

PRESERVATION OF DEPTH IN THE LOCAL GEOMETRIC LANGLANDS CORRESPONDENCE

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To Volodya Drinfeld, on the occasion of his sixtieth birthday

ABSTRACT. It is expected that, under mild conditions, the local Langlands correspondence preserves depths of representations. In this article, we formulate a conjectural geometrization of this expectation. We prove half of this conjecture by showing that the depth of a categorical representation of the loop group is greater than or equal to the depth of its underlying geometric Langlands parameter. A key ingredient of our proof is a new definition of the slope of a meromorphic connection, a definition which uses opers.

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1. INTRODUCTION

Let F be a local non-Archimedean field such as \mathbb{Q}_p or $\mathbb{F}_q((t))$ and G a connected reductive group over F . The local Langlands correspondence is a conjectural relationship between two types of data. The first data are, roughly speaking, equivalence classes of homomorphisms from the absolute Galois group of F to G , the Langlands dual group of G . These are sometimes called the *local Langlands parameters*. The second data are the isomorphism classes of smooth irreducible representations of $G(F)$.

Using the upper numbering ramification groups [35], one can define the notion of depth for local Langlands parameters. On the other hand, the notion of depth for smooth representations of $G(F)$ was defined by Moy and Prasad [31], using the Bruhat-Tits Theory [8, 9]. It is expected that in most circumstances, the local Langlands correspondence preserves depth; cf. [37]. This is established in several cases; cf. [2]. The purpose of this paper is to examine the *geometric* analogue of this expectation. To start, we give a leisurely introduction to Frenkel and Gaitsgory's proposal for geometrizing the local Langlands correspondence.

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1.1. Local geometric Langlands. Henceforth, let G be a simple complex group of adjoint type and $\mathfrak{g} = \text{Lie}(G)$. Let \check{G} and $\check{\mathfrak{g}}$ denote the Langlands dual objects. Let $G((t))$ and $\mathfrak{g}((t))$ denote the formal loop group and formal loop algebra. Let $\mathcal{K} = \mathbb{C}((t))$ and let $\mathcal{D}^\times = \text{Spec}(\mathbb{C}((t)))$ denote the punctured formal disk.

It has been known for a long time that there is a deep and mysterious analogy between Galois representations and local systems; cf. appendix of [28]. By definition, a \check{G} -local system on \mathcal{D}^\times is a $\check{G}(\mathcal{K})$ -gauge equivalence class of operators of the form

$$(1) \quad \nabla = \partial_t + A, \quad A \in \check{\mathfrak{g}}(\mathcal{K}) = \check{\mathfrak{g}} \otimes \mathbb{C}((t)).$$

The gauge action of $g \in \check{G}(\mathcal{K})$ is defined by

$$(2) \quad g \cdot (\partial_t + A) = \partial_t + \text{Ad}(g)A - (dg)g^{-1}.$$

We let $\text{Loc}_{\check{G}}(\mathcal{D}^\times)$ denote the set of isomorphism classes of \check{G} -local systems on \mathcal{D}^\times . It is widely accepted that these are the geometric analogues of the local Langlands parameters.

On the other side of the Langlands correspondence, Frenkel and Gaitsgory propose that the geometric analogue of smooth representations should be *categorical representations* of $G((t))$.¹ We refer the reader to [17, §20] and [16, §1.3] for this notion. A good toy model to keep in mind, for our purposes, is the action of G on the category of \mathfrak{g} -modules. In more detail, the group G acts on its Lie algebra \mathfrak{g} via the adjoint action. Therefore, every $g \in G$ acts on the category $\mathfrak{g} - \text{mod}$ by sending a representation $\mathfrak{g} \rightarrow \text{End}(V)$ to the composition

$$\mathfrak{g} \xrightarrow{\text{Ad}(g)} \mathfrak{g} \rightarrow \text{End}(V).$$

This is an example of a categorical action. It is also possible to “decompose” this categorical representation. Namely, let $\mathcal{Z}(\mathfrak{g})$ denote the centre of the universal enveloping algebra of \mathfrak{g} . For every character χ of $\mathcal{Z}(\mathfrak{g})$, let $\mathfrak{g} - \text{mod}_\chi$ denote the full subcategory of $\mathfrak{g} - \text{mod}$ consisting of those modules on which $\mathcal{Z}(\mathfrak{g})$ acts by the character χ . Then $\mathfrak{g} - \text{mod}_\chi$ is preserved under the action of G ; thus, it is a subrepresentation of $\mathfrak{g} - \text{mod}$.

We are interested, however, in categorical representations of the *loop* group; thus, we should look at the action of $G((t))$ on the category of $\mathfrak{g}((t)) = \mathfrak{g} \otimes \mathbb{C}((t))$ -modules. Actually, it is fruitful to consider not the loop algebra itself but its universal central extension known as the *affine Kac-Moody algebra* $\widehat{\mathfrak{g}}$. Recall that representations of $\widehat{\mathfrak{g}}$ have a parameter, an invariant bilinear form κ on \mathfrak{g} , which is called the *level*. We let $\widehat{\mathfrak{g}}_\kappa - \text{mod}$ denote the category (smooth) representations of $\widehat{\mathfrak{g}}$ at the κ . By a similar reasoning as in the previous paragraph, $\widehat{\mathfrak{g}}_\kappa - \text{mod}$ carries a natural action of the loop group $G((t))$.

Representations of $\widehat{\mathfrak{g}}$ corresponding to the bilinear form $\kappa = c$ which is equal to minus one half of the Killing form are called representations at the *critical level*. The advantage of the critical level is that here the (completed) universal enveloping algebra of $\widehat{\mathfrak{g}}$ acquires a large centre. Thus, we may “decompose” the representation $\widehat{\mathfrak{g}}_c - \text{mod}$ using the characters of the centre.

More precisely, according to a remarkable theorem of Feigin and Frenkel [15], the centre \mathcal{Z}_c of the completed universal enveloping algebra of the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ at the critical level identifies canonically with the algebra of functions on

¹There are other proposals for geometrizing the local Langlands correspondence; cf. [4], [32].

the ind-scheme $\mathrm{Op}_{\check{G}}(\mathcal{D}^\times)$ of \check{G} -opers over the formal punctured disk. Thus, every point $\chi \in \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) = \mathrm{Spec}(\mathcal{Z}_c)$ defines a character of the centre, and therefore a categorical representation $\widehat{\mathfrak{g}}_c - \mathrm{mod}_\chi$ of the loop group. These are supposed to be the categorical analogues of smooth representations of p -adic groups. In particular, the Grothendieck group of $\widehat{\mathfrak{g}}_c - \mathrm{mod}_\chi$ should “look like” a smooth representation.²

Having defined the geometric analogue of Langlands parameters and smooth representations, let us now relate them to each other. To this end, we recall that opers are local systems plus additional data (see §1.2 and §2 for more information on opers). Hence, one has a canonical forgetful map

$$(3) \quad p : \mathrm{Op}_{\check{G}}(\mathcal{D}^\times) \rightarrow \mathrm{Loc}_{\check{G}}(\mathcal{D}^\times).$$

The main result of Frenkel and Zhu [22] states that this map is surjective.

It follows that for every geometric Langlands parameter $\sigma \in \mathrm{Loc}_{\check{G}}(\mathcal{D}^\times)$, one has, in principle, many categorical representations of $G((t))$, namely, the representations $\widehat{\mathfrak{g}}_c - \mathrm{mod}_\chi$ where $\chi \in p^{-1}(\sigma)$. Frenkel and Gaitsgory conjecture that these categorical representations are equivalent. Thus, given σ , there exists a canonical category \mathcal{C}_σ equipped with the action of $G((t))$; moreover, \mathcal{C}_σ is equivalent to $\widehat{\mathfrak{g}}_c - \mathrm{mod}_\chi$ for every $\chi \in p^{-1}(\sigma)$. This is Frenkel and Gaitsgory’s conjectural geometrization of the local Langlands correspondence [17].³

In a series of papers [17–20], Frenkel and Gaitsgory examined the unramified and tamely ramified parts of the local geometric Langlands correspondence. These cases correspond to σ being trivial or regular singular with unipotent monodromy. As far as we know, little is known about the correspondence for general σ .⁴ We hope that the point of view of this text will be useful for further investigations of these cases.

1.2. Main conjecture. We now explain how to geometrize the expectation that the local Langlands correspondence preserves depth. It turns out that \check{G} -local systems have a numerical invariant, called *slope*, which is a natural candidate for the geometric analogue of depth of Langlands parameters. This notion goes back to the work of Katz and Deligne in the early 1970s. We refer the reader to Section 2 for a thorough discussion of various definitions of slope and the history of this invariant. For now, we give a definition of slope which we learned from [21]. A \check{G} -local system σ has slope a/b if the following holds. Pass to the extension given by adjoining the b^{th} root of t : $u^b = t$. Then the local system, written using the parameter u in the extension, should have in its gauge equivalence class a representative which has a pole of order $a + 1$, and its top polar part should not be nilpotent. We denote the slope of σ by $s(\sigma)$.

On the other side of the Langlands correspondence, it is straightforward to generalise Moy and Prasad’s definition of depth to the categorical setting. Let us first recall the classical definition. In [8, 9], Bruhat and Tits associated to G a combinatorial object known as the *Bruhat-Tits building* $\mathcal{B}(G)$. For every $x \in \mathcal{B}(G)$ and $r \in \mathbb{R}_{\geq 0}$, Moy and Prasad [31] defined a subgroup $G_{x,r^+} \subset G(F)$. In addition, they

²According to [17], these categorical representations also have descriptions in terms of (twisted) D -modules on generalised flag varieties. We will not use this alternative description.

³Frenkel and Gaitsgory make their conjecture precise by exploiting connections with the global geometric Langlands correspondence. We will not consider this global characterisation.

⁴For some progress in the irregular case, cf. [23], [24], [25].

defined the depth of a smooth representation of $G(F)$ by

$$\text{depth}(V) := \inf\{r \in \mathbb{R}_{\geq 0} \mid \exists x \in \mathcal{B}(G) \text{ such that } V^{G_{x,r^+}} \text{ is non-trivial}\},$$

where $V^{G_{x,r^+}} \subseteq V$ denotes the subspace V consisting of vectors fixed under G_{x,r^+} .

It is easy to categorify the above definition. First of all, thanks to [36], one knows that G_{x,r^+} comes equipped with a canonical *smooth model*. In particular, this means that one can realise G_{x,r^+} as the group of \mathbb{C} -points of a pro-algebraic group over \mathbb{C} . Now if \mathcal{C} is a categorical representation of the loop group, we define the depth \mathcal{C} by

$$(4) \quad d(\mathcal{C}) := \inf\{r \in \mathbb{R}_{\geq 0} \mid \exists x \in \mathcal{B}(G) \text{ such that } \mathcal{C}^{G_{x,r^+}} \text{ is non-trivial}\}.$$

Here $\mathcal{C}^{G_{x,r^+}}$ denotes the “category of G_{x,r^+} strongly equivariant objects” of \mathcal{C} ; cf. [17, §20], [16, §10].

In view of the above discussions, the following conjecture is the geometric analogue of the expectation that the local Langlands correspondence preserves depth.

Conjecture 1. *Let $\sigma \in \text{Loc}_{\check{G}}(\mathcal{D}^\times)$ and let $\chi \in \text{Op}_{\check{G}}(\mathcal{D}^\times)$ be an oper whose underlying local system is σ (i.e., $p(\chi) = \sigma$). Then*

$$s(\sigma) = d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi).$$

1.3. Main results. In this paper, we prove one-half of the above conjecture. The key ingredient is a new definition of the slope of a local system suggested by Xinwen Zhu, a definition which usesopers. For now, the only fact we need to know aboutopers is that one can represent an oper $\chi \in \text{Op}_{\check{G}}(\mathcal{D}^\times)$ with an ordered ℓ -tuple (v_1, \dots, v_ℓ) where $v_i \in \mathbb{C}((t))$ and ℓ is the rank of G .⁵ Let us write $v_i = t^{-n_i} \cdot h_i$ where $h_i \in \mathbb{C}[[t]]^\times$ if $v_i \neq 0$ and set $n_i = -\infty$ if $v_i = 0$. Let $d_i, i = 1, \dots, \ell$, denote the exponents of the Lie algebra $\widehat{\mathfrak{g}}$.

Definition 2. The slope of χ is defined by

$$s(\chi) := \sup\{0, \sup_{i=1, \dots, \ell} \{-\frac{n_i}{d_i + 1} - 1\}\}.$$

We let $\text{Op}_{\check{G}}^r \subset \text{Op}_{\check{G}}(\mathcal{D}^\times)$ denote the subscheme ofopers of slope less than or equal to r . Note that if n is a positive integer, then $\text{Op}_{\check{G}}^n$ equals the space $\text{Op}_{\check{G}}^{\text{ord}^n}$ ofopers on \mathcal{D} with singularity less than or equal to n ; cf. [5, §3.7.7], [17]. The following result states that the slope of an oper equals the slope of its underlying connection. In particular, it shows that Definition 2 is well-defined.

Proposition 3. *Let $\chi \in p^{-1}(\sigma)$. Then $s(\chi) = s(\sigma)$.*

In view of the Frenkel-Zhu Theorem, the following corollary is an immediate consequence. It appears to be a new result in the theory of meromorphic connections.⁶

Corollary 4. *For every $\sigma \in \text{Loc}_{\check{G}}(\mathcal{D}^\times)$, the denominator of the slope $s(\sigma)$ divides a fundamental degree of $\widehat{\mathfrak{g}}$.*

We refer the reader to Section 2 for a thorough discussion of the slope.

⁵Actually, this ℓ -tuple is not unique, even for simple groups of adjoint type; see Remark 19. However, as we shall see, these choices do not matter.

⁶We note, however, that this result would follow from the Bremer-Sage Theory together with an unpublished result of Yu; see Remark 17. For smooth representations of p -adic groups, the analogous result is proved in [34].

We are now ready to state our main result.

Theorem 5. *For all $\chi \in \text{Op}_{\widehat{G}}(\mathcal{D}^\times)$, we have $s(\chi) \leq d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi)$.*

1.4. Idea of the proof. Let us briefly explain the main ingredient in the proof of this theorem. First of all, one can show (cf. §10 [16]) that in the present situation, the categorical depth of $\widehat{\mathfrak{g}}_c - \text{mod}_\chi$ can be alternatively defined by

$$d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi) = \inf\{r \in \mathbb{R}_{\geq 0} \mid \exists x \in \mathcal{B}(G) \text{ such that } \widehat{\mathfrak{g}}_c - \text{mod}_\chi \text{ contains a } G_{x,r+}\text{-integrable module}\}.$$

Suppose $d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi) \leq r$. Then $\widehat{\mathfrak{g}}_c - \text{mod}_\chi$ contains a $G_{x,r+}$ -integrable module W . To show that $s(\chi) \leq r$, it is enough to show that W is centrally supported on the subscheme $\text{Op}_{\widehat{G}}^r$; i.e., if a central character χ acts non-trivially on W , then we must have $\chi \in \text{Op}^r$.

It follows from Kolchin’s theorem (cf. Remark 25) that we have a canonical non-zero morphism of $\mathbb{U}_{x,r} \rightarrow W$, where

$$(5) \quad \mathbb{U}_{x,r} := \text{Ind}_{\mathfrak{g}_{x,r+} \oplus \mathbb{C}}^{\widehat{\mathfrak{g}}_c}(\mathbb{C}).$$

Thus, Theorem 5 follows from

Theorem 6. *The natural morphism $\mathbb{C}[\text{Op}_{\widehat{G}}(\mathcal{D}^\times)] \simeq \mathcal{Z}_c \rightarrow \text{End}_{\widehat{\mathfrak{g}}_c}(\mathbb{U}_{x,r})$ factors through the quotient $\mathbb{C}[\text{Op}_{\widehat{G}}(\mathcal{D}^\times)] \rightarrow \mathbb{C}[\text{Op}_{\widehat{G}}^r]$.*

If x is the canonical hyperspecial vertex (that is, the one corresponding to $\mathfrak{g}[[t]]$) and n is a non-negative integer, then $\mathfrak{g}_{x,n+} = t^{n+1}\mathfrak{g}[[t]]$. In this case, the above theorem is due to Beilinson and Drinfeld [5, §3.8.7]. In view of the previous discussion, we can rephrase the theorem of Beilinson and Drinfeld as stating that

$$(6) \quad s(\chi) \leq \lceil d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi) \rceil,$$

where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x . Our main theorem, therefore, sharpens Beilinson and Drinfeld’s theorem by removing $\lceil - \rceil$.

We prove Theorem 6 by using basic properties of Segal-Sugawara vectors along with some general properties of Fourier coefficients of vertex fields. We refer the reader to Section 3 for the details of the proof.

1.5. Towards establishing the converse. Let χ be an oper with slope less than or equal to r . How should one prove the inequality $d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi) \leq r$? Suppose we can produce a module $\mathbb{V} \in \widehat{\mathfrak{g}}_c - \text{mod}$ such that

- (i) \mathbb{V} is $G_{x,r+}$ -integrable (thus, centrally supported on $\text{Op}_{\widehat{G}}^r$);
- (ii) the module $\mathbb{V}(\chi) := \mathbb{V} \otimes_{\mathcal{Z}_c} \chi$ is non-zero.

Then $\mathbb{V}(\chi)$ is a $G_{x,r+}$ -integrable object of $\widehat{\mathfrak{g}}_c - \text{mod}_\chi$, implying that $d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi) \leq r$.

As a motivating example, consider the module $\mathbb{V}_n = \text{Ind}_{t^n \mathfrak{g}[[t]] \oplus \mathbb{C}}^{\widehat{\mathfrak{g}}_c}(\mathbb{C})$. According to Lemma 7.2.2 in [17], this module is *free* over $\text{Op}_{\widehat{G}}^n = \text{Op}_{\widehat{G}}^{\text{ord}_n}$. Thus, for all $\chi \in \text{Op}_{\widehat{G}}^{\text{ord}_n}$, we have $\mathbb{V}_n(\chi) \neq 0$, which implies that $d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi) \leq n$. From this, it follows easily that

$$(7) \quad d(\widehat{\mathfrak{g}}_c - \text{mod}_\chi) \leq \lceil s(\chi) \rceil.$$

Our challenge (Conjecture 1) is to sharpen the above inequality by removing the $\lceil - \rceil$.

The aforementioned freeness result was first pointed out by Drinfeld, who deduced it from the flatness of Hitchin's map. Subsequently, Eisenbud and Frenkel gave a purely local proof using results of Mustata on singularities of jet schemes [33, Appendix A]. At the moment, we do not know how to extend this purely local approach to a more general setting.

2. SLOPE OF MEROMORPHIC CONNECTIONS

2.1. Overview. Let G be a connected reductive group over the complex numbers. The notion of slope for G -local systems on \mathcal{D}^\times has a long and complicated history. In §11.9 of [26], Katz explains what irregular connections on *vector bundles* over \mathcal{D}^\times are, building on earlier works of Fuchs, Turrittin and Lutz. In particular, he explains how to attach a canonical rational number to every irregular connection. For this reason, the slope is sometimes known as the *Katz invariant*. The same concept also appears in [11], Section II, §1.

One of the characterisations of the slope of flat vector bundles (the one involving gauge transformation by elements in $G((t^{1/b}))$) can be generalised, verbatim, to the case of connections on G -bundles. It seems that this generalisation was first considered in [3]. We review the Katz-Deligne-Babbitt-Varadarajan definition of slope in §2.2 and give a short proof of the fact that it is well-defined by using opers. This proof, however, uses a non-trivial theorem of Frenkel and Zhu [22].

As mentioned in the introduction, there is a deep analogy between Galois representation and flat connections. Guided by this analogy, Katz [28] defined the *differential Galois group* by employing the Tannakian structure on the category of connections. It is clear from this formulation that the notion of slope of a flat vector bundle, defined in [28] via filtration subgroups, extends to flat G -bundles. We note, however, that the structure of the differential Galois group of \mathcal{D}^\times and its filtration subgroups are not easy to discern. We review the Tannakian definition of slope in §2.3.

In [7], the authors define the slope of flat bundles using Moy-Prasad Theory. In more detail, Bremer and Sage define what it means for a flat bundle σ to contain a "stratum". They prove that the slope of σ is the minimum of depths of a fundamental stratum contained in it. In addition, they provide an algorithm for determining the slope and define a canonical form for connections over the base field (as opposed to going to a field extension of the form $\mathbb{C}((t^{1/b}))$). Their approach makes clear the analogy between slopes of local systems and the depth of smooth representations (or categorical representations). We don't know, however, how to prove any relationship between depths of (categorical) representations and slopes of local systems using their definition.

We use the notion of oper to define the slope of flat G -bundles. One of the advantages of our definition is that it will be *obvious* that the denominator of the slope of a flat G -bundle is a divisor of a fundamental degree of the Lie algebra of G . Another advantage is that we can use this definition to make progress on Conjecture 1. The disadvantage is that one does not have an algorithm for putting a connection in its oper form. (The proof in [22] is non-constructive.)

2.2. First definition of the slope. We start by recalling some basic definitions. Let G be a connected reductive group over \mathbb{C} . Let $\sigma \in \text{Loc}_G(\mathcal{D}^\times)$. By definition, σ consists of a pair (\mathcal{F}, ∇) , where \mathcal{F} is a G -bundle on \mathcal{D}^\times and ∇ is a meromorphic connection on \mathcal{F} . More precisely, we assume ∇ is meromorphic on the disk and

holomorphic on the punctured disk. One knows that every bundle on \mathcal{D}^\times is trivial. Choosing a trivialisation for \mathcal{F} , we can write the connection ∇ as

$$(8) \quad \nabla = \partial_t + A, \quad A = A_{-n}t^{-n} + A_{-n+1}t^{-n+1} + \dots, \quad A_i \in \mathfrak{g}, \quad A_{-n} \neq 0.$$

The integer n and the element A_{-n} are called the *order of singularity* and the (top) *polar part* of this trivialisation, respectively. Changing the trivialisation of \mathcal{F} by $g \in G((t))$ ⁷ corresponds to a gauge transformation of the above expression

$$\nabla \mapsto g \cdot \nabla := \partial_t + gAg^{-1} - (\partial_t g)g^{-1}.$$

Thus, one can alternatively define σ as a $G((t))$ -gauge equivalence class of operators of the form (8). After [26] and [11], one says that $\sigma = (\mathcal{F}, \nabla)$ is *regular* (resp. *regular singular*) if in a particular trivialisation of \mathcal{F} , the order of singularity of the connection ∇ is zero (resp. one). Otherwise, we say that σ is *irregular*.

Note that the order of singularity of ∇ is not invariant under gauge transformation: given an operator in the form (8), it may be possible to find a different trivialisation in which the order of singularity of σ is less than n . The following lemma states that this cannot happen if A_{-n} is *nilpotent*.

Lemma 7. *Assume that the order of singularity of ∇ is $n \geq 2$ and the polar part of ∇ is non-nilpotent. Then every operator in the gauge equivalence class of ∇ will have order of singularity greater than or equal to n .*

Proof. Recall the Cartan decomposition $G((t)) = \bigsqcup_{\lambda \in \mathbb{X}^+} G[[t]]t^\lambda G[[t]]$. Note that gauge transformation by elements of $G[[t]]$ will not change the order of singularity and the non-nilpotency of the polar part; therefore, we are reduced to showing that the order of singularity of $t^\lambda \cdot \nabla$ is $\geq n$. Let

$$A_{-n} = A_{\bar{n}} \oplus A_{\mathfrak{t}} \oplus A_{\mathfrak{n}}$$

be the decomposition induced by the triangular decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{t} \oplus \mathfrak{n}$. If $A_{\mathfrak{t}} \neq 0$, then the operator $t^\lambda \cdot \nabla$ will contain the summand $A_{\mathfrak{t}}/t^n$; hence, the order of singularity of $t^\lambda \cdot \nabla$ is at least n . If $A_{\mathfrak{t}} = 0$, then since A_{-n} is non-nilpotent, it follows that $A_{\bar{n}} \neq 0$. Finally, write

$$A_{\bar{n}} = \bigoplus_{\alpha \in \Delta^+} A_{\bar{n}, -\alpha}.$$

Then every non-zero $A_{\bar{n}, -\alpha}$ will contribute a summand $A_{\bar{n}, -\alpha}t^{-n-\langle \lambda, \alpha \rangle}$ in $t^\lambda \cdot \nabla$. Hence, the order of singularity of $t^\lambda \cdot \nabla$ is $\geq n$. □

The above lemma motivates the following definition.

Definition 8. The operator (8) is in the *reduced form* if A_{-n} is not nilpotent.

It is not always possible to put a connection in a reduced form using $G((t))$ -gauge transformation. This is possible, however, if we allow ourselves to go to the field extension $\mathbb{C}((t^{1/b}))$ for some positive integer b . This is the content of the following lemma, which was originally proved in [11] and [28] for $G = \text{GL}_n$ and [3] for general G . We give a proof in §2.7 using opers.

⁷Following a common abuse of notation, we are writing $G((t))$ when we really mean $G(\mathbb{C}((t)))$.

Lemma 9. *Let σ be an irregular local system on \mathcal{D}^\times . Then there exists a positive integer b such that the $G((t^{1/b}))$ -gauge equivalence class of σ contains an operator in reduced form.*

Next, observe that if we set $u = t^{1/b}$, then we can write the operator (8) as

$$\nabla = A_{-n}u^{-nb} + \dots .$$

This motivates the definition of slope:

Definition 10 (cf. [21]). The slope $s(\sigma)$ of a local system σ is defined as follows: $s(\sigma) = 0$ if σ is regular singular; otherwise, $s(\sigma) = a/b$ if the connection σ is $G((t^{1/b}))$ -gauge equivalent to a reduced operator with order of singularity $a + 1$.

Lemma 11. *The definition of slope is well-defined.*

Proof. We can assume σ is irregular. Let l be another positive integer such that ∇ is $G((t^{1/l}))$ -gauge equivalent to a reduced operator with a pole of order $k + 1$. We have to show that $a/b = k/l$. Passing to the extension $\mathbb{C}(t^{1/bl})$, we see that the connection ∇ is $G((t^{1/bl}))$ -gauge equivalent to both ∇_1 and ∇_2 , where ∇_1 (resp. ∇_2) is a reduced operator having a pole of order $la + 1$ (resp. $bk + 1$). We claim that

$$la + 1 = bk + 1.$$

Clearly, this implies $a/b = k/l$, hence finishes the proof the lemma. Now observe that the operators ∇_1 and ∇_2 are $G((t^{1/bl}))$ -gauge equivalent. Thus, the claim follows from Lemma 7. □

We refer the reader to Example 22 for a local system with non-integral slope.

Remark 12. Let $\sigma = (\mathcal{F}, \nabla)$ be a local system on a bundle \mathcal{F} . Let $\rho : G \rightarrow \text{GL}(V)$ be a faithful representation of G , and let σ^V be the induced connection on the associated vector bundle. Since being reduced is preserved under ρ , we see that the slope of σ is equal to the slope of the connection σ^V .

2.3. Tannakian formulation. Let us first recall the definition of differential Galois group of \mathcal{D}^\times following [28, §2]. Let $\text{Conn}(\mathcal{D}^\times)$ be the category of connections on \mathcal{D}^\times . By definition, the objects of this category are pairs (V, ∇) consisting of a finite dimensional vector space V over $\mathbb{C}((t))$ and a connection ∇ on V . Note that if $\dim(V) = n$, then (V, ∇) is an element of $\text{Loc}_{\text{GL}_n(V)}(\mathcal{D}^\times)$. The category $\text{Conn}(\mathcal{D}^\times)$ has a natural notion of internal Homs and tensor products, giving it the structure of a rigid abelian tensor category with $\text{End}_1 = \mathbb{C}$. It has, moreover, an evident $\mathbb{C}((t))$ -valued fibre functor, namely, the functor which sends the pair (V, ∇) to V .

Using the results of Turrittin and Levelt, Katz constructed a canonical \mathbb{C} -valued fibre functor $F : \text{Conn}(\mathcal{D}^\times) \rightarrow \text{Vect}_{\mathbb{C}}$. Thus, he showed that $\text{Conn}(\mathcal{D}^\times)$ is, in fact, a *neutral* Tannakian category over \mathbb{C} . The differential Galois group $I = \text{Aut}(F)$ is the group of automorphisms of this fibre functor. It is a pro-algebraic group over \mathbb{C} whose finite dimensional representations are identified with objects of $\text{Conn}(\mathcal{D}^\times)$.

The group I has an ‘‘upper numbering filtration’’ defined as follows. For every non-negative real number r , let I^r be the kernel of $I \rightarrow \text{Aut}(F|_{\text{Conn}_{<r}(\mathcal{D}^\times)})$, where $\text{Conn}_{<r}(\mathcal{D}^\times)$ is the subcategory of connections of slope less than r . Similarly, one defines I^{r+} to be the kernel of $I \rightarrow \text{Aut}(F|_{\text{Conn}_{\leq r}(\mathcal{D}^\times)})$. For every $0 < x < y$ we have

$$I^y \subset I^{x+} \subset I^x \subset I.$$

We claim that the data of a G -local system on \mathcal{D}^\times is the same as a homomorphism $I \rightarrow G$. Indeed, suppose we are given $(E, \nabla) \in \text{Loc}_G(\mathcal{D}^\times)$. Then the induction functor $E \rightarrow E \times^G V$ defines a tensor functor

$$\nabla : \text{Rep}(G) \rightarrow \text{Conn}(\mathcal{D}^\times).$$

Composing with Katz’s fibre functor, we obtain a tensor functor $\text{Rep}(G) \rightarrow \text{Vect}_{\mathbb{C}}$, which by Tannaka duality gives us a homomorphism $\phi_\nabla : I \rightarrow G$. The converse construction is also evident.

We now turn our attention to defining the slope of a local system using this formalism.

Definition 13. Let $\sigma = (E, \nabla) \in \text{Loc}_G(\mathcal{D}^\times)$. Define

$$s'(\sigma) = \inf\{r \geq 0 \mid I^{r^+} \subset \ker(\phi_\nabla)\}.$$

Lemma 14. For all $\sigma \in \text{Loc}_G(\mathcal{D}^\times)$, we have $s'(\sigma) = s(\sigma)$.

Proof. Let (r, V) be a faithful representation of G and let σ^V be the induced connection. Then from the definition of s' we see that $s'(\sigma)$ is equal to $s'(\sigma^V)$. By Remark 12, we are reduced to the case of vector bundles, and the lemma follows from the definition of the upper numbering filtration group. \square

2.4. Recollections on Moy-Prasad Theory. Let G be a connected reductive group over \mathbb{C} , and let Z denote the centre of G . Let \mathfrak{g} denote the Lie algebra of G . We fix a maximal torus $T \subseteq G$ with the corresponding Cartan subalgebra \mathfrak{t} . Let Φ denote the set of roots of G with respect to T . For ease of notation, we set

$$\Phi^* = \Phi \sqcup \{0\}.$$

For $\alpha \in \Phi^*$ let $\mathfrak{u}_\alpha \subset \mathfrak{g}$ denote the weight space for T corresponding to α (so $\mathfrak{u}_0 = \mathfrak{t}$). Let $G((t))$ and $\mathfrak{g}((t))$ denote the corresponding loop group and loop algebra.

Let $\bar{\mathcal{B}}$ be the Bruhat-Tits building of $G((t))$ and let $\mathcal{B} = \bar{\mathcal{B}} \times (X_*(Z) \otimes_{\mathbb{Z}} \mathbb{R})$ denote the enlarged building. Let $\mathcal{A} = \mathcal{A}(G, T)$ denote the standard apartment of \mathcal{B} . This is an affine space isomorphic to $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We have an isomorphism $\mathcal{A} \simeq \mathfrak{t}_{\mathbb{R}}$; thus, points in \mathcal{A} may be viewed as elements of \mathfrak{t} .

For every $x \in \mathcal{B}(G)$ and $r \in \mathbb{R}_{\geq 0}$, Moy and Prasad defined subgroups $G_{x,r}$ and G_{x,r^+} inside $G((t))$. In the case $r = 0$, $G_x := G_{x,0}$ is the usual parahoric subgroup associated to x and $G_{x^+} := G_{x,0^+}$ is the pro-unipotent radical of G_x . Recall that $G((t))$ acts on $\mathcal{B}(G)$. If $g.x = y$, for $g \in G((t))$ and $x, y \in \mathcal{B}(G)$, then $\text{Ad}(g).G_{x,r} = G_{y,r}$. Ditto for G_{x,r^+} and its Lie algebra. Since every $x \in \mathcal{B}(G)$ has an element of \mathcal{A} in its orbit, in the applications we have in mind, it suffices to consider $x \in \mathcal{A}$.

It will be convenient for us to have an explicit description of the Lie algebras of G_{x,r^+} , where $x \in \mathcal{A}$. This is given by

$$(9) \quad \mathfrak{g}_{x,r^+} = \bigoplus_{\alpha \in \Phi^*} \mathfrak{u}_\alpha(\mathcal{P}^{1-\lceil \alpha(x)-r \rceil}).$$

Here, $\mathcal{P} = t\mathbb{C}[[t]]$ denotes the maximal ideal of $\mathbb{C}[[t]]$.

Let \mathfrak{g}^* denote the dual of \mathfrak{g} . Moy and Prasad also defined filtration subalgebras of $\mathfrak{g}_{x,r}^*$ and $\mathfrak{g}_{x,-r}^*$, where now $r \in \mathbb{R}_{\leq 0}$. One has a canonical isomorphism

$$(10) \quad (\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+})^* \simeq \mathfrak{g}_{x,-r}^*/\mathfrak{g}_{x,-r^+}^*.$$

Finally, let us make some remarks regarding optimal points. Fix a chamber $C \subset \mathcal{A}$. Let $O \subset \overline{C}$ denote the set of optimal points (see [31] for the definition of optimal points). The set O has many good properties. For example, the set $\{r \in \mathbb{R}_{\geq 0} \mid G_{x,r} \neq G_{x,r^+}\}$ is a discrete subset of \mathbb{Q} . Elements in the above set are called optimal numbers. Also, for any $(y, r) \in \mathcal{A} \times \mathbb{R}_{\geq 0}$, there are $x, z \in O$ such that

$$(11) \quad G_{x,r^+} \subset G_{y,r^+} \subset G_{z,r^+}.$$

In many applications of Bruhat-Tits and Moy-Prasad Theory, it is enough to consider the optimal points of the building, as opposed to arbitrary points.

2.5. Slopes via Moy-Prasad Theory. We are now ready to give Bremer and Sage’s definition of the slope [7]. Let (\mathcal{F}, ∇) be a pair consisting of a G -bundle \mathcal{F} on \mathcal{D}^\times equipped with a connection ∇ . Choosing a trivialisation ϕ of \mathcal{F} , we can write ∇ in terms of a one-form with coefficients in $\mathfrak{g}((t))$. We denote this one-form by $[\nabla]_\phi \in \Omega^1(\mathfrak{g}((t)))$. Recall that a point $x \in \mathcal{A}$ defines an element in \mathfrak{t} which, by an abuse of notation, is also denoted by x . Therefore, $x \frac{d}{dt}$ is an element of $\Omega^1(\mathfrak{g}) \subset \Omega^1(\mathfrak{g}((t)))$, and so $[\nabla]_\phi - x \frac{d}{dt}$ makes sense as an element of $\Omega^1(\mathfrak{g}((t)))$. Now the residue pairing defines a canonical isomorphism

$$\Omega^1(\mathfrak{g}((t))) \simeq \mathfrak{g}^*((t)).$$

Thus, we may think of the one-form $[\nabla]_\phi - x \frac{d}{dt}$ as an element of $\mathfrak{g}^*((t))$. Recall that $\mathfrak{g}_{x,-r}^*$ is a lattice inside $\mathfrak{g}^*((t))$ for $r \in \mathbb{R}_{\geq 0}$.

A stratum is a triple (x, r, β) consisting of a point $x \in \mathcal{A}(G)$, a number $r \in \mathbb{R}_{\geq 0}$, and a functional $\beta \in (\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+})^*$.

Definition 15. We say that the flat G -bundle (\mathcal{F}, ∇) contains the stratum (x, r, β) with respect to the trivialisation ϕ if $[\nabla]_\phi - x \frac{d}{dt} \in \mathfrak{g}_{x,-r}^*$ and the coset $([\nabla]_\phi - x \frac{d}{dt}) + \mathfrak{g}_{x,-r^+}^*$ equals the coset determined by the functional

$$\beta \in (\overline{\mathfrak{g}}_{x,r})^* = (\mathfrak{g}_{x,r}/\mathfrak{g}_{x,r^+})^* \simeq \mathfrak{g}_{x,-r}^*/\mathfrak{g}_{x,-r^+}^*.$$

It is proved in [7] that every σ contains a stratum. In particular, the following definition makes sense:

$$(12) \quad s_{BS}(\sigma) := \min\{r \in \mathbb{R}_{\geq 0} \mid \sigma \text{ contains a stratum of the form } (x, r, \beta)\}.$$

Bremer and Sage establish the following result.

Theorem 16. *For every $\sigma \in \text{Loc}_G(\mathcal{D}^\times)$, we have $s_{BS}(\sigma) = s(\sigma)$.*

Remark 17. Bremer and Sage prove that the slope may also be characterised as the minimum depth of a triple (x, r, β) contained in (\mathcal{F}, ∇) for which x is an optimal point. We have been informed that J. K. Yu has proved that the denominators of the critical numbers at the optimal points are divisors of the fundamental degrees of \mathfrak{g} (unpublished). From these considerations, it follows that the denominator of the slope of a flat connection divides a fundamental degree of \mathfrak{g} . This fact will be evident from our definition of slope via opers.

2.6. Recollections on opers. Let G be a reductive group of rank ℓ . We fix a Borel subgroup B and let $N = [B, B]$ and $T = B/N$, and let W denote the Weyl group of G and let Z be the centre of G . Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}$, and \mathfrak{t} denote the corresponding Lie algebras. Let $\check{\rho}$ be the half-sum of positive coroots.

Choose generators f_i for the root subgroups corresponding to the negative simple roots of \mathfrak{g} . Let

$$f = \sum_{i=1}^{\ell} f_i.$$

The notion of opers is due to Beilinson and Drinfeld [6], building on earlier work by Drinfeld and Sokolov [13]. For us, the following description, given in terms of a local coordinate, suffices: a G -oper (on the punctured disk \mathcal{D}^\times) is a $B((t))$ -gauge equivalence class of operators of the form

$$(13) \quad \nabla = \partial_t + \sum \phi_i f_i + v, \quad \phi_i \in \mathbb{C}((t))^\times, \quad v \in \mathfrak{b}((t)).$$

Let $\text{Op}_G(\mathcal{D}^\times)$ denote the set of G -opers on \mathcal{D}^\times . We will now give an explicit description of this set.

Let e denote the unique element in \mathfrak{n} such that $\{f, 2\check{\rho}, e\}$ is an \mathfrak{sl}_2 -triple. Let V_{can} denote the $\text{ad } e$ invariants in \mathfrak{g} . Note that V_{can} is invariant under ad . Let $p_i, i = 1, \dots, \ell$, be a basis of V_{can} and also eigenvectors of $\text{ad } \check{\rho}$. Let $d_i \in \mathbb{Z}_{\geq 0}, i = 1, \dots, \ell$, be the corresponding eigenvalues. We may order the basis so that $d_i \leq d_{i+1}$. The set $\{d_1, \dots, d_\ell\}$ is called the exponents of \mathfrak{g} . Let

$$V_{\text{can}} := \bigoplus_{i=1}^{\ell} V_{\text{can},i}$$

be the decomposition according to the basis $p_i, i = 1, \dots, \ell$. That is, $V_{\text{can},i} = \mathbb{C}p_i$. According to a theorem of Kostant (cf. [30] or [12]) the composition

$$(14) \quad V_{\text{can}} \xrightarrow{v \mapsto v+f} \mathfrak{g} \rightarrow \mathfrak{g} // G \simeq \mathfrak{t} // W$$

is an isomorphism. In particular, V_{can} is an affine space of dimension ℓ . The following lemma is due to Drinfeld and Sokolov [13]; see also Lemma 4.2.2 of [16].

Lemma 18.

- (i) *Every $B((t))$ -gauge equivalence class of operators (13) contains an operator of the form*

$$(15) \quad \nabla = \partial_t + f + v,$$

where $v \in V_{\text{can}}((t))$.

- (ii) *If G is adjoint, the gauge action of $B((t))$ on the space of operators of the form (13) is free and each gauge equivalence class contains a unique operator of the form in (15).*

The expression (15) is called a *canonical form* of the oper. The above lemma implies that every oper has a canonical form. If G is adjoint, every oper admits a unique canonical form and we have an isomorphism $\text{Op}_G(\mathcal{D}^\times) \simeq V_{\text{can}}((t))$. In fact, this is an isomorphism of ind-schemes of ind-infinite type over \mathbb{C} . However, this isomorphism is not canonical since it depends on a choice of coordinate t .

2.7. Slope via opers.

2.7.1. Let $\chi \in \text{Op}_G(\mathcal{D}^\times)$ and write χ in a canonical form $\partial_t + f + v$, where $v \in V_{\text{can}}(\!(t)\!)$. Let $p_i, i = 1, \dots, \ell$, be the basis in §2.6 and write

$$v(t) = \sum_{i=1}^{\ell} v_i p_i, \quad v_i \in \mathbb{C}(\!(t)\!).$$

Let us write $v_i = t^{-n_i} \cdot h_i, h_i \in \mathbb{C}[[t]]^\times$ if $v_i \neq 0$ and set $n_i = \infty$ if $v_i = 0$. In the introduction, we defined the slope of χ defined by

$$s(\chi) := \sup\{0, \sup_{i=1, \dots, \ell} \left\{ \frac{n_i}{(d_i + 1)} - 1 \right\}\}.$$

Recall that Proposition 3 states that the slope of an oper equals the slope of its underlying connection.

Proof of Proposition 3. Let $\sigma = (\mathcal{F}, \nabla)$ be the underlying connection of the oper χ . Let $\nabla = \partial_t + f + v$ be a canonical form coming from the oper structure. Write $v_i = t^{-n_i} \cdot h_i$ where $h_i \in \mathbb{C}[[t]]^\times$. Assume that $n_k/(d_k + 1)$ is maximal among all $\{n_i/(d_i + 1)\}_{i=1, \dots, \ell}$. Now taking the covering $s = t^{1/(d_k+1)}$, the connection becomes

$$\nabla = \partial_s + (f + s^{-n_k(d_k+1)} h_k p_k + \bigoplus_{i \neq k} s^{-n_i(d_i+1)+c_i} h_i p_i)(d_k + 1) s^{d_k}$$

where $c_i \in \mathbb{Z}_+$. Conjugating with $s^{n_k \cdot \check{\rho}}$ we get

$$\nabla = \partial_s + (d_k + 1) s^{d_k - n_k} (f + h_k p_k + \bigoplus_{i \neq k} s^{c_i} h_i p_i) + n_k \check{\rho} s^{-1}.$$

If $s(\chi) = 0$, then we have $n_k - d_k \leq 1$; thus the connection has regular singularity and it implies $s(\sigma) = 0$. If $s(\chi) > 0$, then we have $n_k - d_k > 1$. Since $h_k = z_0 + z_1 s + \dots \in \mathbb{C}[[s]]^\times$ and $c_i \in \mathbb{Z}_+$, we see that the order of the singularity of the connection ∇ is $n_k - d_k$ and the polar part is $(d_k + 1)(f + z_0 p_k)$, which is not nilpotent by the theorem of Kostant [30]. Hence from the definition of slope in §2.2, we have

$$s(\sigma) = (n_k - d_k - 1)/(d_k + 1) = n_k/(d_k + 1) - 1 = s(\chi).$$

□

Remark 19. By [30, Remark 19’], the isomorphism in (14) remains true if we replace V_{can} by any $\text{ad} \check{\rho}$ invariant subspace $V \subset \mathfrak{b}$ such that $\mathfrak{b} = V \oplus [f, \mathfrak{n}]$. Moreover, using this fact, one can check that Lemma 18 and Proposition 3 remain true if we replace V_{can} by any such V . In particular, in the case of $G = GL(n)$, they apply to the special case of the space $V = V_{\text{com}}$ of companion matrices.

Remark 20. A flat $GL(n)$ -bundle on \mathcal{D}^\times is the same as a rank n bundle E on \mathcal{D}^\times with a flat connection ∇ . An oper structure on (E, ∇) is equivalent to the existence of a cyclic vector of (E, ∇) ; see, for instance, [14, §16]. The existence of a cyclic vector is proved in [11]. Furthermore, in [11, p. 49] Deligne gives a definition of slope using the cyclic vector. It follows from Remark 19 that his definition is equivalent to our Definition 2 in the case $G = GL(n)$.

Recall the definition of reduced form of an operator (Definition 8). The following is a corollary of the proof of the above proposition.

Corollary 21. *Every irregular flat G -bundle on \mathcal{D}^\times is $G((t^{1/b}))$ -gauge equivalent to a reduced operator for some $b \in \mathbb{Z}_{>0}$.*

Proof. Indeed, during the proof of the above proposition we showed that every oper whose underlying flat G -bundle is irregular is $G((t^{1/b}))$ -gauge equivalent to a reduced operator. Since every flat G -bundle has an oper structure, the result follows. \square

Example 22 (cf. [21]). Let G be a simple group of adjoint type. Let us compute the slope of the operator

$$\nabla = \partial_t + \frac{f}{t} + \frac{p_k}{t^2}.$$

Recall that $f = \sum_{i=1}^\ell f_i$ and $p_k \in V_{can,k}$. Let us write $m = d_k + 1$. Passing to the extension $s = t^{1/m}$, the connection becomes

$$\partial_s + \frac{mf}{s} + \frac{mp_k}{s^{m+1}}.$$

Gauge transforming this operator with $g = s^{\check{\rho}}$, where $\check{\rho}$ is the half-sum of positive coroots, the connection becomes

$$\partial_s + \frac{m(f + p_k)}{s^2} - \frac{\check{\rho}}{s}.$$

By Kostant’s theorem, the element $f + p_k$ is non-nilpotent; hence, the slope is equal to $\frac{1}{m}$. Alternatively, we compute the slope of ∇ using the canonical form of oper. Conjugating by $t^{-\check{\rho}}$, the connection ∇ becomes

$$\partial_t + f + \frac{\check{\rho}}{t} + \frac{p_k}{t^{d_k+2}}.$$

To get rid of $\check{\rho} \cdot t^{-1}$ we conjugate by $\exp(\frac{-f}{2t})$ and obtain the operator

$$\partial_t + f - \frac{p_1}{4t^2} + \frac{p_k}{t^{d_k+2}}.$$

This operator has the form of an oper. Now using Definition 2, one can easily see that its slope (as an oper) equals $\frac{d_k+2}{d_k+1} - 1 = \frac{1}{m}$.

3. REPRESENTATIONS OF AFFINE KAC-MOODY ALGEBRAS

The main purpose of this section is to prove Theorem 6. In the first subsection, we define the notion of depth for irreducible smooth modules of affine Kac-Moody algebras. In §3.2, we collect some basic information regarding vertex operators. In §3.3, we apply these general considerations to affine vertex algebras. We recall some basic properties of Segal-Sugawara vectors and operators in §3.4. Our treatment of these topics is not self-contained; for more information on affine vertex algebras and Segal-Sugawara operators, we refer the reader to [14], [10], or [24, §2]. Armed with these preliminaries, we prove Theorem 6 in §3.5.

3.1. Depths of smooth modules. Let \mathfrak{g} be a simple Lie algebra over \mathbb{C} . Let κ be an invariant bilinear form on \mathfrak{g} . The affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa$ at level κ is defined to be the central extension

$$(16) \quad 0 \rightarrow \mathbb{C} \cdot \mathbf{1} \rightarrow \widehat{\mathfrak{g}}_\kappa \rightarrow \mathfrak{g}((t)) \rightarrow 0,$$

with the two-cocycle defined by the formula

$$(17) \quad (x \otimes f(t), y \otimes g(t)) \mapsto -\kappa(x, y) \cdot \text{Res}_{t=0} f dg.$$

A module V over $\widehat{\mathfrak{g}}_\kappa$ is *smooth* if for every $v \in V$ there exists $N_v \geq 0$ such that $t^{N_v} \mathfrak{g}[[t]]$ annihilates v and such that $\mathbf{1} \in \mathbb{C} \cdot \mathbf{1} \subset \widehat{\mathfrak{g}}_\kappa$ acts on V as the identity. Thus, every vector in a smooth module V is annihilated by a bounded subalgebra. The notion of depth measures, in some sense, the largest bounded subalgebra which annihilates a vector in V .

It will be convenient to have the following notation. Let Φ denote the set of roots of \mathfrak{g} with respect to a Cartan subalgebra \mathfrak{t} , and let $\Phi^* := \Phi \sqcup \{0\}$. Then we have the root decomposition

$$(18) \quad \mathfrak{g} = \bigoplus_{\alpha \in \Phi^*} \mathfrak{g}^\alpha,$$

where we set $\mathfrak{g}^0 := \mathfrak{t}$.

Recall the description of Moy-Prasad subalgebras $\mathfrak{g}_{x,r^+} \subset \mathfrak{g}((t))$ given in (9).

Lemma 23. *For all $(x, r) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$, the central extension (16) is split over \mathfrak{g}_{x,r^+} .*

Proof. It is enough to prove the lemma for $x \in \mathcal{A}$ and $r = 0$. Consider the two-cocycle (17). Suppose $x \otimes f(t)$ and $y \otimes g(t)$ are in \mathfrak{g}_x and $x \in \mathfrak{g}^\alpha$. Then if we want $\kappa(x, y) \neq 0$ we must have $y \in \mathfrak{g}^{-\alpha}$. The case of $\alpha = 0$ is immediate, so assume $\alpha \neq 0$. Then, we have that $f \in \mathcal{P}^{1-\lfloor \alpha(x) \rfloor}$ and $dg \in \mathcal{P}^{-\lfloor -\alpha(x) \rfloor}$. It follows that fdg is regular; hence, its residue is zero. \square

Definition 24. Let $V \in \widehat{\mathfrak{g}}_\kappa - \text{mod}$ be an irreducible module. Define

$$d(V) := \inf\{r \in \mathbb{R}_{\geq 0} \mid \exists x \in \mathcal{B}(G) \text{ such that } V^{\mathfrak{g}_{x,r^+}} \text{ is non-empty}\}.$$

Using optimal points [31], one can show that the depth of every smooth module is a rational number. We shall not need this result in what follows.

Remark 25. If V is G_{x,r^+} integrable, then $V^{\mathfrak{g}_{x,r^+}}$ is non-empty. Indeed, this is true if one replaces G_{x,r^+} by any pro-unipotent subgroup of the loop group. The proof is an easy application of Kolchin’s theorem.

Remark 26. Let $V \in \widehat{\mathfrak{g}} - \text{mod}$ be a smooth module at the critical level with central character χ and depth r . Then it follows from Theorem 6 (proved below) that $\chi \in \text{Op}_{\widehat{G}}^r$. Indeed, by assumption, there exists $x \in \mathcal{B}(G)$ such that V has a vector v annihilated by \mathfrak{g}_{x,r^+} . By the previous remark, we have a non-trivial morphism $\mathbb{U}_{x,r} \rightarrow V$, sending the generating vector of $\mathbb{U}_{x,r}$ to v . The result follows from Theorem 6.

3.2. Recollections on vertex algebras. We make some general (and obvious) observations about fields in vertex algebras. Let V be a vertex algebra. For $A \in V$, let

$$A(z) = Y(A, z) = \sum_m A_m z^{-m-1}$$

be the corresponding field. Recall that $A_m \in \text{End}(V)$. We write $[A(z)]_m$ for the m th Fourier coefficient of a field $A(z)$; that is, $[A(z)]_m = A_m$. For example, consider the normally ordered product

$$: A(z)B(z) : = \sum_{s \in \mathbb{Z}} \left(\sum_{r < 0} A_r B_s z^{-r-1} + \sum_{r \geq 0} B_s A_r z^{-r-1} \right) z^{-s-1}.$$

The coefficient of z^{-m-2} is equal to

$$\sum_{r+s=m, r<0} A_r B_s + \sum_{r+s=m, r \geq 0} B_s A_r.$$

We conclude that the coefficient $[: A(z)B(z) :]_m$ is a linear combination of the elements $A_r B_s$ or $B_s A_r$ where $r + s = m - 1$. The following lemma is an obvious generalisation of this statement.

Lemma 27. *If $A_1(z), \dots, A_k(z)$ are fields, then $[: A_1(z) \cdots A_k(z) :]_m$ can be written as a linear combination of elements of $[A_{\sigma(1)}(z)]_{m_1} \cdots [A_{\sigma(k)}(z)]_{m_k}$, where σ is a permutation of $\{1, \dots, k\}$ and $m_1 + \dots + m_k = m - (k - 1)$.*

We apply the above considerations to the fields for the affine vertex algebra $V_c(\mathfrak{g})$ at the critical level. Our exposition will be brief here. For more details, see, for instance, [24, §2]. Let $x \in \mathfrak{g}$ and suppose $n < 0$. Then, in view of the isomorphism of vector spaces $V_c(\mathfrak{g}) \simeq U(t^{-1}\mathfrak{g}[t^{-1}])$, we can think of $x_n = x \otimes t^n$ as an element of $V_c(\mathfrak{g})$. The field corresponding to this element is

$$x_n(z) = \frac{1}{(-n-1)!} \partial_z^{-n-1} \sum_{m \in \mathbb{Z}} x_m z^{-m-1}.$$

The Fourier coefficients can then be considered as elements of the completed universal enveloping algebra $\tilde{U}_c(\widehat{\mathfrak{g}})$ at the critical level; cf. [16, §3]. Now, we have that $[x_n(z)]_m$ is a multiple of x_{m+n+1} . Thus, if $n = -1$, then $[x_{-1}(z)]_m = x_m$.

Next, suppose x^1, \dots, x^k are elements of \mathfrak{g} . Let $x = x_{n_1}^1 \cdots x_{n_k}^k$ where $n_j < 0$ for all j . As above, we can think of x as an element of $V_c(\mathfrak{g})$. The corresponding field $x[z]$ is the normally ordered product $: x_{n_1}^1(z) \cdots x_{n_k}^k(z) :$. Write $x_{[m]} = [x(z)]_m$ for the m th Fourier coefficient of the field $x[z]$. Then by the previous lemma, $x_{[m]} \in \tilde{U}_c(\widehat{\mathfrak{g}})$ is a linear combination of elements of the form

$$[x_{n_{\sigma(1)}}^{\sigma(1)}(z)]_{m_1} \cdots [x_{n_{\sigma(k)}}^{\sigma(k)}(z)]_{m_k}.$$

Thus, we obtain:

Corollary 28. *The operator $x_{[m]}$ is a linear combination of monomials of the form*

$$x_{m_1+n_{\sigma(1)}+1}^{\sigma(1)} \cdots x_{m_k+n_{\sigma(k)}+1}^{\sigma(k)},$$

where σ is some permutation of $\{1, \dots, k\}$ and m_i 's are integers with $m_1 + \dots + m_k = m - (k - 1)$.

3.3. Fourier coefficients acting on smooth modules. Recall the convention of (18). We write x^α for an element of \mathfrak{g}^α and x_n^α for the element $x^\alpha \otimes t^n$ in $\widehat{\mathfrak{g}}_c$.

Lemma 29. *Let V be a $\widehat{\mathfrak{g}}_c$ -module and let $v \in V$. For $\alpha \in \Phi^*$, let $r_\alpha \in \mathbb{Z}$ be such that*

- (1) $x_n^\alpha.v = 0$, for all $n \geq r_\alpha$;
- (2) $r_\alpha + r_\beta \geq r_{\alpha+\beta}$,⁸ and
- (3) $r(0) > 0$.

Let $x = x_{n_1}^{\alpha_1} \cdots x_{n_k}^{\alpha_k} \in \tilde{U}_c(\widehat{\mathfrak{g}})$ and suppose $\sum_{j=1}^k n_j \geq \sum_{j=1}^k r_{\alpha_j}$. Then $x.v = 0$.

⁸By convention, we set $\alpha + \beta = 0$ if $\alpha + \beta$ is not a root.

Proof. The case $k = 1$ being obvious, we first show how one establishes the case $k = 2$. So suppose $x = x_{n_1}^{\alpha_1} x_{n_2}^{\alpha_2}$. If $n_2 \geq r(\alpha_2)$, then clearly $x.v_0 = 0$. Otherwise, we have that $n_2 < r(\alpha_2)$, which implies that $n_1 > r(\alpha_1)$. Now

$$\begin{aligned} x &= x_{n_1}^{\alpha_1} x_{n_2}^{\alpha_2} = x_{n_2}^{\alpha_2} x_{n_1}^{\alpha_1} + [x_{n_1}^{\alpha_1}, x_{n_2}^{\alpha_2}] \\ &= x_{n_2}^{\alpha_2} x_{n_1}^{\alpha_1} + x_{n_1+n_2}^{\alpha_1+\alpha_2} - \kappa(x^{\alpha_1}, x^{\alpha_2}) \text{Res}(n_2 t^{n_1+n_2-1}) \mathbf{1}. \end{aligned}$$

As $n_1 > r(\alpha_1)$ and $n_1 + n_2 \geq r(\alpha_1) + r(\alpha_2) \geq r(\alpha_1 + \alpha_2)$, the first two terms annihilate v_0 . On the other hand, we claim that the last term is always zero. Indeed, $\kappa(x^{\alpha_1}, x^{\alpha_2})$ is non-zero only if $\alpha_1 = -\alpha_2$. Now using condition (3), we have

$$n_1 + n_2 \geq r(\alpha_1) + r(-\alpha_1) > 0,$$

which ensures that the residue is zero.

We now would like to prove the general result using induction. Let j be the largest integer smaller than n such that $n_j \geq \alpha_j$. Note that if $j = k$ we are done. We apply induction on j and k (the PBW degree of x); that is, we assume the result is known for all $k' < k$ and that if $k' = k$, then the result is known for all $j' > j$.

Using the commutation relations in $\widehat{\mathfrak{g}}_\kappa$, we have

$$\begin{aligned} (19) \quad &x_{n_1}^{\alpha_1} \cdots (x_{n_j}^{\alpha_j} x_{n_{j+1}}^{\alpha_{j+1}}) \cdots x_{n_k}^{\alpha_k} \\ &= x_{n_1}^{\alpha_1} \cdots (x_{n_{j+1}}^{\alpha_{j+1}} x_{n_j}^{\alpha_j}) \cdots x_{n_k}^{\alpha_k} \\ &\quad + x_{n_1}^{\alpha_1} \cdots [x_{n_j}^{\alpha_j}, x_{n_{j+1}}^{\alpha_{j+1}}] \cdots x_{n_k}^{\alpha_k} + x_{n_1}^{\alpha_1} \cdots [n_{j+1} \kappa(x^{\alpha_j}, x^{\alpha_{j+1}}) \text{Res}_{t=0} t^{n_j+n_{j+1}-1}] \cdots x_{n_k}^{\alpha_k}. \end{aligned}$$

But note that $[x_{n_j}^{\alpha_j}, x_{n_{j+1}}^{\alpha_{j+1}}] = x_{n_j+n_{j+1}}^{\alpha_j+\alpha_{j+1}}$. Moreover, by assumption

$$\begin{aligned} (n_1 + \cdots + n_k) &\geq r_{\alpha_1} + \cdots + r_{\alpha_j} + r_{\alpha_{j+1}} + \cdots + r_{\alpha_n} \\ &\geq r_{\alpha_1} + \cdots + r_{\alpha_j+\alpha_{j+1}} + \cdots + r_{\alpha_n}. \end{aligned}$$

In the first term, the $x_{n_j}^{\alpha_j}$ factor has moved to the $j + 1$ place; thus, by induction assumption this term annihilates v . On the other hand, the second and third terms have PBW degree $k - 1$ and $k - 2$, respectively; thus, they also annihilate v . \square

Corollary 30. *Suppose we are in the setup of the previous lemma, and assume $x = x_{n_1}^{\alpha_1} \cdots x_{n_k}^{\alpha_k}$ is in $V_c(\widehat{\mathfrak{g}})$; that is, assume all $n_i < 0$. Then, we have*

$$x_{[m]}.v = 0 \quad \forall m \geq \sum_{i=1}^k (r_{\alpha_i} - n_i) - k.$$

Proof. We know that $x_{[m]}$ is a linear combination of elements of the form

$$x_{m_1+n_{\sigma(1)}+1}^{\alpha_{\sigma(1)}} \cdots x_{m_k+n_{\sigma(k)}+1}^{\alpha_{\sigma(k)}},$$

where $m_1 + \cdots + m_k = m - k + 1$. Now observe that

$$\sum_{i=1}^k (m_i + n_{\sigma(i)} + 1) = \sum_{i=1}^k m_i + \sum_{i=1}^k n_{\sigma(i)} + k = m + 1 + \sum_{i=1}^k n_i.$$

The assumption on m implies that the above sum is greater than or equal to $\sum r_{\alpha_i} - k + 1$. The result follows from the previous lemma. \square

Example 31. Suppose we are in the situation of the corollary and $r_\alpha = n$ for all α . In other words, v is killed by $t^n \mathfrak{g}[[t]]$. Let x be an element of degree N ; that is, $N := -\sum_i n_i \geq k$. Then the previous corollary implies that

$$x_{[m]} \cdot v = 0, \quad \forall m \geq k \cdot n + N - k = k(n - 1) + N.$$

Note that we always have $k \leq N$. Now suppose $n \geq 1$. Then $N(n - 1) \geq k(n - 1)$ and so

$$nN \geq k(n - 1) + N.$$

Hence, in this case, we see that $x_{[m]} \cdot v = 0$ for all $m \geq nN$. On the other hand, if $n = 0$, then we get that $x_{[m]} \cdot v = 0$ for all $m \geq N - k$. This is not the sharpest result one has. Indeed, it follows immediately from the vacuum axiom that $x_{[m]} \cdot v = 0$ for all $m \geq 0$. The reason we don't obtain this sharp result from our method is that we are not keeping track of the coefficients of the monomials in $S_{i, [n_i]}$.

3.4. Segal-Sugawara operators. Let us recall some basic facts about Segal-Sugawara vectors. A Segal-Sugawara vector in $V_\kappa(\mathfrak{g})$ is an element B such that for all $A_{(n)} \cdot B = 0$ for all $n \geq 0$ and all $A \in V_\kappa(\widehat{\mathfrak{g}})$. Note that such vectors exist if and only if κ is the critical level.

As mentioned above, the affine vertex algebra $V_\kappa(\widehat{\mathfrak{g}})$ is isomorphic to the universal enveloping algebra $U(\mathfrak{g}_-)$, where

$$\mathfrak{g}_- := \mathfrak{g} \otimes t^{-1} \mathbb{C}[[t^{-1}]].$$

Given $S \in U(\mathfrak{g}_-)$, we write \bar{S} for its image in the associated graded algebra

$$\text{gr}(U(\mathfrak{g}_-)) \simeq S(\mathfrak{g}_-).$$

Note that we have an embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}_-$ given by $x \mapsto x_{-1} = x \otimes t^{-1}$ which induces an embedding $S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_-)$. The following definition is due to Chervov and Molev [10].

Definition 32. A complete set of Segal-Sugawara vectors is a set of elements

$$S_1, S_2, \dots, S_n \in U(\mathfrak{g}_-), \quad n = \text{rk} \mathfrak{g},$$

where S_i are in the centre and $\bar{S}_1, \dots, \bar{S}_n$ coincide with the images of some algebraically independent generators of the algebra of invariants $S(\mathfrak{g})^{\mathfrak{g}}$ under the imbedding $S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_-)$.

Note that the elements S_i are by no means unique. This is related to the fact that there are many choices for generators of the polynomials algebra $S(\mathfrak{g})^{\mathfrak{g}}$.

Let $\{S_1, \dots, S_\ell\} \in V_c(\mathfrak{g})$ be a complete set of Segal-Sugawara vectors. Note that the Feigin-Frenkel centre is invariant under the action of the degree operator $t\partial_t$. Therefore, each homogenous component of any Segal-Sugawara vector is again a Segal-Sugawara vector. Therefore, without loss of generality, we may assume that S_i is homogenous of degree $d_i + 1$.

Lemma 33. *The operators S_i can be written as a linear combination of elements of the form $x_{n_1}^{\alpha_1} x_{n_2}^{\alpha_2} \dots x_{n_k}^{\alpha_k}$ satisfying the following properties:*

- (i) $\sum_{j=1}^k n_j = -(d_i + 1)$;
- (ii) $k \leq d_i + 1$;
- (iii) $\sum_{j=1}^k \alpha_j = 0$.

Proof. Part (i) is immediate from the fact that S_i has degree $d_i + 1$. Part (ii) follows from (i), because all n_i 's are negative. Part (iii) holds since every Segal-Sugawara vector is annihilated, in particular, by elements $x \in \mathfrak{g}$. When x runs over the Cartan subalgebra, this means that the weight of each vector is zero. \square

Let v denote the generating vector of $\mathbb{U}_{x,r}$ (see (5)). According to (9), v is subject to the relation

$$x_s^\alpha \cdot v = 0, \quad \forall s \geq 1 - \lceil \alpha(x) - r \rceil.$$

We will need the following lemma in what follows.

Lemma 34. $S_{i,[m]} \cdot v = 0, \quad m \geq - \left(\sum_{j=1}^k \lceil \alpha_j(x) - r \rceil \right) + d_i + 1.$

Proof. It is easy to check that we always have $1 - \lceil \alpha(x) - r \rceil + 1 - \lceil \beta(x) - r \rceil \geq 1 - \lceil \alpha(x) + \beta(x) - r \rceil$. Therefore, we can use Corollary 30 to conclude that $S_{i,[m]} \cdot v = 0$ for all

$$m \geq \sum_{j=1}^k (r_{\alpha_j} - n_j) - k = \sum_{j=1}^k (1 - \lceil \alpha(x) - r \rceil - n_j) - k.$$

By the previous lemma, the RHS equals $-\left(\sum_{j=1}^k \lceil \alpha_j(x) - r \rceil\right) + d_i + 1$, as required. \square

3.5. Proof of the main theorem. Let $V_c(\mathfrak{g})$ denote the affine vertex algebra at the critical level associated to \mathfrak{g} . Let $S_i \in V_c(\mathfrak{g}), i = 1, \dots, \ell$, be a complete set of Segal-Sugawara vectors. Let $S_{i,[n_i]}$ denote the corresponding Segal-Sugawara operators. (For a quick introduction to these objects see [10, §2.2].) Feigin and Frenkel's Theorem states that the center at the critical level is a completion of the polynomial algebra freely generated on the variables $S_{i,[n_i]}$. It is easy to see that Theorem 6 is equivalent to the following statement:

For $(x, r) \in \mathcal{B}(G) \times \mathbb{R}_{\geq 0}$ and all integers $n_i \geq (d_i + 1)(r + 1)$, the operator $S_{i,[n_i]}$ acts trivially on the vacuum vector $v \in \mathbb{U}_{x,r}$.

It is enough to prove this statement for x in the standard apartment \mathcal{A} . Note that

$$-\lceil \alpha_j(x) - r \rceil \leq -\alpha_j(x) + r.$$

Since $\sum_{j=1}^k \alpha_j(x) = 0$, we conclude that

$$-\sum_{j=1}^k \lceil \alpha_j(x) - r \rceil \leq kr.$$

By the previous corollary, $S_{i,[m]}$ annihilates v for $m \geq kr + (d_i + 1)$. Since $k \leq d_i + 1$, we see that $S_{i,[m]}$ annihilates v for all $m \geq (d_i + 1)(r + 1)$, as required. \square

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