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A WEITZENBÖCK FORMULA FOR CANONICAL METRICS ON FOUR-MANIFOLDS

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Abstract. We first provide an alternative proof of the classical Weitzenböck formula for Einstein four-manifolds using Berger curvature decomposition, motivated by which we establish a unified framework for a Weitzenböck formula for a large class of canonical metrics on four-manifolds. As applications, we classify Einstein four-manifolds and conformally Einstein four-manifolds with half two-nonnegative curvature operator, which in some sense provides a characterization of Kähler-Einstein metrics and Hermitian, Einstein metrics with positive scalar curvature on four-manifolds, respectively. We also discuss the classification of four-dimensional gradient shrinking Ricci solitons with half two-nonnegative curvature operator and half harmonic Weyl curvature.

1. Introduction

This is a sequel to the author's Ph.D. thesis [46] and earlier work [47, 48]. For an oriented Riemannian four-manifold $(M,\ g)$, the Hodge star operator $\star : \wedge^2 TM \to \wedge^2 TM$ induces an eigenspace decomposition $\wedge^2 TM = \wedge^+ M \oplus \wedge^- M$, where $\wedge^\pm M = \{\omega \in \wedge^2 TM : \star\omega = \pm\omega\}$ are eigenspaces of the Hodge star operator. Elements in $\wedge^\pm M$ are called self-dual and anti-self-dual 2-forms. This decomposition further induces the duality decomposition of the curvature operator $\mathfrak{R} : \wedge^2 TM \to \wedge^2 TM$,

$$\mathfrak{R} = \begin{pmatrix} \frac{R}{12} g + W^+ & \overset{\circ}{\mathrm{Ric}} \\ \overset{\circ}{\mathrm{Ric}} & \frac{R}{12} g + W^- \end{pmatrix}, \tag{1}$$

where R is the scalar curvature and $W^\pm$ are called self-dual and anti-self-dual parts of the Weyl curvature tensor. $(M,\ g)$ is called (anti-)self-dual, or half conformally flat, if $W^- = 0$ ($W^+ = 0$).

For Einstein four-manifolds, $\mathfrak{R}$, W, $\mathfrak{R}^\pm$, and $W^\pm$ are all harmonic. Using the harmonicity, Derdziński [19] derived the following Weitzenböck formula.

Theorem 1.1 ([19]). *Let (M, g) be an oriented Einstein four-manifold. Then*

$$\Delta |W^\pm|^2 = 2|\nabla W^\pm|^2 + R|W^\pm|^2 - 36 \det W^\pm,$$

where $\langle S, T\rangle = \frac{1}{4} S_{ijkl} T^{ijkl}$ for any $(0,4)$-tensor S, T.

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The Weitzenböck formula, together with Hitchin's classification of half conformally flat Einstein four-manifolds [3], plays a key role in the classification of Einstein four-manifolds of positive scalar curvature; see for example [26, 46, 48, 50]. In [26], Gursky and LeBrun obtained an optimal gap theorem for $W^{\pm}$ and classified Einstein four-manifolds with nonnegative sectional curvature operator and positive intersection form. In [50], Yang classified Einstein four-manifolds with $\mathrm{Ric} = g$ and sectional curvature bounded below by $\frac{\sqrt{1249}-23}{120}$. In the author's Ph.D. thesis [46], based on an observation on Berger curvature decomposition [2], the author investigated interesting relations between three-positive, four-positive curvature operator, positive isotropic curvature, and sectional curvature, and in [48] the author classified Einstein four-manifolds of three-nonnegative curvature operator, which in particular classified Einstein four-manifolds with $\mathrm{Ric} = g$ and $K \geq \frac{1}{12}$.

In this paper, first following from an argument in [46], we provide an alternative proof of the Weitzenböck formula in Theorem 1.1 by combining an argument of Hamilton (Lemma 7.2 in [27]) and Berger curvature decomposition [2]. As an application, using a similar argument in [48] we classify Einstein four-manifolds of half two-nonnegative curvature operator (half nonnegative isotropic curvature).

A Riemannian metric is said to have k-positive (k-nonnegative) curvature operator if the sum of any k eigenvalues is positive (nonnegative). A Riemannian metric on a four-manifold is said to have half two-positive (two-nonnegative) curvature operator if the self-dual curvature operator $\mathfrak{R}^+ = \frac{R}{12}g + W^+$ or the anti-self-dual curvature operator $\mathfrak{R}^- = \frac{R}{12}g + W^-$ is two-positive (two-nonnegative). A Riemannian metric g is said to have positive isotropic curvature if for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$,

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} > 2R_{1234}. \tag{2}$$

Similarly a Riemannian metric g on a four-manifold is said to have half positive isotropic curvature if condition (2) holds for every orthonormal four-frame $\{e_1, e_2, e_3, e_4\}$ of a fixed orientation.

By the duality decomposition, it is easy to check that half two-positive curvature operator and half positive isotropic curvature are equivalent, which are also equivalent to $W^{\pm} < \frac{R}{6}$; see also [43]. It is obvious that if a four-manifold $(M,\ g)$ is half conformally flat and has positive scalar curvature, then $\mathfrak{R}$ is half two-positive.

In [4], Brendle proved that Einstein manifolds with positive isotropic curvature are isometric to (S^n, g_0) and with nonnegative isotropic curvature are locally symmetric. We prove the following.

Theorem 1.2. *Let $(M,\ g)$ be an Einstein four-manifold with positive scalar curvature.*

(1) *If $\mathfrak{R}^+$ or $\mathfrak{R}^-$ is two-positive, then it is isometric to (S^4, g_0) or $(\mathbb{C}P^2, g_{FS})$.*

(2) *If $\mathfrak{R}^+$ or $\mathfrak{R}^-$ is two-nonnegative, then it is isometric to (S^4, g_0) or a Kähler-Einstein surface.*

(3) *If $\mathfrak{R}^+$ or $\mathfrak{R}^-$ is two-nonnegative and $\mathfrak{R}$ is four-nonnegative, then it is isometric to (S^4, g_0), $(\mathbb{C}P^2, g_{FS})$, or $(S^2 \times S^2,\ g_0 \oplus g_0)$, or their finite quotients.*

Remark. Theorem 1.2 was also proved by Richard and Seshadri [43]. They first proved that the cone of half nonnegative isotropic curvature is preserved along the Ricci flow, then applied an argument of Brendle in [4]. It also follows from the optimal gap theorem of Gursky and LeBrun [26]; see Fine, Krasnov, and Panov [23].

Recall that for any Kähler metric on a four-manifold, W^+ has eigenvalues $\{\frac{R}{6}, -\frac{R}{12}, -\frac{R}{12}\}$; hence any Kähler metric with nonnegative scalar curvature has half two-nonnegative curvature operator (or half nonnegative isotropic curvature). Derdzinski [19] proved that if a Riemannian metric on a four-manifold satisfies $\delta W^+ = 0$ and W^+ has at most two distinct eigenvalues, then the metric is locally conformally Kähler. If in addition the scalar curvature is constant, then the metric itself is Kähler. It is interesting to point out that part (2) of Theorem 1.2 in fact provides a characterization of Kähler-Einstein metrics of positive scalar curvature on four-manifolds: any Einstein metric which is not locally conformally flat and has half two-nonnegative curvature operator (half nonnegative isotropic curvature) is Kähler-Einstein.

Motivated by our alternative proof of the Weitzenböck formula for Einstein four-manifolds, we establish a unified framework for the Weitzenböck formula for a large class of metrics on four-manifolds which are called generalized m-quasi-Einstein metrics. Let (M^n, g) be a Riemannian manifold; g is called a *generalized m-quasi-Einstein metric* [11, 13] if

$$\mathrm{Ric}_f^m = \mathrm{Ric} + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{3}$$

for some $f, \lambda \in C^\infty(M)$ and $m \in \mathbb{R} \cup \{\pm\infty\}$.

Notice that Ric_f^m is exactly the m-Bakry-Emery Ricci curvature, introduced by Bakry and Emery [1], for smooth metric measure space $(M^n, g, m, e^{-f} dv)$; therefore a generalized m-quasi-Einstein metric on a Riemannian manifold should be considered as "an Einstein metric" on a smooth metric measure space. In fact, as is well known, Einstein metrics on manifolds are critical points of a normalized Einstein-Hilbert functional. Similarly when λ=const, "Einstein metrics" on smooth metric measure spaces are critical points of $\mathcal{F}$, the $\mathcal{W}$ functional of Perelman [39] and the $\mathcal{W}^+$ functional of Feldman, Ilmanen and Ni [21] (when $m = \infty$ and $\lambda = 0, > 0, < 0$ respectively, i.e., gradient Ricci solitons) and the $\mathcal{W}^m$ functional of Case [8, 9] (when $|m| < \infty$). On the other hand, Obata characterized critical points of the Yamabe functional on Einstein manifolds; similarly Perelman characterized critical points of the $\mathcal{W}$ functional on compact shrinking gradient Ricci solitons. Recently Miao, Tam [34, 35] and Case [10] investigated weighted Einstein metrics, which are critical points of the weighted Yamabe functional defined in [10].

In particular, "Einstein metrics" on smooth metric measure spaces contain at least the following interesting special cases:

(1) when f =const, it is an Einstein metric;

(2) when $m = \infty$ and λ =const, it is a gradient Ricci soliton;

(3) when $0 < m < \infty$ and λ =const, it is an m-quasi-Einstein metric, and $(M^n \times F^m, g \oplus e^{-\frac{2f}{m}} g_F)$ is a warped product Einstein manifold, where F^m is an m-dimensional Einstein manifold whose Einstein constant is determined by m, f, and λ (see [11]);

(4) when $m = 1$, it is a static metric in general relativity;

(5) when $m = 2 - n$, it is a conformally Einstein metric, and $\bar{g} = e^{\frac{2}{2-n} f} g$ is an Einstein metric;

(6) when $m = 2 - k - n$ with $k \in \mathbb{N}$ and λ =const, $(M^n \times F^k, g \oplus g_F)$ is a conformally Einstein manifold, where F^k is a k-dimensional Einstein manifold

whose Einstein constant is λ, and $\bar{g} = e^{\frac{2}{2-k-n}}(g \oplus g_F)$ is an Einstein metric (see [8]);

(7) when $\lambda = \frac{R_f^m + (m+n-2)\mu}{2(m+n-1)}$ for some $\mu \in \mathbb{R}$, where $R_f^m = R + 2\Delta f - \frac{m+1}{m}|\nabla f|^2$ is the weighted scalar curvature, it is a weighted Einstein metric (see [10]).

The Weitzenböck formula can be stated as follows.

Theorem 1.3. *Let (M, g) be a generalized m-quasi-Einstein four-manifold with $\mathrm{Ric}_f^m = \lambda g$. Then*

$$\begin{aligned}\Delta_f|W^{\pm}|^2 =& 2|\nabla W^{\pm}|^2 + (4\lambda + \frac{2}{m}|\nabla f|^2)|W^{\pm}|^2 - 36\det W^{\pm} \\ & -(1+\frac{2}{m})\langle(\nabla^2 f \circ \nabla^2 f)^{\pm}, W^{\pm}\rangle,\end{aligned}$$

where $T^{\pm}(\alpha, \beta) = T(\alpha^{\pm}, \beta^{\pm})$ for $\alpha, \beta \in \wedge^2 M$.

As special cases, we get the Weitzenböck formula for conformally Einstein four-manifolds and four-dimensional gradient Ricci solitons.

Corollary 1.1. *Let (M, g, f) be a conformally Einstein four-manifold with* $\mathrm{Ric} + \nabla^2 f + \frac{1}{2} df \otimes df = \lambda g$. *Then*

$$\begin{aligned}\Delta_f|W^{\pm}|^2 =& 2|\nabla W^{\pm}|^2 + (4\lambda - |\nabla f|^2)|W^{\pm}|^2 - 36\det W^{\pm}, \\ \Delta(e^{-f}|W^{\pm}|^2) =& 2|\nabla(e^{-\frac{f}{2}}W^{\pm})|^2 + e^{-f}(R|W^{\pm}|^2 - 36\det W^{\pm}).\end{aligned}$$

Corollary 1.2. *Let (M, g, f) be a four-dimensional gradient Ricci soliton with* $\mathrm{Ric} + \nabla^2 f = \lambda g$. *Then*

$$\Delta_f|W^{\pm}|^2 = 2|\nabla W^{\pm}|^2 + 4\lambda|W^{\pm}|^2 - 36\det W^{\pm} - \langle(\mathrm{Ric} \circ \mathrm{Ric})^{\pm}, W^{\pm}\rangle.$$

In [16], Chang, Gursky, and Yang derived an integral Weitzenböck formula for compact four-manifolds,

$$0 = \int_M 2|\nabla W^{\pm}|^2 - 8|\delta W^{\pm}|^2 + R|W^{\pm}|^2 - 36\det W^{\pm},$$

and they also derived an integral Weitzenböck formula for Bach-flat metrics, with the help of which they proved a very interesting conformally invariant sphere theorem in four dimensions.

The Weitzenböck formula for conformally Einstein four-manifolds in Corollary 1.1 can also be derived by using the property of the conformal change of $\delta W^{\pm}$ and the Weitzenböck formula for Einstein four-manifolds; see for example [19, 25, 33].

For applications of the Weitzenböck formula, we first classify conformally Einstein four-manifolds of half two-nonnegative isotropic curvature. The proof is based on an observation for half two-nonnegative curvature operator; see Lemma 4.1 in Section 4.

Theorem 1.4. *Let (M,g) be a conformally Einstein four-manifold with* $\mathrm{Ric}+\nabla^2 f+\frac{1}{2}df\otimes df=\lambda g$*; i.e.* $\bar{g}=e^{-f}g$ *is an Einstein metric.*

(1) *If g has half positive isotropic curvature, then $(M,\bar{g})$ is isometric to either (S^4,g_0) or $(\mathbb{C}P^2,g_{FS})$.*

(2) *If g has half nonnegative isotropic curvature, then $(M,\bar{g})$ is isometric to either (S^4,g_0) or a Hermitian, Einstein manifold.*

By the work of LeBrun [32], in part (2) of Theorem 1.4, $(M,\bar{g})$ is either Kähler-Einstein ($f=$const) or isometric to $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ with Page metric [38] or $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ with Chen-LeBrun-Weber metric [17].

Similarly to Theorem 1.2, part (2) of Theorem 1.4 in fact provides a characterization of Hermitian-Einstein metrics of positive scalar curvature on four-manifolds; that is, for any Einstein metric which is not locally conformally flat, if there exists a metric in its conformal class that has half NIC, then the Einstein metric itself is Hermitian-Einstein.

Next we observe the following from Theorem 1.1.

Theorem 1.5. *Let (M,g) be a compact four-dimensional Riemannian manifold. If $\delta W^{\pm}=0$ and $\mathfrak{R}^{\pm}$ is two-positive, then g is either self-dual or anti-self-dual. If $\delta W^{\pm}=0$ and $\mathfrak{R}^{\pm}$ is two-nonnegative, then either g is self-dual or anti-self-dual or g is a Kähler metric with constant scalar curvature.*

If in addition g is a gradient Ricci soliton, then we get the following triviality result.

Corollary 1.3. *Let (M,g,f) be a compact four-dimensional gradient shrinking Ricci soliton. If $\delta W^{\pm}=0$ and $\mathfrak{R}^{\pm}$ is two-nonnegative, then (M,g) is isometric to (S^4,g_0) or Kähler-Einstein.*

Gradient Ricci solitons were introduced by Hamilton [28]; they played an important role in the Ricci flow and Perelman's resolution to the Poincaré conjecture and the geometrization conjecture [39–41]. In the past three decades, there has been lots of work on the classification of gradient shrinking Ricci solitons. In dimensions 2 and 3, by [6, 28, 30, 37, 39], the classification is complete. In dimensions greater than or equal to 4, by [5, 7, 20, 22, 36, 37, 42, 51] and references therein, the classifications of gradient shrinking Ricci solitons with vanishing Bach tensor or harmonic Weyl curvature are complete. In particular in dimension 4, half conformally flat gradient shrinking Ricci solitons have been completely classified in [5, 18], which can be considered as an analogue of Hitchin's classical classification of half conformally flat Einstein four-manifolds.

Further applications of the Weitzenböck formula to gradient Ricci solitons and quasi-Einstein manifolds will be addressed in subsequent work [45, 49].

The paper is organized as following. In Section 2 we discuss Berger curvature decomposition, provide an alternative proof of the Weitzenböck formula for Einstein four-manifolds using Berger curvature decomposition, and classify Einstein four-manifolds of half two-nonnegative curvature operator. In Section 3, we prove the Weitzenböck formula for generalized quasi-Einstein manifolds. In Section 4, we prove Theorem 1.4 and Theorem 1.5. In the appendix we provide Berger's proof of Berger curvature decomposition.

2. Berger curvature decomposition and proofs of Theorems 1.1 and 1.2

We start from an interesting observation of Berger [2].

Lemma 2.1. *Let (M, g) be an oriented Einstein four-manifold. Then for any $p \in M$ and any orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of T_pM,*

$$K(e_1, e_2) = K(e_3, e_4),$$
$$K(e_1, e_3) = K(e_2, e_4),$$
$$K(e_1, e_4) = K(e_2, e_3).$$

In other words, Lemma 2.1 says that for an Einstein four-manifold,

$$R_{ijkl} = R_{i'j'k'l'},$$

where $(i'j')$ is the dual of the pair (ij), i.e., the pair such that $e_i \wedge e_j \pm e_{i'} \wedge e_{j'} \in \wedge^{\pm} M$. In other words, $(iji'j') = \sigma(1234)$ for some even permutation $\sigma \in S_4$.

Using Lemma 2.1 and basic symmetries of curvature tensor, Berger obtained the following curvature decomposition [2] for Einstein four-manifolds (see also [44]). See the appendix for the proof.

Proposition 2.1. *Let (M, g) be an Einstein four-manifold with $\mathrm{Ric} = \lambda g$. Then at any $p \in M$, there exists an orthonormal basis $\{e_i\}_{1 \le i \le 4}$ of T_pM such that relative to the corresponding basis $\{e_i \wedge e_j\}_{1 \le i < j \le 4}$ of $\wedge^2 T_pM$, $\mathfrak{R}$ takes the form*

$$\mathfrak{R} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where $A = diag\{a_1, a_2, a_3\}$, $B = diag\{b_1, b_2, b_3\}$ satisfying the following properties:

(1) $a_1 = K(e_1, e_2) = K(e_3, e_4) = \min\{K(\sigma) : \sigma \in \wedge^2 T_pM, ||\sigma|| = 1\}$,
$a_3 = K(e_1, e_4) = K(e_2, e_3) = \max\{K(\sigma) : \sigma \in \wedge^2 T_pM, ||\sigma|| = 1\}$,
$a_2 = K(e_1, e_3) = K(e_2, e_4)$, $a_1 + a_2 + a_3 = \lambda$;
(2) $b_1 = R_{1234}$, $b_2 = R_{1342}$, $b_3 = R_{1423}$;
(3) $|b_2 - b_1| \le a_2 - a_1$, $|b_3 - b_1| \le a_3 - a_1$, $|b_3 - b_2| \le a_3 - a_2$.

As observed in [46], Berger curvature decomposition is in fact a special case of the duality decomposition, as it is easy to see that the eigenvalues of $\mathfrak{R}$ are

$$a_1 + b_1 \le a_2 + b_2 \le a_3 + b_3,$$
$$a_1 - b_1 \le a_2 - b_2 \le a_3 - b_3,$$

with corresponding eigenvectors $\omega_1^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$, $\omega_2^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \pm e_4 \wedge e_2)$, $\omega_3^{\pm} = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3)$. In other words, for Einstein four-manifolds,

$$\mathfrak{R} = \begin{pmatrix} \frac{R}{12}g + W^+ & 0 \\ 0 & \frac{R}{12}g + W^- \end{pmatrix} = \begin{pmatrix} a_i + b_i & 0 \\ 0 & a_i - b_i \end{pmatrix}. \tag{4}$$

Therefore for Einstein four-manifolds, $\mathfrak{R}$ is half two-positive if and only if $(a_1 + a_2) \pm (b_1 + b_2) = \lambda - (a_3 \pm b_3) > 0$.

Huisken [29] observed the following.

Lemma 2.2. *Berger curvature decomposition works for every algebraic curvature tensor with constant trace on four-manifolds.*

Proof. The proof is the same as Berger's proof of Proposition 2.1. For the reader's convenience, we provide the proof in the appendix. $\square$

Let us first provide an alternative proof of the Weitzenböck formula for Einstein metrics on four-manifolds.

Proof of Theorem 1.1. The proof follows from Section 2.3 of [46]. Recall that in Lemma 7.2 of [27], Hamilton proved that for an Einstein manifold (M^n, g) with $\mathrm{Ric} = g$,

$$\Delta R_{ijkl} + 2Q(R)_{ijkl} = 2\lambda R_{ijkl}, \tag{5}$$

where $Q(R)_{ijkl} = B(R)_{ijkl} - B(R)_{ijlk} + B(R)_{ikjl} - B(R)_{iljk}$, and $B(R)_{ijkl} = g^{mn}g^{pq}R_{imjp}R_{knlq}$.

For an Einstein four-manifold, by the standard curvature decomposition, $W = \mathrm{Rm} - \frac{\lambda}{6} g \circ g$. Since W is traceless, we get

$$\begin{aligned}\Delta|W|^2 =& 2\langle \Delta W, W\rangle + 2|\nabla W|^2\\ =& 2\langle \Delta \mathrm{Rm}, W\rangle + 2|\nabla W|^2\\ =& 4\langle \lambda\mathrm{Rm} - Q(\mathrm{Rm}), W\rangle + 2|\nabla W|^2\\ =& 4\lambda|W|^2 - 4\langle Q(\mathrm{Rm}), W\rangle + 2|\nabla W|^2.\end{aligned}$$

For self-dual and anti-self-dual Weyl curvature, similarly we get

$$\Delta|W^{\pm}|^2 = 4\lambda|W^{\pm}|^2 - 4\langle Q(\mathrm{Rm})^{\pm}, W^{\pm}\rangle + 2|\nabla W^{\pm}|^2.$$

Using Berger curvature decomposition, it is a direct computation that (see [46])

$$\begin{aligned}Q(\mathrm{Rm})_{1212} = Q(\mathrm{Rm})_{3434} =& a_1^2 + b_1^2 + 2a_2a_3 + 2b_2b_3,\\ Q(\mathrm{Rm})_{1234} =& 2a_1b_1 + 2a_2b_3 + 2a_3b_2,\\ Q(\mathrm{Rm})_{1313} = Q(\mathrm{Rm})_{2424} =& a_2^2 + b_2^2 + 2a_1a_3 + 2b_1b_3,\\ Q(\mathrm{Rm})_{1342} =& 2a_2b_2 + 2a_1b_3 + 2a_3b_1,\\ Q(\mathrm{Rm})_{1414} = Q(\mathrm{Rm})_{2323} =& a_3^2 + b_3^2 + 2a_1a_2 + 2b_1b_2,\\ Q(\mathrm{Rm})_{1423} =& 2a_3b_3 + 2a_1b_2 + 2a_2b_1,\\ Q(\mathrm{Rm})_{ijik} =& 0, \text{ if } i \neq j \neq k.\end{aligned}$$

Recall that B has symmetries $B_{ijkl} = B_{jilk} = B_{klij}$, so we compute

$$\langle Q(\mathrm{Rm})^{\pm}, W^{\pm}\rangle = 9(\bar{a}_1 \pm b_1)(\bar{a}_2 \pm b_2)(\bar{a}_3 \pm b_3) = 9\det W^{\pm},$$

where $\bar{a}_i = a_i - \frac{\lambda}{3}$, which finishes the proof. $\square$

Now using the Weitzenböck formula and Berger curvature decomposition, we classify Einstein four-manifolds of half two-nonnegative curvature operator.

Proof of Theorem 1.2. We follow the arguments in [48]. Without loss of generality, we assume $\mathfrak{R}^+$ is half two-nonnegative.

If $R = 0$, then $\mathfrak{R}^+ = W^+$. Hence $\mathfrak{R}^+$ two-nonnegative implies $W^+ \equiv 0$.

If $R > 0$, without loss of generality, we assume $\mathrm{Ric} = g$. Integrate the Weitzenböck formula,

$$0 = \int_M \Delta|W^+|^2 = \int_M 2|\nabla W^+|^2 + (4|W^+|^2 - 36\det W^+). \tag{6}$$

Let $a = a_1 + b_1 - \frac{1}{3},\ b = a_2 + b_2 - \frac{1}{3},\ c = a_3 + b_3 - \frac{1}{3}$ be eigenvalues of W^+. By Berger curvature decomposition we have

$$f = 4|W^+|^2 - 36 \det W^+ = 8(a^2 + ac + c^2) + 36ac(a + c).$$

(1) If $\mathfrak{R}^+$ is two-positive, then $0 \le c < \frac{2}{3},\ -2c \le a \le -\frac{c}{2}$. Taking the first derivative of f, we get $f_a = (2a + c)(8 + 36c) \le 0$. Hence the minimum of f is attained at $a = -\frac{c}{2}$, at which

$$f = 6c^2(2 - 3c) \ge 0, \tag{7}$$

with equality if and only if $c = 0$, i.e., $W^+ = 0$.

Therefore by equation (6) we get $W^+ \equiv 0$, i.e., $(M,\ g)$ is anti-self-dual.

(2) If $\mathfrak{R}^+$ is two-nonnegative and $W^+ \not\equiv 0$, then by (6) and (7) we get

$$\nabla W^+ = 0, \qquad a = b = -\frac{1}{3},\ c = \frac{2}{3}. \tag{8}$$

Therefore by a theorem of Derdzinski [19], $(M,\ g)$ is a Kähler-Einstein manifold.

(3) Let $x = a_1 - b_1 - \frac{1}{3},\ y = a_2 - b_2 - \frac{1}{3},\ z = a_3 - b_3 - \frac{1}{3}$ be eigenvalues of W^-. If $\mathfrak{R}^+$ is two-nonnegative and $\mathfrak{R}$ is four-nonnegative, assuming $W^+ \not\equiv 0$, then by equation (8),

$$a + b + x + y + \frac{4}{3} = x + y + \frac{2}{3} \ge 0,$$

so $\mathfrak{R}^-$ is also two-nonnegative. By the same argument as above, if $W^- \not\equiv 0$, then

$$\nabla W^- = 0, \qquad x = y = -\frac{1}{3},\ z = \frac{2}{3}.$$

Therefore $\nabla R = 0$ (hence (M, g) is locally symmetric) and $\mathfrak{R}$ has eigenvalues $\{0,\ 0,\ 1,\ 0,\ 0,\ 1\}$. By the classification of four-dimensional symmetric spaces, it is isometric to $(S^2 \times S^2, g_0 \oplus g_0)$ or its finite quotient. □

3. Weitzenböck formula for "Einstein" smooth metric measure spaces

Similar to the proof of Theorem 1.1, we first derive a Weitzenböck formula for the curvature tensor. We start from the following basic lemma (see [11]).

Lemma 3.1. *Let (M^n, g) be a generalized quasi-Einstein manifold with $\mathrm{Ric}_f^m = \lambda g$. Then*

$$\begin{aligned}\nabla_i R_{jk} - \nabla_j R_{ik} =& (\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik}) - R_{ijkl} \nabla_l f \\ &+ \frac{1}{m}(\lambda g_{ik} \nabla_j f - \lambda g_{jk} \nabla_i f + R_{jk} \nabla_i f - R_{ik} \nabla_j f), \\ \nabla_i R =& (n-1)\nabla_i \lambda + 2R_{ij}\nabla_j f - \frac{2}{m} R_{ij} \nabla_j f + \frac{2}{m}[R - (n-1)\lambda]\nabla_i f.\end{aligned}$$

Proof. It follows directly from the Ricci identity that

$$\begin{aligned}
&\nabla_i R_{jk} - \nabla_j R_{ik}\\
=&\nabla_i(\lambda g_{jk} + \frac{1}{m}\nabla_j f \nabla_k f - \nabla_j\nabla_k f) - \nabla_j(\lambda g_{ik} + \frac{1}{m}\nabla_i f\nabla_k f - \nabla_i\nabla_k f)\\
=&(\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik}) + \frac{1}{m}(\nabla_i\nabla_k f\nabla_j f - \nabla_j\nabla_k f\nabla_i f) + (\nabla_j\nabla_i\nabla_k f - \nabla_i\nabla_j\nabla_k f)\\
=&(\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik}) - R_{ijkl}\nabla_l f + \frac{1}{m}(\nabla_i\nabla_k f\nabla_j f - \nabla_j\nabla_k f\nabla_i f)\\
=&(\nabla_i \lambda g_{jk} - \nabla_j \lambda g_{ik}) - R_{ijkl}\nabla_l f + \frac{1}{m}(\lambda g_{ik}\nabla_j f - \lambda g_{jk}\nabla_i f + R_{jk}\nabla_i f - R_{ik}\nabla_j f).
\end{aligned}$$

Taking the trace we get the second equation. □

Using Lemma 3.1 and Hamilton's argument we get the following.

Proposition 3.1. *Let* (M^n, g) *be a generalized quasi-Einstein manifold with* $\mathrm{Ric}_f^m = \lambda g$. *Then*

$$\begin{aligned}
\Delta_f R_{ijkl} =& 2\lambda R_{ijkl} - 2Q(R)_{ijkl} + (\nabla^2\lambda \circ g)_{ijkl} + \frac{1}{m}[(Ric - \lambda g)\circ \nabla^2 f]_{ijkl}\\
&+\frac{1}{m^2}[(Ric-\lambda g)\circ df\otimes df]_{ijkl} + \frac{1}{m}(\nabla\lambda\otimes\nabla f\circ g)_{ijkl}\\
&+\frac{1}{m}[R_{ipkl}\nabla_j f\nabla^p f + R_{ijkp}\nabla_l f\nabla^p f - R_{jpkl}\nabla_i f\nabla^p f - R_{ijlp}\nabla_k f\nabla^p f].
\end{aligned}$$

Proof. By the Ricci identity, we get (see Lemma 7.2 in Hamilton [27])

$$\begin{aligned}
\Delta R_{ijkl} =& \nabla^p\nabla_p R_{ijkl}\\
=& \nabla^p\nabla_i R_{pjkl} - \nabla_p\nabla_j R_{pikl}\\
=& \nabla_i(\nabla_k R_{jl} - \nabla_l R_{jk}) - \nabla_j(\nabla_k R_{il} - \nabla_l R_{ik})\\
&-2Q(R)_{ijkl} + (R_i{}^q R_{qjkl} - R_j{}^q R_{qikl}).
\end{aligned}$$

Applying Lemma 3.1 repeatedly to the first two terms on the right hand side, we get

$$\begin{aligned}
&\nabla_i(\nabla_k R_{jl} - \nabla_l R_{jk}) - \nabla_j(\nabla_k R_{il} - \nabla_l R_{ik})\\
=&\nabla_i[(\nabla_k\lambda g_{jl} - \nabla_l\lambda g_{jk}) - R_{kljp}\nabla^p f\\
&+\frac{1}{m}(\lambda g_{jk}\nabla_l f - \lambda g_{jl}\nabla_k f + R_{jl}\nabla_k f - R_{jk}\nabla_l f)]\\
&-\nabla_j[(\nabla_k\lambda g_{il} - \nabla_l\lambda g_{ik}) - R_{klip}\nabla^p f\\
&+\frac{1}{m}(\lambda g_{ik}\nabla_l f - \lambda g_{il}\nabla_k f + R_{il}\nabla_k f - R_{ik}\nabla_l f)]\\
=&(-\nabla_i R_{jpkl}\nabla_p f - \nabla_j R_{pikl}\nabla_p f) + (R_{ipkl}\nabla_j\nabla_p f - R_{jpkl}\nabla_i\nabla_p f)\\
&+\frac{1}{m}[\lambda g_{jk}\nabla_i\nabla_l f - \lambda g_{jl}\nabla_i\nabla_k f + \nabla_i R_{jl}\nabla_k f + R_{jl}\nabla_i\nabla_k f - \nabla_i R_{jk}\nabla_l f - R_{jk}\nabla_i\nabla_l f]\\
&-\frac{1}{m}[\lambda g_{ik}\nabla_j\nabla_l f - \lambda g_{il}\nabla_j\nabla_k f + \nabla_j R_{il}\nabla_k f + R_{il}\nabla_j\nabla_k f - \nabla_j R_{ik}\nabla_l f - R_{ik}\nabla_j\nabla_l f]\\
=&(\nabla^2\lambda\circ g)_{ijkl} - (\nabla_i R_{jpkl}\nabla^p f + \nabla_j R_{pikl}\nabla^p f) + (R_{ipkl}\nabla_j\nabla^p f - R_{jpkl}\nabla_i\nabla^p f)\\
&+\frac{1}{m}[(\mathrm{Ric}-\lambda g)\circ\nabla^2 f]_{ijkl} + \frac{1}{m}[(\nabla_i R_{jl} - \nabla_j R_{il})\nabla_k f - (\nabla_i R_{jk} - \nabla_j R_{ik})\nabla_l f]
\end{aligned}$$

$$
\begin{aligned}
=&(-\nabla_i R_{jpkl}\nabla_p f - \nabla_j R_{pikl}\nabla_p f) + (R_{ipkl}\nabla_j\nabla_p f - R_{jpkl}\nabla_i\nabla_p f)\\
&+\frac{1}{m}[(\mathrm{Ric}-\lambda g)\circ\nabla^2 f]_{ijkl} + \frac{1}{m}(R_{kpij}\nabla_l f\nabla_p f - R_{lpij}\nabla_k f\nabla_p f)\\
&+\frac{1}{m^2}[\lambda g_{il}\nabla_j f\nabla_k f - \lambda g_{jl}\nabla_i f\nabla_k f + R_{jl}\nabla_i f\nabla_k f - R_{il}\nabla_j f\nabla_k f]\\
&-\frac{1}{m^2}[\lambda g_{ik}\nabla_j f\nabla_l f - \lambda g_{jk}\nabla_i f\nabla_l f + R_{jk}\nabla_i f\nabla_l f - R_{ik}\nabla_j f\nabla_l f]\\
=&(\nabla^2\lambda\circ g)_{ijkl} + \nabla_p R_{ijkl}\nabla^p f + (R_{ipkl}\nabla_j\nabla^p f - R_{jpkl}\nabla_i\nabla^p f)\\
&+\frac{1}{m}(R_{kpij}\nabla_l f\nabla^p f - R_{lpij}\nabla_k f\nabla^p f) + \frac{1}{m}[(\mathrm{Ric}-\lambda g)\circ\nabla^2 f]_{ijkl}\\
&+\frac{1}{m}(\nabla\lambda\otimes\nabla f\circ g)_{ijkl} + \frac{1}{m^2}[(\mathrm{Ric}-\lambda g)\circ df\otimes df]_{ijkl}.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&\Delta_f R_{ijkl}\\
=&-2Q(R)_{ijkl} + (R_{ipkl}R_j{}^p - R_{jpkl}R_i{}^p) + (\nabla^2\lambda\circ g)_{ijkl}\\
&+(R_{ipkl}\nabla_j\nabla^p f - R_{jpkl}\nabla_i\nabla^p f)\\
&+\frac{1}{m}(R_{kpij}\nabla_l f\nabla^p f - R_{lpij}\nabla_k f\nabla^p f) + \frac{1}{m}[(\mathrm{Ric}-\lambda g)\circ\nabla^2 f]_{ijkl}\\
&+\frac{1}{m}(\nabla\lambda\otimes\nabla f\circ g)_{ijkl} + \frac{1}{m^2}[(\mathrm{Ric}-\lambda g)\circ df\otimes df]_{ijkl}\\
=&2\lambda R_{ijkl} - 2Q(R)_{ijkl} + (\nabla^2\lambda\circ g)_{ijkl} + \frac{1}{m}(\nabla\lambda\otimes\nabla f\circ g)_{ijkl}\\
&+\frac{1}{m}[(\mathrm{Ric}-\lambda g)\circ\nabla^2 f]_{ijkl} + \frac{1}{m^2}[(\mathrm{Ric}-\lambda g)\circ df\otimes df]_{ijkl}\\
&+\frac{1}{m}[R_{ipkl}\nabla_j f\nabla^p f - R_{jpkl}\nabla_i f\nabla^p f + R_{kpij}\nabla_l f\nabla^p f - R_{lpij}\nabla_k f\nabla^p f]. \qquad \square
\end{aligned}
$$

Applying the standard curvature decomposition and Berger curvature decomposition, we prove the Weitzenböck formula for $W^{\pm}$.

Proof of Theorem 1.3. We need to express $\Delta_f R_{ijkl}$ in terms of Weyl curvature using the standard curvature decomposition,

$$
R_{ijkl} = -\frac{R}{2(n-1)(n-2)}(g\circ g)_{ijkl} + \frac{1}{n-2}(Ric\circ g)_{ijkl} + W_{ijkl}.
$$

First we have (see Catino and Mantegazza [14])

$$
\begin{aligned}
2Q(\mathrm{Rm})_{ijkl} = &\, 2Q(W)_{ijkl} + \frac{2(n-1)|\mathrm{Ric}|^2 - 2R^2}{2(n-1)(n-2)^2}(g\circ g)_{ijkl}\\
&-+\frac{2}{n-2}(R_{ik}R_{lj} - R_{il}R_{jk})\\
&-\frac{2}{(n-2)^2}(\mathrm{Ric}^2\circ g)_{ijkl} + \frac{2R}{(n-1)(n-2)^2}(\mathrm{Ric}\circ g)_{ijkl}\\
&+\frac{2}{n-2}(W_{ipkq}R^{pq}g_{jl} - W_{jpkq}R^{pq}g_{il} + W_{jplq}R^{pq}g_{ik} - W_{iplq}R^{pq}g_{jk}).
\end{aligned}
$$

Similarly we compute

$$
\begin{aligned}
&\frac{1}{m}[R_{ipkl}\nabla_j f\nabla^p f - R_{jpkl}\nabla_i f\nabla^p f + R_{kpij}\nabla_l f\nabla^p f - R_{lpij}\nabla_k f\nabla^p f]\\
=&\frac{1}{m}[W_{ipkl}\nabla_j f\nabla_p f + W_{ijkp}\nabla_l f\nabla_p f - W_{jpkl}\nabla_i f\nabla_p f - W_{ijlp}\nabla_k f\nabla_p f]\\
+&\frac{1}{m}[\frac{1}{n-2}(R_{ik}\nabla_j f\nabla_l f - R_{il}\nabla_j f\nabla_k f + g_{ik}R_{lp}\nabla_j f\nabla_p f - g_{il}R_{kp}\nabla_j f\nabla_p f)\\
&-\frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f\nabla_l f - g_{il}\nabla_j f\nabla_k f)]\\
-&\frac{1}{m}[\frac{1}{n-2}(R_{jk}\nabla_i f\nabla_l f - R_{jl}\nabla_i f\nabla_k f + g_{jk}R_{lp}\nabla_i f\nabla_p f - g_{jl}R_{kp}\nabla_i f\nabla_p f)\\
&-\frac{R}{(n-1)(n-2)}(g_{jk}\nabla_i f\nabla_l f - g_{jl}\nabla_i f\nabla_k f)]\\
+&\frac{1}{m}[\frac{1}{n-2}(R_{ik}\nabla_j f\nabla_l f - R_{jk}\nabla_i f\nabla_l f + g_{ik}R_{jp}\nabla_l f\nabla_p f - g_{jk}R_{ip}\nabla_l f\nabla_p f)\\
&-\frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f\nabla_l f - g_{jk}\nabla_i f\nabla_l f)]\\
-&\frac{1}{m}[\frac{1}{n-2}(R_{il}\nabla_j f\nabla_k f - R_{jl}\nabla_i f\nabla_k f + g_{il}R_{jp}\nabla_k f\nabla_p f - g_{jl}R_{ip}\nabla_k f\nabla_p f)\\
&-\frac{R}{(n-1)(n-2)}(g_{il}\nabla_j f\nabla_k f - g_{jl}\nabla_i f\nabla_k f)]\\
=&\frac{1}{m}[W_{ipkl}\nabla_j f\nabla^p f + W_{ijkp}\nabla_l f\nabla^p f - W_{jpkl}\nabla_i f\nabla^p f - W_{ijlp}\nabla_k f\nabla^p f]\\
-&\frac{2R}{m(n-1)(n-2)}(g\circ df\otimes df)_{ijkl}\\
+&\frac{2}{m(n-2)}\Big[(\mathrm{Ric}\circ df\otimes df)_{ijkl} + [\nabla f\otimes \mathrm{Ric}(\nabla f)\circ g]_{ijkl}\Big].
\end{aligned}
$$

Since W is traceless, $\langle \alpha\circ g, W\rangle = 0$ for any $(0,2)$-tensor α. Therefore we get the following Weitzenböck formula for Weyl curvature:

$$
\begin{aligned}
&\Delta_f|W|^2\\
=&2\langle\Delta_f W, W\rangle + 2|\nabla W|^2\\
=&2\langle\Delta_f \mathrm{Rm}, W\rangle + 2|\nabla W|^2\\
=&2|\nabla W|^2 + 4\lambda|W|^2 - W^{ijkl}Q(W)_{ijkl} - \frac{1}{2(n-2)}(\mathrm{Ric}\circ\mathrm{Ric})_{ijkl}W^{ijkl}\\
+&\frac{1}{2m}W^{ijkl}[W_{ipkl}\nabla_j f\nabla^p f + W_{ijkp}\nabla_l f\nabla^p f - W_{jpkl}\nabla_i f\nabla^p f - W_{ijlp}\nabla_k f\nabla^p f] \qquad (9)\\
+&\frac{1}{2m}(\mathrm{Ric}\circ\nabla^2 f)_{ijkl}W^{ijkl} + \Big[\frac{1}{2m^2} + \frac{1}{m(n-2)}\Big](\mathrm{Ric}\circ df\otimes df)_{ijkl}W^{ijkl}\\
=&2|\nabla W|^2 + 4\lambda|W|^2 - 4\langle W, Q(W)\rangle - \frac{2}{n-2}\langle \mathrm{Ric}\circ\mathrm{Ric}, W\rangle\\
+&\frac{8}{m}|\iota_{\nabla f}W|^2 + \frac{2}{m}\langle \mathrm{Ric}\circ\nabla^2 f, W\rangle + \Big[\frac{2}{m^2} + \frac{4}{m(n-2)}\Big]\langle \mathrm{Ric}\circ df\otimes df, W\rangle.
\end{aligned}
$$

Next we derive the Weitzenböck formula for $W^{\pm}$ on four-manifolds. For any $\sigma \in S_4$, denote $\sigma(ijkl) = (\sigma_i\sigma_j\sigma_k\sigma_l)$. It is a direct computation that

$$\langle(\alpha \circ g)_{ijkl}, W^{\sigma_i\sigma_j\sigma_k\sigma_l}\rangle = \frac{1}{4}(\alpha_{ik}g_{jl} + \alpha_{jl}g_{ik} - \alpha_{il}g_{jk} - \alpha_{jk}g_{il})W^{\sigma_i\sigma_j\sigma_k\sigma_l} = 0,$$

that is,

$$\langle(\alpha \circ g)^{\pm}, W^{\pm}\rangle = 0.$$

Recall that for any (ij), $(i'j')$ is defined to be the pair such that $e_i \wedge e_j \pm e_{i'} \wedge e_{j'} \in \wedge^{\pm}M$, and for any $(0,4)$-tensor T,

$$\begin{aligned} T^{\pm}_{ijkl} =& \frac{1}{4}T(e_i \wedge e_j \pm e_{i'} \wedge e_{j'}, e_k \wedge e_l \pm e_{k'} \wedge e_{l'}) \\ =& \frac{1}{4}(T_{ijkl} \pm T_{ijk'l'} \pm T_{i'j'kl} + T_{i'j'k'l'}). \end{aligned}$$

So we get

$$\begin{aligned} &\Delta|W^{\pm}|^2 \\ =&2\langle\Delta W^{\pm}, W^{\pm}\rangle + 2|\nabla W^{\pm}|^2 \\ =&2\langle\Delta R^{\pm}, W^{\pm}\rangle + 2|\nabla W^{\pm}|^2 \\ =&\frac{1}{8}\langle\Delta(R_{ijkl} \pm R_{ijk'l'} \pm R_{i'j'kl} + R_{i'j'k'l'}), (W^{ijkl} \pm W^{ijk'l'} \pm W^{i'j'kl} + W^{i'j'k'l'})\rangle \\ &+ 2|\nabla W^{\pm}|^2. \end{aligned}$$

Similarly as in the proof of Theorem 1.1, using the Berger curvature decomposition for $W^{\pm}$, we get

$$W^{\pm\, ijkl}Q(W)^{\pm}_{ijkl} = 36 \det W^{\pm}.$$

Therefore from equation (9), we have

$$\begin{aligned} &\Delta_f|W^{\pm}|^2 \\ =&2|\nabla W^{\pm}|^2 + 4\lambda|W^{\pm}|^2 - 36\det W^{\pm} - \langle(\mathrm{Ric}\circ\mathrm{Ric})^{\pm}, W^{\pm}\rangle \\ &+\frac{2}{m}\langle(\mathrm{Ric}\circ\nabla^2 f)^{\pm}, W^{\pm}\rangle + (\frac{2}{m} + \frac{2}{m^2})\langle(\mathrm{Ric}\circ df\otimes df)^{\pm}, W^{\pm}\rangle \\ &+\frac{1}{2m}W^{\pm\, ijkl}\Big[W_{ipkl}\nabla_j f\nabla^p f + W_{ijkp}\nabla_l f\nabla^p f - W_{jpkl}\nabla_i f\nabla^p f - W_{ijlp}\nabla_k f\nabla^p f\Big]^{\pm} \\ =&2|\nabla W^{\pm}|^2 + 4\lambda|W^{\pm}|^2 - 36\det W^{\pm} \\ &+(1 + \frac{2}{m})\langle(\mathrm{Ric}\circ\mathrm{Ric})^{\pm}, W^{\pm}\rangle + (2 + \frac{4}{m})\langle(\mathrm{Ric}\circ\nabla^2 f)^{\pm}, W^{\pm}\rangle \\ &+\frac{1}{2m}W^{\pm\, ijkl}\Big[W_{ipkl}\nabla_j f\nabla^p f \pm W_{i'pkl}\nabla_{j'} f\nabla^p f \\ &\qquad\qquad \pm W_{ipk'l'}\nabla_j f\nabla^p f + W_{i'pk'l'}\nabla_{j'} f\nabla^p f\Big] \\ =&2|\nabla W^{\pm}|^2 + 4\lambda|W^{\pm}|^2 - 36\det W^{\pm} - (1 + \frac{2}{m})\langle(\nabla^2 f\circ\nabla^2 f)^{\pm}, W^{\pm}\rangle \\ &+\frac{1}{2m}W^{\pm\, ijkl}\Big[W_{ipkl}\nabla_j f\nabla^p f \pm W_{i'pkl}\nabla_{j'} f\nabla^p f \\ &\qquad\qquad \pm W_{ipk'l'}\nabla_j f\nabla^p f + W_{i'pk'l'}\nabla_{j'} f\nabla^p f\Big]. \end{aligned}$$

By the symmetry $W^{\pm}_{ijkl} = \pm W^{\pm}_{i'j'kl} = \pm W^{\pm}_{ijk'l'} = W^{\pm}_{i'j'k'l'}$, we have

$$\begin{aligned}
&\frac{1}{2m} W^{\pm\, ijkl}[W_{ipkl}\nabla_j f \nabla^p f \pm W_{i'pkl}\nabla_{j'} f \nabla^p f \pm W_{ipk'l'}\nabla_j f \nabla^p f + W_{i'pk'l'}\nabla_{j'} f \nabla^p f] \\
&= \frac{2}{m} W^{\pm\, ijkl} W_{ipkl} \nabla_j f \nabla^p f \\
&= \frac{8}{m} \langle \iota_{\nabla f} W^{\pm}, \iota_{\nabla f} W \rangle.
\end{aligned}$$

By definition $\langle W^+, W^- \rangle = 0$. Using Berger curvature decomposition, it is easy to verify the following.

Lemma 3.2. *Let (M, g) be a four-manifold. Then for any $f \in C^\infty(M)$,*

$$\begin{aligned}
\langle \iota_{\nabla f} W^+, \iota_{\nabla f} W^- \rangle &= \frac{1}{4} W^{+\, ipkl} W^-_{iqkl} \nabla_p f \nabla^q f = 0, \\
|\iota_{\nabla f} W^{\pm}|^2 &= \frac{1}{4} W^{\pm\, ipkl} W^{\pm}_{iqkl} \nabla_p f \nabla^q f = \frac{1}{4} |W^{\pm}|^2 |\nabla f|^2.
\end{aligned}$$

Therefore we obtain

$$\frac{8}{m} \langle \iota_{\nabla f} W^{\pm}, \iota_{\nabla f} W \rangle = \frac{8}{m} \langle \iota_{\nabla f} W^{\pm}, \iota_{\nabla f} (W^+ + W^-) \rangle = \frac{2}{m} |W^{\pm}|^2 |\nabla f|^2.$$

This completes the proof of Theorem 1.3. □

4. Applications

Motivated by the proof of Theorem 1.2, we observe the following for any four-manifold of half two-nonnegative curvature operator.

Lemma 4.1. *Let $(M^4,\ g)$ be a four-manifold. If $\mathfrak{R}^{\pm}$ is two-nonnegative, then*

$$R|W^{\pm}|^2 - 36 \det W^{\pm} \geq 0.$$

Furthermore, if $\mathfrak{R}^{\pm}$ is two-positive, the equality holds if and only if $W^{\pm} = 0$. If $\mathfrak{R}^{\pm}$ is two-nonnegative, the equality holds if and only if $W^{\pm} = 0$ or $W^{\pm}$ has eigenvalues $\{-\frac{R}{12},\ -\frac{R}{12},\ \frac{R}{6}\}$.

Proof of Lemma 4.1. The proof is similar to Theorem 1.2. Without loss of generality, we assume $\mathfrak{R}^-$ is two-nonnegative. Let $x \leq y \leq z$ be eigenvalues of $\mathfrak{R}^-$ and let

$$f = R|W^-|^2 - 36 \det W^- = 2R(x^2 + xz + z^2) + 36xz(x + z).$$

Taking the derivative, we get $f_x = 2(2x + z)(R + 18z) \leq 0$, so the minimum of f is attained at $x = -\frac{z}{2}$, at which

$$f = \frac{3}{2} z^2 (R - 6z).$$

If $\mathfrak{R}^-$ is two-nonnegative, then $x + y + \frac{R}{6} \geq 0$, so $\frac{R}{6} - z \geq 0$. Therefore $f \geq 0$.

Moreover, if $\mathfrak{R}^-$ is two-positive, $f = 0$ if and only if $W^- = 0$. If $\mathfrak{R}^-$ is two-nonnegative, $f = 0$ if and only if $W^- = 0$ or W^- has eigenvalues $\{-\frac{R}{12},\ -\frac{R}{12},\ \frac{R}{6}\}$. □

Proof of Theorem 1.4. Taking the trace of the conformally Einstein equation,

$$R + \Delta f + \frac{1}{2} |\nabla f|^2 = 4\lambda,$$

so we have

$$\begin{aligned}\Delta_f e^f =&(4\lambda - R - \frac{1}{2}|\nabla f|^2)e^f\\ =&2|\nabla e^{\frac{1}{2}f}|^2 + (4\lambda - R - |\nabla f|^2)e^f.\end{aligned}$$

Letting $u = e^{\frac{1}{2}f}$ we get

$$\Delta_f u^2 = 2|\nabla u|^2 + (4\lambda - R - |\nabla f|^2)u^2.$$

Therefore we have

$$\begin{aligned}\Delta_f\Big(\frac{|W^\pm|^2}{u^2}\Big) =&\frac{\Delta_f|W^\pm|^2}{u^2} - \frac{|W^\pm|^2\Delta_f u^2}{u^4} - 2\frac{\nabla|W^\pm|^2\nabla u^2}{u^4} + 2\frac{|W^\pm|^2|\nabla u^2|^2}{u^6}\\ =&\frac{2}{u^4}|u\nabla W^\pm - W^\pm\nabla u|^2 - \Big\langle\nabla\Big(\frac{|W^\pm|^2}{u^2}\Big), \nabla\ln u^2\Big\rangle\\ &+\frac{1}{u^2}(R|W^\pm|^2 - 36\det W^\pm),\end{aligned}$$

that is,

$$\Delta\Big(\frac{|W^\pm|^2}{u^2}\Big) = \frac{2}{u^4}|u\nabla W^\pm - W^\pm\nabla u|^2 + \frac{1}{u^2}(R|W^\pm|^2 - 36\det W^\pm). \tag{10}$$

Therefore if g has half positive isotropic curvature, then by equation (10), $W^\pm \equiv 0$. Since $W^\pm$ is conformally invariant, $\bar{g}$ is a half conformally flat Einstein metric; hence by the classical result of Hitchin [3], $(M, \bar{g})$ is isometric to (S^4, g_0) or $(\mathbb{C}P^2, g_{FS})$.

If g has half nonnegative isotropic curvature, then either $W^\pm \equiv 0$ or $W^\pm$ has eigenvalues $\{\frac{R}{6}, -\frac{R}{12}, -\frac{R}{12}\}$, and $u\nabla W^\pm - W^\pm\nabla u = 0$, which implies that $R \equiv ce^{\frac{1}{2}f}$.

If $W^\pm \equiv 0$, then $(M, \bar{g})$ is isometric to (S^4, g_0) or $(\mathbb{C}P^2, g_{FS})$.

If $W^\pm \not\equiv 0$, without loss of generality, assume $W^+ \not\equiv 0$. Then W^+ has eigenvalues $\{\frac{R}{6}, -\frac{R}{12}, -\frac{R}{12}\}$. Since W^+ is conformally invariant, eigenvalues of $\overline{W}^+$ are

$$\frac{R}{6}e^f, -\frac{R}{12}e^f, -\frac{R}{12}e^f. \tag{11}$$

Therefore $\bar{\delta}\,\overline{W}^+ = 0$ and $\overline{W}^+$ has only two distinct eigenvalues, so by Proposition 5 in [19], $\bar{g}$ is conformal to a Kähler metric. In fact $\tilde{g} = (24\|\overline{W}^+\|^2)^{\frac{1}{3}}\bar{g} = e^f\bar{g} = g$ is a Kähler metric. Therefore $\bar{g}$ is a Hermitian-Einstein metric, and by the work of LeBrun [32], $(M, \bar{g})$ is either Kähler-Einstein ($f =$const) or $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ with Page metric or $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ with Chen-LeBrun-Weber metric. □

Proof of Theorem 1.5. Assume for example W^+ is harmonic and $\mathfrak{R}^+$ is two-nonnegative. By the integral Weitzenböck formula (1), we get

$$0 = \int_M 2|\nabla W^+|^2 + R|W^+|^2 - 36\det W^+.$$

Therefore by Lemma 4.1,

$$\nabla W^+ \equiv 0 \quad\text{and}\quad R|W^+|^2 - 36\det W^+ \equiv 0.$$

If $\mathfrak{R}^+$ is two-positive, then $W^+ \equiv 0$.

If $\mathfrak{R}^+$ is two-nonnegative, then by Lemma 4.1, either $W^+ \equiv 0$ or W^+ has eigenvalues $\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}$. If $W^+ \not\equiv 0$, then $\nabla W^+ \equiv 0$ implies that $R \equiv$const, and again by Proposition 5 in [19], g is a cscK metric.

If in addition (M,g,f) is a gradient shrinking Ricci soliton and if $W^{\pm}=0$, then by the work of Chen-Wang [18] or Cao-Chen [5], (M,g,f) must be isometric to $(S^4,\ g_0)$ or $(\mathbb{C}P^2,\ g_{FS})$.

If $W^{\pm}\not\equiv 0$, then by the soliton equation $R+\Delta f=4\lambda$, $R\equiv$const implies $f\equiv$const; therefore g is a Kähler-Einstein metric. □

Appendix A. Proof of Berger curvature decomposition

The proof is translated directly from Berger's paper [2].

Let $P\subset T_pM$ be the 2-plane such that the sectional curvature attains its minimum on P. Let $P^{\perp}$ be the 2-plane that is orthogonal to P; choose $e_1\in P,\ e_2\in P^{\perp}$ such that $K(e_1,e_2)\geq K(X,Y)$ for any $X\in P,\ Y\in P^{\perp}$. Expand $\{e_1,\ e_2\}$ to an orthonormal basis $\{e_1,\ e_2,\ e_3,\ e_4\}$ such that $e_3\in P,\ e_4\in P^{\perp}$.

By the choice of P, $K(X,Y)\geq K(e_1,e_3)$ for any $X,\ Y\in T_pM$; in particular, $K(X,e_3)\geq K(e_1,e_3)$, $K(Y,e_1)\geq K(e_1,e_3)$ for any $X,\ Y\in T_pM$.

Let $X=e_1\cos t+e_2\sin t$, $Y=e_3\cos t+e_2\sin t$, $0\leq t\leq\delta$. By the variation principle we get

$$0=\frac{d}{dt}\Big|_{t=0}K(X,e_3)=2R_{1323},\qquad 0=\frac{d}{dt}\Big|_{t=0}K(Y,e_1)=2R_{1312}.$$

Similarly let $X=e_1\cos t+e_4\sin t$, $Y=e_3\cos t+e_4\sin t$. We get $R_{1343}=0$, $R_{1314}=0$.

By Lemma 2.1, $K(e_2,e_4)=K(e_1,e_3)$, so by the same argument as above, we have $R_{2124}=R_{2324}=R_{4142}=R_{4243}=0$.

On the other hand, we have $K(e_1,e_2)\geq K(X,Y)$ for any $X\in P,\ Y\in P^{\perp}$; in particular, $K(e_1,e_2)\geq K(e_1,X)$ for any $X\in P^{\perp}$. Let $X=e_2\cos t+e_4\sin t$. By the variation principle we have $R_{1214}=0$; also $K(e_1,e_2)\geq K(X,e_2)$ for any $X\in P$. Let $X=e_1\cos t+e_3\sin t$. We get $R_{2123}=0$.

Again by Lemma 2.1, $K(e_3,e_4)=K(e_1,e_2)\geq K(X,Y)$ for any $X\in P,\ Y\in P^{\perp}$, so we get $R_{3432}=R_{4341}=0$. Therefore we have proved (1) and (2).

Since $K(X,Y)\geq K(e_1,e_3)$ for any $X,\ Y\in T_pM$, we obtain

$$K(ae_1+be_2,ce_3+de_4)\geq K(e_1,e_3)$$

for any a,b,c,d such that $a^2+b^2=1,\ c^2+d^2=1$.

Choose $a=b=c=d=\frac{1}{\sqrt{2}}$. We get

$$K(e_1,e_3)\leq K(\frac{1}{\sqrt{2}}e_1+\frac{1}{\sqrt{2}}e_2,\frac{1}{\sqrt{2}}e_3+\frac{1}{\sqrt{2}}e_4)=\frac{1}{2}[R_{1313}+R_{1414}+R_{1324}+R_{1423}].$$

Therefore

$$R_{1423}-R_{1342}\geq R_{1313}-R_{1414}.$$

Similarly choosing $a=b=c=\frac{1}{\sqrt{2}},\ d=-\frac{1}{\sqrt{2}}$ we get $R_{1342}-R_{1423}\geq R_{1313}-R_{1414}$; therefore

$$|R_{1342}-R_{1423}|\leq R_{1414}-R_{1313}.$$

Applying the same argument to $K(ae_1+be_3,ce_2+de_4)\leq K(e_1,e_2)$ and $K(ae_1+be_4,ce_2+de_3)\geq K(e_1,e_3)$, we get

$$|R_{1234}-R_{1342}|\leq R_{1313}-R_{1212},$$
$$|R_{1423}-R_{1234}|\leq R_{1414}-R_{1212}.$$

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References

[1] D. Bakry and Michel Émery, *Diffusions hypercontractives* (French), Séminaire de probabilités, XIX, 1983/84, Lecture Notes in Math., vol. 1123, Springer, Berlin, 1985, pp. 177–206, DOI 10.1007/BFb0075847. MR889476

[2] Marcel Berger, *Sur quelques variétés d'Einstein compactes* (French), Ann. Mat. Pura Appl. (4) **53** (1961), 89–95. MR0130659

[3] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 10, Springer-Verlag, Berlin, 1987. MR867684

[4] Simon Brendle, *Einstein manifolds with nonnegative isotropic curvature are locally symmetric*, Duke Math. J. **151** (2010), no. 1, 1–21, DOI 10.1215/00127094-2009-061. MR2573825

[5] Huai-Dong Cao and Qiang Chen, *On Bach-flat gradient shrinking Ricci solitons*, Duke Math. J. **162** (2013), no. 6, 1149–1169, DOI 10.1215/00127094-2147649. MR3053567

[6] Huai-Dong Cao, Bing-Long Chen, and Xi-Ping Zhu, *Recent developments on Hamilton's Ricci flow*, Surveys in differential geometry. Vol. XII. Geometric flows, Surv. Differ. Geom., vol. 12, Int. Press, Somerville, MA, 2008, pp. 47–112, DOI 10.4310/SDG.2007.v12.n1.a3. MR2488948

[7] Xiaodong Cao, Biao Wang, and Zhou Zhang, *On locally conformally flat gradient shrinking Ricci solitons*, Commun. Contemp. Math. **13** (2011), no. 2, 269–282, DOI 10.1142/S0219199711004191. MR2794486

[8] Jeffrey S. Case, *Smooth metric measure spaces and quasi-Einstein metrics*, Internat. J. Math. **23** (2012), no. 10, 1250110, 36, DOI 10.1142/S0129167X12501108. MR2999055

[9] Jeffrey S. Case, *The energy of a smooth metric measure space and applications*, J. Geom. Anal. **25** (2015), no. 1, 616–667, DOI 10.1007/s12220-013-9441-6. MR3299295

[10] Jeffrey S. Case, *A Yamabe-type problem on smooth metric measure spaces*, J. Differential Geom. **101** (2015), no. 3, 467–505. MR3415769

[11] Jeffrey Case, Yu-Jen Shu, and Guofang Wei, *Rigidity of quasi-Einstein metrics*, Differential Geom. Appl. **29** (2011), no. 1, 93–100, DOI 10.1016/j.difgeo.2010.11.003. MR2784291

[12] Giovanni Catino, *A note on four-dimensional (anti-)self-dual quasi-Einstein manifolds*, Differential Geom. Appl. **30** (2012), no. 6, 660–664, DOI 10.1016/j.difgeo.2012.09.005. MR2996860

[13] Giovanni Catino, *Generalized quasi-Einstein manifolds with harmonic Weyl tensor*, Math. Z. **271** (2012), no. 3-4, 751–756, DOI 10.1007/s00209-011-0888-5. MR2945582

[14] Giovanni Catino and Carlo Mantegazza, *The evolution of the Weyl tensor under the Ricci flow* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **61** (2011), no. 4, 1407–1435 (2012), DOI 10.5802/aif.2644. MR2951497

[15] Giovanni Catino, P. Mastrolia, D. Monticelli, and M. Rigoli, *On the geometry of gradient Einstein-type manifolds*, arXiv:math.DG/1402.3453, 2014.

[16] Sun-Yung A. Chang, Matthew J. Gursky, and Paul C. Yang, *A conformally invariant sphere theorem in four dimensions*, Publ. Math. Inst. Hautes Études Sci. **98** (2003), 105–143, DOI 10.1007/s10240-003-0017-z. MR2031200

[17] Xiuxiong Chen, Claude Lebrun, and Brian Weber, *On conformally Kähler, Einstein manifolds*, J. Amer. Math. Soc. **21** (2008), no. 4, 1137–1168, DOI 10.1090/S0894-0347-08-00594-8. MR2425183

[18] Xiuxiong Chen and Yuanqi Wang, *On four-dimensional anti-self-dual gradient Ricci solitons*, J. Geom. Anal. **25** (2015), no. 2, 1335–1343, DOI 10.1007/s12220-014-9471-8. MR3319974

[19] Andrzej Derdziński, *Self-dual Kähler manifolds and Einstein manifolds of dimension four*, Compositio Math. **49** (1983), no. 3, 405–433. MR707181

[20] Manolo Eminenti, Gabriele La Nave, and Carlo Mantegazza, *Ricci solitons: the equation point of view*, Manuscripta Math. **127** (2008), no. 3, 345–367, DOI 10.1007/s00229-008-0210-y. MR2448435

[21] Michael Feldman, Tom Ilmanen, and Lei Ni, *Entropy and reduced distance for Ricci expanders*, J. Geom. Anal. **15** (2005), no. 1, 49–62, DOI 10.1007/BF02921858. MR2132265

[22] Manuel Fernández-López and Eduardo García-Río, *Rigidity of shrinking Ricci solitons*, Math. Z. **269** (2011), no. 1-2, 461–466, DOI 10.1007/s00209-010-0745-y. MR2836079

[23] Joel Fine, Kirill Krasnov, and Dmitri Panov, *A gauge theoretic approach to Einstein 4-manifolds*, New York J. Math. **20** (2014), 293–323. MR3193955

[24] A. Rod Gover and Paweł Nurowski, *Obstructions to conformally Einstein metrics in n dimensions*, J. Geom. Phys. **56** (2006), no. 3, 450–484, DOI 10.1016/j.geomphys.2005.03.001. MR2171895

[25] Matthew J. Gursky, *Four-manifolds with $\delta W^+ = 0$ and Einstein constants of the sphere*, Math. Ann. **318** (2000), no. 3, 417–431, DOI 10.1007/s002080000130. MR1800764

[26] Matthew J. Gursky and Claude Lebrun, *On Einstein manifolds of positive sectional curvature*, Ann. Global Anal. Geom. **17** (1999), no. 4, 315–328, DOI 10.1023/A:1006597912184. MR1705915

[27] Richard S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306. MR664497

[28] Richard S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988, pp. 237–262, DOI 10.1090/conm/071/954419. MR954419

[29] Gerhard Huisken, *Ricci deformation of the metric on a Riemannian manifold*, J. Differential Geom. **21** (1985), no. 1, 47–62. MR806701

[30] Thomas Ivey, *Ricci solitons on compact three-manifolds*, Differential Geom. Appl. **3** (1993), no. 4, 301–307, DOI 10.1016/0926-2245(93)90008-O. MR1249376

[31] Carlos N. Kozameh, Ezra T. Newman, and K. P. Tod, *Conformal Einstein spaces*, Gen. Relativity Gravitation **17** (1985), no. 4, 343–352, DOI 10.1007/BF00759678. MR788800

[32] Claude LeBrun, *On Einstein, Hermitian 4-manifolds*, J. Differential Geom. **90** (2012), no. 2, 277–302. MR2899877

[33] Claude LeBrun, *Einstein metrics, harmonic forms, and symplectic four-manifolds*, Ann. Global Anal. Geom. **48** (2015), no. 1, 75–85, DOI 10.1007/s10455-015-9458-0. MR3351078

[34] Pengzi Miao and Luen-Fai Tam, *On the volume functional of compact manifolds with boundary with constant scalar curvature*, Calc. Var. Partial Differential Equations **36** (2009), no. 2, 141–171, DOI 10.1007/s00526-008-0221-2. MR2546025

[35] Pengzi Miao and Luen-Fai Tam, *Einstein and conformally flat critical metrics of the volume functional*, Trans. Amer. Math. Soc. **363** (2011), no. 6, 2907–2937, DOI 10.1090/S0002-9947-2011-05195-0. MR2775792

[36] Ovidiu Munteanu and Natasa Sesum, *On gradient Ricci solitons*, J. Geom. Anal. **23** (2013), no. 2, 539–561, DOI 10.1007/s12220-011-9252-6. MR3023848

[37] Lei Ni and Nolan Wallach, *On a classification of gradient shrinking solitons*, Math. Res. Lett. **15** (2008), no. 5, 941–955, DOI 10.4310/MRL.2008.v15.n5.a9. MR2443993

[38] D. Page, *A compact rotating gravitational instanton*, Phys. Lett. **79** B (1979), 235–238.

[39] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159.

[40] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109.

[41] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math.DG/0307245.

[42] Peter Petersen and William Wylie, *On the classification of gradient Ricci solitons*, Geom. Topol. **14** (2010), no. 4, 2277–2300, DOI 10.2140/gt.2010.14.2277. MR2740647

[43] Thomas Richard and Harish Seshadri, *Positive isotropic curvature and self-duality in dimension 4*, Manuscripta Math. **149** (2016), no. 3-4, 443–457, DOI 10.1007/s00229-015-0790-2. MR3458177

[44] I. M. Singer and J. A. Thorpe, *The curvature of* 4*-dimensional Einstein spaces*, Global Analysis (Papers in Honor of K. Kodaira), Univ. Tokyo Press, Tokyo, 1969, pp. 355–365. MR0256303

[45] J.-Y. Wu, P. Wu, and W. Wylie, *Gradient shrinking Ricci solitons of half harmonic Weyl curvature*, arXiv:math.DG/1410.7303.

[46] P. Wu, *Studies on Einstein manifolds and Ricci solitons*, ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–University of California, Santa Barbara. MR3094048

[47] P. Wu, *A note on Einstein four-manifolds with positive sectional curvature*, preprint, 2012.

[48] P. Wu, *Einstein four-manifolds with three-positive curvature operator are half conformally flat*, preprint, 2013.

[49] P. Wu, *Rigidity of quasi-Einstein four-manifolds with half harmonic Weyl curvature*, preprint, 2015.

[50] DaGang Yang, *Rigidity of Einstein* 4*-manifolds with positive curvature*, Invent. Math. **142** (2000), no. 2, 435–450, DOI 10.1007/PL00005792. MR1794068

[51] Zhu-Hong Zhang, *Gradient shrinking solitons with vanishing Weyl tensor*, Pacific J. Math. **242** (2009), no. 1, 189–200, DOI 10.2140/pjm.2009.242.189. MR2525510

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