

EXTRAPOLATION AND WEIGHTED NORM INEQUALITIES IN THE VARIABLE LEBESGUE SPACES

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ABSTRACT. We extend the theory of Rubio de Francia extrapolation, including off-diagonal, limited range and A_∞ extrapolation, to the weighted variable Lebesgue spaces $L^{p(\cdot)}(w)$. As a consequence we are able to show that a number of different operators from harmonic analysis are bounded on these spaces. The proofs of our extrapolation results are developed in a way that outlines a general approach to proving extrapolation theorems on other Banach function spaces.

1. INTRODUCTION

The variable Lebesgue space $L^{p(\cdot)}$ is a generalization of the classical Lebesgue spaces, replacing the constant exponent p with an exponent function $p(\cdot)$. It is a Banach function space with the norm

$$(1.1) \quad \|f\|_{p(\cdot)} = \|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n \setminus \mathbb{R}_\infty^n} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} dx + \|f\|_{L^\infty(\mathbb{R}_\infty^n)} \leq 1 \right\},$$

where $\mathbb{R}_\infty^n = \{x : p(x) = \infty\}$. These spaces have been the subject of considerable interest since the early 1990s, both as function spaces with intrinsic interest and for their applications to problems arising in PDEs and the calculus of variations. For a thorough discussion of these spaces and their history, see [13, 24].

Recently there has been interest in extending the theory of Muckenhoupt A_p weights to this setting. Recall that given a non-negative, measurable function w , for $1 < p < \infty$, $w \in A_p$ if

$$[w]_{A_p} = \sup_B \left(\int_B w(x) dx \right) \left(\int_B w(x)^{1-p'} dx \right)^{p-1} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$, and $\int_B w dx = |B|^{-1} \int_B w dx$. We say $w \in A_1$ if

$$[w]_{A_1} = \sup_B \frac{\int_B w(x) dx}{\operatorname{ess\,inf}_{x \in B} w(x)} < \infty.$$

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These weights characterize the weighted norm inequalities for the Hardy-Littlewood maximal operator,

$$Mf(x) = \sup_B \int_B |f(y)| dy \cdot \chi_B(x).$$

More precisely, $w \in A_p$, $1 < p < \infty$, if and only if $M : L^p(w) \rightarrow L^p(w)$. The Muckenhoupt weights also govern the weighted norm inequalities for a large number of operators in harmonic analysis, including singular integrals, commutators and square functions. For details, see [19, 28, 32].

Weighted norm inequalities for the maximal operator in the variable Lebesgue spaces were proved in [12, 16, 24] (see also [25] for related results). To show the connection with the classical results we restate them by replacing the weight w by w^p in the definition of A_p . In this case we say that $w \in A_p$, $1 < p \leq \infty$, if

$$\sup_B |B|^{-1} \|w\chi_B\|_p \|w^{-1}\chi_B\|_{p'} < \infty,$$

and this is equivalent to the norm inequality

$$\|(Mf)w\|_p \leq C\|fw\|_p.$$

Remark 1.1. Note that in this formulation the inequality holds in the case $p = \infty$; this fact is not well known but was first proved by Muckenhoupt [37].

In this form the definition immediately generalizes to the variable Lebesgue spaces. (See below for precise definitions.) We say that a weight w is in the class $A_{p(\cdot)}$ if

$$\sup_B |B|^{-1} \|w\chi_B\|_{p(\cdot)} \|w^{-1}\chi_B\|_{p'(\cdot)} < \infty.$$

When $p(\cdot)$ is log-Hölder continuous ($p(\cdot) \in LH$) and is bounded and bounded above 1 ($1 < p_- \leq p_+ < \infty$), then $w \in A_{p(\cdot)}$ if and only if

$$\|(Mf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

In this paper we further develop the theory of weighted norm inequalities on the variable Lebesgue spaces. We show that the $A_{p(\cdot)}$ weights govern the weighted norm inequalities for a wide variety of operators in harmonic analysis, including singular and fractional integrals and the Riesz transforms associated to elliptic operators in divergence form. To do this we show that the theory of Rubio de Francia extrapolation holds in this setting. As an immediate consequence we prove, with very little additional work, norm inequalities in weighted $L^{p(\cdot)}$ spaces for any operator that satisfies estimates on $L^p(w)$ when w is a Muckenhoupt A_p weight. The classical theory of extrapolation is a powerful tool in harmonic analysis; for a detailed treatment, see [19]. Extrapolation in the scale of the variable Lebesgue spaces was originally developed in [15] to prove unweighted inequalities (see also [13, 19]). It has found wide application since (see, for instance, [22, 26, 30, 36]), and the results we present here should be equally useful. We note that our work has already been applied to the study of greedy approximation algorithms on variable Lebesgue spaces in [17].

The remainder of this paper is organized as follows. In Section 2 we state our extrapolation results, including the precise definitions needed. In Section 3 we show how to apply extrapolation to prove weighted norm inequalities for several different kinds of operators. Our examples are not exhaustive; rather, they were chosen to illustrate the applicability of extrapolation. In Section 4 we give a general overview

of our approach to proving extrapolation theorems. These ideas are not new—they were implicit in [19]. However, we think it is worthwhile to make them explicit here, for two reasons. First, they will motivate the technical details in our proofs, particularly Theorem 2.14. Second, they will be helpful to others attempting to prove extrapolation theorems in different settings. Finally, in Section 5 we prove our extrapolation theorems. By following the schema outlined in the previous section, we actually prove more general theorems which yield our main results as special cases.

2. MAIN THEOREMS

We begin with some definitions related to the variable Lebesgue spaces. Throughout we will follow the conventions established in [13]. Let $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ be the collection of all measurable functions $p(\cdot) : \mathbb{R}^n \rightarrow [1, \infty]$. Given a set $E \subset \mathbb{R}^n$, we define

$$p_-(E) = \operatorname{ess\,inf}_{x \in E} p(x), \quad p_+(E) = \operatorname{ess\,sup}_{x \in E} p(x).$$

If $E = \mathbb{R}^n$, then for brevity we write p_- and p_+ . Given $p(\cdot)$, the conjugate exponent $p'(\cdot)$ is defined pointwise,

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,$$

with the convention that $1/\infty = 0$.

For our results we need to impose some regularity on the exponent functions $p(\cdot)$. The most important condition, one widely used in the study of variable Lebesgue spaces, is log-Hölder continuity. Given $p(\cdot) \in \mathcal{P}$, we say $p(\cdot) \in LH_0$ if there exists a constant C_0 such that

$$(2.1) \quad |p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}, \quad x, y, \in \mathbb{R}^n, \quad |x - y| < 1/2,$$

and $p(\cdot) \in LH_\infty$ if there exists p_∞ and $C_\infty > 0$, such that

$$(2.2) \quad |p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}, \quad x \in \mathbb{R}^n.$$

If $p(\cdot)$ satisfies both of these conditions we write $p(\cdot) \in LH$. It is immediate that if $p(\cdot) \in LH$ and $p_- > 1$, then $p'(\cdot) \in LH$. A key consequence of log-Hölder continuity is the fact that if $1 < p_-$ and $p(\cdot) \in LH$, then the maximal operator is bounded on $L^{p(\cdot)}$.

Theorem 2.1. *Given $p(\cdot) \in \mathcal{P}$, suppose $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH$. Then $\|Mf\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$.*

However, this condition is not necessary, and there exist exponents $p(\cdot)$ which are not log-Hölder continuous but for which the maximal operator is still bounded on $L^{p(\cdot)}$. (See [13, 24] for further details.)

We define a weight w to be a measurable function such that $0 < w(x) < \infty$ almost everywhere. Given $p(\cdot) \in \mathcal{P}$ define the weighted variable Lebesgue space $L^{p(\cdot)}(w)$ to be the set of all measurable functions f such that $fw \in L^{p(\cdot)}$, and we write $\|f\|_{L^{p(\cdot)}(w)} = \|fw\|_{p(\cdot)}$. We say that an operator T is bounded on $L^{p(\cdot)}(w)$ if $\|(Tf)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}$ for all $f \in L^{p(\cdot)}(w)$. We are interested in weights in $A_{p(\cdot)}$; we restate their definition here.

Definition 2.2. Given an exponent $p(\cdot) \in \mathcal{P}$ and a weight w , we say that $w \in A_{p(\cdot)}$ if

$$[w]_{A_{p(\cdot)}} = \sup_B |B|^{-1} \|w\chi_B\|_{p(\cdot)} \|w^{-1}\chi_B\|_{p'(\cdot)} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Remark 2.3. Definition 2.2 has two immediate consequences. First, if $w \in A_{p(\cdot)}$, then $w \in L_{loc}^{p(\cdot)}$ and $w^{-1} \in L_{loc}^{p'(\cdot)}$. Second, if $w \in A_{p(\cdot)}$, then $w^{-1} \in A_{p'(\cdot)}$.

For our results we will need to assume that the maximal operator is bounded on weighted variable Lebesgue spaces. The following result is essentially from [16]; in the converse statement, we add an additional necessary condition that $p_- > 1$.

Theorem 2.4. *Given $p(\cdot) \in \mathcal{P}$, $1 < p_- \leq p_+ < \infty$, suppose $p(\cdot) \in LH$. Then for every $w \in A_{p(\cdot)}$,*

$$(2.3) \quad \|Mf\|_{L^{p(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

Conversely, given any $p(\cdot)$ and w , if (2.3) holds for $f \in L^{p(\cdot)}(w)$, then $p_- > 1$ and $w \in A_{p(\cdot)}$.

Proof. The proof of (2.3), as well as the part of the converse that $w \in A_{p(\cdot)}$, is in [16]. We need to show that given (2.3) we also have $p_- > 1$. This follows by adapting the proof from the unweighted case in [13, Theorem 3.19], and we sketch the details.

Arguing by contradiction, suppose $p_- = 1$ but (2.3) holds. Then for each $k \in \mathbb{N}$, let s_k be a constant such that $1 < s_k < n(n - \frac{1}{k+1})^{-1}$, and observe that by assuming $p_- = 1$, the set $E_k = \{x : p(x) < s_k\}$ has positive measure, that is, $|E_k| > 0$. Then since $0 < w(x) < \infty$ almost everywhere, there exists $N_k > 1$ such that

$$E_k^w = \{x \in \mathbb{R}^n : p(x) < s_k, w(x) \leq N_k\}$$

also has positive measure. Then we proceed as in [13]: by the Lebesgue differentiation theorem, for each k we can choose balls B_k (with small radius) such that

$$\frac{|B_k \cap E_k^w|}{|B_k|} > 1 - 2^{-n(k+1)}.$$

If we define f_k by

$$f_k(x) = \frac{\chi_{B_k \cap E_k^w}(x)}{|x - x_k|^{n - \frac{1}{k+1}}},$$

then $f_k \in L^{p(\cdot)}(w)$ since its modular is finite:

$$\rho(f_k w) = \int_{B_k \cap E_k^w} \frac{w(x)^{p(x)}}{|x - x_k|^{p(x)(n - \frac{1}{k+1})}} dx \leq \int_{B_k \cap E_k^w} \frac{N_k^{p_+}}{|x - x_k|^{s_k(n - \frac{1}{k+1})}} < \infty.$$

Given that $f_k w \in L^{p(\cdot)}(w)$, the rest of the argument is identical to the unweighted case: we have the pointwise estimate $Mf_k(x) \geq c(n)(k+1)f_k(x)$ almost everywhere, which shows that for every k ,

$$\|Mf_k w\|_{p(\cdot)} \geq c(n)(k+1)\|f_k w\|_{p(\cdot)}.$$

But this inequality contradicts (2.3). □

For the majority of our extrapolation results we prefer to state the regularity of $p(\cdot)$ and w in terms of the boundedness of the maximal operator. Therefore, given $p(\cdot) \in \mathcal{P}$ and a weight w , we will say $(p(\cdot), w)$ is an M -pair if the maximal operator is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$. By Theorem 2.4 we necessarily have $w \in A_{p(\cdot)}$ (equivalently, $w^{-1} \in A_{p'(\cdot)}$). Furthermore, we have that $1 < p_- \leq p_+ < \infty$ (the second inequality since $p'(\cdot)_- = (p_+)'$). Conversely, if $p(\cdot) \in LH$, with $p_- > 1$, then for any $w \in A_{p(\cdot)}$, $(p(\cdot), w)$ is an M -pair. Hereafter we will use both of these facts implicitly.

Remark 2.5. By a very deep result of Diening [23, 24], if $1 < p_- \leq p_+ < \infty$, M is bounded on $L^{p(\cdot)}$ if and only if it is bounded on $L^{p'(\cdot)}$. We conjecture that the same “duality” result holds in the weighted Lebesgue spaces, that is, it suffices to define an M -pair only by the boundedness of M on $L^{p(\cdot)}(w)$. We also conjecture (see [16, 25]) that if M is bounded on $L^{p(\cdot)}$ and $w \in A_{p(\cdot)}$, then M is bounded on $L^{p(\cdot)}(w)$. If these two conjectures are true, then the hypotheses of our results below would be simpler.

Remark 2.6. After this paper was completed, Lerner [35] essentially proved the first conjecture. He proved that if $1 < p_- \leq p_+ < \infty$, $w \in A_{p(\cdot)}$ and $w^{p(\cdot)} \in A_\infty$, then M is bounded on $L^{p(\cdot)}(w)$ if and only if it is bounded on $L^{p'(\cdot)}(w^{-1})$.

Though our goal is to use extrapolation to prove specific operators are bounded on $L^{p(\cdot)}(w)$, we will state our results more abstractly. Following the approach established in [15] (see also [13, 19]) we will write our extrapolation theorems for pairs of functions (f, g) contained in some family \mathcal{F} . Hereafter, if we write

$$\|f\|_X \leq C\|g\|_Y, \quad (f, g) \in \mathcal{F},$$

where X and Y are Banach function spaces (e.g., weighted classical or variable Lebesgue spaces), then we mean that this inequality is true for every pair $(f, g) \in \mathcal{F}$ such that the left-hand side of this inequality is finite. We will make the utility of this formulation clear in Section 3.

We can now state our main results. The first is a direct generalization of the classical Rubio de Francia extrapolation theorem and an extension of [15, Theorem 1.3] to weighted variable Lebesgue spaces.

Theorem 2.7. *Suppose that for some p_0 , $1 < p_0 < \infty$, and every $w_0 \in A_{p_0}$,*

$$(2.4) \quad \int_{\mathbb{R}^n} f(x)^{p_0} w_0(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x) dx, \quad (f, g) \in \mathcal{F}.$$

Then for any M -pair $(p(\cdot), w)$,

$$(2.5) \quad \|f\|_{L^{p(\cdot)}(w)} \leq C\|g\|_{L^{p(\cdot)}(w)}, \quad (f, g) \in \mathcal{F}.$$

If condition (2.4) holds for $p_0 = 1$, then (2.5) holds if we assume only that the maximal operator is bounded on $L^{p'(\cdot)}(w^{-1})$.

Remark 2.8. The case when $p_0 = 1$ is a special case of Theorem 2.21 below.

Our second result yields off-diagonal inequalities between two different weighted variable Lebesgue spaces. In the constant exponent case this result was first proved in [33], and it was proved in unweighted $L^{p(\cdot)}$ spaces in [15, Theorem 1.8]. To state it, we first define the appropriate weight classes that generalize the A_p weights. In the classical case these weights were introduced in [38].

Definition 2.9. Given $1 < p \leq q < \infty$, we say that $w \in A_{p,q}$ if

$$\sup_B \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{1/q} \left(\frac{1}{|B|} \int_B w(x)^{-p'} dx \right)^{1/p'} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. If $p = 1$, then $w \in A_{1,q}$ if

$$\sup_B \frac{\int_B w(x)^q dx}{\operatorname{ess\,inf}_{x \in B} w(x)^q} < \infty.$$

Definition 2.10. Let $p(\cdot), q(\cdot) \in \mathcal{P}$ be such that for some $\gamma, 0 < \gamma < 1$,

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \gamma.$$

Given w such that $0 < w(x) < \infty$ almost everywhere, we say that $w \in A_{p(\cdot),q(\cdot)}$ if

$$\sup_B |B|^{\gamma-1} \|w\chi_B\|_{q(\cdot)} \|w^{-1}\chi_B\|_{p'(\cdot)} < \infty,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Theorem 2.11. Suppose that for some $p_0, q_0, 1 < p_0 \leq q_0 < \infty$, and every $w_0 \in A_{p_0,q_0}$,

$$(2.6) \quad \left(\int_{\mathbb{R}^n} f(x)^{q_0} w_0(x)^{q_0} dx \right)^{1/q_0} \leq C \left(\int_{\mathbb{R}^n} g(x)^{p_0} w_0(x)^{p_0} dx \right)^{1/p_0}, \quad (f, g) \in \mathcal{F}.$$

Given $p(\cdot), q(\cdot) \in \mathcal{P}$, suppose

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$

Define $\sigma \geq 1$ by $1/\sigma' = 1/p_0 - 1/q_0$. If $w \in A_{p(\cdot),q(\cdot)}$ and $(q(\cdot)/\sigma, w^\sigma)$ is an M -pair, then

$$(2.7) \quad \|f\|_{L^{q(\cdot)}(w)} \leq C \|g\|_{L^{p(\cdot)}(w)}, \quad (f, g) \in \mathcal{F}.$$

If condition (2.6) holds for $p_0 = 1$, then (2.7) holds if we assume only that the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(w^{-q_0})$.

Remark 2.12. When $\sigma = 1$, Theorem 2.11 reduces to Theorem 2.7. Therefore, in proving it we will assume that $\sigma > 1$.

Our third result extends the theory of limited range extrapolation to the weighted variable Lebesgue spaces. This concept was introduced by Auscher and Martell [5] and independently by Duoandikoetxea *et al.* [29] in a somewhat different form. We generalize both of their results. To state our main result we recall a definition: we say $w \in RH_s$ for some $s > 1$ if

$$[w]_{RH_s} = \sup_B \frac{\left(\int_B w(x)^s dx \right)^{1/s}}{\int_B w(x) dx} < \infty.$$

Given a weight w , $w \in A_p$ for some $p \geq 1$ if and only if there exists $s > 1$ such that $w \in RH_s$ (see [28]). As given in [5], limited range extrapolation in the constant exponent case is the following.

Theorem 2.13. *Given $1 < q_- < q_+ < \infty$, suppose there exists p_0 , $q_- < p_0 < q_+$, such that for every $w_0 \in A_{p_0/q_-} \cap RH_{(q_+/p_0)'}$,*

$$(2.8) \quad \int f(x)^{p_0} w_0(x) dx \leq c \int g(x)^{p_0} w_0(x) dx, \quad (f, g) \in \mathcal{F}.$$

Then for every p , $q_- < p < q_+$, and every $w \in A_{p/q_-} \cap RH_{(q_+/p)'}$,

$$\int f(x)^p w(x) dx \leq c \int g(x)^p w(x) dx, \quad (f, g) \in \mathcal{F}.$$

In the variable exponent case we have a very different result which does not reduce to the constant case, Theorem 2.13.

Theorem 2.14. *Given $1 < q_- < q_+ < \infty$, suppose there exists p_0 , $q_- < p_0 < q_+$, such that for every $w_0 \in A_{p_0/q_-} \cap RH_{(q_+/p_0)'}$, (2.8) holds. Then for every $p(\cdot) \in LH$ with $q_- < p_- \leq p_+ < q_+$,*

$$(2.9) \quad \|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}, \quad (f, g) \in \mathcal{F}.$$

More generally, there exists p_ , $q_- < p_* < q_+$, such that if we let $\sigma = \frac{p_* q_-}{p_* - q_-}$, then there exists a constant $c = c(p_-, p_+, q_-, q_+, p_*) \in (0, 1)$, so that for every weight w with $w^\sigma \in A_{\frac{p(\cdot)}{c\sigma}}$, we have*

$$(2.10) \quad \|fw\|_{p(\cdot)} \leq C \|gw\|_{p(\cdot)}.$$

Remark 2.15. The two inequalities (2.9) and (2.10) follow from two special cases of a more general version of the above result, which we present as Proposition 5.8. However, the constant exponent result in Theorem 2.13 is from a third special case, and this reduction is not immediately obvious; see Remark 5.10 for details. We discuss the relationship between these cases in Remark 5.11.

Remark 2.16. A weaker version of the unweighted inequality (2.9) in Theorem 2.14 was implicit in Fiorenza *et al.* [30].

Remark 2.17. The regularity assumption on $p(\cdot)$ in Theorem 2.14 can be weakened. For example, it follows from the proof of (2.9) that there exists $s = s(q_-, q_+, p_-, p_+)$ such that it suffices to assume that M is bounded on $L^{p(\cdot)}$ and $L^{(p(\cdot)/s)'}$. By the duality property of the maximal operator (see Remark 2.5) the second assumption is equivalent to assuming M is bounded on $L^{p(\cdot)/s}$. Depending on whether $s > 1$ or $s < 1$, one of these assumptions implies the other, since if M is bounded on $L^{p(\cdot)}$, it is bounded on $L^{rp(\cdot)}$ for all $r > 1$ ([13, Theorem 3.38]). Regarding the constants in the conclusion: c depends on p_* , and as we will see from the proof, the existence of p_* is guaranteed if we take it sufficiently close to q_- .

Remark 2.18. The hypotheses on the weight w for inequality (2.10) to hold is restrictive, but there exist weights that satisfy them. We have shown that if $p(\cdot) \in LH$ and $0 \leq a < n/p_+$, then $w(x) = |x|^{-a} \in A_{p(\cdot)}$. (This result will appear in [21].) Hence, if $0 \leq a < cn/p_+$, $|x|^{-a\sigma} \in A_{\frac{p(\cdot)}{c\sigma}}$. This result can also be used to construct non-trivial examples of weights that satisfy the hypotheses of our other results.

We can also generalize the version of limited range extrapolation from [29].

Corollary 2.19. *Given δ , $0 < \delta \leq 1$, suppose that for every $w \in A_2$,*

$$(2.11) \quad \int f(x)^2 w(x)^\delta dx \leq c \int g(x)^2 w(x)^\delta dx, \quad (f, g) \in \mathcal{F}.$$

Then for every $p(\cdot) \in LH$ such that

$$(2.12) \quad \frac{2}{1+\delta} < p_- \leq p_+ < \frac{2}{1-\delta},$$

we have that

$$(2.13) \quad \|f\|_{p(\cdot)} \leq C \|g\|_{p(\cdot)}, \quad (f, g) \in \mathcal{F}.$$

More generally, in the variable weighted case, if $p(\cdot) \in LH$ and σ is defined as in Theorem 2.14, then there exists a constant $c \in (0, 1)$ such that for every weight w with $w^\sigma \in A_{\frac{p(\cdot)}{c\sigma}}$, we have

$$(2.14) \quad \|fw\|_{p(\cdot)} \leq C \|gw\|_{p(\cdot)}.$$

Remark 2.20. For simplicity we have stated Corollary 2.19 only assuming a weighted L^2 estimate. A more general result is possible; see [19, Remark 3.39]. An unweighted version of corollary (2.12) that includes this generalization has recently been proved by Gogatishvili and Kopaliani [31].

Finally, we give two variants of classical extrapolation. We first consider extrapolation from A_1 weights. This result is a generalization of the original extrapolation theorem for variable Lebesgue spaces in [15, Theorem 1.3]. It shows that we can weaken the hypotheses of Theorem 2.7 when $p_0 = 1$ and also prove results for exponents function such that $p_- \leq 1$. To state our result we introduce a more general class of exponents: we say $p(\cdot) \in \mathcal{P}_0$ if $p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty)$. For such $p(\cdot)$ we define the “norm” $\|\cdot\|_{p(\cdot)}$ (actually a quasi-norm; see [22]) exactly as we do for $p(\cdot) \in \mathcal{P}$.

Theorem 2.21. *Suppose that for some $p_0 > 0$ and every $w_0 \in A_1$,*

$$(2.15) \quad \int_{\mathbb{R}^n} f(x)^{p_0} w_0(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x) dx, \quad (f, g) \in \mathcal{F}.$$

Given $p(\cdot) \in \mathcal{P}_0$ such that $p_- \geq p_0$, suppose that $w \in A_{p(\cdot)/p_0}$ and M is bounded on $L^{(p(\cdot)/p_0)'}(w^{-p_0})$. Then

$$\|f\|_{L^{p(\cdot)}(w)} \leq C \|g\|_{L^{p(\cdot)}(w)}, \quad (f, g) \in \mathcal{F}.$$

Remark 2.22. There is an important difference between Theorem 2.21 (and [15, Theorem 1.3]) and Theorem 2.7. With the latter we can extrapolate both “up” and “down”, i.e., we can get results for $p(\cdot)$ irrespective of whether p_- is larger or smaller than p_0 . With A_1 extrapolation, however, we have the restriction that $p_- \geq p_0$. The same situation holds in the constant exponent case and is to be expected, since the A_1 case often governs “endpoint” inequalities. This weaker conclusion is balanced by the weaker hypothesis: we do not require $(p(\cdot)/p_0, w^{p_0})$ to be an M -pair, since in the proof we will only need the “dual” inequality for the maximal operator.

Remark 2.23. The hypotheses of Theorem 2.21 are redundant, since if M is bounded on $L^{(p(\cdot)/p_0)'}(w^{-p_0})$, then $w^{-p_0} \in A_{(p(\cdot)/p_0)'}$, which in turn implies that $w^{p_0} \in A_{p(\cdot)/p_0}$. Conversely, if we take $p(\cdot) \in LH$, then it is enough to assume $w^{p_0} \in A_{p(\cdot)/p_0}$.

Extrapolation can also be applied to inequalities governed by the larger class $A_\infty = \bigcup_{p>1} A_p$. The following result was first proved in [18].

Theorem 2.24. *If for some $p_0 > 0$ and every $w_0 \in A_\infty$,*

$$(2.16) \quad \int_{\mathbb{R}^n} f(x)^{p_0} w_0(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} w_0(x) dx, \quad (f, g) \in \mathcal{F},$$

then the same inequality holds with p_0 replaced by any $p, 0 < p < \infty$.

A_∞ extrapolation in variable Lebesgue spaces has the following form.

Theorem 2.25. *Suppose that for some $p_0 > 0$ and every $w_0 \in A_\infty$, inequality (2.16) holds. Then given $p(\cdot) \in \mathcal{P}_0$, suppose there exists $s \leq p_-$ such that $w^s \in A_{p(\cdot)/s}$ and M is bounded on $L^{p(\cdot)/s}'(w^{-s})$. Then*

$$\|f\|_{L^{p(\cdot)}(w)} \leq C \|g\|_{L^{p(\cdot)}(w)}, \quad (f, g) \in \mathcal{F}.$$

Remark 2.26. There is a close connection between A_1 and A_∞ extrapolation; see [19, Section 3.3]. We will exploit this fact in our proof.

To make the connection between Theorems 2.24 and 2.25 clearer, we introduce the notation $A_{p(\cdot)}^{var}$ for the weights that satisfy the variable exponent Muckenhoupt condition. Then if $p(\cdot) = p$ is a constant, the hypothesis in Theorem 2.25 is $w^s \in A_{p/s}^{var}$. It follows at once from Definition 2.2 that this is equivalent to $w^p \in A_{p/s} \subset A_\infty$. Conversely, the hypothesis in Theorem 2.24 is that $w^p \in A_\infty$, i.e., for some $t > 1, w^p \in A_t$. Fix $s < p$ such that $t = p/s$; then $w^p \in A_{p/s}$, or equivalently, $w^s \in A_{p/s}^{var}$.

As the next proposition shows, the hypotheses of Theorem 2.25 are weaker than those of Theorem 2.7.

Proposition 2.27. *Given $p(\cdot) \in \mathcal{P}$, suppose $w \in A_{p(\cdot)}$. Then for every $s, 0 < s < 1, w^s \in A_{p(\cdot)/s}$.*

3. NORM INEQUALITIES FOR OPERATORS

In this section we use extrapolation to prove norm inequalities for a variety of operators on the weighted variable Lebesgue spaces. We will first discuss how to prove that an operator T is bounded on $L^{p(\cdot)}(w)$ using Theorem 2.7. These same ideas can be used to apply the other theorems, and the details are left to the reader. Following this, we will give applications to some specific operators. Our goal is not to be exhaustive, but rather to illustrate the utility of extrapolation by concentrating on some key examples. For additional applications, see [13, 15, 19].

Applying extrapolation. The key to applying Theorem 2.7 is to construct the appropriate family \mathcal{F} . This generally requires an approximation argument since we need pairs (f, g) such that f lies in both the appropriate weighted space to apply the hypothesis and in the target weighted variable Lebesgue space. The dense subsets of $L^p(w)$ are well known, e.g., smooth functions and bounded functions of compact support. These sets are also dense in $L^{p(\cdot)}(w)$. This is known in the unweighted case; see, e.g., [13, Theorem 2.72, Corollary 2.73]. The proofs in the weighted case are similar; for completeness we include the details.

Lemma 3.1. *Given $p(\cdot) \in \mathcal{P}$ with $p_+ < \infty$, and a weight $w \in L_{loc}^{p(\cdot)}$, then L_c^∞ , bounded functions of compact support, and C_c^∞ , smooth functions of compact support, are dense in $L^{p(\cdot)}(w)$.*

Proof. We first prove that L_c^∞ is dense. The proof is essentially the same as in the unweighted case [13, Theorem 2.72]; for the convenience of the reader we sketch the details. Given $f \in L^{p(\cdot)}(w)$, define $f_n = \text{sgn}(f) \min(|f|, n) \chi_{B(0,n)}$. Then $f_n \rightarrow f$ pointwise as $n \rightarrow \infty$, and $|f_n|w \leq |f|w$. Since $p_+ < \infty$, we can apply the dominated convergence theorem [13, Theorem 2.62] to conclude that $f_n w \rightarrow f w$ in $L^{p(\cdot)}$; equivalently, $f_n \rightarrow f$ in $L^{p(\cdot)}(w)$.

The density of C_c^∞ now follows from this. By Lusin’s theorem, given $f \in L_c^\infty$, for every $\epsilon > 0$ there exists a continuous function of compact support g_ϵ such that $\|g\|_\infty \leq \|f\|_\infty$ and $|D_\epsilon| = |\{x : g(x) \neq f(x)\}| < \epsilon$. But then

$$\|f - g_\epsilon\|_{L^{p(\cdot)}(w)} \leq 2\|f\|_\infty \|\chi_{D_\epsilon} w\|_{p(\cdot)}.$$

Since $w \in L_{loc}^{p(\cdot)}$, again by the dominated convergence theorem in $L^{p(\cdot)}$, the right-hand term tends to 0 as $\epsilon \rightarrow 0$. Hence, continuous functions of compact support are dense in $L^{p(\cdot)}(w)$. Since every continuous function of compact support can be approximated uniformly by smooth functions, we also have C_c^∞ is dense. \square

Now suppose that for every $w_0 \in A_{p_0}$ and $f \in L^{p_0}(w)$, an operator T satisfies

$$(3.1) \quad \int_{\mathbb{R}^n} |Tf(x)|^{p_0} w_0(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_0} w_0(x) \, dx.$$

We want to show that given an M -pair $(p(\cdot), w)$, T is bounded on $L^{p(\cdot)}(w)$. Since $w \in L_{loc}^{p(\cdot)}$, by a standard argument (cf. [13, Theorem 5.39]) it will suffice to show that $\|(Tf)w\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}$ for all $f \in L_c^\infty$. Intuitively, we want to define the family \mathcal{F} by

$$\mathcal{F} = \{(|Tf|, |f|) : f \in L_c^\infty\}.$$

However, we do not know *a priori* that $Tf \in L^{p(\cdot)}(w)$. To overcome this we make a second approximation and define $\langle Tf \rangle_n = \min(|Tf|, n) \chi_{B(0,n)}$. Again since $w \in L_{loc}^{p(\cdot)}$, we have that $\langle Tf \rangle_n \in L^{p(\cdot)}(w)$. Furthermore, it is immediate that (3.1) holds with $|Tf|$ replaced by $\langle Tf \rangle_n$. Therefore, if we define

$$\mathcal{F} = \{(\langle Tf \rangle_n, |f|) : f \in L_c^\infty, n \geq 1\},$$

then we can apply Theorem 2.7 and Fatou’s lemma in the variable Lebesgue spaces ([13, Theorem 2.61]) to conclude that for all $f \in L_c^\infty$,

$$\|(Tf)w\|_{p(\cdot)} \leq \liminf_{n \rightarrow \infty} \|\langle Tf \rangle_n w\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)}.$$

Similar arguments hold if we need to take $f \in C_c^\infty$ or in some other dense set.

The Hardy-Littlewood maximal operator. Although we must assume the boundedness of the maximal operator to apply extrapolation, as an immediate consequence we get vector-valued inequalities for it. It is well known that for all $p, q, 1 < p, q < \infty$, and all $w \in A_p$,

$$\left\| \left(\sum_{k=1}^\infty (Mf_k)^q \right)^{1/q} \right\|_{L^p(w)} \leq C \left\| \left(\sum_{k=1}^\infty |f_k|^q \right)^{1/q} \right\|_{L^p(w)}.$$

(See, for instance, [1].) From this we immediately get the following inequality.

Corollary 3.2. *Given an M -pair $(p(\cdot), w)$ and $1 < q < \infty$,*

$$\left\| \left(\sum_{k=1}^\infty (Mf_k)^q \right)^{1/q} \right\|_{L^{p(\cdot)}(w)} \leq C \left\| \left(\sum_{k=1}^\infty |f_k|^q \right)^{1/q} \right\|_{L^{p(\cdot)}(w)}.$$

This result is not particular to the maximal operator: such vector-valued inequalities are an immediate consequence of extrapolation defined in terms of ordered pairs of functions. This is proved in the constant exponent case in [19, Corollary 3.12], and the same proof works in our more general setting.

Remark 3.3. In the same way, though we do not discuss them here, weak type inequalities can be proved using extrapolation. See [19, Corollary 3.11] and [13, Corollary 5.33] for details.

Remark 3.4. Vector-valued inequalities for the maximal operator play an important role in studying functions spaces in the variable exponent setting; see, for example, [22, 26].

Singular integral operators. Let T be a convolution type singular integral: $Tf = K * f$, where K is defined on $\mathbb{R}^n \setminus \{0\}$ and satisfies $\hat{K} \in L^\infty$ and

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, \quad x \neq 0.$$

More generally, we can take T to be a Calderón-Zygmund singular integral of the type defined by Coifman and Meyer. Then for all p , $1 < p < \infty$, and $w \in A_p$,

$$(3.2) \quad \int_{\mathbb{R}^n} |Tf(x)|^p w(x) \, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx.$$

(See [28, 32].) As an immediate consequence we get that singular integrals are bounded on weighted Lebesgue spaces.

Corollary 3.5. *Let T be a Calderón-Zygmund singular integral operator. Then for any M -pair $(p(\cdot), w)$,*

$$\|Tf\|_{L^{p(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

We can also use extrapolation to prove norm inequalities for operators that are more singular. Given $1 < r \leq \infty$, let $\Omega \in L^r(S^{n-1})$ satisfy $\int_{S^{n-1}} \Omega(y) \, d\sigma(y) = 0$, where S^{n-1} is the unit sphere and σ the surface measure on S^{n-1} . Given the kernel K ,

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n},$$

define $T_\Omega f = K * f$. Then for all $p > r'$ and $w \in A_{p/r'}$, (3.2) holds for T_Ω [27, 40]. This is a limiting case of Theorem 2.14, with $q_- = r'$ and $q_+ = \infty$. However, it is more straightforward to apply Theorem 2.7 by rescaling. If we rewrite (3.2) as

$$(3.3) \quad \int_{\mathbb{R}^n} (|T_\Omega f(x)|^{r'})^{p/r'} w(x) \, dx \leq C \int_{\mathbb{R}^n} (|f(x)|^{r'})^{p/r'} w(x) \, dx,$$

then for any M -pair $(p(\cdot), w)$,

$$\| |T_\Omega f|^{r'} w \|_{L^{p(\cdot)}} \leq C \| |f|^{r'} w \|_{L^{p(\cdot)}}.$$

In particular, if we replace w by $w^{r'}$ and $p(\cdot)$ by $p(\cdot)/r'$, then by dilation we get a variable exponent analog of inequality (3.3).

Corollary 3.6. *Given $p(\cdot)$ and w such that $(p(\cdot)/r', w^{r'})$ is an M -pair, we have*

$$\|T_\Omega f\|_{L^{p(\cdot)}(w)} \leq C \|f\|_{L^{p(\cdot)}(w)}.$$

Remark 3.7. Weighted norm inequalities for singular integral operators were proved using a different approach by Bernardis, Dalmasso and Pradolini [6, Theorem 3.2].

Off-diagonal operators. Given α , $0 < \alpha < n$, the fractional integral operator of order α (also referred to as the Riesz potential) is the positive integral operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$

The associated fractional maximal operator M_α is defined by

$$M_\alpha f(x) = \sup_B |B|^{\alpha/n} \int_B |f(y)| dy \cdot \chi_B(x).$$

Weighted inequalities for both of these operators are governed by the $A_{p,q}$ weights in Definition 2.9: given p , $1 < p < n/\alpha$, and q such that $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, then for all $w \in A_{p,q}$,

$$\left(\int_{\mathbb{R}^n} |I_\alpha f(x)w(x)|^q dx \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)w(x)|^p dx \right)^{1/p};$$

the same inequality holds if I_α is replaced by M_α [38]. Therefore, we can apply Theorem 2.11 (using the obvious variant of the technical reduction discussed at the beginning of this section) to get the following result.

Corollary 3.8. *Given α , $0 < \alpha < n$, suppose exponents $p(\cdot)$, $q(\cdot)$ are such that $p_+ < n/\alpha$ and $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{\alpha}{n}$. Let $\sigma = (n/\alpha)'$. Then for all M -pairs $(q(\cdot)/\sigma, w^\sigma)$,*

$$\begin{aligned} \|I_\alpha f\|_{L^{q(\cdot)}(w)} &\leq C \|f\|_{L^{p(\cdot)}(w)}, \\ \|M_\alpha f\|_{L^{q(\cdot)}(w)} &\leq C \|f\|_{L^{p(\cdot)}(w)}. \end{aligned}$$

Remark 3.9. The restriction $p_+ < n/\alpha$ is natural for the fractional integral operator, since in the constant exponent case I_α does not map $L^{n/\alpha}$ to L^∞ . On the other hand, M_α does; moreover, in the unweighted case, if $p_+ = n/\alpha$, then $\|M_\alpha f\|_{q(\cdot)} \leq C \|f\|_{p(\cdot)}$. (See [11, 13].) Therefore, we conjecture that the same is true in the weighted case; this question is still open even for $\alpha = 0$ and $p_+ = \infty$.

Remark 3.10. Corollary 3.8 was proved independently in [6, Theorems 2.2, 3.12] using a different approach. They also prove analogous results for more general off-diagonal operators.

Coifman-Fefferman type inequalities. There are a variety of norm inequalities that compare two operators, usually of the form

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |Sf(x)|^p w(x) dx,$$

where $w \in A_\infty$. The first such inequality, due to Coifman and Fefferman [9], compared singular integrals and the Hardy-Littlewood maximal operator, and there have been a number of results proved since; see [19, Chapter 9]. We can use Theorem 2.24 to extend such inequalities to the weighted variable Lebesgue spaces.

We illustrate this by considering one such inequality in particular, the Fefferman-Stein inequality for the sharp maximal operator. (See [28].) Recall that the sharp maximal function is defined by

$$M^\# f(x) = \sup_B \int_B |f(y) - f_B| dy \cdot \chi_B(x),$$

where $f_B = \int_B f(x) dx$. Though pointwise smaller than the maximal operator, we have that for all $p, 0 < p < \infty$ and $w \in A_\infty$,

$$\int_{\mathbb{R}^n} Mf(x)^p w(x) dx \leq C \int_{\mathbb{R}^n} M^\# f(x)^p w(x) dx.$$

Then by Theorem 2.25 we immediately get the following.

Corollary 3.11. *Given $p(\cdot) \in \mathcal{P}$ and a weight w , suppose there exists $s < p_-$ such that $w^s \in A_{p(\cdot)/s}$ and M is bounded on $L^{(p(\cdot)/s)'}(w^{-s})$. Then*

$$\|f\|_{L^{p(\cdot)}(w)} \leq C \|M^\# f\|_{L^{p(\cdot)}(w)}.$$

In exactly the same way other Coifman-Fefferman type inequalities can be extended to the variable Lebesgue space setting.

Operators with a restricted range of exponents. Certain types of operators are not bounded on L^p for every $p, 1 < p < \infty$, but only for p in some interval, say $q_- < p < q_+$. In this case it is natural to conjecture that such operators are bounded on $L^{p(\cdot)}$ provided that $q_- < p_- \leq p_+ < q_+$, and that weighted inequalities hold in the same range for suitable weights w . Here we consider two operators: the spherical maximal operator and the Riesz transforms associated with certain elliptic operators.

The spherical maximal operator is defined by

$$\mathcal{M}f(x) = \sup_{t>0} \left| \int_{S^{n-1}} f(x - ty) d\sigma(y) \right|,$$

where S^{n-1} is the unit sphere in \mathbb{R}^n and $d\sigma$ is the surface measure on the sphere. Stein [39] proved that for $n \geq 3$, \mathcal{M} is bounded on L^p if and only if $p > \frac{n}{n-1}$. Weighted norm inequalities are true for the same values of p but require strong conditions on the weight. Cowling, *et al.* [10] proved that if

$$\frac{n}{n-1} < p < \infty \quad \text{and} \quad \max\left(0, 1 - \frac{p}{n}\right) \leq \delta \leq \frac{n-2}{n-1},$$

and if

$$(3.4) \quad w = u_1^\delta u_2^{\delta(n-1)-(n-2)}, \quad u_1, u_2 \in A_1,$$

then $\mathcal{M} : L^p(w) \rightarrow L^p(w)$.

If we combine this result with Theorem 2.14 we get the following estimates in the variable Lebesgue spaces.

Corollary 3.12. *Fix $n \geq 3$. Given $p(\cdot) \in LH$ such that $\frac{n}{n-1} < p_- \leq p_+ < (n-1)p_-$, then*

$$(3.5) \quad \|\mathcal{M}f\|_{p(\cdot)} \leq C \|f\|_{p(\cdot)}.$$

Moreover, if for some $\sigma > \frac{n-1}{n-2}p_-$, $w^\sigma \in A_{\frac{p(\cdot)}{c\sigma}}$, where $c \in (0, 1)$ is as in the statement of Theorem 2.14, then

$$(3.6) \quad \|(\mathcal{M}f)w\|_{p(\cdot)} \leq C \|fw\|_{p(\cdot)}.$$

Proof. To apply Theorem 2.14 we need to restate the hypotheses of the above weighted norm inequality. By the information encoded in the factorization of A_p weights (see [20, Theorems 2.1, 2.3, 5.1]), if w is given by (3.4), then $w \in A_t \cap RH_{1/\delta}$,

where $1 - t = \delta(n - 1) - (n - 2)$ or $t = (n - 1)(1 - \delta)$. Therefore, if we fix any $p_0 > \frac{n}{n-1}$, we have that $w \in A_{p_0/q_-} \cap RH_{(q_+/p_0)'}$, where

$$q_- = \frac{p_0}{(n-1)(1-\delta)}, \quad q_+ = \frac{p_0}{1-\delta} = (n-1)q_-.$$

Conversely, if we take any $w \in A_{p_0/q_-} \cap RH_{(q_+/p_0)'}$, then it can be written in the form (3.4).

Given this reformulation we can apply Theorem 2.14. To prove the unweighted inequality (3.5), fix $p(\cdot)$ such that $\frac{n}{n-1} < p_- \leq p_+ < (n-1)p_-$. Note that if we fix $\delta = \frac{n-2}{n-1}$, then $q_- = p_0$, so if we take $p_0 = p_-$ and take values of δ close to $\frac{n-2}{n-1}$, we see that we can get q_- as close to p_- as desired. In particular, we can get $p_+ < (n-1)q_- = q_+$. Inequality (3.5) now follows from inequality (2.10) in Theorem 2.14.

To prove the weighted inequality (3.6), we argue similarly. Fix $p(\cdot)$ and $\sigma > \frac{n-1}{n-2}p_-$. Now choose a value of p_0 and fix q_-, q_+ as before. Then we have that

$$\sigma > \frac{n-1}{n-2}q_- = \frac{(n-1)q_-^2}{(n-1)q_- - q_-}.$$

We now apply limited range extrapolation in the constant exponent case, Theorem 2.13. This shows that we can now take *a posteriori* any value $p_0, q_- < p_0 < q_+ = (n-1)q_-$. In particular, we can take p_0 as close to $(n-1)q_-$ as we want. Fix p_0 so that

$$(3.7) \quad \sigma \geq \frac{p_0 q_-}{p_0 - q_-}.$$

By Proposition 2.27, if $w^\sigma \in A_{\frac{p(\cdot)}{\sigma}}$, then the same inclusion holds for any smaller value of σ , so we may assume without loss of generality that equality holds in (3.7). But then we can apply Theorem 2.14 starting from our new value of p_0 and using this value of σ to get (3.6). \square

Inequality (3.5) in Corollary 3.12 was originally proved by Fiorenza *et al.* [30]; their proof relied on a extrapolation argument which was a slightly weaker, unweighted version of Theorem 2.14.

A surprising feature of this result is that while there are weighted inequalities for any value of $p > \frac{n}{n-1}$, variable Lebesgue space bounds only hold for exponents with bounded oscillation. This is not an artifact of the proof: in [30] they also proved that if the spherical maximal operator is bounded on $L^{p(\cdot)}$, then $p_+ \leq np_-$; it is conjectured that this bound is sharp. To prove this via extrapolation it suffices to show that in the above weighted norm inequality we could replace the upper bound on δ by $\frac{n-1}{n}$. It is unclear if this is possible, though we note that in [10, p. 83] they conjectured that one could take weights of the form $w = u_1^{\frac{n-1}{n}}$, which is a special case.

A second kind of operator that satisfies norm inequalities with a limited range of exponents is the Riesz transform associated to complex elliptic operators in divergence form. We sketch the basic properties of these operators; for complete information, see Auscher [2].

Let A be an $n \times n$, $n \geq 3$, matrix of complex-valued measurable functions, and assume that A satisfies the ellipticity conditions

$$\lambda|\xi|^2 \leq \operatorname{Re}\langle A\xi, \xi \rangle, \quad |\langle A\xi, \eta \rangle| \leq \Lambda|\xi||\eta|, \quad \xi, \eta \in \mathbb{C}^n \quad 0 < \lambda < \Lambda.$$

Let $L = -\operatorname{div}A\nabla$. Then L satisfies an L^2 functional calculus so that the square root operator $L^{1/2}$ is well defined. The Kato conjecture asserted that this operator satisfies

$$\|L^{1/2}f\|_2 \approx \|\nabla f\|_2, \quad f \in W^{1,2}.$$

This was proved by Auscher *et al.* [3]. As a consequence of this we have that the Riesz transform associated to L , $\nabla L^{-1/2}$, also satisfies L^2 bounds:

$$\|\nabla L^{-1/2}f\|_2 \leq C\|f\|_2.$$

This operator also satisfies weighted L^p bounds for p close to 2. Auscher and Martell [4] proved that there exist constants $q_- = q_-(L) < \frac{2n}{n+2} < 2$ and $q_+ = q_+(L) > 2$ such that if $q_- < p < q_+$ and $w \in A_{p/q_-} \cap RH_{(q_+/p)'}$, then

$$\|\nabla L^{-1/2}f\|_{L^p(w)} \leq C\|f\|_{L^p(w)}.$$

By Theorem 2.14 we can extend this result to the variable Lebesgue spaces.

Corollary 3.13. *Given an elliptic operator L as defined above, suppose the exponent $p(\cdot) \in LH$ is such that $q_-(L) < p_- \leq p_+ < q_+(L)$. Then*

$$\|\nabla L^{-1/2}f\|_{p(\cdot)} \leq C\|f\|_{p(\cdot)},$$

and for any weight w such that $w^n \in A_{\frac{p(\cdot)}{cn}}$, then

$$\|(\nabla L^{-1/2}f)w\|_{p(\cdot)} \leq C\|fw\|_{p(\cdot)}.$$

Proof. The unweighted inequality is immediate. For the weighted inequality we take $p_0 = 2$, and we take a larger value for σ (possible by Proposition 2.27) by replacing q_- by the upper bound $\frac{2n}{n+2}$. This gives $\sigma = n$. □

Remark 3.14. Bongioanni *et al.* [8] introduced a class of weights that generalize the Muckenhoupt A_p weights and are the appropriate class for studying weighted norm inequalities for the Riesz transforms related to Schrödinger operators which in many cases satisfy limited range inequalities. They also showed that the theory of extrapolation could be extended to these weight classes [7]. It would be of interest to determine if their results could be extended to the appropriate scale of weighted variable Lebesgue spaces.

4. THE GENERAL APPROACH TO EXTRAPOLATION

In this section we give a broad overview of the way in which we prove each of our extrapolation theorems. We have chosen to organize the arguments in a way which does not yield the most elegant proof but which does make clearer the process by which we found the proof. This discussion should be seen as a complement to the overview of extrapolation given in [19, Chapter 2]; we believe that it will be useful for attempts to prove extrapolation theorems in other contexts.

All of our proofs use five basic tools: dilation, duality, Hölder’s inequality, reverse factorization and the Rubio de Francia algorithm. By dilation we mean the property that for any exponent $p(\cdot)$ and any $s > 0$, $\|f\|_{p(\cdot)}^s = \||f|^s\|_{p(\cdot)/s}$. For constant exponents this is trivial, and even for general exponent functions it is an immediate consequence of the definition (1.1). By duality (more precisely, by using

the associate space norm; see [13, Section 2.8]) we have that given $f \in L^{p(\cdot)}$, there exists $h \in L^{p'(\cdot)}$, $\|h\|_{p'(\cdot)} = 1$, such that

$$\|f\|_{p(\cdot)} \leq C \int_{\mathbb{R}^n} f(x)h(x) dx.$$

Conversely, by Hölder's inequality [13, Section 2.4], if $f \in L^{p(\cdot)}$ and $h \in L^{p'(\cdot)}$, then

$$\int_{\mathbb{R}^n} |f(x)h(x)| dx \leq C \|f\|_{p(\cdot)} \|h\|_{p'(\cdot)}.$$

(In both cases the constant depends only on $p(\cdot)$.) To construct the weight $w \in A_{p_0}$ needed to apply the hypothesis, we use reverse factorization: the property that if $\mu_1, \mu_2 \in A_1$, then $w_0 = \mu_1 \mu_2^{1-p_0} \in A_{p_0}$. (See [28, Prop. 7.2].) Finally, to find the A_1 weights we apply the Rubio de Francia extrapolation algorithm in the following form.

Proposition 4.1. *Given $r(\cdot) \in \mathcal{P}$, suppose μ is a weight such that M is bounded on $L^{r(\cdot)}(\mu)$. For a positive function $h \in L^1_{loc}$, with $Mh(x) < \infty$ almost everywhere, define*

$$\mathcal{R}h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L^{r(\cdot)}(\mu)}^k}.$$

Then: (1) $h(x) \leq \mathcal{R}h(x)$; (2) $\|\mathcal{R}h\|_{L^{r(\cdot)}(\mu)} \leq 2\|h\|_{L^{r(\cdot)}(\mu)}$; (3) $\mathcal{R}h \in A_1$, with $[\mathcal{R}h]_{A_1} \leq 2\|M\|_{L^{r(\cdot)}(\mu)}$. More generally, for fixed constants $\alpha > 0$ and $\beta \in \mathbb{R}$, and another weight w , define the operator

$$H = Hh = \mathcal{R}(h^\alpha w^\beta)^{1/\alpha} w^{-\beta/\alpha}.$$

Then: (1) $h(x) \leq H(x)$; (2) if $v = w^{\beta/\alpha} \mu^{1/\alpha}$, then H is bounded on $L^{\alpha r(\cdot)}(v)$, with $\|H\|_{L^{\alpha p(\cdot)}(v)} \leq 2\|h\|_{L^{\alpha p(\cdot)}(v)}$; (3) $H^\alpha w^\beta \in A_1$, with $[H^\alpha w^\beta]_{A_1} \leq 2\|M\|_{L^{r(\cdot)}(\mu)}$.

Proof. The proof is straightforward and essentially the same as in the constant exponent case (see [19, Chapter 2]): property (1) for \mathcal{R} is immediate; property (2) follows from our assumption that M is bounded; and property (3) follows from the fact that M is sublinear. The properties of H are immediate consequences of dilation and those for \mathcal{R} . \square

To prove our extrapolation theorems we use these tools to reduce the quantity we want to estimate (e.g., the left-hand term in (2.5), (2.7), (2.9), (2.10)) to something we can apply our hypothesis to (e.g., a weighted integral in the form of the left-hand side of (2.4), (2.6), (2.8)). Let us use Theorem 2.7 as an example. We first fix a weight w satisfying our hypotheses and a pair $(f, g) \in \mathcal{F}$. For technical reasons we introduce a new function h_1 that depends on both f and g : intuitively, $h_1 = g$, but we introduce a term involving f so that we can prove that the integral corresponding to the left-hand side of the weighted norm inequality in the hypothesis is finite. We also define it to have uniformly bounded norm. We majorize it by an operator H_1 with constants α_1 and β_1 to be determined. If we first apply dilation with an exponent $s > 0$ and then duality, we get a function h_2 , also with uniformly bounded norm, which we majorize by a second operator H_2 with constants α_2 and β_2 . We multiply and divide by H_1^γ , $\gamma > 0$, and apply Hölder's inequality to get, for example,

$$\|f\|_{L^{p(\cdot)}(w)}^s \leq \left(\int_{\mathbb{R}^n} f^{p_0} H_1^{-\gamma(p_0/s)} H_2 w^s dx \right)^{s/p_0} \left(\int_{\mathbb{R}^n} H_1^{\gamma(p_0/s)'} H_2 w^s dx \right)^{1/(p_0/s)'}$$

Our goal is to show that the second integral is uniformly bounded and that the first is bounded by the right-hand side of our desired conclusion. To do so we need to find appropriate values for the six undetermined parameters: $\alpha_j, \beta_j, 1 \leq j \leq 2, s$ and γ . These parameters are subject to the following constraints:

- (1) Since we know which (unweighted) variable Lebesgue space h_2 belongs to (e.g., $h_2 \in L^{(\frac{p(\cdot)}{s})'}$), we will assume that $H_2 = \mathcal{R}_2(h_2^{\alpha_2} w^{\beta_2})^{1/\alpha_2} w^{-\beta_2/\alpha_2}$ is bounded there too. We can then use Proposition 4.1 “backwards” (i.e., set $v = 1, (\frac{p(\cdot)}{s})' = \alpha_2 r(\cdot)$ and solve for μ) to deduce that we need the maximal operator M bounded on $L^{(p(\cdot)/s)'/\alpha_2}(w^{-\beta_2})$. This gives constraints on α_2 and β_2 .
- (2) Similarly, we want $H_1 = \mathcal{R}_1(h_1^{\alpha_1} w^{\beta_1})^{1/\alpha_1} w^{-\beta_1/\alpha_1}$ to be bounded on the same space in which h_1 is contained, and again by Proposition 4.1 (taking $v = w$ and $p(\cdot) = \alpha_1 r(\cdot)$) this means that we need M to be bounded on $L^{p(\cdot)/\alpha_1}(w^{\alpha_1 - \beta_1})$. This gives constraints on α_1, β_1 and γ .
- (3) Lastly, to apply our hypothesis, we need $H_1^{-\gamma(p_0/s)} H_2 w^s$ to satisfy the A_{p_0} condition. To apply reverse factorization (since H_1 and H_2 both yield A_1 weights) we get more constraints on all the parameters (in particular on s).

If we combine all of these constraints we are able to find sufficient conditions on the exponent $p(\cdot)$ and the weight w to get the desired conclusion.

In each of the proofs in Section 5 below, we follow this schema. Some of the parameters described above have their values determined, but others are still free. For our first three theorems we will prove a (seemingly) more general result, in the sense that we will show that the desired weighted norm inequality holds for a family of weight classes parameterized by β_1 (the constant from H_1) and s (the constant that determines the dual space). We will get the stated result by choosing appropriate values for these parameters.

For Theorem 2.7 one can see the choice of the parameters as simply what is necessary to get the result that is the obvious analog of the classical Rubio de Francia extrapolation theorem. However, we will also show, in the special case of power weights, that our choice of parameters is in some sense optimal. The proof of off-diagonal extrapolation, Theorem 2.11, will follow the same pattern. However, the proof has some technical difficulties related to the variable Lebesgue space norm and requires more care in choosing the parameters.

For both Theorems 2.7 and 2.11, the proofs would be simpler if we had simply fixed our parameters initially, without motivating our choices. Indeed, we admit that when we first proved each result we chose our parameters in an *ad hoc* fashion, justifying our choices by the fact that we got the desired outcome. However, in proving limited range extrapolation, Theorem 2.14, we discovered that the “right” parameters were not obvious: none of our initial choices led to a meaningful result, let alone one analogous to the constant exponent case. Ultimately we used the approach outlined above in order to discover what was actually going on. We have chosen to retain it here since it both illuminates our final result and makes clear why the constant exponent theorem does not immediately generalize to the variable space setting. But then, in order to help the reader understand our approach, we chose to write the previous two proofs in this more general fashion.

Finally, extrapolation with A_∞ and A_1 weights, Theorems 2.25 and 2.21, requires some minor modification to our general approach; we will make these clear in the course of the proofs.

5. PROOF OF THE THEOREMS

In this section we give the proofs of all the results in Section 2.

Proof of Theorem 2.7. When $p_0 = 1$, Theorem 2.7 is a special case of Theorem 2.21, so here we will assume $p_0 > 1$. We will prove the following proposition.

Proposition 5.1. *Suppose (2.4) holds for some $p_0 > 1$. Fix $p(\cdot) \in \mathcal{P}$, $\beta_1 \in \mathbb{R}$ and choose any s such that*

$$(5.1) \quad \max(0, p_0 - p_-(p_0 - 1)) < s < \min(p_-, p_0).$$

Let $\alpha_1 = \frac{p_0 - s}{p_0 - 1}$ and $\beta_2 = s - \beta_1(1 - p_0)$. If M is bounded on $L^{p(\cdot)/\alpha_1}(w^{\alpha_1 - \beta_1})$ and $L^{(p(\cdot)/s)'}(w^{-\beta_2})$, then $\|fw\|_{p(\cdot)} \leq C\|gw\|_{p(\cdot)}$.

The constant s comes from duality and the constants α_j and β_j are from using Proposition 4.1 to define H_1 and H_2 . The values and constraints are the only ones which arise in applying the method outlined in Section 4.

To prove Theorem 2.7 it is enough to take $s = 1$ and $\beta_1 = 0$. Then (5.1) holds (since $p_0 > 1$) and the conditions on the maximal operator reduce to saying that M is bounded on $L^{p(\cdot)}(w)$ and $L^{p'(\cdot)}(w^{-1})$, that is, that $(p(\cdot), w)$ is an M -pair. We will consider other choices of parameters in Remark 5.2 below.

Proof. Let $(f, g) \in \mathcal{F}$ with $\|f\|_{L^{p(\cdot)}(w)} < \infty$. Without loss of generality we may assume $\|f\|_{L^{p(\cdot)}(w)} > 0$ and $\|g\|_{L^{p(\cdot)}(w)} < \infty$ since otherwise there is nothing to prove. We may also assume $\|g\|_{L^{p(\cdot)}(w)} > 0$. Otherwise, $g(x) = 0$ almost everywhere, and so by our assumption (2.4) (perhaps via an approximation argument like the one in Section 3) we get that $f(x) = 0$ a.e. Define

$$h_1 = \frac{f}{\|f\|_{L^{p(\cdot)}(w)}} + \frac{g}{\|g\|_{L^{p(\cdot)}(w)}}.$$

Then $h_1 \in L^{p(\cdot)}(w)$ and $\|h_1\|_{L^{p(\cdot)}(w)} \leq 2$.

We will use Proposition 4.1 to define the two operators H_1 and H_2 ,

$$(5.2) \quad H_1 = \mathcal{R}_1(h_1^{\alpha_1} w^{\beta_1})^{1/\alpha_1} w^{-\beta_1/\alpha_1}, \quad H_2 = \mathcal{R}_2(h_2^{\alpha_2} w^{\beta_2})^{1/\alpha_2} w^{-\beta_2/\alpha_2},$$

where h_2 will be fixed momentarily. Fix s , $0 < s < \max(p_0, p_-)$. By dilation, duality and Hölder's inequality, there exists $h_2 \in L^{(p(\cdot)/s)'}$, $\|h_2\|_{(p(\cdot)/s)'} = 1$, such that for any $\gamma > 0$,

(5.3)

$$\begin{aligned} \|fw\|_{p(\cdot)}^s &\leq C \int_{\mathbb{R}^n} f^s w^s h_2 dx \leq \int_{\mathbb{R}^n} f^s H_1^\gamma H_1^{-\gamma} H_2 w^s dx \\ &\leq C \left(\int_{\mathbb{R}^n} f^{p_0} H_1^{-\gamma(p_0/s)} H_2 w^s dx \right)^{s/p_0} \left(\int_{\mathbb{R}^n} H_1^{\gamma(p_0/s)'} H_2 w^s dx \right)^{1/(p_0/s)'} \\ &= I_1^{s/p_0} I_2^{1/(p_0/s)'}. \end{aligned}$$

We will first find assumptions that let us show that I_2 is uniformly bounded. Since $h_1 \in L^{p(\cdot)}(w)$ and $h_2 \in L^{(p(\cdot)/s)'}$, we must have that H_1 and H_2 are bounded on these spaces. To get the norm of H_2 in $L^{(p(\cdot)/s)'}$ we apply Hölder’s inequality with exponent $p(\cdot)/s$ to get

$$I_2 \leq C \|H_1^{\gamma(p_0/s)'} w^s\|_{p(\cdot)/s} \|H_2\|_{(p(\cdot)/s)'}$$

To use our assumption that H_1 is bounded on $L^{p(\cdot)}(w)$ we need to fix $\gamma = \frac{s}{(p_0/s)'}$. Then by dilation and the properties of H_1 and H_2 in Proposition 4.1 we have that

$$\|H_1^{\gamma(p_0/s)'} w^s\|_{p(\cdot)/s} = \|H_1 w\|_{p(\cdot)}^s \leq 2^s \|h_1 w\|_{p(\cdot)}^s \leq 4^s$$

and

$$\|H_2\|_{(p(\cdot)/s)'} \leq 2 \|h_2\|_{(p(\cdot)/s)'} = 2.$$

For H_1 and H_2 to be bounded on these spaces, by Proposition 4.1, we must have that the maximal operator satisfies

$$(5.4) \quad M \text{ bounded on } L^{p(\cdot)/\alpha_1}(w^{\alpha_1-\beta_1}) \text{ and } L^{(p(\cdot)/s)'/\alpha_2}(w^{-\beta_2}).$$

By Theorem 2.4, a necessary condition for this is that $p_-/\alpha_1 > 1$ and $[(p(\cdot)/s)'/\alpha_2]_- > 1$, or equivalently,

$$p_- > \alpha_1, \quad (p_+/s)' > \alpha_2.$$

We must now estimate I_1 ; with our choice of γ it can be written as

$$I_1 = \int_{\mathbb{R}^n} f^{p_0} H_1^{s-p_0} H_2 w^s \, dx.$$

In order to apply (2.4), we must show that I_1 is finite. Since $h_1 \leq H_1$, by Hölder’s inequality

$$\begin{aligned} I_1 &\leq \int_{\mathbb{R}^n} f^{p_0} \left(\frac{f}{\|fw\|_{p(\cdot)}} \right)^{s-p_0} H_2 w^s \, dx \\ &= \|fw\|_{p(\cdot)}^{p_0-s} \int_{\mathbb{R}^n} f^s w^s H_2 \, dx \leq \|fw\|_{p(\cdot)}^{p_0-s} \|fw\|_{p(\cdot)}^s \|H_2\|_{(p(\cdot)/s)'} < \infty. \end{aligned}$$

Suppose for the moment that $w_0 = H_1^{s-p_0} H_2 w^s \in A_{p_0}$; then we can use (2.4) to estimate I_1 . Again since $h_1 \leq H_1$ and by Hölder’s inequality,

$$\begin{aligned} I_1 &\leq C \int_{\mathbb{R}^n} g^{p_0} H_1^{s-p_0} H_2 w^s \, dx \leq C \int_{\mathbb{R}^n} g^{p_0} \left(\frac{g}{\|gw\|_{p(\cdot)}} \right)^{s-p_0} H_2 w^s \, dx \\ &= C \|gw\|_{p(\cdot)}^{p_0-s} \int_{\mathbb{R}^n} g^s H_2 w^s \, dx \leq C \|gw\|_{p(\cdot)}^{p_0-s} \|gw\|_{p(\cdot)}^s \|H_2\|_{(p(\cdot)/s)'} \leq C \|gw\|_{p(\cdot)}^{p_0}. \end{aligned}$$

If we combine this with the previous inequalities we get the desired norm inequality.

To complete the proof we must determine constraints on the parameters so that $H_1^{s-p_0} H_2 w^s \in A_{p_0}$. By reverse factorization and Proposition 4.1 we need to fix our parameters so that

$$H_1^{s-p_0} H_2 w^s = \left[H_1^{\frac{p_0-s}{p_0-1}} w^{\beta_1} \right]^{1-p_0} H_2 w^{s-\beta_1(1-p_0)} = [H_1^{\alpha_1} w^{\beta_1}]^{1-p_0} H_2^{\alpha_2} w^{\beta_2}.$$

Equating the exponents we get that

$$(5.5) \quad \alpha_1 = \frac{p_0 - s}{p_0 - 1}, \quad \beta_1 \in \mathbb{R}, \quad \alpha_2 = 1, \quad \beta_2 = s - \beta_1(1 - p_0).$$

(In other words, there is no constraint on β_1 .) Since above we assumed $s < p_0$, we have $\alpha_1 > 0$ as required in Proposition 4.1. Above, we required that $p_- > \alpha_1$; combining this with the new constraint we have that $s > p_0 - p_-(p_0 - 1)$. With $\alpha_2 = 1$, the second restriction from above (that $(p_+/s)' > \alpha_2$) always holds.

To summarize: we have shown that given a constant s such that (5.1) holds, and constants α_j, β_j as in (5.5), and if the maximal operator satisfies (5.4), then the desired weighted norm inequality holds. This completes the proof. \square

Remark 5.2. As we noted above, if $s = 1, \beta_1 = 0$, then we get a result analogous to the classical extrapolation theorem. This is enough to motivate our choice of these parameters. But in some sense this choice is also optimal.

To see this for β_1 , we will construct power weights that satisfy the boundedness conditions on the maximal operator in (5.4). By Remark 2.18 above, if $p(\cdot) \in LH$ and $0 \leq a < n/p_+$, then $w(x) = |x|^{-a} \in A_{p(\cdot)}$. Using this, we get from (5.4) that $w^{\alpha_1 - \beta_1} \in A_{p(\cdot)/\alpha_1}$ and $w^{\beta_2} \in A_{p(\cdot)/s}$. Assume that $\alpha_1 \geq \beta_1$. Then the weight $|x|^{-a}$ satisfies these inclusions if

$$a(\alpha_1 - \beta_1) < \frac{\alpha_1 n}{p_+}, \quad a(s + \beta_1(p_0 - 1)) < \frac{sn}{p_+}.$$

Clearly, we get the same range for $a, a < n/p_+$, in each inequality if $\beta_1 = 0$, and if $\beta_1 \neq 0$ one of the ranges will be smaller than this. Therefore, to maximize the range of exponents we should take $\beta_1 = 0$.

When $\beta_1 = 0$ we then have $w^{\alpha_1} \in A_{p(\cdot)/\alpha_1}$ and $w^s \in A_{p(\cdot)/s}$. If $\alpha_1 = s$, then $s = 1$ and we get the single condition $w \in A_{p(\cdot)}$. If $\alpha_1 > s$, then $s < 1$ and so $\alpha_1 > 1$, and by Proposition 2.27 we get that $w^{\alpha_1} \in A_{p(\cdot)/\alpha_1}$ implies $w \in A_{p(\cdot)}$. If $\alpha_1 < s$, then $s > 1$, and we again get a condition stronger than $w \in A_{p(\cdot)}$. So we have that the choice $s = 1$ is in some sense optimal.

Proof of Theorem 2.11. For the proof we need a few propositions. The first gives the relationship between Muckenhoupt A_p weights and $A_{p,q}$ weights. It was first observed in [38]; the proof follows immediately from the definition.

Proposition 5.3. *Given $p, q, 1 \leq p < q < \infty$, suppose $w \in A_{p,q}$. Then $w^q \in A_r$ when $r = 1 + q/p'$.*

The next result is not strictly necessary to our proof, but we include it as it is the variable exponent version of Proposition 5.3.

Proposition 5.4. *Given $p(\cdot), q(\cdot) \in \mathcal{P}, 1 < p(x) \leq q(x) < \infty$, suppose there exists $\sigma > 1$ such that $\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{\sigma}$. Then $w \in A_{p(\cdot),q(\cdot)}$ if and only if $w^\sigma \in A_{q(\cdot)/\sigma}$.*

Proof. First note that $\sigma r'(\cdot) = p'(\cdot)$. Indeed, taking the reciprocal, we have

$$\frac{1}{\sigma r'(\cdot)} = \frac{1}{\sigma} - \frac{1}{\sigma r(\cdot)} = 1 - \frac{1}{\sigma'} - \frac{1}{q(\cdot)} = 1 - \frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{p'(\cdot)}.$$

The equivalence then follows by dilation and the definition of $A_{r(\cdot)}$ and $A_{p(\cdot),q(\cdot)}$:

$$\begin{aligned} &|B|^{-1} \|w^\sigma \chi_B\|_{r(\cdot)} \|w^{-\sigma} \chi_B\|_{r'(\cdot)} \\ &= |B|^{-1} \|w \chi_B\|_{q(\cdot)}^\sigma \|w^{-1} \chi_B\|_{p'(\cdot)}^\sigma = (|B|^{\frac{1}{\sigma}-1} \|w \chi_B\|_{q(\cdot)} \|w^{-1} \chi_B\|_{p'(\cdot)})^\sigma. \end{aligned}$$

□

To state the next result recall that given $p(\cdot) \in \mathcal{P}$, the modular is defined by

$$\rho_{p(\cdot)}(f) = \rho(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

In the case of constant exponents, the L^p norm and the modular differ only by an exponent. In the variable Lebesgue spaces their relationship is more subtle, as the next result shows. For a proof see [13, Prop. 2.21, Cor. 2.23].

Proposition 5.5. *Given $p(\cdot) \in \mathcal{P}$, suppose $p_+ < \infty$. Then:*

- (1) $\|f\|_{p(\cdot)} = 1$ if and only if $\rho(f) = 1$;
- (2) if $\rho(f) \leq C$, then $\|f\|_{L^{p(\cdot)}} \leq \max(C^{1/p_-}, C^{1/p_+})$;
- (3) if $\|f\|_{p(\cdot)} \leq C$, then $\rho(f) \leq \max(C^{p_+}, C^{p_-})$.

We can now prove Theorem 2.11. As we noted above, when $\sigma = 1$ Theorem 2.11 reduces to Theorem 2.7, so we will assume $\sigma > 1$. The proof when $p_0 = 1$ is more similar to that of Theorem 2.21, and so we will defer this case to below after the proof of Theorem 2.21. Here we will assume that $p_0 > 1$. We will actually prove the following more general proposition.

Proposition 5.6. *Let p_0, q_0, σ and exponents $p(\cdot), q(\cdot)$ be as in the statement of Theorem 2.11. Fix $\beta_1 \in \mathbb{R}$ and choose any s such that*

$$(5.6) \quad q_0 - q_- \left(\frac{q_0}{\sigma} - 1 \right) < s < \min(q_0, q_-).$$

Let $r_0 = q_0/s$, and define $\alpha_1 = s$ and $\beta_2 = s - \beta_1(1 - r_0)$. Then if w is a weight such that M is bounded on $L^{q(\cdot)/s}(w^{\alpha_1 - \beta_1})$ and $L^{(q(\cdot)/s)'}(w^{-\beta_2})$, we have that $\|fw\|_{q(\cdot)} \leq C \|gw\|_{p(\cdot)}$.

To prove Theorem 2.11, we take $\beta_1 = 0$ and $s = \sigma$. Since

$$1 - \frac{1}{\sigma} = \frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_-} - \frac{1}{q_-},$$

we have that the second inequality in (5.6) holds. The first inequality is equivalent to $\sigma^2 - (q_0 + q_-)\sigma + q_-q_0 > 0$, which follows from the second inequality. The requirement on the weight w reduces to M being bounded on $L^{q(\cdot)/\sigma}(w^\sigma)$ and $L^{(q(\cdot)/\sigma)'}(w^{-\sigma})$, or equivalently, $(q(\cdot)/\sigma, w^\sigma)$ is an M -pair.

Proof. The proof follows an outline similar to that of Theorem 2.7; we will concentrate on details that are different. Fix a pair $(f, g) \in \mathcal{F}$; as before we may assume without loss of generality that $0 < \|f\|_{L^{q(\cdot)}(w)}, \|g\|_{L^{p(\cdot)}(w)} < \infty$. Moreover, if (f, g) satisfies (2.7), then so does $(\lambda f, \lambda g)$ for any $\lambda > 0$, so without loss of generality we may assume that $\|g\|_{L^{p(\cdot)}(w)} = 1$. Then by Proposition 5.5 it will suffice to prove that $\|fw\|_{q(\cdot)} \leq C$.

Define

$$h_1 = \frac{f}{\|fw\|_{q(\cdot)}} + g^{\frac{p(\cdot)}{q(\cdot)}} w^{\frac{p(\cdot)}{q(\cdot)} - 1};$$

we claim that $\|h_1 w\|_{q(\cdot)} \leq C$. This follows from Proposition 5.5:

$$\rho_{q(\cdot)}(h_1 w) \leq 2^{q_+} \int_{\mathbb{R}^n} \left(\frac{f(x)w(x)}{\|fw\|_{q(\cdot)}} \right)^{q(x)} dx + 2^{q_+} \int_{\mathbb{R}^n} (g(x)w(x))^{p(x)} dx \leq 2^{q_++1}.$$

We again use Proposition 4.1 to define two operators H_1 and H_2 as in (5.2). Let $r_0 = q_0/s$, and fix s , $0 < s < \min(q_0, q_-)$. Then there exists $h_2 \in L^{(q(\cdot)/s)'}$, $\|h_2\|_{(q(\cdot)/s)'} = 1$, such that for any $\gamma > 0$,

$$\begin{aligned} \|fw\|_{q(\cdot)}^s &\leq C \int_{\mathbb{R}^n} f^s w^s h_2 dx \leq C \int_{\mathbb{R}^n} f^s H_1^\gamma H_1^{-\gamma} H_2 w^s dx \\ &\leq C \left(\int_{\mathbb{R}^n} f^{q_0} H_1^{-\gamma(q_0/s)} H_2 w^s dx \right)^{1/r_0} \left(\int_{\mathbb{R}^n} H_1^{\gamma r'_0} w^s H_2 dx \right)^{1/r'_0} = I_1^{1/r_0} \cdot I_2^{1/r'_0}. \end{aligned}$$

We start by finding conditions to insure that I_2 is uniformly bounded. Since $h_1 \in L^{q(\cdot)}(w)$ and $h_2 \in L^{(q(\cdot)/s)'}$, we require H_1 and H_2 to be bounded on these spaces. We apply Hölder's inequality with exponent $q(\cdot)/s$ to get

$$I_2 \leq C \|H_1^{\gamma(q_0/s)'} w^s\|_{q(\cdot)/s} \|H_2\|_{(q(\cdot)/s)'}$$

If we let $\gamma = \frac{s}{(q_0/s)^\gamma}$, then by dilation,

$$\begin{aligned} \|H_1^{\gamma r'_0} w^s\|_{q(\cdot)/s} &= \|H_1 w\|_{q(\cdot)}^s \leq 2^s \|h_1 w\|_{q(\cdot)}^s \leq C, \\ \|H_2\|_{(q(\cdot)/s)'} &\leq 2 \|h_1\|_{(q(\cdot)/s)'} = 2. \end{aligned}$$

For H_1 and H_2 to be bounded on these spaces, by Proposition 4.1 we must have that the maximal operator satisfies

$$M \text{ bounded on } L^{q(\cdot)/\alpha_1}(w^{\alpha_1 - \beta_1}) \text{ and } L^{(q(\cdot)/s)'/\alpha_2}(w^{-\beta_2}).$$

By Theorem 2.4, for these to hold we must have that

$$(5.7) \quad q_- > \alpha_1 \quad \text{and} \quad (q_+/s)' > \alpha_2.$$

It remains to estimate I_1 ; with our value of γ we now have that

$$I_1 = \int_{\mathbb{R}^n} f^{q_0} H_1^{-q_0/r'_0} H_2 w^s dx.$$

In order to apply (2.6) we need to show that I_1 is finite. However, this follows from Hölder's inequality and the above estimates for H_1 and H_2 :

$$\begin{aligned} I_1 &\leq \|f\|_{L^{q(\cdot)}(w)}^{q_0} \int_{\mathbb{R}^n} H_1^{q_0} H_1^{-q_0/r'_0} H_2 w^s dx \\ &= \|f\|_{L^{q(\cdot)}(w)}^{q_0} \int_{\mathbb{R}^n} H_1^s H_2 w^s dx \leq \|f\|_{L^{q(\cdot)}(w)}^{q_0} \|H_1^s w^s\|_{q(\cdot)/s} \|H_2\|_{(q(\cdot)/s)'} < \infty. \end{aligned}$$

To apply our hypothesis (2.6) we need the weight $w_0 = (H_1^{-\gamma(q_0/s)} H_2 w^s)^{1/q_0}$ to be in A_{p_0, q_0} , or equivalently by Proposition 5.3, $w^{q_0} = H_1^{-(q_0-s)} H_2 w^s \in A_{r_1}$, where

$$r_1 = 1 + \frac{q_0}{p'_0} = \frac{q_0}{\sigma}.$$

To apply reverse factorization we write

$$w^{q_0} = \left(H_1^{\frac{q_0-s}{r_1-1}} w^{\beta_1} \right)^{1-r_1} H_2 w^{s-\beta_1(1-r_1)}.$$

By Proposition 4.1 this gives the following constraints on α_j, β_j :

$$\alpha_1 = \frac{q_0 - s}{\frac{q_0}{\sigma} - 1}, \quad \beta_1 \in \mathbb{R}, \quad \alpha_2 = 1, \quad \beta_2 = s - \beta_1(1 - q_0/\sigma).$$

If we combine these with the constraints in (5.7) we see that the second one there always holds and that the first one holds if

$$s > q_0 - q_- \left(\frac{q_0}{\sigma} - 1 \right).$$

We can now apply (2.6): by the definition of h_1 and by Hölder’s inequality with respect to the undetermined exponent $\alpha(\cdot)$, we get

$$\begin{aligned} I_1^{1/q_0} &\leq C \left(\int_{\mathbb{R}^n} g^{p_0} [H_1^{-q_0/r'_0} w^s H_2]^{p_0/q_0} dx \right)^{1/p_0} \\ &\leq C \left(\int_{\mathbb{R}^n} \left(h_1^{\frac{q(\cdot)}{p(\cdot)}} w^{\frac{q(\cdot)}{p(\cdot)} - 1} \right)^{p_0} H_1^{-p_0/r'_0} H_2^{p_0/q_0} w^{sp_0/q_0} dx \right)^{1/p_0} \\ &\leq C \left(\int_{\mathbb{R}^n} H_1^{p_0(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0})} H_2^{p_0/q_0} w^{p_0(\frac{s}{q_0} + \frac{q(\cdot)}{p(\cdot)} - 1)} dx \right)^{1/p_0} \\ &\leq C \|H_1\|_{\alpha'(\cdot)}^{p_0(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0})} \|w\|_{\alpha'(\cdot)}^{p_0(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0})} \|H_2\|_{\alpha(\cdot)}^{p_0/q_0} \|w\|_{\alpha(\cdot)}^{1/p_0} \\ &= C J_1^{1/p_0} J_2^{1/p_0}. \end{aligned}$$

If we let $\alpha(\cdot) = \frac{q_0(q(\cdot)/s)'}{p_0}$, then by dilation J_2 is uniformly bounded. To show that J_1 is uniformly bounded we first note that

$$p_0 \left(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0} \right) \alpha'(\cdot) = q(\cdot).$$

(This is given without proof in the constant case in [19, Section 3.5]. It follows by a tedious but straightforward computation. Though r_0 depends on s , the argument only uses the fact that $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{p_0} - \frac{1}{q_0}$ and does not depend on the value of s .) Given this, then

$$\rho_{\alpha'(\cdot)}(H_1^{p_0(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0})} w^{p_0(\frac{q(\cdot)}{p(\cdot)} - \frac{1}{r'_0})}) = \int_{\mathbb{R}^n} H_1^{q(\cdot)} w^{q(\cdot)} dx = \rho_{q(\cdot)}(H_1 w).$$

If we apply Proposition 5.5 twice, since $\|H_1\|_{L^{q(\cdot)}(w)} \leq 2\|h_1\|_{L^{q(\cdot)}(w)}$ is uniformly bounded, $\rho_{q(\cdot)}(H_1 w)$ is as well, and hence, J_1 is uniformly bounded. This completes the proof. □

Proof of Theorem 2.14. For the proof we will need a lemma due to Johnson and Neugebauer [34].

Lemma 5.7. *Given a weight w , then $w \in A_p \cap RH_s$, $1 < p, s < \infty$, if and only if $w^s \in A_\tau$, where $\tau = s(p - 1) + 1$.*

We again prove a more general result.

Proposition 5.8. *Given that the hypotheses of Theorem 2.14 hold, suppose $p(\cdot) \in LH$ with $q_- < p_- \leq p_+ < q_+$. Then there exists p_* , $q_- < p_* < q_+$ and $s > 0$ such that*

$$(5.8) \quad \max \left(p_- - p_* \left(\frac{p_-}{q_-} - 1 \right), \frac{p_* p_+}{q_+} \right) < s < \min(p_-, p_*).$$

Define

$$\tau_0 = \left(\frac{q_+}{p_*}\right)' \left(\frac{p_*}{q_-} - 1\right).$$

Let $\beta_1 \in \mathbb{R}$ be any constant and define

$$\alpha_1 = q_- \left(\frac{p_* - s}{p_* - q_-}\right), \quad \alpha_2 = \left(\frac{q_+}{p_*}\right)', \quad \beta_2 = s \left(\frac{q_+}{p_*}\right)' - \beta_1(1 - \tau_0).$$

Then for any weight w such that

$$(5.9) \quad w^{\alpha_1 - \beta_1} \in A_{p(\cdot)/\alpha_1} \quad \text{and} \quad w^{-\beta_2} \in A_{(p(\cdot)/s)'/\alpha_2},$$

we have that

$$\|f\|_{L^{p(\cdot)}(w)} \leq C \|g\|_{L^{p(\cdot)}(w)}, \quad (f, g) \in \mathcal{F}.$$

Remark 5.9. It will follow from the proof that the values of p_* and s are not unique. We will also see that the $A_{p(\cdot)}$ conditions in (5.9) are well defined.

To prove Theorem 2.14, note first that if we take $w = 1$, then (5.9) holds since $p(\cdot) \in LH$ and $p_- > 1$ implies $p(\cdot)$ has the K_0 condition (see Corollary 4.50 in [14]), and so we get the unweighted inequality (2.9).

To prove the weighted norm inequality (2.10), let p_* and s be any values satisfying (5.8). We want $\beta_2 = 0$ so that the second condition in (5.9) always holds. This is the case if we let

$$\beta_1 = \frac{s(q_+/p_*)'}{1 - \tau_0} = \frac{sq_-}{q_- - p_*} = -\frac{s\sigma}{p_*} < 0,$$

where $\sigma = \frac{p_*q_-}{p_* - q_-}$. Then $\alpha_1 - \beta_1 = \sigma$, and if we let $c = 1 - \frac{s}{p_*}$, the first condition in (5.9) reduces to $w^\sigma \in A_{\frac{p(\cdot)}{c\sigma}}$.

Proof. Fix an exponent $p(\cdot) \in LH$, $q_- < p_- \leq p_+ < q_+$, and fix a pair $(f, g) \in \mathcal{F}$. As before, without loss of generality we may assume that $0 < \|f\|_{L^{p(\cdot)}(w)}, \|g\|_{L^{p(\cdot)}(w)} < \infty$. Define $h_1 \in L^{p(\cdot)}(w)$, $\|h_1\|_{L^{p(\cdot)}(w)} \leq 2$, by

$$h_1 = \frac{f}{\|f\|_{L^{p(\cdot)}(w)}} + \frac{g}{\|g\|_{L^{p(\cdot)}(w)}}.$$

We will use Proposition 4.1 to define two operators H_1 and H_2 as in (5.2). By dilation and duality, there exists $h_2 \in L^{(p(\cdot)/s)'}$, $\|h_2\|_{(p(\cdot)/s)'} = 1$, such that

$$\begin{aligned} \|fw\|_{p(\cdot)}^s &= \|f^s w^s\|_{p(\cdot)/s} \\ &\leq C \int_{\mathbb{R}^n} f(x)^s h_2(x) w^s dx \leq C \int_{\mathbb{R}^n} f(x)^s H_1^{-\gamma} H_1^\gamma H_2 w^s dx \\ &\leq C \left(\int_{\mathbb{R}^n} f^{p_0} H_1^{-\gamma r_0} H_2 w^s dx \right)^{1/r_0} \left(\int_{\mathbb{R}^n} H_1^{\gamma r'_0} H_2 w^s dx \right)^{1/r'_0} = C I_1^{1/r_0} I_2^{1/r'_0}, \end{aligned}$$

where $r_0 = p_0/s$.

We first show that I_2 is uniformly bounded. As in the proof of Theorem 2.7, we want H_1 to be bounded on $L^{p(\cdot)}(w)$ and H_2 to be bounded on $L^{(p(\cdot)/s)'}$. Then by Hölder's inequality and dilation,

$$\begin{aligned} I_2 &\leq C \|H_1^{\gamma r'_0} w^s\|_{p(\cdot)/s} \|H_2\|_{(p(\cdot)/s)'} \\ &\leq C \|H_1 w\|_{\gamma r'_0 p(\cdot)/s}^{\gamma r'_0} \|H_2\|_{(p(\cdot)/s)'} \leq C \|h_1 w\|_{\gamma r'_0 p(\cdot)/s}^{\gamma r'_0} \|h_2\|_{(p(\cdot)/s)'} \end{aligned}$$

The last term will be uniformly bounded if we let $\gamma = s/r'_0$. For H_1 and H_2 to be bounded on these spaces, by Proposition 4.1 we must have that

$$M \text{ bounded on } L^{p(\cdot)/\alpha_1}(w^{\alpha_1-\beta_1}) \text{ and } L^{(p(\cdot)/s)'/\alpha_2}(w^{-\beta_2}).$$

By Theorem 2.4, since $p(\cdot) \in LH$, this will be the case if

$$(5.10) \quad p_- > \alpha_1, \quad (p_+/s)' > \alpha_2.$$

To bound I_1 , we want to apply our hypothesis (2.8); to do so we need to show that it is finite. But by our assumptions on H_1 and H_2 and the definition of h_1 , we have that

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n} f^{p_0} H_1^{-(p_0-s)} H_2 w^s dx \leq \int_{\mathbb{R}^n} (\|fw\|_{p(\cdot)} H_1)^{p_0} H_1^{-(p_0-s)} H_2 w^s dx \\ &= \|fw\|_{p(\cdot)}^{p_0} \int_{\mathbb{R}^n} H_1^s H_2 w^s dx \leq C \|fw\|_{p(\cdot)}^{p_0} \|H_1 w\|_{p(\cdot)}^s \|H_2\|_{(p(\cdot)/s)'} < \infty. \end{aligned}$$

Assume for the moment that $w_0 = H_1^{-(p_0-s)} H_2 w^s \in A_{p_0/q_-} \cap RH_{(q_+/p_0)'}$. Then by (2.8) and arguing as we did in the previous inequality, we get that

$$\begin{aligned} \int_{\mathbb{R}^n} f^{p_0} H_1^{-(p_0-s)} H_2 w^s dx &\leq C \int_{\mathbb{R}^n} g^{p_0} H_1^{-(p_0-s)} H_2 w^s dx \\ &\leq C \|gw\|_{p(\cdot)}^{p_0} \int_{\mathbb{R}^n} H_1^{p_0} H_1^{-(p_0-s)} H_2 w^s dx \leq C \|gw\|_{p(\cdot)}^{p_0}. \end{aligned}$$

If we combine this with the previous estimates we get the desired weighted norm inequality.

We can complete the proof if our various assumptions hold. However, as we will see, this may not be possible with our given value of p_0 , and so we will introduce a new parameter p_* . We first consider the weight w_0 . We want $w_0 = H_1^{-(p_0-s)} H_2 w^s$ to be in $A_{p_0/q_-} \cap RH_{(q_+/p_0)'}$, which by Lemma 5.7 is equivalent to $w_0^{(q_+/p_0)'} \in A_{\tau_0}$, where $\tau_0 = \left(\frac{q_+}{p_0}\right)' \left(\frac{p_0}{q_-} - 1\right) + 1$. To apply reverse factorization, we rewrite w_0 as

$$\begin{aligned} w_0^{(q_+/p_0)'} &= \left[H_1^{-(p_0-s)} H_2 w^s \right]^{(q_+/p_0)'} \\ &= \left[H_1^{(q_-) \frac{p_0-s}{p_0-q_-}} w^{\beta_1} \right]^{1-\tau_0} H_2^{(q_+/p_0)'} w^{s(q_+/p_0)' - \beta_1(1-\tau_0)}. \end{aligned}$$

Therefore, by Proposition 4.1 we must have that

$$\alpha_1 = q_- \left(\frac{p_0 - s}{p_0 - q_-} \right), \quad \beta_1 \in \mathbb{R}, \quad \alpha_2 = \left(\frac{q_+}{p_0} \right)', \quad \beta_2 = s \left(\frac{q_+}{p_0} \right)' - \beta_1(1 - \tau_0).$$

If we combine this with the first constraint in (5.10) we see that we need

$$\frac{p_-}{q_-} \left(\frac{p_0 - q_-}{p_0 - s} \right) > 1;$$

equivalently, we must have that

$$s > p_- - p_0 \left(\frac{p_-}{q_-} - 1 \right) > 0.$$

Similarly, the second constraint in (5.10) implies that we also need

$$s > \frac{p_0 p_+}{q_+}.$$

However, it need not be the case that we can find such an s that also satisfies $s < \min(p_-, p_0)$. We can overcome this problem by changing the value p_0 . By limited range extrapolation in the constant exponent case, Theorem 2.13, we have that our hypothesis (2.8) holds with p_0 replaced by any p_* , $q_- < p_* < q_+$ provided that $w_0 \in A_{p_*/q_-} \cap RH_{(q_+/p_*)}$.

We can, therefore, repeat the entire argument above with p_0 replaced by p_* , and we will get our desired conclusion if we can find p_* and $s > 0$ such that (5.8) holds. (The constants $\alpha_j, \beta_j, \tau_0$ are also redefined as in the statement of Proposition 5.8.) This is equivalent to the following four inequalities being true:

$$\begin{aligned}
 (1) \quad p_* &> \frac{p_* p_+}{q_+}, & (3) \quad p_- &> p_- - p_* \left(\frac{p_-}{q_-} - 1 \right), \\
 (2) \quad p_* &> p_- - p_* \left(\frac{p_-}{q_-} - 1 \right), & (4) \quad p_- &> \frac{p_* p_+}{q_+}.
 \end{aligned}$$

Inequalities (1) and (3) always hold. Inequality (2) is equivalent to $p_- \left(\frac{p_*}{q_-} \right) > p_-$, which is always true. Inequality (4) holds if p_* is such that

$$q_- < p_* < \frac{q_+}{p_+} p_- < q_+;$$

such a p_* exists since $\frac{p_+}{p_-} < \frac{q_+}{q_-}$. Therefore, we can find the desired value of p_* and s , and this completes the proof of Proposition 5.8. \square

Remark 5.10. The limited-range extrapolation theorem with constant exponents does not follow from Theorem 2.14. However, it does follow from Proposition 5.8 by choosing a different set of parameters. We need to prove that if we let $p(\cdot) = p$, $q_- < p < q_+$, then the norm inequality $\|fw\|_p \leq C\|gw\|_p$ holds provided that the weight $w^p \in A_{p/q_-} \cap RH_{(q_+/p)}$, which by Lemma 5.7 is equivalent to $w^{p(q_+/p)'} \in A_{\tau_p}$, where $\tau_p = \left(\frac{q_+}{p}\right)' \left(\frac{p}{q_-} - 1\right) + 1$. Restating this condition in terms of our variable weight condition, we need that the norm inequality holds provided w satisfies

$$(5.11) \quad w^{p(q_+/p)' / \tau_p} \in A_{\tau_p}^{var}.$$

(See the comments just before Proposition 2.27 for this notation.) For the two conditions in (5.9) to reduce to this one requirement, we must have that:

- (1) The first condition must be the same as (5.11). This is the case if $\alpha_1 - \beta_1 = p(q_+/p)' / \tau_p$, and $p/\alpha_1 = \tau_p$, or $\alpha_1 = p/\tau_p$ and $\beta_1 = \frac{p}{\tau_p} (1 - (q_+/p)')$. Therefore, s and β_2 must satisfy

$$s = \frac{p}{\tau_p} \left(1 - \frac{p_0}{q_-} \right) + p_0, \quad \beta_2 = s(q_+/p_0)' - \beta_1(1 - \tau_{p_0}).$$

- (2) The second condition must be the ‘dual’ of (5.11), i.e., $w^{-p(q_+/p)' / \tau_p} \in A_{\tau_p'}^{var}$.

Thus we must have that

$$\frac{(p/s)'}{\alpha_2} = \tau_p', \quad \beta_2 = \frac{p}{\tau_p} (q_+/p)'.$$

A lengthy but straightforward computation shows that these two pairs of values for s and β_2 are exactly the same.

Finally, we also need to show that s satisfies (5.8); that is, with $p_- = p = p_+$, if we have

$$\max \left(p - p_0 \left(\frac{p}{q_-} - 1 \right), \frac{p_0 p}{q_+} \right) < s < \min(p, p_0).$$

This actually follows from the above computations. First note that by the first condition in (1), we have $s < p_0$ since $p_0 > q_-$. By the first condition in (2) we must have $p/s > 1$ for $(p/s)'$ to be defined. To prove the lower inequalities, it is easier to look back to the proof to see where these come from. The first comes from the requirement that $p/\alpha_1 > 1$, which follows from the fact that in this case we have $p/\alpha_1 = \tau_p > 1$. The second condition comes from the requirement that $(p/s)'/\alpha_2 > 1$, which comes from the fact that this equal to τ'_p .

Remark 5.11. The computations in the previous remark also show why our extrapolation theorem is stated in a way that is quite different from the constant exponent case. In our reduction we need to choose the constants so that the two conditions on the weight in (5.9) are actually the same, i.e., $\alpha_1 - \beta_1 = \beta_2$ and $(p(\cdot)/\alpha_1)' = (p(\cdot)/s)'/\alpha_2$. But this last equality reduces to

$$p(\cdot) = \frac{s\alpha_2 - \alpha_1}{\alpha_2 - 1} = \frac{s(q_+ - q_-) + q_-(p_* - q_+)}{p_* - q_-},$$

and this can only hold if $p(\cdot) = p$ is a constant. However, in obtaining (2.10), we did have two separate conditions from (5.9), namely $w^\sigma \in A_{\frac{p(\cdot)}{c\sigma}}$ and $1 \in A_{(p(\cdot)/s)'/\alpha_2}$, which always hold. It would be of interest to find a different version of Theorem 2.14 that does reduce immediately to the constant exponent theorem.

Proof of Corollary 2.19. Given $\delta \in (0, 1]$ we can restate our hypothesis (2.11) as follows:

$$\int_{\mathbb{R}^n} f(x)^2 w_0(x) dx \leq c \int_{\mathbb{R}^n} g(x)^2 w_0(x) dx,$$

for all weights w_0 such that $w_0^{1/\delta} \in A_2$. By Lemma 5.7 this is equivalent to $w_0 \in A_{2/q_-} \cap RH_{(q_+/2)'}$, where $q_- = \frac{2}{1+\delta}$ and $q_+ = \frac{2}{1-\delta}$. This is the hypothesis (2.8) of Theorem 2.14, and applying this theorem, we get (2.13) and (2.14) for all $p(\cdot)$ satisfying (2.12).

Proof of Theorem 2.21 and Theorem 2.11 when $p_0 = 1$. To prove Theorem 2.21 we need to modify the general approach outlined in Section 4. To see why, first consider the proof of Theorem 2.7. If we take $p_0 = 1$, then the proof fails, because in order to apply Hölder’s inequality we require $s < 1$, but later we need the constraint $s > 1$ for the maximal operator to be bounded on $L^{p(\cdot)/\alpha_1}(w^{\alpha_1 - \beta_1})$. This suggests that we should not use Hölder’s inequality and not introduce the operator H_1 (which leads to this condition on the boundedness of the maximal operator). We can still dualize if we take $s = 1$, and this gives us the correct exponent to apply our hypothesis. We can then introduce the operator H_2 and argue as before to determine the appropriate values for α_2 and β_2 .

This same approach works for general p_0 . Fix $p(\cdot) \in \mathcal{P}_0$, $p_- \geq p_0$, and $(f, g) \in \mathcal{F}$. As before, we may assume without loss of generality that $0 < \|f\|_{L^{p(\cdot)}(w)}$ and $\|g\|_{L^{p(\cdot)}(w)} < \infty$. We will use Proposition 4.1 to define an operator $H_2 = \mathcal{R}_2(h_2^{\alpha_2} w^{\beta_2})^{1/\alpha_2} w^{-\beta_2/\alpha_2}$. By dilation and duality, there exists $h_2 \in L^{(p(\cdot)/p_0)'}$, $\|h_2\|_{(p(\cdot)/p_0)'} = 1$, such that

$$\|fw\|_{p(\cdot)}^{p_0} \leq C \int_{\mathbb{R}^n} f^{p_0} h_2 w^{p_0} dx \leq C \int_{\mathbb{R}^n} f^{p_0} H_2 w^{p_0} dx.$$

To apply our hypothesis (2.15) we need the right-hand term to be bounded. Since $h_2 \in L^{(p(\cdot)/p_0)'}$, if we assume that H_2 is bounded on the same space, then by Hölder's inequality and dilation we have that

$$\int_{\mathbb{R}^n} f^{p_0} H_2 w^{p_0} dx \leq \|fw\|_{p(\cdot)}^{p_0} \|H_2\|_{(p(\cdot)/p_0)'} \leq 2\|fw\|_{p(\cdot)} \|h_2\|_{(p(\cdot)/p_0)'} < \infty.$$

For H_2 to be so bounded, we need M to be bounded on $L^{(p(\cdot)/p_0)'/\alpha_2}(w^{-\beta_2})$. Furthermore, to apply our hypothesis we also need $H_2 w^{p_0} \in A_1$, so we must have that $\alpha_2 = 1$ and $\beta_2 = p_0$.

Therefore, if M is bounded on $L^{(p(\cdot)/p_0)'}(w^{-p_0})$, we have that

$$\int_{\mathbb{R}^n} f^{p_0} H_2 w^{p_0} dx \leq C \int_{\mathbb{R}^n} g^{p_0} H_2 w^{p_0} dx \leq C \|gw\|_{p(\cdot)}^{p_0} \|H_2\|_{(p(\cdot)/p_0)'} \leq C \|gw\|_{p(\cdot)}^{p_0}.$$

This completes the proof.

Remark 5.12. In this endpoint case we do not have any flexibility in choosing our parameters: at each stage our choice is completely determined by the requirements of the proof.

The proof of Theorem 2.11 when $p_0 = 1$ is nearly identical to the proof of Theorem 2.21 and can be motivated by exactly the same analysis as we made of the proof of Theorem 2.7. If we apply dilation and duality with p_0 replaced by q_0 , we get

$$\|fw\|_{q(\cdot)}^{q_0} \leq C \int_{\mathbb{R}^n} f^{q_0} H_2 w^{q_0} dx.$$

Checking the required conditions we see that we can apply our hypothesis if $H_2 w^{q_0} \in A_1$, which is equivalent to $H_2^{1/q_0} w \in A_{1, q_0}$, and this follows if the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(w^{-q_0})$. The rest of the proof now continues exactly as before.

Proof of Theorem 2.25 and Proposition 2.27. We could prove Theorem 2.25 by an analysis similar to that used to prove Theorem 2.21. However, we can also derive it directly from this result using the connection between A_1 and A_∞ extrapolation (cf. [19, Proposition 3.20]). Fix $p(\cdot)$ and $s \leq p_-$ as in our hypotheses. Then by Theorem 2.24, we have that (2.16) holds with p_0 replaced by s and for any $w_0 \in A_\infty$. In particular, we can take $w_0 \in A_1$, and this gives us the hypothesis (2.15) in Theorem 2.21 with p_0 replaced by s . The desired conclusion now follows from this result.

Finally, we prove Proposition 2.27. Fix a ball B . Define the exponent function $r(\cdot) = \frac{1}{1-s}$. Then it is immediate that

$$\frac{1}{(p(\cdot)/s)'} = \frac{s}{p'(\cdot)} + \frac{1}{r(\cdot)}.$$

Therefore, by dilation and the generalized Hölder's inequality [13, Corollary 2.28],

$$\begin{aligned} |B|^{-1} \|w^s \chi_B\|_{p(\cdot)/s} \|w^{-s} \chi_B\|_{(p(\cdot)/s)'} &\leq |B|^{-1} \|w \chi_B\|_{p(\cdot)}^s \|w^{-s} \chi_B\|_{p'(\cdot)/s} \|\chi_B\|_{r(\cdot)} \\ &= |B|^{-1} \|w \chi_B\|_{p(\cdot)}^s \|w^{-1} \chi_B\|_{p'(\cdot)}^s |B|^{1-s} \leq [w]_{A_{p(\cdot)}}^s. \end{aligned}$$

Since this is true for all B , $w^s \in A_{p(\cdot)/s}$.

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