

## ERRATUM TO “A CHARACTERIZATION OF VIRTUALLY EMBEDDED SUBSURFACES IN 3-MANIFOLDS”

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ABSTRACT. The purpose of this note is to report an incorrect formula in the author’s paper *A characterization of virtually embedded subsurfaces in 3-manifolds*, *Trans. Amer. Math. Soc.* **369** (2017), no. 2, 1237–1264.

The asserted Formula 7.2 in [L] is not true in general and there is a gap in its argument. Consequently, the statement of Corollary 1.3 is in fact false, and the construction of Example 8.1 has to be modified to meet some different requirements (see [Sun] for a more general treatment). Apart from those damages, the results of [L] remain true as they were stated. The error has been recently pointed out to the author by Hongbin Sun.

The error can be pinned down at the sentence that “ $\ell(l'_\delta)$  depends only on the carrying vertex  $v$ ” [L, p. 1258], where Formula 7.2 is derived. We notice that throughout Section 7 there, the flow-time length  $\ell$  is supposed to be understood with respect to the prescribed Nielsen–Thurston suspension flow structure. However, for the quoted sentence to be true, the length should be taken rather with respect to the suspension flow associated with  $S'_v$ . In general, the two suspension flows are isotopic as 1-foliations (unparametrized flows), but they are not necessarily the same flow in the parametrized sense (even up to rescaling).

The following proposition describes the exact situation when the quoted sentence holds true.

**Proposition 1.** *Let  $F_v$  be a (connected) compact oriented surface and let  $\theta_v : F_v \rightarrow F_v$  be a pseudo-Anosov or periodic automorphism. Denote by  $J_v$  the mapping torus of  $\theta_v$  equipped with the suspension flow of  $\theta_v$ . Suppose that  $S_v \looparrowright J_v$  is an oriented properly immersed virtual fiber transverse to the  $\theta_v$ -suspension flow. Let  $J'_v$  be a finite cover of  $J_v$  which contains an embedded lift  $S'_v$  of  $S_v$ , such that the monodromy is isotopic to a pseudo-Anosov or periodic automorphism of  $S'_v$  which fixes all the periodic points on  $\partial S'_v$ . Then the following statements are equivalent:*

- (1) *For all degeneracy slopes  $l'_\delta$  on  $\partial J'_v$ , the flow-time length  $\ell(l'_\delta)$  with respect to the  $\theta_v$ -suspension flow is constant independent of  $\delta$ .*
- (2) *There exist a nonzero integer  $m_v$  and a properly embedded taut subsurface  $K'_v$  of  $J'_v$  with every component of  $\partial K'_v$  parallel to degeneracy slopes, such that the equality*

$$m_v \cdot [S'_v] = [F'_v] + [K'_v]$$

*holds in  $H_2(J'_v, \partial J'_v; \mathbb{Z})$ , where  $F'_v$  is any elevation of  $F_v$  in  $J'_v$ .*

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Received by the editors August 30, 2016.

2010 *Mathematics Subject Classification.* Primary 57M05.

- (3) For any finite cover  $J''_v$  of  $J_v$  into which  $S_v$  and  $F_v$  can be elevated to be embedded fibers  $S''_v$  and  $F''_v$ , the class  $[S''_v] \in H_2(J''_v, \partial J''_v; \mathbb{Q})$  is a nonzero rational multiple of  $[F''_v]$  plus some class  $\eta$  with  $\partial\eta \in H_1(\partial J''_v; \mathbb{Q})$  contained in the subspace spanned by the degeneracy slopes of  $J''_v$ .

*Proof.* To see that the first statement implies the second, observe that the preimage  $\mathcal{F}'_v$  of  $F_v$  in  $J'_v$  is a union of parallel copies of  $F'_v$ . By definition,

$$\ell(l'_\delta) = \mathbf{i}(l'_\delta, \partial\mathcal{F}'_v) = \frac{[J'_v : J_v]}{[F'_v : F_v]} \cdot \mathbf{i}(l'_\delta, \partial F'_v) = \frac{[J'_v : J_v]}{[F'_v : F_v]} \cdot \frac{\mathbf{i}(l'_\delta, \partial F'_v)}{\mathbf{i}(l'_\delta, \partial S'_v)}.$$

We may assume without loss of generality that  $S_v$  is oriented by the positive direction of the  $\theta_v$ -flow; then the geometric intersection numbers agree with the algebraic intersection numbers on  $H_1(\partial J'_v; \mathbb{Z})$ . As  $\ell(l'_\delta)$  is constant independent of  $\delta$ , there exists a constant  $m_v$  such that  $\partial([F'_v] - m_v \cdot [S'_v])$  intersects every  $[l'_\delta]$  algebraically zero times. In fact,  $m_v$  is a nontrivial integer since the monodromy assumption for  $S'_v$  implies that  $\mathbf{i}(S'_v, l'_\delta) = 1$ . The claim equality holds for any taut subsurface  $K'_v$  that realizes  $[F'_v] - m_v \cdot [S'_v]$ .

To see that the second statement implies the third, denote by  $V''$  the preimage of the subspace of  $H_1(\partial J''_v; \mathbb{Q})$  generated by degeneracy slopes, for the homomorphism  $\partial : H_2(J''_v, \partial J''_v; \mathbb{Q}) \rightarrow H_1(\partial J''_v; \mathbb{Q})$ . We must show that  $[S''_v] \in \mathbb{Q} \cdot [F''_v] + V''$ . When  $J''_v$  is a regular finite cover of  $J_v$  factoring through  $J'_v$ , the property follows directly from the second statement. In general, we take a regular finite cover  $\tilde{J}_v$  of  $J_v$  that factors through both  $J''_v$  and  $J'_v$ , then the transfer homomorphism gives rise to an embedding of  $H_2(J''_v, \partial J''_v; \mathbb{Q})$  into  $H_2(\tilde{J}_v, \partial \tilde{J}_v; \mathbb{Q})$ , so that the image intersects  $\mathbb{Q} \cdot [\tilde{F}_v] + \tilde{V}$  in exactly  $\mathbb{Q} \cdot [\tilde{F}_v] + V''$ . As  $S''_v$  is a fiber, the transferred image of  $[S''_v]$  is a multiple of  $[\tilde{S}_v]$ , which lies in  $\mathbb{Q} \cdot [\tilde{F}_v] + \tilde{V}$  as we have argued. So  $[S''_v] \in \mathbb{Q} \cdot [F''_v] + V''$ .

To see that the third statement implies the first, we may take  $J''_v$  to be  $J'_v$ . Then the ratio between the geometric intersection number  $\mathbf{i}(l'_\delta, \partial F'_v) / \mathbf{i}(l'_\delta, \partial S'_v)$  is the constant  $m_v$ , independent of  $\delta$ , so the calculation in the first part shows that  $\ell(l'_\delta)$  is constant independent of  $l'_\delta$ . □

For a periodic  $\theta_v$ , the necessary and sufficient conditions of Proposition 1 are always satisfied: Indeed,  $K_v$  can be taken to be a collection of vertical annuli. In this case, [L, Formula 7.2] recovers the formula of Rubinstein–Wang [RW]. For a pseudo-Anosov  $\theta_v$ , [L, Formula 7.2] only works for a more restricted class of flow-transverse subsurfaces: If every vertex subsurface of the almost fiber part is known to be constructed by virtually homologically perturbing the prescribing fiber class in a “pseudo-vertical direction” (the third condition of Proposition 1), Formula 7.2 remains valid. However, in the pseudo-Anosov case using methods such as [PW, Proposition 3.11], it is always possible to find some properly immersed flow-transverse subsurface  $S_v$  that violates the homological requirement of Proposition 1. We conclude that it is generally impossible to determine the spirality character for flow-transverse subsurfaces merely from the intersection pattern of curves on the JSJ tori.

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