

SEMICLASSICAL CAUCHY ESTIMATES AND APPLICATIONS

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ABSTRACT. In this note, we consider the solutions to semiclassical analytic Schrödinger equations and prove a semiclassical version of the Cauchy estimate. As an application, we study the asymptotics for the Hausdorff measures of the nodal sets of the solutions on a compact analytic manifold as $h \rightarrow 0$.

1. INTRODUCTION

In this note, we study the solutions to semiclassical Schrödinger equations on a real analytic manifold M of dimension $n > 1$:

$$(1.1) \quad (-h^2 \Delta_g + V(x) - E(h))u(h) = 0,$$

where V and g are the real-analytic function and metric on M , respectively, and $E(h) \rightarrow E_0$ as $h \rightarrow 0$. We also consider more general differential operators and, when $M = \mathbb{R}^n$, analytic pseudodifferential operators satisfying a suitable ellipticity condition.

The analyticity of V and of the metric g imply that solutions are real-analytic (see Hörmander [12, Theorem 8.6.1, 9.5.1 (for hyperfunctions)]) and in particular Cauchy estimates hold:

$$\sup_K |\partial^\alpha u(x)| \leq C_h^{|\alpha|} |\alpha|^{|\alpha|} \|u\|_{L^\infty}, \quad \forall K \Subset M,$$

for some constants C_h depending on h . The semiclassical Cauchy estimate provides details about the constants C_h :

Theorem 1.1. *Let (M, g, V) be as above. Then the solutions $u = u(h)$ to (1.1) satisfy the estimates*

$$(1.2) \quad \sup_K |\partial^\alpha u(x)| \leq C^{|\alpha|} (h^{-1} + |\alpha|)^{|\alpha|} \|u\|_{L^\infty}, \quad \forall K \Subset M,$$

where C depends only on g , V and K .

The theorem follows immediately from Theorem 2.6 which applies to more general classically elliptic differential operators satisfying certain analytic conditions. We refer to Section 2.3 for the details of such conditions. The proof of Theorem 2.6 is based on the FBI transform approach to analytic semiclassical theory developed by Sjöstrand [16]. We refer to Martinez [15] for a more accessible presentation.

We should remark that for differential operators one can obtain estimates equivalent to (1.2) (see Proposition 2.2) by using Hörmander's approach to analytic hypoellipticity and rescaling; see [5, Lemma 7.1]. In fact, we learned about this

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after proving (1.2) directly using the FBI transform, and the study of the Donnelly-Fefferman paper [5] led to the following applications to the volume of nodal sets (zero set of $u(h)$) in the semiclassical setting.

Theorem 1.2. *Let (M, g) be a real-analytic compact Riemannian manifold and let V be a real analytic function on M . Then the nodal set N of the solution $u = u(h)$ of (1.1) has $(n - 1)$ -dimensional Hausdorff measure of order h^{-1} . In other words, there exist constants $c, C > 0$ depending only on (M, g, V) such that*

$$(1.3) \quad ch^{-1} \leq \mathcal{H}^{n-1}(N) \leq Ch^{-1}.$$

Remark 1.3. In our proof, the lower bound comes from the classically allowed region $\{V < E_0\}$ which is nonempty. However, our argument does not give any lower bounds in the classically forbidden region. The results in [14] suggest that the behavior of nodal sets in the classically forbidden region could be very different from the one in the classically allowed region.

An illustration of the level sets of eigenfunctions is shown in Figure 1: the zero sets occur in the regions where the eigenfunction is “small” and, in particular, are indistinguishable from the classically forbidden regions.

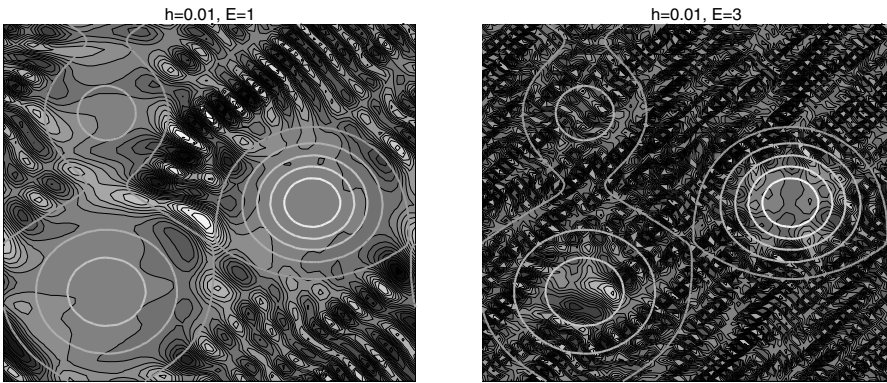


FIGURE 1. Level sets of eigenfunctions of $-h^2\Delta + V$ on a torus, $[0, 1] \times [0, 1]$ where $h = 0.01$ and V is a periodized sum of three bumps: $5e^{-10((x-0.75)^2+(y-0.5)^2)} + 2e^{-10((x+0.25)^2+(y-0.75)^2)} + 3e^{-5((x+0.25)^2+(y+0.25)^2)}$ (with level sets shown). In the first picture the energy level is close to 1 and, in the second, to 3, so that the difference in classically forbidden regions is clearly visible.

When $V \equiv 0$, i.e., u is an eigenfunction of the Laplacian operator on M with eigenvalues Eh^{-2} , this is the analytic case of Yau’s conjecture [19] and is proved by Donnelly-Fefferman [5]. In the smooth setting, this is still an open problem. Exponential types of upper and lower bounds were first established by Hardt and Simon [10]; see the notes [9] for a detailed study on nodal sets and [3], [17], [11], [18] for recent progress on Yau’s conjecture. Also see [14] for nodal sets of semiclassical harmonic oscillators and [2] for the physics perspective of nodal sets for solutions of Schrödinger operators.

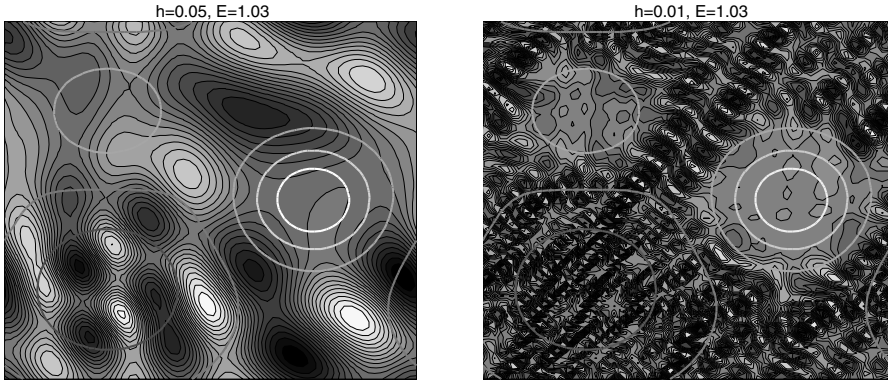


FIGURE 2. Level sets of eigenfunctions of $-h^2\Delta + V$ on a torus, $[0, 1] \times [0, 1]$ where V is a periodized sum of two bumps and one well: $5e^{-10((x-0.75)^2+(y-0.5)^2)} + 2e^{-10((x+0.25)^2+(y-0.75)^2)} - 3e^{-5((x+0.25)^2+(y+0.25)^2)}$ (with level sets shown). The energy level is now fixed at $E = 1$ but $h = 0.05$ and $h = 0.01$.

Structure of the paper. In Section 2, we prove the semiclassical Cauchy estimate. We start by reviewing the definition and basic properties of the Fourier-Bros-Iagolnitzer (FBI) transform in Section 2.1. Next, in Section 2.2, we give a characterization of analyticity in the semiclassical setting using the FBI transform. Then in Section 2.3, we use Agmon estimates on the FBI transform to conclude the proof of Theorem 2.6 and thus Theorem 1.1.

Section 3 contains a proof of the doubling property of solutions to (1.1) on compact manifolds where we allow the manifold and the potential to be merely smooth. These types of results have been proved in a more general setting (e.g., [1] for C^1 -potentials) and are closely related to the unique continuation problems. There are two different ways to achieve such results: the usual approach is through the Carleman-type estimates which establish a priori estimates with a weight; another approach was developed by Garofalo and Lin [7] based on a combination of geometric and variational ideas. We follow the usual approach based on a Carleman estimate which is presented in Section 3.1, and use the ideas of Donnelly-Fefferman [5] to prove the tunneling estimate (3.1) and then the doubling estimate (3.2).

Finally, in Section 4, we use the estimates in Section 3 to study the vanishing property of solutions to (1.1). We first prove in Section 4.1 that a nonzero solution can only vanish to order $O(h^{-1})$; then we deduce Theorem 1.2 on the size of the nodal sets of the solutions in Sections 4.2 and 4.3. Our proof of Theorem 1.2 is very close to the approach of Donnelly-Fefferman [5] and uses some basic estimates for holomorphic functions from [5]. We also use some of the ideas from [3].

2. SEMICLASSICAL CAUCHY ESTIMATES AND ANALYTIC CONTINUATION

2.1. Fourier-Bros-Iagolnitzer transform. In this section we review some basic facts of the Fourier-Bros-Iagolnitzer transform. For $h > 0$, we define $T_h : \mathcal{S}'(\mathbb{R}^n) \rightarrow \mathcal{S}'(\mathbb{R}^{2n})$ as

$$(2.1) \quad T_h u(x, \xi) = 2^{-n/2}(\pi h)^{-3n/4} \int e^{i(x-y)\xi/h - (x-y)^2/2h} u(y) dy.$$

In other words, $T_h u(x, \xi) = \langle \phi_{x, \xi}, u \rangle_{\mathcal{S}, \mathcal{S}'}$ where $\phi_{x, \xi}$ is the so-called coherent state centered at (x, ξ) , so $T_h u(x, \xi)$ captures the microlocal property of u at $(x, \xi) \in \mathbb{R}^{2n}$. We state some basic properties of the FBI transform:

(1) If $u \in \mathcal{S}'(\mathbb{R}^n)$, then $e^{\xi^2/2h} T_h u(x, \xi)$ is a holomorphic function of $z = x - i\xi \in \mathbb{C}^n$. In fact, $T_h(\mathcal{S}'(\mathbb{R}^n)) = \mathcal{S}'(\mathbb{R}^{2n}) \cap e^{-\xi^2/2h} \mathcal{H}(\mathbb{C}_{x-i\xi}^n)$ where $\mathcal{H}(\mathbb{C}_{x-i\xi}^n)$ is the space of entire functions on \mathbb{C}^n . This also shows $hD_x T_h u = (\xi + ihD_\xi) T_h u$.

(2) For every $u \in \mathcal{S}'(\mathbb{R}^n)$, $u = T_h^* T_h u$ where T_h^* is defined as

$$T_h^* v(y) = 2^{-n/2} (\pi h)^{-3n/4} \int e^{-i(x-y)\xi/h - (x-y)^2/2h} v(x, \xi) dx d\xi$$

(interpreted as an oscillatory integral with respect to ξ).

(3) If $u \in L^2(\mathbb{R}^n)$, then $T_h u \in L^2(\mathbb{R}^{2n})$ and $\|T_h u\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)}$. Moreover, $T_h T_h^*$ is the orthogonal projection from $L^2(\mathbb{R}^{2n})$ onto $T_h(L^2(\mathbb{R}^n)) = L^2(\mathbb{R}^{2n}) \cap e^{-\xi^2/2h} \mathcal{H}(\mathbb{C}_{x-i\xi}^n)$.

(4) Let $p \in S_{2n}(1)$; then $\tilde{p}(x, \xi, x^*, \xi^*) = p(x - \xi^*, x^*)$ belongs to $S_{4n}(1)$ and we have

$$T_h \circ p(x, hD_x) = \tilde{p}(x, \xi, hD_x, hD_\xi) \circ T_h,$$

where x^* and ξ^* are the dual variables of x and ξ respectively. This formula is exact and \tilde{p} does not depend on μ or which quantization we are using.

Bros and Iagolnitzer first used this type of transform to characterize the analytic wavefront set: $(x_0, \xi_0) \notin \text{WF}_a(u)$ if and only if $T_h u(x, \xi) = O(e^{-c/h})$ uniformly in a neighborhood of (x_0, ξ_0) for some $c > 0$. In general, we can define the FBI transform with a phase which “looks like” the standard phase above and an elliptic analytic symbol. All such FBI transforms can be used to characterize the analytic wavefront set; see [4], [16] and [20] for this general approach. For convenience, we shall only consider the standard FBI transform and the following modification.

Lemma 2.1 (Change of FBI by an analytic symbol). *Suppose that $a = a(x, y, \xi)$ is a symbol defined for $x, y \in \mathbb{C}^n, \xi \in \mathbb{R}^n$, holomorphic in x, y , analytic in ξ and of tempered growth in x, y, ξ . Let $\mu > 1$ be fixed; then we define a new FBI transform T'_h as*

$$(2.2) \quad T'_h u(x, \xi) = 2^{-n/2} (\pi h)^{-3n/4} \int e^{i(x-y)\xi/h - \mu(x-y)^2/2h} a(x, y, \xi) u(y) dy.$$

Then if $|T_h u(x, \xi)| = O(e^{-c/h})$ in a real neighborhood U of (x_0, ξ_0) , we have $|T'_h u(x, \xi)| = O(e^{-c'/h})$ in a neighborhood V of (x_0, ξ_0) , where c' and V only depend on c , the growth of a and the size of U .

Proof. We shall write $T'_h u = T'_h(T_h^* T_h u) = (T'_h T_h^*) T_h u$ as follows:

$$\begin{aligned} & T'_h u(x, \xi) \\ &= \iint\!\!\!\int e^{i(x-y)\xi/h - \mu(x-y)^2/2h - i(\tilde{x}-y)\tilde{\xi}/h - (\tilde{x}-y)^2/2h} a(x, y, \xi) T_h u(\tilde{x}, \tilde{\xi}) d\tilde{x} d\tilde{\xi} dy. \end{aligned}$$

Therefore

$$(2.3) \quad T'_h u(x, \xi) = \iint b(x, \xi, \tilde{x}, \tilde{\xi}) T_h u(\tilde{x}, \tilde{\xi}) d\tilde{x} d\tilde{\xi}$$

where

$$b(x, \xi, \tilde{x}, \tilde{\xi}) = e^{\frac{1}{\mu+1} \frac{i}{h} (x-\tilde{x})(\xi+\mu\tilde{\xi}) - \frac{\mu}{\mu+1} \frac{1}{2h} (x-\tilde{x})^2 - \frac{1}{\mu+1} \frac{1}{2h} (\xi-\tilde{\xi})^2} \times \int e^{-\frac{\mu+1}{2h} (y - \frac{\mu x + \tilde{x}}{\mu+1} + i \frac{\xi - \tilde{\xi}}{\mu+1})^2} a(x, y, \xi) dy.$$

Now if we change the contour to $y \mapsto y + \frac{\mu x + \tilde{x}}{\mu+1} - i \frac{\xi - \tilde{\xi}}{\mu+1}$,

$$b(x, \xi, \tilde{x}, \tilde{\xi}) = e^{\frac{1}{\mu+1} \frac{i}{h} (x-\tilde{x})(\xi+\mu\tilde{\xi}) - \frac{\mu}{\mu+1} \frac{1}{2h} (x-\tilde{x})^2 - \frac{1}{\mu+1} \frac{1}{2h} (\xi-\tilde{\xi})^2} \times \int e^{-\frac{\mu+1}{2h} y^2} a(x, y + \frac{\mu x + \tilde{x}}{\mu+1} - i \frac{\xi - \tilde{\xi}}{\mu+1}, \xi) dy,$$

and use the assumption that a is of tempered growth in x, y, ξ , we get the estimate

$$|b(x, \xi, \tilde{x}, \tilde{\xi})| \leq C e^{-\delta(x-\tilde{x})^2/h - \delta(\xi-\tilde{\xi})^2/h},$$

where δ, C depend only on the growth of a . Now the theorem follows easily from (2.3) by separating the integral into two parts: $(\tilde{x}, \tilde{\xi})$ close to (x, ξ) and $(\tilde{x}, \tilde{\xi})$ far away from (x, ξ) . □

2.2. Equivalence between Cauchy estimates and decay of the FBI transform. In this section, we prove the equivalence between the semiclassical Cauchy estimate and the uniform exponential decay for the FBI transform when $|\xi|$ is large. Comparing to [15], we use different parameters for the FBI transform and the function u itself, so we can capture both the microlocal and semiclassical properties of u . The idea of the proof is similar to the proof of the fact the projection of the analytic wavefront set is the analytic singular support; see [16].

Proposition 2.2. *Let $u = u(h), 0 < h \leq h_0$, be a family of functions on a neighborhood X of $x_0 \in \mathbb{R}^n$ such that $\|u\|_{L^\infty(X)} = O(h^{-N})$. Then the following are equivalent:*

(i) *There exist an open neighborhood $V \Subset X$ of x_0 and constants $C_0, C_1, \delta > 0$ such that for every $0 < \tilde{h} \leq h \leq h_0, x \in V$ and $|\xi| \geq C_0$,*

$$(2.4) \quad |T_{\tilde{h}} u(x, \xi)| \leq C_1 e^{-\delta/\tilde{h}} \|u\|_{L^\infty}.$$

(ii) *There exist a complex neighborhood $W \Subset X + i\mathbb{R}^n$ of x_0 and constants $C, C_2 > 0$ such that $u(h)$ can be extended holomorphically to W and*

$$(2.5) \quad \sup_W |u(h)| \leq C_2 e^{C/h} \|u\|_{L^\infty}.$$

(iii) *There exist an open neighborhood $U \Subset X$ of x_0 and $C_3 > 0$ such that for all $x \in U$,*

$$(2.6) \quad |(hD)^\alpha u(x)| \leq C_3^{|\alpha|} (1 + h|\alpha|)^{|\alpha|} \|u\|_{L^\infty}.$$

Proof. First we notice that all of the statements are local, so we can extend u to functions on \mathbb{R}^n , say by setting $u = 0$ outside X , or better, to a family of functions in C_0^∞ since each condition implies that u is smooth (in fact, analytic) near x_0 . Also if (ii) is true, then by Hadamard’s three line theorem, there exist new constants $C, C_2 > 0$ such that

$$(2.7) \quad |u(z)| \leq C_2 h^{-N} e^{\frac{C|\operatorname{Im} z|}{h}} \|u\|_{L^\infty}.$$

To prove that (ii) and (iii) are equivalent, we need the following elementary inequalities:

$$(2.8) \quad \forall t, s > 0, \quad \left(1 + \frac{s}{t}\right)^t \leq e^s \leq \left(1 + \frac{s}{t}\right)^{t+s},$$

$$(2.9) \quad \forall \alpha \in \mathbb{N}^n, \quad (ne)^{-|\alpha|} |\alpha|^{|\alpha|} \leq \alpha! \leq |\alpha|^{|\alpha|}.$$

Proof of (ii) \Rightarrow (iii): We can find a real neighborhood $U \Subset X$ of x_0 and a constant $r_0 > 0$ such that for all $x \in U$, the polydisc $D(x, r_0) \Subset W$. Then by Cauchy’s inequality (see [13], Theorem 2.2.7) on $D(x, r)$ we have for a new constant $C > 0$,

$$(2.10) \quad |(hD)^\alpha u(x)| \leq Ch^{-N} e^{Cr/h} h^{|\alpha|} \alpha! r^{-|\alpha|}, \quad 0 < r \leq r_0, x \in U.$$

Case 1. If $r_0 \geq h|\alpha|/C$, then we take $r = h|\alpha|/C$ in (2.10) and get

$$|(hD)^\alpha u(x)| \leq Ch^{-N} e^{|\alpha|} \alpha! C^{|\alpha|} |\alpha|^{-|\alpha|}.$$

Now by (2.9), we have for a new constant $C > 0$,

$$|(hD)^\alpha u(x)| \leq C^{|\alpha|} h^{-N}.$$

This implies (2.6).

Case 2. If $r_0 < h|\alpha|/C$, then we take $r = r_0$ in (2.10) and get

$$|(hD)^\alpha u(x)| \leq Ch^{-N} e^{Cr_0/h} h^{|\alpha|} \alpha! r_0^{-|\alpha|}.$$

We use (2.8) for $s = Cr_0, t = h|\alpha|$ and (2.9) and get

$$\begin{aligned} |(hD)^\alpha u(x)| &\leq Ch^{-N} \left(1 + \frac{Cr_0}{h|\alpha|}\right)^{Cr_0/h+|\alpha|} h^{|\alpha|} |\alpha|^{|\alpha|} r_0^{-|\alpha|} \\ &= Ch^{-N} \left(1 + \frac{Cr_0}{h|\alpha|}\right)^{Cr_0/h} (Cr_0 + h|\alpha|)^{|\alpha|} r_0^{-|\alpha|} \end{aligned}$$

which also implies (2.6) by our assumption $Cr_0/h < |\alpha|$.

Proof of (iii) \Rightarrow (ii): For $\delta > 0$ small enough, $B(x_0, \delta) = \{|x - x_0| < \delta\} \subset U$. Then for $x \in B(x_0, \delta)$, by Taylor’s theorem,

$$u(x) = \sum_{0 \leq |\alpha| \leq k-1} \frac{\partial^\alpha u(x_0)}{\alpha!} (x - x_0)^\alpha + R_k$$

where

$$R_k = \sum_{|\alpha|=k} \frac{1}{\alpha!} (x - x_0)^\alpha \int_0^1 k(1-t)^{k-1} \partial^\alpha u(x_0 + t(x - x_0)) dt.$$

Therefore by (2.6),

$$|R_k| \leq \sum_{|\alpha|=k} \frac{1}{\alpha!} \delta^k C_3^k (1 + hk)^k h^{-k}.$$

We use (2.8) and (2.9) again to get

$$|R_k| \leq (k + 1)^n (C_3 ne \delta)^k \left(1 + \frac{1}{hk}\right)^k \leq e^{\frac{1}{h}} (k + 1)^n (C_3 ne \delta)^k.$$

Therefore as long as $\delta < (C_3 ne)^{-1}$, $R_k \rightarrow 0$ as $k \rightarrow \infty$, so u is analytic on $B(x_0, \delta)$. Now we can extend u holomorphically to $W = \{z \in \mathbb{C}^n : |z - x_0| < \delta\}$ by

$$(2.11) \quad u(z) = \sum_{\alpha} \frac{\partial^\alpha u(x_0)}{\alpha!} (z - x_0)^\alpha.$$

Since

$$\left| \frac{\partial^\alpha u(x_0)}{\alpha!} (z - x_0)^\alpha \right| \leq \frac{C_3^{|\alpha|} (1 + h|\alpha|)^{|\alpha|}}{h^{|\alpha|} \alpha!} \delta^{|\alpha|},$$

we apply (2.8) for $s = 1, t = h|\alpha|$ and (2.9) to get

$$\left| \frac{\partial^\alpha u(x_0)}{\alpha!} (z - x_0)^\alpha \right| \leq (C_3 n e \delta)^{|\alpha|} \left(1 + \frac{1}{h|\alpha|}\right)^{|\alpha|} \leq (C_3 n e \delta)^{|\alpha|} e^{\frac{1}{h}}.$$

Thus

$$|u(z)| \leq \sum_\alpha \left| \frac{\partial^\alpha u(x_0)}{\alpha!} (z - x_0)^\alpha \right| \leq e^{\frac{1}{h}} \sum_\alpha (C_3 n e \delta)^{|\alpha|},$$

which gives (2.5) since $\delta < (C_3 n e)^{-1}$.

Now we turn to the proof of (i) ⇔ (ii). We use the same type of deformation of the integral contour as in the proof that the projection of the analytic wavefront set is the analytic singular support (see [16]).

Proof of (ii) ⇒ (i): We have (2.7) for z in a neighborhood of x_0 , say $\{z = y + it : |y - x_0| < 2r, |t| < r\}$. For $|x - x_0| < r$, in the formula of FBI transform (2.1),

$$T_{\tilde{h}} u(x, \xi) = 2^{-n/2} (\pi \tilde{h})^{-3n/4} \int e^{i(x-y)\xi/\tilde{h} - (x-y)^2/2\tilde{h}} u(y) dy,$$

we deform the contour to

$$(2.12) \quad \Gamma_x : y \mapsto z = y + i\epsilon\chi(y) \frac{\xi}{|\xi|},$$

where $\chi \in C_0^\infty(\mathbb{R}^n), 0 \leq \chi \leq 1, \chi = 1$ on $|y - x| < \frac{r}{2}, \text{supp } \chi \subset \{|y - x| < r\}$ and $\epsilon \in (0, r)$. Then along Γ_x ,

$$\left| e^{i(x-z)\xi/\tilde{h} - (x-z)^2/2\tilde{h}} u(z) \right| \leq C h^{-N} e^{C\epsilon\chi(y)/h - \epsilon\chi(y)|\xi|/\tilde{h} + \epsilon^2\chi(y)^2/2\tilde{h} - |x-y|^2/2\tilde{h}}.$$

Since

$$C\epsilon\chi(y)/h - \epsilon\chi(y)|\xi|/\tilde{h} + \epsilon^2\chi(y)^2/2\tilde{h} - |x-y|^2/2\tilde{h} \leq \epsilon\chi(y) \left[C + \frac{\epsilon}{2} - |\xi| \right] / \tilde{h} - |x-y|^2/2\tilde{h},$$

we have that if $|\xi| > C_0 = C + \epsilon/2 + \delta/\epsilon$,

$$\left| e^{i(x-z)\xi/\tilde{h} - (x-z)^2/2\tilde{h}} u(z) \right| \leq \begin{cases} C h^{-N} e^{-\delta/\tilde{h}}, & \text{when } |y - x| < \frac{r}{2}, \\ C h^{-N} e^{-r^2/8\tilde{h}}, & \text{when } |y - x| \geq \frac{r}{2}, \end{cases}$$

which shows (2.4).

Proof of (i) ⇒ (ii): We use the following formula for the δ -function in the sense of the oscillatory integral:

$$\delta(x) = (2\pi h)^{-n} \int e^{\frac{i}{h} x \xi} d\xi.$$

Following Lebeau, we deform to the complex contour

$$(2.13) \quad \tilde{\Gamma}_x : \xi \mapsto \zeta = \xi + \frac{i}{2} |\xi| x.$$

Along $\tilde{\Gamma}_x$

$$d\zeta = a(x, \xi) d\xi, \quad a(x, \xi) = 1 + \frac{i}{2} \sum_1^n \frac{x_j \xi_j}{|\xi|}.$$

Therefore in the sense of the oscillatory integral,

$$(2.14) \quad \delta(x) = (2\pi h)^{-n} \int e^{ix\xi/h - |\xi|x^2/2h} a(x, \xi) d\xi.$$

Now we can write u in the form of

$$(2.15) \quad u(x) = (2\pi h)^{-n} \iint e^{i(x-y)\xi/h - |\xi|(x-y)^2/2h} a(x-y, \xi) u(y, h) dy d\xi.$$

Let

$$I(x, \xi, h) = \int e^{i(x-y)\xi/h - |\xi|(x-y)^2/2h} a(x-y, \xi) u(y, h) dy.$$

We claim that for some constants $C, c > 0$,

$$(2.16) \quad I(x + it, \xi, h) = O(e^{C/h - c|\xi|/h})$$

uniformly for $x + it$ in a complex neighborhood of x_0 and $0 < h \leq h_0$. In fact,

$$(2.17) \quad I(x + it, \xi, h) = e^{-t\xi/h + t^2|\xi|/2h} \int e^{i(x-y)(\xi - |\xi|t)/h - |\xi|(x-y)^2/2h} a(x + it - y, \xi) u(y) dy.$$

It is easy to see $I(x + it, \xi, h) = O(e^{C/h})$ when $|\xi| < C'$, so we have (2.16). Now we assume that $|\xi| > C'$ where we shall choose C' to be large later. Then since $|t| < \epsilon$, we have

$$(1 - \epsilon)|\xi| < |\xi - |\xi|t| < (1 + \epsilon)|\xi|.$$

Let $\tilde{h} = \mu h / |\xi|$, where μ is large and to be fixed later. Then we can rewrite (2.17) as

$$I(x + it, \xi, h) = e^{-t\xi/h + t^2|\xi|/2h} \int e^{i(x-y) \cdot [(\tilde{h}/h)(\xi - |\xi|t)]/\tilde{h} - \mu(x-y)^2/2\tilde{h}} a(x + it - y, \xi) u(y) dy.$$

Now if we choose $C' > \mu > C_0(1 - \epsilon)^{-1}$, then $\tilde{h} \leq h$ and

$$\left| \frac{\tilde{h}}{h} (\xi - |\xi|t) \right| = \mu \frac{|\xi - |\xi|t|}{|\xi|} \geq (1 - \delta)\mu > C_0.$$

By Lemma 2.1 (we notice that $y \mapsto a(x + it - y, \xi)$ has uniform tempered growth when $|t|$ is small) and (2.4), we have uniform exponential decay for the integral in (2.17) when $x + it$ is in a small complex neighborhood of x_0 , and $|\xi| > C'$,

$$I(x + it, \xi, h) \leq C e^{-t\xi/h + t^2|\xi|/2h} e^{-\delta'/\tilde{h}} = C e^{-t\xi/h + t^2|\xi|/2h} e^{-\delta'|\xi|/\mu h} \leq C e^{-c|\xi|/h},$$

if we assume $|t| < \epsilon$ is small enough. This finishes the proof of (2.16).

Now we can extend u holomorphically to a complex neighborhood of x_0 simply by

$$(2.18) \quad u(z) = (2\pi h)^{-n} \int I(z, \xi, h) d\xi$$

since $I(z, \xi, h)$ is holomorphic and the integral is uniformly convergent. Furthermore,

$$(2.19) \quad |u(z)| \leq C(2\pi h)^{-n} \int e^{C/h - c|\xi|/h} d\xi \leq C_2 e^{C/h},$$

which gives (2.5). □

Remark 2.3. Since $e^{\xi^2/2\hbar}T_{\hbar}u(x, \xi; h)$ is holomorphic, we can even replace condition (i) by the exponential decay of the local L^2 -norm of $T_{\hbar}u(h)$.

2.3. Agmon estimates for the FBI transform. We shall follow the approach in [15]. First we recall the following theorem of the microlocal exponential estimate from [15, Corollary 3.5.3, $f = 1$]:

Theorem 2.4. *Suppose $p \in S_{2n}(1)$ can be extended holomorphically to*

$$\Sigma(a) = \{(x, \xi) \in \mathbb{C}^{2n} : |\operatorname{Im} x| < a, |\operatorname{Im} \xi| < a\}$$

such that

$$\forall \alpha \in \mathbb{N}^{2n}, \partial^\alpha p = O(1), \text{ uniformly in } \Sigma(a).$$

Assume also that the real-valued function $\psi \in S_{2n}(1)$ satisfies

$$\sup_{\mathbb{R}^{2n}} |\nabla_x \psi| < a, \sup_{\mathbb{R}^{2n}} |\nabla_\xi \psi| < a.$$

Then

$$\|e^{\psi/h}T_{\hbar}P(x, hD)u\|^2 = \|p(x - 2\partial_z \psi, \xi + 2i\partial_z \psi)e^{\psi/h}T_{\hbar}u\|^2 + O(h)\|e^{\psi/h}T_{\hbar}u\|^2$$

uniformly for $u \in L^2(\mathbb{R}^n)$, $h > 0$ small enough. Here $\partial_z = \frac{1}{2}(\partial_x + i\partial_\xi)$ is the holomorphic derivative with respect to $z = x - i\xi$.

Remark 2.5. From the argument in [15], we can also see that this estimate only depends on the seminorms of p and ψ in $S_{2n}(1)$. In other words, if p and ψ vary in a way such that every $\sup_{\Sigma(a,b)} |\partial^\alpha p|$ and $\sup_{\mathbb{R}^{2n}} |\partial^\alpha \psi|$ is uniformly bounded, then the estimate is uniform in p and ψ . Furthermore, we only need that p can be extended holomorphically to the set $\{(y, \eta) \in \mathbb{C}^{2n} : \exists (x, \xi) \in \operatorname{supp} \psi, |y - x| < \sup |\nabla \psi|, |\eta - \xi| < \sup |\nabla \psi|\}$. Also here $P(x, hD)$ can be replaced by any quantization as in [15], [20].

Now we consider a semiclassical differential operator $P = P(x, hD_x)$ of order m with analytic coefficients, defined in a neighborhood X of x_0 . We assume the symbol

$$p(x, \xi; h) = \sum_{|\alpha| \leq m} a_\alpha(x; h)\xi^\alpha$$

can be extended holomorphically to a fixed complex neighborhood

$$\Sigma_\delta = \{(x, \xi) \in \mathbb{C}^n \times \mathbb{C}^n : |\operatorname{Re} x - x_0| < \delta, |\operatorname{Im} x| < \delta, |\operatorname{Im} \xi| < \delta\}$$

and also that P is classically elliptic in Σ_δ , in the sense that the principal symbol

$$p_0(x, \xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

satisfies

$$|p_0(x, \xi)| \geq \frac{1}{C_0} \langle \operatorname{Re} \xi \rangle^m, \text{ for } (x, \xi) \in \Sigma_\delta, |\operatorname{Re} \xi| > C.$$

Theorem 2.6. *Let P be as above and assume $\{u(h)\}_{0 < h \leq h_0}$ is a family of functions defined in X such that*

$$P(x, hD_x)u(h) = 0 \text{ in } X$$

and

$$\|u(h)\|_{L^2(X)} \leq Ch^{-N}.$$

Then there exists an open neighborhood $U \Subset X$ of x_0 , such that for all $x \in U$,

$$(2.20) \quad |(hD)^\alpha u(x)| \leq C^{|\alpha|} (1 + h|\alpha|)^{|\alpha|} \|u(h)\|_{L^\infty}.$$

Remark 2.7. Also by the standard semiclassical elliptic estimates (e.g. [20, Lemma 7.10]), we know $\|u\|_{L^\infty} \leq Ch^{-n/2} \|u\|_{L^2}$. So we also have $\|u(h)\|_{L^\infty} \leq Ch^{-M}$.

Proof. First for $0 < \tilde{h} \leq h$, we write

$$\tilde{P}(x, \tilde{h}D_x) = (\tilde{h}/h)^m \chi_1(x) P(x, hD_x)$$

so

$$\tilde{p}(x, \xi) = \sum_{|\alpha| \leq m} (\tilde{h}/h)^{m-|\alpha|} \chi_1(x) a_\alpha(x; h) \xi^\alpha$$

where $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function satisfying $\chi_1(x) = 1$ for $|x - x_0| < \frac{\delta}{4}$; 0 for $|x - x_0| > \frac{\delta}{2}$. Therefore $\tilde{p}(x, \xi)$ can still be extended holomorphically to $\Sigma_{\frac{\delta}{4}}$ and is classically elliptic in $\Sigma_{\frac{\delta}{4}}$. Now let $Q = \langle \tilde{h}D_x \rangle^{-m} \circ \tilde{P}(x, \tilde{h}D_x)$; then we can write $Q(x, \tilde{h}D_x) = \text{Op}_{\tilde{h}}^1(q)$ where

$$q(y, \xi) = \langle \xi \rangle^{-m} \sum_{|\alpha| \leq m} \sum_{\beta \leq \alpha} (\tilde{h}D_y)^\beta (\chi_1(y) a_\alpha(y; h)) (-\xi)^{\alpha-\beta}.$$

In fact,

$$\begin{aligned} Qv &= (2\pi\tilde{h})^{-n} \iint e^{\frac{i}{\tilde{h}}(x-y)\xi} \langle \xi \rangle^{-m} (\tilde{P}v)(y) dy d\xi \\ &= (2\pi\tilde{h})^{-n} \iint e^{\frac{i}{\tilde{h}}(x-y)\xi} q(y, \xi) v(y) dy d\xi. \end{aligned}$$

Therefore q can also be extended holomorphically to $\Sigma_{\frac{\delta}{4}}$ and

$$|\partial^\alpha q| = O_\alpha(1), \text{ in } \Sigma_{\frac{\delta}{4}}.$$

Furthermore q is elliptic in $\Sigma_{\frac{\delta}{3}}$ and thus for \tilde{h} small,

$$|q(x, \xi)| \geq \frac{1}{C}, \text{ for } (x, \xi) \in \Sigma_{\frac{\delta}{4}}, |\xi| > C_0.$$

Now let $v(x) = \chi_2(x)u(x)$, where $\chi_2(x) \in C_0^\infty(\mathbb{R}^n)$ is a cut-off function satisfying $0 \leq \chi_2 \leq 1$, $\chi_2(x) = 1$ for $|x - x_0| < \frac{\delta}{2}$; 0 for $|x - x_0| > \frac{3\delta}{4}$. Therefore $v \in L^2(\mathbb{R}^n)$ and

$$Qv = 0,$$

$$\|v\|_{L^2(\mathbb{R}^n)} \leq \|u\|_{L^2(X)} \leq Ch^{-N}.$$

Now we choose $\psi = \psi(x, \xi) \in S_{2n}(1)$ such that $\text{supp } \psi \subset U_1 = \{|x - x_0| < \frac{\delta}{8}, |\xi| > 2C_0\}$ and $\psi = c > 0$ on $U_2 = \{|x - x_0| < \frac{\delta}{16}, |\xi| > 3C_0\}$ and $\sup_{\mathbb{R}^{2n}} |\nabla_{(x,\xi)} \psi| < \frac{\delta}{16}$. Then if $(x, \xi) \in \text{supp } \psi$, we have $(y, \eta) \in \Sigma_{\frac{\delta}{4}}$ for any $(y, \eta) \in \mathbb{C}^{2n}$ satisfying $|y - x| < \frac{\delta}{16}, |\eta - \xi| < \frac{\delta}{16}$. This allows us to apply the microlocal exponential estimate for ψ and q :

$$0 = \|e^{\psi/\tilde{h}} T_{\tilde{h}} Qv\|^2 = \|q(x - 2\partial_z \psi, \xi + 2i\partial_z \psi) e^{\psi/\tilde{h}} T_{\tilde{h}} v\|^2 + O(\tilde{h}) \|e^{\psi/\tilde{h}} T_{\tilde{h}} v\|^2.$$

Therefore

$$\|q(x - 2\partial_z \psi, \xi + 2i\partial_z \psi) e^{\psi/\tilde{h}} T_{\tilde{h}} v\|_{L^2(U_1)}^2 = O(\tilde{h}) \|e^{\psi/\tilde{h}} T_{\tilde{h}} v\|^2.$$

For $(x, \xi) \in U_1$, $(x - 2\partial_z\psi, \xi + 2i\partial_z\psi) \in \Sigma_{\frac{\delta}{4}}$ and $|\operatorname{Re}(\xi + 2i\partial_z\psi)| \geq 2C_0 - \frac{\delta}{16} > C_0$, so

$$|q(x - 2\partial_z\psi, \xi + 2i\partial_z\psi)| \geq \frac{1}{C}.$$

Therefore

$$\|e^{\psi/\tilde{h}}T_{\tilde{h}}v\|_{L^2(U_1)}^2 = O(\tilde{h})\|e^{\psi/\tilde{h}}T_{\tilde{h}}v\|^2.$$

When \tilde{h} is small, we have

$$\|e^{\psi/\tilde{h}}T_{\tilde{h}}v\|_{L^2(U_1)}^2 = O(\tilde{h})\|e^{\psi/\tilde{h}}T_{\tilde{h}}v\|_{L^2(\mathbb{R}^{2n}\setminus U_1)}^2.$$

Since $\psi = 0$ outside U_1 , we have

$$\|e^{\psi/\tilde{h}}T_{\tilde{h}}v\|_{L^2(U_1)}^2 \leq C\tilde{h}\|T_{\tilde{h}}v\|_{L^2(\mathbb{R}^{2n}\setminus U_1)}^2 \leq C\tilde{h}\|T_{\tilde{h}}v\|^2 = C\tilde{h}\|v\|^2 \leq C\tilde{h}\|u\|^2.$$

Since $\psi = c > 0$ on $U_2 \subset U_1$,

$$\|T_{\tilde{h}}v\|_{L^2(U_2)} \leq Ce^{-c/\tilde{h}}\|u\|.$$

Now by Proposition 2.2 and the remark after it, we can conclude the proof of (2.20). □

Remark 2.8. The same argument can also be applied to elliptic pseudodifferential operators on \mathbb{R}^n with symbol

$$(2.21) \quad p(x, \xi; h) \sim \sum_{j=0}^{\infty} a_m(x, \xi/|\xi|; h)|\xi|^{m-j}$$

which can be extended holomorphically to $\Sigma(a)$ for some $a > 0$ and is classically elliptic in $\Sigma(a)$. In this case, we do not need any cut-off function and the weight function ψ can be chosen to only depend on ξ . Then the solutions of $P(x, hD_x)u(h) = 0$ in \mathbb{R}^n also satisfy the semiclassical Cauchy estimates (2.20).

3. DOUBLING PROPERTY

In this section, we use a Carleman-type estimate to prove the so-called doubling property of solutions of semiclassical Schrödinger equations on a compact Riemannian manifold. We do not assume the analyticity of either the manifold or the potential. See [1] for a general setting where the potential is only assumed to be C^1 . From now on, for simplicity, we shall use $\|u\|_U$ to represent the L^2 -norm of the function u in the set U .

Theorem 3.1. *Suppose (M, g) is a compact Riemannian manifold and $V : M \rightarrow \mathbb{R}$ is a smooth function. Let $P(h) = -h^2\Delta_g + V(x)$ and $P(h)u(h) = E(h)u(h)$ where $E(h) \rightarrow E_0$ as $h \rightarrow 0$. Then*

(i) *(Tunneling) For every $r > 0$, there exists $c = c(r) > 0$ depending on r (and $(M, g), V$) such that*

$$(3.1) \quad \|u\|_{L^2(B(p,r))} \geq e^{-c(r)/h}\|u\|_{L^2(M)}$$

for every $p \in M$, $0 < h < h_0$.

(ii) *(Doubling property) There exists $c_0 > 0$ depending only on (M, g) and V and such that for every $c_1 > 0$,*

$$(3.2) \quad \|u\|_{L^2(B(p,r))} \geq e^{-c_0/h}\|u\|_{L^2(B(p,2r))}$$

uniformly for $p \in M, r > c_1h$ and $0 < h < h_1$.

Remark 3.2. We can remove the condition $r > c_1h$ in part (ii) by carefully constructing a weight involving logarithmic terms near the origin in Carleman estimates. For the details, see [1]. For our purpose, the weak version above will be sufficient. From now on, in this section, every constant will depend on (M, g) and V , but we shall not write it out explicitly.

3.1. Carleman estimates. We start by writing the equation in the local coordinates. Let r_1 be the injective radius of M ; then for any $p \in M$, we write $P(h)u(h) = 0$ on $B(p, r_1)$ in the normal geodesic coordinates centered at p still as

$$(3.3) \quad [-h^2\Delta_g + V(x) - E]u = 0, x \in B(0, r_1).$$

Let $p = |\xi|_g^2 + V(x) - E$ be the symbol of $P = -h^2\Delta_g + V(x) - E$. We wish to conjugate P by a weight $e^{\varphi/h}$ to get an operator $P_\varphi = e^{\varphi/h}Pe^{-\varphi/h}$ whose symbol $p_\varphi = |\xi + i\partial\varphi(x)|_{g(x)}^2 + V(x) - E$ satisfies Hörmander’s hypoelliptic condition:

$$(3.4) \quad \text{if } p_\varphi = 0, \text{ then } \frac{i}{2}\{p_\varphi, \bar{p}_\varphi\} = \{\text{Re } p_\varphi, \text{Im } p_\varphi\} > 0,$$

on $B(0, R) \setminus B(0, r)$. Here $\{\cdot, \cdot\}$ denotes the Poisson bracket.

Since

$$\text{Re } p_\varphi = |\xi|_g^2 - |\partial\varphi|_g^2 + V - E, \text{Im } p_\varphi = 2\langle \xi, \partial\varphi \rangle_g,$$

we have

$$\begin{aligned} \{\text{Re } p_\varphi, \text{Im } p_\varphi\} &= \langle \partial_\xi(\text{Re } p_\varphi), \partial_x(\text{Im } p_\varphi) \rangle - \langle \partial_x(\text{Re } p_\varphi), \partial_\xi(\text{Im } p_\varphi) \rangle \\ &= 4\langle g\xi, \partial^2\varphi g\xi \rangle + 4\langle g\partial\varphi, \partial^2\varphi g\partial\varphi \rangle - 2\langle g\partial\varphi, \partial V \rangle \\ &\quad + 4\langle g\xi, \langle \xi, \partial\varphi \rangle_{\partial g} \rangle - 2\langle g\partial\varphi, |\xi|_{\partial g}^2 - |\partial\varphi|_{\partial g}^2 \rangle. \end{aligned}$$

We set $\varphi = \tau e^{\mu\psi}$, where $\tau, \mu \geq 1$ is a large constant to be chosen later. Then

$$\partial\varphi = \tau\mu e^{\mu\psi}\partial\psi, \partial^2\varphi = \tau e^{\mu\psi}(\mu^2\partial\psi \otimes \partial\psi + \mu\partial^2\psi).$$

Therefore

$$\begin{aligned} \frac{i}{2}\{p_\varphi, \bar{p}_\varphi\} &= 4\tau\mu^2 e^{\mu\psi}\langle g\xi, \partial\psi \rangle^2 + 4\tau\mu e^{\mu\psi}\langle g\xi, \partial^2\psi g\xi \rangle \\ &\quad + 4\tau^3\mu^4 e^{3\mu\psi}\langle \partial\psi, g\partial\psi \rangle^2 + 4\tau^3\mu^3 e^{3\mu\psi}\langle g\partial\psi, \partial^2\psi g\partial\psi \rangle \\ &\quad - 2\tau\mu e^{\mu\psi}\langle g\partial\psi, \partial V \rangle + 4\tau\mu e^{\mu\psi}\langle g\xi, \langle \xi, \partial\psi \rangle_{\partial g} \rangle \\ &\quad - 2\tau\mu e^{\mu\psi}\langle g\partial\psi, |\xi|_{\partial g}^2 \rangle + 2\tau^3\mu^3 e^{3\mu\psi}\langle g\partial\psi, |\partial\psi|_{\partial g}^2 \rangle. \end{aligned}$$

When $\text{Re } p_\varphi = \text{Im } p_\varphi = 0$, we have

$$\begin{aligned} |\xi|_g^2 &= |\partial\varphi|_g^2 + V(x) - E = \tau^2\mu^2 e^{2\mu\psi}|\partial\psi|_g^2 + V - E, \\ \langle \xi, \partial\varphi \rangle_g &= 2\tau\mu e^{\mu\psi}\langle g\xi, \partial\psi \rangle = 0. \end{aligned}$$

Thus

$$\xi = \tau\mu e^{\mu\psi}|\partial\psi|_g\eta, \text{ where } C_1^{-1} \leq |\eta| \leq C_1.$$

We shall choose ψ to be a radial and radially decreasing function which equals $A - |x|$ on $B(0, R) \setminus B(0, r)$ so that

$$\psi \geq 1, C_2^{-1} \leq |\partial\psi|_g \leq C_2 \text{ and } |\partial^2\psi| \leq C_2 \text{ on } B(0, R) \setminus B(0, r).$$

Hence

$$\frac{i}{2}\{p_\varphi, \bar{p}_\varphi\} \geq 4\tau^3\mu^4 e^{3\mu\psi}[|\partial\psi|_g^4 - C_3\mu^{-1}] \geq C_4\tau^{-1}e^{-\mu\psi}\langle \psi \rangle^4$$

where C_3, C_4 is a constant depending only on ψ when μ and τ are large depending on ψ . Now we can prove the basic Carleman estimate:

Lemma 3.3. *For any $v \in C_0^\infty(B(0, R) \setminus B(0, r))$,*

$$(3.5) \quad C_5 \tau^{\frac{1}{2}} \|P_\varphi v\| \geq h^{\frac{1}{2}} \|v\|_{H_h^2},$$

where C_5 is a constant only depending on μ, ψ .

Proof. The proof is based on the standard commutator argument. First,

$$\begin{aligned} \|P_\varphi v\|^2 &= \langle P_\varphi v, P_\varphi v \rangle = \langle P_\varphi^* P_\varphi v, v \rangle \\ &= \langle P_\varphi^* P_\varphi v, v \rangle + \langle [P_\varphi^*, P_\varphi]v, v \rangle = \|P_\varphi^* v\|^2 + \langle [P_\varphi^*, P_\varphi]v, v \rangle. \end{aligned}$$

For any $M > 1$ and h small enough, this implies

$$\begin{aligned} \|P_\varphi v\|^2 &\geq Mh \|P_\varphi^* v\|^2 + \langle [P_\varphi^*, P_\varphi]v, v \rangle \\ &= h \langle \text{Op}_h(M|p_\varphi|^2 + i\{p_\varphi, \bar{p}_\varphi\})u, u \rangle - O(h^2) \|u\|_{H_h^2}. \end{aligned}$$

From the construction above, we can find M large enough so that

$$M|p_\varphi|^2 + i\{p_\varphi, \bar{p}_\varphi\} \geq C_6 \tau^{-1} \langle \xi \rangle^4$$

where C_6 is a constant only depending on μ, ψ . Now we can use the sharp Gårding's inequality to conclude the lemma. \square

Now we prove a Carleman estimate on different shells.

Proposition 3.4 (Carleman estimate on shells). *Let φ be as above. We have the following estimate for solutions u to the equation (3.3):*

$$(3.6) \quad \begin{aligned} \|e^{\varphi/h} u\|_{B(0, \frac{R}{2}) \setminus B(0, 2r)} &\leq C_7 \tau^{\frac{1}{2}} [R^{-2} e^{\varphi(\frac{R}{2})/h} \|u\|_{B(0, R) \setminus B(0, \frac{R}{2})} \\ &\quad + r^{-2} e^{\varphi(2r)/h} \|u\|_{B(0, 2r) \setminus B(0, r)}], \end{aligned}$$

where $0 < 8r < R < r_1$, $C_7 > 0$ only depend on μ, ψ .

Proof. We shall take $v = e^{\varphi/h} \chi u$ in (3.5) where χ is a function supported in $C_0^\infty(B(0, R) \setminus B(0, r))$, such that $\chi \equiv 1$ on $B(0, \frac{R}{2}) \setminus B(0, 2r)$, and

$$|\nabla \chi| \leq Cr^{-1}, |\nabla^2 \chi| \leq Cr^{-2}.$$

Then

$$\|v\| = \|e^{\varphi/h} \chi u\| \geq \|e^{\varphi/h} u\|_{B(0, \frac{R}{2}) \setminus B(0, 2r)}$$

and

$$\|P_\varphi v\| = \|e^{\varphi/h} P(\chi u)\| = \|e^{\varphi/h} [P, \chi]u\|$$

where $[P, \chi] = \langle a, hD \rangle + b$ is a first order differential operator supported on $\text{supp } \nabla \chi \subset (B(0, R) \setminus B(0, \frac{R}{2})) \cup (B(0, 2r) \setminus B(0, r))$ with coefficients $a = O(hr^{-1}), b = O(h^2 r^{-2})$ on $B(0, 2r) \setminus B(0, r)$ and with r replaced by R on $B(0, R) \setminus B(0, \frac{R}{2})$. Therefore by standard elliptic estimates (e.g. [20, Chapter 7]), we have

$$\begin{aligned} \|P_\varphi v\| &\leq C_8 [e^{\varphi(\frac{R}{2})/h} \|u\|_{H_h^1(B(0, R) \setminus B(0, \frac{R}{2}))} + e^{\varphi(2r)/h} \|u\|_{H_h^1(B(0, 2r) \setminus B(0, r))}] \\ &\leq C_9 [R^{-2} h^2 e^{\varphi(\frac{R}{2})/h} \|u\|_{B(0, R) \setminus B(0, \frac{R}{2})} + r^{-2} h^2 e^{\varphi(r)/h} \|u\|_{B(0, 2r) \setminus B(0, r)}], \end{aligned}$$

which finishes the proof. \square

3.2. Proof of Theorem 3.1. We shall use Proposition 3.4 to prove Theorem 3.1. The tunneling estimate follows from the standard overlapping chains of balls argument introduced by Donnelly and Fefferman [5] while the doubling property is a corollary of the tunneling and the Carleman estimates on shells.

Proof of (3.1). Without loss of generality, we can assume $\|u\|_M = 1$ and we only need to prove (3.1) for $r < \frac{1}{100}r_1$. We shall fix τ large in the expression of φ and replace r by $\frac{r}{4}$ and take $R = 32r$ in Proposition 3.4. Then we get

$$(3.7) \quad e^{\varphi(2r)/h} \|u\|_{B(x,2r) \setminus B(x,r)} \leq C_{10} [e^{\varphi(4r)/h} \|u\|_{B(x,8r) \setminus B(x,4r)} + e^{\varphi(\frac{r}{4})/h} \|u\|_{B(0,\frac{r}{2}) \setminus B(0,\frac{r}{4})}]$$

for any $x \in M$. It is obvious that there exists a point $q \in M$ such that

$$\|u\|_{B(q,r)} \geq cr^{-n} \geq e^{-A_0/h}.$$

For any $p \in M$, we can find a sequence $x_0 = q, x_1, \dots, x_m = p$ such that $d(x_j, x_{j+1}) < r$ and $m \leq C_{11}r^{-1}$. We shall prove by induction that there exist $A_j > 0$ and $h_j > 0$ only depending on r such that for $0 < h < h_j$

$$(3.8) \quad \|u\|_{B(x_j,r)} \geq e^{-A_j/h}.$$

We already know this is true for $j = 0$. Suppose this is true for j . Then for $j + 1$, since $B(x_j, r) \subset B(x_{j+1}, 2r)$, either

$$\|u\|_{B(x_{j+1},r)} \geq e^{-(A_j+1)/h}$$

or

$$\|u\|_{B(x_{j+1},2r) \setminus B(x_{j+1},r)} \geq e^{-(A_j+1)/h}.$$

For the first case, there is nothing to prove, for the latter, we let $x = x_{j+1}$ in (3.7) to get

$$(3.9) \quad \|u\|_{B(0,\frac{r}{2}) \setminus B(0,\frac{r}{4})} \geq e^{-\varphi(\frac{r}{4})/h} [C_{12}^{-1} h^{-\frac{3}{2}} e^{\varphi(2r)/h} e^{-(A_j+1)/h} - e^{\varphi(4r)/h}].$$

We only need to choose μ and A large enough in the expression of φ so that $\varphi(2r) - A_j - 1 > \varphi(4r)$ to get (3.8). Now since m is bounded, we get the desired tunneling estimates (3.1). \square

Proof of (3.2). Again, we only need to prove (3.2) for $r \in (c_1h, \frac{r_1}{100})$. Now we shall fix $R = \frac{r}{2}$ in Proposition 3.4 and replace r by $\frac{r}{2}$. Then for any $p \in M$,

$$(3.10) \quad \begin{aligned} & e^{\varphi(\frac{R}{4})/h} \|u\|_{B(p,\frac{R}{4}) \setminus B(p,\frac{R}{8})} + e^{\varphi(2r)/h} \|u\|_{B(p,2r) \setminus B(p,r)} \\ & \leq C_{13} \tau^{\frac{1}{2}} [e^{\varphi(\frac{R}{2})/h} \|u\|_{B(p,R) \setminus B(p,\frac{R}{2})} + h^{-2} e^{\varphi(\frac{r}{2})/h} \|u\|_{B(p,r) \setminus B(p,\frac{r}{2})}]. \end{aligned}$$

By the tunneling estimates (3.1) and the fact that there exists a ball of radius $\frac{R}{8}$ inside $B(p, \frac{R}{4}) \setminus B(p, \frac{R}{8})$,

$$\|u\|_{B(p,\frac{R}{4}) \setminus B(p,\frac{R}{8})} \geq e^{-C_{14}/h} \|u\| \geq e^{-C_{14}/h} \|u\|_{B(p,R) \setminus B(p,\frac{R}{2})}.$$

By choosing μ and A large enough only depending on R and $\tau = C_{15}r^{-1} \leq C_{16}h^{-1}$, we can make $\varphi(\frac{R}{4}) - C_{14} \geq \varphi(\frac{R}{2})$. Therefore for $h < h_0$,

$$e^{\varphi(\frac{R}{4})/h} \|u\|_{B(p,\frac{R}{4}) \setminus B(p,\frac{R}{8})} \geq C_{13} \tau^{\frac{1}{2}} e^{\varphi(\frac{R}{2})/h} \|u\|_{B(p,R) \setminus B(p,\frac{R}{2})}.$$

Therefore from (3.10) we see that

$$C_{13} \tau^{\frac{1}{2}} h^{-2} e^{\varphi(\frac{r}{2})/h} \|u\|_{B(p,r) \setminus B(p,\frac{r}{2})} \geq e^{\varphi(2r)/h} \|u\|_{B(p,2r) \setminus B(p,r)}.$$

Since

$$\varphi\left(\frac{r}{2}\right) - \varphi(2r) = \tau e^{\mu(A-2r)}(e^{3\mu r/2} - 1) \geq \frac{3}{2}\mu\tau r \geq C_{17}$$

and $\tau \leq C_{16}h^{-1}$, we can get that for $h < h_1$,

$$\|u\|_{B(p,r)\setminus B(p,\frac{r}{2})} \geq e^{-C_{18}/h} \|u\|_{B(p,2r)\setminus B(p,r)}.$$

Now (3.2) is a simple consequence of this estimate. □

4. NODAL SETS FOR SOLUTIONS TO SEMICLASSICAL SCHRÖDINGER EQUATIONS

In this section, we assume that (M, g) is a real analytic compact Riemannian manifold, and V is a real analytic function on M . Let $u = u(h)$ be the solution to the semiclassical Schrödinger equation $(-h^2\Delta_g + V(x) - E(h))u = 0$ and we study the vanishing properties of u .

4.1. Order of vanishing. First, we prove a result on the order of vanishing of u .

Proposition 4.1. *There exists a constant $C > 0$ such that if u vanishes at x_0 to the order k , then $k \leq Ch^{-1}$.*

Proof. Without loss of generality, we can assume $\|u(h)\| = 1$. By Taylor’s formula, for $|x - x_0| < \epsilon$,

$$(4.1) \quad |u(h)(x)| \leq \frac{\epsilon^k}{k!} \sup_{|\alpha|=k} \sup_{|y-x_0|<2\epsilon} |D^\alpha u(y)|.$$

Now we can apply semiclassical Cauchy estimates to get

$$(4.2) \quad |u(h)(x)| \leq (C\epsilon)^k \left(1 + \frac{1}{hk}\right)^k$$

where C is a constant only depending on (M, g) and V as long as ϵ is small enough. If $k < \frac{1}{h}$, then there is nothing to prove. Otherwise, we can take ϵ small (not depending on h) to get

$$|u(h)(x)| \leq e^{-k}, |x - x_0| < \epsilon.$$

On the other hand, by the Carleman estimate

$$(4.3) \quad \|u\|_{B(x_0,\epsilon)} \geq e^{-\frac{C}{h}}.$$

Again we have $k \leq Ch^{-1}$. □

Remark 4.2. In [1], it is proved that this is true even when (M, g) and V are only smooth and the constant C only depends on (M, g) and the C^1 -norm of V . Moreover, this result is optimal; see [20, Section 7.3].

4.2. Nodal set: Upper bounds. In this section, we prove the upper bound on the size of the nodal set in Theorem 1.2.

Proposition 4.3. *The $(n - 1)$ -dimensional Hausdorff measure of the nodal set $N = \{x \in M, u(x) = 0\}$ satisfies the estimate*

$$(4.4) \quad \mathcal{H}^{n-1}(N) \leq Ch^{-1},$$

for some constant $C > 0$.

Proof. From above, we know, for each $x \in M$, there exists $r = r(x)$ (independent of h) such that in local geodesic coordinates, u can be analytically continued from $B(x, r)$ to $B_{\mathbb{C}^n}(x, \frac{r}{2})$ with $\sup_{B_{\mathbb{C}^n}(x, \frac{r}{2})} |u| \leq e^{C/h} \sup_{B(x, r)} |u|$. We also recall the following technical lemma from [5].

Lemma 4.4. *There exists some constant $C > 0$ such that if $H(z)$ is a holomorphic function in $|z| \leq 2, z \in \mathbb{C}^n$, and for some $\alpha > 1$*

$$\sup_{B(x, \frac{1}{5})} |H| \geq e^{-\alpha} \max_{B_{\mathbb{C}^n}(0, 2)} |H(z)|, x \in \mathbb{R}^n, |x| < \frac{1}{10},$$

then $H^{n-1}(\{x \in \mathbb{R}^n, |x| < \frac{1}{20}, H(x) = 0\}) \leq C\alpha$.

Now from this and the doubling property (3.2), it is easy to see

$$H^{n-1}(\{y \in B(x, \frac{r}{40}) : u(y) = 0\}) \leq Ch^{-1}.$$

By compactness, we can cover M by finitely many $B(x, \frac{r}{40})$ and conclude the proof. □

4.3. Nodal set: Lower bounds. Now we prove the lower bound on the size of the nodal set in Theorem 1.2.

Proposition 4.5. *The $(n - 1)$ -dimensional Hausdorff measure of the nodal set $\mathcal{H}^{n-1}(N) \geq ch^{-1}$. More precisely, for some constant $c > 0$, we have*

$$(4.5) \quad \mathcal{H}^{n-1}(N_a) \geq ch^{-1},$$

where $N_a = \{x \in M, u(x) = 0, V(x) < E\}$ is the nodal set in the classically allowed region.

First we prove a lemma which shows that there are enough nodal points in the classically allowed region. We fix a small constant $\delta > 0$ such that $M_\delta := \{x \in M : V(x) < E - \delta\}$ is a nonempty open set.

Lemma 4.6. *There exists $C > 0$ such that $u(h)$ has a zero in every ball $B(x_0, Ch)$ for any $p \in M_\delta$ and small h such that $B(p, Ch)$ is contained in the classically allowed region.*

Proof. We shall work in the normal geodesic coordinate centered at p , so that $p = 0$. First, by Cheng’s eigenvalue comparison theorem, the first Dirichlet eigenvalue of $-h^2\Delta_g$ on $B(0, Ch)$ is at most $\delta/2$ if C is chosen to be large enough. Therefore the first Dirichlet eigenvalue of $-h^2\Delta_g + V(x)$ on $B(0, Ch)$ is at most $V(0) + \delta/2 + c’h$. Let u_0 be the corresponding eigenfunction; we can assume that $u_0 > 0$ on $B(0, Ch)$. Now suppose $u > 0$ on $B(0, c_1h)$. We set $w = u_0/u$; then $w = 0$ on $\partial B(0, c_1h)$ and $w > 0$ in $B(0, c_1h)$. Therefore w achieves the maximum at some $x_1 \in B(0, c_1h)$. We have at the point x_1 ,

$$0 = \partial_i w = \frac{u\partial_i u_0 - u_0\partial_i u}{u^2}$$

and

$$\begin{aligned} 0 &\leq -h^2 \Delta_g w = \frac{(-h^2 \Delta_g u_0)u - (-h^2 \Delta_g u)u_0}{u^2} \\ &= \frac{(-h^2 \Delta_g + V)u_0 u - (-h^2 \Delta_g + V)u u_0}{u^2} \\ &\leq \frac{(V(0) + \delta/2 + c'h - E(h))u u_0}{u^2} < 0 \end{aligned}$$

if h is small enough, since $V(0) + \delta/2 < E$. This contradiction shows that u must have a zero in the ball $B(p, Ch)$. \square

The second lemma is a generalization of the mean-value formula for the Euclidean Laplacian. The ideas of our proof are from [3].

Lemma 4.7. *There exist $c > 0$ and $c_0 \in (0, 1)$, such that for every $h \in (0, h_0)$, $p \in M$ such that $u(p) = 0$,*

$$\left| \int_{B(p, ch)} u \right| \leq c_0 \int_{B(p, ch)} |u|.$$

As a corollary, we have for some $c_1 > 0$,

$$\min \left\{ \int_{B(p, ch)} u^+, \int_{B(p, ch)} u^- \right\} \geq c_1 \int_{B(p, ch)} |u|.$$

Proof. First, we define for any function v ,

$$I_v(s) = s^{1-n} \int_{\partial B(p, s)} v.$$

Then

$$I'_v(s) = s^{1-n} \int_{B(p, s)} \Delta v + s^{1-n} \int_{\partial B(p, s)} v \left(\Delta d + \frac{1-n}{d} \right),$$

where $d(\cdot) = \text{dist}(p, \cdot)$ denote the distance to the center of the ball. On M , when d is small depending on $\sup |K_M|$ and the injective radius of M , by the Hessian comparison theorem we have

$$\left| \Delta d + \frac{1-n}{d} \right| \leq k(d)$$

where $k : [0, \infty) \rightarrow \mathbb{R}$ is a continuous monotone nondecreasing function such that $k(0) = 0$.

Let $l(s) = \max_{x \in B(p, s)} |V(x) - V(p)|$; then $l : [0, \infty)$ is also a continuous monotone nondecreasing function such that $l(0) = 0$. Moreover, $l(s) \leq (\max_M |\nabla V|)s$.

Since

$$\int_{B(p, s)} u = \int_0^s t^{n-1} I_u(t) dt,$$

we have

$$\left| \int_{B(p, s)} u \right| \leq \frac{s^n}{n} f(s)$$

where $f(s) = \max_{t \leq s} |I_u(t)|$. f is also a continuous monotone nondecreasing function such that $f(0) = 0$. Moreover f is Lipschitz. Then for almost every s , we have the following estimate:

$$\begin{aligned} f'(s) &\leq |I'_u(s)| \leq h^{-2} s^{1-n} \left| \int_{B(p,s)} (V(x) - E)u \right| + s^{1-n} k(s) \int_{\partial B(p,s)} |u| \\ &\leq h^{-2} s^{1-n} |V(p) - E| \left| \int_{B(p,s)} u \right| + h^{-2} s^{1-n} l(s) \int_{B(p,s)} |u| + k(s) I_{|u|}(s) \\ &\leq Ch^{-2} s f(s) + h^{-2} s^{1-n} l(s) \int_0^s t^{n-1} I_{|u|}(t) dt + k(s) I_{|u|}(s) \\ &\leq Ch^{-2} s f(s) + h^{-2} l(s) \int_0^s I_{|u|}(t) dt + k(s) I_{|u|}(s). \end{aligned}$$

Therefore for $s \leq ch$,

$$f'(s) \leq Ch^{-1} f(s) + Ch^{-1} \int_0^s I_{|u|}(t) dt + k(s) I_{|u|}(s).$$

By Gronwall's inequality, we have

$$f(s) \leq e^{Ch^{-1}s} \int_0^s \left[Ch^{-1} \int_0^t I_{|u|}(t') dt' + k(t) I_{|u|}(t) \right] dt \leq C \int_0^s I_{|u|}(t) dt.$$

Hence,

$$(4.6) \quad \left| \int_{B(p,ch)} u \right| \leq \frac{(ch)^n}{n} f(ch) \leq Ch^n \int_0^{ch} I_{|u|}(t) dt.$$

By rescaling $B(p, ch)$ to the ball of unit radius, we get a family of functions $\tilde{u}(x) = u(p + chx)$ on $B(0, 1)$ solving a family of uniform elliptic equations

$$\sum_{i,j=1}^n a_{ij} \partial_i \partial_j \tilde{u} + \sum_{i=1}^n b_i \partial_i \tilde{u} + c \tilde{u} = 0.$$

Here

$$C^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \leq C |\xi|^2, |b_i(x)| \leq C, |c(x)| \leq C.$$

By a standard elliptic estimate (e.g. [8, Theorem 9.20]), we have

$$\sup_{B(0, \frac{1}{2})} |\tilde{u}| \leq C \int_{B(0,1)} |\tilde{u}|$$

for some $C > 0$ uniformly in \tilde{u} . Back to u , we have

$$\sup_{B(p, \frac{1}{2}ch)} |u| \leq Ch^{-n} \int_{B(p,ch)} |u|.$$

Therefore we have

$$\begin{aligned} \int_0^{ch} I_{|u|}(t)dt &= \int_0^{ch/2} \left(t^{1-n} \int_{\partial B(p,t)} |u| \right) dt + \int_{ch/2}^{ch} \left(t^{1-n} \int_{\partial B(p,t)} |u| \right) dt \\ &\leq \frac{ch}{2} \left(\sup_{B(p, \frac{1}{2}ch)} |u| \right) \left(\sup_{t \leq ch/2} t^{1-n} \text{Vol}(B(p, t)) \right) \\ &\quad + \left(\frac{ch}{2} \right)^{1-n} \int_{ch/2}^{ch} \left(\int_{\partial B(p,t)} |u| \right) dt \\ &\leq Ch^{1-n} \int_{B(p, ch)} |u|. \end{aligned}$$

The last inequality follows from the volume comparison theorem:

$$\sup_{t \leq ch/2} t^{1-n} \text{Vol}(B(p, t))$$

and is bounded by a constant only depending on the curvature of M . □

Now following the idea of [5], we prove the theorem.

Proof. Assume $U \Subset \{V < E\}$ is a coordinate patch. Then we can cover U by cubes Q_ν of side a_1h such that there exists a nodal point x_ν with $B_\nu = B(x_\nu, ch) \Subset \frac{1}{2}Q_\nu$ (the cube with the same center and sides of half length of Q_ν). Then by [5, Proposition 5.11] and the same argument as [5, Lemma 7.3,7.4], we can conclude that for at least half of the Q_ν ,

$$\text{Av}_{B_\nu} u^2 \geq c \text{Av}_{Q_\nu} u^2,$$

where $\text{Av}_U f$ denotes the average of f on U :

$$\text{Av}_U f = |U|^{-1} \int_U f.$$

Now by the standard elliptic theory, we have

$$\|u\|_{L^\infty(B_\nu)} \leq C(\text{Av}_{Q_\nu} u^2)^{\frac{1}{2}} \leq C(\text{Av}_{B_\nu} u^2)^{\frac{1}{2}}.$$

Therefore

$$(\text{Av}_{B_\nu} u^2)^{\frac{1}{2}} \leq C|B_\nu|^{-1} \int_{B_\nu} |u|.$$

Let $B_\nu^+ = \{x \in B_\nu : u > 0\}$, $B_\nu^- = \{x \in B_\nu, u < 0\}$; then

$$\int_{B_\nu^\pm} u \leq \|u\|_{B_\nu^\pm} |B_\nu^\pm|^{\frac{1}{2}} \leq C \frac{|B_\nu^\pm|^{\frac{1}{2}}}{|B_\nu|^{\frac{1}{2}}} (\text{Av}_{B_\nu} u^2)^{\frac{1}{2}}.$$

Combining this with the previous lemma, we have

$$\min\{|B_\mu^+|, |B_\mu^-|\} \geq c|B_\mu|.$$

The isoperimetric inequality shows that

$$H^{n-1}(B_\mu \cap N) \geq c \min\{|B_\mu^+|, |B_\mu^-|\}^{\frac{n-1}{n}} \geq ch^{n-1}.$$

Since we have at least ch^{-n} such cubes Q_ν , we conclude that

$$H^{n-1}(N \cap U) \geq ch^{-1}.$$

□

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