

FINITE PRIMITIVE GROUPS AND REGULAR ORBITS OF GROUP ELEMENTS

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ABSTRACT. We prove that if G is a finite primitive permutation group and if g is an element of G , then either g has a cycle of length equal to its order, or for some r , m and k , the group $G \leq \text{Sym}(m) \text{wrSym}(r)$ preserves the product structure of r direct copies of the natural action of $\text{Sym}(m)$ on k -sets. This gives an answer to a question of Siemons and Zalesski and a solution to a conjecture of Giudici, Praeger and the second author.

1. INTRODUCTION

Let G be a finite primitive permutation group on a set Ω and let $H \leq G$. A *regular orbit* of H in Ω is an H -orbit of cardinality $|H|$. In particular, when $H = \langle g \rangle$, a regular orbit of H is the point set of a g -cycle of length equal to the order $|g|$ of g in its disjoint cycle representation; we call such a cycle a *regular cycle*. Motivated by a problem in representation theory, Siemons and Zalesski [26] asked for conditions under which subgroups of primitive groups could be guaranteed to have regular orbits.

Following the preliminary work of Emmett, Siemons and Zalesski [12, 26, 27], Giudici, Praeger and the second author [14] have investigated this question for cyclic subgroups and have noticed that (surprisingly) primitive groups admitting elements with no regular orbits are very uncommon.

The alternating groups $\text{Alt}(m)$ (with $m \geq 7$) and the symmetric groups $\text{Sym}(m)$ (with $m \geq 5$) are examples of groups admitting elements with no regular orbit. (For instance, the permutation $g = (1\ 2\ 3)(4\ 5)(6\ 7) \in \text{Alt}(m)$ has no cycle of length $|g| = 6$ in its natural action on m points.) We denote the collection of the k -subsets of $\{1, \dots, m\}$ by $\binom{m}{k}$ and we refer to the natural action of $\text{Sym}(m)$ on $\binom{m}{k}$ as its k -set action. It is easy to see that, for k fixed and for m large compared to k , the group $\text{Sym}(m)$ in its k -set action is another permutation group admitting elements with no regular orbit. Actually, it is possible to specify precisely the k -set actions of finite symmetric groups for which all elements have regular cycles [14, Theorem 1.1].

Loosely speaking the main result of this paper shows that the groups above are the building blocks of every primitive permutation group admitting elements with no regular cycle.

Received by the editors June 5, 2014 and, in revised form, December 29, 2014 and February 2, 2015.

2010 *Mathematics Subject Classification*. Primary 20B15, 20H30.

Key words and phrases. Cycle lengths, element orders, primitive groups.

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Theorem 1.1. *Let G be a finite primitive permutation group on Ω and let $g \in G$. Then either g has a regular cycle, or there exist integers $k \geq 1$, $r \geq 1$ and $m \geq 5$ such that G preserves a product structure on $\Omega = \Delta^r$ and $\text{Alt}(m)^r \trianglelefteq G \leq \text{Sym}(m) \text{ wr } \text{Sym}(r)$, where $\Delta = \binom{m}{k}$ and $\text{Sym}(m)$ induces its natural k -set action on Δ .*

Our work gives a complete answer to the question: “When do all elements of a finite primitive permutation group G have at least one regular cycle?” In particular, we offer a precise answer to the question of Siemons and Zalesski [26] in the case of cyclic subgroups.

Theorem 1.1 was conjectured [14, Conjecture 1.2] by Giudici, Praeger and the second author. Using the O’Nan-Scott reduction theorem for finite primitive permutation groups, the authors of [14] made a significant contribution towards a proof of Theorem 1.1. In fact, in [14, Theorem 1.3] they have shown that Theorem 1.1 holds true, if it holds true for each almost simple primitive group with socle a non-abelian simple classical group. Therefore Theorem 1.1 follows from [14, Theorem 1.3] and the following.

Theorem 1.2. *Let G be a finite almost simple primitive group on Ω with socle G_0 a non-abelian simple classical group, and let $g \in G$. Then either g has a regular cycle, or (G_0, Ω) is isomorphic to one of the following: $(\text{Alt}(5), \{1, \dots, 5\})$, $(\text{Alt}(6), \{1, \dots, 6\})$ or $(\text{Alt}(8), \{1, \dots, 8\})$.*

Theorem 1.1 is yet another result showing that the primitive group $\text{Sym}(m) \text{ wr } \text{Sym}(r)$ endowed with its natural product action has rather peculiar properties (see for example [9, Section 6]). Since it is computationally quite cheap to determine the order and the cycle lengths of a permutation, Theorem 1.1 has some practical interest and can be easily implemented as part of a recognition algorithm for primitive permutation groups [2]. In practice Theorem 1.1 can be used to determine how much effort should be invested into running the “Jellyfish algorithm” [21, 24] on a primitive group.

Theorem 1.1 also has the following striking corollary.

Corollary 1.3. *Let G be a finite primitive permutation group on Ω and let $g \in G$. Then either $|g| \leq |\Omega|$, or there exist integers $k \geq 1$, $r \geq 1$ and $m \geq 5$ such that G preserves a product structure on $\Omega = \Delta^r$ and $\text{Alt}(m)^r \trianglelefteq G \leq \text{Sym}(m) \text{ wr } \text{Sym}(r)$, where $\Delta = \binom{m}{k}$ and $\text{Sym}(m)$ induces its natural k -set action on Δ .*

2. PRELIMINARY RESULTS

We start our analysis with some elementary (but very useful) lemmas.

Lemma 2.1 ([14, Lemma 2.1]). *Let G be a permutation group on Ω . Assume that for every $g \in G$, with $|g|$ square-free, g has a regular cycle. Then, for every $g \in G$, the element g has a regular cycle.*

In the proof of Theorem 1.2, the previous lemma allows us to consider only elements of square-free order. Before proceeding, we introduce a definition.

Definition 2.2. Let G be a finite permutation group on the finite set Ω and let $x \in G$. We let $\text{Fix}_\Omega(x) = \{\omega \in \Omega \mid \omega^x = \omega\}$ and $\text{fpr}_\Omega(x) = |\text{Fix}_\Omega(x)|/|\Omega|$. (The label “fpr” stands for “fixed-point-ratio”.)

We recall a basic lemma on fixed-point-ratios.

Lemma 2.3 ([22, Lemma 2.5]). *Let G be a transitive permutation group on Ω , let H be a point stabilizer and let $g \in G$. Then*

$$\text{fpr}_\Omega(g) = \frac{|g^G \cap H|}{|g^G|}.$$

The next lemma is one of the main tools used in this paper.

Lemma 2.4 ([14, Lemma 2.4]). *Let G be a transitive permutation group on Ω and let g be in $G \setminus \{1\}$. The element g has a regular cycle if and only if*

$$\bigcup_{\substack{r||g| \\ r \text{ prime}}} \text{Fix}_\Omega(g^{|g|/r}) \subsetneq \Omega.$$

In particular, if

$$\sum_{\substack{r||g| \\ r \text{ prime}}} \text{fpr}_\Omega(g^{|g|/r}) < 1,$$

then g has a regular cycle.

In light of Lemma 2.4, we define $S(g, \Omega)$ to be

$$S(g, \Omega) := \sum_{\substack{r||g| \\ r \text{ prime}}} \text{fpr}_\Omega(g^{|g|/r})$$

and in order to prove Theorem 1.2, we will show that (in most cases) $S(g, \Omega) < 1$ for every $g \in G \setminus \{1\}$.

Unless specified otherwise, for the rest of this paper we will use the following notation.

Notation 2.5. We let G denote a finite almost simple classical group defined over the finite field of size q and with socle G_0 . For twisted groups our notation for q is such that $\text{PSU}_n(q)$ and $\text{P}\Omega_{2m}^-(q)$ are the twisted groups contained in $\text{PSL}_n(q^2)$ and $\text{P}\Omega_{2m}^+(q^2)$, respectively.

We write $q = p^e$, for some prime p and some $e \geq 1$, and we define

$$q_0 := \begin{cases} q^2 & \text{if } G \text{ is unitary,} \\ q & \text{otherwise.} \end{cases}$$

We let V be the natural module defined over the field \mathbb{F}_{q_0} of size q_0 for the covering group of G_0 , and we let n be the dimension of V over \mathbb{F}_{q_0} . Finally, we denote by m the Witt index of G_0 , that is, the maximal dimension of a totally singular subspace of V .

We now recall a fundamental and pioneering result of Liebeck and Saxl [22, Theorem 1]:

Theorem 2.6. *Let G and q be as in Notation 2.5. If G acts transitively and faithfully on a set Ω , then for every $x \in G \setminus \{1\}$ we have*

$$\text{fpr}_\Omega(x) \leq \frac{4}{3q}$$

apart from the exceptions listed in [22, Table 1].

Remark 2.7. Unfortunately (for our application) the exceptions in [22, Table 1] are all genuine. For each such case, however, [22, Table 1] lists all possible G -sets Ω and all possible group elements x with $\text{fpr}_\Omega(x) > 4/(3q)$. Furthermore, the third column of [22, Table 1] gives an exact formula for $\text{fpr}_\Omega(x)$ in each case.

2.1. Number theory. We recall that in elementary number theory, for a positive integer n , the number of distinct prime divisors of n is denoted by $\omega(n)$. We will find the following result of Robin useful.

Lemma 2.8 ([25, Theorem 13]). *For $n \geq 26$, we have*

$$\omega(n) \leq \log(n)/(\log(\log(n)) - 1.1714).$$

The next lemma is much more elementary and we omit its proof.

Lemma 2.9. *For $q \geq 2$, we have*

$$\sum_{\ell=0}^{\infty} \ell q^{-\ell} = \frac{q}{(q-1)^2}.$$

3. CLASSICAL GROUPS OF SMALL DIMENSION

We start by introducing some more notation that will prove useful for the proof of Theorems 3.1 and 4.3. Let $|\text{Aut}(G_0)|_{p'}$ denote the p' -part of $|\text{Aut}(G_0)|$ and set

$$a(n, q) := 1 + \frac{\log(|\text{Aut}(G_0)|_{p'})}{\log(\log(|\text{Aut}(G_0)|_{p'}) - 1.1714)}.$$

Comprehensive information on $|G_0|$ and $|\text{Aut}(G_0)|$ can be found in [11, Table 5, page xvi]. Observe that, once the Lie type of G_0 is fixed, $a(n, q)$ is indeed an explicit function of n and q . Suppose that $G_0 \not\cong \text{PSL}_2(4)$, $\text{PSL}_2(7)$, and observe that $|\text{Aut}(G_0)|_{p'} \geq 26$. By Lemma 2.8, we have

$$(1) \quad \omega(|\text{Aut}(G_0)|) = 1 + \omega(|\text{Aut}(G_0)|_{p'}) \leq 1 + \frac{\log(|\text{Aut}(G_0)|_{p'})}{\log(\log(|\text{Aut}(G_0)|_{p'}) - 1.1714)} = a(n, q),$$

where the summand 1 corresponds to the prime divisor p of $|\text{Aut}(G_0)|$.

Now we prove a preliminary theorem that will simplify some of our arguments later and will also provide a taste of the arguments used in the rest of the paper.

Theorem 3.1. *Let G, G_0 and n be as in Notation 2.5 with $n \leq 4$. Suppose that G acts primitively on Ω and let $g \in G$. Then either g has a regular cycle, or (G_0, Ω) is isomorphic to $(\text{Alt}(5), \{1, \dots, 5\})$, $(\text{Alt}(6), \{1, \dots, 6\})$ or $(\text{Alt}(8), \{1, \dots, 8\})$.*

Proof. Using the various isomorphisms between classical groups, we may assume that G_0 is either $\text{PSL}_2(q)$ with $q \geq 5$, or $\text{PSL}_3(q)$ with $q \geq 3$, or $\text{PSL}_4(q)$ with $q \geq 2$, or $\text{PSU}_3(q)$ with $q \geq 3$, or $\text{PSU}_4(q)$ with $q \geq 2$, or $\text{PSP}_4(q)$ with $q \geq 4$.

We start by explaining our strategy. Let $A = \text{Aut}(G_0)$ so that, in particular, we may assume that $G_0 \leq G \leq A$.

For now we suppose that $G_0 \not\cong \text{PSL}_2(q)$, $\text{PSL}_4(2)$, or $\text{PSU}_4(2)$. In particular, from [22, Table 1], we have $\text{fpr}_\Omega(g^{|g|/r}) \leq 4/(3q)$, for each prime r dividing $|g|$. Thus from (1) we get

$$(2) \quad S(g, \Omega) \leq \sum_{\substack{r||g| \\ r \text{ prime}}} \frac{4}{3q} = \omega(|g|) \frac{4}{3q} \leq \omega(|A|) \frac{4}{3q} \leq a(n, q) \frac{4}{3q}.$$

TABLE 1. Exceptional cases in the proof of Theorem 3.1

Group G_0	Comments
$\mathrm{PSL}_n(q)$	$(n, q) \in \{(2, 5), (2, 7), (2, 8), (2, 9), (2, 11), (2, 16), (2, 19), (3, 3), (3, 4), (3, 5), (4, 2), (4, 3), (4, 4), (4, 5), (4, 8)\}$
$\mathrm{PSU}_n(q)$	$(n, q) \in \{(3, 3), (3, 4), (3, 5), (4, 2), (4, 3), (4, 4), (4, 5), (4, 8)\}$
$\mathrm{PSP}_{2m}(q)$	$(m, q) \in \{(2, 4), (2, 5)\}$

In particular, if $4a(n, q) < 3q$, then Lemma 2.1 implies that g has a regular cycle on Ω . Now we check with a computer (implementing the function $a(n, q)$) that there are only a finite number of pairs (n, q) such that $4a(n, q) \geq 3q$. For each of the remaining groups, we compute the exact value of $\omega(|A|)$ and show that either $4\omega(|A|) < 3q$ or G_0 is one of the groups in Table 1.

When $G_0 \cong \mathrm{PSL}_2(q)$, $\mathrm{PSL}_4(2)$, or $\mathrm{PSU}_4(2)$, we substitute the upper bound $4/(3q)$ with the amended upper bound in [22, Table 1] and follow the calculation above. For each of these groups, we show that either $S(g, \Omega) < 1$ or G_0 is one of the groups in Table 1.

The remaining groups G_0 in Table 1 are rather small and we can compute the exact value of $\max\{\omega(|g|) \mid g \in \mathrm{Aut}(G_0)\}$. Using this value in (2) (in place of $a(n, q)$) and using the upper bound for $\mathrm{fpr}_\Omega(g^{|g|/r})$ in [22, Theorem 1], we have $S(g, \Omega) < 1$ and g has a regular orbit, or G_0 is isomorphic to one of $\mathrm{PSL}_2(5)$, $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(9)$, $\mathrm{PSL}_4(2)$, $\mathrm{PSL}_4(4)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_4(4)$.

Finally, for each group G_0 in this short list, we compute for every G with $G_0 \leq G \leq \mathrm{Aut}(G_0)$ all the primitive permutation representations of G and check whether every conjugacy class representative has a regular orbit: the only exceptions are the examples in the statement of this theorem. \square

4. NON-SUBSPACE ACTIONS OF CLASSICAL GROUPS

In studying actions of classical groups, it is rather natural to distinguish between those actions which permute the subspaces of the natural module and those which do not. The stabilizers of subspaces are generally rather large (every parabolic subgroup falls into this class) and therefore the fixed-point-ratio in these cases also tends to be rather large. As the culmination of an important series of papers [4–7], Burness obtained remarkably good upper bounds on the fixed-point-ratio for each finite almost simple classical group in *non-subspace actions*. For future reference, we first need to make the definition of non-subspace action precise.

Definition 4.1. Assume Notation 2.5. A subgroup H of G is a *subspace subgroup* if, for each maximal subgroup M of G_0 that contains $H \cap G_0$, one of the following conditions hold:

- (a): M is the stabilizer in G_0 of a proper non-zero subspace U of V , where U is totally singular, or non-degenerate, or, if G_0 is orthogonal and $p = 2$, a non-singular 1-subspace (U can be any subspace if $G_0 = \mathrm{PSL}(V)$);
- (b): $M = \mathrm{O}_{2m}^\pm(q)$ and $(G_0, p) = (\mathrm{Sp}_{2m}(q)', 2)$.

A transitive action of G on a set Ω is a *subspace action* if the point stabilizer G_ω of the element $\omega \in \Omega$ is a subspace subgroup of G ; *non-subspace subgroups* and *actions* are defined accordingly.

For the convenience of the reader we report [4, Theorem 1].

Theorem 4.2 ([4, Theorem 1]). *Let G be a finite almost simple classical group acting transitively and faithfully on a set Ω with point stabilizer $G_\omega \leq H$, where H is a maximal non-subspace subgroup of G . Let G_0 be the socle of G . Then, for every $x \in G \setminus \{1\}$, we have*

$$\text{fpr}_\Omega(x) < |x^G|^{-\frac{1}{2} + \frac{1}{n} + \iota},$$

where x^G is the conjugacy class of x in G , n is as defined in [4, Definition 2] and either $\iota = 0$ or (G_0, H, ι) is listed in [4, Table 1].

In Theorem 4.2, apart from the two rather natural exceptions $\text{PSL}_3(2)$ and $\text{PSp}_4(2)'$ (where $n = 2$), n is exactly as in Notation 2.5.

The upper bound in Theorem 4.2 is quite sharp when n is large and is extremely useful for our application. However, for small values of n , Theorem 4.2 loses all of its power. For instance, when $G_0 = \text{PSL}_4(q)$, we see from [4, Table 1] that $\iota = 1/4$ and hence the upper bound in Theorem 4.2 only says $\text{fpr}_\Omega(x) \leq |x^{G_0}| = 1$. In these cases, we will often revert to the bounds in Theorem 2.6. However, we point out that recently Burness and the first author [8] have strengthened Theorem 4.2 for linear groups of very small rank.

We are now ready to make our first important contribution towards the proof of Theorem 1.2.

Theorem 4.3. *Assume Notation 2.5. Let G be acting primitively with a non-subspace action on a set Ω and let $g \in G$. Then either g has a regular cycle, or (G_0, Ω) is isomorphic to $(\text{Alt}(5), \{1, \dots, 5\})$, $(\text{Alt}(6), \{1, \dots, 6\})$ or $(\text{Alt}(8), \{1, \dots, 8\})$.*

Proof. As in Theorem 3.1, the proof makes heavy use of a computer. Observe that by Theorem 3.1, we may assume that $n \geq 5$.

Let $m(G_0)$ denote the minimal degree of a non-trivial transitive permutation representation of G_0 . Clearly, for every $x \in G \setminus \{1\}$, we have

$$|x^G| = |G : \mathbf{C}_G(x)| \geq |G_0 \mathbf{C}_G(x) : \mathbf{C}_G(x)| = |G_0 : \mathbf{C}_{G_0}(x)| \geq m(G_0).$$

The explicit value for $m(G_0)$ (as a function of n and q) can be found in [20, Table 5.2.A]. (We note that the rows corresponding to the classical groups $\text{P}\Omega_{2m}^+(q)$ and $\text{PSU}_{2m}(2)$ in [20, Table 5.2.A] are incorrect as brilliantly observed by Vasil'ev and Mazurov in [23]. For these rows we refer to the amended values in [23]. A table listing $m(G_0)$ for each non-abelian simple group and taking into account all the known corrections can be found in [16].)

Denote by $t(G_0)$ the upper bound for $\text{fpr}_\Omega(x)$ obtained in Theorem 2.6 (so, typically $t(G_0) = 4/(3q)$ and the exceptional cases, together with the corresponding amended upper bound, are found in [22, Table 1]; see Remark 2.7). Let

$$t(n, q) := \min \left\{ t(G_0), m(G_0)^{-\frac{1}{2} + \frac{1}{n} + \iota} \right\},$$

and observe that, for each almost simple classical group, $t(n, q)$ is an explicit function of n and q .

TABLE 2. Exceptional cases in the proof of Theorem 4.3

Group G_0	Comments
$\mathrm{PSL}_n(q)$	$(n, q) \in \{(5, 2), (5, 3), (5, 4), (6, 2), (6, 3), (6, 4), (6, 5), (6, 7), (6, 8), (6, 9), (6, 11), (7, 2), (8, 2), (8, 3), (10, 2)\}$
$\mathrm{PSU}_n(q)$	$(n, q) \in \{(6, 2), (6, 3)\}$
$\mathrm{PSp}_{2m}(q)$	$(m, q) \in \{(3, 2), (3, 3), (3, 4), (3, 5), (3, 7), (3, 8), (3, 9), (4, 2), (5, 2)\}$
$\mathrm{P}\Omega_{2m+1}(q)$	$(m, q) = (3, 3)$
$\mathrm{P}\Omega_{2m}^+(q)$	$(m, q) \in \{(4, 2), (4, 3), (5, 2)\}$
$\mathrm{P}\Omega_{2m}^-(q)$	$(m, q) \in \{(4, 2), (4, 3), (4, 4), (5, 2)\}$

Let $A = \mathrm{Aut}(G_0)$. By the definition of $t(n, q)$ and Theorems 2.6 and 4.2, we have $\mathrm{fpr}_\Omega(g^{|g|/r}) \leq t(n, q)$ for each prime divisor r of $|g|$. Thus by (1) we get

$$(3) \quad S(g, \Omega) \leq \sum_{\substack{r||g| \\ r \text{ prime}}} t(n, q) = \omega(|g|)t(n, q) \leq \omega(|A|)t(n, q) \leq a(n, q)t(n, q).$$

In particular, if $a(n, q)t(n, q) < 1$, then Lemma 2.1 implies that g has a regular cycle in its action on Ω . Using a computer we find the pairs (n, q) such that $a(n, q)t(n, q) \geq 1$. For each of these remaining groups, we compute the exact value of $\omega(|A|)$ and we find that either $\omega(|A|)t(n, q) < 1$ or G_0 is one of the groups in Table 2.

Using **magma** [2], for each G_0 in Table 2, we can compute $\max\{\omega(|g|) \mid g \in \mathrm{Aut}(G_0)\}$. By using this value as an upper bound for $\omega(|g|)$ in (3) and by arguing as in the previous paragraph, we see that either g has a regular orbit on Ω or G_0 is one of the following groups:

$$\mathrm{PSL}_6(2), \mathrm{PSL}_6(3), \mathrm{PSL}_6(4), \mathrm{PSL}_6(5), \mathrm{PSU}_6(2), \\ \mathrm{PSp}_6(2), \mathrm{PSp}_6(3), \mathrm{P}\Omega_8^+(2), \mathrm{P}\Omega_8^+(3), \mathrm{P}\Omega_8^-(2).$$

For $G_0 = \mathrm{P}\Omega_8^+(3)$ we use the full strength of Theorem 4.2: for each prime divisor r of $|g|$ we have $\mathrm{fpr}_\Omega(g^{|g|/r}) \leq |x_r^G|^{-1/2+1/n+\iota}$ where $x_r := g^{|g|/r}$. In particular, using [4, Table 1], for any non-subspace action we have $\iota \leq 0.219$ and thus $\mathrm{fpr}_\Omega(g^{|g|/r}) \leq |x_r^G|^{-0.156}$. Let $\alpha(g, r) := |x_r^G|^{-0.156}$. We can check in **magma** that for every almost simple G with socle $\mathrm{P}\Omega_8^+(3)$, and every $g \in G$ of square-free order, the sum $\sum \alpha(g, r)$ (over all prime divisors r of $|g|$) is less than 1. Therefore the case $G_0 = \mathrm{P}\Omega_8^+(3)$ is eliminated. The group $G_0 = \mathrm{PSL}_6(5)$ is dealt with in exactly the same fashion.

Finally, for each remaining group G_0 , we construct in **magma** all of the primitive permutation representations of every almost simple group G with socle G_0 and check directly that every conjugacy class representative has a regular cycle. \square

5. A REDUCTION FOR SUBSPACE ACTIONS

In view of Theorems 3.1 and 4.3 we may now consider only subspace actions of almost simple classical groups G and we may assume that $n \geq 5$.

Remark 5.1. Let G be an almost simple group as above acting on the set Ω and let $\alpha \in \Omega$. Now there are two main cases to consider: either $G_\alpha \cap G_0$ is a maximal subgroup of G_0 , or $G_\alpha \cap G_0$ is not maximal in G_0 . In the first case, $G_\alpha \cap G_0$ is

one of the subgroups given in Definition 4.1 and hence we may identify the action of G on Ω with the natural extension to G of the action of G_0 on the coset space $G_0/(G_\alpha \cap G_0)$. (We give an example of this case: $G = \mathrm{PGL}_n(q)$, $G_0 = \mathrm{PSL}_n(q)$ and $G_\alpha \cap G_0$ is a maximal parabolic subgroup of G_0 stabilizing a k -dimensional subspace of the natural module $V = \mathbb{F}_q^n$. Clearly, we may identify the action of G on Ω with the action of G on the k -dimensional subspaces of V .) In particular, in this first case the action of G is perfectly understood.

Now suppose that $G_\alpha \cap G_0$ is not maximal in G_0 and let M be a maximal subgroup of G_0 with $G_\alpha \cap G_0 < M$. Clearly $\mathbf{N}_G(G_\alpha \cap G_0) = G_\alpha \not\leq \mathbf{N}_G(M)$ and hence (using the terminology in [20, page 66]) the subgroup $G_\alpha \cap G_0$ is a G -novelty. All such novelties were classified for $n \geq 13$ in [20, Table 3.5H]; we will make use of this table to identify the group G_α and the action of G on Ω .

Next, we state a definition, which will play a crucial role in this paper.

Definition 5.2. Let G be a finite group, and let $\mathrm{Irr}(G)$ denote the set of complex irreducible characters of G and let $\langle \cdot, \cdot \rangle_G$ denote the natural inner-product on the class functions of G . Given complex characters π and π' of G , we write $\pi \leq \pi'$ if, for each $\eta \in \mathrm{Irr}(G)$, we have $\langle \pi, \eta \rangle_G \leq \langle \pi', \eta \rangle_G$. In other words, writing

$$\pi = \sum_{\eta \in \mathrm{Irr}(G)} a_\eta \eta \quad \text{and} \quad \pi' = \sum_{\eta \in \mathrm{Irr}(G)} a'_\eta \eta,$$

we have $\pi \leq \pi'$ if and only if $a_\eta \leq a'_\eta$ for every $\eta \in \mathrm{Irr}(G)$.

Given a subgroup H of G and a character π of G , we write $\pi|_H$ for the restriction of π to H .

Lemma 5.3. *Let G be a finite group with two permutation representations on the finite sets Ω_1 and Ω_2 . For $i \in \{1, 2\}$, let π_i be the permutation character for the action of G on Ω_i , and suppose that $\pi_1 \leq \pi_2$. If $g \in G$ has ℓ cycles of length $|g|$ on Ω_1 , then g has at least ℓ cycles of length $|g|$ on Ω_2 .*

Proof. The proof of this result is somehow implicit in [26, Theorem 3.4]. Let $g \in G$, let ℓ be the number of cycles of g of length $|g|$ on Ω_1 and let ρ be the regular character of the cyclic group $\langle g \rangle$.

By [26, Corollary 2.5], we see that $\ell\rho \leq (\pi_1)|_{\langle g \rangle}$. Since $\pi_1 \leq \pi_2$, we have $\ell\rho \leq (\pi_2)|_{\langle g \rangle}$. Now another application of [26, Corollary 2.5] implies that g has at least ℓ orbits of length $|g|$ on Ω_2 . \square

In a nutshell, Lemma 5.3 shows that in order to prove Theorem 1.2 we need only consider those actions that have permutation character that is minimal with respect to the partial ordering in Definition 5.2.

In the following, we use the term “totally singular” instead of separating into “totally isotropic” or “totally singular” subspaces according to the type of V . We use the term “non-singular” to mean having zero radical. For the benefit of the reader we report here [13, Corollary 1.2].

Theorem 5.4 ([13, Corollary 1.2]). *Let G be an almost simple classical group with natural module V of dimension n . If G is linear, assume that G does not contain a graph automorphism. Let $k \in \{2, \dots, n-1\}$, and let K be the stabilizer of a*

non-degenerate or totally singular k -subspace of V . Let P be the stabilizer of a singular 1-subspace of V . Then $1_P^G \leq 1_K^G$ unless one of the following holds:

- (1): K is the stabilizer of a maximal totally singular subspace of V .
- (2): K is the stabilizer of a 2-subspace containing no singular points or the orthogonal complement of such.

Theorems 4.3, 5.4, Lemma 5.3 and Remark 5.1 imply the following reduction.

Proposition 5.5. *Assume Notation 2.5. Theorem 1.2 holds if it holds for the following cases of subspace G -actions on Ω :*

- Case (i):** Ω consists of totally singular 1-subspaces;
- Case (ii):** Ω consists of non-degenerate 1-subspaces;
- Case (iii):** Ω consists of maximal totally singular subspaces;
- Case (iv):** Ω consists of non-degenerate 2-subspaces containing no singular points;
- Case (v):** $G_0 = \text{PSL}_n(q)$ and G contains a graph automorphism;
- Case (vi):** $G_0 = \text{Sp}_n(q)$ with q even, and the stabilizer in G_0 of a point is isomorphic to $O_n^\pm(q)$;
- Case (vii):** $G_\alpha \cap G_0$ is a G -novelty and G_α is a subspace subgroup, where G_α is the stabilizer of the point α .

Remark 5.6. Recall that we are assuming $n \geq 5$. As $\text{P}\Omega_5(q) \cong \text{P}\text{Sp}_4(q)$, from Theorem 3.1 we may assume in the rest of this paper that $n \geq 7$ when $G_0 = \text{P}\Omega_n(q)$. Similarly, as $\text{P}\Omega_6^+(q) \cong \text{PSL}_4(q)$ and $\text{P}\Omega_6^-(q) \cong \text{PSU}_4(q)$, we may assume that $n \geq 8$ when $G_0 = \text{P}\Omega_n^\varepsilon(q)$ with $\varepsilon \in \{+, -\}$.

Remark 5.7. We recall a few standard facts on the subspace actions in **Cases (i)–(iii)**; for some more detailed information see for example [10].

The group G_0 acts primitively on the totally singular 1-subspaces of V . In particular, in **Case (i)**, the set Ω is the whole set of totally singular 1-subspaces.

The same comment applies in **Case (ii)** except when G_0 is orthogonal and q is odd. When G_0 is orthogonal and q is odd, G_0 has two orbits on non-degenerate 1-subspaces. Following [18, Section 3.2], we denote these two G_0 -orbits by NS_1^+ and NS_1^- , and we write $\text{NS}_1 = \text{NS}_1^+ \cup \text{NS}_1^-$. If Q is the quadratic form on V for G_0 , then $\text{NS}_1^+ = \{\langle v \rangle \mid v \in V \setminus \{0\}, Q(v) \text{ is a square in } \mathbb{F}_{q_0}\}$ and $\text{NS}_1^- = \{\langle v \rangle \mid v \in V \setminus \{0\}, Q(v) \text{ is not a square in } \mathbb{F}_{q_0}\}$. Moreover,

$$|\text{NS}_1^+| = \begin{cases} q^{(n-1)/2}(q^{(n-1)/2} + 1)/2 & \text{if } G_0 = \text{P}\Omega_n(q), nq \text{ odd,} \\ q^{n/2-1}(q^{n/2} - 1)/2 & \text{if } G_0 = \text{P}\Omega_n^+(q), n \text{ even, } q \text{ odd,} \\ q^{n/2-1}(q^{n/2} + 1)/2 & \text{if } G_0 = \text{P}\Omega_n^-(q), n \text{ even, } q \text{ odd,} \end{cases}$$

$$|\text{NS}_1^-| = \begin{cases} q^{(n-1)/2}(q^{(n-1)/2} - 1)/2 & \text{if } G_0 = \text{P}\Omega_n(q), nq \text{ odd,} \\ q^{n/2-1}(q^{n/2} - 1)/2 & \text{if } G_0 = \text{P}\Omega_n^+(q), n \text{ even, } q \text{ odd,} \\ q^{n/2-1}(q^{n/2} + 1)/2 & \text{if } G_0 = \text{P}\Omega_n^-(q), n \text{ even, } q \text{ odd.} \end{cases}$$

When n is odd, $G_0 = \text{P}\Omega_n(q)$ and $\eta \in \{+, -\}$, the stabilizer of a point in NS_1^η is isomorphic to $O_1(q) \perp O_{2m}^\eta(q)$ (these points are usually referred to as η points, as in [11]). When n is even, $G_0 = \text{P}\Omega_n^\varepsilon(q)$, $\varepsilon \in \{+, -\}$ and q is odd, we have $|\text{NS}_1^+| = |\text{NS}_1^-|$ and the setwise stabilizer H in $\text{PGO}_n^\varepsilon(q)$ of NS_1^+ and NS_1^- has index 2 in $\text{PGO}_n^\varepsilon(q)$. Moreover, $H = \text{Inndiag}(G_0)$ and $H \cap \text{PO}_n^\varepsilon(q) = \text{PSO}_n^\varepsilon(q)$. In particular, in this latter case we must have $G \leq \langle H, \mathcal{F} \rangle$ (where \mathcal{F} is the group

of field automorphisms); otherwise G would be transitive on $\mathbf{NS}_1^+ \cup \mathbf{NS}_1^- = \mathbf{NS}_1$ with $\{\mathbf{NS}_1^+, \mathbf{NS}_1^-\}$ being a G -invariant system of imprimitivity, and hence we do not have a primitive action.

Finally we consider **Case** (iii). The group G_0 acts primitively on the maximal totally singular subspaces of V , except when $G_0 = \text{P}\Omega_n^+(q)$. When $G_0 = \text{P}\Omega_n^+(q)$, the group G_0 has two orbits of equal size on the set of maximal totally singular subspaces and acts primitively on both orbits. As in **Case** (ii), elements in $\text{PGO}_n^+(q) \setminus \text{Inndiag}(G_0)$ interchange these two G_0 -orbits. In particular, we have $G \leq \langle \text{Inndiag}(G_0), \mathcal{F} \rangle$ (where \mathcal{F} is the group of field automorphisms); otherwise G would be imprimitive on the set of all maximal totally singular subspaces.

Recall the definition of q_0 in Notation 2.5 and let

$$S_1(g, \Omega) := \sum_{\substack{r||g| \\ r \text{ prime} \\ r \nmid ep(q_0-1)}} \text{fpr}_\Omega(g^{|g|/r}) \quad \text{and} \quad S_2(g, \Omega) := \sum_{\substack{r||g| \\ r \text{ prime} \\ r \nmid ep(q_0-1)}} \text{fpr}_\Omega(g^{|g|/r})$$

so that

$$S(g, \Omega) = \sum_{\substack{r||g| \\ r \text{ prime}}} \text{fpr}_\Omega(g^{|g|/r}) = S_1(g, \Omega) + S_2(g, \Omega).$$

For some of the G -sets in Proposition 5.5, it will be useful to have stronger upper bounds on fixed-point-ratios for elements of certain prime orders.

Suppose that r is a prime with $r \nmid pe(q_0 - 1)$. Let ℓ be the minimum positive integer with $r \mid (q_0^\ell - 1)$ and let x be an element of G of order r . Clearly, $\ell \geq 2$. Moreover, x lies in the inner-diagonal subgroup of G and is semisimple. Let x' be a semisimple element of $\text{GL}(V)$ projecting to x . Since x' is semisimple, by Maschke's theorem V can be written as a direct sum $V = [V, x'] \oplus \mathbf{C}_V(x')$, where $[V, x'] = \{-v + vx' \mid v \in V\}$. Let $\ell' := \dim_{\mathbb{F}_{q_0}} [V, x']$ and observe that $\ell' \geq \ell$ because every non-trivial irreducible $\mathbb{F}_{q_0}\langle x' \rangle$ -submodule of V has dimension ℓ .

Proposition 5.8. *Suppose that G and Ω are as in Proposition 5.5 **Cases** (i), (ii). Let $x \in G$ have prime order r with $r \nmid ep(q_0 - 1)$, and let ℓ and ℓ' be as above. Then*

$$\text{fpr}_\Omega(x) \leq \begin{cases} 1/q_0^{\ell'} & \text{in **Case** (i) with } G \text{ linear;} \\ 2/q_0^{\ell'} & \text{in **Case** (i) with } G \text{ unitary or orthogonal;} \\ 2/q_0^{\ell'} & \text{in **Case** (ii) with } G \text{ unitary;} \\ 36/(13q_0^{\ell'}) & \text{in **Case** (ii) with } G \text{ orthogonal and } n \geq 7. \end{cases}$$

Proof. Let $v \in V \setminus \{0\}$ with $\langle v \rangle^x = \langle v \rangle$. We can write $v = w + t$, with $w \in [V, x']$ and $t \in \mathbf{C}_V(x')$. Since $\langle v \rangle^x = \langle v \rangle$, we have $vx' = \lambda v$, for some $\lambda \in \mathbb{F}_{q_0}$. Since $[V, x']$ is x' -invariant, we obtain $wx' = \lambda w$. Thus $\langle w \rangle$ is a 1-dimensional $\mathbb{F}_{q_0}\langle x \rangle$ -module. As every non-trivial irreducible $\mathbb{F}_{q_0}\langle x' \rangle$ -submodule of V has dimension ℓ and as $\ell \geq 2$, it follows that x' acts trivially on w and $w \in \mathbf{C}_V(x') \cap [V, x'] = 0$. This proves that

$$(4) \quad \text{if } \langle v \rangle^x = \langle v \rangle, \text{ then } v \in \mathbf{C}_V(x').$$

Recall that $\dim_{\mathbb{F}_{q_0}} \mathbf{C}_V(x') = n - \ell'$.

First suppose that G is linear so that **Case** (i) holds and Ω is the set of totally singular 1-subspaces. From (4), we have $\text{fpr}_\Omega(x) = (q_0^{n-\ell'} - 1)/(q_0^n - 1) \leq q_0^{-\ell'}$.

Suppose that G is unitary. Then, the Hermitian form on V restricts to a non-degenerate Hermitian form on $\mathbf{C}_V(x')$. Therefore, in **Case** (i) from (4) we get that the number of totally singular 1-subspaces fixed by x is

$$(q^{n-\ell'} - (-1)^{n-\ell'})(q^{n-\ell'-1} - (-1)^{n-\ell'-1})/(q^2 - 1)$$

and hence

$$\text{fpr}_\Omega(x) = \frac{(q^{n-\ell'} - (-1)^{n-\ell'})(q^{n-\ell'-1} - (-1)^{n-\ell'-1})}{(q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})} \leq \frac{2}{q^{2\ell'}} = \frac{2}{q_0^{\ell'}}$$

(the inequality follows from a direct computation distinguishing the parity of n and ℓ'). Similarly, in **Case** (ii) from (4) we get that the number of non-degenerate 1-subspaces fixed by x is $(q^{n-\ell'} - (-1)^{n-\ell'})q^{n-\ell'-1}/(q + 1)$ and hence

$$\text{fpr}_\Omega(x) = \frac{(q^{n-\ell'} - (-1)^{n-\ell'})q^{n-\ell'-1}}{(q^n - (-1)^n)q^{n-1}} \leq \frac{2}{q^{2\ell'}} = \frac{2}{q_0^{\ell'}}.$$

Now suppose that G is orthogonal. From Remark 5.7, in **Case** (i) the set Ω consists of all singular 1-subspaces of V , and hence

$$|\Omega| = \begin{cases} (q^{n-1} - 1)/(q - 1) & \text{if } G_0 = \text{P}\Omega_n(q), \\ (q^{n/2} - 1)(q^{n/2-1} + 1)/(q - 1) & \text{if } G_0 = \text{P}\Omega_n^+(q), \\ (q^{n/2-1} - 1)(q^{n/2} + 1)/(q - 1) & \text{if } G_0 = \text{P}\Omega_n^-(q). \end{cases}$$

We denote by Ω_n° , Ω_n^+ and Ω_n^- these three G -sets, respectively. Note that for every n and q we have

$$(5) \quad |\Omega_n^-| \leq |\Omega_n^\circ| \leq |\Omega_n^+|.$$

The quadratic form Q on V restricts to a non-degenerate quadratic form Q' on $\mathbf{C}_V(x')$. Observe that the type of Q' depends on both Q and $\mathbf{C}_V(x')$. (In particular, if q is even, then $n - \ell'$ is even.) A maximal totally isotropic subspace of $\mathbf{C}_V(x')$ has dimension $(n - \ell' - 1)/2$ (if $n - \ell'$ is odd), $(n - \ell')/2$ (if $n - \ell'$ is even and Q' is of “plus type”), and $(n - \ell')/2 - 1$ (if $n - \ell'$ is even and Q' is of “minus type”). Therefore the number of totally singular 1-subspaces fixed by x is either $(q^{n-\ell'-1} - 1)/(q - 1)$, or $(q^{(n-\ell')/2} - 1)(q^{(n-\ell')/2-1} + 1)$, or $(q^{(n-\ell')/2-1} - 1)(q^{(n-\ell')/2} + 1)/(q - 1)$ in the three respective cases described above. Observe that when $\dim_{\mathbb{F}_q} \mathbf{C}_V(x') \leq 1$ (that is, $\ell' \geq n - 1$), the space $\mathbf{C}_V(x')$ contains no singular points, and hence $\text{fpr}_\Omega(x) = 0$. For $\ell' \leq n - 2$, using (5), with an easy case-by-case analysis studying the various possibilities for Q and Q' gives

$$\text{fpr}_\Omega(x) = \frac{|\text{Fix}_\Omega(x)|}{|\Omega|} \leq \frac{|\text{Fix}_\Omega(x)|}{|\Omega_n^-|} \leq \frac{(q^{(n-\ell')/2} - 1)(q^{(n-\ell')/2-1} + 1)}{(q^{n/2-1} - 1)(q^{n/2} + 1)} \leq \frac{2}{q^{\ell'}}.$$

Finally we consider **Case** (ii). Assume that q is even. In particular, $\Omega = \mathbf{NS}_1$ and $|\Omega| = q^{n/2-1}(q^{n/2} - \varepsilon \cdot 1)$ when $G_0 = \text{P}\Omega_n^\varepsilon(q)$, $\varepsilon \in \{+, -\}$. Observe that ℓ' is even. Now the number of non-degenerate 1-subspaces fixed by x is either

$$q^{(n-\ell')/2-1}(q^{(n-\ell')/2} - 1) \quad (\text{when } Q' \text{ has Witt index } (n - \ell')/2)$$

or

$$q^{(n-\ell')/2-1}(q^{(n-\ell')/2} + 1) \quad (\text{when } Q' \text{ has Witt index } (n - \ell')/2 - 1).$$

For $\ell' = n$, we have $\text{fpr}_\Omega(x) = 0$, and for $\ell' < n$, we have

$$\text{fpr}_\Omega(x) \leq \frac{|\text{Fix}_\Omega(x)|}{|\Omega|} \leq \frac{q^{(n-\ell')/2-1}(q^{(n-\ell')/2} + 1)}{q^{n/2-1}(q^{n/2} - 1)} \leq \frac{2}{q^{\ell'}}.$$

Finally, assume that q is odd. By looking at the various possibilities for $|\Omega|$ in Remark 5.7, we get $|\Omega| \geq q^{(n-1)/2}(q^{(n-1)/2} - 1)/2$. Now the number of non-degenerate 1-subspaces fixed by x is $q^{n-\ell'-1}$ (when $n - \ell'$ is odd), or $q^{(n-\ell')/2-1}(q^{(n-\ell')/2} - 1)$ (when $n - \ell'$ is even and Q' has Witt index $(n - \ell')/2$) or $q^{(n-\ell')/2-1}(q^{(n-\ell')/2} + 1)$ (when $n - \ell'$ is even and Q' has Witt index $(n - \ell')/2 - 1$). If $\ell' = n$, then $\text{Fix}_\Omega(x) = \emptyset$ and $\text{fpr}_\Omega(x) = 0$. If $\ell' = n - 1$, then $\dim_{\mathbb{F}_q} \mathbf{C}_V(x') = 1$ and hence $\mathbf{C}_V(x')$ is the only non-degenerate 1-subspace fixed by x . Thus $\text{fpr}_\Omega(x) = 1/|\Omega| < 36/(13q_0^{n-1})$ (where the inequality follows with a computation). Assume that $n - \ell' \geq 2$. Thus

$$\text{fpr}_\Omega(x) \leq \frac{|\text{Fix}_\Omega(x)|}{|\Omega|} \leq \frac{q^{(n-\ell')/2-1}(q^{(n-\ell')/2} + 1)}{q^{(n-1)/2}(q^{(n-1)/2} - 1)/2} = \frac{2}{q^{\ell'}} \left(\frac{1 + \frac{1}{q^{(n-\ell')/2}}}{1 - \frac{1}{q^{(n-1)/2}}} \right) \leq \frac{36}{13q^{\ell'}},$$

where the last inequality follows by noticing that the maximum of the second factor in the above product is achieved for $q = 3$, $n = 7$ and $n - \ell' = 2$. □

We conclude this section by introducing a notation that will be used frequently in our arguments. Let t be a prime power and let $\ell \geq 1$; then

$$\omega_t(t^\ell - 1)$$

denotes the number of primitive prime divisors of $t^\ell - 1$, that is, the number of prime divisors r of $t^\ell - 1$ with $r \nmid t^i - 1$ for every $1 \leq i \leq \ell - 1$.

6. Case (iii) OF PROPOSITION 5.5: ACTION ON MAXIMAL TOTALLY SINGULAR SUBSPACES

As usual we assume Notation 2.5, and we start our analysis by considering **Case (iii)** of Proposition 5.5. We let G be a primitive group with socle G_0 acting on a set Ω of totally singular m -subspaces of V . In particular, Ω is the set of all maximal totally singular subspaces of V when $G_0 \neq \text{P}\Omega_{2m}^+(q)$, and Ω is a G_0 -orbit on the set of maximal totally singular subspaces of V when $G_0 = \text{P}\Omega_{2m}^+(q)$. We need the following lemma.

Lemma 6.1 ([18, Lemma 3.14]). *Suppose that $m \geq 3$. Then, for $g \in G \setminus \{1\}$, we have $\text{fpr}_\Omega(g) < 2/q_0^{m^*} + 1/q_0^{m^\sharp}$, where m^* and m^\sharp are given in Table 3.*

TABLE 3. Definitions of m^* and m^\sharp

G_0	$\text{PSp}_{2m}(q)$	$\text{P}\Omega_{2m}^+(q)$	$\text{P}\Omega_{2m+1}(q)$	$\text{P}\Omega_{2m+2}^-(q)$	$\text{PSU}_{2m}(q)$	$\text{PSU}_{2m+1}(q)$
m^*	m	$m - 1$	m	$m + 1$	$m - \frac{1}{2}$	$m + \frac{1}{2}$
m^\sharp	$m - 1$	$m - 2$	$m - 1$	m	$m - 1$	m

Proposition 6.2. *If $g \in G$, then either g has a regular cycle, or $(G_0, \Omega) \cong (\text{Alt}(5), \{1, \dots, 5\})$.*

Proof. Denote by o the maximum order of an element of G . In particular, $|g| \leq o$. (Recall that from Remark 5.6 we may assume that $n \geq 7$ when G_0 is orthogonal.)

Assume first that $m \geq 3$. (We use the notation introduced in Lemma 6.1 and Table 3.) From Lemma 6.1 we get

$$(6) \quad S(g, \Omega) < \sum_{\substack{r \text{ prime} \\ r \parallel |g|}} \left(\frac{2}{q_0^{m^*}} + \frac{1}{q_0^{m^\sharp}} \right) = \omega(|g|) \left(\frac{2}{q_0^{m^*}} + \frac{1}{q_0^{m^\sharp}} \right) \\ \leq \log_2(|g|) \left(\frac{2}{q_0^{m^*}} + \frac{1}{q_0^{m^\sharp}} \right) \leq \log_2(o) \left(\frac{2}{q_0^{m^*}} + \frac{1}{q_0^{m^\sharp}} \right).$$

Now, an explicit upper bound for o as a function of q and m can be found in [16, Table 3]. It follows that $\log_2(o)(2/q_0^{m^*} + 1/q_0^{m^\sharp}) < 1$ except when G_0 belongs to the following list:

$$(†) \quad \text{PSp}_6(2), \text{PSp}_8(2), \text{PSP}_6(3), \text{P}\Omega_8^+(2), \text{P}\Omega_{10}^+(2), \\ \text{P}\Omega_{12}^+(2), \text{P}\Omega_8^+(4), \text{P}\Omega_8^+(3), \text{P}\Omega_8^-(2), \text{P}\Omega_7(3).$$

In particular, apart from this handful of exceptions, the proof follows as usual from Lemma 2.4. When $G_0 = \text{P}\Omega_8^+(4)$, we replace $\log_2(o)$ by 6 (the exact number of primes dividing $|G|$) in (6), which shows that $S(g, \Omega) < 1$. The same argument works for $G_0 = \text{P}\Omega_{12}^+(2), \text{P}\Omega_8^+(3)$, where we can replace $\log_2(o)$ by 7 and 5, respectively. For $G_0 = \text{P}\Omega_{10}^+(2)$, we can calculate in **magma** all the possible element orders of $\text{Aut}(G_0) = \text{PGO}_{10}^+(2)$, and check that $\omega(|g|) \leq 3$, for every $g \in G$. This refinement of (6) implies that $S(g, \Omega) < 1$. For each of the remaining groups G arising from G_0 in (†), we can construct in **magma** the permutation representation of G on the maximal totally singular subspaces of V and check directly that every conjugacy class representative has a regular cycle.

Suppose that $m \leq 2$. In particular, G_0 is $\text{PSp}_2(q) \cong \text{PSL}_2(q), \text{PSp}_4(q), \text{PSU}_3(q), \text{PSU}_4(q)$, or $\text{PSU}_5(q)$. Observe that by Theorem 3.1, we may assume that $G_0 = \text{PSU}_5(q)$. By Theorem 2.6, we have $\text{fpr}_\Omega(x) \leq 4/(3q)$ for every $x \in G \setminus \{1\}$. In particular, the argument in the previous paragraph implies that

$$(7) \quad S(g, \Omega) = \sum_{\substack{r \text{ prime} \\ r \parallel |g|}} \text{fpr}_\Omega(g^{|g|/r}) \leq \frac{4\omega(|g|)}{3q} \leq \frac{4\log_2(o)}{3q}.$$

Now an easy computation (using the value of o in [16, Table 3]) shows that $S(g, \Omega) < 1$ except when $q \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 23\}$. Now observe that $\omega(|g|) \leq \omega(|\text{Aut}(G_0)|)$ and that $\omega(|\text{Aut}(G_0)|)$ can be explicitly computed for each remaining G_0 . In particular, we see that $4\omega(|\text{Aut}(G_0)|)/(3q) < 1$ except when $q \in \{2, 3, 4, 5, 7, 8\}$. We note that if x has prime order r and r does not divide $|G_\alpha|$, for $\alpha \in \Omega$, then $\text{fpr}_\Omega(x) = 0$ (see Lemma 2.3). For $q = 7$, the only primes dividing $|G_\alpha|$ are 2, 3, 5, 7 (see [20, Proposition 4.1.18] for example) so we can replace $\omega(|g|)$ by 4 in (7) and we obtain $S(g, \Omega) < 1$, the desired bound. Similarly, for $q = 8$ we replace $\omega(|g|)$ by 5, which eliminates this case. The remaining cases are $q = 2, 3, 4, 5$, and these are small enough to eliminate directly in **magma** by computing the permutation representation of $\text{PGU}_5(q)$ on maximal totally singular subspaces. \square

7. Case (i) of Proposition 5.5: Action on totally singular 1-subspaces

Here we let G be a primitive group with socle G_0 acting on the set Ω of totally singular 1-subspaces of V . The following lemma is in [18, Propositions 3.1 (ii) and 3.15].

Lemma 7.1. *Suppose that $m \geq 3$ and $g \in G \setminus \{1\}$. Then*

$$\text{fpr}_\Omega(g) < \begin{cases} \min\{1/2, 1/q + 1/q^{n-1}\} & \text{if } G_0 = \text{PSL}_n(q), \\ 2/q_0^{m^*} + 1/q^{m^\sharp} + 1/q_0 & \text{otherwise,} \end{cases}$$

where m^* and m^\sharp are given in Table 3.

Proposition 7.2. *If $g \in G$, then either g has a regular cycle, or $(G_0, \Omega) \cong (\text{Alt}(5), \{1, \dots, 5\})$.*

Proof. When $m = 1$, a maximal totally singular subspace of V has dimension 1 and hence, by Proposition 6.2, we may assume that $m \geq 2$. Observe that when G_0 is $\text{PSL}_n(q)$ or $\text{PSP}_n(q)$ every 1-subspace of V is totally singular (recall that when $G_0 = \text{PSL}_n(q)$ the space V is endowed with the trivial form). In particular, $\mathbf{N}_{\text{Sym}(\Omega)}(\text{PSP}_n(q)) \leq \mathbf{N}_{\text{Sym}(\Omega)}(\text{PSL}_n(q)) = \text{PGL}_n(q)$. Therefore, in this proof, we may omit the symplectic groups from our analysis.

We now split the proof into various cases.

Case $G_0 = \text{PSL}_n(q)$. In this case $q_0 = q$. Observe that when q is prime, we have $G \leq \text{PGL}_n(q)$ and hence the proof follows from [26, Theorem 2.3]. In particular, we may assume that $e \geq 2$.

Recall that if $x \in G \setminus \{1\}$, then $\text{fpr}_\Omega(x) \leq 4/(3q)$. Therefore

$$S_1(g, \Omega) \leq \omega(ep(q-1)) \frac{4}{3q}.$$

From Proposition 5.8, we see that if x has order r , with $r \nmid ep(q-1)$, then $\text{fpr}_\Omega(x) \leq q^{-\ell}$ where ℓ is the smallest positive integer with $r \mid q^\ell - 1$. The number $\omega_q(q^\ell - 1)$ of primitive prime divisors of $q^\ell - 1$ is at most $\log_2(q^\ell - 1) \leq \ell \log_2(q)$. Thus

$$S_2(g, \Omega) \leq \sum_{\ell=2}^{\infty} \frac{\omega_q(q^\ell - 1)}{q^\ell} \leq \sum_{\ell=2}^{\infty} \frac{\ell \log_2(q)}{q^\ell} = \log_2(q) \left(\frac{q}{(q-1)^2} - \frac{1}{q} \right)$$

where the last equality follows from Lemma 2.9. Observe that $ep \leq q$ and hence $\omega(ep(q-1)) \leq \log_2(q^2)$.

Using these upper bounds for $S_1(g, \Omega)$ and $S_2(g, \Omega)$, it is easy to see that $S(g, \Omega) < 1$ for $q \geq 11$. For $q \in \{8, 9\}$, we can compute $\omega(ep(q-1))$ explicitly from which it follows that $S(g, \Omega) < 1$. Finally, for $q = 4$, we have $\omega(ep(q-1)) = 2$ and by Lemma 7.1 we have $\text{fpr}_\Omega(x) < 1/4 + 1/4^{n-1} \leq 1/4 + 1/4^4$ (recall that $n \geq 5$). Thus $S_1(g, \Omega) < 2 \cdot (1/4 + 1/4^4)$ and $S(g, \Omega) < 1$.

Case $G_0 = \text{PSU}_n(q)$. In this case $q_0 = q^2$ and $|\Omega| = (q^n - (-1)^n)(q^{n-1} - (-1)^{n-1})/(q^2 - 1)$.

From Proposition 5.8, we see that if x has order r , with $r \nmid ep(q-1)$, then $\text{fpr}_\Omega(x) \leq 2q_0^{-\ell} \leq 2q^{-2\ell}$ and, arguing as in the case $G_0 = \text{PSL}_n(q)$, we get

$$(8) \quad S_2(g, \Omega) \leq \sum_{\ell=2}^{\infty} \frac{2\omega_{q^2}(q^{2\ell} - 1)}{q^{2\ell}} \leq 4 \log_2(q) \left(\frac{q^2}{(q^2 - 1)^2} - \frac{1}{q^2} \right).$$

Suppose that $n \geq 6$, that is, $m \geq 3$. Using Lemma 7.1, we have $S_1(g, \Omega) \leq \omega(ep(q^2 - 1))(2q^{-2m^*} + q^{-m^\sharp} + q^{-2})$. It follows that $S(g, \Omega) < 1$ for $q \neq 2$. If $q = 2$, then $q^2 - 1 = 3$, $q^4 - 1 = 15 = 3 \cdot 5$ and $q^6 - 1 = 63 = 3^2 \cdot 7$. Hence $\omega_{q^2}(q^4 - 1) = 1$

and $\omega_{q^2}(q^6 - 1) = 1$, and so for the first two terms of the summation in (8) we can take $2/q^4$ and $2/q^6$. Therefore

$$S_2(g, \Omega) \leq 4 \left(\frac{q^2}{(q^2 - 1)^2} - \frac{1}{q^2} - \frac{2}{q^4} - \frac{3}{q^6} \right) + \frac{2}{q^4} + \frac{2}{q^6}.$$

For $q = 2$, it follows that $S(g, \Omega) < 1$ unless $n \in \{6, 7, 8\}$. We verify the cases $G_0 = \text{PSU}_6(2)$, $\text{PSU}_7(2)$ and $\text{PSU}_8(2)$ directly in `magma`.

It remains to consider $G_0 = \text{PSU}_5(q)$. Here we observe that $S_1(g, \Omega) \leq \omega(ep(q^2 - 1)) \frac{4}{3q}$ by Theorem 2.6. Using (8), it is easy to check that $S(g, \Omega) < 1$ for $q \geq 5$. The cases $q = 2, 3, 4$ follow by direct computations in `magma`.

Case G_0 is orthogonal. From Remark 5.6, we have $n \geq 7$. Using Lemma 7.1 and arguing exactly as in the linear and unitary case, we have $S_1(g, \Omega) \leq \omega(ep(q - 1))(2/q^{m^*} + 1/q^{m^{\ddagger}} + 1/q)$ and

$$(9) \quad S_2(g, \Omega) \leq \sum_{\ell=2}^{\infty} \frac{2\omega_q(q^\ell - 1)}{q^\ell} \leq 2 \log_2(q) \left(\frac{q}{(q - 1)^2} - \frac{1}{q} \right).$$

It follows that $S(g, \Omega) < 1$ for $q \geq 7$. If $q = 5$, then $\omega_q(q^2 - 1) = 1$ and hence for the first term in the summation in (9) we can take $2/q^2$. In particular,

$$S_2(g, \Omega) \leq 2 \log_2(q) \left(\frac{q}{(q - 1)^2} - \frac{1}{q} - \frac{2}{q^2} \right) + \frac{2}{q^2}.$$

With this upper bound on $S_2(g, \Omega)$, it follows that $S(g, \Omega) < 1$. For $q = 4$, we have $\omega_q(q^2 - 1) = \omega_q(q^3 - 1) = 1$ and the usual argument shows that $S(g, \Omega) < 1$. Similarly, if $q = 3$, then $\omega_q(q^2 - 1) = 0$ and $\omega_q(q^3 - 1) = \omega_q(q^4 - 1) = 1$ and it follows that $S(g, \Omega) < 1$ unless $G_0 = \text{P}\Omega_7(3), \text{P}\Omega_8^+(3)$. We eliminate the remaining two cases with the invaluable help of `magma`.

It remains to deal with the case $q = 2$. In particular, n is even. If $n \geq 10$, then [20, Tables 2.1C and D] shows that $\text{Aut}(G_0)$ is the isometry group $I(V)$ of the orthogonal space V , and hence $G \leq I(V)$. When $n = 8$, the triality automorphism of $\Omega_8^+(2)$ does not preserve the action of G_0 on the totally singular subspaces, and hence $G \leq I(V)$ in this case as well. Therefore the proof follows from [12, Theorem 1.2]. □

8. Case (ii) OF PROPOSITION 5.5: ACTION ON NON-DEGENERATE 1-SUBSPACES

Here we let G be a primitive group with socle G_0 acting on a set Ω consisting of non-degenerate 1-subspaces of V . The various possibilities for Ω (depending on the type of G_0) are discussed in Remark 5.7.

Proposition 8.1. *If $g \in G$, then g has a regular orbit on Ω .*

Proof. We subdivide the proof depending on the type of G_0 . Here G_0 is either unitary or orthogonal.

Case $G_0 = \text{PSU}_n(q)$. We may assume that $n \geq 5$ by Theorem 3.1. First suppose that $n = 5$. Let ω be the number of prime divisors of $|\text{Aut}(G_0)|$. Clearly, it suffices to show that $4\omega/(3q) < 1$. Now $\omega \leq \log_2(|\text{Aut}(G_0)|)$ and we see that the inequality $4 \log_2(|\text{Aut}(G_0)|)/(3q) < 1$ is satisfied for $q \geq 261$. For $q \leq 260$, we can compute the exact value of ω and we see that $4\omega/(3q) < 1$ when $q \geq 9$. For $q \in \{7, 8\}$, replacing ω with $\omega' = \max(\omega(|h|) \mid h \in \text{Aut}(G_0))$, we obtain $4\omega'/(3q) < 1$. For

$q = 5$, we check in `magma` that the number of prime divisors of $|g|$ is at most 3, unless $|g| \in \{210, 630\}$, in which case the prime divisors of $|g|$ are 2, 3, 5, 7. By Proposition 5.8, for $r = 7$ we have $\text{fpr}_\Omega(g^{|g|/7}) < 2/25^3$ and with this refinement, $S(g, \Omega) \leq 3 \cdot 4/(3q) + 2/25^3 < 1$. For $q \in \{2, 3, 4\}$, we construct the explicit permutation representation of the action of $\text{PFU}_5(q)$ on $\Omega = \mathbf{NS}_1$ and check that every element has a regular orbit.

We now suppose that $n \geq 6$ so that $m \geq 3$. For every element $x \in G \setminus \{1\}$, we see from [18, Proposition 3.16] that

$$(10) \quad \begin{aligned} \text{fpr}_\Omega(x) &< f(n, q) \\ &:= \begin{cases} 2/q^{2(m-2)} + 1/q^{2m-1} + 1/q^{2(m-1)} + 1/q^2 & \text{if } n = 2m \text{ and } m \geq 3, \\ 1/q^{2m+1} + 1/q^{2m+1} + 1/q^{2m} + 1/q^2 & \text{if } n = 2m + 1 \text{ and } m \geq 3. \end{cases} \end{aligned}$$

By (10), we have $S_1(g, \Omega) \leq \omega(ep(q^2 - 1))f(n, q)$ and, from Proposition 5.8, we have

$$(11) \quad S_2(g, \Omega) \leq \sum_{\ell=2}^{\infty} \frac{2\omega_{q^2}(q^{2\ell} - 1)}{q^{2\ell}} \leq \sum_{\ell=2}^{\infty} \frac{2\ell \log_2(q^2)}{q^{2\ell}} = 4 \log_2(q) \left(\frac{q^2}{(q^2 - 1)^2} - \frac{1}{q^2} \right).$$

It follows that $S(g, \Omega) < 1$ for $q \neq 2$. (To see this, for $q \geq 17$, observe that $ep(q^2 - 1) \leq q^3$ and hence $\omega(ep(q^2 - 1)) \leq \log_2(q^3)$. For $q < 17$, use the explicit value of $\omega(ep(q^2 - 1))$.) Now suppose that $q = 2$. Then $q^2 - 1 = 3$, $q^4 - 1 = 3 \cdot 5$ and $q^6 - 1 = 3^2 \cdot 7$. Hence $\omega_{q^2}(q^4 - 1) = 1$ and $\omega_{q^2}(q^6 - 1) = 1$, and so for the first two terms of the summation in (11) we can take $2/q^4$ and $2/q^6$. Thus

$$S_2(g, \Omega) \leq 4 \left(\frac{q^2}{(q^2 - 1)^2} - \frac{1}{q^2} - \frac{2}{q^4} - \frac{3}{q^6} \right) + \frac{2}{q^4} + \frac{2}{q^6},$$

and now $S(g, \Omega) < 1$ for every $n \geq 9$. Finally for the groups G with socle $G_0 = \text{PSU}_n(2)$ and $n = 6, 7, 8$, the proof follows by calculating $S(g, \Omega)$ precisely in `magma`.

Case G_0 is orthogonal. From Remark 5.6, $n \geq 7$. For every $x \in G \setminus \{1\}$, [18, Proposition 3.16] (applied with $k = n - 1$) gives

$$(12) \quad \text{fpr}_\Omega(x) < f(n, q) \quad \text{where} \quad f(n, q) := \frac{2}{q^{m^*}} + \frac{2}{q^{m^\sharp}} + \frac{1}{q}$$

(where m^\sharp and m^* are as in Table 3). Using (12) and Proposition 5.8 and arguing as in the unitary case we find that $S_1(g, \Omega) \leq \omega(ep(q - 1))f(n, q)$ and

$$(13) \quad S_2(g, \Omega) \leq \sum_{\ell=2}^{\infty} \frac{36\omega_q(q^\ell - 1)}{13q^\ell} \leq \frac{36}{13} \log_2(q) \left(\frac{q}{(q - 1)^2} - \frac{1}{q} \right).$$

It follows that $S(g, \Omega) < 1$ for $q \geq 7$, unless $G_0 \in \{\text{P}\Omega_7(7), \text{P}\Omega_8^+(7)\}$. In the case $G_0 \in \{\text{P}\Omega_7(7), \text{P}\Omega_8^+(7)\}$, we note that the only primes dividing $|G|$ but not $ep(q - 1)$ are 5, 19, 43. Thus $S_2(g, \Omega) \leq 3 \cdot \frac{36}{13q^2}$ and $S(g, \Omega) < 1$.

Now suppose that $q = 5$. We have $\omega(ep(q - 1)) = 2$ and $\omega_q(q^2 - 1) = 1$, so $S_1(g, \Omega) \leq 2 \cdot 4/(3q) = 8/15$ and

$$S_2(g, \Omega) \leq \frac{36}{13} \log_2(q) \left(\frac{q}{(q - 1)^2} - \frac{1}{q} - \frac{2}{q^2} \right) + \frac{36}{13q^2} < \frac{7}{15}.$$

Thus $S(g, \Omega) < 1$.

If $q = 4$, then $\omega(ep(q - 1)) = 2$ and $\omega_q(q^2 - 1) = \omega_q(q^3 - 1) = \omega_q(q^4 - 1) = 1$, so $S_1(g, \Omega) \leq 2 \cdot 4/(3q) = 2/3$ and

$$S_2(g, \Omega) \leq \frac{36}{13} \log_2(q) \left(\frac{q}{(q - 1)^2} - \frac{1}{q} - \frac{2}{q^2} - \frac{3}{q^3} - \frac{4}{q^4} \right) + \frac{36}{13q^2} + \frac{36}{13q^3} + \frac{36}{13q^4} < \frac{1}{3}.$$

For $q = 3$, we have $\omega(ep(q - 1)) = 2$ and $S_1(g, \Omega) \leq 2 \cdot f(n, q)$. Moreover, $\omega_q(q^2 - 1) = 0$ and $\omega_q(q^i - 1) = 1$, for $i = 3, 4, 5, 6$. The usual argument shows that $S(g, \Omega) < 1$, unless $G_0 \in \{P\Omega_8^+(3), P\Omega_{10}^+(3), P\Omega_7(3), P\Omega_9(3), P\Omega_8^-(3)\}$. For these remaining groups, the proof follows by calculating $S(g, \Omega)$ precisely in **magma**.

Finally, suppose that $q = 2$, and hence n is even. If $n \geq 10$, then [20, Tables 2.1C and D] shows that $\text{Aut}(G_0)$ is the isometry group $I(V)$ of the orthogonal space V , and hence $G \leq I(V)$. When $n = 8$, the triality automorphism of $\Omega_8^+(2)$ does not preserve the action of G_0 on $\Omega = \mathbf{NS}_1$, and hence $G \leq I(V)$ in this case as well. Therefore the proof follows from [12, Theorem 1.2]. \square

9. **Case (iv) OF PROPOSITION 5.5: ACTION ON NON-DEGENERATE 2-SUBSPACES WITH NO SINGULAR POINT**

Here we let G be a primitive group with socle G_0 acting on the set Ω consisting of all non-degenerate 2-subspaces of V that contain no singular point. This means that the quadratic form on V restricted to each of these 2-dimensional subspaces is anisotropic (see [10, Section 6.3] for example). In particular, G_0 is an orthogonal group. Moreover, in the light of Remark 5.6, G_0 is either $P\Omega_{2m+1}(q)$ (with $m \geq 3$), or $P\Omega_{2m}^+(q)$ (with $m \geq 4$) or $P\Omega_{2m+2}^-(q)$ (with $m \geq 3$). We recall that

$$|\Omega| = \begin{cases} \frac{1}{2}q^{2(m-1)}(q^m - 1)(q^{m-1} - 1)(q + 1)^{-1} & \text{if } G_0 = P\Omega_{2m}^+(q), \\ \frac{1}{2}q^{2m}(q^{m+1} + 1)(q^m + 1)(q + 1)^{-1} & \text{if } G_0 = P\Omega_{2m+2}^-(q), \\ \frac{1}{2}q^{2m-1}(q^{2m} - 1)(q + 1)^{-1} & \text{if } G_0 = P\Omega_{2m+1}(q) \end{cases}$$

(these formulae can be deduced from the index of $G_0 \cap (O_2^-(q) \perp O_{n-2}^\epsilon(q))$ in G_0). When necessary we write Ω_n^+ , Ω_n^- and Ω_n° for Ω , according to the corresponding type of G_0 . By writing $|\Omega|$ as a function of n and q (so that $m = n/2$, $n/2 - 1$ or $(n - 1)/2$ respectively), we get

$$(14) \quad |\Omega_n^-| > |\Omega_n^\circ| > |\Omega_n^+|.$$

Lemma 9.1. *For $x \in G \setminus \{1\}$, we have*

$$\text{fpr}_\Omega(x) \leq f(n, q) \quad \text{where} \quad f(n, q) := 3/q^{n/2-2} + 1/q^{n/2-1} + 1/q^2.$$

Proof. This follows from [18, Proposition 3.16] (and the comment following its proof) applied with $k = d - 2$ and by writing m as a function of n in each case. It is straightforward to see that the worst upper bound for $\text{fpr}_\Omega(x)$ arises when $n = 2m$ and $G_0 = P\Omega_{2m}^+(q)$. \square

Proposition 9.2. *If $g \in G$, then g has a regular orbit on Ω .*

Proof. Let $x \in G$ of prime order r and suppose that r is coprime to $ep(q^2 - 1)$; in particular $x \in G_0$ is semisimple. Let ℓ be the minimum positive integer such that $r \mid (q^\ell - 1)$. In particular we have $\ell \geq 3$. Let x' be a semisimple element of $\text{GL}(V)$ projecting to x . As in Proposition 5.8 we observe that $V = [V, x'] \oplus \mathbf{C}_V(x')$ and $\ell' = \dim_{\mathbb{F}_q}[V, x'] \geq \ell$.

Let $U \in \Omega$ with $U^x = U$. Since $\dim U = 2$ and U is x' -invariant, we see that x' acts trivially on U and $U \leq \mathbf{C}_V(x')$. This proves that

$$(15) \quad \text{if } U^x = U, \text{ then } U \leq \mathbf{C}_V(x').$$

The quadratic form Q on V restricts to a non-degenerate quadratic form Q' on $\mathbf{C}_V(x')$. Therefore from (15) we obtain that the number of elements of Ω fixed by x is $|\Omega_{n-\ell'}^\varepsilon|$, where $\varepsilon \in \{+, -, \circ\}$. In particular, (14) implies that for fixed ℓ' , the largest fixed-point-ratio occurs when $G_0 = \text{P}\Omega_{2m}^+(q)$ and Q' is of $-$ type (that is, when Q' has Witt index $(n - \ell')/2 - 1$). If $\ell' \geq n - 1$, then $\dim_{\mathbb{F}_q} \mathbf{C}_V(x') \leq 1$ and $\text{fpr}_\Omega(x) = 0$, and if $\ell' \leq n - 2$, it follows that

$$\text{fpr}_\Omega(x) \leq \frac{q^{n-\ell'-2}(q^{(n-\ell')/2} + 1)(q^{(n-\ell')/2-1} + 1)}{q^{n-2}(q^{n/2} - 1)(q^{n/2-1} - 1)} \leq \frac{4}{q^{2\ell'}}.$$

With an argument similar to the cases that we discussed so far, we have

$$\begin{aligned} S(g, \Omega) &= \sum_{\substack{r||g| \\ r|ep(q^2-1)}} \text{fpr}_\Omega(g^{|g|/r}) + \sum_{\substack{r||g| \\ r \nmid ep(q^2-1)}} \text{fpr}_\Omega(g^{|g|/r}) \\ &\leq \omega(ep(q^2 - 1))f(n, q) + \sum_{\ell=3}^\infty \frac{4\omega_q(q^\ell - 1)}{q^{2\ell}} \\ &\leq \omega(ep(q^2 - 1))f(n, q) + 4\log_2(q) \left(\frac{q^2}{(q^2 - 1)^2} - \frac{1}{q^2} - \frac{2}{q^4} \right), \end{aligned}$$

where $f(n, q)$ is as in Lemma 9.1.

Since $\omega(ep(q^2 - 1)) \leq \log_2(q^3)$ and $n \geq 7$, we have $S(g, \Omega) < 1$ for $q > 9$. For $q \leq 9$, we compute the exact value of $\omega(ep(q^2 - 1))$ and we see that $S(g, \Omega) < 1$, unless $q = 2$ or $(n, q) = (7, 3)$. The case $(n, q) = (7, 3)$ can be easily eliminated using magma. Finally if $q = 2$, then G is contained in the isometry group of $I(V)$ and hence we can apply [12, Theorem 1.2]. \square

10. Case (vi) OF PROPOSITION 5.5: EXCEPTIONAL SUBSPACE ACTIONS OF $\text{Sp}_n(2^e)'$

In this section $G_0 = \text{Sp}_n(q)'$ with $q = 2^e$, and we let G be a primitive permutation group with socle G_0 such that the stabilizer in G_0 of a point is isomorphic to either $\text{O}_n^+(q)$ or $\text{O}_n^-(q)$. We denote these two G -sets by Ω^+ and Ω^- , respectively. From Theorem 3.1, we may assume that $n \geq 6$. In particular, $G_0 = \text{Sp}_n(q)$ and $m = n/2 \geq 3$. We have the following useful lemma.

Lemma 10.1. *Suppose that $m \geq 3$ and $q = 2^e$. Let $x \in \text{Sp}_{2m}(q) \setminus \{1\}$ be semisimple and set $c = \dim_{\mathbb{F}_q} \mathbf{C}_V(x)$. If $c > 0$ or $\varepsilon = +$, then $\text{fpr}_{\Omega^\varepsilon}(x) \leq 4q^{-2m+c}$, and if $c = 0$ and $\varepsilon = -$, then $\text{fpr}_{\Omega^\varepsilon}(x) \leq 4/(q^m(q^m - 1))$.*

Proof. Replacing x by a suitable power, we may assume that x has prime order r (and r is odd since q is even). Let ℓ be the smallest positive integer such that $r \mid q^\ell - 1$. Let $1, \omega_1, \dots, \omega_{r-1}$ be the r th roots of unity in \mathbb{F}_{q^ℓ} and note that the Frobenius automorphism $\tau : \omega \mapsto \omega^q$ of \mathbb{F}_{q^ℓ} induces a permutation of the set $\{1, \omega_1, \dots, \omega_{r-1}\}$. Let $\{1\}, \Delta_1, \dots, \Delta_t$ be the orbits of $\langle \tau \rangle$ on $\{1, \omega_1, \dots, \omega_{r-1}\}$. Note that the multiset of eigenvalues of x is a (multiset) union of these orbits. Observe that c denotes the multiplicity of $\{1\}$ since $c = \dim_{\mathbb{F}_q} \mathbf{C}_V(x)$, and let a_i

denote the multiplicity of Δ_i in the multiset of eigenvalues of x . Thus the multiset of eigenvalues of x can be parametrized by the $(t + 1)$ -tuple (c, a_1, \dots, a_t) .

Let H denote the stabilizer in G of a point of Ω^ε . We claim that

$$(16) \quad |x^G| = |x^{\text{Sp}_n(q)}| \quad \text{and} \quad |x^G \cap H| \leq 2^{\delta_{c,0}} |x^{\text{O}_n^\varepsilon(q)}|,$$

where the Kronecker delta $\delta_{c,0} = 1$ if $c = 0$ and $\delta_{c,0} = 0$ otherwise. Observe that to prove the inequality in (16) we need to show that $x^G \cap H$ is an $\text{O}_n^\varepsilon(q)$ -conjugacy class (when $c \geq 1$) and is the union of at most two $\text{O}_n^\varepsilon(q)$ -conjugacy classes (when $c = 0$). Let \mathbb{F} be the algebraic closure of \mathbb{F}_q , let X denote the algebraic group $\text{Sp}_{2m}(\mathbb{F})$ and Y the algebraic group $\text{PSO}_{2m}(\mathbb{F})$. Note that [5, Lemma 3.34] implies that $\mathbf{C}_X(x)$ and $\mathbf{C}_Y(x)$ are both connected. By [5, Corollary 3.7], we see that two elements g and h of $Y_\sigma = \text{Inndiag}(\text{P}\Omega_{2m}^\varepsilon(q))$ are Y_σ -conjugate if and only if they are Y -conjugate and, in turn, by [15, 4.2.2(j)], this is true if and only if g and h are $\text{P}\Omega_{2m}^\varepsilon(q)$ -conjugate. Furthermore, it follows from [5, Lemma 3.39] that if g and h in Y have the same multiset of eigenvalues and if 1 is an eigenvalue of g , then g and h are Y -conjugate and hence $\text{P}\Omega_{2m}^\varepsilon(q)$ -conjugate. Again from [5, Lemma 3.39], if g and h in Y have the same multiset of eigenvalues and if 1 is not an eigenvalue of g , then there are at most two Y -classes to which g and h can belong. This proves (16).

Combining Lemma 2.3 and (16), we have

$$(17) \quad \text{fpr}_{\Omega^\varepsilon}(x) = \frac{|x^G \cap H|}{|x^G|} \leq \frac{2^{\delta_{c,0}} |x^{\text{O}_{2m}^\varepsilon(q)}|}{|x^{\text{Sp}_{2m}(q)}|} = \frac{2^{\delta_{c,0}} |\text{O}_{2m}^\varepsilon(q)|}{|\text{Sp}_{2m}(q)|} \cdot \frac{|\mathbf{C}_{\text{Sp}_{2m}(q)}(x)|}{|\mathbf{C}_{\text{O}_{2m}^\varepsilon(q)}(x)|}.$$

Explicit formulae for the order of the centralizers can be found in [5, Table 3.6]. If ℓ is odd, then

$$\frac{|\mathbf{C}_{\text{Sp}_{2m}(q)}(x)|}{|\mathbf{C}_{\text{O}_{2m}^\varepsilon(q)}(x)|} = \frac{|\text{Sp}_c(q)|}{2^{\delta_{c,0}-1} |\text{O}_c^\varepsilon(q)|}$$

and

$$\frac{|\text{O}_{2m}^\varepsilon(q)|}{|\text{Sp}_{2m}(q)|} = \frac{2q^{m^2-m}}{q^{m^2}(q^m + \varepsilon \cdot 1)} = \frac{2q^{-m}}{(q^m + \varepsilon \cdot 1)}.$$

Therefore, for $c > 0$, the right hand side of (17) becomes

$$\frac{2q^{-m}}{(q^m + \varepsilon \cdot 1)} \cdot \frac{(q^{c/2} + \varepsilon \cdot 1)}{q^{-c/2}} \leq 4q^{-2m+c}$$

(for the inequality observe that $(q^{c/2} + \varepsilon \cdot 1)/(q^m + \varepsilon \cdot 1) \leq (q^{c/2} + 1)/(q^m - 1) \leq 2q^{c/2-m}$). When ℓ is even we have very similar calculations: the only difference is that

$$\frac{|\mathbf{C}_{\text{Sp}_{2m}(q)}(x)|}{|\mathbf{C}_{\text{O}_{2m}^\varepsilon(q)}(x)|} = \frac{|\text{Sp}_c(q)|}{2^{\delta_{c,0}-1} |\text{O}_c^{\varepsilon'}(q)|},$$

where ε' can be either of $+$ or $-$. For $c > 0$, the right hand side of (17) then becomes

$$\frac{2q^{-m}}{(q^m + \varepsilon \cdot 1)} \cdot \frac{(q^{c/2} + \varepsilon' \cdot 1)}{q^{-c/2}} \leq \frac{2q^{-m}}{(q^m - 1)} \cdot \frac{(q^{c/2} + 1)}{q^{-c/2}} \leq 4q^{-2m+c}$$

as before.

For $c = 0$, (17) gives $\text{fpr}_{\Omega^\varepsilon}(x) \leq 2|\text{O}_{2m}^\varepsilon(q)|/|\text{Sp}_{2m}(q)| = 4q^{-m}/(q^m + \varepsilon \cdot 1)$, and the lemma follows. \square

Proposition 10.2. *For $\varepsilon \in \{+, -\}$, if $g \in G$, then g has a regular orbit on Ω^ε .*

Proof. If $q = 2$, then $G = G_0$ is simple and hence the proof follows from [12]. So we assume that $q \geq 4$. The usual computations yield $S_1(g, \Omega^\varepsilon) \leq \omega(2e(q-1))\frac{4}{3q}$. Let $x \in G$ of prime order r and suppose that r is coprime to $2e(q-1)$, so that, in particular, we have $x \in G_0$.

Let ℓ be the minimum positive integer with $r \mid (q^\ell - 1)$. In particular $\ell \geq 2$. Since r is odd, we see that x is semisimple, and arguing as usual we see that $\mathbf{C}_V(x)$ has dimension at most $2m - \ell$. It follows from Lemma 10.1 that $\text{fpr}_{\Omega^\varepsilon}(x) \leq 4q^{-\ell}$ when $\ell < 2m$, and $\text{fpr}_{\Omega^\varepsilon}(x) \leq 4/(q^m(q^m - 1)) = 4q^{-2m} + 4(q^{2m}(q^m - 1))^{-1}$ when $\ell = 2m$. The usual computations yield

$$\begin{aligned}
 S_2(g, \Omega^\varepsilon) &\leq 4 \sum_{\ell=2}^{2m} \frac{\omega_q(q^\ell - 1)}{q^\ell} + \frac{4\omega_q(q^{2m} - 1)}{q^{2m}(q^m - 1)} \\
 &\leq 4 \sum_{\ell=2}^{\infty} \frac{\omega_q(q^\ell - 1)}{q^\ell} + \frac{4\omega_q(q^{2m} - 1)}{q^{2m}(q^m - 1)} \\
 (18) \qquad &\leq 4 \log_2(q) \left(\frac{q}{(q-1)^2} - \frac{1}{q} \right) + \frac{4\omega_q(q^{2m} - 1)}{q^{2m}(q^m - 1)}.
 \end{aligned}$$

Since $\omega(2e(q-1)) \leq \log_2(q^2) = 2e$ we see that $S(g, \Omega^\varepsilon) < 1$ for $e \geq 4$. If $e = 3$, then $q = 8$, and $\omega(2e(q-1)) = 3$, which is enough to show that $S(g, \Omega^\varepsilon) < 1$. Finally, suppose that $e = 2$. Now, the only primes dividing $ep(q-1)$ are 2 and 3. If $x \in G$ has order 3, then $\dim_{\mathbb{F}_q} \mathbf{C}_V(x) \geq 2m - 2$ and hence $\text{fpr}_{\Omega^\varepsilon}(x) \leq 4/q^2$ by Lemma 10.1; thus $S_1(g, \Omega^\varepsilon) \leq 4/(3q) + 4/q^2 = 7/12$. Also $\omega_q(q^2 - 1) = \omega_q(q^3 - 1) = \omega_q(q^4 - 1) = 1$ so for the first three terms of the summation in (18), we can take $4/q^2, 4/q^3$ and $4/q^4$. This shows that $S(g, \Omega) < 1$ since we have

$$S_2(g, \Omega) \leq \frac{4}{q^2} + \frac{4}{q^3} + \frac{4}{q^4} + 8 \left(\frac{q}{(q-1)^2} - \frac{1}{q} - \frac{2}{q^2} - \frac{3}{q^3} - \frac{4}{q^4} \right) + \frac{4\omega_q(q^{2m} - 1)}{q^{2m}(q^m - 1)} < \frac{5}{12}.$$

(For the last inequality observe that $\omega_q(q^{2m} - 1) \leq \log_2(q^{2m}) = 4m$ and that $4m/(q^{2m}(q^m - 1))$ achieves its maximum at $m = 3$.) \square

Our proof of Proposition 10.2 is in line with the main techniques used in this paper, and is very much different from the arguments used in [27] to deal with the case $q = 2$. With some effort, an entirely different proof (in the same spirit as [27]) can be obtained by following the arguments in [27, Section 3] and replacing [27, Proposition 3.1] with [17, Theorem 1].

11. Case (v) of Proposition 5.5: $G_0 = \text{PSL}_n(q)$, G contains a graph automorphism and a point stabilizer is of type $P_{k,n-k}$ or $\text{GL}_k(q) \oplus \text{GL}_{n-k}(q)$

In this section, we let G be a primitive group on Ω with socle $G_0 = \text{PSL}_n(q)$ and containing a graph automorphism. For the time being, we are interested in only two subspace actions of G . We suppose that, for some k with $1 \leq k < n/2$, we have

$$\Omega = \Omega_{k, \leq}$$

where

$$\Omega_{k, \leq} := \{ \{W, U\} \mid W, U \leq V, \dim W = k, \dim U = n - k, W \leq U \},$$

or

$$\Omega = \Omega_{k, \perp}$$

where

$$\Omega_{k,\perp} := \{\{W, U\} \mid W, U \leq V, \dim W = k, \dim U = n - k, V = W \oplus U\}.$$

In both cases, $\mathrm{P}\Gamma\mathrm{L}_n(q)$ acts transitively on Ω and, using the notation in [20], the stabilizer of a point of Ω is a subspace subgroup of type $P_{k,n-k}$ (when $\Omega = \Omega_{k,\leq}$) or $\mathrm{GL}_k(q) \oplus \mathrm{GL}_{n-k}(q)$ (when $\Omega = \Omega_{k,\perp}$).

We endow V with a non-degenerate bilinear form and we denote by τ the involutory automorphism of the projective geometry mapping W to W^\perp , for $W \leq V$. Observe that τ normalizes $\mathrm{P}\Gamma\mathrm{L}_n(q)$ and acts on G_0 as a graph automorphism.

Proposition 11.1. *If $g \in G$, then g has a regular cycle.*

Proof. Suppose that $g \in G \cap \mathrm{P}\Gamma\mathrm{L}_n(q)$. By Propositions 5.5, 6.2 and 7.2, there exists a k -subspace U of V with $\langle g \rangle$ inducing in its action on k -subspaces a regular orbit on $\{U^h \mid h \in \langle g \rangle\}$. Let $\omega = \{U, W\} \in \Omega$, for some $(n - k)$ -subspace W of V . Clearly, $\langle g \rangle$ induces a regular orbit on ω .

Suppose that $g \notin \mathrm{P}\Gamma\mathrm{L}_n(q)$. In particular, g has even order. From Lemma 2.1, we may assume that g has square-free order. Write $g = xy = yx$, with $|x| = 2$ and $|y|$ odd, that is, $y \in \mathrm{P}\Gamma\mathrm{L}_n(q)$. As $g \notin \mathrm{P}\Gamma\mathrm{L}_n(q)$, we see that $x = \tau z$, for some $z \in \mathrm{P}\Gamma\mathrm{L}_n(q)$. In particular, x maps k -subspaces to $(n - k)$ -subspaces. Let W be a k -subspace with y inducing a regular orbit on $\{W^h \mid h \in \langle y \rangle\}$. Note that since y has odd order, we have $\{y^{1+2t} \mid t \geq 0\} = \langle y \rangle$.

Assume that there exists $U \notin \{W^{xy^i} \mid i \geq 0\}$ with $\omega = \{W, U\} \in \Omega$. If $\omega^{g^i} = \omega$ with $1 \leq i < |g|$, then i is odd because $g^2 = y^2$ acts regularly on $\{W^h \mid h \in \langle y \rangle\}$. Therefore, $i = 1 + 2t$, for some $t \geq 0$. Now, $\omega^{g^i} = \{W^{xy^{1+2t}}, U^{xy^{1+2t}}\} = \omega = \{W, U\}$ and hence $U = W^{xy^{1+2t}}$, a contradiction. This yields that $\langle g \rangle$ induces a regular orbit on ω . In particular, we may assume that every subspace U of V with $\{W, U\} \in \Omega$ is of the form W^{xy^i} , for some $i \geq 0$. This yields that every subspace U of V with $\{W^y, U\} \in \Omega$ is in $(\{W^{xy^i} \mid i \geq 0\})^y = \{W^{xy^i} \mid i \geq 0\}$. When $\Omega = \Omega_{k,\perp}$, it follows that $V = W \oplus U$ if and only if $V = W^y \oplus U$, and when $\Omega = \Omega_{k,\leq}$, it follows that $W \leq U$ and $\dim U = n - k$ if and only if $W^y \leq U$ and $\dim U = n - k$.

Let t_1, \dots, t_κ be a basis of $W \cap W^y$, let $t_1, \dots, t_\kappa, w_1, \dots, w_{k-\kappa}$ be a basis of W , let $t_1, \dots, t_\kappa, w'_1, \dots, w'_{k-\kappa}$ be a basis of W^y and let $t_1, \dots, t_\kappa, w_1, \dots, w_{k-\kappa}, w'_1, \dots, w'_{k-\kappa}, v_1, \dots, v_{n-2k+\kappa}$ be a basis of V .

Suppose $\Omega = \Omega_{k,\perp}$ and set $U = \langle w'_1, \dots, w'_{k-\kappa}, v_1, \dots, v_{n-2k+\kappa} \rangle$. Clearly, $V = W \oplus U$, and hence $V = W^y \oplus U$. This gives $W^y \cap U = 0$ and hence (by the definition of U) we get $k = \kappa$ and $W = W^y$.

Suppose $\Omega = \Omega_{k,\leq}$ and set $U = \langle t_1, \dots, t_\kappa, w_1, \dots, w_{k-\kappa}, v_1, \dots, v_{n-2k} \rangle$. Clearly, $W \leq U$ and $\dim U = n - k$, and hence $W^y \leq U$. The definition of U gives $k = \kappa$ and hence $W^y = W$.

The previous two paragraphs give $W = W^y$. Since y induces a regular orbit on $\{W^h \mid h \in \langle y \rangle\}$, we get $y = 1$ and $g = x$ has order 2. Now, clearly g has a regular cycle. \square

12. Case (vii) OF PROPOSITION 5.5: NOVELTIES

Here we may assume that $G_\alpha \cap G_0$ is non-maximal in G_0 and since G_α is a subspace subgroup, the only maximal subgroups of G_0 containing $G_\alpha \cap G_0$ are either reducible subgroups (as in Definition 4.1(a)) or of type $O_{2m}^\pm(2^f)$ when $G_0 = \mathrm{Sp}_{2m}(2^f)'$ (as in Definition 4.1(b)). The case of $G_0 = \mathrm{P}\Omega_8^+(q)$ is special and the

TABLE 4. Notation for Section 12 (from [20])

G_0	Ω	Γ	A
$\mathrm{PSL}_n(q)$	$\mathrm{SL}_n(q)$	$\mathrm{GL}_n(q)$	$\Gamma\langle\iota\rangle$
$\mathrm{PSU}_n(q)$	$\mathrm{SU}_n(q)$	$\mathrm{GU}_n(q)$	Γ
$\mathrm{PSp}_n(q)$	$\mathrm{Sp}_n(q)$	$\mathrm{GSp}_n(q)$	Γ
$\mathrm{P}\Omega_n^\varepsilon(q)$	$\Omega_n^\varepsilon(q)$	$\mathrm{GO}_n^\varepsilon(q)$	Γ

novelties have been obtained explicitly by Kleidman [19]. We postpone this case until the end of this section and assume for now that $G_0 \neq \mathrm{P}\Omega_8^+(q)$. Moreover, by Theorem 3.1, we may assume that $n \geq 5$ and in particular, that $G_0 \neq \mathrm{PSp}_4(q)$.

We use the notation of [20] quite liberally in this section. In particular, we let $H = G_\alpha$ be a maximal subgroup of G . By Aschbacher’s theorem, H (which is sometimes denoted by H_G in [20]) belongs to one of the nine collections $\mathcal{C}_i(G)$ of subgroups of G (where $i \in \{1, \dots, 9\}$). We define Ω , Γ and A depending on G_0 according to Table 4. The reader will notice an abuse of notation here, since we have hitherto used Ω to denote a primitive G -set. For the remainder of this section, we will denote this primitive G -set by Ω_{set} .

For a group X such that $\Omega \leq X \leq A$ we define \overline{X} to be the quotient group $\overline{X} = XZ/Z$, where $Z = \mathbf{Z}(\mathrm{GL}_n(q_0))$ (and where q_0 is as in Notation 2.5). In particular $\overline{\Omega}$ is the simple group G_0 ($G_0 \neq \mathrm{Sp}_4(2)'$ since $n \geq 5$) and we have $G = \overline{X}$ for some X as above. The subgroups $H \in \mathcal{C}_9(G)$ are almost simple, and each has a socle whose covering group L acts absolutely irreducibly on the natural module V and satisfies various other conditions to prevent H from being naturally contained in a subgroup in $\bigcup_{i=1}^8 \mathcal{C}_i(G)$. For these groups we define the corresponding subgroup of $\overline{\Omega}$ to be $H_{\overline{\Omega}} = H \cap \overline{\Omega}$.

For $i = 1, \dots, 8$, we define the subgroup $H_\Gamma \in \mathcal{C}_i(\Gamma)$ to be the stabilizer in Γ of some geometrical structure on V . For example, when $i = 2$, we have $V = V_1 \oplus \dots \oplus V_k$ where $k = \dim V_j$ for all j and $H_\Gamma = \mathbf{N}_\Gamma(\{V_1, \dots, V_k\})$. The corresponding subgroup of Ω , denoted H_Ω , is then defined to be $H_\Gamma \cap \Omega$. Similarly, for X as above, we define the corresponding subgroup H_X of X to be

$$H_X = \begin{cases} H_\Gamma \cap X & \text{if } X \leq \Gamma; \\ \mathbf{N}_X(H_\Gamma) & \text{otherwise.} \end{cases}$$

Further, we define $H_{\overline{X}}$ to be $H_X Z/Z$. In particular $H_{\overline{\Omega}} = H_{\overline{\Gamma}} \cap \overline{\Omega}$. Moreover, in almost all situations, we have $H_{\overline{\Omega}} = H_G \cap \overline{\Omega}$. (Indeed, this is clear from the definitions above if $G \leq \overline{\Gamma}$ since $H_{\overline{\Omega}} = H_{\overline{\Gamma}} \cap \overline{\Omega}$ and $H_G = H_{\overline{\Gamma}} \cap G$. Otherwise G_0 is linear, $G \not\leq \overline{\Gamma}$ and we have $H_G \cap \overline{\Omega} = \mathbf{N}_G(H_{\overline{\Gamma}}) \cap \overline{\Omega} = \mathbf{N}_{\overline{\Omega}}(H_{\overline{\Gamma}})$. As $H_{\overline{\Omega}} = H_{\overline{\Gamma}} \cap \overline{\Omega}$, we have $H_{\overline{\Omega}} \leq H_G \cap \overline{\Omega}$; however it may occur that equality does not hold when (and only when) H_G is a type $\mathrm{GL}_1(q)$ wr S_n group. Moreover this only happens when $q = 2$ and n is even or when $(n, q) = (2, 5), (4, 3)$, or $(3, 4)$ (see [20, Proposition 3.1.3]). For example, when $q = 2$, n is even and $G = \mathrm{GL}_n(2)\langle\iota\rangle$, we have $H_{\overline{\Omega}} \cong \mathrm{GL}_1(2)$ wr $S_n \cong S_n$ but $H_G \cap \overline{\Omega} = \mathbf{N}_{\overline{\Omega}}(H_{\overline{\Gamma}}) = 2 \times S_n$.)

Usually, $H_{\overline{\Omega}}$ is maximal in G_0 , in which case H is not a novelty and hence H does not concern us. Thus we may assume that $H_{\overline{\Omega}}$ is strictly contained in some subgroup $K_{\overline{\Omega}} \in \mathcal{C}_i(\overline{\Omega})$, for some $i \in \{1, \dots, 9\}$. It is very important to observe that,

since $H = H_G$ is a subspace subgroup, we have

$$K_{\overline{\Omega}} \in \mathcal{C}_1(\overline{\Omega}), \text{ or } G_0 = \text{Sp}_n(2^f) \text{ and } K_{\overline{\Omega}} \in \mathcal{C}_8(\overline{\Omega}).$$

Complete information on the novelties for $n \geq 13$ is contained in [20, Table 3.5.H]. Reading off the possibilities for $H_{\overline{\Omega}}$ and $K_{\overline{\Omega}}$ in [20, Table 3.5.H] we have

- (row 6) $H_{\overline{\Omega}}$ of type $P_{k,n-k}$, $G_0 = \text{PSL}_n(q)$ and G contains graph automorphisms. This case has been eliminated already in Section 11.
- (row 7) $H_{\overline{\Omega}}$ of type $\text{GL}_k(q) \oplus \text{GL}_{n-k}(q)$, $G_0 = \text{PSL}_n(q)$ and G contains graph automorphisms. This case also has been eliminated already in Section 11.
- (row 12) $H_{\overline{\Omega}}$ of type $P_{n/2-1}$, G_0 orthogonal. This case is eliminated in Section 12.1.
- (row 14) $H_{\overline{\Omega}}$ of type $O_2^+(3) \perp O_{n-2}(3)$, $G_0 = \text{P}\Omega_n^\pm(q)$ (n even) and G contains similarities. This case is eliminated in Section 12.1.
- (row 22) $H_{\overline{\Omega}}$ of type $\text{GL}_{n/2}(q).2$, G_0 orthogonal. This case is eliminated in Section 12.2.

We note that the examples in row 15 of [20, Table 3.5.H], where $H_{\overline{\Omega}}$ of type $O_2^+(2)$ wr $S_{n/2}$ is contained in $K_{\overline{\Omega}}$ of type $P_{n/2}$, do not in fact yield maximal subgroups of G since $\mathbf{N}_G(H_{\overline{\Omega}}) < \mathbf{N}_G(K_{\overline{\Omega}})$ (see [20, page 68] and row 12 of [20, Table 3.5.H]).

When $n \leq 12$, again we recall that Aschbacher’s theorem [1] asserts that since $H = G_\alpha$ is maximal in G , it must be a subgroup of type $\mathcal{C}_i(G)$ for some $i \in \{1, \dots, 9\}$. Loosely speaking, we have two strategies: first we adapt some of the arguments in [20] for our purposes, and second we use the complete classification of the low dimensional classical groups in [3]. Moreover our preliminary arguments will simplify the checking of the tables of maximal subgroups in [3].

Proposition 12.1. *Suppose that $n \geq 5$, $G_0 \neq \text{P}\Omega_8^\pm(q)$, $H = G_\alpha$ is a subspace subgroup as in Definition 4.1 and that $G_0 \cap H$ is not maximal in G_0 (so that H is a G -novelty). Then H is not a subgroup of type $\mathcal{C}_i(G)$ for any $i \in \{4, \dots, 9\}$.*

Proof. First suppose that $H \in \mathcal{C}_4(G)$ so that H_Γ stabilizes a tensor decomposition $V = V_1 \otimes V_2$ as described in [20, Section 4.4]. We claim that H_Ω acts irreducibly on V . H_Ω contains a subgroup of the form $\Omega_1 \otimes \Omega_2$ (by [20, (4.4.13)]), which acts absolutely irreducibly on V provided Ω_i acts absolutely irreducibly on V_i for $i = 1$ and 2 by [20, Proposition 4.4.3(vi)]. This is true as long as $\Omega_i \neq \Omega_2^\pm(q)$ by [20, Proposition 2.10.6]; but $\Omega_i \neq \Omega_2^\pm(q)$ in all cases by definition of \mathcal{C}_4 [20, Table 4.4.A]. Thus H_Ω acts irreducibly on V and hence H cannot be a subspace subgroup as in Definition 4.1 (note that when $G_0 = \text{Sp}_n(2^f)$ there are no \mathcal{C}_4 -subgroups [20, Table 4.4.A]).

Similarly suppose that $H \in \mathcal{C}_7(G)$ so that H_Γ stabilizes a subspace decomposition $V = V_1 \otimes \dots \otimes V_t$, with $\dim V_i = k$ and $n = \dim V = k^t$. Again there are no \mathcal{C}_7 -subgroups when $G_0 = \text{Sp}_n(2^f)$ so it suffices to show that H_Ω acts irreducibly on V . Now H_Ω contains a subgroup L of the form $\Omega'_1 \otimes \dots \otimes \Omega'_t$, and Ω'_i acts absolutely irreducibly on V_i by [20, Corollary 2.10.7] (each Ω'_i is quasisimple by definition of \mathcal{C}_7 [20, page 156]). Thus L and H_Ω act absolutely irreducibly on V by [20, Lemma 4.4.3(vi)].

Next, suppose that H is a subfield subgroup in $\mathcal{C}_5(G)$. Now H_Ω contains a subgroup $\Omega_{\mathbb{F}}$ by [20, (4.5.5)], which acts absolutely irreducibly on the corresponding

vector space $V_{\#}$ over the corresponding subfield by [20, Proposition 2.10.6], and hence $\Omega_{\#}$ and H_{Ω} act irreducibly on $V = V_{\#} \otimes \mathbb{F}_q$. Thus we may assume that q is even, G_0 is symplectic and H_{Ω} preserves a quadratic form Q . However this is absurd. Indeed, let $w \in V_{\#} \setminus \{0\}$ with $Q(w) = 0$. Now $\Omega_{\#} = \text{Sp}_n(q_0)$ is transitive on the non-zero vectors of $V_{\#}$ by [20, Lemma 2.10.5], which implies that $Q(v) = 0$ for all $v \in V_{\#}$. It follows that the associated bilinear form f_Q vanishes on V , which is a contradiction since f_Q is non-degenerate (by definition). Thus H_{Ω} is irreducible and not contained in a type $O_n^{\pm}(2^f)$ subgroup when $G_0 = \text{Sp}_n(2^f)$.

Now suppose that $H \in \mathcal{C}_6(G)$. Note that if G_0 is symplectic, then q is odd (by [20, Table 3.5.C]), so it suffices to show that H_{Ω} acts irreducibly on V . Certainly the r -group R in the definition of $\mathcal{C}_6(\Gamma)$ acts (absolutely) irreducibly on V by definition [20, page 150]. Moreover, R is contained in H_{Ω} (see the proof of [20, Proposition 4.6.4]) and hence H_{Ω} also acts irreducibly on V .

Now let us assume that H is of classical type in $\mathcal{C}_8(G)$ so that G_0 is either linear or symplectic (see [20, Table 4.8.A]). In all these cases, since $n \geq 3$, H_{Ω} acts absolutely irreducibly on V by [20, Proposition 2.10.6]. Thus we may assume that $G_0 = \text{Sp}_n(2^f)$ and that H is of type $O_n^{\pm}(q)$. We claim that these groups do not give rise to novelties since by [3, Chapter 8] these subgroups are maximal in $G_0 = \text{Sp}_n(2^f)$ for $n \leq 12$.

Finally, suppose that H is an almost simple subgroup of type \mathcal{C}_9 . Then, by definition, H_{Ω} acts absolutely irreducibly on V [20, page 3]; thus we may assume that $G_0 = \text{Sp}_n(2^f)$ and $H \cap G_0$ is only contained in a type $O_n^{\pm}(2^f)$ subgroup. In particular, the socle of H_{Ω} fixes a quadratic form, which cannot happen when $G_0 = \text{Sp}_n(2^f)$ by [20, page 3]. □

The next two lemmas prove results for \mathcal{C}_2 -subgroups and \mathcal{C}_3 -subgroups.

Lemma 12.2. *Suppose that $n \geq 5$, that $G_0 \neq P\Omega_8^+(q)$, that $H = G_{\alpha}$ is a subspace subgroup as in Definition 4.1 and that $G_0 \cap H$ is not maximal in G_0 (so that H is a G -novelty). Then H is not a \mathcal{C}_2 -subgroup of type $\text{GL}_k^{\pm}(q) \text{ wr } S_t$, $\text{Sp}_k(q) \text{ wr } S_t$ or $O_k^{\zeta}(q) \text{ wr } S_t$.*

Proof. We argue by contradiction and we assume that H is a \mathcal{C}_2 -subgroup of type $\text{GL}_k^{\pm}(q) \text{ wr } S_t$, $\text{Sp}_k(q) \text{ wr } S_t$ or $O_k^{\zeta}(q) \text{ wr } S_t$. Here H_{Γ} is the stabilizer $\text{N}_{\Gamma}(\{V_1, \dots, V_t\})$ where $V = V_1 \oplus \dots \oplus V_t$ is a decomposition \mathcal{D} of V into subspaces of equal dimension k . Observe that we are assuming that the V_i s are non-degenerate when G_0 is not linear.

Observe that when q is even and G_0 is symplectic we have $q \geq 4$, since otherwise $\text{Out}(G_0) = 1$ and there are no novelties.

We claim that subgroups H of type $\text{Sp}_k(2^f) \text{ wr } S_t$ do not fix a quadratic form Q (when $q \geq 4$). In fact, if this is not the case, then $H_{\Omega} \leq K$ with $K = O_n^{\pm}(q)$. Now, note that $\text{Sp}_k(2^f)^t$ is contained in H_{Ω} . As $\text{Sp}_k(2^f)^t$ acts irreducibly on V_1 , we see that V_1 is either totally singular or non-degenerate with respect to Q . Suppose that V_1 is totally singular. Let $g \in \text{Sp}_k(2^f)$ with $g \neq 1$ and set $g_1 = (g, 1, \dots, 1) \in \text{Sp}_k(2^f)^t$. Clearly, g_1 acts non-trivially on V_1 and trivially on V_i for $i \geq 2$. Therefore g_1 acts non-trivially on V_1 and trivially on $V/V_1^{\perp Q}$. However, this contradicts [20, Lemma 4.1.9]: if g_1 acts non-trivially on V_1 , then it acts non-trivially also on $V/V_1^{\perp Q}$. This shows that V_1 is non-degenerate with respect to Q and hence the natural image K_{V_1} of K in $\text{GL}(V_1)$ is of type $O_k^{\pm}(2^f)$. Now observe that K_{V_1} contains the natural image H_{V_1} of H_{Ω} in $\text{GL}(V_1)$ and that H_{V_1} is of type $\text{Sp}_k(2^f)$.

This can only be true if $k = q = 2$, which contradicts our assumption that $q \geq 4$. Thus $\text{Sp}_k(2^f)$ wr S_t does not fix a quadratic form (when $q \geq 4$).

Now the result follows by showing that H_Ω acts irreducibly on V . By [20, Proposition 2.10.13], H_Ω acts irreducibly on V if the following three conditions hold:

- (i) $(H_\Omega)_{(\mathcal{D})}$, the subgroup of H_Ω fixing setwise each V_i , acts irreducibly on each V_i ;
- (ii) the V_i are pairwise non-isomorphic $(H_\Omega)_{(\mathcal{D})}$ -modules; and
- (iii) H_Ω acts transitively on $\{V_1, \dots, V_t\}$.

To prove (i), we may assume that $\dim V_i \geq 2$, otherwise the result is trivial. By [20, Proposition 2.10.6], (i) holds unless $\Omega_i = \Omega(V_i)$ is of type $\Omega_2^\pm(q)$, since $(H_\Omega)_{(\mathcal{D})}$ contains Ω_i . In this exceptional case, G_0 is orthogonal and n is even so we may assume that $n \geq 8$ and $t \geq 4$. Now by [20, (6.2.3)], for each $g' \in O_m^c(V_i)$, there exists $g \in (H_\Omega)_{(\mathcal{D})}$ that acts on V_i in the same way as g' . In particular, $(H_\Omega)_{(\mathcal{D})}$ acts absolutely irreducibly on V_i in this case as well, unless $q = 2, 3$ and $\Omega_i(q) = \Omega_2^+(q)$ by [20, Proposition 2.10.6(iii)].

For (ii) we simply observe that if $\Omega_i = \Omega(V_i)$ is non-trivial, then $(H_\Omega)_{(\mathcal{D})}$ contains an element g that acts non-trivially on V_i and trivially on V_j for all $j \neq i$. So (ii) holds unless $\Omega_i = 1$, which is only true if $\Omega_i = \text{SL}_1(2), \Omega_2^+(2)$ or $\Omega_2^+(3)$.

Finally (iii) always holds from [20, Lemma 4.2.2(iii)] (see also Proposition 4.2.9, page 106, page 109 and (4.2.12) of [20]).

The lemma now follows from [20, Proposition 2.10.13] unless H is of type

$$\text{GL}_1(2) \text{ wr } S_n, O_2^+(2) \text{ wr } S_{n/2} \text{ or } O_2^+(3) \text{ wr } S_{n/2}.$$

In the first case, H stabilizes the decomposition $V = V_1 \oplus \dots \oplus V_n$ where $V_i = \langle v_i \rangle$ and (v_1, \dots, v_n) is a basis of V . If n is odd, then H is contained in $\mathbf{N}_G(K)$ where K is a subgroup of type $\text{GL}_1(2) \oplus \text{GL}_{n-1}(2)$; therefore this case does not occur. Suppose n is even, define $f(v_i, v_j) = 1 + \delta_{ij}$ and extend f bilinearly to V . It is easy to show that f is a symmetric non-degenerate bilinear form on V ; thus H_Ω is contained in $\text{Sp}_n(2)$ and H is not a subspace subgroup.

In the second case, subgroups H of this type are never maximal, for they are contained in $\mathbf{N}_G(K)$ where K is a subgroup of type $P_{n/2-1}$ (see [20, page 68]); so these cases do not concern us.

The examples in the third case are eliminated since [3, Chapter 8] shows that these subgroups do not give rise to novelties. □

Lemma 12.3. *Let H be a \mathcal{C}_3 field extension subgroup of G of type $X_t(q^r)$ and suppose that $G_0 \neq \text{P}\Omega_8^+(q)$ and $n \geq 5$. Then H_Ω acts irreducibly on V when $t \geq 2$ (which is always true when G_0 is symplectic or orthogonal). If G_0 is linear or unitary, then $H_\Gamma \cap \text{GL}_n^\pm(q)$ acts irreducibly on V in all cases.*

Proof. See [20, Lemma 4.3.2 and Proposition 4.3.3(i)]. □

Proposition 12.4. *Suppose that $n \geq 5$, that $G_0 \neq \text{P}\Omega_8^+(q)$, that $H = G_\alpha$ is a subspace subgroup as in Definition 4.1 and that $G_0 \cap H$ is not maximal in G_0 (so that H is a G -novelty). Then H is one of the generic examples in rows 6, 7, 12, 14, 22 of [20, Table 3.5H].*

Proof. By Proposition 12.1 and Lemmas 12.2 and 12.3, for $n \leq 12$ we just need to check in [3, Chapter 8] that subgroups of G_0 of the following types are indeed maximal in G_0 :

- (1) type $C_1(\overline{\Omega})$,
- (2) type $GL_1^\epsilon(q^n)$ when n is prime and $G_0 = PSL_n^\epsilon(q)$,
- (3) type $GL_{n/2}(q^u).2$ when $G_0 = PSU_n(q)$, $P\Omega_{2m}^+(q)$ or when $G_0 = PSp_{2m}(q)$ and q is odd,
- (4) type $O_{n/2}(q)^2$ when q is odd and $n/2$ is odd.

Note that when G_0 is symplectic, we may assume that $q \geq 3$ since otherwise $\text{Out}(G_0) = 1$ and there are no novelties. The result now follows easily by inspection unless (i) $G_0 = PSU_6(2)$, $PSU_5(2)$, or (ii) $G_0 = P\Omega_n^\pm(2)$ or $P\Omega_n^\epsilon(3)$ and H is a C_1 -subgroup of type $O_2^+(q) \perp O_{n-2}(q)$.

For case (i), there are no novelties when $G_0 = PSU_5(2)$ by [3, Table 8.20] and we verify the case $G_0 = PSU_6(2)$ in magma.

For case (ii) with $q = 2$, we note that no novelties occur by [3, Tables 8.31, 8.33, 8.50, 8.52, 8.66, 8.68, 8.82, 8.84]. For case (ii) with $q = 3$, the case n even is in row 14 of [20, Table 3.5H] and hence we may assume that n is odd and $G = PSO_n(3)$. Now [3, Tables 8.22, 8.39, 8.58, 8.74] show that there are no novelties here. \square

Proposition 12.4 shows that, when $G_0 \neq P\Omega_8^+(q)$, for the proof of Theorem 1.2 the only G -novelties we need to consider are the generic examples in rows 6, 7, 12, 14, 22 of [20, Table 3.5H].

Next, we turn our attention to $G_0 = P\Omega_8^+(q)$. We check [19] carefully and we find that the only possibilities are the following:

- (i) $H \cap G_0$ is the stabilizer of a totally singular 3-subspace and G does not contain a triality automorphism (as in row 12 of [20, Table 3.5H]);
- (ii) G does contain a triality automorphism and H is a parabolic subgroup P_2 corresponding to the removal of the three outer nodes of the Dynkin diagram.

For the proof of Theorem 1.2, case (i) is considered in Section 12.1. We conclude this section by dealing with case (ii). Here we know that $|H|$ divides $q^{11}(q-1)^2 |PGL_2(q)| |\text{Sym}(4)|e$; thus for $x \in G$ of prime order r we have $\text{fpr}_\Omega(x) = 0$ unless $r \mid 6ep(q^2 - 1)$, by Lemma 2.3. Now the usual calculations show that

$$S(g, \Omega) \leq \omega(6ep(q^2 - 1)) \frac{4}{3q},$$

which is less than 1 unless $q = 2, 4$. We eliminate the case $q \in \{2, 4\}$ with the help of magma.

12.1. Novelties: Lines 12 and 14 in [20, Table 3.5H]. Here $G_0 = P\Omega_{2m}^+(q)$ and $G_0 \triangleleft G \leq \text{Aut}(G_0)$. The group G acts primitively on the set Ω_{set} of totally isotropic subspaces of V of dimension $m - 1$ or on the set Ω_{set} of non-degenerate 2-subspaces of $+$ type.

Proposition 12.5. *If $g \in G$, then g has a regular orbit on Ω_{set} .*

Proof. By Theorem 5.4, the permutation character of G in its action on Ω_{set} is contained in the permutation character of G in its action on the totally singular 1-subspaces of V . Now the result follows from Lemma 5.3 and Proposition 7.2. \square

12.2. **Novelties: Line 22 in** [20, Table 3.5H]. Here $G_0 = \mathrm{P}\Omega_{2m}^+(q)$ with m odd and $G_0 \triangleleft G \leq \mathrm{Aut}(G_0)$. The group G acts primitively on the set

$$\Omega_{\mathrm{set}} = \{\{U, W\} \mid V = U + W, \dim U = \dim W = m, \text{ and } U, W \text{ totally singular}\}.$$

Proposition 12.6. *If $g \in G$, then g has a regular orbit on Ω_{set} .*

Proof. The proof is similar to the proof of Proposition 11.1. From Remark 5.6, we have $m \geq 4$, and hence $m \geq 5$ because m is odd.

From Proposition 6.2, the element g has a regular orbit in its action on the totally singular subspaces of V of dimension m . Let U be an element in such a regular orbit. Suppose that there exists $W \notin \{U^{g^t} \mid t \geq 0\}$ with $\{U, W\} \in \Omega_{\mathrm{set}}$. Then, clearly, g has a regular orbit on $\{U, W\}$. In particular, we may assume that the only totally singular subspaces W of V with $V = U + W$ are of the form U^{g^i} , for some positive integer i . An easy computation shows that there are exactly $q^{m(m-1)/2}$ totally singular complements W for U in V . Therefore $|g| \geq q^{m(m-1)/2}$. However this contradicts [16, Theorem 2.16], which states that $|g| \leq q^{m+1}/(q-1)$. \square

ACKNOWLEDGEMENT

We thank the referee for carefully reading the manuscript and for several helpful comments. In particular, for pointing out that [3] is now available, which has simplified some of our arguments.

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