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ON THE SLOPE OF HYPERELLIPTIC FIBRATIONS WITH POSITIVE RELATIVE IRREGULARITY

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ABSTRACT. Let $f : S \to B$ be a locally non-trivial relatively minimal fibration of hyperelliptic curves of genus $g \geq 2$ with relative irregularity q_f. We show a sharp lower bound on the slope λ_f of f. As a consequence, we prove a conjecture of Barja and Stoppino on the lower bound of λ_f as an increasing function of q_f in this case, and we also prove a conjecture of Xiao on the ampleness of the direct image of the relative canonical sheaf if $\lambda_f < 4$.

1. INTRODUCTION

Let $f : S \to B$ be a fibration (or a family) of curves of genus $g \geq 2$, i.e., f is a proper surjective morphism from a smooth complete surface S to a smooth complete curve B with connected fibers over a complex number, and the general fiber is a smooth complete curve of genus g. If the general fiber is a hyperelliptic curve, then we call f a *hyperelliptic fibration*. The fibration f is called *relatively minimal* if there is no (-1)-curve contained in the fibers of f. Here a curve C is called a $(-k)$-*curve* if it is a smooth rational curve with self-intersection $C^2 = -k$. Without other statements, we always assume that fibrations in this paper are relatively minimal. The fibration f is called *smooth* if all its fibers are smooth, *isotrivial* if all its smooth fibers are isomorphic to each other, *locally trivial* if it is both smooth and isotrivial, and *semi-stable* if all its singular fibers are semi-stable. Here a singular fiber F of f is called *semi-stable* if it is a reduced nodal curve.

Let ω_S (resp. K_S) be the canonical sheaf (resp. the canonical divisor) of S. Denote by $\omega_{S/B} = \omega_S \otimes f^*\omega_B^\vee$ (resp. $K_f = K_{S/B} = K_S - f^*K_B$) the relative canonical sheaf (resp. the relative canonical divisor) of f. If f is relatively minimal, K_f is numerical effective (nef), i.e., $K_f \cdot C \geq 0$ for any curve $C \subseteq S$. Set $b = g(B)$, $p_g = h^0(S, \omega_S)$, $q = h^1(S, \omega_S)$, $\chi(\mathcal{O}_S) = p_g - q + 1$, and let $\chi_{\rm top}(S)$ be the topological Euler characteristic of S. We consider the following relative invariants of f:

$$\begin{aligned}\chi_f &= \deg f_*\omega_{S/B} = \chi(\mathcal{O}_S) - (g-1)(b-1),\\ K_f^2 &= \omega_{S/B}\cdot\omega_{S/B} = K_S^2 - 8(g-1)(b-1),\\ e_f &= \chi_{\rm top}(S) - 4(g-1)(b-1).\end{aligned}$$

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They satisfy the Noether's formula

$$12\chi_f = K_f^2 + e_f. \tag{1-1}$$

If f is relatively minimal, then these invariants are non-negative, and $\chi_f = 0$ (equivalently, $K_f^2 = 0$) if and only if f is locally trivial (see [1]). Note also that $e_f = 0$ iff f is smooth.

The *relative irregularity* q_f of f is defined to be

$$q_f = q - b.$$

It is clear that $0 \leq q_f \leq g$. The equality $q_f = g$ holds if and only if S is birational to $B \times F$ by [8]. For $b \geq 1$, $q_f = 0$ if and only if f is the Albanese map of S.

If f is not locally trivial, the *slope* of f is defined to be

$$\lambda_f = \frac{K_f^2}{\chi_f}.$$

It follows immediately that $0 < \lambda_f \leq 12$. It turns out that the slope of a fibration is sensible to a lot of geometric properties, both of the fibers of f and of the surface S itself (cf. [3]). We are mainly concerned with a lower bound of the slope. The main known result in this direction is the slope inequality

$$\text{If } g \geq 2 \text{ and } f \text{ is not locally trivial, then } \lambda_f \geq \frac{4(g-1)}{g}. \tag{1-2}$$

This was first proven by Horikawa and Persson for hyperelliptic fibrations. Xiao gave a proof for general fibrations (cf. [29]), and independently, Cornalba and Harris proved it for semi-stable fibrations (cf. [11]).

We would like to concentrate on the influence of the relative irregularity q_f on the slope λ_f of f. It seems that the lower bound of λ_f should be an increasing function of q_f. The main influence of q_f is the following Fujita decomposition (cf. [15, 16, 19]; see also Catanese and Dettweiler's recent paper [10] for a more detailed proof of the decomposition and a discussion on the unitary factor):

$$f_*\omega_{S/B} = \mathcal{A} \oplus \mathcal{F} \oplus \mathcal{O}_B^{\oplus q_f}, \tag{1-3}$$

with $\mathcal{A}$ ample, $\mathcal{F}$ unitary and $\dim H^1(B, \Omega_B^1(\mathcal{F})) = 0$. The first result in this direction is due to Xiao ([29]):

If $q_f > 0$, then $\lambda_f \geq 4$ and the equality holds only if $q_f = 1$.

In particular, $q_f = 0$ if $\lambda_f < 4$. He made the following conjecture ([29, Conjecture 2]):

Conjecture 1.1 (Xiao)**.** *For any locally non-trivial fibration f, $f_*\omega_{S/B}$ has no locally free quotient of degree zero (i.e., $f_*\omega_{S/B}$ is ample) if $\lambda_f < 4$.*

The above conjecture was confirmed to be true by Barja and Zucconi (cf. [6]) under the assumption that f is non-hyperelliptic or g (or b) is small.

Related to the influence of q_f on the lower bound of λ_f, Xiao asked the following question ([30, Problem 4]):

Problem 1.2 (Xiao)**.** Let $f : S \to B$ be a fibration of genus $g \geq 2$, which is not locally trivial. Find a good relationship between λ_f, q_f and g.

After that, there are many important results in this direction; see for instance, [5, 7, 12, 20, 21, 23]. Some explicit lower bounds depending on q_f are also given in these literatures. Among these, we would like to highlight the recent result obtained by Barja and Stoppino in [5, Theorem 1.3]. They proved that

$$\lambda_f \geq \frac{4(g-1)}{g-[m/2]}, \tag{1-4}$$

where $m := \min\{\mathrm{Cliff}(f),\, q_f\}$ and $\mathrm{Cliff}(f)$ is defined to be the Clifford index of the general fiber of f. When $\mathrm{Cliff}(f)$ is big enough, we see from (1-4) that the lower bound of λ_f is indeed an increasing function of q_f. In fact, based on much evidence, they made the following conjecture ([5, Conjecture 1.1]).

Conjecture 1.3 (Barja and Stoppino)**.** *Let $f : S \to B$ be as in Problem* 1.2*. If $q_f < g-1$, then*

$$\lambda_f \geq \frac{4(g-1)}{g-q_f}. \tag{1-5}$$

The bound (1-4) obtained by Barja and Stoppino is very close to the conjectured bound (1-5) once $\mathrm{Cliff}(f)$ is big. In [12] a bound for fibrations which are double covers of fibrations of genus γ was proved. This bound is analogous to the conjectured bound (1-5), but this second bound is not proven there. In [5, Example 4.1] it was proved that $\gamma = q_f$ and the two bounds coincide for double covers of trivial fibrations whose associated line bundle is ample. In [23], we proved (1-5) for semi-stable hyperelliptic fibrations. We remark that if $q_f = g-1$, (1-5) is known to be false (cf. [5, 25]).

We are mainly interested in the lower bound of the slope of hyperelliptic fibrations, especially those with positive relative irregularity. The main result is the following.

Theorem 1.4. *Let $f : S \to B$ be a locally non-trivial fibration of hyperelliptic curves of genus $g \geq 2$ with relative irregularity q_f. Then $q_f \leq \frac{g+1}{2}$ and*

$$\lambda_f \geq \lambda_{g,q_f}, \tag{1-6}$$

where

$$\lambda_{g,q_f} = \begin{cases} 8 - \dfrac{4(g+1)}{(q_f+1)(g-q_f)}, & \text{if } q_f \leq \dfrac{g-1}{2}; \\ \dfrac{8(g-1)}{g}, & \text{if } g \text{ is even and } q_f = \dfrac{g}{2}; \\ 8, & \text{if } g \text{ is odd and } q_f = \dfrac{g+1}{2}. \end{cases} \tag{1-7}$$

We will present examples to show that the bound (1-6) is sharp. It is not difficult to show that $\lambda_{g,q_f} \geq \frac{4(g-1)}{g-q_f}$. Therefore, we obtain

Corollary 1.5. *For a locally non-trivial hyperelliptic fibration f, Conjecture* 1.3 *is true. Moreover, the inequality* (1-5) *can become an equality only if $q_f = 0$, $\frac{g-1}{2}$, $\frac{g}{2}$ or $\frac{g+1}{2}$. In particular, $g \leq 3$ if $q_f = 1$ and $\lambda_f = 4$.*

Note that a fibration f is hyperelliptic if and only if $\mathrm{Cliff}(f) = 0$. Hence our result is somewhat orthogonal to the cases treated in [5]. In particular, this seems to indicate that the Clifford index has no essential influence on the relation between the slope and the relative irregularity.

According to [23, Theorem 4.7] (see also Theorem A.1), if f is hyperelliptic, then $\mathcal{F} = 0$ (i.e., there is no non-trivial unitary part) in (1-3) after a suitable finite étale base change. Hence by our theorem, Conjecture 1.1 is true when f is hyperelliptic. Combining with the result of Barja and Zucconi (cf. [6]), we prove

Corollary 1.6. *Conjecture* 1.1 *is true.*

Our paper is organized as follows. In Section 2, we review some basic properties about a hyperelliptic fibration $f : S \to B$ mainly due to Xiao Gang. By blowing up the isolated fixed points of the hyperelliptic involution, we get a double cover $\pi : \widetilde{S} \to \widetilde{P}$ of smooth projective surfaces. We then define the local relative invariants s_i for $2 \leq i \leq g+2$, and show in Theorem 2.6 that the global relative invariants of f can be expressed by those local invariants. In Section 3, we restrict ourselves to the case where the relative irregularity is positive, and prove an inequality (3-1) involving these invariants s_i's. The proof starts from the observation that the double cover $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$ is fibred. In Section 4, we prove Theorem 1.4 and its corollaries. When $q_f = 0$, (1-6) is nothing new but (1-2). If $q_f > 0$, (1-6) follows from (3-1) and the formulas given in Theorem 2.6. In Section 5, we present examples to show that the bound (1-6) is sharp. Finally in the Appendix, we provide a proof of a theorem on the unitary part of a semi-stable hyperelliptic fibration, which is used in the proof of Corollary 1.6.

2. Preliminaries

2.1. **Double covers.** In this subsection, we review some basic properties of double covers (cf. [9, §V.22] and [31, §2]).

A double cover $\pi : X \to Y$ of a smooth projective surface Y is determined by a line bundle L over Y and a section $s \in H^0(Y, L^2)$ where

$$X = \mathrm{Proj}\left(\bigoplus_{i=0}^{+\infty} L^{-i}\right) \Big/ \left\langle s^\vee \right\rangle.$$

Let R be the zero divisor of s. Then $L^{\otimes 2} \cong \mathcal{O}_Y(R)$, the map π is completely determined by the pair (R, L), and X is smooth if and only if R is smooth. To obtain a smooth double cover from a double cover $\pi : X \to Y$ of a smooth projective surface Y, we perform the *canonical resolution* (cf. [9, §III.7])

$$\begin{array}{ccccccccccccc}
\widetilde{X} & = & X_t & \xrightarrow{\phi_t} & X_{t-1} & \xrightarrow{\phi_{t-1}} & \cdots & \xrightarrow{\phi_2} & X_1 & \xrightarrow{\phi_1} & X_0 & = & X \\
 & & \downarrow{\scriptstyle \tilde{\pi}=\pi_t} & & \downarrow{\scriptstyle \pi_{t-1}} & & & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0=\pi} & & \\
\widetilde{Y} & = & Y_t & \xrightarrow{\psi_t} & Y_{t-1} & \xrightarrow{\psi_{t-1}} & \cdots & \xrightarrow{\psi_2} & Y_1 & \xrightarrow{\psi_1} & Y_0 & = & Y
\end{array}$$

where $\widetilde{X} = X_t$ is smooth and the ψ_i's are successive blowing-ups resolving the singularities of R. Here $\pi_i : X_i \to Y_i$ is the double cover determined by (R_i, L_i) with

$$R_i = \psi_i^*(R_{i-1}) - 2[m_{i-1}/2]\,\mathcal{E}_i, \qquad L_i = \psi_i^*(L_{i-1}) \otimes \mathcal{O}_{Y_i}\left(\mathcal{E}_i^{-[m_{i-1}/2]}\right), \tag{2-1}$$

where $\mathcal{E}_i$ is the exceptional divisor of ψ_i, m_{i-1} is the multiplicity of the singular point y_{i-1} in R_{i-1}, [ ] stands for the integral part, $R_0 = R$ and $L_0 = L$.

The morphism

$$\psi \triangleq \psi_1 \circ \cdots \circ \psi_t : \ \widetilde{Y} \longrightarrow Y$$

is also called a *minimal even resolution* of R. We call a singularity $y_j \in R_j \subseteq Y_j$ infinitely close to $y_{i-1} \in R_{i-1} \subseteq Y_{i-1}$ $(j \geq i)$ if $\psi_i \circ \cdots \circ \psi_j(y_j) = y_{i-1}$.

Definition 2.1. The singularity $y_{i-1} \in R_{i-1} \subseteq Y_{i-1}$ above is said to be negligible if $[m_{i-1}/2] = 1$, and $[m_j/2] \leq 1$ for any $y_j \in R_j \subseteq Y_j$ $(j \geq i)$ infinitely close to y_{i-1}. In this case, the blowing-up $\psi_i : Y_i \to Y_{i-1}$ is called a negligible blowing-up.

It is easy to see that ψ can be decomposed into $\tilde{\psi} : \widetilde{Y} \to \hat{Y}$ and $\hat{\psi} : \hat{Y} \to Y$, where $\tilde{\psi}$ and $\hat{\psi}$ are composed of negligible and non-negligible blowing-ups respectively. We call $\hat{\psi}$ the *minimal even resolution of non-negligible singularities* of R.

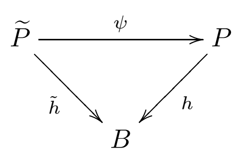
2.2. **Invariants of hyperelliptic fibrations.** In this subsection, we review some results about hyperelliptic fibrations mainly due to Xiao (cf. [31, §2] and [33, §5.1]).

Let $f : S \to B$ be a relatively minimal hyperelliptic fibration. The relative canonical map of f is generically of degree 2. This map determines an involution σ on S whose restriction on a general fiber F of f is the hyperelliptic involution of F. The involution σ is called the hyperelliptic involution associated to f.

Let $\vartheta : \widetilde{S} \to S$ be the composition of all the blowing-ups of isolated fixed points of the hyperelliptic involution, and let $\tilde{\sigma}$ be the induced involution on $\widetilde{S}$. The quotient space $\widetilde{P} = \widetilde{S}/\langle\tilde{\sigma}\rangle$ is a smooth surface, and f induces a ruling $\tilde{h} : \widetilde{P} \to B$ on $\widetilde{P}$. The quotient map $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$ is a double cover which is determined by the pair $(\widetilde{R}, \widetilde{L})$, where $\widetilde{R}$ is the branch locus of $\tilde{\pi}$ and $\widetilde{L}$ is a line bundle such that $\mathcal{O}_{\widetilde{P}}(\widetilde{R}) \cong \widetilde{L}^{\otimes 2}$.

For any contraction $\varphi : \widetilde{P} \to P'$ and $R' = \varphi(\widetilde{R})$, the double cover $\tilde{\pi}$ induces a double cover $S' \to P'$, which is assumed to be determined by the pair (R', L'). For convenience, we simply call (R', L') the image of $(\widetilde{R}, \widetilde{L})$.

Lemma 2.2 ([31, 33])**.** *There exists a contraction of ruled surfaces $\psi : \widetilde{P} \to P$:*



such that P is a geometrical ruled surface (i.e., any fiber of h is $\mathbb{P}^1$), the singularities of R are at most of multiplicity $g+2$, and the self-intersection R^2 is the smallest among all such choices, where (R, L) is the image of $(\widetilde{R}, \widetilde{L})$ in P.

Note that ψ can be decomposed into $\tilde{\psi} : \widetilde{P} \to \hat{P}$ and $\hat{\psi} : \hat{P} \to P$ in the following diagram, where $\hat{\psi} : \hat{P} \to P$ is a minimal even resolution of non-negligible singularities of R:

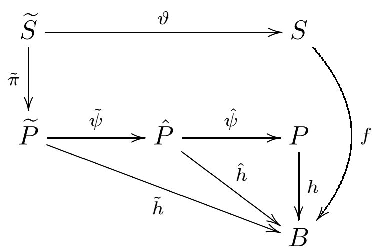


FIGURE 1. Hyperelliptic fibration.

Let $(\hat{R}, \hat{L})$ be the image of $(\widetilde{R}, \widetilde{L})$ in $\hat{P}$. Let $\hat{\psi} = \hat{\psi}_1 \circ \cdots \circ \hat{\psi}_t$ be the decomposition of $\hat{\psi}$, where $\hat{\psi}_i : \hat{P}_i \to \hat{P}_{i-1}$ is a blowing-up at y_{i-1}, $\hat{P}_0 = P$ and $\hat{P}_t = \hat{P}$. Let $\hat{R}_i$ be the image of $\hat{R}$ in $\hat{P}_i$. It could happen that there is one or more singular points of $\hat{R}_i$ over the exceptional curve $\hat{\mathcal{E}}_i$ of $\hat{\psi}_i$. We remark that the decomposition of $\hat{\psi}$ is not unique. If y_{i-1} is a singular point of $\hat{R}_{i-1}$ of odd multiplicity $2k+1$ $(k \geq 1)$ and there is a unique singular point y of $\hat{R}_i$ on the exceptional curve $\hat{\mathcal{E}}_i$ of multiplicity $2k+2$, then we always assume that $\hat{\psi}_{i+1} : \hat{P}_{i+1} \to \hat{P}_i$ is the standard blowing-up at $y_i = y$. We call such a pair (y_{i-1}, y_i) a *singularity of R of type* $(2k+1 \to 2k+1)$, and call y_{i-1} (resp. y_i) the first (resp. second) component of such a singularity.

Definition 2.3. For any singular fiber F of f and $3 \leq i \leq g+2$, the i-th singularity index of F is defined as follows (with respect to the contraction ψ):

- if i is odd, $s_i(F)$ equals the number of $(i \to i)$ type singularities of R over the image $f(F)$;
- if i is even, $s_i(F)$ equals the number of singularities of multiplicity i or $i+1$ of R over the image $f(F)$, neither belonging to the second component of $(i-1 \to i-1)$ type singularities or to the first component of $(i+1 \to i+1)$ type singularities.

We remark that the infinitely close singularities of R should also be taken into consideration when defining $s_i(F)$ above for even i. Let $K_{\hat{P}/B} = K_{\hat{P}} - \hat{h}^* K_B$ and $R' = \hat{R} \setminus \hat{V}$, where $\hat{V}$ is the union of isolated vertical (-2)-curves in $\hat{R}$. Here a curve $C \subseteq \hat{R}$ is said to be *isolated* in $\hat{R}$ if there is no other curve $C' \subseteq \hat{R}$ such that $C \cap C' \neq \emptyset$. We define

$$s_2 \triangleq (K_{\hat{P}/B} + R') \cdot R' \quad \text{and} \quad s_i \triangleq \sum_{F \text{ is singular}} s_i(F), \quad 3 \leq i \leq g+2.$$

By definition, s_i is non-negative for $i \geq 3$, but it is not clear whether s_2 is non-negative or not.

Lemma 2.4 ([31, 33])**.** *These singularity indices s_i defined above are independent on the choices of ψ in Lemma* 2.2.

We just remark that the independence of s_2 on ψ is contained implicitly in [31, Lemma 8], which proves the independence of $\tilde{\psi}$ and s_i on ψ for $i \geq 3$.

Remark 2.5. In [22], it is proven that if f is semi-stable, then $s_{g+2}=0$ and $s_2 + 2\sum_{k=2}^{[g/2]} s_{2k}$ $\big($resp. s_{2k+1} for $1\le k\le [g/2]\big)$ is the number of nodes of type 0 (resp. k) contained in the fibers of f.

Theorem 2.6 ([31, 33]). *Let $f: S\to B$ be a fibration of hyperelliptic curves of genus $g\ge 2$, and let s_i be the singularity indices as above. Then*

$$(2g+1)K_f^2 = (g-1)\Big(s_2+(3g+1)s_{g+2}\Big)+\sum_{k=1}^{[\frac{g}{2}]} a_k s_{2k+1}+\sum_{k=2}^{[\frac{g+1}{2}]} b_k s_{2k},$$

$$(2g+1)\chi_f = \frac{gs_2+(g^2-2g-1)s_{g+2}}{4}+\sum_{k=1}^{[\frac{g}{2}]} k(g-k)s_{2k+1}+\sum_{k=2}^{[\frac{g+1}{2}]}\frac{k(g-k+1)}{2}s_{2k},$$

where $a_k = 12k(g-k)-2g-1$ and $b_k = 6k(g-k+1)-4g-2$.

For convenience, we reproduce a proof of Theorem 2.6. To start it, we need

Lemma 2.7 ([31,33]). *Let F be a singular fiber of the fibration f, and let $\widetilde{F}$ (resp. $\hat{\Gamma}$) be the corresponding fiber in $\widetilde{S}$ (resp. $\hat{P}$). Then the (-1)-curves in $\widetilde{F}$ are in one-to-one correspondence to the isolated (-2)-curves of $\hat{R}$, which are also contained in $\hat{\Gamma}$, and the number of these (-1)-curves is equal to*

$$2s_{g+2}(F)+\sum_{k=1}^{[\frac{g}{2}]} s_{2k+1}(F).$$

Proof of Theorem 2.6. Let

$$n=\frac{L^2}{g+1}. \tag{2-2}$$

Then $K_{P/B}\cdot L=-n$, where $K_{P/B}=K_P-h^*K_B$. As $\hat{\psi}:\hat{P}\to P$ is a minimal even resolution of non-negligible singularities of R, by Definition 2.3 and the formula (2-1), one gets

$$\hat{L}^2 = L^2-\frac{g^2+4g+5}{2}s_{g+2}-\sum_{k=1}^{[\frac{g}{2}]}(2k^2+2k+1)s_{2k+1}-\sum_{k=2}^{[\frac{g+1}{2}]}k^2 s_{2k},$$

$$K_{\hat{P}/B}\cdot\hat{L} = K_{P/B}\cdot L+(g+2)s_{g+2}+\sum_{k=1}^{[\frac{g}{2}]}(2k+1)s_{2k+1}+\sum_{k=2}^{[\frac{g+1}{2}]}ks_{2k}.$$

(We should remark that the minimality of the self-intersection R^2 in Lemma 2.2 implies that $s_{g+2}=0$ if g is even; otherwise, there is a singularity p of R with multiplicity $g+2$. In this case, an elementary transformation of P centered at p gives another contraction $\psi':\widetilde{P}\to P'$ with $\big(\psi'(\widetilde{R})\big)^2<R^2$, which is a contradiction to the minimality of R^2.) So by the definition of s_2 and Lemma 2.7, one has

$$s_2-4s_{g+2}-2\sum_{k=1}^{[\frac{g}{2}]}s_{2k+1} = (K_{\hat{P}/B}+\hat{R})\cdot\hat{R} = (K_{\hat{P}/B}+2\hat{L})\cdot 2\hat{L}$$

$$=(4g+2)n-2(g^2+3g+3)s_{g+2}-\sum_{k=1}^{[\frac{g}{2}]}2(4k^2+2k+1)s_{2k+1}-\sum_{k=2}^{[\frac{g+1}{2}]}2k(2k-1)s_{2k}.$$

Hence

$$n = \frac{1}{2(2g+1)}s_2 + \frac{g^2+3g+1}{2g+1}s_{g+2} + \sum_{k=1}^{[\frac{g}{2}]}\frac{4k^2+2k}{2g+1}s_{2k+1} + \sum_{k=2}^{[\frac{g+1}{2}]}\frac{2k^2-k}{2g+1}s_{2k}. \tag{2-3}$$

Let $b = g(B)$ be the genus of B and $K_{\hat{P}/B} = K_{\hat{P}} - \hat{h}^*K_B$. Note that all singular points of $\hat{R} \subseteq \hat{P}$ are negligible (cf. Definition 2.1) by construction. According to the standard formulas for double covers (cf. [9, § V.22]), we get

$$\begin{aligned}
\chi_{\tilde{f}} &= 2\chi(\mathcal{O}_{\hat{P}}) + \frac{1}{2}(\hat{L}^2 + K_{\hat{P}}\cdot\hat{L}) - (g-1)(b-1) = \frac{1}{2}(\hat{L}^2 + K_{\hat{P}/B}\cdot\hat{L})\\
&= \frac{gs_2 + (g^2-2g-1)s_{g+2}}{4(2g+1)} + \sum_{k=1}^{[\frac{g}{2}]}\frac{k(g-k)}{2g+1}s_{2k+1} + \sum_{k=2}^{[\frac{g+1}{2}]}\frac{k(g-k+1)}{2(2g+1)}s_{2k},\\
K_{\tilde{f}}^2 &= 2(\hat{L}+K_{\hat{P}})^2 - 8(g-1)(b-1) = 2(\hat{L}+K_{\hat{P}/B})^2\\
&= \frac{g-1}{2g+1}s_2 + \frac{3(g^2-2g-1)}{2g+1}s_{g+2} + \sum_{k=1}^{[\frac{g}{2}]}\left(\frac{a_k}{2g+1}-1\right)s_{2k+1} + \sum_{k=2}^{[\frac{g+1}{2}]}\frac{b_k}{2g+1}s_{2k}.
\end{aligned}$$

Note that $\chi_f = \chi_{\tilde{f}}$, and $K_f^2 = K_{\tilde{f}}^2 + 2s_{g+2} + \sum\limits_{k=1}^{[g/2]} s_{2k+1}$ by Lemma 2.7. The theorem follows immediately. □

3. Hyperelliptic fibrations with positive relative irregularity

The purpose of this section is to prove the following inequality for a locally non-trivial hyperelliptic fibration with positive relative irregularity.

Proposition 3.1. *Let $f: S \to B$ be a hyperelliptic fibration of genus g, which is not locally trivial. Let s_i $(2 \leq i \leq g+2)$ be the i-th singularity index of f defined in Definition 2.3. Assume that the relative irregularity $q_f > 0$. Then*

$$\begin{aligned}
&s_2 + \sum_{k=1}^{q_f-1}4k(2k+1)s_{2k+1} + \sum_{k=2}^{q_f}2k(2k-1)s_{2k}\\
&\leq \sum_{k=q_f}^{[\frac{g}{2}]}\frac{(2k+1)(2g+1-2k)}{g+1}s_{2k+1} + \sum_{k=q_f+1}^{[\frac{g+1}{2}]}\frac{2k(g+1-k)}{g+1}s_{2k} + (g+1)s_{g+2}.
\end{aligned} \tag{3-1}$$

In order to prove the above proposition, we always assume that $q_f > 0$ in this section. Let $\tilde{\pi}: \widetilde{S} \to \widetilde{P}$ be the induced double cover with branch divisor $\widetilde{R} \subseteq \widetilde{P}$ as in Figure 1. The strategy of the proof is as follows. The starting point is the observation that the double cover $\tilde{\pi}$ is fibred (see [18] or [23, Definition 6.2] for the definition). Based on this observation, we prove that there exists a fibration $\tilde{h}': \widetilde{P} \to \mathbb{P}^1$ such that the branch divisor $\widetilde{R}$ of $\tilde{\pi}$ is contained in exactly $2q_f+2$ fibers of $\tilde{h}'$. Let ψ be the contraction in Lemma 2.2. We decompose it into $\bar{\psi}: \widetilde{P} \to \overline{P}$ and $\check{\psi}: \overline{P} \to P$ such that there is an induced fibration $\bar{h}': \overline{P} \to \mathbb{P}^1$ and any (-1)-curve contracted by $\check{\psi}$ is not contracted by $\bar{h}'$. Then Proposition 3.1 follows from the facts that the contraction $\check{\psi}$ contributes only to s_i with $i > 2q_f$ and that $\overline{R} = \bar{\psi}(\widetilde{R})$ is semi-negative definite.

The fact that the double cover $\tilde{\pi}$ is fibred, is proved in [23] under the extra assumption that $f : S \to B$ is semi-stable. But the proof there does not use this assumption. Hence we have

Proposition 3.2 ([23, Propositions 6.4 and 6.5])**.** *The double cover $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$ in Figure 1 is fibred, i.e., there exists a double cover $\pi' : B' \to \mathbb{P}^1$ of smooth projective curves and two morphisms $\tilde{f}' : \widetilde{S} \to B'$ and $\tilde{h}' : \widetilde{P} \to \mathbb{P}^1$, such that the diagram*

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\ \tilde{\pi}\ } & \widetilde{P} \\ {\scriptstyle \tilde{f}'}\downarrow & & \downarrow{\scriptstyle \tilde{h}'} \\ B' & \xrightarrow{\ \pi'\ } & \mathbb{P}^1 \end{array}$$

is commutative, $\widetilde{R}$ is contained in the fibers of $\tilde{h}'$ and

$$q_f = q(\widetilde{S}) - q(\widetilde{P}) = g(B') \leq \frac{g+1}{2}. \tag{3-2}$$

Remark 3.3. Let $f : S \to B$ be as in Proposition 3.1 with $g(B) \geq 1$, and let $d \geq 2$ be the degree of the Albanese map $S \to \mathrm{Alb}\,(S)$. Xiao ([32]) proved a more precise description on q_f:

$$\frac{g+1}{d} - 1 \leq q_f \leq \frac{g-1}{d} + 1.$$

Based on Proposition 3.2, we would like to show that the branch divisor $\widetilde{R}$ of $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$ has a very special form which is described in Lemma 3.4.

Let $\tilde{f}' : \widetilde{S} \to B'$ be the fibration in Proposition 3.2. Since $g(B') = q_f \geq 1$, it follows that any (-1)-curve in $\widetilde{S}$ is contracted by $\tilde{f}'$. Hence $\tilde{f}'$ factors through $\vartheta : \widetilde{S} \to S$. Let $f' : S \to B'$ be the induced map. Note that the fibration $\tilde{f}' : \widetilde{S} \to B'$ in Proposition 3.2 is clearly unique. Hence the hyperelliptic involution σ induces an involution σ' on B' such that $B'/\langle\sigma'\rangle \cong \mathbb{P}^1$ with the following diagram:

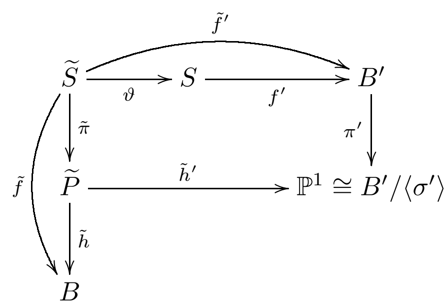


FIGURE 2. Hyperelliptic fibration with positive relative irregularity.

Assume $\pi' : B' \to \mathbb{P}^1$ is branched over $\Delta \subseteq \mathbb{P}^1$. Applying the Hurwitz formula to the double cover π', one sees that $|\Delta| = 2q_f + 2$. For any $y \in \Delta$, let $\widetilde{\Gamma}'_y = \sum \tilde{n}'_C C$ be the fiber of $\tilde{h}'$ over y, and

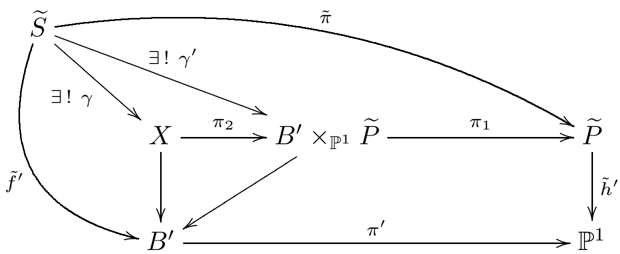
$$\widetilde{\Gamma}'_{y,\mathrm{o}} \triangleq \sum_{\substack{C \subseteq \widetilde{\Gamma}'_y \\ \tilde{n}'_C \text{ is odd}}} C \subseteq \widetilde{\Gamma}'_y.$$

According to Proposition 3.2, $\widetilde{R}$ is contained in the fibers of $\tilde{h}'$. In fact, we can prove an explicit expression of $\widetilde{R}$.

Lemma 3.4.

$$\widetilde{R} = \sum_{y \in \Delta} \widetilde{\Gamma}'_{y,\mathrm{o}}.$$

Proof. Let $B' \times_{\mathbb{P}^1} \widetilde{P}$ be the fiber-product, and let $X \to B' \times_{\mathbb{P}^1} \widetilde{P}$ be the normalization. By the universal property of the fiber-product (cf. [17, § II-2]), there exists a unique morphism $\gamma' : \widetilde{S} \to B' \times_{\mathbb{P}^1} \widetilde{P}$. Since $\widetilde{S}$ is smooth, there also exists a unique morphism $\gamma : \widetilde{S} \to X$ such that the following diagram is commutative:



Clearly the composition $\pi_1 \circ \pi_2 : X \to \widetilde{P}$ is a double cover branched exactly over

$$\sum_{y \in \Delta} \widetilde{\Gamma}'_{y,\mathrm{o}}.$$

Therefore, it suffices to prove that γ is an isomorphism.

As $\deg \tilde{\pi} = \deg(\pi_1 \circ \pi_2)$, we get $\deg \gamma = 1$, i.e., $\gamma : \widetilde{S} \to X$ is a contraction of curves. Note that $\tilde{\pi}$ does not contract any curve. Neither does γ, because any curve contracted by γ must be also contracted by $\tilde{\pi}$. This completes the proof. □

The contraction $\psi : \widetilde{P} \to P$ in Lemma 2.2 is composed of several blowing-ups. We divide those blowing-ups as $\psi = \check{\psi} \circ \bar{\psi}$, where $\bar{\psi} : \widetilde{P} \to \overline{P}$ is the largest contraction such that $\tilde{h}'$ factors through $\bar{\psi}$. So we have the following diagram:

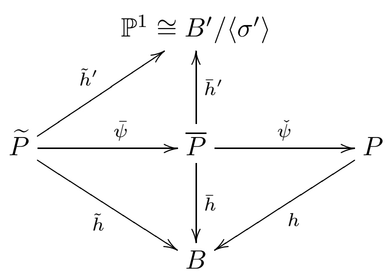


FIGURE 3. Decomposition of ψ.

Next we want to show that the contraction $\check{\psi}$ contributes only to s_i with $i > 2q_f$. For this purpose, we first prove

Lemma 3.5. *Let $\mathcal{E} \subseteq \overline{P}$ be any (-1)-curve contracted by $\check{\psi}$. Then $\mathcal{E} \cdot \overline{R}$ is even and*

$$\mathcal{E} \cdot \overline{R} \geq 2(q_f + 1).$$

Proof. Let $(\overline{R},\,\overline{L};\,\overline{\Gamma}'_y)$ be the image of $(\widetilde{R},\,\widetilde{L};\,\widetilde{\Gamma}'_y)$ on $\overline{P}$, where $y\in\Delta$. Let $\overline{m}=\mathcal{E}\cdot\overline{R}$, and let $\sigma:\,\overline{P}\to\overline{P}_1$ be the contraction of $\mathcal{E}$, x the image of $\mathcal{E}$, and $(\overline{R}_1,\,\bar{L}_1)$ the image of $(\overline{R},\,\bar{L})$ on $\overline{P}_1$. Then x is a singularity of $\overline{R}_1$ of multiplicity $\overline{m}$, and $\mathcal{E}$ is mapped surjectively onto $\mathbb{P}^1$ by $\bar{h}'$ according to the construction of $\bar{h}'$ in Figure 3. Let

$$\overline{\Gamma}'_{y,\mathrm{o}}=\sum_{\substack{C\subseteq\overline{\Gamma}'_y\\ \bar{n}'_C\text{ is odd}}}C\quad\text{and}\quad\overline{\Gamma}'_{y,\mathrm{r}}=\sum_{\substack{C\subseteq\overline{\Gamma}'_y\\ \bar{n}'_C=1}}C,\qquad\text{if }\overline{\Gamma}'_y=\sum\bar{n}'_CC.$$

By Lemma 3.4,

$$\overline{R}=\bar{\psi}(\widetilde{R})=\sum_{y\in\Delta}\overline{\Gamma}'_{y,\mathrm{o}}$$

is contained in the fibers of $\bar{h}'$. Hence $\mathcal{E}\not\subseteq\overline{R}$, from which it follows that $\overline{m}=\mathcal{E}\cdot\overline{R}$ is even. Let

$$\overline{R}_{\mathrm{all}}=\sum_{y\in\Delta}\overline{\Gamma}'_y=(\bar{h}')^*(\Delta)\quad\text{and}\quad\overline{R}_{\mathrm{r}}=\sum_{y\in\Delta}\overline{\Gamma}'_{y,\mathrm{r}}.$$

Then $\overline{R}_{\mathrm{r}}\subseteq\overline{R}\subseteq\overline{R}_{\mathrm{all}}$. To complete the proof, it is enough to prove that

$$\mathcal{E}\cdot\overline{R}_{\mathrm{r}}\geq 2(q_f+1).$$

Note that the restricted morphism $\bar{h}'|_{\mathcal{E}}:\,\mathcal{E}\to\mathbb{P}^1$ is surjective. For any $p\in\mathcal{E}\cap\overline{R}_{\mathrm{all}}$, let $r_p=I_p(\mathcal{E},\,\overline{R}_{\mathrm{all}})$ be the local intersection number. Since $\overline{R}_{\mathrm{all}}=(\bar{h}')^*(\Delta)$ consists of $|\Delta|=2q_f+2$ fibers of $\bar{h}'$, one has

$$\sum_{p\in\mathcal{E}\cap\overline{R}_{\mathrm{all}}}r_p=\mathcal{E}\cdot\overline{R}_{\mathrm{all}}=\deg(\bar{h}'|_{\mathcal{E}})\cdot(2q_f+2).$$

By definition, $r_p\geq 2$ for any $p\in\big(\mathcal{E}\cap\overline{R}_{\mathrm{all}}\big)\setminus\big(\mathcal{E}\cap\overline{R}_{\mathrm{r}}\big)$. On the other hand, as $\mathcal{E}$ is a (-1)-curve, the ramification number of $\bar{h}'|_{\mathcal{E}}$ is $2\deg(\bar{h}'|_{\mathcal{E}})-2$. So

$$\begin{aligned}2\deg(\bar{h}'|_{\mathcal{E}})-2\quad&\geq\sum_{p\in\mathcal{E}\cap\overline{R}_{\mathrm{all}}}(r_p-1)=\sum_{p\in(\mathcal{E}\cap\overline{R}_{\mathrm{all}})\setminus(\mathcal{E}\cap\overline{R}_{\mathrm{r}})}(r_p-1)\;+\sum_{p\in\mathcal{E}\cap\overline{R}_{\mathrm{r}}}(r_p-1)\\&\geq\sum_{p\in(\mathcal{E}\cap\overline{R}_{\mathrm{all}})\setminus(\mathcal{E}\cap\overline{R}_{\mathrm{r}})}\frac{r_p}{2}+\sum_{p\in\mathcal{E}\cap\overline{R}_{\mathrm{r}}}\frac{(r_p-1)}{2}\\&=\frac{1}{2}\sum_{p\in\mathcal{E}\cap\overline{R}_{\mathrm{all}}}r_p-\frac{\big|\mathcal{E}\cap\overline{R}_{\mathrm{r}}\big|}{2}=\deg(\bar{h}'|_{\mathcal{E}})\cdot(q_f+1)-\frac{\big|\mathcal{E}\cap\overline{R}_{\mathrm{r}}\big|}{2}.\end{aligned}$$

Therefore

$$\mathcal{E}\cdot\overline{R}_{\mathrm{r}}\geq\big|\mathcal{E}\cap\overline{R}_{\mathrm{r}}\big|\geq 2\deg(\bar{h}'|_{\mathcal{E}})\cdot(q_f-1)+4\geq 2(q_f-1)+4=2(q_f+1).$$

The proof is complete. □

We assume that $\check{\psi}=\check{\psi}_1\circ\cdots\circ\check{\psi}_u$, where $\check{\psi}_i:\,\check{P}_i\to\check{P}_{i-1}$ is a blowing-up at $\check{x}_{i-1}\in\check{P}_{i-1}$ with exceptional curve $\check{\mathcal{E}}_i\subseteq\check{P}_i$, $\check{P}_0=P$ and $\check{P}_u=\overline{P}$. Let $\check{R}_i$ be the image of $\overline{R}$ in $\check{P}_i$, and let $\check{x}_i$ be a singularity of $\check{R}_i$ of multiplicity $\check{m}_i$.

Lemma 3.6. *For $1\leq i\leq u-1$, we have either $\check{m}_i\geq 2(q_f+1)$, or $\check{m}_i=2q_f+1$ and $\check{x}_{i+1}$ is the unique singular point on $\check{\mathcal{E}}_{i+1}\subseteq\check{P}_{i+1}$ of multiplicity $\check{m}_{i+1}=2q_f+2$.*

Proof. First, we show that $\check{m}_i \geq 2q_f + 1$ for any $1 \leq i \leq u-1$. Since $\check{\psi}$ is a part of the even resolution of $R = \check{R}_0$, one has

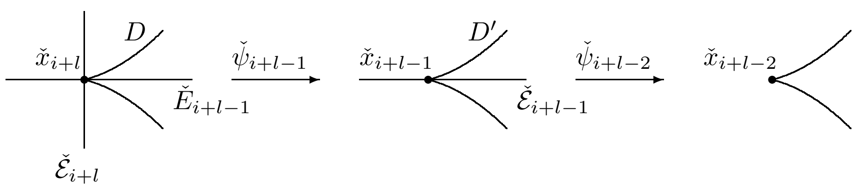
$$\text{(3-3)} \qquad \begin{cases} \text{if } \check{m}_i \text{ is even, then } \check{\mathcal{E}}_{i+1} \not\subseteq \check{R}_{i+1}, \text{ and so } \check{m}_{i+1} \leq \check{m}_i; \\ \text{if } \check{m}_i \text{ is odd, then } \check{\mathcal{E}}_{i+1} \subseteq \check{R}_{i+1}, \text{ and so } \check{m}_{i+1} \leq \check{m}_i + 1. \end{cases}$$

By induction, for any singularity $\check{x}_{i+j}$, infinitely close to $\check{x}_i$, we have $\check{m}_{i+j} \leq \check{m}_i$ if $\check{m}_i$ is even, and $\check{m}_{i+j} \leq \check{m}_i + 1$ if $\check{m}_i$ is odd. By Lemma 3.5, $\check{m}_{i+j_i} \geq 2(q_f+1)$ for the last infinitely close singularity $\check{x}_{i+j_i}$ introduced by $\check{\psi}$. Thus $\check{m}_i \geq 2q_f + 1$ as required.

Now we assume $\check{m}_i = 2q_f + 1$. Note that we have already proved that $\check{m}_{i+1} \geq 2q_f + 1$ in the above. If $\check{m}_{i+1} = 2q_f + 2$, then $\check{x}_{i+1}$ must be the unique singular point of $\check{R}_{i+1}$ on $\check{\mathcal{E}}_{i+1} \subseteq \check{P}_{i+1}$, and we are done. Therefore it is enough to derive a contradiction if $\check{m}_{i+1} = 2q_f + 1$.

Let l be the smallest number such that $\check{m}_{i+l} = 2(q_f+1)$, where we assume that $\check{x}_{i+j}$ is infinitely close to $\check{x}_{i+j-1}$ for $j = 1, \cdots, l$. Such an l exists by Lemma 3.5, and $l \geq 2$ if $\check{m}_{i+1} = 2q_f + 1$. Note that the exceptional curve $\check{\mathcal{E}}_{i+j}$ is contained in $\check{R}_{i+j}$ for $1 \leq j \leq l$, since $\check{m}_{i+j-1}$ is odd. Because $\check{m}_{i+l} = \check{m}_{i+l-1} + 1$, $\check{x}_{i+l}$ must be the unique singular point of $\check{R}_{i+l}$ on the exceptional curve $\check{\mathcal{E}}_{i+l}$.

Let $\check{E}_{i+l-1} \subseteq \check{P}_{i+l}$ be the strict transform of $\check{\mathcal{E}}_{i+l-1} \subseteq \check{P}_{i+l-1}$, $D = \check{R}_{i+l} - (\check{E}_{i+l-1} + \check{\mathcal{E}}_{i+l})$, and let D' be the image of D in $\check{P}_{i+l-1}$. Then $\check{x}_{i+l} \in \check{E}_{i+l-1}$, since $\check{x}_{i+l}$ is the unique singular point of $\check{R}_{i+l}$ on the exceptional curve $\check{\mathcal{E}}_{i+l}$ and $\check{E}_{i+l-1} \cap \check{\mathcal{E}}_{i+l}$ is a singularity of $\check{R}_{i+l}$. Thus



Since $\check{m}_{i+l} = 2(q_f+1)$, D has multiplicity $\check{m}_{i+l} - 2 = 2q_f$ at $\check{x}_{i+l}$. Hence the local intersection

$$I_{\check{x}_{i+l}}(\check{E}_{i+l-1}, D) \geq 2q_f, \qquad I_{\check{x}_{i+l}}(\check{\mathcal{E}}_{i+l}, D) \geq 2q_f.$$

Note that D is nothing but the strict transform of D', and $(\check{\psi}_{i+l-1})^*(\check{\mathcal{E}}_{i+l-1}) = \check{E}_{i+l-1} + \check{\mathcal{E}}_{i+l}$. Thus

$$I_{\check{x}_{i+l-1}}\big(\check{\mathcal{E}}_{i+l-1}, D'\big) = I_{\check{x}_{i+l}}\big(\check{\psi}^*_{i+l-1}(\check{\mathcal{E}}_{i+l-1}), D\big) = I_{\check{x}_{i+l}}(\check{E}_{i+l-1} + \check{\mathcal{E}}_{i+l}, D) \geq 4q_f.$$

Note that $\check{\psi}_{i+l-2}(D') = \check{R}_{i+l-2}$, and the multiplicity $\check{m}_{i+l-2}$ of $\check{R}_{i+l-2}$ at $\check{x}_{i+l-2}$ equals the intersection number $\check{\mathcal{E}}_{i+l-1} \cdot D'$. Hence

$$2q_f + 1 = \check{m}_{i+l-2} = \check{\mathcal{E}}_{i+l-1} \cdot D' \geq I_{\check{x}_{i+l-1}}(\check{\mathcal{E}}_{i+l-1}, D') \geq 4q_f,$$

which is a contradiction, since $q_f \geq 1$. So we finish the proof. □

By the above lemma, it is easy to get the following:

Corollary 3.7. *The contraction $\check{\psi}$ is composed of several blowing-ups of singularities of R of type $(2k+1 \to 2k+1)$ with $k \geq q_f$, or of singularities with multiplicity at least $2(q_f+1)$, i.e., it contributes only to s_i with $i > 2q_f$.*

Now we are in the position to prove Proposition 3.1.

Proof of Proposition 3.1. Let $\check{s}_{2k+1}$ ($k \geq 1$) be the number of the singularities of R of type $(2k+1 \to 2k+1)$ introduced by $\check{\psi}$, and $\check{s}_{2k}$ ($k \geq 2$) be the number of the singularities of R with multiplicity $2k$ or $2k+1$ introduced by $\check{\psi}$, neither belonging to the second component of $(2k-1 \to 2k-1)$ type singularities or to the first component of $(2k+1 \to 2k+1)$ type singularities. Then $\check{s}_i \geq 0$, and by Corollary 3.7 one has

$$\text{(3-4)} \qquad \check{s}_{2k+1} = 0, \quad \forall\, 1 \leq k \leq q_f - 1; \qquad \check{s}_{2k} = 0, \quad \forall\, 1 \leq k \leq q_f.$$

Let $\bar{s}_i = s_i - \check{s}_i$. Then for $i > 2$, $\bar{s}_i$ is nothing but the number of the singularities of R introduced by $\bar{\psi}$ with the corresponding multiplicity or types. Hence $\bar{s}_i \geq 0$ for $i > 2$, and one has by (3-4) that

$$\text{(3-5)} \qquad s_{2k+1} = \bar{s}_{2k+1}, \quad \forall\, 1 \leq k \leq q_f - 1; \qquad s_{2k} = \bar{s}_{2k}, \quad \forall\, 1 \leq k \leq q_f.$$

According to Lemma 3.4 and the decomposition of ψ in Figure 3, we see that $\overline{R}$ is contained in the fibers of $\bar{h}'$, hence it is semi-negative definite. By the definition of the $\check{s}_i$'s and Lemma 2.7, there are $\sum_{k=q_f}^{[g/2]} \check{s}_{2k+1} + 2\check{s}_{g+2}$ isolated (-2)-curves contained in $\overline{R}$. Thus

$$\overline{R}^2 \leq -2\left(\sum_{k=q_f}^{[\frac{g}{2}]} \check{s}_{2k+1} + 2\check{s}_{g+2}\right).$$

On the other hand, by definition,

$$\overline{R}^2 = R^2 - \sum_{k=q_f}^{[\frac{g}{2}]} 4(2k^2+2k+1)\check{s}_{2k+1} - \sum_{k=q_f+1}^{[\frac{g+1}{2}]} 4k^2\check{s}_{2k} - 2(g^2+4g+5)\check{s}_{g+2}.$$

As $R^2 = 4L^2 = 4(g+1)n$ by (2-2), we get

$$\text{(3-6)} \quad (g+1)n \leq \sum_{k=q_f}^{[\frac{g}{2}]} \left(2k^2+2k+\frac{1}{2}\right)\check{s}_{2k+1} + \sum_{k=q_f+1}^{[\frac{g+1}{2}]} k^2\check{s}_{2k} + \frac{(g+1)(g+3)}{2}\check{s}_{g+2}.$$

Hence

$$\begin{aligned}
& s_2 + \sum_{k=1}^{q_f-1} 4k(2k+1)s_{2k+1} + \sum_{k=2}^{q_f} 2k(2k-1)s_{2k} \\
\leq\; & \bar{s}_2 + \sum_{k=1}^{[\frac{g}{2}]} 4k(2k+1)\bar{s}_{2k+1} + \sum_{k=2}^{[\frac{g+1}{2}]} 2k(2k-1)\bar{s}_{2k} + 2(g^2+3g+1)\bar{s}_{g+2} \\
\leq\; & \sum_{k=q_f}^{[\frac{g}{2}]} \frac{(2k+1)(2g+1-2k)}{g+1}\check{s}_{2k+1} + \sum_{k=q_f+1}^{[\frac{g+1}{2}]} \frac{2k(g+1-k)}{g+1}\check{s}_{2k} + (g+1)\check{s}_{g+2} \\
\leq\; & \sum_{k=q_f}^{[\frac{g}{2}]} \frac{(2k+1)(2g+1-2k)}{g+1}s_{2k+1} + \sum_{k=q_f+1}^{[\frac{g+1}{2}]} \frac{2k(g+1-k)}{g+1}s_{2k} + (g+1)s_{g+2}.
\end{aligned}$$

The first and last inequalities above follow immediately from the non-negativity of $\bar{s}_i$'s for $i > 2$ and (3-5), and the second one follows from (2-3) and (3-6). The proof is complete. ☐

4. Proof of Theorem 1.4 and its corollaries

This section aims to prove our main result, Theorem 1.4, and its corollaries. It is based on (3-1) given in Proposition 3.1 and the formulas given in Theorem 2.6.

Proof of Theorem 1.4. According to (3-2), it is known that $q_f \leq \frac{g+1}{2}$. Recall the definition of λ_{g,q_f} in (1-7). If $q_f = 0$, then $\lambda_{g,0} = \frac{4(g-1)}{g}$, and so (1-6) holds by (1-2). Thus we assume $q_f \geq 1$ in the following.

First we prove that

$$\begin{aligned} K_f^2 \ &\geq \lambda_{g,q_f} \cdot \chi_f + \alpha s_{g+2} \\ &\qquad + \sum_{k=1}^{q_f-1} \alpha_k s_{2k+1} + \sum_{k=2}^{q_f} \beta_k s_{2k} + \sum_{k=q_f}^{[\frac{g}{2}]} \gamma_k s_{2k+1} + \sum_{k=q_f+1}^{[\frac{g+1}{2}]} \delta_k s_{2k}, \end{aligned} \tag{4-1}$$

where

$$\begin{cases} \alpha = \dfrac{(g-1)(8-\lambda_{g,q_f})}{4}, & \\ \alpha_k = k^2\lambda_{g,q_f} - (2k-1)^2, & \forall\, 1 \leq k \leq q_f - 1, \\ \beta_k = \dfrac{(k-1)\big(k\lambda_{g,q_f} - 4(k-1)\big)}{2}, & \forall\, 2 \leq k \leq q_f, \\ \gamma_k = \dfrac{8(4k(g-k)-1) - (4k(g-k)+g)\cdot\lambda_{g,q_f}}{4(g+1)}, & \forall\, q_f \leq k \leq [\dfrac{g}{2}], \\ \delta_k = \dfrac{k(g+1-k)(8-\lambda_{g,q_f}) - 4(g+1)}{2(g+1)}, & \forall\, q_f + 1 \leq k \leq [\dfrac{g+1}{2}]. \end{cases}$$

Indeed, it is clear that $\lambda_{g,q_f} \geq \frac{4(g-1)}{g}$. Hence by Theorem 2.6 and (3-1), one obtains

$$\begin{aligned} &(2g+1)\left(K_f^2 - \lambda_{g,q_f}\cdot\chi_f\right) \\ = \ & \left(g-1-\frac{g}{4}\cdot\lambda_{g,q_f}\right)s_2 + \left((3g^2-2g-1) - (g^2-2g-1)\cdot\frac{\lambda_{g,q_f}}{4}\right)s_{g+2} \\ &+ \sum_{k=1}^{[\frac{g}{2}]}\left(a_k - k(g-k)\cdot\lambda_{g,q_f}\right)s_{2k+1} + \sum_{k=2}^{[\frac{g+1}{2}]}\Big(b_k - \frac{1}{2}k(g+1-k)\cdot\lambda_{g,q_f}\Big)s_{2k} \\ \geq \ & \left(g-1-\frac{g}{4}\cdot\lambda_{g,q_f}\right)\cdot\Lambda_h + \left((3g^2-2g-1) - (g^2-2g-1)\cdot\frac{\lambda_{g,q_f}}{4}\right)s_{g+2} \\ &+ \sum_{k=1}^{[\frac{g}{2}]}\left(a_k - k(g-k)\cdot\lambda_{g,q_f}\right)s_{2k+1} + \sum_{k=2}^{[\frac{g+1}{2}]}\Big(b_k - \frac{1}{2}k(g+1-k)\cdot\lambda_{g,q_f}\Big)s_{2k} \\ = \ & (2g+1)\left(\alpha s_{g+2} + \sum_{k=1}^{q_f-1}\alpha_k s_{2k+1} + \sum_{k=2}^{q_f}\beta_k s_{2k} + \sum_{k=q_f}^{[\frac{g}{2}]}\gamma_k s_{2k+1} + \sum_{k=q_f+1}^{[\frac{g+1}{2}]}\delta_k s_{2k}\right), \end{aligned}$$

where

$$\begin{aligned}\Lambda_h \;=\; & \sum_{k=q_f}^{[\frac{g}{2}]} \frac{(2k+1)(2g+1-2k)}{g+1} s_{2k+1} + \sum_{k=q_f+1}^{[\frac{g+1}{2}]} \frac{2k(g+1-k)}{g+1} s_{2k} + (g+1)s_{g+2} \\ & - \sum_{k=1}^{q_f-1} 4k(2k+1)s_{2k+1} - \sum_{k=2}^{q_f} 2k(2k-1)s_{2k}.\end{aligned}$$

Therefore, (4-1) follows.

To prove (1-6), it suffices to prove that those coefficients α, α_k, β_k, δ_k and γ_k in (4-1) are all non-negative by noting that $s_i \geq 0$ for any $i \geq 3$. It is clear that

$$\text{(4-2)} \qquad \alpha \geq 0; \qquad \alpha_k > 0,\ \forall\, 1 \leq k \leq q_f - 1; \qquad \text{and } \beta_k > 0,\ \forall\, 2 \leq k \leq q_f.$$

If $q_f \leq \frac{g-1}{2}$, then

$$\begin{aligned}\gamma_k \;=\; & \frac{4k(g-k)+g}{(q_f+1)(g-q_f)} - 2 \\ \geq\; & \frac{4q_f(g-q_f)+g}{(q_f+1)(g-q_f)} - 2 = \frac{2(q_f-1)(g-q_f)+g}{(q_f+1)(g-q_f)} > 0, \qquad \forall\ q_f \leq k \leq [\frac{g}{2}],\end{aligned}$$

$$\begin{aligned}\delta_k \;=\; & 2\left(\frac{k(g+1-k)}{(q_f+1)(g-q_f)} - 1\right) \\ \geq\; & 2\left(\frac{(q_f+1)\big(g+1-(q_f+1)\big)}{(q_f+1)(g-q_f)} - 1\right) = 0, \qquad \forall\ q_f+1 \leq k \leq [\frac{g+1}{2}].\end{aligned}$$

Hence (1-6) follows from (4-1) when $q_f \leq \frac{g-1}{2}$.

If g is even and $q_f = \frac{g}{2}$, then the last summation in (4-1) disappears automatically. So it suffices to show $\gamma_{g/2} \geq 0$. By definition,

$$\gamma_{g/2} = \frac{1}{4(g+1)} \cdot \left(8 \cdot \Big(4 \cdot \frac{g}{2} \cdot \big(g - \frac{g}{2}\big) - 1\Big) - \Big(4 \cdot \frac{g}{2} \cdot \big(g - \frac{g}{2}\big) + g\Big) \cdot \frac{8(g-1)}{g}\right) = 0.$$

Hence, (1-6) also holds in this case.

Finally, if g is odd and $q_f = \frac{g+1}{2}$, then the last two summations in (4-1) disappear, hence we have already showed that $K_f^2 \geq \lambda_{g,q_f} \cdot \chi_f$ by (4-2). So (1-6) holds in this case too. This completes the proof. □

Proof of Corollary 1.5. If g is even and $q_f = \frac{g}{2}$, or g is odd and $q_f = \frac{g+1}{2}$, then by definition,

$$\lambda_{g,q_f} = \frac{4(g-1)}{g-q_f}.$$

If $q_f \leq \frac{g-1}{2}$, i.e., $g - 2q_f - 1 \geq 0$, then

$$\lambda_{g,q_f} - \frac{4(g-1)}{g-q_f} = \frac{4q_f\big(g-2q_f-1\big)}{(q_f+1)(g-q_f)} \geq 0,$$

and '=' holds only if $q_f = 0$ or $\frac{g-1}{2}$. Therefore, our corollary is a consequence of (1-6). □

Remark 4.1. When $q_f = 0$, Corollary 1.5 is just (1-2). When $q_f = \frac{g+1}{2}$, $\frac{g}{2}$ or $\frac{g-1}{2}$, Corollary 1.5 was already obtained by Xiao in [32].

Proof of Corollary 1.6. Let $f : S \to B$ be a fibration of genus $g \geq 2$, which is not locally trivial and $\lambda_f < 4$. We need to prove that $f_*\omega_{S/B}$ has no locally free quotient of degree zero. By [6, Theorem 1], we may assume that f is a hyperelliptic fibration. Recall that by (1-3) we have the following decomposition:

$$f_*\omega_{S/B} = \mathcal{A} \oplus \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k \oplus \mathcal{O}_B^{\oplus q_f}, \tag{4-3}$$

with $\mathcal{A}$ ample, $\mathcal{F}_i$ irreducible unitary and $\dim H^1(B, \Omega_B^1(\mathcal{F}_i)) = 0$ for $1 \leq i \leq k$. Since $\lambda_f < 4$, we have that $q_f = 0$ by Corollary 1.5. Clearly $\mathcal{A}$ has no non-trivial locally free quotient of degree zero. Hence it suffices to prove that $\mathcal{F}_i = 0$ in the above decomposition (4-3).

Assume that $\mathcal{F}_i \neq 0$ for some i. By construction, $\mathcal{F}_i$ corresponds to a unitary representation of the fundamental group

$$\rho_i : \ \pi_1(B) \longrightarrow U(r_i), \qquad \text{where } r_i = \operatorname{rank} \mathcal{F}_i.$$

If the image of ρ_i is finite, then, after a suitable finite étale base change, $\mathcal{F}_i$ becomes trivial, which implies that $q_f > 0$ after such a finite étale base change. However, it is a contradiction by Corollary 1.5, since the slope does not change under any finite étale base change. Hence we may assume that ρ_i has infinite image.

By the stable reduction theorem (cf. [2, 14]), there exists a base change $\phi : \widetilde{B} \to B$ of finite degree, possibly ramified, such that the pull-back fibration $\tilde{f} : \widetilde{S} \to \widetilde{B}$ is semi-stable. According to [23, Theorem 4.7] (see also Theorem A.1 in the appendix), applying possibly a further base change, we may assume that the Fujita decomposition of $\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}}$ is as follows:

$$\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} = \widetilde{\mathcal{A}} \oplus \mathcal{O}_{\widetilde{B}}^{\oplus q_{\tilde{f}}}, \quad \text{with } \widetilde{\mathcal{A}} \text{ ample}. \tag{4-4}$$

Here the pull-back fibration $\tilde{f} : \widetilde{S} \to \widetilde{B}$ is constructed as follows. Let S_1 be the resolution of singularities of $S \times_B \widetilde{B}$. Then $\tilde{f} : \widetilde{S} \to \widetilde{B}$ is just the relatively minimal model of S_1.

$$\begin{array}{ccccccc}
\widetilde{S} & \xleftarrow{\ \theta\ } & S_1 & \xrightarrow{\ \Phi_1\ } & S \times_B \widetilde{B} & \xrightarrow{\ \Phi_0\ } & S \\
\downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f_1} & & \downarrow & & \downarrow{\scriptstyle f} \\
\widetilde{B} & = & \widetilde{B} & = & \widetilde{B} & \xrightarrow{\ \phi\ } & B
\end{array}$$

It is known that (cf. [28, p. 231]) there is an inclusion $\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} \subseteq \phi^* f_*\omega_{S/B}$. As

$$\operatorname{rank} \tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} = \operatorname{rank} \phi^* f_*\omega_{S/B} = g,$$

the quotient $Q := (\phi^* f_*\omega_{S/B}) / (\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}})$ is a torsion sheaf. By projection, we get a morphism

$$\mathrm{pr}_i : \ \tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}} \longrightarrow \phi^*\mathcal{F}_i.$$

Hence the quotient $Q_i := (\phi^*\mathcal{F}_i)/\mathrm{pr}_i(\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}})$ is also a torsion sheaf. Note that $\deg\left(\mathrm{pr}_i(\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}})\right) \geq 0$, since it is a quotient of $\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}}$. Note also that $\deg Q_i \geq 0$ and

$$0 = \deg \phi^*\mathcal{F}_i = \deg Q_i + \deg\left(\mathrm{pr}_i(\tilde{f}_*\omega_{\widetilde{S}/\widetilde{B}})\right).$$

Hence we obtain that Q_i is zero and pr_i is surjective.

By construction, $\phi^*\mathcal{F}_i$ comes from the following unitary representation:

$$\begin{array}{ccc} \pi_1(\widetilde{B}) & \xrightarrow{\quad\tilde{\rho}_i\quad} & U(r_i) \\ & {\scriptstyle \phi_*}\searrow \quad \nearrow{\scriptstyle \rho_i} & \\ & \pi_1(B) & \end{array}$$

It follows that $\tilde{\rho}_i$ has infinite image, since ρ_i has infinite image and $\phi_*\Big(\pi_1(\widetilde{B})\Big)$ has finite index in $\pi_1(B)$.

Since $\widetilde{\mathcal{A}}$ in (4-4) is ample, it maps to zero by pr_i. Therefore we have a surjective morphism

$$\mathrm{pr}_i:\ \mathcal{O}_{\widetilde{B}}^{\oplus q_{\tilde{f}}} \longrightarrow \phi^*\mathcal{F}_i.$$

Note that the trivial bundle $\mathcal{O}_{\widetilde{B}}^{\oplus q_{\tilde{f}}}$ corresponds to the trivial representation (hence also the unitary representation) of $\pi_1(\widetilde{B})$. Hence by [24], $\phi^*\mathcal{F}_i$ is a direct summand of $\mathcal{O}_{\widetilde{B}}^{\oplus q_{\tilde{f}}}$, which implies that the representation $\tilde{\rho}_i$ corresponding to $\phi^*\mathcal{F}_i$ is also trivial. This gives a contradiction, since the representation $\tilde{\rho}_i$ has infinite image by construction. This completes the proof. □

5. Examples

In this section, we construct examples to show that the bound (1-6) is sharp.

Example 5.1. We construct a hyperelliptic fibration f of genus g with relative irregularity q_f satisfying $g+1=m(q_f+1)$ for some $m\geq 2$ and

$$\lambda_f=\frac{K_f^2}{\chi_f}=\lambda_{g,q_f},\qquad \text{where } \lambda_{g,q_f} \text{ is defined in (1-7).}$$

Let $P=\mathbb{P}_{\mathbb{P}^1}\big(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1}(e)\big)$ be the rational ruled surface with invariant $e\geq 1$. Let

$$h:\ P\longrightarrow B\triangleq\mathbb{P}^1$$

be the ruling, $\Gamma\subseteq P$ a general fiber of h, and $C_0\subseteq P$ the unique section with self-intersection $C_0^2=-e$. According to [17, § V-2], the divisor $mC_0+b\Gamma$ is very ample if and only if $b>me$. Let $mC_0+b_0\Gamma$ be a very ample divisor. Then by Bertini's theorem (cf. [17, § II-8]), a general member $D\in|mC_0+b_0\Gamma|$ is smooth and any two general members $D_1,\,D_2\in|mC_0+b_0\Gamma|$ intersect with each other transversely. Let D, D' be two general members in $|mC_0+b_0\Gamma|$, and Λ the pencil generated by D and D'. Then Λ defines a rational map $\varphi_\Lambda:\ P\dashrightarrow\mathbb{P}^1$. By blowing up the base points of Λ, we get a fibration $\tilde{h}':\ \widetilde{P}\to\mathbb{P}^1$,

$$\begin{array}{ccc} \widetilde{P} & \xrightarrow{\quad\psi\quad} & P \\ & {\scriptstyle \tilde{h}'}\searrow \quad \swarrow{\scriptstyle \varphi_\Lambda} & \\ & \mathbb{P}^1 & \end{array}$$

where $\psi:\ \widetilde{P}\to P$ is composed of blowing-ups centered at the base points of Λ. Let $\widetilde{\Gamma}'$ be a general fiber of $\tilde{h}'$, and let $K_{\widetilde{P}}$ be the canonical divisor of $\widetilde{P}$. Then

$$K_{\widetilde{P}}^2=8-x,\qquad K_{\widetilde{P}}\cdot\widetilde{\Gamma}'=\frac{(m-1)x}{m}-2m,\qquad \Big(\widetilde{\Gamma}'\Big)^2=0, \tag{5-1}$$

where $x=(mC_0+b_0\Gamma)^2$ is the number of blowing-ups contained in ψ. Let $\Delta\subseteq\mathbb{P}^1$ be a set of $2(q_f+1)$ general points, and let $\widetilde{R}=(\tilde{h}')^*(\Delta)$ be the corresponding fibers of $\tilde{h}'$. Let $\pi':B'\to\mathbb{P}^1$ be the double cover ramified over Δ, and $\widetilde{S}$ be the normalization of the fiber-product $\widetilde{P}\times_{\mathbb{P}^1}B'$:

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\ \tilde{\pi}\ } & \widetilde{P} \\ {\scriptstyle \tilde{f}'}\downarrow & & \downarrow{\scriptstyle \tilde{h}'} \\ B' & \xrightarrow{\ \pi'\ } & \mathbb{P}^1 \end{array}$$

Note that if Δ is general on $\mathbb{P}^1$, then $\widetilde{R}$ is both reduced and smooth. Hence $\widetilde{S}$ is also smooth. Note that $\widetilde{R}\equiv(2q_f+2)\widetilde{\Gamma}'$, where $\widetilde{\Gamma}'$ is a general fiber of $\tilde{h}'$. So by the standard formulas for double covers (cf. [9, §V.22]) and (5-1), one gets

$$\begin{aligned} K_{\widetilde{S}}^2 &= 2\left(K_{\widetilde{P}}+\frac{1}{2}\widetilde{R}\right)^2=\left(\frac{4(m-1)(q_f+1)}{m}-2\right)x-\big(8m(q_f+1)-16\big),\\ \chi(\mathcal{O}_{\widetilde{S}}) &= 2\chi(\mathcal{O}_{\widetilde{P}})+\frac{1}{2}\left(K_{\widetilde{P}}+\frac{1}{2}\widetilde{R}\right)\cdot\frac{\widetilde{R}}{2}=\frac{(m-1)(q_f+1)}{2m}x-\big(m(q_f+1)-2\big). \end{aligned}$$

The ruling $h:P\to B\cong\mathbb{P}^1$ induces a fibration $\tilde{h}:\widetilde{P}\to B$ and hence a fibration $f:\widetilde{S}\to B$:

$$\begin{array}{ccccc} \widetilde{S} & \xrightarrow{\ \tilde{\pi}\ } & \widetilde{P} & \xrightarrow{\ \psi\ } & P \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \tilde{h}} & \swarrow{\scriptstyle h} & \\ & & B\cong\mathbb{P}^1 & & \end{array}$$

It is easy to show that the induced map $\tilde{h}|_{\widetilde{R}}:\widetilde{R}\to B$ is of degree $m\cdot 2(q_f+1)=2(g+1)$, hence f is a hyperelliptic fibration of genus g. By construction, the relative irregularity of f is just $q_f=g(B')$, and

$$\begin{aligned} K_f^2 &= K_{\widetilde{S}}^2-8(g-1)(g(B)-1)=\left(\frac{4(m-1)(q_f+1)}{m}-2\right)x,\\ \chi_f &= \chi(\mathcal{O}_{\widetilde{S}})-(g-1)(g(B)-1)=\frac{(m-1)(q_f+1)}{2m}x. \end{aligned}$$

Actually, f is relatively minimal. To see this, let R be the image of $\widetilde{R}$ in P. Then the singular points of R are all of multiplicity $2(q_f+1)$. Hence $s_{2k+1}=0$ for all $k\geq 1$, which implies that $\widetilde{S}$ is relatively minimal by Lemma 2.7. Hence f is a relatively minimal locally non-trivial hyperelliptic fibration of genus g with relative irregularity $q_f=\frac{g+1}{m}-1\leq\frac{g-1}{2}$, and

$$\lambda_f=\frac{K_f^2}{\chi_f}=8-\frac{4m}{(m-1)(q_f+1)}=8-\frac{4(g+1)}{(g-q_f)(q_f+1)}=\lambda_{g,q_f}.$$

Remarks 5.2. (i) Let $g=3$ and $m=2$ in the above example. Then we get a hyperelliptic fibration f of genus 3 with relative irregularity $q_f=1$ and slope $\lambda_f=4$.

(ii) According to Bertini's theorem (cf. [17, §II-8]), for a general member $D\in|mC_0+b_0\Gamma|$, the projection of $h|_D:D\to B$ has at most simply ramified points. Hence if Δ is general enough, then the fibration f obtained in the above example is semi-stable.

Example 5.3. We construct a hyperelliptic fibration f of genus g with

$$\begin{cases} q_f = \dfrac{g}{2}, & \lambda_f = \dfrac{K_f^2}{\chi_f} = \dfrac{8(g-1)}{g}, \qquad \text{if } g \text{ is even;} \\ q_f = \dfrac{g+1}{2}, & \lambda_f = \dfrac{K_f^2}{\chi_f} = 8, \qquad \text{if } g \text{ is odd.} \end{cases}$$

Let F be the hyperelliptic curve of genus g defined by $u^2 = v^{2g+2} - 1$, and τ_1 be an involution of F defined by $\tau_1(u,v) = (-u,-v)$. Then τ_1 has exactly two fixed points if g is even and τ_1 has no fixed point if g is odd. Let $\phi : \widetilde{B} \to B$ be a double cover between two projective curves of genus $\tilde{b} = g(\widetilde{B})$ and $b = g(B)$ respectively, let $\Sigma \subseteq B$ be the branch divisor, let $\widetilde{\Sigma} = \phi^{-1}(\Sigma)$, and let τ_2 be the induced involution of $\widetilde{B}$ such that $B = \widetilde{B}/\langle\tau_2\rangle$. Let $\tau = (\tau_1, \tau_2)$ be an involution of $X = F \times \widetilde{B}$ defined by $\tau(p,q) = \big(\tau_1(p), \tau_2(q)\big)$, where $p \in F$ and $q \in \widetilde{B}$. Then $X/\langle\tau\rangle$ has a natural fibration of genus g over B. Let $f : S \to B$ be the relatively minimal smooth model of $X/\langle\tau\rangle$ as follows:

$$\begin{array}{ccccc} X & \xrightarrow{\ \Phi\ } & X/\langle\tau\rangle & \dashleftarrow\dashrightarrow & S \\ \Big\downarrow{\scriptstyle \tilde{f}} & & \Big\downarrow{\scriptstyle f'} & \swarrow{\scriptstyle f} & \\ \widetilde{B} & \xrightarrow{\ \phi\ } & B & & \end{array}$$

Assume $\Sigma \neq \emptyset$. Then we see that f is a non-trivial hyperelliptic fibration. If g is even, then τ has exactly two fixed points over each fiber in $\tilde{f}^*(\widetilde{\Sigma})$, and so $X/\langle\tau\rangle$ has only rational singularities of type A_1 (cf. [9, §III-3]). If g is odd, then τ is fixed-point-free, and hence $S = X/\langle\tau\rangle$ is already smooth and relatively minimal. Let $|\Sigma|$ be the number of points in Σ. Then one can compute that

$$\begin{aligned} K_f^2 &= 2(g-1)\cdot|\Sigma|, \\ \chi_f &= \begin{cases} \dfrac{g}{4}\cdot|\Sigma|, & \text{if } g \text{ is even;} \\ \dfrac{g-1}{4}\cdot|\Sigma|, & \text{if } g \text{ is odd.} \end{cases} \end{aligned}$$

Therefore, we obtain hyperelliptic fibrations with the required slopes. To compute the relative irregularity, we consider another projection of X, i.e., $\tilde{h} : X \to F$. It induces a fibration $h' : X/\langle\tau\rangle \to B' = F/\langle\tau_1\rangle$, and hence also a fibration $h : S \to B'$ with

$$g(B') = \frac{g}{2}, \text{ if } g \text{ is even,} \qquad \text{and} \qquad g(B') = \frac{g+1}{2}, \text{ if } g \text{ is odd.}$$

In particular, $q_f \geq g(B')$. Combining this with (3-2), we see that $q_f = g(B')$ as required.

Remark 5.4. Taking $g = 2$ in the above example, we get a hyperelliptic fibration of genus 2 with relative irregularity $q_f = 1$ and slope $\lambda_f = 4$.

Appendix

In this appendix, we would like to prove the following theorem on the unitary part of a semi-stable hyperelliptic fibration:

Theorem A.1. *Let $f: S \to B$ be a semi-stable hyperelliptic fibration of curves of genus $g \geq 2$ with relative irregularity q_f. Consider the following Fujita decomposition (cf.* [10, 16, 19]*):*

$$f_*\omega_{S/B} = \mathcal{A}^{1,0} \oplus \mathcal{F}^{1,0}, \quad \text{where } \mathcal{A}^{1,0} \text{ is ample and } \mathcal{F}^{1,0} \text{ is unitary.}$$

Then after passing to a suitable finite étale base change, one has

$$\operatorname{rank} \mathcal{F}^{1,0} = q_f. \tag{A-1}$$

Note that the above theorem has been proven in [23, Theorem 4.7]. The proof there used a general lemma (cf. [23, Lemma 7.1]) regarding the global invariant cycle with unitary locally constant coefficient, which may be of independent interest. However, the theorem is purely on fibred surfaces. Hence we would like to give another proof without using such a lemma. Instead, we prove a weak result about the lifting property of the unitary part $\mathcal{F}^{1,0}$ (cf. Lemma A.2), which is enough for our purpose. The rest of the argument is parallel to that of [23, Theorem 4.7].

First reduction. We first prove that Theorem A.1 can be reduced to the following theorem.

Theorem A.1′. *Let f be the same as in Theorem* A.1. *Then after passing to a suitable finite (not necessarily étale) base change, one has*

$$\operatorname{rank} \mathcal{F}^{1,0} = q_f. \tag{A-2}$$

Proof of Theorem A.1 *based on Theorem* A.1′. Note that the bundle $\mathcal{F}^{1,0}$ comes from a unitary representation of the fundamental group

$$\rho: \ \pi_1(B) \longrightarrow U(r), \qquad \text{with } r = \operatorname{rank} \mathcal{F}^{1,0}.$$

It follows that $\operatorname{rank} \mathcal{F}^{1,0} = q_f$ if and only if the representation ρ is trivial. Note that for any finite subset $\Sigma \subseteq B$, the restriction $\mathcal{F}^{1,0}\big|_{B\setminus\Sigma}$ corresponds to the unitary representation ρ_Σ:

$$\begin{array}{ccc} \pi_1\big(B\setminus\Sigma\big) & \xrightarrow{\quad\rho_\Sigma\quad} & U(r) \\ & {\scriptstyle i_*}\searrow \quad \nearrow{\scriptstyle \rho} & \\ & \pi_1(B) & \end{array}$$

By Theorem A.1′, there exists a subset Σ such that ρ_Σ has finite image. Because ρ_Σ factors through $\pi_1(B)$ and i_* is surjective, one gets that ρ has also finite image. It implies that $\mathcal{F}^{1,0}$ becomes trivial after a suitable finite étale base change, i.e., the equality (A-1) holds after a suitable finite étale base change. □

By the above reduction, it suffices to prove Theorem A.1′. We first recall some general knowledge about the variation of a Hodge structure for a fibration f and prove a lifting property about $\mathcal{F}^{1,0}$ in Lemma A.2.

Let $\Upsilon \subset S$ be the singular fibers over the degeneration locus $\Delta \subset B$. Consider the weight one variation of Hodge structures given by $R^1f_*\mathbb{Z}_{S\setminus\Upsilon}$. Taking the graded sheaves, one obtains the (logarithmic) Higgs bundle (cf. [27, § 4])

$$(E,\ \theta) = (E^{1,0} \oplus E^{0,1},\ \theta)$$

with $E^{1,0} = f_*\Omega^1_{S/B}(\log \Upsilon) \cong f_*\omega_{S/B}$ and $E^{0,1} = R^1 f_*\mathcal{O}_S$. The Higgs field θ is given by the edge morphism

$$f_*\Omega^1_{S/B}(\log \Upsilon) \longrightarrow R^1 f_*\mathcal{O}_S \otimes \Omega^1_B(\log \Delta)$$

of the tautological sequence

$$0 \longrightarrow f^*\Omega^1_B(\log \Delta) \longrightarrow \Omega^1_S(\log \Upsilon) \longrightarrow \Omega^1_{S/B}(\log \Upsilon) \longrightarrow 0. \tag{A-3}$$

By [10, 16, 19], E can be decomposed as a direct sum $\mathcal{A} \oplus \mathcal{F}$ of Higgs bundles with $E^{1,0} \cap \mathcal{A}$ ample and $\mathcal{F}$ flat; hence for $\mathcal{A}^{i,j} = E^{i,j} \cap \mathcal{A}$ and $\mathcal{F}^{i,j} = E^{i,j} \cap \mathcal{F}$, the Higgs bundle E decomposes as

$$(E^{1,0} \oplus E^{0,1},\ \theta) = \big(\mathcal{A}^{1,0} \oplus \mathcal{A}^{0,1},\ \theta|_{\mathcal{A}^{1,0}}\big) \oplus \big(\mathcal{F}^{1,0} \oplus \mathcal{F}^{0,1},\ 0\big).$$

In particular, $\mathcal{F}^{1,0} \subseteq M := \mathrm{Ker}\,(\theta) \subseteq f_*\Omega^1_{S/B}(\log \Upsilon)$, and we have the following exact sequence induced by (A-3):

$$0 \longrightarrow \Omega^1_B(\log \Delta) \longrightarrow f_*\Omega^1_S(\log \Upsilon) \stackrel{\varsigma}{\longrightarrow} M \longrightarrow 0. \tag{A-4}$$

By [15], one has a further decomposition of $\mathcal{F}^{1,0}$:

$$f_*\Omega^1_{S/B}(\log \Upsilon) \cong f_*\omega_{S/B} = \mathcal{A}^{1,0} \oplus \mathcal{F}^{1,0} = \mathcal{A}^{1,0} \oplus \mathcal{F}_u \oplus \mathcal{O}_B^{\oplus q_f}, \tag{A-5}$$

with $\mathcal{A}^{1,0}$ ample, $\mathcal{F}_u$ unitary and $h^0(B, \mathcal{F}_u^\vee) = h^1(B, \Omega^1_B(\mathcal{F}_u)) = 0$. Note that $h^0(B, \mathcal{F}_u^\vee) = 0$ if and only if $h^0(B, \mathcal{F}_u) = 0$, since $\mathcal{F}_u$ is unitary.

Lemma A.2. *Let $i : \mathcal{F}_u \hookrightarrow M$ be the induced inclusion. Then there exists an injective sheaf map $\iota : \mathcal{F}_u \hookrightarrow f_*\Omega^1_S(\log \Upsilon)$ such that $i = \varsigma \circ \iota$, i.e., we have the following commutative diagram:*

$$\begin{array}{ccccccccc} & & & & & & \mathcal{F}_u & & \\ & & & & & \swarrow^{\exists\,\iota} & \big\downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & \Omega^1_B(\log \Delta) & \longrightarrow & f_*\Omega^1_S(\log \Upsilon) & \stackrel{\varsigma}{\longrightarrow} & M & \longrightarrow & 0 \end{array}$$

Proof. The exact sequence (A-4) gives an element α in $\mathrm{Ext}^1(M, \Omega^1_B(\log \Delta))$. The existence of ι such that $i = \varsigma \circ \iota$ is equivalent to the vanishing of α under the natural map

$$\mathrm{Ext}^1(M, \Omega^1_B(\log \Delta)) \stackrel{i^\vee}{\longrightarrow} \mathrm{Ext}^1(\mathcal{F}_u, \Omega^1_B(\log \Delta)).$$

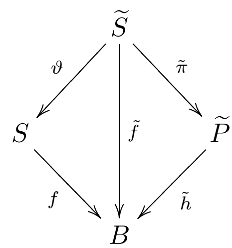
Note that

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{F}_u, \Omega^1_B(\log \Delta)) &\cong \mathrm{Ext}^1(\mathcal{O}_B, \mathcal{F}_u^\vee \otimes \Omega^1_B(\log \Delta)) \\ &\cong H^1(B, \mathcal{F}_u^\vee \otimes \Omega^1_B(\log \Delta)) \\ &\cong H^0(B, \mathcal{F}_u(-\Delta))^\vee = 0. \end{aligned}$$

The last equality is due to the fact that $h^0(B, \mathcal{F}_u) = 0$ and Δ is effective. Therefore, we obtain a map ι such that $i = \varsigma \circ \iota$. The injectivity of ι follows from that of i. □

Remark A.3. The above lemma is weaker than [23, Corollary 7.2], in which it is proven based on [23, Lemma 7.1] that there exists a lifting $\tilde{\iota} : \mathcal{F}^{1,0} \hookrightarrow f_*\Omega^1_S(\log \Upsilon)$ for the inclusion $\mathcal{F}^{1,0} \hookrightarrow M$ and the image of $\tilde{\iota}$ is actually contained in $f_*\Omega^1_S \subseteq f_*\Omega^1_S(\log \Upsilon)$.

Now we restrict ourselves to the situation that $f : S \to B$ is a semi-stable hyperelliptic fibration. Let σ be the hyperelliptic involution of f, let $\vartheta : \widetilde{S} \to S$ be the composition of all the blowing-ups of isolated fixed points of σ, and let $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$ be the induced double cover with the following diagram:



Lemma A.4. *Let $\widetilde{\Upsilon}$ be the strict inverse image of Υ in $\widetilde{S}$. Then*

$$\vartheta^*\big(\Omega^1_S(\log\Upsilon)\big) \subseteq \Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon}).$$

Proof. It is enough to consider the case when $\vartheta : \widetilde{S} \to S$ consists of exactly one blowing-up. Let E be the exceptional curve. Then

$$\vartheta^*\big(\Omega^1_S(\log\Upsilon)\big) = \Omega^1_{\widetilde{S}}\big(\log(\widetilde{\Upsilon}+E)\big)(-E) \subseteq \Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon}). \qquad \square$$

The lifting ι obtained in Lemma A.2 gives rise to a morphism

$$f^*\mathcal{F}_u \hookrightarrow f^*f_*\Omega^1_S(\log\Upsilon) \longrightarrow \Omega^1_S(\log\Upsilon).$$

Hence by Lemma A.4, one obtains a morphism

$$\tilde{f}^*\mathcal{F}_u = \vartheta^* f^*\mathcal{F}_u \longrightarrow \Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon}).$$

Equivalently, we obtain an element $\eta \in H^0\big(\widetilde{S}, \Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon}) \otimes \tilde{f}^*\mathcal{F}_u^\vee\big)$, and hence also an element

$$\tilde{\pi}_*\eta \in H^0\Big(\widetilde{P}, \tilde{\pi}_*\big(\Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon}) \otimes \tilde{f}^*\mathcal{F}_u^\vee\big)\Big) \cong H^0\Big(\widetilde{P}, \tilde{\pi}_*\big(\Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon})\big) \otimes \tilde{h}^*\mathcal{F}_u^\vee\Big).$$

So one gets a sheaf map

$$\tilde{h}^*\mathcal{F}_u \longrightarrow \tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon})\right).$$

The Galois group $\mathrm{Gal}(\widetilde{S}/\widetilde{P}) \cong \mathbb{Z}_2$ acts on $\tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon})\right)$. One therefore obtains the eigenspace decomposition

$$\tilde{h}^*(\mathcal{F}_u) \longrightarrow \tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon})\right)_1, \qquad \tilde{h}^*(\mathcal{F}_u) \longrightarrow \tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon})\right)_{-1}.$$

Lemma A.5. *Assume that the fixed locus of the hyperelliptic involution σ consists of $2g+2$ disjoint sections and possibly some isolated points. Then the image of the map*

$$\varrho : \tilde{h}^*(\mathcal{F}_u) \longrightarrow \tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log\widetilde{\Upsilon})\right)_{-1}$$

is an invertible subsheaf $\mathcal{M}$ such that $\mathcal{M}$ is nef and $\mathcal{M}^2 = 0$. Let D be any component of the branch divisor $\widetilde{R} \subseteq \widetilde{P}$ of the double cover $\tilde{\pi} : \widetilde{S} \to \widetilde{P}$. Then $\mathcal{M} \cdot D = 0$.

Proof. First of all, we show that $\varrho \neq 0$. By assumption, the strict inverse image $\tilde{\pi}^{-1}(\widetilde{R})$, which is the fixed locus of the hyperelliptic involution $\tilde{\sigma}$ on $\widetilde{S}$, consists of $2g+2$ disjoint sections and the exceptional curves of ϑ. Hence it is easy to see that $\widetilde{\Upsilon}$ intersects $\pi^{-1}(\widetilde{R})$ transversely. Let $\Upsilon' \subseteq \widetilde{P}$ be the image of $\widetilde{\Upsilon}$, and $\widetilde{L} \in \mathrm{Pic}(\widetilde{P})$ such that $\widetilde{L}^{\otimes 2} \cong \mathcal{O}_{\widetilde{P}}(\widetilde{R})$ defines the double cover $\tilde{\pi}$. Then

$$\text{(A-6)} \qquad \begin{cases} \tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log \widetilde{\Upsilon})\right)_1 = \Omega^1_{\widetilde{P}}(\log \Upsilon'), \\ \tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log \widetilde{\Upsilon})\right)_{-1} = \Omega^1_{\widetilde{P}}\left(\log(\Upsilon' + \widetilde{R})\right) \otimes \widetilde{L}^{-1}. \end{cases}$$

Note that $\tilde{h}_*\left(\Omega^1_{\widetilde{P}}(\log \Upsilon')\right) = 0$, since $h^0\left(\widetilde{\Gamma}, \Omega^1_{\widetilde{P}}(\log \Upsilon')\big|_{\widetilde{\Gamma}}\right) = 0$ for any general fiber $\widetilde{\Gamma}$ of $\tilde{h}$. Note also that the induced map

$$\begin{aligned} \mathcal{F}_u = \tilde{h}_*\tilde{h}^*\mathcal{F}_u = \tilde{f}_*\tilde{f}^*\mathcal{F}_u \longrightarrow \tilde{f}_*\Omega^1_{\widetilde{S}}(\log \widetilde{\Upsilon}) &= \tilde{h}_*\left(\tilde{\pi}_*(\Omega^1_{\widetilde{S}}(\log \widetilde{\Upsilon}))\right) \\ &= \tilde{h}_*\left(\Omega^1_{\widetilde{P}}(\log \Upsilon')\right) \oplus \tilde{h}_*\left(\Omega^1_{\widetilde{P}}\left(\log(\Upsilon' + \widetilde{R})\right) \otimes \widetilde{L}^{-1}\right) \end{aligned}$$

is the composition of the inclusions $\mathcal{F}_u \stackrel{\iota}{\hookrightarrow} f_*\Omega^1_S(\log \Upsilon) \hookrightarrow \tilde{f}_*\Omega^1_{\widetilde{S}}(\log \widetilde{\Upsilon})$, where the latter inclusion follows from Lemma A.4. On the other hand, the second component of the above map is actually given by $\tilde{h}_*\varrho$. Hence in particular, $\varrho \neq 0$.

Second, we prove that the image of ϱ is a subsheaf of rank one. Otherwise, it is of rank two, and so the second wedge product

$$\bigwedge^2 \tilde{h}^*\mathcal{F}_u \xrightarrow{\bigwedge^2 \varrho} \bigwedge^2\left(\tilde{\pi}_*\left(\Omega^1_{\widetilde{S}}(\log \widetilde{\Upsilon})\right)_{-1}\right) = \omega_{\widetilde{P}}(\Upsilon')$$

is a non-zero map. On the other hand, the image $\mathcal{C}$ of this map is a quotient sheaf of $\bigwedge^2 \tilde{h}^*\mathcal{F}_u$ coming from a unitary representation. Note that for any morphism $\alpha : C \to \widetilde{P}$ from a smooth complete curve C, $\alpha^*\left(\bigwedge^2 \tilde{h}^*\mathcal{F}_u\right)$ is poly-stable of slope zero since it comes from a unitary representation (cf. [24]), which implies that $\bigwedge^2 \tilde{h}^*\mathcal{F}_u$ is semi-positive. Therefore, as a quotient of $\bigwedge^2 \tilde{h}^*\mathcal{F}_u$, $\mathcal{C}$ is also semi-positive. Here we recall that a locally free sheaf $\mathcal{E}$ on $\widetilde{P}$ is called semi-positive if for any morphism $\alpha : C \to \widetilde{P}$ from a smooth complete curve C, the pulling-back $\alpha^*\mathcal{E}$ has no quotient line bundle of negative degree. Since $\omega_{\widetilde{P}}(\Upsilon') \cdot \widetilde{\Gamma} = -2$ for any general fiber $\widetilde{\Gamma}$ of $\tilde{h}$, $\omega_{\widetilde{P}}(\Upsilon')$ cannot contain any non-zero semi-positive subsheaf. So the image of ϱ is a rank one subsheaf $\mathcal{M} \otimes I_Z$, where $\mathcal{M}$ is an invertible subsheaf and $\dim Z = 0$.

We claim that $Z = \emptyset$, and hence the image of ϱ is an invertible subsheaf. Indeed, if $Z \neq \emptyset$, by a suitable blowing-up $\psi : X \to \widetilde{P}$, we may assume that the image $\psi^*\varrho\left(\psi^*\tilde{h}^*(\mathcal{F}_u)\right)$ is $\psi^*(\mathcal{M}) \otimes (-E)$, where $E(\neq 0)$ is a suitable combination of the exceptional curves of ψ. As $\mathcal{F}_u$ comes from a unitary representation, we get that $\psi^*(\mathcal{M}) \otimes (-E)$ is semi-positive and hence

$$0 \leq (\psi^*(\mathcal{M}) - E)^2 = \mathcal{M}^2 + E^2.$$

So $\mathcal{M}^2 \geq -E^2 > 0$. On the other hand, the invertible sheaf $\mathcal{M}$ is also a quotient sheaf of $\tilde{h}^*\mathcal{F}_u$, which comes from a unitary representation, and hence it is semi-positive. This implies that the Kodaira dimension of $\mathcal{M}$ is 2. By (A-6), we get the following inclusion of sheaves:

$$\widetilde{L}\otimes\mathcal{M}\subseteq\Omega^1_{\widetilde{P}}\left(\log(\Upsilon'+\widetilde{R})\right).$$

As $2\widetilde{L}\equiv\widetilde{R}$ is effective, the Kodaira dimension of $\widetilde{L}\otimes\mathcal{M}$ is also 2, which is impossible by the Bogomolov lemma (cf. [26, Lemma 7.5]). Hence the image of ϱ is an invertible subsheaf $\mathcal{M}$, which is semi-positive since it is a quotient sheaf of a vector bundle coming from a unitary representation.

Finally, one can prove similarly that $\mathcal{M}\cdot D=0$ for any component $D\in\widetilde{R}$; otherwise, $\widetilde{L}\otimes\mathcal{M}$ will be of Kodaira dimension two, which is impossible by the Bogomolov lemma (cf. [26, Lemma 7.5]). The proof is complete. □

Now we prove Theorem A.1′ and hence complete the proof of Theorem A.1.

Proof of Theorem A.1′. After a suitable base change, we may assume that the fixed locus of the hyperelliptic involution σ consists of $2g+2$ disjoint sections and possibly some isolated points; actually, one achieves this by base change unbranched over $B\setminus\Delta$ as f is semi-stable. To prove our theorem, it suffices to show that $\mathcal{F}_u$ in (A-5) becomes trivial after a suitable base change. We prove this by induction on $r=\operatorname{rank}\mathcal{F}_u$.

If $r=1$, then by [13, §4.2] or [4, Theorem 3.4], we get that $\mathcal{F}_u$ is torsion in $\operatorname{Pic}^0(B)$. So after a suitable finite étale base change, $\mathcal{F}_u$ will be trivial as required.

Now we assume that $r>1$. If $\mathcal{F}_u=\mathcal{F}_u'\oplus\mathfrak{L}$ with $\operatorname{rank}\mathcal{F}_u'=r-1$ and $\operatorname{rank}\mathfrak{L}=1$, then again by [13, §4.2] or [4, Theorem 3.4], one gets that the pull-back of $\mathfrak{L}$ becomes trivial after a suitable finite étale base change; hence our theorem follows from the induction. Therefore, it is enough to prove that $\mathcal{F}_u$ contains a direct summand $\mathfrak{L}$ of rank one.

Let $s: B\to\widetilde{P}$ be a section of $\tilde{h}$ contained in $\widetilde{R}$, and let $D=s(B)$, $j: D\hookrightarrow\widetilde{P}$ be the inclusion. By applying Lemma A.5, one obtains that $\mathcal{M}\cdot D=0$ with $\mathcal{M}=\varrho\left(\tilde{h}^*\mathcal{F}_u\right)$, i.e., $\deg\mathcal{O}_D(\mathcal{M})=0$. Note that $s^*j^*\tilde{h}^*\mathcal{F}_u\cong\mathcal{F}_u$ and $s^*j^*\mathcal{M}=s^*\mathcal{O}_D(\mathcal{M})$, where we view s as a morphism from B to D. Hence we may view $\mathfrak{L}:=s^*\mathcal{O}_D(\mathcal{M})$ as an invertible sheaf on B, which is a quotient of $\mathcal{F}_u$ since $\mathcal{M}$ is a quotient of $\tilde{h}^*\mathcal{F}_u$. Moreover, $\deg\mathfrak{L}=\deg\mathcal{O}_D(\mathcal{M})=0$ since s is an isomorphism. As $\mathcal{F}_u$ comes from a unitary representation, $\mathcal{F}_u$ is poly-stable (cf. [24]). Thus $\mathcal{F}_u=\mathfrak{L}\oplus\mathcal{F}_u'$ contains a direct summand $\mathfrak{L}$ of rank one as required. This completes the proof. □

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