ the lattice of compactly generated localizing subcategories of $D(R)$, and by $\boldsymbol{fgCGLoc}(D(R))$ the subposet of finite elements. We shall see shortly that $\boldsymbol{fgCGLoc}(D(R))$ is a distributive lattice and that $\boldsymbol{CGLoc}(D(R))$ is a coherent frame. We first characterize the finite elements:

Lemma 2.1.15. *The finite elements in $\boldsymbol{CGLoc}(D(R))$ are the localizing subcategories of $D(R)$ that can be generated by a single compact object.*

Proof. A localizing subcategory generated by a single compact object is easily seen to be finite by Proposition 1.1.2. Conversely if S is a set of compact objects and $\mathrm{Loc}(S)$ is a finite element in $\boldsymbol{CGLoc}(D(R))$, then the equality $\mathrm{Loc}(S) = \bigvee_{s\in S} \mathrm{Loc}(s)$ implies that also $\mathrm{Loc}(S) = \bigvee_{s\in K} \mathrm{Loc}(s) = \mathrm{Loc}(K)$ for some finite subset $K \subset S$. Now for each $s \in K$ we have $\mathrm{Loc}(s) = \mathrm{Loc}(R/I_s)$ for some finitely generated ideal I_s, and Proposition 2.1.10 shows that then $\mathrm{Loc}(K) = \mathrm{Loc}(R/J)$ where J is the product of the ideals I_s. ☐

We analyze the join and meet inside $\boldsymbol{fgCGLoc}(D(R))$: in case the generating set is just a single compact object, we may replace it by a cyclic module R/I. For the join operation, Proposition 2.1.10 gives

$$\mathrm{Loc}(R/I) \vee \mathrm{Loc}(R/J) = \mathrm{Loc}(R/I, R/J) = \mathrm{Loc}(R/(I \cdot J)).$$

For the meet operation, the following shows that the intersection of two localizing subcategories each generated by one compact object is again a compactly generated localizing subcategory generated by a single compact, so in this case meet is just intersection:

Lemma 2.1.16. *If I and J are finitely generated ideals in R, then*

$$\mathrm{Loc}(R/I) \cap \mathrm{Loc}(R/J) = \mathrm{Loc}(R/(I + J)).$$

Proof. For $E \in \mathrm{Loc}(R/I) \cap \mathrm{Loc}(R/J)$ and $x \in H_*(E)$, by Lemma 2.1.5 there exist $n, m \in \mathbb{N}$ such that $I^n x = 0 = J^m x$. A direct computation shows that $(I+J)^{mn}x = 0$, and therefore $E \in \mathrm{Loc}(R/(I+J))$. Conversely, as $R/(I+J)$ is both I and J-torsion we conclude that $\mathrm{Loc}(R/I) \cap \mathrm{Loc}(R/J) \supset \mathrm{Loc}(R/(I+J))$. □

The following result is a key point.

Proposition 2.1.17. *The lattice* $\boldsymbol{fgCGLoc}(D(R))$ *of localizing subcategories generated by a single compact object is isomorphic to the opposite of the Zariski lattice:*

$$\begin{array}{rcl} \boldsymbol{fgCGLoc}(D(R)) & \simeq & \boldsymbol{RadfgId}(R)^{op} \\ \mathrm{Loc}(R/I) & \leftrightarrow & \sqrt{I}. \end{array}$$

Proof. The assignment from right to left, $I \mapsto \mathrm{Loc}(R/I)$, is well defined by Proposition 2.1.2: $\mathrm{Loc}(R/I) = \mathrm{Loc}(K(I))$ which is compactly generated since I is the radical of a finitely generated ideal, and $\mathrm{Loc}(K(I))$ is insensitive to taking radical by Corollary 2.1.7. The assignment $\mathrm{Loc}(R/I) \mapsto I$ is well defined since by Proposition 2.1.13 for any perfect complex C there is a finitely generated ideal I such that $\mathrm{Loc}(C) = \mathrm{Loc}(R/I)$, and by Corollary 2.1.6, this finitely generated ideal is uniquely determined up to taking radical. Having established that the two assignments are well defined, it is obvious from the description that they constitute an inclusion-reversing bijection. □

We now extend this isomorphism to $\boldsymbol{CGLoc}(D(R))$. We first give a rather formal argument, which relies on some results in Section 3, then give a more elementary proof of a more geometric flavor. By Corollary 1.1.3 we have $\boldsymbol{CGLoc}(D(R)) = \boldsymbol{Thick}(D^\omega(R))$, and by Lemmas 1.1.8 and 3.1.7 all thick subcategories are radical thick tensor ideals, so that $\boldsymbol{CGLoc}(D(R))$ is a coherent frame by Theorem 3.1.9.

Theorem 2.1.18. *The isomorphism of Proposition* 2.1.17 *extends to an isomorphism of frames*

$$\boldsymbol{CGLoc}(D(R)) = \boldsymbol{RadId}(R)^\vee.$$

Proof. The two frames are precisely the ideal frames (join completions) of the distributive lattices in Proposition 2.1.17. □

The description of the isomorphism in Proposition 2.1.17, and hence Theorem 2.1.18, relies on Proposition 2.1.2 and Corollary 2.1.7. We provide a more geometrical reformulation that lifts this dependence:

Theorem 2.1.19. *There is a natural inclusion-preserving bijection*

$$\left\{ \begin{array}{c} \textit{Compactly generated} \\ \textit{localizing subcategories of } D(R) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{Hochster open} \\ \textit{sets in } \mathrm{Spec}(R) \end{array} \right\}.$$

The bijection is given from left to right by

$$f : \mathrm{Loc}(S) \longmapsto \bigcup_{\substack{R/I \in \mathrm{Loc}(S) \\ I \textit{ fin. gen.}}} V(I),$$

and from right to left by

$$\mathrm{Loc}(K(I) \mid V(I) \subset U, I \textit{ fin. gen.}) \longleftarrow U : g.$$

Proof. Note first that the new description of f agrees with that of Proposition 2.1.17: given a perfect complex C, the subset

$$\bigcup_{R/J\in \mathrm{Loc}(C),\ J \text{ f.g.}} V(J)$$

is of the form $V(I)$ for some finitely generated ideal I. By Proposition 2.1.13, there is a finitely generated ideal I such that $\mathrm{Loc}(C) = \mathrm{Loc}(R/I)$, and for any finitely generated ideal J, by Corollary 2.1.6,

$$R/J \in \mathrm{Loc}(R/I) \Leftrightarrow \sqrt{J} \supset \sqrt{I} \Leftrightarrow V(J) \subset V(I).$$

So indeed $f(\mathrm{Loc}(R/I)) = V(I)$.

We now check that $f \circ g = \mathrm{Id}$. Given an arbitrary Hochster open set U, to show that $f \circ g(U) = U$, it is enough to prove that if J is a finitely generated ideal in R such that $R/J \in \mathrm{Loc}\big(K(I) \mid V(I) \subset U, I \text{ fin. gen.}\big)$, then $V(J) \subset U$. Choose a Koszul complex $K(J)$ for the ideal J. By hypothesis $K(J) \in \mathrm{Loc}\big(K(I) \mid V(I) \subset U, I \text{ fin. gen.}\big)$, and since it is compact there exist finitely many ideals $J_1, \dots, J_k$ such that $V(J_k) \in U$ and $K(J) \in \mathrm{Loc}(K(J_1), \dots, K(J_k)) = \mathrm{Loc}(R/J_1, \dots, R/J_k)$. But $\mathrm{Loc}(R/J_1, \dots, R/J_k) = \mathrm{Loc}(R/(J_1 \cdots J_k))$, so by Corollary 2.1.6,

$$\sqrt{J} \supset \sqrt{J_1 \cdots J_k},$$

hence

$$V(J) \subset V(J_1 \cdots J_k) = \bigcup_{i=1}^{k} V(J_i) \subset U.$$

Finally we establish that $g \circ f = \mathrm{Id}$. Since by Proposition 2.1.13 any perfect complex is cellularly equivalent to a finitely generated cyclic module, it is clear that for any compactly generated category $\mathrm{Loc}(S)$, we have $(g \circ f)(\mathrm{Loc}(S)) \supset \mathrm{Loc}(S)$. For the reverse inclusion we argue as follows. Given a compactly generated localizing subcategory $\mathrm{Loc}(S)$, let J be a finitely generated ideal such that $V(J) \subset \bigcup_{\substack{R/I\in\mathrm{Loc}(S)\\ I \text{ fin. gen.}}} V(I)$. Then, because Hochster opens of the form $V(J)$ are finite elements in the Hochster frame, there exist finitely many ideals $I_1, \dots, I_n$ such that $R/I_j \in \mathrm{Loc}(S)$ for $1 \leq j \leq n$ and $V(J) \subset \bigcup_{j=1}^{n} V(I_j)$. Again by Proposition 2.1.17,

$$\mathrm{Loc}(K(J)) \subset \mathrm{Loc}(K(I_1), \dots, K(I_n)) \subset \mathrm{Loc}(S). \qquad \square$$

2.2. **Hochster duality in $D(R)$ and $\mathrm{Spec}_Z R$.** Usually in algebraic geometry the topology of interest on $\mathrm{Spec}\, R$ is the Zariski topology (or closely related Grothendieck topologies such as the étale topology), not the Hochster dual topology. As discussed in the Introduction, it is somewhat mysterious that compactly generated localizing subcategories in $D(R)$ yield the Hochster dual topology on $\mathrm{Spec}\, R$ (Theorem 2.1.18), in spite of the fact that it is actually a Zariski-like construction, as we shall see in Section 3.

It is natural to ask whether also the Zariski frame itself can be realized as a sublattice inside $D(R)$. For the lattice of finite elements $\mathrm{Loc}(R/I)$, the dual lattice can be obtained by passing to the right orthogonal categories. We shall show how to describe the join completion of this lattice inside $D(R)$.

By Proposition 1.1.5 localizing subcategories generated by sets of compact objects admit Bousfield localizations, so for any set of compact objects S, we have

${}^{\perp}(\mathrm{Loc}(S)^{\perp}) = \mathrm{Loc}(S)$. In particular [20, Prop. 4.9.1(6)], we have the following order-reversing bijection of lattices:

$$\big\{\,\mathrm{Loc}(C) \mid C \text{ compact}\,\big\} \underset{{}^{\perp}(-)}{\overset{(-)^{\perp}}{\rightleftarrows}} \big\{\,\mathrm{Loc}(C)^{\perp} \mid C \text{ compact}\,\big\}.$$

A priori on the right-hand side what we get are colocalizing subcategories as explained in 1.1.4, but in this specific case we get categories that are also localizing:

Proposition 2.2.1. *Let S be a set of compact objects in a tensor triangulated category $\mathcal{T}$ admitting arbitrary sums. Then $\mathrm{Loc}(S)^{\perp}$ is both a colocalizing and a localizing subcategory.*

Proof. As the subcategory $\mathrm{Loc}(S)^{\perp}$ is colocalizing, it is triangulated; we just have to prove that it is closed under arbitrary sums. Consider an arbitrary sum $\coprod_{j\in J} N_j$ of objects $N_j \in \mathrm{Loc}(S)^{\perp}$. Given a compact generator C of $\mathrm{Loc}(S)$, consider any map $f \in \mathrm{Hom}_{\mathcal{T}}(C, \coprod_{j\in J} N_j)$. Since C is compact, there exists a finite subset $K \subset J$ such that f factors via $f_K : C \to \coprod_{j\in K} N_j = \prod_{j\in K} N_j$. But then $f_K \in \mathrm{Hom}_{\mathcal{T}}(C, \prod_{j\in K} N_j) = \prod \mathrm{Hom}_{\mathcal{T}}(C, N_j) = 0$, and $f = 0$. In particular $\coprod_{j\in J} N_j \in \mathrm{Loc}(S)^{\perp}$ as we wanted. □

In the Zariski spectrum of a ring there is a basis of principal open sets given by complements of Zariski closed sets defined by a single element of the ring. We first determine their corresponding localizing subcategories.

Proposition 2.2.2. *For an element $f \in R$, we have $\mathrm{Loc}(R/(f))^{\perp} = \mathrm{Loc}(R_f)$. Moreover $\mathrm{Loc}(R_f)$ is the essential image of the functor $D(R_f) \to D(R)$ induced by restriction of scalars along the canonical map $R \to R_f$.*

Proof. From [20, Props. 4.9.1 and 4.10.1], $\mathrm{Loc}(R/(f))^{\perp}$ is the essential image of the Bousfield localization functor associated to the compactly generated $\mathrm{Loc}(R/(f))$, and since $\mathrm{Loc}(R/(f))$ is a tensor ideal (Lemma 1.1.8), by Theorem 1.1.9, this localization functor is isomorphic to $L_{R/(f)}(R) \otimes -$. We proceed to compute the fundamental triangle:

$$\Gamma_{R/(f)}(R) \longrightarrow R \longrightarrow L_{R/(f)}(R) \longrightarrow \Sigma\Gamma_{R/(f)}(R).$$

In [11], Dwyer and Greenlees showed how to compute the complex $\Gamma_{R/(f)}(R)$. For each power f^k, the Koszul complex $K(f^k)$ is given by $R \xrightarrow{f^k} R$, and we may form an inductive system $K(f^k) \to K(f^{k+1})$ via the commutative diagram:

$$\begin{array}{ccc} R & = & R \\ {\scriptstyle f^k}\downarrow & & \downarrow{\scriptstyle f^{k+1}} \\ R & \xrightarrow{f} & R. \end{array}$$

The homotopy colimit of these complexes is by definition the complex denoted by $K^{\bullet}(f^{\infty})$. It is shown in [11, Proposition 6.10] that $K^{\bullet}(f^{\infty})$ is quasi-isomorphic to $\Gamma_{R/(f)}(R)$ and that it is also quasi-isomorphic to the complex $R \to R_f$, where R is in degree 0 and the map $K^\bullet(f^\infty) \to R$ is simply the map

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & R_f \arrow[d] \\ R \arrow[r] & 0. \end{tikzcd}$$

By direct computation using the long exact sequence in homology we get that $L_{R/(f)}(R)$ is quasi-isomorphic to the complex R_f concentrated in degree 0.

Finally, following the discussion in the beginning of this proof, a complex M belongs to $\mathrm{Loc}(R/(f))^\perp$ if and only if it is quasi-isomorphic to $L_{R/(f)}(R) \otimes M = R_f \otimes M \in \mathrm{Loc}(R_f)$. The reverse inclusion is trivial.

To prove the last assertion, just notice that the functor $D(R_f) \to D(R)$ induced by restriction of scalars along $R \to R_f$ is exact, commutes with both products and sums, and that R_f generates $D(R_f)$ as a triangulated category with infinite sums. □

From this result it is easy to extract a criterion for a complex to be in $\mathrm{Loc}(R/(f))^\perp$, much in the spirit of Lemma 2.1.5:

Lemma 2.2.3. *For $f \in R$, a complex belongs to* $\mathrm{Loc}(R/(f))^\perp = \mathrm{Loc}(R_f)$ *if and only if its homology modules are R_f-modules.*

Proof. By Bousfield localization we know that $\mathrm{Loc}(R_f)$ is the essential image of the functor $R_f \otimes -$. Given an arbitrary complex M, the Künneth spectral sequence that computes the homology of $R_f \otimes M$ collapses onto the horizontal edge at the page E_2 because R_f is flat. We conclude that $M \to R_f \otimes M$ is a quasi-isomorphism if and only if for each $n \in \mathbb{Z}$ we have $H_n(M) \simeq R_f \otimes H_n(M)$, and this happens precisely when the homology modules of M are R_f-modules. □

We also get a description of $\mathrm{Loc}(R/I)^\perp$ for an arbitrary finitely generated ideal I:

Theorem 2.2.4. *For a finitely generated ideal $I = (f_1, \dots, f_n)$ in R, we have*

$$\mathrm{Loc}(R/I)^\perp = \mathrm{Loc}(R_{f_1}, \dots, R_{f_n}).$$

Proof. First, we have $\mathrm{Loc}(R/I)^\perp = (\mathrm{Loc}(R/(f_1)) \cap \cdots \cap \mathrm{Loc}(R/(f_n)))^\perp$, so $R_{f_1}, \dots, R_{f_n}$ all belong to $\mathrm{Loc}(R/I)^\perp$, and since this is a localizing subcategory, we have $\mathrm{Loc}(R_{f_1}, \dots, R_{f_n}) \subset \mathrm{Loc}(R/I)^\perp$. For the reverse inclusion consider again the localization triangle:

$$\Gamma_{R/I}(R) \longrightarrow R \longrightarrow L_{R/I}(R) \longrightarrow \Sigma\Gamma_{R/I}(R).$$

By (the proof of) Proposition 2.2.2, a model for $L_{R/I}(R)$ is $K^\bullet(f_1^\infty) \otimes \cdots \otimes K^\bullet(f_n^\infty)$. Since each of these complexes is quasi-isomorphic to a flat complex, $K^\bullet(f_i^\infty) \simeq (R \to R_{f_i})$, the derived tensor may be computed using the ordinary tensor product of complexes. This is then a complex of the following form:

$$R \longrightarrow R_{f_1} \oplus \cdots \oplus R_{f_n} \longrightarrow \cdots \longrightarrow R_{f_1 \cdots f_n}$$

where in degree $-p+1$ we have the direct sum of the $\binom{n}{p}$ modules obtained by choosing p elements among the n generators and localizing R at their product.

Then the cone of the map of complexes

$$\begin{tikzcd} R \arrow[r] \arrow[d, equal] & R_{f_1} \oplus \cdots \oplus R_{f_n} \arrow[r] & \cdots \arrow[r] & R_{f_1 \cdots f_n} \\ R \end{tikzcd}$$

is quasi-isomorphic to the suspension of the complex

$$R_{f_1} \oplus \cdots \oplus R_{f_n} \longrightarrow \cdots \longrightarrow R_{f_1 \cdots f_n}$$

and this is clearly an element in $\mathrm{Loc}(R_{f_1}, \ldots, R_{f_n})$. Since this localizing subcategory is a tensor ideal, we find that $\mathrm{Loc}(R/I)^{\perp}$, the essential image of the Bousfield localization functor $L_{R/I}(M) = L_{R/I}(R) \otimes M$, is contained in $\mathrm{Loc}(R_{f_1}, \ldots, R_{f_n})$. □

Observe that for any finite set $f_1, \ldots, f_n$, the localizing subcategory $\mathrm{Loc}(R_{f_1}, \ldots, R_{f_n})$ only depends on the radical ideal generated by $f_1, \ldots, f_n$. For future reference we record two immediate consequences:

Corollary 2.2.5. *For any finitely generated ideal $J = (f_1, \ldots, f_n)$ we have*

$$\mathrm{Loc}(R_{f_1}, \ldots, R_{f_n}) = \mathrm{Loc}(R_f \mid f \in \sqrt{J}).$$

Corollary 2.2.6. *Let I and J be finitely generated ideals in R. Then*

$$\mathrm{Loc}(R_f \mid f \in \sqrt{I}) \subset \mathrm{Loc}(R_f \mid f \in \sqrt{J}) \Leftrightarrow \sqrt{I} \subset \sqrt{J}.$$

Proof. This follows readily from the fact that taking right orthogonal is an order-reversing operation and from Corollary 2.1.6. □

Summing up we have proved the following result.

Proposition 2.2.7. *The poset of localizing categories of the form*

$$\mathrm{Loc}(R_f \mid f \in \sqrt{J}),$$

where J is a finitely generated ideal, is isomorphic to the poset of radicals of finitely generated ideals in R. In particular since the latter is a distributive lattice so is the former.

Recalling that radicals of finitely generated ideals correspond to quasi-compact open sets in $\mathrm{Spec}_Z R$, we have the following topological formulation.

Proposition 2.2.8. *There is a natural isomorphism of lattices between the lattice of localizing subcategories of $D(R)$ generated by finitely many localizations of the ring R and the lattice of quasi-compact open sets in* $\mathrm{Spec}_Z R$ *given by*

$$\mathrm{Loc}(R_{f_1}, \ldots, R_{f_n}) \longmapsto \bigcup_{i=1}^{n} D(f_i)$$

and

$$\mathrm{Loc}(R_f \mid D(f) \subset U) \longleftarrow U.$$

The join in the lattice of localizing subcategories is given by

$$\begin{aligned} \mathrm{Loc}(R_f \mid f \in \sqrt{J}) \vee \mathrm{Loc}(R_f \mid f \in \sqrt{I}) &= \mathrm{Loc}(R_f \mid f \in \sqrt{I+J}) \\ &= \mathrm{Loc}(R_f \mid f \in \sqrt{I} \cup \sqrt{J}). \end{aligned}$$

For the meet operation, we find

$$\mathrm{Loc}(R_f \mid f \in \sqrt{J}) \wedge \mathrm{Loc}(R_f \mid f \in \sqrt{I}) \quad = \quad \mathrm{Loc}(R_f \mid f \in \sqrt{I} \cdot \sqrt{J}),$$

and this is in fact intersection:

Lemma 2.2.9. *For any two finitely generated ideals I and J in R we have*

$$\mathrm{Loc}(R_f \mid f \in \sqrt{I} \cdot \sqrt{J}) = \mathrm{Loc}(R_f \mid f \in \sqrt{J}) \cap \mathrm{Loc}(R_f \mid f \in \sqrt{I}).$$

Proof. Compute using the fact that we are dealing with right orthogonals:

$$\begin{aligned} \mathrm{Loc}(R_f \mid f \in \sqrt{I} \cdot \sqrt{J}) &= \mathrm{Loc}(R_f \mid f \in \sqrt{I \cdot J}) \\ &= \mathrm{Loc}(R/(I \cdot J))^{\perp} \\ &= \mathrm{Loc}(R/I, R/J)^{\perp} \\ &= \{R/I, R/J\}^{\perp} \\ &= \{R/I\}^{\perp} \cap \{R/J\}^{\perp} \\ &= \mathrm{Loc}(R_f \mid f \in \sqrt{I}) \cap \mathrm{Loc}(R_f \mid f \in \sqrt{J}). \end{aligned}$$

□

Just as for the case of compactly generated localizing subcategories (Theorem 2.1.18), we proceed to establish that the correspondence of Proposition 2.2.8 extends by join-completion to a frame isomorphism, realizing the whole Zariski frame inside $D(R)$. Again this amounts to dropping the "finite generation" assumption, considering now localizing categories generated by an arbitrary number of localizations of the ring R.

We consider now subcategories $\mathrm{Loc}(R_f \mid f \in J)$, where a priori $J \subset R$ is an arbitrary subset. The following lemma tells us that this is no more general than requiring J to be a radical ideal.

Lemma 2.2.10. *Let J be an arbitrary subset of R and $\sqrt{J}$ the radical ideal it generates. Then*

$$\mathrm{Loc}(R_f \mid f \in J) = \mathrm{Loc}(R_f \mid f \in \sqrt{J}).$$

Proof. The inclusion $\mathrm{Loc}(R_f \mid f \in J) \subset \mathrm{Loc}(R_f \mid f \in \sqrt{J})$ is clear. To establish the other inclusion, we proceed as follows. Let $g \in \sqrt{J}$ be an arbitrary element. Then there exists a finite subset $K \subset J$ such that $g \in \sqrt{K}$. By Corollary 2.2.5, we know that $R_g \in \mathrm{Loc}(R_f \mid f \in \sqrt{K})$. Again by Corollary 2.2.5 we have that $\mathrm{Loc}(R_f \mid f \in \sqrt{K}) = \mathrm{Loc}(R_f \mid f \in K) \subset \mathrm{Loc}(R_f \mid f \in J)$, whence the desired inclusion. □

From now on we will parametrize our categories by radical ideals, still denoted as $\sqrt{J}$ to emphasize the *radical* property. Notice that in the poset $\{\mathrm{Loc}(R_f \mid f \in \sqrt{J})\}$, for any family of objects $\{\mathrm{Loc}(R_f \mid f \in \sqrt{J_i})\}_{i \in I}$, the join is

$$\bigvee_{i \in I} \mathrm{Loc}(R_f \mid f \in \sqrt{J_i}) = \mathrm{Loc}(R_f \mid f \in \sqrt{\Sigma_{i \in I} J_i}).$$

Concerning the meet operation, all we can say is that

$$\begin{aligned} &\mathrm{Loc}(R_f \mid f \in \sqrt{J}) \wedge \mathrm{Loc}(R_f \mid f \in \sqrt{I}) \\ &\qquad\qquad \subset \mathrm{Loc}(R_f \mid f \in \sqrt{J}) \cap \mathrm{Loc}(R_f \mid f \in \sqrt{I}), \end{aligned}$$

with equality if and only if $\mathrm{Loc}(R_f \mid f \in \sqrt{J}) \cap \mathrm{Loc}(R_f \mid f \in \sqrt{I})$ is an element in our poset; this is because $\mathrm{Loc}(R_f \mid f \in \sqrt{J}) \cap \mathrm{Loc}(R_f \mid f \in \sqrt{I})$ is the largest localizing subcategory contained in both $\mathrm{Loc}(R_f \mid f \in \sqrt{J})$ and $\mathrm{Loc}(R_f \mid f \in \sqrt{I})$. The last difficulty we have to cope with is to show that this frame has as finite elements precisely the localizing subcategories parametrized by radicals of finitely generated ideals.

To prove this we need to be a little bit more precise in our description of a localizing subcategory generated by a set of objects S. Recall that for any ordinal α, its cardinal $|\alpha|$ is the initial ordinal in the set of ordinals that can be put in bijection with α, so we can write $|\alpha| \leq \alpha$ as *ordinals*. Let us define a filtration of $\mathrm{Loc}(S)$ as follows:

1. $\mathrm{Loc}^0(S)$ is the full subcategory consisting of the zero object.
2. $\mathrm{Loc}^1(S)$ is the full subcategory whose objects are isomorphic to an arbitrary suspension of elements in S.
3. If $\alpha \geq 1$ is a successor ordinal, then $\mathrm{Loc}^{\alpha+1}(S)$ is the full subcategory consisting of objects that are
   - either isomorphic to an arbitrary suspension of direct sums of less than $|\alpha|$ objects in $\mathrm{Loc}^{\alpha}(S)$ or
   - isomorphic to an arbitrary suspension of a cone between two objects in $\mathrm{Loc}^{\alpha}(S)$.
4. If β is a limit ordinal, then $\mathrm{Loc}^{\beta}(S) = \bigcup_{\alpha<\beta} \mathrm{Loc}^{\alpha}(S)$.

Notice that with this definition $\mathrm{Loc}^{\alpha}(S)$ is always an essentially small category, and if $\alpha \leq \beta$, then $\mathrm{Loc}^{\alpha}(S) \subset \mathrm{Loc}^{\beta}(S)$. Moreover for any ordinal α, $\mathrm{Loc}^{\alpha}(S) \subset \mathrm{Loc}(S)$. For any object M, if α is the least ordinal such that $M \in \mathrm{Loc}^{\alpha}(S)$, then we will say that M can be constructed in α steps from S or has length α.

Lemma 2.2.11. *For any set of objects S, we have*

$$\bigcup_{\alpha} \mathrm{Loc}(S) = \mathrm{Loc}(S).$$

.

Proof. By construction $\bigcup_{\alpha} \mathrm{Loc}(S)$ is closed under suspension; we first prove that it is triangulated. For this consider two objects $M, N \in \bigcup_{\alpha} \mathrm{Loc}(S)$. By definition there exists an ordinal α such that $M, N \in \mathrm{Loc}^{\alpha}(S)$; for instance take any ordinal that is larger than the lengths of M and N. For any morphism $f : M \to N$, the cone of f is in $\mathrm{Loc}^{\alpha+1}(S)$. It remains to show that $\bigcup_{\alpha} \mathrm{Loc}(S)$ is closed under arbitrary sums. For this, let $\{M\}_{i\in I}$ be a set of objects in $\bigcup_{\alpha} \mathrm{Loc}(S)$, and let $\{\alpha_i\}_{i\in I}$ be the set of lengths of these elements. Then there exists an ordinal β such that $\forall i \in I,\ \alpha_i \leq \beta$, and without loss of generality we may also assume that β is larger than the cardinality of I. Then, by definition,

$$\coprod_{i\in I} M_i \in \mathrm{Loc}^{\beta}(S).$$

□

From this we can understand the meet of any two objects in our category, starting with the meet with our (potential) finite elements:

Proposition 2.2.12. *Let $\{f_j\}_{j\in J}$ be a family of elements in R, and let g be an element in R. Then*

$$\mathrm{Loc}(R_g) \cap \mathrm{Loc}(R_{f_j} \mid j \in J) = \mathrm{Loc}(R_{gf_j} \mid j \in J).$$

Proof. Since R_g is flat, for any $j \in J$ we have $R_g \otimes R_{f_j} \simeq R_{gf_j}$, and since localizing subcategories in $D(R)$ are all tensor ideals by Lemma 1.1.8, from this we get the inclusion

$$\mathrm{Loc}(R_g) \cap \mathrm{Loc}(R_{f_j} \mid j \in J) \supset \mathrm{Loc}(R_{gf_j} \mid j \in J).$$

To get the reverse inclusion, consider an object $M \in \mathrm{Loc}(R_g) \cap \mathrm{Loc}(R_{f_j} \mid j \in J)$. Because $M \in \mathrm{Loc}(R_g)$, we know by Lemma 2.2.3 that $M \simeq R_g \otimes M$. Let α be the length of M in $\mathrm{Loc}(R_{f_j} \mid j \in J)$. We prove by transfinite induction on α that $M \in \mathrm{Loc}(R_{gf_j} \mid j \in J)$.

If $\alpha = 0$, there is nothing to prove.

If $\alpha = 1$, then there exists an integer $n \in \mathbb{Z}$ and an element $j \in J$ such that $M = \Sigma^n R_{f_j}$, but then:

$$M = R_g \otimes M = \Sigma^n (R_{gf_j}) \in \mathrm{Loc}(R_{gf_j} \mid j \in J).$$

Let β be an ordinal ≥ 1 and assume by induction that the statement has been proved for all objects of length $\alpha < \beta$. If β is a limit ordinal, then by construction, for any object $M \in \mathrm{Loc}^{\beta}(R_{gf_j} \mid j \in J)$, there exists $\alpha < \beta$ such that $M \in \mathrm{Loc}^{\alpha}(R_{gf_j} \mid j \in J)$, and so by induction hypothesis $M \in \mathrm{Loc}(R_{f_j g} \mid j \in J)$. If β is a successor ordinal, say $\beta = \alpha + 1$, then there are two cases. Either there exists an integer $n \in \mathbb{Z}$, a set Y of cardinality $< |\beta|$, a family of objects $N_y \in \mathrm{Loc}(R_{gf_j} \mid j \in J)^{\alpha}$ such that $M = \Sigma^n \coprod_{y \in Y} N_y$, and then $M = R_g \otimes M = \Sigma^n \coprod_{y \in Y} (R_g \otimes N_y)$. In this case, by induction hypothesis for all $y \in Y$ we have $R_g \otimes N_y \in \mathrm{Loc}(R_{gf_j} \mid j \in J)$, and because this class is localizing we conclude that $M \in \mathrm{Loc}(R_{gf_j} \mid j \in J)$ or there exists an integer $n \in \mathbb{Z}$ and two objects N_1 and N_2 in $\mathrm{Loc}^{\alpha}(R_{f_j} \mid j \in J)$ such that we have a triangle

$$\Sigma^n N_1 \longrightarrow \Sigma^n N_2 \longrightarrow \Sigma^n M \longrightarrow \Sigma^{n+1} N_1.$$

Tensoring this triangle with R_g, which is flat, we get the exact triangle

$$R_g \otimes \Sigma^n N_1 \longrightarrow R_g \otimes \Sigma^n N_2 \longrightarrow R_g \otimes \Sigma^n M \longrightarrow R_g \otimes \Sigma^{n+1} N_1,$$

and by induction hypothesis N_1 and $R_g \otimes N_2$ belong to $\mathrm{Loc}(R_{gf_j} \mid j \in J)$, which is triangulated, so $M \in \mathrm{Loc}(R_{gf_j} \mid j \in J)$ as we wanted. ☐

Corollary 2.2.13. *Let $\{f_j\}_{j \in J}$ be a family of elements in R, and let $\{g_i\}_{1 \leq i \leq n}$ be a finite family of elements in R. Then*

$$\mathrm{Loc}(R_{f_j} \mid j \in J) \cap \mathrm{Loc}(R_{g_i} \mid 1 \leq i \leq n) = \mathrm{Loc}(R_{f_j g_i} \mid j \in J,\ 1 \leq i \leq n).$$

Proof. The same proof works, but instead of tensoring with the flat module R_g one tensors with the flat complex $L_{R/(g_i, 1 \leq i \leq n)}(R)$ described in the proof of Theorem 2.2.4. ☐

As an immediate consequence we get that if J is an arbitrary ideal and I a finitely generated ideal, then:

$$\begin{aligned} \mathrm{Loc}(R_f \mid f \in \sqrt{J}) \wedge \mathrm{Loc}(R_f \mid f \in \sqrt{I}) &= \mathrm{Loc}(R_f \mid f \in \sqrt{I} \cdot \sqrt{J}) \\ &= \mathrm{Loc}(R_f \mid f \in \sqrt{J}) \cap \mathrm{Loc}(R_f \mid f \in \sqrt{I}). \end{aligned}$$

Proposition 2.2.14. *In the poset of localizing categories of the form* $\mathrm{Loc}(R_f \mid f \in \sqrt{J})$*, ordered by inclusion, the finite elements are exactly those of the form* $\mathrm{Loc}(R_f \mid f \in \sqrt{J})$ *with J a finitely generated ideal.*

Proof. First notice that if $\mathrm{Loc}(R_f \mid f \in \sqrt{J})$ cannot be generated by finitely many localizations of the ring R, then it is not finite, for then we know that the equality $\mathrm{Loc}(R_f \mid f \in \sqrt{J}) = \bigvee_{f\in J} \mathrm{Loc}(R_f)$ cannot factor through a join over any finite subset of J.

Also, if one proves that for any $g \in R$, $\mathrm{Loc}(R_g)$ is finite, then it is immediate that for any finite set $K \subset R$, $\mathrm{Loc}(R_g \mid g \in K)$ is finite, as it would be the join of finitely many finite elements.

It remains to prove that if $\mathrm{Loc}(R_g) \subset \mathrm{Loc}(R_f \mid f \in J)$, with J infinite, then there exists a finite subset $K \subset J$ such that $\mathrm{Loc}(R_g) \subset \mathrm{Loc}(R_f \mid f \in K)$. By Proposition 2.2.12, we know that $\mathrm{Loc}(R_g) = \mathrm{Loc}(R_g) \cap \mathrm{Loc}(R_f \mid f \in J) = \mathrm{Loc}(R_{fg} \mid f \in J)$, and since $\mathrm{Loc}(R_g)$ is equivalent to $D(R_g)$ we may assume without loss of generality that $R = R_g$. We now have to prove that if $\mathrm{Loc}(R_f \mid f \in J) = D(R)$, then there exists a finite set $K \in J$ such that $\mathrm{Loc}(R_f \mid f \in K) = D(R)$. There are two cases to consider.

If the ideal generated by J is R, then there exists a finite subset $K \in J$ such that this finite subset already generates R as an ideal. In this case we apply Theorem 2.2.4 to get $\mathrm{Loc}(R_f \mid f \in K) = \mathrm{Loc}(R/(K))^\perp = \mathrm{Loc}(\{0\})^\perp = D(R)$ and we have $D(R) = \mathrm{Loc}(R_f \mid f \in K) \subset \mathrm{Loc}(R_f \mid f \in J) \subset D(R)$.

On the other hand, if the ideal generated by J, call it again J, is a proper ideal, then $\mathrm{Loc}(R_f \mid f \in J)$ is a proper subcategory of $D(R)$, in contradiction with the assumption. Indeed, consider an injective envelope $E(R/J)$ of R/J. Then a direct computation shows that for all $f \in J$ we have $\mathrm{Hom}_{D(R)}(R_f, E(R/J)) = 0$, so $\mathrm{Loc}(R_f \mid f \in J)^\perp \ni E(R/J) \neq 0$. In particular $\mathrm{Loc}(R_f \mid f \in J) \neq D(R)$. $\square$

Definition 2.2.15. Denote by $\boldsymbol{RfGLoc}(D(R))$ the poset of localizing subcategories of $D(R)$ generated by sets of localizations of R, and by $\boldsymbol{fgRfGLoc}(D(R))$ the sublattice of finite elements (i.e. localizing subcategories of $D(R)$ generated by finite sets of localizations of R).

The following theorem shows that $\boldsymbol{RfGLoc}(D(R))$ is in fact a coherent frame (and therefore $\boldsymbol{fgRfGLoc}(D(R))$ is a distributive lattice).

Theorem 2.2.16. *There is a natural isomorphism of posets*

$$\boldsymbol{RfGLoc}(D(R)) \simeq \boldsymbol{RadId}(R)$$

given by

$$\mathrm{Loc}(R_f \mid f \in J) \xmapsto{f} \sqrt{(f \mid f \in J)}$$
$$\mathrm{Loc}(R_f \mid f \in I) \xleftarrow{g} I.$$

Moreover, when restricted to their finite parts this isomorphism induces an isomorphism of distributive lattices:

$$\boldsymbol{fgRfGLoc}(D(R)) \simeq \mathbf{fgRadId}(R).$$

Proof. It is obvious that $f \circ g$ is the identity, so we just have to prove that $g \circ f$ is the identity, namely that if $J \subset R$, then $\mathrm{Loc}(R_f \mid f \in J) = \mathrm{Loc}(R_f \mid f \in \sqrt{J})$, where $\sqrt{J}$ stands for the radical ideal generated by J. It is clear that $\mathrm{Loc}(R_f \mid f \in J) \subset \mathrm{Loc}(R_f \mid f \in \sqrt{J})$. Let $g \in \sqrt{J}$; then some power of g, say g^n, is a linear combination of finitely many elements in J, say $j_1, \dots, j_n$. In particular we know that in this case $\mathrm{Loc}(R(g)) \supset \mathrm{Loc}(R/(j_1, \dots, j_n))$, and taking right orthogonals we get $\mathrm{Loc}(R_g) \subset \mathrm{Loc}(R_{j_1}, \dots, R_{j_n}) \subset \mathrm{Loc}(R_f \mid f \in J)$. So $\mathrm{Loc}(R_f \mid f \in J) \supset \mathrm{Loc}(R_f \mid f \in \sqrt{J})$, as we wanted. $\square$

As for the Hochster frame, this poset isomorphism shows that on the left-hand side we have a coherent frame. It is straightforward to check that the join operation on the left

$$\mathrm{Loc}(R_f \mid f \in I) \wedge \mathrm{Loc}(R_g \mid g \in J) = \mathrm{Loc}(R_{fg} \mid (f,g) \in I \times J)$$

is given by taking "localization closure". We do not know whether the meet operation is always given by intersection or not. Corollary 2.2.13 shows that "meet is intersection" if just one of the localizing subcategories is generated by a finite number of objects (i.e. is a finite element in the lattice). We suspect that in general the meet may be strictly smaller than the intersection.

2.2.17. *Colocalizing subcategories.* In Theorem 2.2.16 above, the involved localizing subcategories are also colocalizing. Neeman [24] has recently proved a theorem classifying colocalizing subcategories of $D(R)$ in the case where R is a noetherian ring: they are in inclusion-preserving one-to-one correspondence with arbitrary subsets of the prime spectrum $\operatorname{Spec} R$. This result does not involve any topology at all. As a corollary to Theorem 2.2.16 we obtain an interesting addendum to Neeman's colocalizing classification in the noetherian case, namely a characterization of those colocalizing subcategories that correspond to Zariski open subsets: they are precisely the right orthogonals to the subcategories of the form $\mathrm{Loc}(R/I)$.

2.2.18. *Functoriality.* To a commutative ring R, we can associate the Zariski frame or the Hochster frame. These assignments are the object part of two covariant functors $\boldsymbol{Ring} \to \boldsymbol{Frm}$. In this subsection we describe these two functors in a way fitting our description as frames embedded in the derived category of a ring.

Fix a ring homomorphism $\phi : S \to R$. Extension of scalars functor induces a triangulated functor

$$\begin{array}{rcl} \phi_* : D(S) & \longrightarrow & D(R) \\ E & \longmapsto & E \otimes_S R. \end{array}$$

Since extension of scalars sends the module S onto R and commutes with arbitrary sums, the derived functor preserves compact objects and sends localizing subcategories to localizing subcategories. In particular,

$$\forall E \in D(S),\ \mathrm{Loc}(E) \otimes_S R \subset \mathrm{Loc}(E \otimes_S R).$$

Hence we have a canonical map of frames:

$$\mathrm{Loc}(C_i \mid i \in I) \quad \longmapsto \quad \mathrm{Loc}(C_i \otimes_S R \mid i \in I).$$

For the Zariski spectrum the situation is similar. Recall that if I is a subset of S and $V(I)$ is the Zariski *closed* set associated to I, then the preimage of $V(I)$ via $\phi^* : \operatorname{Spec}_Z R \to \operatorname{Spec}_Z S$ is $V(\phi(I))$. At the level of open sets, this means that the preimage of the Zariski open set $\bigcup_{f \in I} D(f)$ is $\bigcup_{f \in I} D(\phi(f))$. But the extension of scalars is also compatible with localization; in fact since for any element $f \in S$, the module S_f is flat and is a ring it is straightforward to check using the universal property of localization that

$$S_f \otimes_S R = R_{\phi(f)}.$$

In particular we have an induced map of frames

$$\begin{array}{rcl} \boldsymbol{RfGLoc}(D(S)) & \longrightarrow & \boldsymbol{RfGLoc}(D(R)) \\ \mathrm{Loc}(S_f \mid f \in I) & \longmapsto & \mathrm{Loc}(R_{\phi(f)} \mid f \in I) \end{array}$$

which coincides with the map induced by the map $\operatorname{Spec}_Z R \to \operatorname{Spec}_Z S$.

2.3. **Points in** Spec$R$**.** We digress to give point-set characterizations of the Zariski and Hochster open sets.

Proposition 2.3.1. *Given a prime ideal $\mathfrak{p} \subset R$ and a finitely generated ideal I, the following conditions are equivalent:*

i) The point $\mathfrak{p}$ belongs to the Zariski open set corresponding to $\mathrm{Loc}(R_f \mid f \in I)$,

ii) $\exists f \in I$ *such that* $\kappa(\mathfrak{p}) \in \mathrm{Loc}(R_f)$.

Proof. Condition i) is clearly equivalent to the condition: $\exists f \in I$ such that $f \notin \mathfrak{p}$. But $f \notin \mathfrak{p}$ if and only if multiplication by f is an isomorphism in the residue field $\kappa(\mathfrak{p})$; this happens if and only if $\kappa(\mathfrak{p})$ is canonically an R_f-module, and we conclude by Lemma 2.2.3. □

Proposition 2.3.2. *Given a finitely generated ideal I in R and a prime ideal $\mathfrak{p}$, the following are equivalent:*

i) The point $\mathfrak{p}$ belongs to the Hochster open set $\mathrm{Loc}(R/I)$,

ii) as ideals in R, we have $I \subset \mathfrak{p}$,

iii) $R/\mathfrak{p} \in \mathrm{Loc}(R/I)$,

iv) $R/I \otimes R_{\mathfrak{p}} \neq 0$,

v) $\exists E \in \mathrm{Loc}(R/I)$ *such that* $E \otimes R_{\mathfrak{p}} \neq 0$.

Proof. The equivalences $i) \Leftrightarrow ii) \Leftrightarrow iii)$ are immediate consequences of the characterization of the objects in $\mathrm{Loc}(R/I)$ given in Lemma 2.1.5 and the fact that $\mathfrak{p}$ is a prime ideal.

Let us prove that condition $ii)$ implies $iv)$. Since $I \subset \mathfrak{p}$, we have a surjection $R/I \twoheadrightarrow R/\mathfrak{p}$. Tensoring with $R_{\mathfrak{p}}$ we get a surjective map $R/I \otimes R_{\mathfrak{p}} \twoheadrightarrow \kappa_{\mathfrak{p}}$ onto the residue field at $\mathfrak{p}$, hence $R/I \otimes R_{\mathfrak{p}} \neq 0$.

Conversely, let us prove by contraposition that $iv) \Rightarrow iii)$: If $I \not\subset \mathfrak{p}$, then $R/I \otimes R_{\mathfrak{p}} = 0$. For this consider the exact sequence of R-modules:

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Tensoring with the flat module $R_{\mathfrak{p}}$ gives the exact sequence

$$0 \longrightarrow I \otimes R_{\mathfrak{p}} \longrightarrow R_{\mathfrak{p}} \longrightarrow R/I \otimes R_{\mathfrak{p}} \longrightarrow 0.$$

But as $I \not\subset R_{\mathfrak{p}}$, there is an element in I that becomes invertible in $R_{\mathfrak{p}}$; in particular the first arrow has to be an epimorphism and hence $R/I \otimes R_{\mathfrak{p}} = 0$.

The implication $iv) \Rightarrow v)$ is trivial. To show the converse, observe that since the triangulated functor $- \otimes R_{\mathfrak{p}}$ commutes with arbitrary sums, if R/I belongs to its kernel, then so does the entire localizing subcategory generated by R/I; hence by contraposition $v) \Rightarrow iv)$. □

From this we recover Neeman's description of the correspondence between compactly generated localizing subcategories and Hochster open sets, but extended to the non-noetherian case; see [22, Theorem 2.8]:

Corollary 2.3.3. *Let S be a set of compact objects. Then a point $\mathfrak{p}$ belongs to the Hochster open set* $\mathrm{Loc}(S)$ *if and only if there exists $C \in S$ such that $C \otimes R_{\mathfrak{p}} \neq 0$.*

Proof. We know that $\mathrm{Loc}(S) = \bigvee_{C \in S} \mathrm{Loc}(C)$, and since the join operation corresponds to the union of open sets, the point $\mathfrak{p}$ belongs to $\mathrm{Loc}(S)$ if and only if it belongs to $\operatorname{Loc}(C)$ for some $C \in S$. The condition is then clearly necessary as it fulfills condition v) in Proposition 2.3.2.

Conversely, if for any $C \in S$ we have $C \otimes R_{\mathfrak{p}} = 0$, then given any $E \in \operatorname{Loc}(S)$ and a recipe for E, tensoring this recipe with $R_{\mathfrak{p}}$ we conclude that $E \otimes R_{\mathfrak{p}} = 0$, and again by v) in Proposition 2.3.2 we conclude that the point $\mathfrak{p}$ does not belong to the open set $\operatorname{Loc}(S)$. □

More geometrically we have:

Corollary 2.3.4. *For any perfect complex C in $D(R)$ its homological support*

$$\operatorname{supph} C = \{\mathfrak{p} \in \operatorname{Spec} R \mid C \otimes R_{\mathfrak{p}} \neq 0\}$$

is a Hochster open set.

3. Tensor triangulated categories

In this section we revisit Balmer's theory of spectra and supports of tensor triangulated categories. The point-free approach reveals that this construction and its basic properties are so similar to the ring case that they can be seen as a variation of Joyal's constructive account of the Zariski spectrum in terms of supports, dating back to the early 1970s [17].

3.1. **The Zariski spectrum of a tensor triangulated category.**

Definition 3.1.1. Let S be a set of objects in a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$. Define $G(S)$ to be the set consisting of those objects of the form:

i) an iterated suspension or desuspension of an object in S,
ii) or a finite sum of objects in S,
iii) or an object $s \otimes t$ with $s \in S$ and $t \in \mathcal{T}$,
iv) or an extension of two objects in S,
v) or a direct summand of an object in S.

Clearly, if a thick tensor ideal contains S, then it also contains $G(S)$, and hence by induction it contains $G^{\omega}(S) := \bigcup_{n \in \mathbb{N}} G^n(S)$. On the other hand, it is easy to see that $G^{\omega}(S)$ is itself a thick tensor ideal; hence it is the smallest thick tensor ideal containing S. We denote it by $\langle S \rangle$.

The following result expresses the finiteness in the definition of thick tensor ideal.

Lemma 3.1.2. *Let S be a set of objects and suppose $x \in \langle S \rangle$. Then there exists a finite subset $K \subset S$ such that also $x \in \langle K \rangle$.*

Proof. We have $x \in G^n(S)$ for some $n \in \mathbb{N}$. This means x is obtained by one of the construction steps in G from finitely many objects in $G^{n-1}(S)$. By downward induction, x is then obtained from a finite set of objects $K \subset G^0(S) = S$, hence $x \in \langle K \rangle$. □

3.1.3. *Radical thick tensor ideals.* Fix a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$. To any thick tensor ideal I (cf. Definition 1.1.7) we may associate its radical closure $\sqrt{I}$ just as in the ring case:

$$\sqrt{I} = \{a \in \mathcal{T} \mid \exists n \in \mathbb{N} \text{ such that } a^{\otimes n} \in I\}.$$

A thick tensor ideal I is called *radical* when $I = \sqrt{I}$.

More generally, for any set of objects S we denote by $\sqrt{S}$ the radical of the thick tensor ideal $\langle S \rangle$.

Corollary 3.1.4. *Let S be a set of objects and suppose $x \in \sqrt{S}$. Then there exists a finite subset $K \subset S$ such that also $x \in \sqrt{K}$.*

Proof. Apply Lemma 3.1.2 to a suitable power of x. □

Lemma 3.1.5. *If I is a thick tensor ideal, then $\sqrt{I}$ is a radical thick tensor ideal.*

Balmer proved this [4, Lemma 4.2] by establishing the classical formula $\sqrt{I} = \bigcap_{\mathfrak{p} \supset I} \mathfrak{p}$, valid assuming Zorn's lemma. We offer instead a direct point-free proof:

Proof. It is immediate to check from the definitions that $\sqrt{I}$ is closed under suspension and desuspension, finite sums, direct summands, and tensoring with objects of $\mathcal{T}$. Finally for a triangle $x \to y \to z \to \Sigma x$, the following general lemma shows that if x and y belong to $\sqrt{I}$, then so does z. □

Lemma 3.1.6. *Let I be a tensor ideal in a tensor triangulated category, and consider a triangle $x \to y \to z \to \Sigma x$. If x^p and y^q belong to I, then z^{p+q-1} belongs to I.*

Proof. More generally we show by induction on k that

$$x^i y^j z^k \in I \quad \forall i, j, k \text{ such that } i + j + k = p + q - 1$$

(where for economy we omit the tensor sign between the factors). The case $k = 0$ is clear since I is a tensor ideal. For the monomial $x^i y^j z^{k+1}$ (with $i + j + k + 1 = p + q - 1$), tensor the triangle $x \to y \to z \to \Sigma x$ with $x^i y^j z^k$. By induction the first two vertices in the resulting triangle belong to I, and hence so does the third. □

Radical thick tensor ideals in $\mathcal{T}$ are naturally ordered by inclusion with a top element $\mathcal{T}$ itself and a bottom element

$$\sqrt{0} = \{a \in \mathcal{T} \mid \exists n \in \mathbb{N} \text{ such that } a^{\otimes n} = 0\},$$

the full subcategory of *nilpotent* elements. Although radical thick tensor ideals might not form a set, we have well-defined frame operations:

(1) If I_1 and I_2 are two radical thick tensor ideals, then $I_1 \bigwedge I_2 = I_1 \bigcap I_2$.
(2) If $\{I_j\}_{j \in J}$ is a set of radical thick tensor ideals, then $\bigvee_{j \in J} I_j$ is the radical of the thick tensor ideal generated by the union $\bigcup_{j \in J} I_j$. This is well defined by Definition 3.1.1 and Lemma 3.1.5.

The main theorem in this subsection (Theorem 3.1.9 below) states that the radical thick tensor ideals of a tensor triangulated category $\mathcal{T}$ form a coherent frame. For this to make sense it is necessary that there is only a set of them. The easiest way to ensure this is to assume that $\mathcal{T}$ is essentially small, as in Balmer [4]. A source of examples of this situation comes from starting with a compactly generated triangulated category $\mathcal{T}$, for then (as explained for instance in [23, Chapter 3 and Remark 4.2.6]) the full subcategory of compact objects $\mathcal{T}^\omega$ is essentially small. If we add the assumption that the tensor unit is compact and that the tensor product of two compact objects is again compact, then the full subcategory of compact objects $\mathcal{T}^\omega$ is an essentially small tensor triangulated category. Our main example is when $\mathcal{T}$ is the derived category of a commutative ring or the derived category of a coherent scheme as in Section 4. It follows that $\mathcal{T}^\omega$, the derived category of perfect complexes, is essentially small.

As pointed out by Balmer [4], in many important situations, passage to the radical is a harmless operation. For instance, in $D^\omega(R)$, all thick tensor ideals are radical, which follows from the fact that every perfect complex is strongly dualizable, as we proceed to briefly recall. Observe that in $D(R)$ all thick subcategories are automatically tensor ideals (by an argument similar to the proof of Lemma 1.1.8); hence all thick subcategories are radical thick tensor ideals.

For an object $a \in \mathcal{T}$, put $a^\vee = \mathrm{Hom}(a, \mathbf{1})$. An object a is *strongly dualizable* if and only if the natural transformation

$$- \otimes a^\vee \to \mathrm{Hom}(a, -)$$

is an isomorphism. It is well known [21] that any strongly dualizable object a is a direct summand of $a \otimes a \otimes a^\vee$; hence:

Lemma 3.1.7. *If all compact objects in $\mathcal{T}$ are strongly dualizable, then all thick tensor ideals in $\mathcal{T}^\omega$ are radical.*

Lemma 3.1.8. *In the poset of radical thick tensor ideals, the infinite distributive law holds: for any radical thick tensor ideal J and any set of radical thick tensor ideals $(I_\alpha)_{\alpha\in A}$, we have*

$$\bigvee_\alpha (J \wedge I_\alpha) = J \cap (\bigvee_\alpha I_\alpha).$$

Proof. The inclusion $\subset$ is clear. To get the reverse inclusion fix an object $x \in J \cap (\bigvee_\alpha I_\alpha)$. By radicality it is enough to prove that $x \otimes x \in \bigvee_\alpha (J \wedge I_\alpha)$. Define

$$C_x = \{k \in \bigvee_\alpha I_\alpha \mid x \otimes k \in \bigvee_\alpha (J \wedge I_\alpha)\};$$

we are done if we can prove that C_x is all of $\bigvee_\alpha I_\alpha$. It is trivial to check that C_x is a triangulated category, because tensoring with x preserves triangles.

First we prove that C_x is a thick subcategory. Suppose $a \oplus b \in C_x$; this means that $x \otimes (a \oplus b) \in \bigvee_\alpha (J \wedge I_\alpha)$. But the tensor product distributes over sums, so also $(x \otimes a) \oplus (x \otimes b) \in \bigvee_\alpha (J \wedge I_\alpha)$. As the latter is a thick subcategory, we conclude that already each of $(x \otimes a)$ and $(x \otimes b)$ belong here, which is to say that a and b are in C_x.

We show now C_x is an ideal: let a be an arbitrary object of the triangulated category, and let $k \in C_x \subset \bigvee_\alpha I_\alpha$. Since $\bigvee_\alpha I_\alpha$ is an ideal, we also have $a \otimes k \in \bigvee_\alpha I_\alpha$. For the same reason $x \otimes (a \otimes k) = a \otimes (x \otimes k)$ belongs to $\bigvee_\alpha (J \wedge I_\alpha)$. By definition of C_x we therefore find that $a \otimes k \in C_x$ as required.

Finally, radicality of C_x: Suppose $k^{\otimes n} \in C_x$. This means that $x \otimes k^{\otimes n} \in \bigvee_\alpha (J \wedge I_\alpha)$. But then we can tensor $n-1$ times more with x to conclude that $(x \otimes k)^{\otimes n} \in \bigvee_\alpha (J \wedge I_\alpha)$, and since this is a radical ideal, it then follows that $x \otimes k \in \bigvee_\alpha (J \wedge I_\alpha)$, which is to say that $k \in C_x$.

In conclusion, C_x is a radical thick tensor ideal contained in $\bigvee_\alpha I_\alpha$, and it contains each I_α, so it also contains their join, hence is equal to the whole join. $\square$

Theorem 3.1.9. *The radical thick tensor ideals of a tensor triangulated category $\mathcal{T}$ form a coherent frame, provided there is only a set of them. The finite elements are the principal radical thick tensor ideals, i.e. of the form $\sqrt{a}$ for some $a \in \mathcal{T}$.*

Proof. The proof follows the same lines as the proof that the radical ideals in a commutative ring form a coherent frame, but instead of relying on finiteness of sums in a ring, it uses finiteness of generation of thick tensor ideals. Some of the arguments have a different flavor because of the thickness condition which has no reasonable analogue for commutative rings.

Lemma 3.1.8 establishes that the radical thick tensor ideals form a frame. We now establish that this frame is coherent. We first show that finite elements are generated by a single object. Let K be a finite element in the frame. Since there is only a set of principal radical thick tensor ideals by assumption, there is certainly a set $M(K)$ of those that are contained in K. Then trivially $K = \bigvee_{\sqrt{c} \in M(K)} \sqrt{c}$, and, as K is a finite element in the frame, there exists a finite subset $J \subset M(K)$ such that $K = \bigvee_{c \in J} \sqrt{c}$, so K is generated by a finite set consisting of one generator for each $c \in J$. It is now a direct consequence of the thickness assumption that if K is generated by $c_1, \ldots, c_k$, then it is generated by the single object $c_1 \oplus \cdots \oplus c_k$.

Finally we show each ideal of the form $\sqrt{a}$ is indeed a finite element in the frame. Given a set of radical thick tensor ideals $\{J_\alpha\}_{\alpha \in A}$ such that $\sqrt{a} \leq \bigvee_{\alpha \in A} J_\alpha$, we need to find a finite subset $B \subset A$ such that also $\sqrt{a} \leq \bigvee_{\alpha \in B} J_\alpha$. Since the join in question is a radical thick tensor ideal, it is enough to find $B \subset A$ such that $a \in \bigvee_{\alpha \in B} J_\alpha$. Let S denote the union of the ideals J_α. Then $\bigvee_{\alpha \in A} J_\alpha = \sqrt{S}$. We have $a \in \sqrt{S}$. But then by Corollary 3.1.4, there is a finite subset $K \subset S$ such that $a \in \sqrt{K}$. Finitely many J_α are needed to contain this finite subset K, so take those. □

Definition 3.1.10. The frame of radical thick tensor ideals in $\mathcal{T}$ is denoted $\boldsymbol{Zar}(\mathcal{T})$ and called the *Zariski frame*. The spectral space associated to $\boldsymbol{Zar}(\mathcal{T})$ we call the *Zariski spectrum* of $\mathcal{T}$, denoted $\operatorname{Spec} \mathcal{T}$.

3.2. **Supports of a tensor triangulated category.** Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category, from now on assumed to have only a set of radical thick tensor ideals. Just as in the ring case, the coherent frame $\boldsymbol{Zar}(\mathcal{T})$ comes equipped with a canonical notion of support. The universal property of the Zariski frame of a ring (Theorem 1.3.3) readily carries over to the Zariski frame of $\mathcal{T}$ and yields one of the main theorems of [4], as we proceed to explain. In order to stress the parallel with the classical case, we shall use a slight modification of Balmer's notions:

Definition 3.2.1. A *support* on $(\mathcal{T}, \otimes, \mathbf{1})$ is a pair (F, d) where F is a frame and $d : \mathrm{obj}(\mathcal{T}) \to F$ is a map satisfying:

(1) $d(0) = 0$ and $d(\mathbf{1}) = \mathbf{1}$,
(2) $\forall a \in \mathcal{T} :\ d(\Sigma a) = d(a)$,
(3) $\forall a, b \in \mathcal{T} :\ d(a \oplus b) = d(a) \vee d(b)$,
(4) $\forall a, b \in \mathcal{T} :\ d(a \otimes b) = d(a) \wedge d(b)$,
(5) if $a \to b \to c \to \Sigma a$ is a triangle in $\mathcal{T}$, then $d(b) \leq d(a) \vee d(c)$.

A *morphism of supports* from (F, d) to (F', d') is a frame map $F \to F'$ compatible with the maps d and d'.

Lemma 3.2.2. *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category assumed to have only a set of radical thick tensor ideals. Then the assignment*

$$\begin{array}{ccc} \mathrm{obj}(\mathcal{T}) & \longrightarrow & \boldsymbol{Zar}(\mathcal{T}) \\ a & \longmapsto & \operatorname{supp}(a) := \sqrt{a} \end{array}$$

is a support.

Proof. Items (1), (2) and (5) in Definition 3.2.1 are trivially satisfied. Let us check item (3): given $a, b \in \mathcal{T}$, we have $\sqrt{a \oplus b} = \sqrt{a} \vee \sqrt{b}$. Since we are dealing with thick ideals, $\sqrt{a} \subset \sqrt{a \oplus b}$ and $\sqrt{b} \subset \sqrt{a \oplus b}$, so $\sqrt{a} \vee \sqrt{b} \subset \sqrt{a \oplus b}$. Conversely, $a \oplus b$ certainly is in $\sqrt{a} \vee \sqrt{b}$.

Finally let us check (4). Given $a, b \in \mathcal{T}$, we wish to show that $\sqrt{a \otimes b} = \sqrt{a} \wedge \sqrt{b}$. Certainly $a \otimes b$ belongs to both $\sqrt{a}$ and $\sqrt{b}$, so $\sqrt{a \otimes b} \subset \sqrt{a} \wedge \sqrt{b}$. For the converse we will adapt the proof of Lemma 1.1.8. Let $R(a) = \{x \in \mathcal{T} \mid a \otimes x \in \sqrt{a \otimes b}\}$. Then $R(a)$ is a radical thick tensor ideal that trivially contains b, hence $\sqrt{b} \subset R(a)$. Now fix $c \in \sqrt{b}$ and consider $L(c) = \{x \in \mathcal{T} \mid x \otimes c \in \sqrt{a \otimes b}\}$. Then $L(c)$ is a radical thick tensor ideal that contains a by the previous step. Now, let $y \in \sqrt{a} \cap \sqrt{b}$. From the ideal $L(y)$ we know that $y \otimes y \in \sqrt{a \otimes b}$, so $\sqrt{a} \cap \sqrt{b} \subset \sqrt{a \otimes b}$ as we wanted. $\square$

Theorem 3.2.3. *Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category, assumed to have only a set of radical thick tensor ideals. Then the support*

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \boldsymbol{Zar}(\mathcal{T}) \\ a & \longmapsto & \sqrt{a} \end{array}$$

is initial among supports.

Proof. For an arbitrary support $d : \mathcal{T} \to F$, we need to exhibit a frame map $u : \boldsymbol{Zar}(\mathcal{T}) \to F$, compatible with the maps from $\mathcal{T}$, and check that this map is unique. Since $\boldsymbol{Zar}(\mathcal{T})$ is coherent, every element is a join of finite elements, so u is completely determined by its value on finite elements. The finite elements are those of form $\sqrt{a}$ and there is no choice: we must send $\sqrt{a}$ to $d(a)$. So there is at most one support map u. We only need to check it is well defined; this means to check that

$$\forall a, b \in \mathcal{T},\ \sqrt{a} = \sqrt{b} \Rightarrow d(a) = d(b).$$

For $a \in \mathcal{T}$, define $I(a) = \{c \in \mathcal{T} \mid d(c) \leq d(a)\}$. Then the properties of a support show that $I(a)$ is a radical thick tensor ideal containing a and hence $\sqrt{a}$. If $\sqrt{b} \subset \sqrt{a}$ we deduce that $d(b) \leq d(a)$, and by symmetry we get our result. $\square$

The fact that this support is initial implies functoriality: any triangulated functor $F : \mathcal{T} \to \mathcal{S}$ induces a coherent frame map $\boldsymbol{Zar}(\mathcal{T}) \to \boldsymbol{Zar}(\mathcal{S})$, taking $\sqrt{I}$ to $\sqrt{F(I)}$.

3.3. **Tensor nilpotence.** The tensor nilpotence theorem by Devinatz, Hopkins and Smith [10], one of the deep theorems in stable homotopy, has a version for derived categories, which in the work Neeman [22] and Thomason [27] is a basic tool to analyze localizing subcategories. The theorem says that if a morphism has empty support, then it is tensor nilpotent. As observed by Balmer [4], this is in fact a consequence of general topological properties of the spectrum of a tensor triangulated category, and as we shall see, it comes out very elegantly in the point-free setting.

Let $(\mathcal{T}, \otimes, \mathbf{1})$ be a tensor triangulated category, assumed to have only a set of radical thick tensor ideals. Given a morphism $f : x \to y$ in $\mathcal{T}$, we write $z|f$ to express that there exists a factorization

$$\begin{tikzcd} & z \arrow[dr] & \\ x \arrow[ur] \arrow[rr, "f"'] & & y\,. \end{tikzcd}$$

We define the *support* of f as

$$\operatorname{supp}(f) := \bigwedge_{z|f} \operatorname{supp}(z) = \bigcap_{z|f} \sqrt{z} \ \in \boldsymbol{Zar}(\mathcal{T}).$$

(The notation $z|f$ is inspired by rings: $\sqrt{n} = \bigcap_{p|n} \sqrt{p}$.) Note that strictly speaking, to ensure that the meet is indexed over a set, we should write it as the meet of all radical thick tensor ideals that occur as support of an element $z|f$.

The notion of support for morphisms extends the usual notion of support of objects:

Lemma 3.3.1. *We have* $\operatorname{supp}(\mathrm{Id}_x) = \operatorname{supp}(x)$.

Proof. The trivial factorization of Id_x shows that $\operatorname{supp}(\mathrm{Id}_x) \leq \operatorname{supp}(x)$. Any other factorization $x \to y \to x$ exhibits x as a retract of y, and hence $\operatorname{supp}(x) \leq \operatorname{supp}(y)$ by thickness. ☐

Theorem 3.3.2 (Tensor nilpotence)**.** *If* $\operatorname{supp}(f)$ *is the bottom element of* $\mathbf{Zar}(\mathcal{T})$*, then* f *is tensor nilpotent; i.e. there is* $n \in \mathbb{N}$ *such that* $f^{\otimes n}$ *is the zero map.*

Proof. The premise says that $\operatorname{supp}(0) = \bigwedge_{z|f} \operatorname{supp}(z)$, and by coherence a finite meet of such elements will do: there exist $z_1|f, \dots, z_k|f$ such that $\operatorname{supp}(0) = \bigwedge_{i=1}^k \operatorname{supp}(z_i)$. In other words,

$$\sqrt{0} = \bigcap_{i=1}^{k} \sqrt{z_i} = \sqrt{z_1 \otimes \cdots \otimes z_k}.$$

So there is $n \in \mathbb{N}$ such that $\big(z_1 \otimes \cdots \otimes z_k\big)^{\otimes n} = 0$. Now $f^{\otimes kn}$ factors through $\big(z_1 \otimes \cdots \otimes z_k\big)^{\otimes n}$, hence is the zero map. So f is tensor nilpotent. ☐

3.4. **Points.** In order to relate the results of this section to Balmer's work and the earlier literature, we proceed to extract the points.

According to Balmer [4], the spectrum of an essentially small tensor triangulated category $\mathcal{T}$ is the topological space $X = \operatorname{Spec} \mathcal{T}$ whose set of points is the set of *prime* thick tensor ideals in $\mathcal{T}$. Just as in the ring case, a thick tensor ideal $\mathfrak{p}$ is *prime* when

$$\forall a, b \in \mathcal{T} : [\ a \otimes b \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \text{ or } b \in \mathfrak{p}].$$

The topology on X is defined by taking as a basis of open sets the subsets of the form

$$U(a) := \{\mathfrak{p} \in \operatorname{Spec} \mathcal{T} \mid a \in \mathfrak{p}\}$$

for each object $a \in \mathcal{T}$. We shall see that this is the Hochster dual of the Zariski spectrum.

Recall from 1.2.5 that for a frame point $x : F \to \{0,1\}$, the corresponding frame prime ideal is $\mathcal{P}_x := \{u \in F \mid x(u) = 0\}$, and that the prime ideals in turn are in natural bijection with the prime elements of F: the prime element corresponding to x is $e_x := \bigvee_{b \in \mathcal{P}_x} b$, and then $\mathcal{P}_x = (e_x)$. A point x belongs to the open set corresponding to a frame element $u \in F$ iff $u \notin \mathcal{P}_x$ iff $u \not\leq e_x$. Now specialize to the case $F = \boldsymbol{Zar}(\mathcal{T})$. It is easy to see that the prime elements in $\boldsymbol{Zar}(\mathcal{T})$ are precisely the prime thick tensor ideals, in analogy with Lemma 1.3.1. A point x corresponding to a prime element $e_x = \mathfrak{p}$ belongs to the open set corresponding to $u = \sqrt{a} \in \boldsymbol{Zar}(\mathcal{T})$ iff $\sqrt{a} \not\leq \mathfrak{p}$. Altogether we have:

Proposition 3.4.1. *The frame-theoretic points in $\boldsymbol{Zar}(\mathcal{T})$ correspond bijectively to prime thick tensor ideals in $\mathcal{T}$. Under this correspondence, a finite element $\sqrt{a} \in \boldsymbol{Zar}(\mathcal{T})$ corresponds to the set of prime thick tensor ideals*

$$\{\mathfrak{p} \in \operatorname{Spec}\mathcal{T} \mid a \notin \mathfrak{p}\}.$$

Balmer's $U(a)$ is the complement of this, so we get in particular:

Corollary 3.4.2. *Balmer's spectrum is the Hochster dual of the Zariski spectrum.*

We wish to point out that modulo the passage between frames and point-set spaces and the identification of points just established, our Theorem 3.1.9 subsumes several results from Balmer's seminal paper [4], and in particular his Classification of Radical Thick Tensor Ideals. Balmer proves that the topological space $X = \operatorname{Spec}\mathcal{T}$ of prime ideals in $\mathcal{T}$, with basic open sets $U(a)$ as above, is a spectral space, and he proceeds to set up an order-preserving bijection (his Classification Theorem 4.10) between radical thick tensor ideals in $\mathcal{T}$ and subsets of X of the form "arbitrary unions of closed sets with quasi-compact complement". These are clearly precisely the Hochster dual open sets of X. So after eliminating the implicit double Hochster duality, his Classification Theorem says that there is an order-preserving bijection between radical thick tensor ideals in $\mathcal{T}$ and Zariski open sets in the Zariski spectrum (in our terminology). From the viewpoint of Theorem 3.1.9, this is a tautology.

Finally we express tensor nilpotence in terms of points, recovering the now classical result of Balmer. We first characterize the points of $\operatorname{supp}(f)$:

Lemma 3.4.3. *For f a morphism in $\mathcal{T}$ and $\mathfrak{p} \in \operatorname{Spec}\mathcal{T}$, we have*

$$\mathfrak{p} \in \operatorname{supp}(f) \iff f \neq 0 \mod \mathfrak{p}.$$

Proof. The first two steps of the biimplication

$$\mathfrak{p} \in \operatorname{supp}(f) \Leftrightarrow \forall z|f : \mathfrak{p} \in \operatorname{supp}(z) \Leftrightarrow \forall z|f : z \notin \mathfrak{p}$$

follow by definition of support for morphisms and by a support reformulation of Proposition 3.4.1. The final step is easier to do negated:

$$\exists z|f : z \in \mathfrak{p} \Leftrightarrow f = 0 \mod \mathfrak{p},$$

which is straightforward (see [4, Lemma 2.22]). □

Corollary 3.4.4 (Balmer)**.** *If $f = 0 \mod \mathfrak{p}$ for all prime thick tensor ideals $\mathfrak{p}$, then f is tensor nilpotent.*

Proof. The premise says that there are no points in $\operatorname{supp}(f)$. But since $\boldsymbol{Zar}(\mathcal{T})$ is coherent, the only frame element without points is the bottom element; this is a special case of the fact that a coherent frame has enough points, i.e. is spatial [15, Theorem II.3.4]. We conclude by Theorem 3.3.2. □

4. Reconstruction of coherent schemes

In this section we show how to assemble our results as local data to obtain a new proof of the classical results of Thomason [27] on the classification of thick subcategories of $D^{\omega}_{\mathrm{qc}}(X)$ for X a coherent scheme, but again without having to bother about points. We also reconstruct the structure sheaf of X from its derived category of perfect complexes.

The key ingredients are on one hand our explicit results in the affine case, and on the other hand the result that the Zariski frame of a tensor triangulated category is coherent and is the recipient of the initial support. From this we will establish that the Zariski frame of $D^{\omega}_{\mathrm{qc}}(X)$ is isomorphic to the Hochster dual of the Zariski frame of X; we establish this by checking it in an affine open cover of X. In each such affine open, the isomorphism is essentially Theorem 2.1.18. We then pass from local to global using the fact that coherent schemes are the schemes finitely built from affine schemes.

4.1. **Coherent schemes and the Hochster topology.**

4.1.1. *Coherent schemes.* Recall that a scheme is *coherent* when it is quasi-compact and quasi-separated; this is the terminology recommended in SGA4 [1, exp. VI]. A scheme is coherent precisely when its frame of Zariski open sets is coherent. In terms of distributive lattices, coherent schemes are those ringed lattices which can be covered by a finite number of Zariski lattices; cf. Coquand-Lombardi-Schuster [9], who call such schemes “spectral schemes”. The fact that coherent schemes are thus “finitely built” from affine schemes allows a natural passage from local to global and is encompassed in the following Reduction Principle, which we learned from [7], where a proof can be found.

Lemma 4.1.2 (Reduction Principle)**.** *Let P be a property of schemes. Assume that*

(H0): Property P holds for all affine schemes.

(H1): If X is a scheme and $X = X_1 \cup X_2$ is an open cover with intersection X_{12}, and if property P holds for X_{12}, X_1, and X_2, then property P holds for X.

Then property P holds for all coherent schemes.

4.1.3. *Hochster topology.* For a coherent scheme, we denote by $D_{\mathrm{qc}}(X)$ the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent homology. This is a compactly generated triangulated category (see Bondal and van den Bergh [7, Theorem 3.1.1], and also [2]). Its subcategory $D^{\omega}_{\mathrm{qc}}(X)$ of compact objects is the category of perfect complexes, i.e. locally isomorphic to bounded complexes of finitely generated projective $\mathcal{O}_X$-modules. Bondal and van den Bergh [7, Theorem 3.1.1] show that it is generated by a single compact object.

The following observation shows that one can apply the results from Section 3.

Lemma 4.1.4. *Let $E, F \in D^{\omega}_{qc}(X)$. Then:*

(i) $\mathrm{Hom}(E,F) \in D^{\omega}_{cq}(X)$*.*

(ii) The complex E is strongly dualizable.

Proof. (i) In [7, Lemma 3.3.8], it is shown using the reduction principle that the complex $\mathrm{Hom}(E,F)$ is bounded. So compactness can then be checked locally, and the statement is clearly true on an affine scheme.

(ii) We have to check that the canonical map $F \otimes \mathrm{Hom}(E, \mathcal{O}_X) \to \mathrm{Hom}(F, E)$ is an isomorphism for all compact objects E, F. But isomorphisms can be detected locally, and the statement is clearly true on an affine scheme, where all the involved objects are bounded complexes of finitely projective modules. $\square$

From this lemma we get that $D^{\omega}_{\mathrm{qc}}(X)$ satisfies the assumptions of the theorems in Section 3. In particular by Theorem 3.1.9, radical thick tensor ideals in $D^{\omega}_{\mathrm{qc}}(X)$ form a coherent frame, and the map $C \mapsto \sqrt{C}$ is the initial support by Theorem 3.2.3. We will now compare this with a homologically defined support; cf. Thomason [27, Definition 3.2].

Definition 4.1.5. For $C \in D^{\omega}_{\mathrm{qc}}(X)$, the *homological support* is the subspace $\mathrm{supph}(C) \subset X$ of those points x at which the stalk complex of $\mathcal{O}_{X,x}$-modules C_x is not acyclic.

Lemma 4.1.6. *For any perfect complex C,* $\mathrm{supph}(C)$ *is a Zariski closed set with quasi-compact complement (and in particular a Hochster open set).*

Proof. By quasi-compactness of X, we can cover X by finitely many open affine subschemes on which C is quasi-isomorphic to a bounded complex of finitely generated projective modules. Since an affine scheme $\mathrm{Spec}\, R$ is quasi-compact, it is enough to show that on each of these, $\mathrm{supph}(C) \cap \mathrm{Spec}(R)$ is of the form $V(I)$ for some finitely generated ideal $I \subset R$. But $C|_{\mathrm{Spec}\, R}$ is a perfect complex of R-modules, and the stalk at a point $\mathfrak{p}$ can be computed by tensoring the complex with $R_{\mathfrak{p}}$, so the statement is our Corollary 2.3.4. $\square$

We wish to avoid points as much as possible, so we express instead this notion of support in a more conceptual manner:

Lemma 4.1.7. *The assignment*

$$\begin{array}{ccc} D^{\omega}_{qc}(X) & \longrightarrow & \boldsymbol{Zar}(X)^{\vee} \\ C & \longmapsto & \mathrm{supph}\, C \end{array}$$

is a notion of support in the sense of Definition 3.2.1.

Proof. The fact that $\mathrm{supph}(\Sigma C) = \mathrm{supph}(C)$ is trivial, as is the fact that $\mathrm{supph}\, 0 = \emptyset$ and $\mathrm{supph}\, \mathcal{O}_X = X$. For the thickness property observe that if C_1, C_2 are perfect complexes, then $(C_1 \oplus C_2)|_x = C_1|_x \oplus C_2|_x$, and that $C_1|_x \oplus C_2|_x$ is acyclic if and only if both $C_1|_x$ and $C_2|_x$ are acyclic. For compatibility with the tensor product, observe first that $(C_1 \otimes C_2)|_x = C_1|_x \otimes C_2|_x$. Then, by the Künneth formula we have that $C_1|_x \otimes C_2|_x$ is acyclic if and only if $C_1|_x$ or $C_2|_x$ are acyclic. So indeed $\mathrm{supph}(C_1 \otimes C_2) = \mathrm{supph}(C_1) \cup \mathrm{supph}(C_2)$. For compatibility with triangles, let

$$C_1 \longrightarrow C_2 \longrightarrow C_3 \longrightarrow \Sigma C_1$$

be a triangle of perfect complexes. Since taking stalks is an exact functor, the long exact sequence in homology associated to the triangle of stalks shows immediately that $\mathrm{supph}\, C_3 \subset \mathrm{supph}\, C_1 \cup \mathrm{supph}\, C_2$. $\square$

The next two results can be found in Thomason [27] as Lemmas 3.4 and 3.14 respectively. The difference lies in the fact that we deduce them from an analysis of the affine case. In this way we avoid the use of points in the proof of the first result and avoid both the Tensor Nilpotence and the Absolute Noetherian Approximation theorems in the second proof.

Lemma 4.1.8. *In a coherent scheme X, let $Z \subset X$ be a Zariski closed set with quasi-compact complement. Then there exists $E \in D^{\omega}_{qc}(X)$ with* $\operatorname{supph} E = Z$.

Proof. We apply the Reduction Principle (Lemma 4.1.2) to the property P that asserts that the lemma holds for any $Z \subset X$, a Zariski closed set with quasi-compact complement in a scheme. The affine case is Theorem 2.1.19.

For the induction step we need the following deep result due to Thomason-Trobaugh [28, Lemma 5.6.2a]: Fix a coherent scheme X, U a Zariski open set in X and Z a closed set with quasi-compact complement. Let F be a perfect complex on U, acyclic on $U \setminus U \cap Z$. Then there exists a perfect complex E on X, acyclic on $X \setminus Z$, such that $E_{|U} \simeq F$ if and only if $[F] \in K_0(U \text{ on } U \cap Z)$ is in the image of the map induced by restriction:

$$K_0(X \text{ on } Z) \to K_0(U \text{ on } U \cap Z).$$

Let $X = X_1 \cup X_2$ be a scheme covered by two open subschemes. We assume that P is true on X_i, $i = 1, 2$, so we have a perfect complex F_i on each X_i such that $\operatorname{supph} F_i = Z_i = X_i \cap Z$. Observe that $\operatorname{supph}(F_i \oplus \Sigma F_i) = Z_i$, but also that by definition of the sum in K-theory (see the proof of [27, Theorem 2.1])$[F_i \oplus \Sigma F_i] = 0$ in $K_0(X_i \text{ on } Z_i)$. Therefore we have two perfect complexes E_1 and E_2 on X, with support included in Z such that $E_{i|U_i} \simeq F_i$. We claim that $E_1 \oplus E_2$ is the perfect complex we are looking for. Indeed

$$\operatorname{supph}(E_1 \oplus E_2) \supset \operatorname{supph} E_1 \cup \operatorname{supph} E_2 \supset Z_1 \cup Z_2 = Z,$$

and by construction, $\operatorname{supph}(E_1 \oplus E_2) \subset Z$. $\square$

Lemma 4.1.9. *Let X be a coherent scheme. Given two perfect complexes $E, F \in D^{\omega}_{qc}(X)$, we have*

$$\operatorname{supph}(E) \subset \operatorname{supph}(F) \;\Leftrightarrow\; \sqrt{E} \subset \sqrt{F}.$$

Proof. The implication "$\Leftarrow$" is obvious: if for some $n \geq 1$ we have $E^{\otimes n}$ can be built from F, consider any finite recipe for $E^{\otimes n}$. Then at any point x, if the stalk of F at x is zero, so is the stalk of the recipe and hence $E^{\otimes n}$ itself. Therefore $\operatorname{supph}(E^{\otimes n}) = \operatorname{supph}(E) \subset \operatorname{supph}(F)$.

For the converse implication $\Rightarrow$, we first enlarge a bit the setting and consider $\operatorname{Loc}(F) \subset D_{qc}(X)$. Denote by L_F and Γ_F the Bousfield localization and cellularization functors associated to $\operatorname{Loc}(F)$. Then we have a triangle

$$\Gamma_F E \longrightarrow E \longrightarrow L_F E \longrightarrow \Sigma\Gamma_F E,$$

obtained by tensoring the triangle

$$\Gamma_F \mathcal{O}_X \longrightarrow \mathcal{O}_X \longrightarrow L_F \mathcal{O}_X \longrightarrow \Sigma\Gamma_F \mathcal{O}_X$$

by E. We claim that $L_F E = 0$ in $D_{qc}(X)$, so that the leftmost morphism in the top triangle is an isomorphism. If this is the case, then $E \in \operatorname{Loc}(F)$ by definition of $L_F E$, and as E is compact, $E \in \operatorname{Loc}(F) \cap D^{\omega}_{qc}(X) = \sqrt{F}$. That a complex is quasi-isomorphic to the trivial complex can be checked on the stalks. Since E is perfect we may apply the Künneth spectral sequence to compute the homology of $(L_F E)_x = L_F(\mathcal{O}_X)_x \otimes E_x$. First restrict the triangle to an affine open set $\operatorname{Spec} R$. Since restriction is a triangulated functor that preserves arbitrary sums and respects compact objects we have that $(\Gamma_F \mathcal{O}_X)_{|R} = \Gamma_{F_{|R}}$ and we get the triangle

$$\Gamma_{F_{|R}} R \otimes E_{|R} \longrightarrow E_{|R} \longrightarrow L_{F_{|R}} R \otimes E_{|R} \longrightarrow \Sigma\Gamma_{F_{|R}} R \otimes E_{|R}.$$

In $D(R)$, by Proposition 2.1.13, $F_{|_R}$ is cellularly equivalent to R/I for some finitely generated ideal I. An explicit description of $L_{R/I}(R)$ is provided by Dwyer-Greenlees (see the proof of Theorem 2.2.4), and from this it is immediate to check that for a point $x \in \operatorname{supph} F$, $(L_{F_{|_R}} R)_x = 0$. As a consequence the E_2 page of the Künneth spectral sequence is trivial for these points. Now, if on the contrary $x \notin \operatorname{supph} F$, then as $\operatorname{supph} E \subset \operatorname{supph} F$ we have that $E_x = 0$ by definition, and the spectral sequence is again trivial. □

Theorem 4.1.10. *For X a coherent scheme, the Zariski frame of $D^{\omega}_{\mathrm{qc}}(X)$ is the Hochster dual of the Zariski frame of X itself.*

Proof. By Lemma 4.1.7, $(\boldsymbol{Zar}(X)^{\vee}, \operatorname{supph})$ is a support. Now we invoke the universal property of the Zariski frame of $D^{\omega}_{\mathrm{qc}}(X)$ to get a unique morphism of supports

$$\boldsymbol{Zar}(D^{\omega}_{\mathrm{qc}}(X)) \xrightarrow{u} \boldsymbol{Zar}(X)^{\vee}$$

sending $\sqrt{C}$ to $\operatorname{supph}(C)$. It is surjective by Lemma 4.1.8 and injective by Lemma 4.1.9. □

4.2. **Zariski topology, structure sheaf, and reconstruction of schemes.** Theorem 4.1.10 shows that the underlying topological space of a coherent scheme X can be reconstructed from its derived category. We wish to reconstruct also the structure sheaf $\mathcal{O}_X$. The structure sheaf refers to the Zariski topology on $\operatorname{Spec} X$, not to the Hochster dual topology, so to get it we need to pass to the Hochster dual of the Zariski frame of $D^{\omega}_{\mathrm{qc}}(X)$. The key point is the standard fact that in a tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$, the endomorphism ring of the tensor unit $\operatorname{End}_{\mathcal{T}}(\mathbf{1})$ is a commutative ring, by the Eckmann-Hilton argument.

Recall that a sheaf of rings on a frame F is a functor $F^{\mathrm{op}} \to \boldsymbol{Ring}$ satisfying an exactness condition. For a coherent frame it is enough to specify the values on the finite elements (playing the role of a basis for a topology).

4.2.1. *The affine case.* For an affine scheme $X = \operatorname{Spec}_Z R$, the structure sheaf on the Zariski frame $\boldsymbol{Zar}(X) = \boldsymbol{RadId}(R)$ is completely specified by the assignment

$$\begin{array}{rcl} \boldsymbol{RadId}(R)^{\mathrm{op}} & \longrightarrow & \boldsymbol{Ring} \\ \sqrt{f} & \longmapsto & R_f, \end{array}$$

corresponding to the fact that the principal open sets $D(f) = \operatorname{Spec} R \setminus V(f)$ form a basis for the Zariski topology.

We are concerned with the coherent frame $\boldsymbol{RfGLoc}(D(R))$ and its distributive lattice of finite elements $\boldsymbol{fgRfGLoc}(D(R))$ consisting of localizing subcategories generated by a finite number of modules of the form R_f. These are both localizing and colocalizing (see Proposition 2.2.1), and we have at our disposal a Bousfield localization functor $L_{\operatorname{Loc}(R_{f_1},\ldots,R_{f_n})}$ with values in our categories $\operatorname{Loc}(R_{f_1},\ldots,R_{f_n})$. All these are naturally tensor triangulated categories as they are tensor ideals in $D(R)$ and have as tensor unit the localization of the unit in $D(R)$. The natural presheaf

$$\begin{array}{rcl} \boldsymbol{fgRfGLoc}(D(R))^{\mathrm{op}} & \longrightarrow & \boldsymbol{Ring} \\ \operatorname{Loc}(R_{f_1},\ldots,R_{f_n}) & \longmapsto & \operatorname{End}_{D(R)}(L_{\operatorname{Loc}(R_{f_1},\ldots,R_{f_n})}(R)) \end{array}$$

yields by sheafification a sheaf

$$\mathcal{E}nd : \boldsymbol{RfGLoc}(D(R))^{\mathrm{op}} \longrightarrow \boldsymbol{Ring}.$$

Proposition 4.2.2. *Under the isomorphism*

$$\boldsymbol{RfGLoc}(D(R)) \simeq \boldsymbol{RadId}(R)$$

of Theorem 2.2.16*, the sheaf* $\mathcal{E}nd$ *is canonically isomorphic to the structure sheaf on* $\operatorname{Spec}_Z R$*.*

Proof. It is enough to compute the sheaf on a basis of the topology, and for this we take the lattice of localizing subcategories generated by a single localization, corresponding to the basis of principal open sets in $\operatorname{Spec}_Z R$. We know that as tensor triangulated categories $\operatorname{Loc}(R_f) \simeq D(R_f)$ (Proposition 2.2.2), hence

$$\begin{aligned} \operatorname{End}_{D(R)}(L_{\operatorname{Loc}(R_f)}(R)) &= \operatorname{End}_{D(R_f)}(L_{\operatorname{Loc}(R_f)}(R)) \\ &= \operatorname{End}_{D(R_f)}(R_f) \\ &= R_f, \end{aligned}$$

as rings. But R_f is precisely the value of the structure sheaf of $\operatorname{Spec}_Z R$ on the principal open set $D(f) = \operatorname{Spec} R \setminus V(f)$. □

4.2.3. *Reconstruction of a general coherent scheme.* First we enlarge the framework to that of the whole derived category of complexes of modules with quasi-coherent homology $D_{\mathrm{qc}}(X)$. It follows from Corollary 1.1.3 that we have an isomorphism of posets between the poset of localizing subcategories of $D_{\mathrm{qc}}(X)$ generated by a single perfect complex and the poset of thick subcategories of $D^{\omega}_{\mathrm{qc}}(X)$ generated by a single perfect complex, via the assignment $\mathcal{L} \mapsto \mathcal{L} \cap D^{\omega}_{\mathrm{qc}}(X)$. Since $\mathcal{O}_X$ generates $D_{\mathrm{qc}}(X)$ as a localizing category, all localizing subcategories are tensor ideals by Lemma 1.1.8, and hence also all the thick subcategories are thick tensor ideals. And since all perfect complexes are strongly dualizable, all thick tensor ideals are radical thick tensor ideals. Altogether we have an isomorphism of posets between the localizing subcategories of $D_{\mathrm{qc}}(X)$ generated by a single perfect complex and the Zariski lattice $\boldsymbol{Zar}(D^{\omega}_{\mathrm{qc}}(X))^{\omega} = \{\sqrt{C} \mid C \in D^{\omega}_{\mathrm{qc}}(X)\}$ of principal radical thick tensor ideals, i.e. the distributive lattice of finite elements in $\boldsymbol{Zar}(D^{\omega}_{\mathrm{qc}}(X))$.

We know that the Hochster dual of this lattice is the basis of the topology of X given by the quasi-compact open sets in X. To flip this lattice as in the affine case we take right orthogonals. The relations between right and left orthogonals for (co)localizing subcategories as stated for instance in [20, Prop 4.9.1 and 4.10.1] imply that we have order-reversing inverse bijections:

$$\big\{ \operatorname{Loc}(C) \mid C \in D^{\omega}_{\mathrm{qc}}(X) \big\} \underset{{}^{\perp}(-)}{\overset{(-)^{\perp}}{\rightleftarrows}} \big\{ \operatorname{Loc}(C)^{\perp} \mid C \in D^{\omega}_{\mathrm{qc}}(X) \big\}.$$

We therefore have:

Proposition 4.2.4. *Let* X *be a coherent scheme. There is a canonical isomorphism between the distributive lattice* $\big\{ \operatorname{Loc}(C)^{\perp} \mid C \in D^{\omega}_{\mathrm{qc}}(X) \big\}$ *and the Zariski lattice of* X *(i.e. the lattice of quasi-compact open sets in* X*).*

To reconstruct the sheaf we proceed again as in the affine case. The categories $\operatorname{Loc}(C)^{\perp}$ are localizing as they are the right orthogonal categories to a compact object (cf. Proposition 2.2.1). By Lemma 1.1.8, they are tensor ideals, and we may apply Bousfield localization techniques; see for instance [20]. The localization of the tensor unit $\mathcal{O}_X$ at $\operatorname{Loc}(C)^{\perp}$, namely $L_{C^{\perp}}(\mathcal{O}_X)$, is the tensor unit in this tensor triangulated category; its ring of endomorphisms is a commutative ring, and we get a presheaf

$$\begin{array}{rcl} \left\{ \mathrm{Loc}(C)^{\perp} \mid C \in D^{\omega}_{\mathrm{qc}}(X) \right\}^{\mathrm{op}} & \longrightarrow & \boldsymbol{Ring} \\ \mathrm{Loc}(C)^{\perp} & \longmapsto & \mathrm{End}_{\mathrm{Loc}(C)^{\perp}}(L_{C^{\perp}}(\mathcal{O}_X)). \end{array}$$

Sheafification of this presheaf defines the sheaf $\mathcal{E}nd$.

Finally we get the reconstruction theorem, slightly generalizing that proved by Balmer [3], who did the special case where X is topologically noetherian:

Theorem 4.2.5. *Under the isomorphism $\boldsymbol{Zar}(D^{\omega}_{\mathrm{qc}}(X))^{\vee} \simeq \boldsymbol{Zar}(X)$ of Theorem* 4.1.10*, the sheaf $\mathcal{E}nd$ is canonically isomorphic to the structure sheaf on X.*

Proof. The isomorphism of sheaves can be checked on the subbasis of affine open subsets, whence we reduce to the case of Proposition 4.2.2. ☐

4.2.6. *The domain sheaf.* An affine scheme $X = \mathrm{Spec}\, R$ also has a natural sheaf for the Hochster dual topology, given by sheafification of the presheaf

$$\begin{array}{rcl} (\boldsymbol{RadfgId}^{\mathrm{op}})^{\mathrm{op}} & \longrightarrow & \boldsymbol{Ring} \\ I & \longmapsto & R/I. \end{array}$$

Note that while the usual structure sheaf for the Zariski topology is a local-ring object in the petit Zariski topos, the structure sheaf for the Hochster dual topology is instead a domain object [15, V.4]. (Or in terms of points: the stalk of this sheaf at a prime $\mathfrak{p}$ is the domain $R/\mathfrak{p}$.)

Also the domain sheaf of $\mathrm{Spec}_H\, R$ can be reconstructed from the derived category $D(R)$ simply by copying over the definition of the sheaf as sheafification of the presheaf

$$\begin{array}{rcl} \boldsymbol{fgCGLoc}(D(R))^{\mathrm{op}} & \longrightarrow & \boldsymbol{Ring} \\ \mathrm{Loc}(R/I) & \longmapsto & R/I. \end{array}$$

In principle this local description can be globalized to account for some notion of scheme defined as "ringed space which is locally the domain spectrum of a commutative ring". Having no feeling for this notion, we postpone further investigations of this point.

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