AUSLANDER-REITEN QUIVER OF TYPE A
AND GENERALIZED QUANTUM AFFINE
SCHUR-WEYL DUALITY

SE-JIN OH

Abstract. The quiver Hecke algebra $R$ can be also understood as a generalization of the affine Hecke algebra of type $A$ in the context of the quantum affine Schur-Weyl duality by the results of Kang, Kashiwara and Kim. On the other hand, it is well known that the Auslander-Reiten (AR) quivers $\Gamma_Q$ of finite simply-laced types have a deep relation with the positive root systems of the corresponding types. In this paper, we present explicit combinatorial descriptions for the AR-quivers $\Gamma_Q$ of finite type $A$. Using the combinatorial descriptions, we can investigate relations between finite dimensional module categories over the quantum affine algebra $U'_q(A_n(i)) (i = 1, 2)$ and finite dimensional graded module categories over the quiver Hecke algebra $R_{A_n}$ associated to $A_n$ through the generalized quantum affine Schur-Weyl duality functor.

INTRODUCTION

Let $\Delta_n$ be a rank $n$ Dynkin diagram of finite type $A$, $D$ or $E$, and $Q$ be a Dynkin quiver by orienting edges of $\Delta_n$. Gabriel’s theorem [18] states that isomorphism classes of indecomposables of the category of modules mod$CQ$ over the path algebra $CQ$ are labeled by the set of positive roots $\Phi^+_n$ related to $\Delta_n$.

The Auslander-Reiten quiver (AR-quiver) $\Gamma_Q$ arising from $Q$ encodes the structure of the mod$CQ$ in the following sense: (i) vertices of AR-quiver present the indecomposables, (ii) arrows present the irreducible morphisms between them. Moreover, $\Gamma_Q$ itself provides reduced expressions of the longest element $w_0$ of the Weyl group associated with $\Delta_n$ by reading the levels of vertices in $\Gamma_Q$ in a manner compatible with paths [5]. The AR-quiver also plays an important role in the research area of cluster algebras, cluster-tilted algebras and cluster categories (for example, [13]).

For each reduced expression $\tilde{w}_0$ of $w_0$, we can assign a total order $<_{\tilde{w}_0}$ on $\Phi^+_n$ by [8] Chapter VI, §1.6. Moreover, the total order is convex in the sense that for any $\alpha, \beta \in \Phi^+_n$ with $\alpha + \beta \in \Phi^+_n$, we have that $\alpha <_{\tilde{w}_0} \beta$ implies $\alpha <_{\tilde{w}_0} \alpha + \beta <_{\tilde{w}_0} \beta$ ([16]). By abstracting this concept, we say a partial order $< \Phi^+_n$ convex if $\alpha + \beta \in \Phi^+_n$ implies either $\alpha < \alpha + \beta < \beta$ or $\beta < \alpha + \beta < \alpha$ (see Section 1.3).

On the other hand, Hernandez and Leclerc [22] defined a certain category $\mathcal{C}^{(1)}_Q$ which consists of finite dimensional integrable modules over quantum affine algebras $U'_q(g)$, and depends on the AR-quiver $\Gamma_Q$. They proved that $\mathcal{C}^{(1)}_Q$ categorifies $\mathbb{C}[N]$ as the coordinate ring of the unipotent group $N$ associated with $g_0$. Here $g$ is
an affine Kac-Moody algebra of untwisted simply-laced type and \( g_0 \) is the finite dimensional semisimple Lie subalgebra inside \( g \).

The quiver Hecke algebras \( R_g \), introduced by Khovanov-Lauda \([38,39]\) and Rouquier \([48]\), categorify the negative part of \( U_q(g) \) for all symmetrizable Kac-Moody algebras \( g \). Since \( R_g \) provides quantum versions of Lascoux-Leclerc-Thibon-Ariki theory \((2,41)\), many authors have studied the representation theories on quiver Hecke algebras \( R_g \) very actively since its introduction (for example, \([10,11,24,26,31,42,49]\)). In particular, if \( g \) is a finite classical semisimple Lie algebra, the theory on the construction of simple modules over \( R_g \) is well developed (see \([6,12,23,32,37,40,43]\)). Among these results, the approach of \([37,40,43]\) provides the way of constructing simple modules over \( R_g \) by using PBW-bases which have arisen from a convex order induced by a fixed reduced expression \( \tilde{w}_0 \) of \( w_0 \).

Note that, for \( g_0 \) inside \( g \), the specialization of the integral form \( U_A^-\left(g_0\right) \) of \( U^{-}\left(g_0\right) \) at \( q = 1 \) is isomorphic to \( \mathbb{C}[N] \) and hence \( U_A^-\left(g_0\right) \) is isomorphified via \( R_{g_0} \) (with forgotten grading) and \( \mathcal{C}_Q^{(1)} \) over \( U_q(g) \), simultaneously. Since the result of Brundan and Kleshchev in \([10]\) provides the isomorphism between affine Hecke algebras of type \( A \) and the \( R_{A_n^{(1)}} \) up to a specialization and a localization, Hernandez and Leclerc expected that quiver Hecke algebras might be interpreted as a generalization of an affine Hecke algebra of type \( A \) in the context of the quantum affine Schur-Weyl duality \((14,15,19)\).

In \([27,28]\), Kang, Kashiwara and Kim came up with an answer for the expectation. They constructed the generalized quantum affine Schur-Weyl duality functor

\[
F : \text{Rep}(R_J) \to \mathcal{C}_g
\]

between the finite dimensional integrable modules category \( \mathcal{C}_g \) over \( U_q(g) \) and the category \( \text{Rep}(R_J) \) of finite dimensional graded modules over a quiver Hecke algebra \( R_J \), by observing the zeros of the denominator of normalized \( R \)-matrix \( R_{k,l}^{\text{norm}}(z) \). Their work can be understood as a graded generalization of quantum affine Schur-Weyl duality since \( \text{Rep}(R_J) \) has a grading. Here \( R_J \) is defined by the quiver \( Q_J \), which appears in the process of constructing \( F \). Interestingly, \( R_J \) does depend on the choice of good modules over \( U_q(g) \).

By the results of \([27,28]\) and \([22]\), we have an exact functor \( F_Q^{(1)} \) from \( \text{Rep}(R_{g_0}) \) to \( \mathcal{C}_Q^{(1)} \) over \( U_q(g) \),

\[
F_Q^{(1)} : \text{Rep}(R_{g_0}) \to \mathcal{C}_Q^{(1)},
\]

which sends simples to simples, for

(a) each Dynkin quiver \( Q \) of finite type \( A \) (resp. \( D \)),

(b) \( g \) is of type \( A_n^{(1)} \) (resp. \( D_n^{(1)} \)) and \( g_0 \) is of type \( A_n \) (resp. \( D_n \)).

In this paper, we investigate \( F_Q^{(1)} \) more deeply by using the explicit combinatorial description of an AR-quiver \( \Gamma_Q \) which arises from a chosen Dynkin quiver \( Q \) of finite type \( A \).

Note that all vertices in \( \Gamma_Q \) are labeled by the set of positive roots \( \Phi^+_n \) and each positive root \( \beta \) in \( \Phi^+_n \) of finite type \( A \) can be expressed by the segment \([a,b]\), where \( \beta = \sum_{k=a} b \alpha_k \). We say \( a \) is the first component of \( \beta \) and \( b \) is the second one. Identifying \( \beta \) with \([a,b]\), \( \Gamma_Q \) satisfies the following property: Every positive root appearing in the maximal \( N \)-sectional (resp. \( S \)-sectional) path in \( \Gamma_Q \) has the same
first (resp. second) component \((\text{Theorem }1.11)\). With this property, the facts
\[\Phi_n^+ = \{[a, b] \mid 1 \leq a \leq b \leq n\},\]
provide a way (Remark 1.14) for computing the bijection
\[
\phi^{-1} : \Phi_n^+ \times \{0\} \rightarrow \Gamma
\]
without using the Coxeter element \(\tau\) adapted \(Q\) and the additive property of the
dimension vectors.

The description of \(\mathcal{C}_Q^{(1)}\) (Definition 2.2) is the smallest category containing \(V_Q(\beta)\)
for \(\beta \in \Phi_n^+\) and is stable by certain operations on modules, where
\[V_Q(\beta) := V(\varpi_i)(-q)_{\mathcal{D}}\] is the fundamental \(U_q'(A_n^{(1)})\)-modules for \(\phi^{-1}(\beta, 0) = (i, p)\).

Using the combinatorial descriptions of \(\Gamma_Q\), we can prove that Dorey’s rule in
[14] always holds for every \(\alpha \prec \beta \in \Phi_n^+\) with \(\gamma = \alpha + \beta \in \Phi_n^+\); i.e., the following
surjective homomorphisms exist:
\[V_Q(\beta) \otimes V_Q(\alpha) \rightarrow V_Q(\gamma) \quad \text{and} \quad S_Q(\beta) \circ S_Q(\alpha) \rightarrow S_Q(\gamma),\]
where \(\prec\) is the convex partial order arising from \(\Gamma_Q\) and the simple \(R_{A_n}\)-module
\(S_Q(\beta)\) is the preimage of \(V_Q(\beta)\) under the functor \(\mathcal{F}_Q^{(1)}\) (Theorem 3.2).

Moreover, we prove that every pair \((\alpha, \beta) \in (\Phi_n^+)^2\) with \(\alpha + \beta \in \Phi_n^+\) is indeed a
minimal pair in the sense of McNamara [43], with respect to a suitable total order
which is compatible with the convex partial order \(\prec\) (Theorem 3.4). Thus, for every
pair \((\alpha, \beta)\) with \(\alpha + \beta \in \Phi_n^+\), we have (Corollary 3.7)
\[V_Q(\beta) \otimes V_Q(\alpha) \text{ and } S_Q(\beta) \circ S_Q(\alpha) \text{ have composition length 2.}\]

In [135], it is proved that every finite dimensional simple integrable \(U_q'(\mathfrak{g})\)-module \(M\) appears as a simple head of \(\bigotimes_{i=1}^l V(\varpi_{i_k})_{a_k}\) for some finite sequence
\[(i_1, a_1), \ldots, (i_l, a_l) \text{ in } \{1, \ldots, n\} \times k^I\]
such that the normalized \(R\)-matrices \(R_{i_k,i_{k'}}^{\text{norm}}(z)\) have no pole at \(z = a_{k'}/a_k\), for all
\(1 \leq k < k' \leq l\). Here, \(k = \mathbb{C}(q) \subset \bigcup_{m>0} \mathbb{C}(q^{1/m})\). With this theorem and the
denominators in [41] Appendix A, we can construct various exact sequences and
simple modules over \(R_{A_n}\)-modules, which can be understood as generalizations of the
exact sequences and simple modules in [28] Proposition 4.2.3 and [32] (see Section 4.3). We also obtain an alternative proof for the exactness of \(\mathcal{F}_Q^{(1)}\) (Theorem 4.9). Moreover, we can rephrase one of the main results of [43] with the following forms: For any simple module \(M \in \text{Rep}(R_{A_n})\) and Dynkin quiver \(Q\) of finite type \(A_n\), there exist two \(n\)-tuples of sequences
\[\{(a_{1,i_{k}}, a_{2,i_{k}}, \ldots, a_{k-1,i_{k}}, a_{k,i_{k}}) \mid 1 \leq k \leq n\} \quad \text{and} \quad \{(a_{j_1,j_{l}}, a_{j_{l+1},j_{l}}, \ldots, a_{j_{n-1},j_{n}}) \mid 1 \leq l \leq n\}\]
such that \(a_{s,i_{k}}, a_{j_{l};n} \in \mathbb{Z}_{\geq 0}\) for \(1 \leq s \leq i_{k}, j_{l} \leq u \leq n, 1 \leq k, l \leq n\) and
\[\bigotimes_{k=1}^n S_Q([s, i_{k}])_{a_{s,i_{k}}}) \rightarrow M \quad \text{and} \quad \bigotimes_{l=1}^n S_Q([j_{l}, u])_{a_{j_{l};n}}) \rightarrow M,\]
where \([s, i_{k}], [j_{l}, u] \in \Phi_n^+, (i_1, i_2, \ldots, i_{n-1}, i_{n})\) and \((j_1, j_2, \ldots, j_{n-1}, j_{n})\) are re-indexing of \(\{1, 2, \ldots, n\}\) defined by the combinatorial property we observed in this.
paper (Theorem 4.14). Here, \( \frac{1}{s_1} S_2 \left( \left[ s, i_k \right] \right)^{\cup s, i_k} \) and \( \frac{n}{u} S_2 \left( \left[ j_l, u \right] \right)^{\cup j_l, u} \) are simple, and hence every simple \( R_{A_n} \)-module appears as a head of a convolution product of \( n \)-simple modules.

Lastly, for \( A_n^{(2)} \), we will prove that the description for \( C_{Q}^{(2)} \) in Definition 5.1 can be re-expressed (Theorem 5.2) by using the combinatorial properties and the Dorey type morphisms studied in [44]. The observations on \( C_{Q}^{(2)} \) will be one of the main ingredients for investigating relations between \( \text{Rep}(R_{A_n}) \) and \( C_{Q}^{(2)} \) in [30].

We call a positive root \( \beta = \sum_k n_k \alpha_k \) multiplicity free if \( n_k \leq 1 \) for all \( 1 \leq k \leq n \). Note that all positive roots in \( \Phi^+_n \) of finite type \( A \) are multiplicity free. However, it is not true for \( \Phi^+_n \) of finite type \( D \) and \( E \). Thus we can state the following naturally:

**Conjecture.** For any Dynkin quiver \( Q \) of finite type \( A, D \) or \( E \), every pair \( (\alpha, \beta) \) of a positive root \( \gamma = \alpha + \beta \in \Phi^+_n \) is a minimal with respect to a suitable total order compatible with \( \prec_Q \) if and only if \( \gamma \) is multiplicity free.

In the forthcoming paper [45], we will prove that the above conjecture is true for \( \Phi^+_n \) of finite type \( D \). In other words, there exists a pair \( (\alpha, \beta) \) of a multiplicity non-free positive root \( \gamma = \alpha + \beta \) such that the pair cannot be minimal for any total order compatible with \( \prec_Q \). Furthermore, we will prove that Dorey's rule always holds even though \( (\alpha, \beta) \) is not minimal; i.e., \( S_Q(\beta) \circ S_Q(\alpha) \rightarrow S_Q(\gamma) \) for every Dynkin quiver \( Q \) of finite type \( D \), every positive root \( \gamma \) and its pair \( (\alpha, \beta) \).

The outline of this paper is as follows. In Section 1, we first recall the definition of \( \Gamma_Q \) and their basic properties, briefly review the various orders on \( \Phi^+_n \), and provide explicit descriptions for \( \Gamma_Q \) of finite type \( A \). In Section 2, we recall the backgrounds and theories of the generalized quantum affine Schur-Weyl duality briefly. In the later sections, we investigate the categories \( C_{Q}^{(i)} \) \((i = 1, 2)\) and \( \text{Rep}(R_{A_n}) \) using the generalized quantum affine Schur-Weyl duality and the explicit descriptions of \( \Gamma_Q \) in Section 1.

**1. Combinatorial characterization of AR-quivers of finite type \( A \)**

In this section, we recall Gabriel's Theorem and various orders on the set of positive roots. We also observe the combinatorial feature of AR-quivers \( \Gamma_Q \) of finite type \( A \) and provide an explicit combinatorial description of \( \Gamma_Q \). We refer to [3][4][18] for basic theories on quiver representations and Auslander-Reiten theory.

**1.1. Gabriel's Theorem.** Let \( \Delta_n \) be a rank \( n \) Dynkin diagram of finite type. We denote by \( I_0 = \{1, 2, \ldots, n\} \) the set of indices, \( \Pi_n = \{\alpha_i \mid i \in I_0\} \) the set of simple roots, \( \Phi_n = (\Phi^+_n, \Phi^-_n) \) the set of (positive, negative) roots, \( \mathbf{E} \) the vector space in which \( \Phi_n \) lies, and \( (, ) \) the scalar product defined on \( \mathbf{E} \) which are associated to the Dynkin diagram \( \Delta_n \).

Let \( W_0 \) be the Weyl group associated to \( \Delta_n \), which is generated by the set of simple reflections \( (s_i)_{i \in I_0} \). We denote by \( w_0 \) the unique longest element in \( W_0 \). It is well known that \( w_0 \) induces an involution \( \ast \) on \( I_0 \) given by \( w_0 \alpha_i = -\alpha_{i^*} \) (see [8] PLATE I~IX).

The Dynkin quiver \( Q \) is obtained by assigning orientation on the edges of \( \Delta_n \); i.e., \( Q \) is a data \((Q_0, Q_1)\) where \( Q_0 \) is a set of vertices indexed by \( I_0 \) and \( Q_1 \) is a set of arrows with underlying graph \( \Delta_n \). We say that a vertex \( i \) is a source (resp. sink) if and only if there are only exiting arrows out of it (resp. entering arrows into it).
Let $\text{Mod}\mathbb{C}Q$ be the category of finite dimensional modules over the path algebra $\mathbb{C}Q$. An object $M$ in this category is defined by the following data:

(a) To each $i \in Q_0$ is associated a finite dimensional $\mathbb{C}$-vector space $M_i$.

(b) To each arrow $i \rightarrow j$ in $Q_1$ is associated a linear map $\varphi_{i \rightarrow j} : M_i \rightarrow M_j$.

We define the dimension vector of $M \in \text{Mod}\mathbb{C}Q$ as

$$\text{dim} M := \sum_{i \in Q_0} (\text{dim} M_i)\alpha_i.$$ 

The simple objects in $\text{Mod}\mathbb{C}Q$ are one dimensional vector spaces $S(i)$ ($i \in Q_0$) which can be characterized by $\text{dim} S(i) = \alpha_i$. We denote by $\text{Ind}Q$ the set of isomorphism classes $[M]$ of indecomposable modules in $\text{Mod}\mathbb{C}Q$.

**Theorem 1.1** (Gabriel’s Theorem). For a Dynkin quiver $Q$ of finite type $A$, $D$ or $E$, the map $[M] \mapsto \text{dim} M$ gives a bijection from $\text{Ind}Q$ to $\Phi^+_n$. Thus the set $\text{Ind}Q$ consists of the indecomposable modules $M(\beta)$ ($\beta \in \Phi^+_n$) such that $\text{dim}(M(\beta)) = \beta$ when a Dynkin quiver $Q$ is of finite type $A$, $D$ or $E$.

1.2. Auslander-Reiten quiver. In this subsection, we fix a Dynkin quiver $Q$ whose underlying graph is of finite type $A$, $D$ or $E$. We denote by $s_iQ$ the quiver obtained from $Q$ by reversing the orientation of each arrow that ends at $i$ or starts at $i$. A map $\xi : I_0 \rightarrow \mathbb{Z}$ is called a height function on $Q$ if $\xi_j = \xi_i - 1$ for $i \rightarrow j \in Q_1$. Connectedness of $Q$ implies that any two height functions on $Q$ differ by a constant.

Set

$$\mathbb{Z}Q := \{(i, p) \in \{1, 2, \ldots, n\} \times \mathbb{Z} \mid p - \xi_i \in 2\mathbb{Z}\}.$$ 

We view $\mathbb{Z}Q$ as the quiver with arrows

$$(i, p) \rightarrow (j, p + 1), \ (j, q) \rightarrow (i, q + 1)$$

for which $i$ and $j$ are adjacent in $\Delta_n$ and call it the repetition quiver of $Q$. Note that $\mathbb{Z}Q$ does not depend on the orientation of the quiver $Q$. It is well known that the quiver $\mathbb{Z}Q$ itself has an isomorphism with the AR-quiver of $D^b(\mathbb{C}Q)$-mod, the bounded derived category of $\mathbb{C}Q$-mod (20). In our convention, the injective module $I(i)$ is located on the vertex $(i, \xi_i)$ of $\mathbb{Z}Q$.

For a reduced expression $\bar{w} = s_{i_1}s_{i_2}\cdots s_{i_l}$ of an element $w$ in $W_0$, it is called adapted to $Q$ if

$$i_k \text{ is a source of } s_{i_{k-1}}s_{i_2}\cdots s_{i_l}Q$$

for all $1 \leq k \leq l$.

It is well known that there is a unique Coxeter element $\tau$ whose reduced representations are adapted to $Q$.

Set $\widehat{\Phi}_n := \Phi^+_n \times \mathbb{Z}$. For $i \in I_0$, we define

$$\gamma_i = \sum_{j \in B(i)} \alpha_j \quad \text{and} \quad \theta_i = \sum_{j \in C(i)} \alpha_j \quad \text{where}$$

- $B(i)$ is the set of vertices $j$ in $Q_0$ such that there exists a path from $j$ to $i$,
- $C(i)$ is the set of vertices $j$ in $Q_0$ such that there exists a path from $i$ to $j$. 


The bijection $\phi : \mathbb{Z}Q \to \tilde{\Phi}_n$ defined by $M(\beta)[m] \mapsto (\beta, m)$ is described combinatorially as follows ([22] §2.2):

(a) $\phi(i, \xi_i) = (\gamma_i, 0)$,

(b) for a given $\beta \in \Phi_n^+$ with $\phi(i, p) = (\beta, m)$,

\begin{align*}
(1.2) \quad & \text{if } \tau(\beta) \in \Phi_n^+, \text{ we set } \phi(i, p - 2) = (\tau(\beta), m), \\
& \text{if } \tau(\beta) \in \Phi_n^-, \text{ we set } \phi(i, p - 2) = (-\tau(\beta), m - 1), \\
& \text{if } \tau^{-1}(\beta) \in \Phi_n^+, \text{ we set } \phi(i, p + 2) = (\tau^{-1}(\beta), m), \\
& \text{if } \tau^{-1}(\beta) \in \Phi_n^-, \text{ we set } \phi(i, p + 2) = (-\tau^{-1}(\beta), m + 1).
\end{align*}

The Auslander-Reiten quiver $\Gamma_Q$ is the full subquiver of $\mathbb{Z}Q$ whose set of vertices is $\phi^{-1}(\Phi_n^+ \times \{0\})$. Here the vertex $\phi^{-1}(\beta, 0)$ corresponds to the indecomposable module $M(\beta)$ in $\text{Ind}Q$ and the arrow $\phi^{-1}(\beta, 0) \to \phi^{-1}(\beta', 0)$ corresponds to an irreducible morphism from $M(\beta)$ to $M(\beta')$.

In particular, the injective envelope $\text{I}(i)$ of $S(i)$ corresponds to the vertex $\phi^{-1}(\gamma_i, 0)$ and the projective cover $P(i)$ of $S(i)$ corresponds to the vertex $\phi^{-1}(\theta_i, 0)$. It is well known that

\begin{equation}
(1.3) \quad \theta_i = \tau^{-m_i*}(\gamma_i, 0) \quad \text{where } m_i = \max\{k \geq 0 \mid \tau^k(\gamma_i) \in \Phi_n^+\}.
\end{equation}

For $\beta \in \Phi_n^+$ with $\tau(\beta) \in \Phi_n^+$, we set $\tau M(\beta) := M(\tau(\beta))$. In the Auslander-Reiten quiver $\Gamma_Q$, this map $\tau$ is called the Auslander-Reiten translation.

(i) The dimension vector is an additive function on $\Gamma_Q$ with respect to the map $\tau$; that is, for each vertex $X \in \Gamma_Q$ such that $X = \phi^{-1}(\beta, 0)$ and $\tau(\beta) \in \Phi_n^+$,

\begin{equation}
(1.4) \quad \dim X + \dim X = \sum_{Z \in X^-} \dim Z.
\end{equation}

Here $X^-$ is the set of vertices $Z$ in $\Gamma_Q$ such that there exists an arrow from $Z$ to $X$.

(ii) It is well known that, for $\beta \in \Phi_n^+$,

\begin{equation}
(1.5) \quad \tau(\beta) \in \Phi_n^- \quad \text{if and only if } \beta = \theta_i \quad \text{for some } i \in I_0.
\end{equation}

A subquiver $\Gamma'$ of $\Gamma_Q$ is said to be sectional ([9]) if whenever $x \to y$ and $y \to z$ are arrows in $\Gamma'$, we have $x \neq \tau(z)$.

The following description is one of the characterizations of $\Gamma_Q$ inside $\mathbb{Z}Q$:

\begin{equation}
(1.6) \quad \phi^{-1}(\Phi_n^+ \times \{0\}) = \{(i, p) \in \mathbb{Z}Q \mid \xi_i - 2m_i \leq p \leq \xi_i\}.
\end{equation}

Recall that there is the Nakayama permutation $\nu$ on $\mathbb{Z}Q$ which is given by

\begin{equation}
\nu(i, p) = (i^\ast, p + h_n - 2)
\end{equation}

where $h_n$ is the Coxeter number associated to $\Delta_n$. Since, for all $i \in I_0$, the Nakayama permutation $\nu$ sends vertices corresponding to $P(i)$ to vertices corresponding to $I(i^\ast)$, we have

\begin{equation}
(1.7) \quad \xi_{i^\ast} - 2m_{i^\ast} = \xi_i - h_n + 2.
\end{equation}

1.3. Orders on $\Phi_n^+$. In this section, we recall various orders on $\Phi_n^+$.

A convex order on $\Phi_n^+$ is a partial order $\preceq$ satisfying the following condition ([7]):

For all $\alpha, \beta$ and $\gamma = \alpha + \beta \in \Phi_n^+$, either $\alpha \prec \gamma \prec \beta$ or $\beta \prec \gamma \prec \alpha$.

Note that a coarsest convex partial order on $\Phi_n^+$ determines the commutation class of $w_0$; that is, the equivalence class on reduced expressions of $w_0$ which is given by

$$s_{i_1}s_{i_2} \cdots s_{i_n} \sim s_{j_1}s_{j_2} \cdots s_{j_n} \quad (N := |\Phi_n^+|)$$
if and only if \( s_{j_1} s_{j_2} \cdots s_{j_n} \) is obtained from \( s_{i_1} s_{i_2} \cdots s_{i_N} \) by replacing \( sa_s b \) by \( sb s_a \) for \( a \) and \( b \) not linked in \( \Delta_n \) (see [5]).

A Dynkin quiver \( Q \) defines a convex partial order \( \leq_Q \) on \( \Phi_n^+ \) in the following way [17]:

\[
\alpha \leq_Q \beta \text{ if and only if there is a path from } \beta \text{ to } \alpha \text{ in } \Gamma_Q.
\]

Thus, for a pair \((\alpha, \beta)\) with \( \gamma = \alpha + \beta \in \Phi_n^+ \) and \( \alpha \not< Q \beta \),

\[
\text{there exist paths from } \beta \text{ to } \gamma \text{ and from } \gamma \text{ to } \alpha \text{ in } \Gamma_Q.
\]

On the other hand, it is well known that a reduced expression \( \bar{w}_0 = s_{i_1} s_{i_2} \cdots s_{i_N} \) of \( w_0 \) induces a convex total order \( \leq_{\bar{w}_0} \) on \( \Phi_n^+ \) by (5)

\[
\beta : = s_{i_1} s_{i_2} \cdots s_{i_{z-1}} \alpha_{i_z} \text{ and } \beta_x \prec \bar{w}_0 \beta_y \text{ if and only if } x < y.
\]

Note that any convex total order is obtained in this way by a reduced expression \( \bar{w}_0 \) of \( w_0 \) (10).

The following theorem provides a way of obtaining all reduced expressions of \( w_0 \) adapted to \( Q \) and hence convex total orders compatible with the convex partial order \( \leq_Q \):

\[
ad \alpha \prec Q \beta \text{ implies } \alpha \prec \bar{w}_0 \beta \text{ for any } \bar{w}_0 \text{ adapted to } Q.
\]

**Theorem 1.2** ([5], Theorem 2.17). Let \( Q \) be a quiver of finite type \( A, D \) or \( E \). Then any reduced expression of \( w_0 \) adapted to \( Q \) can be obtained in the following way: We read \( \Gamma_Q \) sequentially, in a manner compatible with arrows; that is, if there exists an arrow \( \beta \rightarrow \alpha, \alpha \) appears before \( \beta \). Replacing a vertex \( \beta \) with \( i \) for \( \phi^{-1}(\beta) = (i, p) \), we have a sequence \( (i_1, i_2, \ldots, i_N) \) giving a reduced expression of \( w_0 \) adapted to \( Q \),

\[
\bar{w}_0 = s_{i_1} s_{i_2} \cdots s_{i_N}.
\]

**Remark 1.3.** In this remark, we suggest two canonical readings of \( \Gamma_Q \) which follow the rule in Theorem 1.2. Thus we can obtain two convex total orders on \( \Phi_n^+ \) compatible with \( \leq_Q \).

(A) We denote by \( \prec^L_Q \) the convex total order induced from the following reading:

\[
(i, p) \text{ appears before } (i', p') \iff \begin{cases} d(1, i) - p < d(1, i') - p' & \text{ or} \\ d(1, i) - p = d(1, i') - p' & \text{ and } i > i'. \end{cases}
\]

(B) We denote by \( \prec^U_Q \) the convex total order induced from the following reading:

\[
(i, p) \text{ appears before } (i', p') \iff \begin{cases} d(1, i) + p > d(1, i') + p' & \text{ or} \\ d(1, i) + p = d(1, i') + p' & \text{ and } i < i'. \end{cases}
\]

Here, \( d(i, j) \) denotes the distance between \( i \) and \( j \) in \( \Delta_n \).

**Definition 1.4** ([33], §2.1]). For a total order \( < \) on \( \Phi_n^+ \), a pair \((\alpha, \beta)\) with \( \alpha < \beta \) is called a minimal pair of \( \gamma \in \Phi_n^+ \) with respect to the total order \( < \) if

- \( \gamma = \alpha + \beta \),
- there exist no pair \((\alpha', \beta')\) such that \( \gamma = \alpha' + \beta' \) and \( \alpha < \alpha' < \gamma < \beta' < \beta \).
1.4. Characterization of Auslander-Reiten quiver of type \(A\). In this subsection, we fix the \(\Delta_n\) as the Dynkin diagram of finite type \(A\). Citing the referee, the combinatorial properties in this subsection are well known to the experts. More precisely, using the linear quiver \(Q\) (see \([4,2]\)), the reflection functor on \(D^b(CQ)\)-mod and the tilting theorem ([4, Chapter VII]), one can observe the descriptions in this subsection. However, we have difficulty finding the explicit statement of Theorem 1.11 in standard textbooks; we shall deal with and derive it from Lemma 1.7 and Remark 1.8 below.

Since \(\Delta_n\) is the Dynkin diagram of finite type \(A\), we have the Coxeter number \(h_n = n + 1\) and the involution induced by \(w_0 \in W_0\) is given by \(i \mapsto i^* = n + 1 - i\). We say that a vertex \(i \in Q\) is a right (resp. left) intermediate if

\[
\cdots \cdots \bullet \bullet \bullet \bullet \cdots \quad \text{ (resp. } \cdots \cdots \bullet \bullet \bullet \bullet \cdots \text{)} \quad \text{in } Q.
\]

Recall that, for every \(1 \leq a \leq b \leq n\), \(\beta = \sum_{a \leq k \leq b} \alpha_k\) is a positive root in \(\Phi_n^+\) and every positive root in \(\Phi_n^+\) is of that form. Thus we sometimes identify \(\beta \in \Phi_n^+\) (and hence a vertex in \(\Gamma_Q\)) with a segment \([a, b]\). For \(\beta = [a, b]\), we say \(a\) is the first component of \(\beta\) and \(b\) the second component of \(\beta\). If \(\beta\) is simple, we write \(\beta\) as \([a]\).

Example 1.5. Consider the quiver \(\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet\) of type \(A_5\). We set \(\xi_1 = 0\). Then the \(\Gamma_Q\) inside \(\mathbb{Z}Q\) can be obtained by using the Coxeter element \(\tau\) \([1,2]\) as the additive property of the dimension vectors \([1,3]\) as follows:

\[
\begin{array}{cccccc}
(i, p) & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
1 & 5 \\
2 & 4, 5 \\
3 & 2, 4 \\
4 & 1, 4 \\
5 & 1, 2 \\
\end{array}
\]

\[
\begin{array}{cccccc}
& \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \\
1 & 2 \\
3 & 4 \\
5 & 3 \\
\end{array}
\]

Definition 1.6. (a) A path \(\rho\) in \(\Gamma_Q\) is \(S\)-sectional (resp. \(N\)-sectional) if the path is a concatenation of arrows whose forms are \((i, p) \rightarrow (i + 1, p + 1)\) (resp. \((i, p) \rightarrow (i - 1, p + 1)\)).

(b) A positive root \(\beta \in \Phi_n^+\) is contained in the path \(\rho\) in \(\Gamma_Q\) if \(\beta\) is an end or a start of some arrow in the path \(\rho\).

(c) An \(S\)-sectional (resp. \(N\)-sectional) path \(\rho\) is maximal if there is no longer an \(S\)-sectional (resp. \(N\)-sectional) path containing all positive roots in \(\rho\).

(d) For \(\beta \in \Phi_n^+\) with \(\phi^{-1}(\beta, 0) = (i, p)\), we denote by \(\phi^{-1}_1(\beta) = i\) and \(\phi^{-1}_2(\beta) = p\).

The main goal of this subsection is to provide an explicit description of an AR-quiver of finite type \(A\) by using the following lemma and the basic properties for an AR-quiver shown in Section 1.2.

Lemma 1.7 ([5, Lemma 2.11], [28, §3.2]). (a) For \(k \in I_0\),

\[
\phi^{-1}(\alpha_k, 0) = \begin{cases} (k, \xi_k) & \text{if } k \text{ is a source,} \\
(n + 1 - k, \xi_k - n + 1) & \text{if } k \text{ is a sink,} \\
(1, \xi_k - k + 1) & \text{if } k \text{ is a left intermediate,} \\
(n, \xi_k - n + k) & \text{if } k \text{ is a right intermediate.} \\
\end{cases}
\]

(b) If $\beta \to \alpha$ is an arrow in $\Gamma_Q$ for $\alpha, \beta \in \Phi_n^+$, then $(\beta, \alpha) = 1$.
(c) We have
\[(1.12) \quad (i, \xi_j - d(i, j)), \ (i, \xi_j - 2m_j + d(i, j)) \in \Gamma_Q \quad \text{for any } i, j \in I_0.\]

Proof. The second and third assertions are identical to [5, Lemma 2.11] and [28, Lemma 3.2.2], respectively. For the first assertion, we can use the [28, Lemma 3.2.3] in the following way: (i) If $k$ is a left intermediate, then take the extremal vertex $i$ as 1 in $I_0$. Then the formula
\[(1.13) \quad \phi^{-1}(\alpha_k, 0) = (i, \xi_k - d(i, k))\]
in [28, Lemma 3.2.1] tells us that $\phi^{-1}(\alpha_k, 0) = (1, \xi_k - k + 1)$. (ii) If $k$ is a right intermediate, then take the extremal vertex $i$ as $n$ in $I_0$. Then [1.13] tells us that $\phi^{-1}(\alpha_k, 0) = (n, \xi_k - n + k)$. It is obvious for $k$ which is a source or a sink. \(\Box\)

 Remark 1.8. For a left (resp. right) intermediate $k$, we can also take the extremal vertex $i$ as $n$ (resp. 1) in $I_0$. Then we have
\[(1.14) \quad \phi^{-1}(\alpha_k, 0) = (i^*, \xi_k^* - 2m_k^* + d(i, k))\]
by the formula [28, page 1568].

**Proposition 1.9.** Let $\rho$ be an $N$-sectional (resp. $S$-sectional) path containing a simple root. Then every positive root contained in $\rho$ has the same first (resp. second) component.

Proof. (1) Let $t$ be a source in $Q$; i.e., \[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
_{t-1} \quad t \quad t+1 \\
\end{array}
\]
Since $t$ is a source,
\[\dim I(t - 1) = [a, t], \ \dim I(t + 1) = [t, b] \quad \text{for some } a \leq t - 1 \text{ and } b \geq t + 1.\]

Note that $t - 1$ or $t + 1 \in I_0$. By Lemma 1.7 (c), the following subquiver is contained in $\Gamma_Q$:
\[
\begin{array}{c}
i = 1 \\
\quad \beta = B_1 \quad B_{t-1} \quad B_t \\
\quad \beta = B_1 \quad \ldots \quad B_{t-1} \\
\quad N_{t-1} \quad \ldots \quad N_t = \dim I(t - 1) \\
\quad C_1 = B_t \quad \ldots \quad C_{n-1} \\
\quad C_1 = B_t \quad \ldots \quad C_{n-1} \\
\eta \quad \ldots \quad M_{n-1} = \dim I(t + 1) \\
i = n \\
\end{array}
\]

By the additive property of the dimension vectors (1.4), we have
- \([t + 1, b] = M_1 - [t] = B_1 - N_1 = B_2 - N_2 = B_{t-1} - N_{t-1} = \beta\) and
- \([a, t - 1] = N_{t-1} - [t] = C_1 - M_1 = C_2 - M_2 = C_{n-t} - M_{n-t} = \eta.\]

Since $\beta + N_2 = B_2 \in \Phi_n^+$, $N_2$ is $[b + 1, d]$ or $[a_2, t]$. If $N_2 = [b + 1, d]$, then $(N_1, N_2) = 0$, which yields a contradiction to Lemma 1.7 (b). Thus $N_2 = [c, t]$. In this way, one can show that $N_u$ for $1 \leq u \leq t - 1$ is of the form $[a_u, t]$. In a similar way, we can prove that $M_v$ for $1 \leq v \leq n - t$ is of the form $[t, b_v]$. (2) Let $t$ be a sink in $Q$; i.e., \[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \\
_{t-1} \quad t \quad t+1 \\
\end{array}
\]
Since $t$ is a sink,
\[\dim P(t - 1) = [a, t], \ \dim P(t + 1) = [t, b] \quad \text{for some } a \leq t - 1 \text{ and } b \geq t + 1.\]
By using a similar argument given in (1), one can prove our assertion.

(3) Let \( t \) be a left intermediate in \( Q \); i.e., \( t \in Q \) and \( t+1 \) is an arrow. Then we have

\[
\dim I(t-1) = [k, t-1], \quad \dim I(t) = [k, t], \quad \dim P(t^*) = [t, l] \quad \text{and} \quad \dim P(t^*-1) = [t+1, l]
\]

for some \( k \leq t-1 \) and \( t+1 \leq l \). Moreover, Lemma 1.7(a) tells us that \( \phi^{-1}(\alpha_t, 0) = (1, \xi_t - t + 1) \). By Lemma 1.7(c), the following subquivers are contained in \( \Gamma_Q \):

\[
\begin{array}{c}
M_1 \rightarrow \cdots \rightarrow M_{t-1} \rightarrow M_t \rightarrow \cdots \rightarrow M_{k-1} \rightarrow M_k = \dim I(t) \\
N_1 \rightarrow \cdots \rightarrow N_{t-1} \rightarrow N_t = \dim I(t-1) \\
L_1 \rightarrow \cdots \rightarrow L_{t-1} \rightarrow L_t = \dim P(t^*-1) \\
J_1 \rightarrow \cdots \rightarrow J_{t-1} \rightarrow J_t = \dim P(t^*)
\end{array}
\]

Since \( [t] + N_k = M_k \in \Phi^+; \) \( N_k \) is \([a_k, t-1]\) or \([t+1, b_k]\). However \((N_{t-1}, N_t) = 1\) implies that \( N_{t-1} = [a_{t-1}, t-1]\) and hence \( M_{t-1} = [a_{t-1}, t]\). Then we can show \( M_k = [a_k, t]\) for \( 1 \leq k \leq t \) by using the additive property of the dimension vectors (1.3).

By applying the same argument, one can prove \( J_k = [t, b_k]\) for \( 1 \leq k \leq t^*\).

(4) Let \( t \) be a right intermediate in \( Q \); i.e., \( t \in Q \) and \( t-1 \) is an arrow. Then we have

\[
\dim I(t) = [t, l], \quad \dim I(t+1) = [t+1, l], \quad \dim P(t) = [k, t] \quad \text{and} \quad \dim P(t+1) = [k, t-1]
\]

for some \( k \leq t-1 \) and \( t+1 \leq l \). By using a similar argument given in (3), one can prove our assertion.

\[\square\]

**Proposition 1.10.** Let \( \rho \) be a maximal \( N \)-sectional path or a maximal \( S \)-sectional path. Then it contains a simple root.

**Proof.** By Proposition 1.9, \( \rho \) contains at most one simple root. Assume that \( \rho \) is a maximal \( N \)-sectional path which does not contain a simple root, and we write \( \rho = N_t \rightarrow N_{t-1} \rightarrow \cdots \rightarrow N_1 \).

Assume first that \( \phi_1^{-1}(N_1) = a \neq 1 \). Hence, \( N_1 = \tau^k(\dim I(a)) \), for some \( k \geq 0 \). If the arrow between \( a \) and \( a-1 \) in \( \Delta_n \) was \( \bullet \rightarrow \bullet \), the irreducible morphism \( I(a) \rightarrow I(a-1) \) would give the arrow \((a, \xi_a) \rightarrow (a-1, \xi_{a+1})\). Applying \( \tau^k \), we would have a contradiction to the maximality of \( \rho \). Thus, \( a \) is either right intermediate or a source:

\[\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \text{ or } \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]

In both cases, \( N_1 \) is the vertex for an injective module; otherwise, we have an arrow \( N_1 \rightarrow \tau^{k-1}(\dim I(a-1)) \) in \( \Gamma_Q \) and it contradicts the maximality of \( \rho \) again. Thus we have \( \phi_1^{-1}(N_1, 0) = (a, \xi_a) \). Hence Lemma 1.7(a) tells us that \( \phi^{-1}(N_t, 0) = (n, \xi_a - n + a) \). Thus Lemma 1.7(a) implies that \( N_t = [a] \) if \( a \) is a right intermediate and \( N_1 = [a] \) if \( a \) is a source, contradicting our assumption that \( \rho \) does not contain a simple root. Hence we have \( \phi_1^{-1}(N_1) = 1 \).

If \( \phi_2^{-1}(N_1) = \xi_1 \), then \( N_1 = \dim I(1) \). Since \( 1 \) is a source or sink, then \( N_1 = [1] \), or \( N_{1m} = [1] \) and \( \phi^{-1}(N_n) = (n, \xi_n - 2m_n) \) contradicting our assumption on \( \rho \) again.

Now we assume that \( \phi_2^{-1}(N_1) < \xi_1 \). If \( \phi_1^{-1}(N_i) = b \neq n \), then

\[\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \text{ or } \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet \]

in \( Q \).
by a similar reason as above. In both cases, \( \phi^{-1}(N_t, 0) = (b, \xi_b - 2m_b) \). Hence Lemma 1.7 (c) tells us that \( \phi^{-1}(N_1) = (1, \xi_b - 2m_b + (b-1)) \). Thus Lemma 1.7 (a) and Remark 1.8 imply that \( N_t = [b^*] \) if \( b \) is a left intermediate, and \( N_t = [b^*] \) if \( b \) is a sink, respectively. Thus we have \( t = n \) and \( \phi^{-1}(N_t) = n \). However, the below inequality

\[
\phi_2^{-1}(N_n) = \phi_2^{-1}(N_1) - n + 1 < \xi_n - 2m_n = \xi_1 - n + 1
\]
yields a contradiction to (1.6). Similarly, one can prove our assertion for a maximal \( S \)-sectional path.

**Theorem 1.11.** Every positive root in an \( N \)-sectional path has the same first component and every root in an \( S \)-sectional path has the same second component.

**Proof.** By the previous proposition, every maximal \( N \)-sectional path (resp. maximal \( S \)-sectional path) contains a simple root. Thus our assertion follows from Proposition 1.9 and the fact that every \( N \)-sectional path (resp. \( S \)-sectional path) is contained in some maximal \( N \)-sectional path (resp. maximal \( S \)-sectional path).

Thus we say that an \( N \)-sectional path \( \rho \) is the maximal \((N, i)\)-sectional path if all positive roots contained in \( \rho \) have \( i \) as a first component. Similarly, we can define a notion of the maximal \((S, i)\)-sectional path.

**Corollary 1.12.** For \( 1 \leq i \leq n \), the Auslander-Reiten quiver \( \Gamma_Q \) of finite type \( A \) contains a maximal \( N \)-sectional path of length \( n - i \) once and exactly once. At the same time, \( \Gamma_Q \) contains a maximal \( S \)-sectional path of length \( i - 1 \) once and exactly once.

**Proof.** For \( i \in I \), there are \((n - i + 1)\)-many positive roots whose first component is \( i \), and \( i \)-many positive roots whose second component is \( i \). Thus our assertion follows from Theorem 1.11.

**Corollary 1.13.** For vertices \( \beta, \beta' \in \Phi^+_n \) with \( \beta \prec_Q \beta' \) and \( \beta + \beta' \in \Phi^+_n \),

\[
(1.15) \quad \text{there exists only one path from } \beta' \text{ to } \beta + \beta', \text{ so is from } \beta + \beta' \text{ to } \beta.
\]

**Proof.** The proof comes from (1.8) and Theorem 1.11.

To get a \( \Gamma_Q \) from \( Q \), we have used \( \tau(\beta) \) (1.2) or the additive property of the dimension vectors (1.4). On the other hand, Lemma 1.7 and Theorem 1.11 provide a combinatorial way for computing the bijection \( \phi^{-1} \) without using the Coxeter element and the additive property of the dimension vectors.

The following remark is well known to experts but we record it here for later use.

**Remark 1.14.** (a) By Lemma 1.7 (a), we know the images of \( \Pi_n \) in \( \Gamma_Q \).
(b) For each \( a \in I_0 \), we draw a hook as follows:

(i) If \( a \) is a source,

\[
\overline{N_n \to \cdots \to N_{a-1} \to N_a} = [a] = S_a \leftarrow S_{a-1} \leftarrow \cdots \leftarrow S_1.
\]

(ii) If \( a \) is a sink,
(ii) If \( a \) is a sink, the maximal \((N,a)\)-sectional path
\[
\overset{\text{maximal } (N,a)-\text{sectional path}}{N_n \leftarrow \cdots N_{n-a} \leftarrow N_{n-a+1}} = [a] = S_{n-a+1} \rightarrow S_{n-a+2} \rightarrow \cdots \rightarrow S_1.
\]

(iii) If \( \bullet \underset{a-1}{\rightarrow} \underset{a}{\rightarrow} \underset{a+1}{\longrightarrow} \bullet \) in \( Q \), the maximal \((N,a)\)-sectional path
\[
\overset{\text{maximal } (N,a)-\text{sectional path}}{N_{n-a+1} \rightarrow \cdots N_2 \rightarrow N_1} = [a] = S_1 \rightarrow S_2 \rightarrow \cdots \rightarrow S_a.
\]

(iv) If \( \bullet \underset{a-1}{\rightarrow} \underset{a}{\rightarrow} \underset{a+1}{\longrightarrow} \bullet \) in \( Q \), the maximal \((N,a)\)-sectional path
\[
\overset{\text{maximal } (N,a)-\text{sectional path}}{N_a \leftarrow \cdots N_{n-1} \leftarrow N_n} = [a] = S_n \leftarrow S_{n-1} \leftarrow \cdots \leftarrow S_{n+1-a}.
\]

Here \( \phi^{-1}_1(N_k) = k \) and \( \phi^{-1}_1(S_i) = l \).

(c) If \((i,p)\) in \( \Gamma_Q \) is located at an intersection of an \((N,a)\)-sectional path and an \((S,b)\)-sectional path,
\[
\phi^{-1}([a,b],0) = (i,p).
\]

Let \( \kappa \) and \( \sigma \) be subsets of \( \Phi^+_n \) defined as follows:
\[
(1.16) \quad \kappa := \{ \beta \in \Phi^+_n \mid \phi^{-1}_1(\beta) = 1 \}, \quad \sigma := \{ \beta \in \Phi^+_n \mid \phi^{-1}_1(\beta) = n \}.
\]

We enumerate the positive roots in \( \kappa = \{ \kappa_1, \ldots, \kappa_r \} \) and \( \sigma = \{ \sigma_1, \ldots, \sigma_s \} \) in the following way:
\[
\phi^{-1}_2(\kappa_{i+1}) + 2 = \phi^{-1}_2(\kappa_i), \quad \phi^{-1}_2(\sigma_{j+1}) - 2 = \phi^{-1}_2(\sigma_j), \quad 1 \leq i < r \text{ and } 1 \leq j < s.
\]

**Corollary 1.15.**

(a) If \( \kappa_i = [a,b] \), then \( \kappa_{i+1} = [b+1,c] \).

(b) If \( \sigma_j = [a,b] \), then \( \sigma_{j+1} = [b+1,c] \).

(c) \( \sum_{i=1}^{r} \kappa_i = [1,n] = \sum_{j=1}^{s} \sigma_j \).

(d) If \( \phi^{-1}_1([1,n]) = k \), then \( r = k = m_1, s = n + 1 - k = m_n \) and hence
\[
\phi^{-1}([1,n]) = (m_1, \xi_1 - m_1 + 1) = (n + 1 - m_n, \xi_n - m_n + 1).
\]

**Proof.** By Remark 1.14, we have a subquiver in \( \Gamma_Q \) which can be described as follows:
\[
\begin{array}{cccccccc}
\kappa_k = [b,n] & \rightarrow & \kappa_{k-1} & \rightarrow & \cdots & \rightarrow & \kappa_2 & \rightarrow & \kappa_1 = [1,a] \\
M_{k-1}^{(1)} & \rightarrow & M_{k}^{(k-1)} & \rightarrow & \cdots & \rightarrow & M_{1}^{(1)} & \rightarrow & \\
(1) & \rightarrow & (k) & \rightarrow & \cdots & \rightarrow & (1) & \rightarrow & \\
i = k & & & & & & & \\
\sigma_1 = [1,a] & \rightarrow & \cdots & \rightarrow & \sigma_{n-k+2} & \rightarrow & N_{n-k+2} & \rightarrow & \\
N_{n-k+1}^{(n-k+2)} & \rightarrow & \cdots & \rightarrow & N_2^{(n-k+2)} & \rightarrow & N_{k}^{(k)} & \rightarrow & \\
& \rightarrow & \sigma_{n+1-k} = [b',n] & \rightarrow & \cdots & \rightarrow & \end{array}
\]

Then our assertions easily follow from observations on the given subquiver. \( \square \)
2. THE GENERALIZED QUANTUM AFFINE SCHUR-WEYL DUALITY

In this section, we briefly recall the backgrounds and theories on the generalized quantum affine Schur-Weyl duality developed in [27, 28]. Thus, for precise statements and definitions referred to in this section, see [1, 22, 27, 28, 35, 44).

2.1. Quantum affine algebras and the category $\mathcal{C}_Q$. Let $I = \{0, 1, \ldots, n\}$ be a set of indices. An affine Cartan datum is a quadruple $(\mathcal{A}, \mathcal{P}, \Pi, \Pi^\vee)$ consisting of

(a) a matrix $A$ of corank 1, called the affine Cartan matrix satisfying

\begin{align*}
(i) & \quad a_{ij} = 2 (i \in I), \\
(ii) & \quad a_{ij} \in \mathbb{Z}_{\leq 0}, \\
(iii) & \quad a_{ij} = 0 \text{ if } a_{ji} = 0
\end{align*}

with $D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$ making $DA$ symmetric,

(b) a free abelian group $\mathcal{P}$ of rank $n + 2$, the weight lattice,

(c) a linearly independent set $\Pi = \{\alpha_i \mid i \in I\} \subset \mathcal{P}$, the set of simple roots,

(d) a linearly independent set $\Pi^\vee = \{h_i \mid i \in I\} \subset \Pi^\vee := \text{Hom}(\mathcal{P}, \mathbb{Z})$, the set of simple coroots,

which satisfy

\begin{align*}
(1) & \quad \langle h_i, \alpha_j \rangle = a_{ij} \quad \text{for all } i, j \in I, \\
(2) & \quad \text{for each } i \in I, \text{ there exists } \Lambda_i \in \mathcal{P} \text{ such that } \langle h_i, \Lambda_j \rangle = \delta_{ij} \quad \text{for all } j \in I.
\end{align*}

We call $Q = \sum_{i \in I} \mathbb{Z} \alpha_i$ the root lattice, and $Q^+ = \sum_{i \in I} \mathbb{Z}_{>0} \alpha_i$ the positive cone of the root lattice. For $\beta = \sum_{i \in I} k_i \alpha_i \in Q^+$, the height of $\beta$ is defined by $|\beta| = \sum_{i \in I} k_i$. We denote by $\delta = \sum_{i \in I} a_{ij} h_i$ the null root and by $c = \sum_{i \in I} c_i h_i$ the center (Chapter 4).

Set $\mathfrak{h} = Q \otimes \mathbb{Z} \Pi^\vee$. Then there exists a symmetric bilinear form $(\ , \ )$ on $\mathfrak{h}^*$ satisfying

\begin{align*}
\langle h_i, \lambda \rangle = \frac{2(\langle \alpha_i, \lambda \rangle)}{(\alpha_i, \alpha_i)} \quad \text{for any } i \in I \text{ and } \lambda \in \mathfrak{h}^*.
\end{align*}

We normalize the bilinear form by $\langle c, \lambda \rangle = (\delta, \lambda)$ for any $\lambda \in \mathfrak{h}^*$.

We choose $0 \in I$ as the leftmost vertex in the tables in [25] pages 54, 55] except for the $A^{(2)}_{2n}$-case in which we take the longest simple root as $\alpha_0$.

We denote $g$ by the affine Kac-Moody algebra associated with the Cartan datum $(\mathcal{A}, \mathcal{P}, \Pi, \Pi^\vee)$ and $g_0$ by the subalgebra of $g$ generated by $e_i, f_i$ and $K_i$ for $i \in I_0$. Note that $g_0$ is a finite-dimensional simple Lie algebra.

Let $\gamma$ be the smallest positive integer such that

\begin{align*}
\gamma(\alpha_i, \alpha_i)/2 \in \mathbb{Z} \quad \text{for any } i \in I.
\end{align*}

Note that $(\alpha_i, \alpha_i)/2$ takes the rational number $1, 2, 1/2, 1/3$ when $g = G_2^{(1)}$.

For an indeterminate $q$, $m, n \in \mathbb{Z}_{\geq 0}$ and $i \in I$, we define $q_i = q^{(\alpha_i, \alpha_i)/2}$ and

\begin{align*}
[n]_i = \frac{q_i^m - q_i^{-n}}{q_i - q_i^{-1}}, & \quad [n]_i! = \prod_{k=1}^{n} [k]_i, & \quad [m]_i! = \frac{[m]_i!}{[m-n]_i![n]_i!}.
\end{align*}

Let $k$ be an algebraic closure of $\mathbb{C}(q)$ in $\bigcup_{m>0} \mathbb{C}((q^1/m))$.

**Definition 2.1.** The quantum affine algebra $U_q(g)$ associated with $(\mathcal{A}, \mathcal{P}, \Pi, \Pi^\vee)$ is the associative $k$-algebra generated by $e_i, f_i$ ($i \in I$) and $q^h$ ($h \in \gamma^{-1}\Pi^\vee$) satisfying the following relations:

\begin{align*}
(1) & \quad q^0 = 1, q^h q^{h'} = q^{h+h'} \quad \text{for } h, h' \in \gamma^{-1}\Pi^\vee, \\
(2) & \quad q^h e_i q^{-h} = q^{(\alpha_i, h)} e_i, \quad q^h f_i q^{-h} = q^{-(\alpha_i, h)} f_i \quad \text{for } h \in \gamma^{-1}\Pi^\vee, i \in I,
\end{align*}
\( (3) \) \( e_i f_j - f_j e_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i^h - q_i^{-h}} \), where \( K_i = q_i^h \),

\[ \sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j^{(k)} = 0 \text{ for } i \neq j, \]

where \( e_i^{(k)} = e_i^k / [k]! \) and \( f_i^{(k)} = f_i^k / [k]! \).

We denote by \( U'_q(\mathfrak{g}) \) the subalgebra of \( U_q(\mathfrak{g}) \) generated by \( e_i, f_i, K_i^{\pm 1} (i \in I) \) and call it also the quantum affine algebra.

We call \( P_{cl} := P / \mathbb{Z} \delta \) the classical weight lattice. For \( cl : P \to P_{cl} \) the canonical projection, we have

\[ P_{cl} = \bigoplus_{i \in I} \mathbb{Z} cl(A_i) \quad \text{and} \quad P_{cl}^\vee := \text{Hom}(P_{cl}, \mathbb{Z}) = \{ h \in P_{cl} | \langle h, \delta \rangle = 0 \} = \bigoplus_{i \in I} \mathbb{Z} h_i. \]

Set \( \Pi_{cl} = cl(\Pi) \) and \( \Pi_{cl}^\vee = \{ h_0, \cdots, h_n \} \). Then we can regard \( U'_q(\mathfrak{g}) \) as the quantum affine algebra associated with the classical affine Cartan datum \((A, P_{cl}, \Pi_{cl}, \Pi_{cl}^\vee)\).

We define the smallest abelian full subcategory \( \mathcal{P}_{cl} \) of \( \mathcal{P} \) as follows:

\[ \mathcal{P}_{cl} := \{ u \in \mathcal{P} | \langle h, u \rangle = 0 \text{ for all } h \in P_{cl} \}. \]

For \( i \in I_0 \), there exists a unique simple \( U'_q(\mathfrak{g}) \)-module \( V(\varpi_i) \) (up to an isomorphism), called the \( i \)-th fundamental module, satisfying the certain properties \((1)\). Moreover, there exist the left dual \( V(\varpi_i)^* \) and the right dual \( V(\varpi_i)^\ast \) of \( V(\varpi_i) \) with the following \( U'_q(\mathfrak{g}) \)-homomorphisms:

\[ V(\varpi_i)^* \otimes V(\varpi_i) \xrightarrow{\text{tr}} k \quad \text{and} \quad V(\varpi_i) \otimes V(\varpi_i)^\ast \xrightarrow{\text{tr}} k \]

where

\[ V(\varpi_i)^* := V(\varpi_i)^{\ast \ast}, \quad V(\varpi_i) := V(\varpi_i)^{\ast \ast \ast}, \quad p := (-1)^{\langle \rho', \delta \rangle} q^{\langle \rho, \delta \rangle}. \]

Here \( \rho \) is defined by \( \langle h_i, \rho \rangle = 1 \) and \( p^\ast \) is defined by \( \langle p^\ast, \alpha_i \rangle = 1 \) for all \( i \in I \) (see \((1)\)).

We say that a \( U'_q(\mathfrak{g}) \)-module \( M \) is good if it has a bar involution, a crystal basis with simple crystal graph, and a global basis (see \(35\) for precise definitions). For instance, every fundamental module is a good module. Note that every good module is a simple \( U'_q(\mathfrak{g}) \)-module.

**Definition 2.2** \((22)\; \text{and} \; (23)\). Let \( Q \) be a Dynkin quiver of finite type \( A_n \) (resp. \( D_n \)) and \( U'_q(\mathfrak{g}) \) be the quantum affine algebra of type \( A_n^{(1)} \) (resp. \( D_n^{(1)} \)). For any positive root \( \beta \) contained in \( \Phi_n^+ \) associated to \( \mathfrak{g}_0 \), we set the \( U'_q(\mathfrak{g}) \)-module \( V_Q(\beta) \) defined as follows:

\[ V_Q(\beta) := V(\varpi_i)^{(-q)^p} \quad \text{where} \quad \phi^{-1}(\beta, 0) = (i, p). \]

We define the smallest abelian full subcategory \( \mathcal{P}_Q^{(1)} \) consisting of finite dimensional integrable \( U'_q(\mathfrak{g}) \)-modules such that

\( a \) it is stable by taking subquotient, tensor product and extension,

\( b \) it contains \( V_Q(\beta) \) for all \( \beta \in \Phi_n^+ \).

For the rest of this subsection, we briefly recall the morphisms between \( U'_q(\mathfrak{g})^{(1)} \)-modules \((t = 1, 2)\):

\[ V(\varpi_i)^{a} \otimes V(\varpi_j)^{b} \quad \text{and} \quad V(\varpi_k)^{c}. \]
These kinds of morphisms are known as *Dorey type morphisms* and are well studied for the classical untwisted quantum affine algebras of types $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ in [14]. Recently, the author investigated such morphisms for the classical twisted quantum affine algebras of types $A_{2n-1}^{(2)}$, $A_{2n}^{(2)}$ and $D_n^{(2)}$ in [44].

**Theorem 2.3** ([14 Theorem 6.1]). Let $U'_q(g)$ be the affine algebra of type $A_n^{(1)}$. For $1 \leq i, j, k \leq n$ and $a, b, c \in \mathbb{Z}$,

$$\text{Hom}_{U'_q(g)}(V(\varpi_i)(-q)^a \otimes V(\varpi_j)(-q)^b, V(\varpi_k)(-q)^c) \neq 0$$

if and only if one of the following holds:

(i) $i + j \leq n$, $i + j = k$, $a - c = -j$ and $b - c = i$,

(ii) $i + j > n$, $k = i + j - n - 1$, $c - a = -n - 1 + j$ and $b - a = n + 1 - i$.

In this paper, we denote the $D_3^{(2)}$ in [25] by $A_3^{(2)}$ with the following enumeration on a Dynkin diagram:

```
\begin{array}{c}
\text{\textbackslashhphantom{m}} \\
-1 \\
2 \\
\text{\textbackslashhphantom{m}}
\end{array}
```

**Theorem 2.4** ([44 Theorem 3.5, Theorem 3.9]). (a) For $i + j = k \leq n$, there exists a surjective $U'_q(A_{2n-1}^{(2)})$-module (resp. $U'_q(A_{2n}^{(2)})$-module) homomorphism

$$p_{i,j}: V(\varpi_i)(-q)^{-j} \otimes V(\varpi_j)(-q)^i \twoheadrightarrow V(\varpi_k).$$

(b) There exists a surjective $U'_q(A_{2n}^{(2)})$-module homomorphism

$$p_{1,n}: V(\varpi_n)(-q)^{-1} \otimes V(\varpi_1)(-q)^n \twoheadrightarrow V(\varpi_n).$$

**2.2. The denominators of normalized $R$-matrices.** For good modules $M$ and $N$, we have an *intertwiner* between $U'_q(g)$-modules

$$R_{M,N}^{\text{norm}}: M^{\text{aff}} \otimes N^{\text{aff}} \rightarrow k(z_M, z_N) \otimes k[z_M^{\pm 1}, z_N^{\pm 1}] \otimes M^{\text{aff}}$$

which satisfies

$$R_{M,N}^{\text{norm}} \circ z_M = z_M \circ R_{M,N}^{\text{norm}}, \quad R_{M,N}^{\text{norm}} \circ z_N = z_N \circ R_{M,N}^{\text{norm}} \quad \text{and} \quad R_{M,N}^{\text{norm}}(V_M \otimes V_N) = V_N \otimes V_M.$$ 

Here $M^{\text{aff}}$ is the affinization of $M$ and $z_M$ is the $U'_q(g)$-automorphism of $M^{\text{aff}}$ of weight $\delta$. We call $R_{M,N}^{\text{norm}}$ the normalized $R$-matrix.

Note that

$$R_{M,N}^{\text{norm}}(M^{\text{aff}} \otimes N^{\text{aff}}) \subset k(z_N/z_M) \otimes k[z_N/z_M^{\pm 1}] \otimes (N^{\text{aff}} \otimes M^{\text{aff}})$$

and there exists a unique monic polynomial $d_{M,N}(u) \in k[u]$ such that

$$d_{M,N}(z_N/z_M)R_{M,N}^{\text{norm}}(M^{\text{aff}} \otimes N^{\text{aff}}) \subset (N^{\text{aff}} \otimes M^{\text{aff}}).$$

We call $d_{M,N}(u)$ the *denominator* of $R_{M,N}^{\text{norm}}$.

**Theorem 2.5** ([1][35]). (1) For good modules $M_1$ and $M_2$, the zeros of $d_{M_1,M_2}(z)$ belong to $\mathbb{C}[[q^{1/m}]]q^{1/m}$ for some $m \in \mathbb{Z}_{>0}$.

(2) $V(\varpi_i)_{a_i} \otimes V(\varpi_j)_{a_j}$ remains simple if and only if

$$d_{i,j}(z) := d_{V(\varpi_i), V(\varpi_j)}(z)$$

does not have zeros at $z = a_i/a_j$ or $a_j/a_i$. 
(3) Let $M$ be a finite dimensional simple integrable $U_q'(g)$-module $M$. Then, there exists a finite sequence

$$(i_1, a_1), \ldots, (i_l, a_l)$$

in $(I_0 \times k^\times)^l$

such that $d_{i_k, i_{k'}}(a_k^t a_{k'}) \neq 0$ for $1 \leq k < k' \leq l$ and $M$ is isomorphic to the head of $\bigoplus_{i=1}^l V(\omega_{i_k})_{a_k}$. Moreover, such a sequence $((i_1, a_1), \ldots, (i_l, a_l))$ is unique up to permutation.

Thus the information on zeros of denominators between fundamental modules is crucial to investigate the finite dimensional integrable $U_q'(g)$-modules. The denominators between fundamental modules are calculated in [16, 28, 44] for all classical quantum affine algebras.

**Theorem 2.6 ([16, 44]).** For $1 \leq k, l \leq n$, we have

$$d_{k,l}(z) = \prod_{s=1}^{\min(k,l,n+1-k,n+1-l)} (z - (-q)^{2s+|k-l|})$$

if $g = A_n^{(1)}$,

$$d_{k,l}(z) = \prod_{s=1}^{\min(k,l)} (z - (-q)^{2s+|k-l|})(z - (p^*)(-q)^{2s-k-l})$$

if $g = A_{2n-1}^{(2)}$ or $A_{2n}^{(2)}$.

2.3. Quiver Hecke algebras and their finite dimensional modules. We now recall the definition of the quiver Hecke algebra. Let $I$ be an index set and $k$ be a field. For any symmetrizable Cartan datum $(A, P, \Pi, \Pi^\vee)$, $Q_{i,j}(u, v) \in k[u, v]$ $(i, j \in I)$ denote polynomials satisfying the following conditions:

$$Q_{i,j}(u, v) = \left\{ \begin{array}{ll}
\sum_{(\alpha_i, \alpha_j) p + (\alpha_j, \alpha_i) q = -2(\alpha_i, \alpha_j)} t_{i,j,p,q} u^p v^q & \text{if } i \neq j, \\
0 & \text{if } i = j,
\end{array} \right.$$

where $t_{i,j,p,q} = t_{j,i,p,q} \in k$ and $t_{i,j,-1,0} \neq 0$. Thus we have $Q_{i,j}(u, v) = Q_{j,i}(v, u)$. Note that the symmetric group on $m$-letters, $S_m = \langle s_1, s_2, \ldots, s_{m-1} \rangle$, acts on $I^m$ by place permutations.

**Definition 2.7 ([38, 39, 48]).** The quiver Hecke algebra $R(m)$ associated with polynomials $(Q_{i,j}(u, v))_{i,j \in I}$ is the $\mathbb{Z}$-graded $k$-algebra defined by three sets of generators

$$\{e(\nu) \mid \nu = (\nu_1, \ldots, \nu_m) \in I^m\}, \{x_k \mid 1 \leq k \leq m\}, \{\tau_l \mid 1 \leq l \leq m-1\}$$

satisfying the following relations:

$$e(\nu)e(\nu') = \delta_{\nu, \nu'}e(\nu), \quad \sum_{\nu \in I^m} e(\nu) = 1, \quad x_k e(\nu) = e(\nu)x_k, \quad x_k x_l = x_l x_k,$$

$$\tau_l e(\nu) = e(s_l(\nu))\tau_l, \quad \tau_k \tau_l = \tau_l \tau_k \text{ if } |k - l| > 1, \quad \tau_k^2 e(\nu) = Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1})e(\nu),$$

$$(\tau_l x_k - x_k \tau_l) e(\nu) = \left\{ \begin{array}{ll}
-e(\nu) & \text{if } l = k \text{ and } \nu_k = \nu_{k+1}, \\
e(\nu) & \text{if } l = k + 1 \text{ and } \nu_k = \nu_{k+1}, \\
0 & \text{otherwise},
\end{array} \right.$$
By assigning $\mathbb{Z}$-grading on a generator as below, $R(m)$ becomes $\mathbb{Z}$-graded:
\[
\deg(e(\nu)) = 0, \quad \deg(x_k e(\nu)) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg(\tau e(\nu)) = -(\alpha_{\nu}, \alpha_{\nu+1}).
\]

We denote the direct sum of the Grothendieck groups of the categories $\text{Rep}(R(m))$ of finite dimensional graded $R(m)$-modules by
\[
[\text{Rep}(R)] = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} [\text{Rep}(R(m))].
\]

Note that $[\text{Rep}(R)]$ has a free $\mathbb{A} := \mathbb{Z}[q, q^{-1}]$-module structure induced from the $\mathbb{Z}$-grading on $R(m)$, i.e., $(qM)_k = M_{k-1}$ for a graded module $M = \bigoplus_{k \in \mathbb{Z}} M_k$.

For $m, n \in \mathbb{Z}_{\geq 0}$, let
\[
R(m) \otimes R(n) \to R(m+n)
\]
be the $k$-algebra homomorphism given by $e(\mu) \otimes e(\nu) \mapsto e(\mu \ast \nu)$ ($\mu \in \mathbb{I}_m, \nu \in \mathbb{I}_n$), $x_k \otimes 1 \mapsto x_k (1 \leq k \leq m)$, $1 \otimes x_k \mapsto x_{m+k} (1 \leq k \leq n)$, $\tau_k \otimes 1 \mapsto \tau_k (1 \leq k < n)$, $1 \otimes \tau_k \mapsto \tau_{m+k} (1 \leq k < n)$, where $\mu \ast \nu$ denotes the concatenation of $\mu$ and $\nu$.

For an $R(m)$-module $M$ and an $R(n)$-module $N$, the induced $R(m+n)$-module
\[
M \circ N := R(m + n) \otimes_{R(m) \otimes R(n)} (M \otimes N)
\]
is called the convolution product of $M$ and $N$. For $M \in \text{Rep}(m)$, $M_k \in \text{Rep}(m_k)$ $(k = 1, \ldots, n)$ and $r \in \mathbb{Z}_{\geq 0}$, let
\[
M^{00} = k, \quad M^{0r} = \underbrace{M \circ \cdots \circ M}_{r}, \quad \overset{n}{\underset{k=1}{\sum}} M_k = M_1 \circ \cdots \circ M_n.
\]

For $\beta \in \mathbb{Q}^+$ with $|\beta| = m$, we define $I^\beta$ and $e(\beta)$ as follows:
\[
I^\beta = \{\nu = (\nu_1, \ldots, \nu_m) \in \mathbb{I}_m \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_m} = \beta\} \quad \text{and} \quad e(\beta) = \sum_{\nu \in I^\beta} e(\nu).
\]
Then $e(\beta)$ becomes a central idempotent. We define the subalgebra $R(\beta)$ of $R(m)$ by
\[
R(\beta) := R(m) e(\beta)
\]
and call it the quiver Hecke algebra at $\beta$.

**Theorem 2.8** ([38, 39, 48]). For any symmetrizable Cartan datum $(\mathbb{A}, \mathbb{P}, \Pi, \Pi^\vee)$, take $(Q_{i,j}(u, v))_{i, j \in I}$ satisfying (2.10). Then there exists an $\mathbb{A}$-algebra isomorphism
\[
U^-_A(g)^\vee \simeq [\text{Rep}(R)]
\]
where the multiplication of $[\text{Rep}(R)]$ is given by the convolution product.

For a module $M$, we denote by $\text{hd}(M)$ the head of $M$ and $\text{soc}(M)$ the socle of $M$, respectively.

**Theorem 2.9** ([12] Theorem 4.7], [43] Theorem 3.1; see also [37, 40]). For a finite simple Lie algebra $\mathfrak{g}_0$, fix an equivalence class $[\bar{w}_0]$ of reduced expression of $w_0$ and take any reduced expression $\tilde{w}_0$ in the class $[\bar{w}_0]$. We fix a convex total order $\leq_{\bar{w}_0}$ induced from the reduced expression $\tilde{w}_0$ (see (1.10)).

Let $R$ be the quiver Hecke algebra corresponding to $\mathfrak{g}_0$. For each positive root $\beta \in \Phi^+_R$, there exists a simple module $S_{\bar{w}_0}(\beta)$ such that:
(a) For all $m \in \mathbb{Z}_{\geq 0}$, $S_{\bar{w}_0}(\beta)^{\circ m}$ is simple.
(b) For every $m = (m_1, \cdots, m_N) \in \mathbb{Z}_{\geq 0}^N$, there exists a non-zero $R$-module homomorphism

$$r_m : \overrightarrow{S_{\tilde{w}_0}(m)} := S_{\tilde{w}_0}(\beta_1)^{\circ m_1} \circ \cdots \circ S_{\tilde{w}_0}(\beta_N)^{\circ m_N}$$

$$\rightarrow S_{\tilde{w}_0}(m) := S_{\tilde{w}_0}(\beta_N)^{\circ m_N} \circ \cdots \circ S_{\tilde{w}_0}(\beta_1)^{\circ m_1},$$

and $\text{Im}(r_m) \simeq \text{hd} \left( \overrightarrow{S_{\tilde{w}_0}(m)} \right) \simeq \text{soc} \left( S_{\tilde{w}_0}(m) \right)$ is simple.

(c) For every simple $R$-module $M$, there exists a unique sequence $m \in \mathbb{Z}_{\geq 0}^N$ such that $M \simeq \text{Im}(r_m) \simeq \text{hd} \left( \overrightarrow{S_{\tilde{w}_0}(m)} \right)$.

(d) For any minimal pair $(\beta_k, \beta_l)$ of $\beta_j = \beta_k + \beta_l$ with respect to the convex total $\leq \tilde{w}_0$, there exists an exact sequence

$$0 \rightarrow S_{\tilde{w}_0}(\beta_j) \rightarrow S_{\tilde{w}_0}(\beta_k) \circ S_{\tilde{w}_0}(\beta_l) \xrightarrow{r_m} S_{\tilde{w}_0}(\beta_l) \circ S_{\tilde{w}_0}(\beta_k) \rightarrow S_{\tilde{w}_0}(\beta_j) \rightarrow 0,$$

where $m \in \mathbb{Z}_{\geq 0}^N$ such that $m_k = m_l = 1$ and $m_i = 0$ for all $i \neq k, l$.

Remark 2.10.  
(i) In this paper, we use the module of $S_{\tilde{w}_0}(\beta)$ in [28]. Thus the statement (c) of the above theorem is different from [12] Theorem 2.9 even though we use the same convention $\leq \tilde{w}_0$ for the convex total order induced from $\tilde{w}_0$ (see [28] Remark 4.3.3).

(ii) For a minimal pair $(\beta_k, \beta_l)$ of $\beta_j$ with respect to $\leq \tilde{w}_0$, the statement (d) of the above theorem tells us that the modules $S_{\tilde{w}_0}(\beta_k) \circ S_{\tilde{w}_0}(\beta_l)$ and $S_{\tilde{w}_0}(\beta_l) \circ S_{\tilde{w}_0}(\beta_k)$ have composition length 2.

(iii) For any pair of reduced expressions $\tilde{w}_0$ and $\tilde{w}_0'$ of $w_0$ which are adapted to $Q$, we have

$$S_{\tilde{w}_0}(\beta) \simeq S_{\tilde{w}_0'}(\beta) \quad \text{for all } \beta \in \Phi^+_n,$$

by (1.11) and Theorem 2.9(d). Thus we denote by $S_Q(\beta)$ the simple $R(\beta)$-module $S_{\tilde{w}_0}(\beta)$ for any reduced expression $\tilde{w}_0$ adapted to $Q$.

Definition 2.11. We say the quiver Hecke algebra $R(\beta)$ at $\beta$ is symmetric if $Q_{i,j}(u,v)$ is a polynomial in $u - v$ for all $i, j \in \text{supp}(\beta)$. Here

$$\text{supp}(\beta) := \{j \in I \mid a_j \neq 0 \text{ where } \beta = \sum_{j \in I} a_j \alpha_j \}.$$

2.4. The generalized quantum affine Schur-Weyl duality functors. Now we briefly review the generalized quantum affine Schur-Weyl duality functors which were studied in [27][28].

Let $S$ be an index set and $\{V_s\}_{s \in S}$ be a family of good $U'_q(\mathfrak{g})$-modules indexed by $S$.

Assume that we have a triple $(J, X, s)$ consisting of an index set $J$ and two maps $X : J \rightarrow \mathbb{k}^\times$, $s : J \rightarrow S$. For a given $(J, X, s)$ and $\{V_s\}_{s \in S}$, we define a quiver $Q^J = (Q_0^J, Q_1^J)$ by observing the order of zeros of denominators as follows:

(1) $Q_0^J = J$.

(2) For $i, j \in J$, we put $d_{ij}$ many arrows from $i$ to $j$, where $d_{ij}$ is the order of the zero of $d_{V_s(i), V_s(j)}(z_2/z_1)$ at $X(j)/X(i)$. 

For a quiver $Q^J$, we can associate

- a symmetric Cartan matrix $A^J = (a_{ij}^J)_{i,j \in J}$ by

$$a_{ij}^J = 2 \quad \text{if } i = j, \quad a_{ij}^J = -d_{ij} - d_{ji} \quad \text{if } i \neq j,$$

- the set of polynomials $(Q_{i,j}^J(u,v))_{i,j \in I}$

$$Q_{i,j}^J(u,v) = (u - v)^{d_{ij}}(v - u)^{d_{ji}} \quad \text{if } i \neq j.$$

We denote by $R_J$ the symmetric quiver Hecke algebra associated with $(Q_{i,j}^J(u,v))_{i,j \in I}$.

**Theorem 2.12** ([27]). There exists a functor

$$\mathcal{F} : \text{Rep}(R_J) \to \mathcal{C}_g$$

where $\mathcal{C}_g$ denotes the category of finite-dimensional integrable $U_q'(g)$-modules. Moreover, the functor enjoys the following properties:

(a) $\mathcal{F}$ is a tensor functor; that is, there exist $U_q'(g)$-module isomorphisms

$$\mathcal{F}(R_J(0)) \simeq \mathbb{k} \quad \text{and} \quad \mathcal{F}(M_1 \circ M_2) \simeq \mathcal{F}(M_1) \otimes \mathcal{F}(M_2)$$

for any $M_1, M_2 \in \text{Rep}(R_J)$.

(b) If the underlying graph of $Q^J$ is a Dynkin diagram of finite type $A$, $D$ or $E$, then the functor $\mathcal{F}$ is exact.

We call the functor $\mathcal{F}$ the generalized quantum affine Schur-Weyl duality functor.

**Theorem 2.13** ([28]). Let $g$ be an affine Kac-Moody algebra of type $A_n^{(1)}$ (resp. $D_n^{(1)}$) and $Q$ be a quiver whose underlying graph is a Dynkin diagram of finite type $A_n$ (resp. $D_n$). Set

$$J := \{(i,p) \in \mathbb{Z}Q \mid \phi(i,p) \in \Pi_n \times \{0\}\},$$

where the map is given in (1.2). We define two maps $s : J \to \{V(\varpi_i) \mid i \in I_0\}$ and $X : J \to \mathbb{k}^\times$ as

$$s(i,p) = V(\varpi_i) \quad \text{and} \quad X(i,p) = (-q)^p \quad \text{for } (i,p) \in J.$$

(a) The underlying graph of the quiver $Q^J$ associated to $(J,X,s)$ is of finite type $A_n$ (resp. $D_n$). Hence the functor

$$\mathcal{F}_Q^{(1)} : \text{Rep}(R_J) \to \mathcal{C}_Q^{(1)}$$

in Theorem 2.12 is exact.

(b) The functor $\mathcal{F}_Q^{(1)}$ sends simples to simples, bijectively. In particular, $\mathcal{F}_Q^{(1)}$ sends $S_Q(\beta)$ to $V_Q(\beta)$.

**Remark 2.14.** In Theorem 4.9 we will provide an alternative proof of (a) in the above theorem by using only the results in this paper. In [28], the authors used the result of [22] to prove (a) in the above theorem.
3. Dorey’s rules and minimal pair

In this section, we prove that $\Gamma_Q$ encodes the information of the Dorey type morphisms for $U'_q(A_n^{(1)})$, and all pair $(\alpha, \beta)$ of $\gamma \in \Phi^+_n$ are minimal with suitable convex total orders, which is compatible with respect to the convex partial order $\preceq_Q$.

**Definition 3.1.** Let $\beta$ be a vertex in $\Gamma_Q$. (Equivalently, $\beta \in \Phi^+_n$.)

(a) The upper ray of $\beta$ is the path consisting of one $S$-sectional path and one $N$-sectional path satisfying the following properties:

- $S_1 \to \cdots S_{a-1} \to S_a = \beta = N_b \to N_{b-1} \to N_1$,
- there is no vertex $S_0 \in \Gamma_Q$ such that $S_0 \to S_1$ is an $S$-sectional path in $\Gamma_Q$,
- there is no vertex $N_0 \in \Gamma_Q$ such that $N_1 \to N_0$ is an $N$-sectional path in $\Gamma_Q$.

(b) The lower ray of $\beta$ is the path consisting of one $S$-sectional path and one $N$-sectional path satisfying the following properties:

- $N_1 \to \cdots N_{a-1} \to N_a = \beta = S_b \to S_{b-1} \to S_1$,
- there is no vertex $N_0 \in \Gamma_Q$ such that $N_0 \to N_1$ is an $N$-sectional path in $\Gamma_Q$,
- there is no vertex $S_0 \in \Gamma_Q$ such that $S_1 \to S_0$ is an $S$-sectional path in $\Gamma_Q$.

In Example 1.5, the upper ray and the lower ray of $[1,4]$ can be described as follows:

$$
\begin{array}{c}
[4] \\
[2,4] \to [1,3] \to [1,4] \\
[1,4] \to [1,5] \to [3,4]
\end{array}
$$

**Theorem 3.2.** For every pair $(\alpha, \beta)$ of $\alpha + \beta = \gamma \in \Phi^*_n$, we write

$$
\phi^{-1}_1(\alpha) = i_\alpha, \quad \phi^{-1}_1(\beta) = i_\beta \quad \text{and} \quad \phi^{-1}_1(\gamma) = i_\gamma.
$$

Then we have

$$
i_\alpha + i_\beta = i_\gamma \quad \text{or} \quad (n + 1 - i_\alpha) + (n + 1 - i_\beta) = (n + 1 - i_\gamma).
$$

**Proof.** We write $\gamma = [a,b]$. By Corollary 1.13 the pair $(\alpha, \beta)$ is contained in the upper ray of $[a,b]$ or the lower ray of $[a,b]$. Assume that it is contained in the upper ray; that is, there exists $a \leq c \leq b$ such that $\alpha = [a,c]$ and $\beta = [c+1,b]$. Using (1.6), Theorem 1.11 and Remark 1.14 the situation in $\Gamma_Q$ can be described
as follows:

By Theorem 1.11 and Remark 1.14, \( \kappa_u = [e, c] \ (e \leq c) \) and \( \kappa_w = [c+1, f] \ (c+1 \leq f) \). Then Corollary 1.15 (a) tells us that \( w = u + 1 \) and hence \( \ell = 1 \). Thus we can conclude

\[
i_\alpha = i_\gamma - i_\beta + 1 - \ell = i_\gamma - i_\beta
\]

which implies our first assertion. If a pair \((\alpha, \beta)\) of \( \gamma \in \Phi_n^+ \) is contained in the lower ray of \( \beta_z \), then one can prove our second assertion by applying a similar argument.

**Remark 3.3.** Using the above theorem, we can observe the following: For pairs \((\alpha, \beta), (\alpha', \beta')\) such that \( \alpha + \beta = \alpha' + \beta' = \gamma \in \Phi_n^+ \) and they are contained in the upper ray (resp. lower ray) of \( \gamma \) together, the number of edges between \( \alpha \) and \( \beta \) in the upper ray (resp. lower ray) coincides with the one between \( \alpha' \) and \( \beta' \).

Now we show that every pair \((\alpha, \beta)\) of \( \alpha + \beta = \gamma \in \Phi_n^+ \) is indeed a minimal pair with respect to a suitable convex total order compatible with the convex partial order \( \preceq_Q \).

**Theorem 3.4.** For every pair \((\alpha, \beta)\) of \( \alpha + \beta = \gamma \), there is a convex total order \(<\) such that

(a) it is compatible with the convex partial order \( \preceq_Q \),

(b) \((\alpha, \beta)\) is a minimal pair of \( \gamma \) with respect to the convex total order.

**Proof.** As we observed, all pair \((\alpha, \beta)\) of \( \gamma \) is located in the upper ray of \( \gamma \) or lower ray of \( \gamma \). Assume that \((\alpha, \beta)\) is in the upper ray. Take any pair \((\alpha', \beta')\) of \( \gamma \).

(i) If \((\alpha', \beta')\) is also contained in the upper ray of \( \gamma \), the observation in Remark 3.3 tells us that we have one of the following two situations in \( \Gamma_Q \):

Thus, with respect to the total convex order \( \prec_U^Q \) in Remark 1.3 we have

\[
\alpha <_Q \alpha' <_Q \gamma <_Q \beta <_Q \beta' \text{ or } \alpha' <_Q \alpha <_Q \gamma <_Q \beta' <_Q \beta.
\]
(ii) Assume that \((\alpha', \beta')\) is contained in the lower ray of \(\gamma\). Then the situation can be expressed as follows:

\[
\begin{array}{c}
\gamma \\
\alpha' \\
\beta' \\
\alpha \\
\end{array}
\]

Hence, with respect to the total convex order \(\prec^U_Q\) in Remark 1.3, we have

\[\alpha' \prec^U_Q \alpha \prec^U_Q \gamma \prec^U_Q \beta' \prec^U_Q \beta.\]

By (i) and (ii), we can conclude that every pair \((\alpha, \beta)\) in the upper ray becomes minimal with respect to the total convex order \(\prec^U_Q\). Similarly, one can see that every pair \((\alpha, \beta)\) in the lower ray becomes minimal with respect to the total convex order \(\prec^L_Q\). \(\square\)

For the rest of this paper, we say that a pair \((\alpha, \beta)\) is minimal when \(\alpha + \beta \in \Phi_n^+\) and the pair is a minimal pair with respect to a suitable convex total order which is induced by some reduced expression \(\tilde{w}_0\) adapted to \(Q\).

**Corollary 3.5.** (a) For every pair \((\alpha, \beta)\) of \(\alpha + \beta = \gamma \in \Phi_n^+\), we have a surjection

\[S_Q(\beta) \circ S_Q(\alpha) \twoheadrightarrow S_Q(\gamma) \quad \text{in } \text{Rep}(R_{A_n}).\]

Hence we have a surjection

\[V_Q(\beta) \otimes V_Q(\alpha) \twoheadrightarrow V_Q(\gamma) \quad \text{in } \mathcal{E}^{(1)}_Q.\]

(b) For every positive root \(\beta \in \Phi_n^+\), there exists a sequence \((i_1, i_2, \ldots, i_{|\beta|}) \in \Pi_{|\beta|}^0\) such that

\[\bigotimes_{i=1}^{|\beta|} V_Q(\alpha_{i_t}) \twoheadrightarrow V_Q(\beta).\]

**Proof.** Note that for a minimal pair \((\alpha, \beta)\) of \(\alpha + \beta = \gamma\) with respect to some convex total order \(\leq_{\tilde{w}_0}\), Theorem 2.9 (d) guarantees the existence of \((\ref{surjection})\) in \(\text{Rep}(R_{A_n})\).

On the other hand, Theorem 3.4 tells us that, for every pair \((\alpha, \beta)\) of \(\gamma\), there exists a reduced expression \(\tilde{w}_0\) adapted to \(Q\) such that the pair is minimal with respect to the total order \(\leq_{\tilde{w}_0}\). Thus our first assertion for \(\text{Rep}(R_{A_n})\) follows.

Recall that the functor \(\mathcal{F}_Q^{(1)}\) sends simples to simples. Thus \(\mathcal{F}_Q^{(1)}(\text{Im}(f))\) is non-zero for any non-zero homomorphism \(f\) in \(\text{Rep}(R_{A_n})\); i.e., \(\mathcal{F}_Q^{(1)}\) is faithful. Then our first assertion for \(\mathcal{E}^{(1)}_Q\) follows since \(\mathcal{F}_Q^{(1)}\) is a tensor functor sending \(S_Q(\beta)\) to \(V_Q(\beta)\). The existence of \((\ref{tensor})\) is also guaranteed by Theorem 2.3 and Theorem 3.2.

The second assertion follows from the first assertion. \(\square\)

From Corollary 3.5 (b), the condition (b) in Definition 2.2 can be modified as follows:

(b') It contains \(V_Q(\alpha_k)\) for all \(\alpha_k \in \Pi_n\).

**Corollary 3.6.** For \(\phi^{-1}(\alpha, 0) = (i, a), \phi^{-1}(\beta, 0) = (j, b)\), assume that \((k, c) \in \Gamma_Q\) such that the triple \(\{(i, a), (j, b), (k, c)\}\) satisfies one of the conditions (i) and (ii) in Theorem 2.3. Then we have

\[\gamma := \phi(k, c) = \alpha + \beta.\]
Proof. By assumption, $\alpha, \beta$ are contained in the upper ray or lower ray of $\gamma$, respectively. Applying the argument in Theorem 3.2, one can prove the assertion. \qed

Recall Example 1.5.

(i) Using the reading in Remark 1.3 (A), the reduced expression $\bar{w}_0$ of $w_0$ associated to the reading is

$$\bar{w}_0 = s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_1$$

and the convex total order on $\Phi^+_5$ is given as follows:

$$[1] < \bar{w}_0 [1, 3] < \bar{w}_0 [2, 3] < \bar{w}_0 [3, 4] < \bar{w}_0 [1, 4] < \bar{w}_0 [2, 4] < \bar{w}_0 [4]$$

$$< \bar{w}_0 [3, 5] < \bar{w}_0 [1, 5] < \bar{w}_0 [2, 5] < \bar{w}_0 [4, 5] < \bar{w}_0 [5] < \bar{w}_0 [1, 2] < \bar{w}_0 [2].$$

With this convex total order, one can observe that every pair in lower rays becomes minimal.

(ii) Using the reading in Remark 1.3 (B), the reduced expression $\bar{w}'_0$ of $w_0$ associated to the reading is

$$\bar{w}'_0 = s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_2 s_1$$

and the convex total order on $\Phi^+_5$ is given as follows:

$$[3] < \bar{w}'_0 [3, 4] < \bar{w}'_0 [3, 5] < \bar{w}'_0 [1] < \bar{w}'_0 [1, 3] < \bar{w}'_0 [1, 4] < \bar{w}'_0 [1, 5] < \bar{w}'_0 [1, 2]$$

$$< \bar{w}'_0 [2, 3] < \bar{w}'_0 [2, 4] < \bar{w}'_0 [2, 5] < \bar{w}'_0 [2] < \bar{w}'_0 [4] < \bar{w}'_0 [4, 5] < \bar{w}'_0 [5].$$

With this convex total order, one can observe that every pair in upper rays becomes minimal.

**Corollary 3.7.** For every pair $(\alpha, \beta)$ of $\alpha + \beta \in \Phi^+$, $V_Q(\beta) \otimes V_Q(\alpha)$ and $S_Q(\beta) \circ S_Q(\alpha)$ are of length 2.

**Proof.** Since every pair $(\alpha, \beta)$ is minimal, our assertions follow from Theorem 2.10 (d) and Theorem 2.13 (b). \qed

**Remark 3.8.** For a reduced expression of the longest element $w_0$ of $A_5$, $\bar{w}_0 = s_1 s_3 s_2 s_5 s_4 s_3 s_1 s_2 s_3 s_4 s_3 s_1 s_2 s_3$, one can easily check that it is not adapted to any Dynkin quiver $Q$ of type $A_5$. The convex total order induced by $\bar{w}_0$ is given as follows:

$$[1] < \bar{w}_0 [1, 2] < \bar{w}_0 [1, 3] < \bar{w}_0 [5] < \bar{w}_0 [1, 5] < \bar{w}_0 [4, 5] < \bar{w}_0 [2] < \bar{w}_0 [2, 5]$$

$$< \bar{w}_0 [2, 3] < \bar{w}_0 [1, 4] < \bar{w}_0 [2, 4] < \bar{w}_0 [4] < \bar{w}_0 [3, 5] < \bar{w}_0 [3, 4] < \bar{w}_0 [3].$$

Thus the pair $(\bar{w}_0 [1, 5], \bar{w}_0 [3, 5])$ is not a minimal pair of $[1, 5]$ with respect to $\prec$, since we have

$$[1, 2] < \bar{w}_0 [1, 3] < \bar{w}_0 [1, 5] < \bar{w}_0 [4, 5] < \bar{w}_0 [3, 5].$$

One can compute that

$$S_{\bar{w}_0 ([1, 2])} = L(21), \quad S_{\bar{w}_0 ([1, 3])} = L(321), \quad S_{\bar{w}_0 ([4, 5])} = L(45), \quad S_{\bar{w}_0 ([3, 5])} = L(345)$$

where $L(21), L(321), L(45)$ and $L(345)$ are one-dimensional modules which are defined in the similar way of (4.11) below. Moreover, we have

- $L(345) \circ L(21)$ and $L(45) \circ L(321)$ have composition length 2,
- $\mathrm{hd}(L(345) \circ L(21)) \simeq \mathrm{soc}(L(45) \circ L(321))$ are simple.
Hence we can conclude that
\[ \text{hd}(L(345) \circ L(21)) \not\simeq \text{hd}(L(45) \circ L(321)) \simeq S_{\omega_0}([1, 5]), \]
by [29, Corollary 3.9].

4. REVISIT OF CONSTRUCTION OF EXACT SEQUENCES AND SIMPLE MODULES

In this section, we revisit the theories on construction of exact sequences and simple modules of \( R_{A_n} \) developed in [6,12,23,28,29,32,40,43] by using the explicit description of \( \Gamma_Q \) in Section 1.

For a \( \beta = [a, b] \in \Phi_n^+ \), we define a one-dimensional simple \( R_{A_n}(\beta) \)-module \( L(a, b) = ku(a, b) \) as follows:

\[
(4.1) \quad x_m u(a, b) = 0 \quad (1 \leq m \leq b - a + 1),
\]
\[
\tau_k u(a, b) = 0 \quad (1 \leq k < b - a + 1),
\]
\[
e(\nu)u(a, b) = \begin{cases} 
    u(a, b) & \text{if } \nu = (a, a + 1, \ldots, b), \\
    0 & \text{otherwise}.
\end{cases}
\]

Lemma 4.1. Fix a Dynkin quiver \( \leftarrow Q \) of type \( A_n \), called the linear quiver, as follows:

\[
\begin{array}{c}
  \bullet \\
  \Rightarrow [1, n] \\
  \Rightarrow [2, n] \\
  \Rightarrow [1, n - 1] \\
  \Rightarrow \vdots \\
  \Rightarrow [1, 2] \\
  \Rightarrow [2, n - 1] \\
  \Rightarrow \vdots \\
  \Rightarrow [n - 1, n] \\
  \Rightarrow [n] \\
\end{array}
\]

For \( \beta = (a, b) \in \Phi_n^+ \), we have
\[ S_Q^- (\beta) \simeq L(a, b). \]

Proof. By Lemma 1.7 (a), we have
\[ \phi^{-1}(\alpha_k, 0) = (n, \xi_n - 2(n - k)) \quad \text{for all } k \in I_0. \]

As is well known, the shape of \( \Gamma_Q^\rightarrow \) is given as follows:

\[
\begin{array}{c}
  1 \\
  \Rightarrow [1, n] \\
  \Rightarrow [2, n] \\
  \Rightarrow [1, n - 1] \\
  \Rightarrow \vdots \\
  \Rightarrow [1, 2] \\
  \Rightarrow [2, n - 1] \\
  \Rightarrow \vdots \\
  \Rightarrow [n - 1, n] \\
  \Rightarrow [n] \\
\end{array}
\]

For a simple root \( \beta \), it is trivial. By an induction hypothesis on a height \( |\beta| \) and Corollary 3.5, we have \( (c + 1, b) < (a, c) \) and a surjection
\[ L(a, c) \circ L(c + 1, b) \twoheadrightarrow S_Q^- (\beta) \quad \text{for } \beta = (a, b) \in \Phi_n^+ \text{ with } |\beta| \geq 2. \]

Since there is a canonical surjection
\[ L(a, c) \circ L(c + 1, b) \twoheadrightarrow L(a, b), \]
and \( L(a, c) \circ L(c + 1, b) \) has a unique simple head, our assertion follows. \( \square \)

Lemma 4.2 ([32]). For \( [a_1, b_1], [a_2, b_2] \in \Phi_n^+ \text{ with } a_1 = a_2 \text{ or } b_1 = b_2, \)
\[ L(a_1, b_1) \circ L(a_2, b_2) \simeq L(a_2, b_2) \circ L(a_1, b_1) \text{ is simple.} \]
Now the following proposition can be regarded as a generalization of the above lemma.

**Proposition 4.3.** For any Dynkin quiver $Q$ of type $A_n$ and $\alpha = [a_1, b_1]$, $\beta = [a_2, b_2] \in \Phi^+_n$ with $a_1 = a_2$ or $b_1 = b_2$,

\[ S_Q(\alpha) \circ S_Q(\beta) \simeq S_Q(\beta) \circ S_Q(\alpha) \text{ is simple.} \]

**Proof.** By Remark 1.14 we have

\[ |\phi_2^{-1}(\alpha) - \phi_2^{-1}(\beta)| = |\phi_1^{-1}(\alpha) - \phi_1^{-1}(\beta)|. \]

By (2.8) and Theorem 2.5 we have

\[ V_Q(\alpha) \otimes V_Q(\beta) \simeq V_Q(\beta) \otimes V_Q(\alpha) \text{ is simple.} \]

Thus our assertion follows from Theorem 2.13. \( \square \)

**Corollary 4.4.** For any Dynkin quiver $Q$ of type $A_n$ and $\beta_k = [a_k, b_k] \in \Phi^+_n$ ($k \in \mathbb{Z}_{>0}$), assume that $a_k = a_l$ (resp. $b_k = b_l$) for all $k \neq l$. Then we have

\[ S_Q(\beta_1) \circ S_Q(\beta_2) \circ \cdots \circ S_Q(\beta_k) \text{ is simple.} \]

**Proposition 4.5.** Let $Q$ be any Dynkin quiver of finite type $A_n$.

(a) For a source or a sink $i$ in $Q$, and $a \leq i \leq b$,

\[ S_Q([a, i]) \circ S_Q([i, b]) \simeq S_Q([i, b]) \circ S_Q([a, i]) \text{ is simple.} \]

(b) For a right intermediate or a left intermediate $i$ in $Q$, and $a < i < b$,

\[ S_Q([a, i]) \circ S_Q([i, b]) \text{ and } S_Q([i, b]) \circ S_Q([a, i]) \text{ are reducible.} \]

**Proof.** (a) Write $\phi^{-1}([i, b]) = (k, p)$ and $\phi^{-1}([a, i]) = (l, q)$. By Remark 1.14 (b-i) and (b-ii), we have

\[ |k - l| \geq |p - q|. \]

Hence, by (2.8), $d_{k, l}(z)$ does not have a zero at $(-q)^{p-q}$. Thus

\[ V_Q([a, i]) \otimes V_Q([i, b]) \simeq V_Q([i, b]) \otimes V_Q([a, i]) \text{ is simple,} \]

which yields our first assertion.

(b) Assume that $i$ is a left intermediate. By Remark 1.14, all positive root of the form $[i, b]$ or $[a, i]$ are contained in the following hook in $\Gamma_Q$:

\[
\text{maximal (}N, i\text{-sectional path)} = \underbrace{N_{n-i+1} \to \cdots \to N_2} \to N_1 = [i] = \underbrace{S_1 \to S_2 \to \cdots \to S_l}_{\text{maximal (}S, i\text{-sectional path)}}.
\]

Thus we can assume that $[i, b] = N_k$ ($2 \leq k \leq n - i + 1$) and $[a, i] = S_l$ ($2 \leq l \leq i$). Then we have

\[
\phi_1^{-1}(N_k) = k, \quad \phi_1^{-1}(S_l) = l \quad \text{and} \quad \phi_2^{-1}(S_l) - \phi_2^{-1}(N_k) = k + l - 2.
\]

In this case, one can check that

- $\min\{k, l, n + 1 - k, n + 1 - k\} = \min\{k, l\}$,
- $|k - l| + 2s = k + l - 2$ when $1 \leq s = \min\{k, l\} - 1$.

Thus

\[ V_Q([a, i]) \otimes V_Q([i, b]) \text{ and } V_Q([i, b]) \otimes V_Q([a, i]) \text{ are reducible,} \]

which yields our second assertion. The case for a right intermediate $i$ can be proved by applying the same argument. \( \square \)
Corollary 4.6. Let $Q$ be a Dynkin quiver of finite type $A_n$ and $i$ be a source or a sink in $Q$. For $\beta_k = [a_k, b_k] \in \Phi_n^+$ ($k \in \mathbb{Z}_{>0}$), assume that $a_k = i$ or $b_k = i$ for all $k$. Then we have

$$S_Q(\beta_1) \circ S_Q(\beta_2) \circ \cdots \circ S_Q(\beta_k)$$

is simple.

Lemma 4.7 (32). For $[a_1, b_1], [a_2, b_2] \in \Phi_n^+$ with $b_1 < a_2 - 1$,

$$L(a_1, b_1) \circ L(a_2, b_2) \simeq L(a_2, b_2) \circ L(a_1, b_1)$$

is simple.

Now the following proposition can be regarded as a generalization of the above lemma.

Proposition 4.8. For any Dynkin quiver $Q$ of finite type $A_n$ and $[a, b], [a', b'] \in \Phi_n^+$, assume that $a < b < a' - 1 < b'$. Then we have

$$S_Q([a, b]) \circ S_Q([a', b']) \simeq S_Q([a', b']) \circ S_Q([a, b])$$

is simple.

Proof. Note that $[a, b'] \in \Phi_n^+$ and there is no $[a', b] \notin \Phi_n^+$. Thus

- $[a, b']$ is located at the intersection of the maximal $(N, a)$-sectional path and the maximal $(S, b')$-sectional path,
- there is no intersection between the maximal $(S, a)$-sectional path and the maximal $(N, b')$-sectional path.

Then the situations can be described as follows:

Write $\phi^{-1}([a, b]) = (i, p)$, $\phi^{-1}([a', b']) = (j, q)$ and $\phi^{-1}([a, b']) = (k, r)$. For the first and second cases, one can easily notice that $|i - j| > |p - q|$. Thus the assertion follows from (2.8).

Assume the third case. Then it is enough to show that, for any $1 \leq s \leq \min\{i, j, n + 1 - i, n + 1 - j\}$,

$$|i - j| + 2s \neq 2k - i - j \quad (4.3)$$

Set $v = \max\{i, j\}$ and $w = \min\{i, j, n + 1 - i, n + 1 - j\}$. Then we have

$$|i - j| + 2s = 2k - i - j \quad \text{if } s = k - v.$$

However, the fact $k > i + j$ implies $k - v > w$. Thus the assertion holds. For the fourth case, it can be proved by applying a similar argument of the third case. \(\square\)

Now, we have an alternative proof of Theorem 2.13 (a) as we emphasized in Remark 2.14.

Theorem 4.9. With the same choice of $(J, X, s)$ as in Theorem 2.13 the underlying graph of $Q^J$ is of finite type $A_n$. 

Proof. By Corollary 3.5 there exists a surjective morphism
\( V_Q(\alpha_k) \otimes V_Q(\alpha_{k+1}) \to V_Q(\alpha_k + \alpha_{k+1}) \).

On the other hand, Proposition 4.11 tells us that
\( V_Q(\alpha_k) \otimes V_Q(\alpha_{k+1}) \) is reducible.

By Corollary 3.5, there exists a surjective morphism
\( V_Q(\alpha_k) \otimes V_Q(\alpha_{k+1}) \to V_Q(\alpha_k + \alpha_{k+1}) \) and hence
\( V_Q(\alpha_k) \otimes V_Q(\alpha_{k+1}) \) is reducible.

\[ (\mathbb{Q}, \mathbf{V}) \]

By Corollary 3.5 and the fact that \( d_{k,l}(z) \) in (2.8) has only zeros of multiplicity 1, our assertion follows. \( \square \)

Definition 4.10. A simple \( R(\beta_k) \)-module \( M \) is called real if \( M \circ M \) is again simple.

For example, every \( S_Q(\beta) \) is real by Theorem 2.9 (a). Now, we recall one of the main results in [29]:

Theorem 4.11 ([29; see also [30]). Let \( M_k \) be a simple \( R(\beta_k) \)-module \( (k = 1, 2) \) and assume one of \( M_1 \) and \( M_2 \) is real, and \( R^\text{norm}_{\mathbb{Q}}(\mathbb{Q}, \mathbf{V})(z) \) has a simple pole at \( z = 1 \). Then \( M_1 \circ M_2 \) has composition length 2. In particular, there exists an exact sequence
\[ 0 \to N \to M_1 \circ M_2 \to M_2 \circ M_1 \to N \to 0, \]
where \( r_{M_1,M_2} \) is non-zero, \( \text{Im}(r_{M_1,M_2}) \) is the unique simple socle of \( M_2 \circ M_1 \) and \( N \) is the unique simple head of \( M_2 \circ M_1 \) (up to isomorphisms).

Proposition 4.12. Let \( Q \) be any Dynkin quiver of finite type \( A_n \). Assume we have the following rectangle subquiver in \( \Gamma_Q \):

\[ \begin{array}{ccc}
\bullet & \bullet & \bullet \\
|a',b'| & |a',b| & |a,b| \\
\bullet & \bullet & \bullet
\end{array} \]

for \( a \neq a' \) and \( b \neq b' \). Then there exists a short exact sequence
\[ 0 \to S_Q([a,b]) \circ S_Q([a',b']) \to S_Q([a,b']) \circ S_Q([a',b]) \to 0, \]
\[ (4.4) \]
such that \( r := r_{S_Q([a,b]),S_Q([a',b'])} \) and
\begin{itemize}
  \item \( S_Q([a,b]) \circ S_Q([a',b']) \to S_Q([a',b']) \circ S_Q([a,b]) \) is simple,
  \item \( S_Q([a',b]) \circ S_Q([a,b']) \) has composition length 2 with the unique simple socle \( \text{Im}(r) \) and the simple head \( S_Q([a',b']) \circ S_Q([a,b]) \) (up to isomorphisms).
\end{itemize}

Proof. Write \( \phi^{-1}([a,b]) = (j,q), \phi^{-1}([a',b']) = (i,p), \phi^{-1}([a,b]) = (k,r) \) and \( \phi^{-1}([a',b']) = (l,s) \). In the rectangle subquiver, we can read that
\[ |k - l| > |r - s| \]
and hence
\[ S_Q([a, b]) \circ S_Q([a', b']) = S_Q([a', b']) \circ S_Q([a, b]) \]
is simple.

Now we claim that \( R_{\text{norm}}^{\mathrm{VQ}([a', b]), \text{VQ}([a, b'])}(z) \) has a pole at \( z = 1 \); that is, there exists \( 1 \leq t \leq \min\{i, j, n + 1 - i, n + 1 - j\} \) satisfying
\[ |i - j| + 2t = 2k - i - j. \]

Set \( v = \max\{i, j\} \) and \( u = \min\{i, j\} \). Then we have \( w = \min\{u, n + 1 - v\} \) and
\[ |i - j| + 2t = 2k - i - j \quad \text{if} \quad t = k - v. \]

Since \( k \leq i + j - 1 \), \( k - v < w \) always. Thus our claim holds.

By Theorem 4.11, it is enough to show there exists a non-zero \( R \)-homomorphism
\[ \psi : S_Q([a', b]) \circ S_Q([a, b']) \twoheadrightarrow S_Q([a, b]) \circ S_Q([a', b']) \simeq S_Q([a', b']) \circ S_Q([a, b]). \]

Note that the following four cases can happen:
\[ a < a' \leq b' < b, \quad a' < a \leq b < b', \quad a < a' \leq b < b' \quad \text{or} \quad a' < a \leq b' < b. \]

Let us consider the first case. From the rectangle quiver, we can read that \([a', b]\) is located at the right part of the upper ray of \([a', b]\) and hence \([b' + 1, b]\) is located at the left part of the upper ray. Thus we have an injective \( R \)-homomorphism
\[ S_Q([a', b]) \hookrightarrow S_Q([a', b']) \otimes S_Q([b' + 1, b]). \]

By taking \(- \circ S_Q([a, b'])\) to (4.6), we have a composition
\[ S_Q([a', b]) \circ S_Q([a, b']) \rightarrow S_Q([a', b']) \circ S_Q([b' + 1, b]) \circ S_Q([a, b'])\]
\[ \rightarrow S_Q([a', b']) \circ S_Q([a, b]), \]
since \(([b' + 1, b], [a, b'])\) is the pair contained in the upper ray of \([a, b]\). By [29] Corollary 3.11, the composition is non-zero. From the fact that \( S_Q([a', b']) \circ S_Q([a, b]) \) is simple, the composition is surjective and hence \( S_Q([a', b']) \circ S_Q([a, b]) \) is a head of \( S_Q([a', b]) \circ S_Q([a, b']) \). By applying a similar argument, one can see that there exists such a surjective homomorphism for the second case.

Now, we assume the third case; that is, \( a < a' \leq b < b' \). Since \(([a, b], [b' + 1, b'])\) is the pair contained in the lower ray of \([a, b]\), we have an injective \( R \)-homomorphism
\[ S_Q([a, b']) \hookrightarrow S_Q([a, b]) \circ S_Q([b + 1, b']). \]

By taking \( S_Q([a', b]) \circ -\) to (4.7), we have
\[ S_Q([a', b]) \circ S_Q([a, b']) \rightarrow S_Q([a', b]) \circ S_Q([a, b]) \circ S_Q([b + 1, b'])\]
\[ \simeq S_Q([a, b]) \circ S_Q([a', b']) \circ S_Q([b + 1, b']) \rightarrow S_Q([a, b]) \circ S_Q([a', b']), \]
by Proposition 4.13 In a similar way, one can do this for the fourth case, also. \( \Box \)

Remark 4.13. When \( Q \) is equal to \( \overline{Q} \), we always have the following inequality in (4.4):
\[ a' < a \leq b < b'. \]
Thus the above proposition can be understood as generalizations of the following statements [27] Proposition 4.2.3:

(a) For \( a' \leq a \leq b < b' \), \( L(a, b) \circ L(a', b') \simeq L(a', b') \circ L(a, b) \) is simple.
(b) For \( a' < a \leq b < b' \), there exists an exact sequence
\[
0 \to L(a, b) \circ L(a', b') \to L(a, b') \circ L(a', b)
\]
\[
\xrightarrow{\mathbf{r}} L(a', b) \circ L(a, b') \to L(a, b) \circ L(a', b') \to 0,
\]
where the image of \( \mathbf{r} \) coincides with the head of \( L(a, b') \circ L(a', b) \) and the socle of \( L(a', b) \circ L(a, b') \).

Theorem 1.11 tells us that all positive roots \( \beta_k \) in a maximal \( S \)-sectional (resp. \( N \)-sectional) path \( \rho \) share the second (resp. first) component and have the same value
\[
\phi_1^{-1}(\beta_k) - \phi_2^{-1}(\beta_k) = \phi_1^{-1}(\beta_{k'}) - \phi_2^{-1}(\beta_{k'})
\]
(resp. \( \phi_1^{-1}(\beta_k) + \phi_2^{-1}(\beta_k) = \phi_1^{-1}(\beta_{k'}) + \phi_2^{-1}(\beta_{k'}) \)).

For each maximal \( S \)-sectional (resp. \( N \)-sectional) path \( \rho \), we assign a value
\[
\chi_S(\rho) = \phi_1^{-1}(\beta) - \phi_2^{-1}(\beta) \quad \text{(resp.} \quad \chi_N(\rho) = \phi_1^{-1}(\beta) + \phi_2^{-1}(\beta)\text{)}
\]
where \( \beta \) is any positive root in \( \rho \).

Note that \( \chi_S(\rho) \neq \chi_S(\rho') \) (resp. \( \chi_N(\rho) \neq \chi_N(\rho') \)) for any distinct pair of maximal \( S \)-sectional (resp. \( N \)-sectional) paths \( (\rho, \rho') \).

We set \( \rho_1 \) as the maximal \( S \)-sectional path such that \( \chi_S(\rho_1) \) is maximum and \( i_1 \) is the second component of \( \rho_1 \). In this way, we define \( \rho_k \) as the maximal \( S \)-sectional path such that \( \chi_S(\rho_k) \) is \( k \)th maximum and \( i_k \) is the second component of \( \rho_k \) for \( 1 \leq k \leq n \). Similarly, we set \( \rho'_k \) as the maximal \( N \)-sectional path such that \( \chi_N(\rho'_k) \) is \( k \)th maximum and \( j_k \) is the first component of \( \rho'_k \) for \( 1 \leq k \leq n \).

In Example 1.5,
\[
(i_1, i_2, i_3, i_4, i_5) = (1, 3, 4, 5, 2) \quad \text{and} \quad (j_1, j_2, j_3, j_4, j_5) = (3, 1, 2, 4, 5).
\]

**Theorem 4.14.** Let \( Q \) be any Dynkin quiver of finite type \( A_n \). Assume that we have a simple \( R_{A_n} \)-module \( M \).

(a) There exists a sequence
\[
(a_{1; i_1}, a_{2; i_1}, \ldots, a_{k; i_k})
\]
for \( 1 \leq s \leq i_k, 1 \leq k \leq n \) and \( a_{s; i_k} \in \mathbb{Z}_{\geq 0} \) such that
\[
\bigoplus_{s=1}^n S_Q([s, i_k])^{\otimes a_{s; i_k}} \to M,
\]
where \( \bigoplus_{s=1}^n S_Q([s, i_k])^{\otimes a_{s; i_k}} \) is a simple module for every \( 1 \leq k \leq n \).

(b) There exists a sequence
\[
(a_{j_k; j_k}, a_{j_k; j_k+1}, \ldots, a_{j_k; n-1}, a_{j_k; n})
\]
for \( j_k \leq s \leq n, 1 \leq k \leq n \) and \( a_{s; j_k} \in \mathbb{Z}_{\geq 0} \) such that
\[
\bigoplus_{s=1}^n S_Q([j_k, s])^{\otimes a_{j_k; s}} \to M,
\]
where \( \bigoplus_{s=1}^n S_Q([j_k, s])^{\otimes a_{j_k; s}} \) is a simple module for every \( 1 \leq k \leq n \).

**Proof.** With Theorem 1.11 and Theorem 2.9 our assertions come from the convex total orders \( <_Q^U, <_Q^L \) and Proposition 4.3. \( \square \)
Remark 4.15. Fix the Dynkin quiver as $Q$. Then we have

$$(j_1, j_2, \ldots, j_{n-1}, j_n) = (n, n-1, \ldots, 2, 1).$$

(a) For a Kashiwara-Nakashima tableau $T$ of shape $\lambda$ of finite type $A_n$ ([36]), we can associate an $(\ell(\lambda))$-tuple of partitions $(\mu(1), \ldots, \mu(\ell(\lambda)-1), \mu(\ell(\lambda)))$ as in [32, Section 1]. Here $\ell(\eta)$ denotes the length of a partition $\eta$. Set $L(a, b) = k$ for $a > b$. Then [32, Theorem 4.8] tells us that the simple head $Δ(T) := \bigoplus_{k=1}^{\ell(\lambda)} S_{Q}^{\ell((\lambda) + 1 - k)}(\ell(\lambda) + 1 - k, \ell(\lambda) + 1 - k + \mu(\ell(\lambda)+1-k)-1)\big)$ is indeed a simple module over $R_{A_n}^\lambda$, the cyclotomic quotient of $R_{A_n}$ with the dominant integral weight $\lambda$. Moreover, the crystal structure on $\{Δ(T)\}$ ([32]) is isomorphic to the crystal structure of $B(\lambda)$ of $V(\lambda)$ over $U_q(A_n)$ ([33],[34]).

(b) If we give a total order on $I_0$ as $j_k > j_{k+1}$ for $1 \leq k \leq n - 1$, we have

$$1 < 2 < \cdots < n - 1 < n.$$  

Then Lemma [4.1] tells us that $S_{\lambda}(\beta)$ coincides with the cuspidal representation $L_{\beta}$ in [23],[40] with respect to the total order (4.8) on $I_0$.

5. THE CATEGORIES $\mathcal{C}_{Q}^{(2)}$ OVER $U_q'(A_{2n-1}^{(2)})$ AND $U_q'(A_{2n}^{(2)})$

In this section, we study the category $\mathcal{C}_{Q}^{(2)}$ over $U_q'(A_{2n-1}^{(2)})$ and $U_q'(A_{2n}^{(2)})$ by using the combinatorial descriptions in Section 1 and the Dorey type morphisms for $U_q'(A_{2n-1}^{(2)})$ and $U_q'(A_{2n}^{(2)})$ studied in [44].

Now, we fix $m \in \mathbb{Z}_{\geq 2}$ as $2n$ or $2n - 1$, and a Dynkin quiver $Q$ of finite type $A_m$. For $1 \leq i \leq m$ and $(-q)^p \in k^\times$, we define

$$\{1, \ldots, n\} \ni i^* := \begin{cases} i & \text{if } 1 \leq i \leq n, \\ m+1-i & \text{if } n < i \leq m. \end{cases}$$

Using (5.1), we can obtain an injective map $\ast$, from the set $\{V(\varpi_i)(-q)^p \mid 1 \leq i \leq m, \ p \in \mathbb{Z}\}$ consisting of fundamental $U_q'(A_{m}^{(1)})$-modules to the set $\{V(\varpi_i)(-q)^p \mid 1 \leq i \leq n, \ p \in \mathbb{Z}\}$ consisting of fundamental $U_q'(A_{m}^{(2)})$-modules, which is given by (see [21])

$$(V(\varpi_i)(-q)^p)^* := V(\varpi_{i^*})(-q)^p.$$ 

For each $(i, p) \in \Gamma_Q$ with $\phi(i, p) = (\beta, 0) \in \Phi_{+} \times \{0\}$, we set

$$V_Q(\beta) := (V_Q(\beta))^* = \begin{cases} V(\varpi_i)(-q)^p & \text{if } 1 \leq i \leq n, \\ V(\varpi_{m+1-i})(-1)^m(-q)^p & \text{if } n < i \leq m, \end{cases}$$

which is a fundamental module over $U_q'(A_{m}^{(2)}).$

By mimicking Definition 2.2 and using (5.2), we can define the category $\mathcal{C}_{Q}^{(2)}$ as follows (see also [21]).
Definition 5.1. Let \( Q \) be a Dynkin quiver of finite type \( A_m \) and \( U'_q(g) \) be the quantum affine algebra of type \( A^{(2)}_m \) for \( m = 2n \) or \( 2n - 1 \in \mathbb{Z}_{\geq 2} \). We define the smallest abelian full subcategory \( \mathcal{C}^{(2)}_Q \) consisting of finite dimensional integrable \( U'_q(A^{(2)}_m) \)-modules such that

(a) it is stable by taking subquotient, tensor product and extension,

(b) it contains \( V_Q(\beta) \) for all \( \beta \in \Phi^+_m \).

Here, \( \Phi^+_m \) is the set of positive roots of \( A_m \).

The goal of this section is to show the following theorem:

**Theorem 5.2.** For every positive root \( \gamma \in \Phi^+_m \) with \( |\gamma| \geq 2 \), there exists a minimal pair \( (\alpha, \beta) \) such that there exists a surjective \( U'_q(A^{(2)}_m) \)-module homomorphism

\[
V_Q(\beta) \otimes V_Q(\alpha) \twoheadrightarrow V_Q(\gamma).
\]

It is known that ([1, (1.7)]), for all \( p \in \mathbb{Z} \),

\[
(5.3) \quad V(\varpi_n)^p \simeq V(\varpi_n)^{-p}
\]

where \( V(\varpi_n) \) is a \( U'_q(A^{(2)}_{2n-1}) \)-module.

In this section, we prove Theorem 5.2 only for \( m = 2n - 1 \). In the way of proving Theorem 5.2 for \( m = 2n - 1 \), we use (5.3) several times. To prove the case when \( m = 2n \), one has to use the same arguments, the same pairs of positive roots in this section, and use the morphism (2.5) instead of (5.3).

Note that the \( p' \) in (2.2) is \((-1)^m(-q)^{m+1}\) for \( g = A^{(2)}_m \) ([14 Appendix A]).

**Lemma 5.3.** For \( \alpha_k + \alpha_{k+1} \) with \( 1 \leq k < 2n - 1 \), we have a surjective \( U'_q(A^{(2)}_{2n-1}) \)-module homomorphism

\[
V_Q(\alpha_k) \otimes V_Q(\alpha_{k+1}) \twoheadrightarrow V_Q(\alpha_k + \alpha_{k+1}) \text{ or } V_Q(\alpha_{k+1}) \otimes V_Q(\alpha_k) \twoheadrightarrow V_Q(\alpha_k + \alpha_{k+1}).
\]

**Proof.**

*Case 1: \( \bullet \overbrace{\bullet \cdots \bullet}^k \overbrace{\bullet \cdots \bullet}^k+1. \) In this case, Lemma 1.14(a) tells us that

\[
\phi^{-1}(\alpha_k) = (k, \xi_k) \quad \text{and} \quad \phi^{-1}(\alpha_{k+1}) = (2n - 1 - k, \xi_k - (2n - 1)).
\]

Thus, by Remark 1.14 we have \( \alpha_k \prec_Q \alpha_{k+1} \),

\[
\phi^{-1}(\alpha_k + \alpha_{k+1}) = (2n - 1, \xi_k - (2n - 1 - k))
\]

and hence \( V_Q(\alpha_k + \alpha_{k+1}) = V(\varpi_1)^{-(-q)\xi_k-(2n-1-k)} \). For each \( k \), \( V_Q(\alpha_k) \) and \( V_Q(\alpha_{k+1}) \) are summarized by the following table:

<table>
<thead>
<tr>
<th>( k )</th>
<th>( V_Q(\alpha_{k+1}) )</th>
<th>( V_Q(\alpha_k) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k \leq n - 1 )</td>
<td>( V(\varpi_{k+1})^{-(-q)\xi_k-(2n-1)} )</td>
<td>( V(\varpi_k)^{(-q)\xi_k} )</td>
</tr>
<tr>
<td>( k \geq n )</td>
<td>( V(\varpi_{2n-1-k})^{-(-q)\xi_k-(2n-1)} )</td>
<td>( V(\varpi_{2n-k})^{-(-q)\xi_k} )</td>
</tr>
</tbody>
</table>

For \( k \leq n - 1 \), we have an injection \( V(\varpi_{k+1}) \hookrightarrow V(\varpi_1)^{-k} \otimes V(\varpi_k)^{-(-q)-1} \) by taking the dual to (2.4). Tensoring the right dual of \( V(\varpi_k)^{-(-q)-1} \), we have

\[
V(\varpi_{k+1}) \otimes V(\varpi_k)^{-(-q)^{2n-1}} \twoheadrightarrow V(\varpi_1)^{-k},
\]

which yields our assertion. Similarly, we can prove for \( k \geq n \).
Case 2: \(\bullet \bullet \bullet \quad k \quad k+1 \quad \bullet \bullet \bullet\). In this case, we have

\[
\phi^{-1}(\alpha_k) = (k, \xi_k), \quad \phi^{-1}(\alpha_{k+1}) = (1, \xi_k - 1 - k), \quad \phi^{-1}(\alpha_k + \alpha_{k+1}) = (k + 1, \xi_k - 1)
\]

and hence \(V_Q(\alpha_{k+1}) = V(\omega_k)_{1-q}\xi_k - 1 - k}\). For each \(k\), \(V_Q(\alpha_k)\) and \(V_Q(\alpha_{k+1})\) are summarized by the following table:

<table>
<thead>
<tr>
<th>(k)</th>
<th>(V_Q(\alpha_k))</th>
<th>(V_Q(\alpha_k + \alpha_{k+1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k \leq n - 1)</td>
<td>(V(\omega_k)_{1-q}\xi_k)</td>
<td>(V(\omega_{k+1})<em>{1-q}\xi</em>{k+1} - 1)</td>
</tr>
<tr>
<td>(k \geq n)</td>
<td>(V(\omega_{2n-k})<em>{1-q}\xi</em>{k-2n} + 2)</td>
<td>(V(\omega_{2n-k})<em>{1-q}\xi</em>{k-2n} + 2)</td>
</tr>
</tbody>
</table>

Using the same technique as in (Case 1), we have

\[
V_Q(\alpha_{k+1}) \otimes V_Q(\alpha_k) \implies V_Q(\alpha_k + \alpha_{k+1}).
\]

Case 3: \(\bullet \bullet \bullet \quad k \quad k+1 \quad \bullet \bullet \bullet\). We have

\[
\phi^{-1}(\alpha_k) = (2n - k, \xi_k - 2n + 2), \quad \phi^{-1}(\alpha_{k+1}) = (k + 1, \xi_k + 1),
\]

\[
\phi^{-1}(\alpha_k + \alpha_{k+1}) = (1, \xi_k + 1 - k)
\]

and hence \(V_Q(\alpha_k + \alpha_{k+1}) = V(\omega_k)_{1-q}\xi_{k+1} - k}\). For each \(k\), \(V_Q(\alpha_k)\) and \(V_Q(\alpha_{k+1})\) are summarized by the following table:

<table>
<thead>
<tr>
<th>(k)</th>
<th>(V_Q(\alpha_k))</th>
<th>(V_Q(\alpha_{k+1}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(k \leq n - 1)</td>
<td>(V(\omega_{2n-k})<em>{1-q}\xi</em>{k-2n} + 2)</td>
<td>(V(\omega_{2n-k})<em>{1-q}\xi</em>{k-2n} + 2)</td>
</tr>
<tr>
<td>(k \geq n)</td>
<td>(V(\omega_{2n-k})<em>{1-q}\xi</em>{k-2n} + 2)</td>
<td>(V(\omega_{2n-k})<em>{1-q}\xi</em>{k-2n} + 2)</td>
</tr>
</tbody>
</table>

In the same way as (Cases 1, 2), one can prove that \(V_Q(\alpha_k) \otimes V_Q(\alpha_{k+1}) \implies V_Q(\alpha_k + \alpha_{k+1})\). For the remaining cases, we have

\[
\begin{array}{c|c|c|c}
Q & \phi^{-1}(\alpha_k, 0) & \phi^{-1}(\alpha_{k+1}, 0) & \phi^{-1}(\alpha_k + \alpha_{k+1}, 0) \\
\hline
\bullet \bullet \bullet \quad k \quad k+1 \quad \bullet \bullet \bullet\ & (2n - k, \xi_k - 2n + 2) & (2n - k - 1, \xi_k - 2n + 3) & (2n - k + 1, \xi_k + 2n + 3) \\
\bullet \bullet \bullet \quad k \quad k+1 \quad \bullet \bullet \bullet\ & (1, \xi_k - k + 1) & (2n - k + 1, \xi_k - 2n + 3) & (2n - k - 1, \xi_k + 2n + 3) \\
\bullet \bullet \bullet \quad k \quad k+1 \quad \bullet \bullet \bullet\ & (2n - k - 1, \xi_k - 2n + 1 + k) & (2n - k, \xi_k + 1) & (2n - k - 2, \xi_k - 2n + 2 + k) \\
\bullet \bullet \bullet \quad k \quad k+1 \quad \bullet \bullet \bullet\ & (2n - k, \xi_k - 2n + 1 + k) & (2n - k - 1, \xi_k + 1) & (2n - k - 2, \xi_k - 2n + 2 + k)
\end{array}
\]

By using the same technique, one can prove our assertion for the remaining cases.

\(\square\)

**Lemma 5.4.** Let \(\gamma = [k, \ell] \in \Phi^+_n\) with \(|\gamma| \geq 3\). We write the upper ray and the lower ray of \([k, \ell]\) as follows:

Upper ray: \(\overbrace{S^u_{1} \to \cdots S^u_{n-1} \to S^u_{n}}^{N_{b_1}^u} = [k, \ell] \to N_{b_1}^u \to N_{b_1-1}^u \to N_1^u\)

Lower ray: \(\overbrace{N_1^l \to \cdots N_{b_2-1}^l \to N_{b_2}^l}^{N_{a_2}^l} = [k, \ell] \to S_{a_2}^l \to S_{a_2-1}^l \to S_1^l\).
Then the following hold:

(a) If $a_1 = 1$ and $b_1 > 1$, then all $N^k_j$ (1 ≤ $j < b_2$) are of the form $[k, t]$ with $t < \ell$ and all $N^u_s$ (1 ≤ $s < b_1$) are of the form $[k, v]$ with $v > l$. Moreover, $N^l_{b_2-1} = [k, \ell - 1]$ and the pair $([\ell], [k, \ell - 1])$ is in the lower ray of $\gamma$.

(b) If $b_1 = 1$ and $a_1 > 1$, then all $S^l_j$ (1 ≤ $j < a_2$) are of the form $[b, \ell]$ with $k < b$ and all $S^u_s$ (1 ≤ $s < a_1$) are of the form $[a, \ell]$ with $a < k$. Moreover, $S^l_{a_2-1} = [k + 1, \ell]$ and the pair $([k + 1, \ell], [k])$ is in the lower ray of $\gamma$.

(c) If $a_2 = 1$ and $b_2 > 1$, then all $N^u_j$ (1 ≤ $j < b_1$) are of the form $[k, t]$ with $t < \ell$ and all $N^l_s$ (1 ≤ $s < b_2$) are of the form $[k, v]$ with $v > l$. Moreover, $N^u_{b_1-1} = [k, \ell - 1]$ and $([\ell], [k, \ell - 1])$ is in the upper ray of $\gamma$.

(d) If $b_2 = 1$ and $a_2 > 1$, then all $S^u_j$ (1 ≤ $j < a_1$) are of the form $[b, \ell]$ with $k < b$ and all $S^l_s$ (1 ≤ $s < a_2$) are of the form $[a, \ell]$ with $a < k$. Moreover, $S^u_{a_1-1} = [k + 1, \ell]$ and the pair $([k + 1, \ell], [k])$ is in the upper ray of $\gamma$.

Proof. We only give a proof of (a). The remaining cases can be proved in a similar way. The second assertion of (a) follows from Theorem 1.11 and 1.13. In particular, $N^i \neq [k]$. Note that

\[ N^l_1 \to \cdots \to N^l_{b_2-1} \to N^l_{b_2} = N^u_{b_1} \to N^u_{b_1-1} \to N^u_1 \] is a maximal $N$-sectional path.

Thus by Proposition 1.10 $N^l_1 = [k]$ and hence $k$ is a sink or a right intermediate. On the other hand, $b_1 > 1$ implies that $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \bullet$ in $Q$. Hence $S^l_1 = [\ell]$.

Thus, by Remark 1.14 the situation in $\Gamma_Q$ can be drawn as follows:

\[ \text{Corollary 1.12 and Corollary 1.13 tell us that for any } u < v < w, \]
\[ \sigma_u = [k, c], \quad \sigma_v = [e, f] \quad \text{and} \quad k \leq c < e \leq f < \ell. \]

By applying Corollary 1.12 once again, we can conclude that $N^l_j = [k, f]$ for $f < \ell$. Furthermore, we can observe that

\[ \sigma_{w-1} = [g, \ell - 1] \quad \text{for some} \quad g \leq \ell - 1 \quad \text{and hence} \quad N^l_{b_2-1} = [k, \ell - 1]. \]

□

Lemma 5.5. Let $\gamma = [k, \ell] \in \Phi^+_n$ with $|\gamma| \geq 3$.

(a) If there is no pair of $\gamma$ in the upper ray, there exists a pair $(\alpha, \beta)$ in the lower ray such that

\[ |\phi^{-1}_1(\alpha) - \phi^{-1}_1(\gamma)| = 1 \quad \text{or} \quad |\phi^{-1}_1(\beta) - \phi^{-1}_1(\gamma)| = 1. \]

(b) If there is no pair of $\gamma$ in the lower ray, there exists a pair $(\alpha, \beta)$ in the upper ray such that

\[ |\phi^{-1}_1(\alpha) - \phi^{-1}_1(\gamma)| = 1 \quad \text{or} \quad |\phi^{-1}_1(\beta) - \phi^{-1}_1(\gamma)| = 1. \]
Proof. (a) Using the notation in Lemma 5.4, Corollary 1.12 implies that
\[ N^i_1 = [k] \quad \text{and} \quad S^i_1 = [\ell] \]
under the assumption of (a).
Thus \( k \) cannot be a source and a right intermediate in \( Q \), and \( \ell \) cannot be a sink and a right intermediate in \( Q \).

*Case 1: \( k \) is a sink and \( \ell \) is a source.* If \( k = 1 \) or \( \ell = n \), then the situation in \( \Gamma_Q \) can be drawn as follows:

For each case, \( \sigma_2 = [2, a] \) and \( \sigma_{a-1} = [b, n-1] \), respectively. Thus \( (S^i_{b-1} = [2, \ell], [1]) \) and \( ([n], N^i_{b-1} = [k, n-1]) \) are desired ones, respectively.

If \( k \neq 1 \) and \( \ell \neq n \), then Remark 1.14 and Corollary 1.15 imply that the situation in \( \Gamma_Q \) can be depicted as follows:

for some \( k < b < c < a < \ell \). Thus \( ([k, b], [b + 1, \ell]) \) is a pair in the upper ray of \( \gamma \) which yields a contradiction to the assumption.

*Case 2: \( k \) is a not sink or \( \ell \) is not a source.* Equivalently,

Thus, using the notation in Lemma 5.4, the situation in \( \Gamma_Q \) can be drawn as follows:

Then \( (S^i_{a-1}, \sigma_{a+1}) = ([k + 1, \ell], [k + 1, c]) \) and \( (N^i_{b-1}, \sigma_{b-1}) = ([k, \ell - 1], [d, \ell - 1]) \), respectively. Hence \( ([k + 1, \ell], [k]) \) and \( ([\ell], [k, \ell - 1]) \) are desired, respectively. The proof for (b) can be obtained by applying a similar argument.

**Proof of Theorem 5.2** For \( \gamma \in \Phi^+_2n-1 \) with \( |\gamma| = 2 \), it is already proved in Lemma 5.3. Thus it suffices to consider when \( |\gamma| = ||[k, \ell|| \geq 3 \). To prove this, we need to observe the situation in \( \Gamma_Q \) for each case.
Case 1: $1 < \phi_1^{-1}(\gamma) = i_\gamma \leq n$. (a) If there is a pair $(\alpha, \beta)$ in the upper ray, then Theorem 5.3 tells us that
\[
(\phi_1^{-1}(\alpha))^* = \phi_1^{-1}(\alpha), \quad (\phi_1^{-1}(\beta))^* = \phi_1^{-1}(\beta) \quad \text{and} \quad \phi_1^{-1}(\alpha) + \phi_1^{-1}(\beta) = \phi_1^{-1}(\gamma).
\]
Then we have a surjective homomorphism
\[
V_Q(\beta) \otimes V_Q(\alpha) \rightarrow V_Q(\gamma)
\]
by (2.4).

(b) Now we deal with the case when there is no pair in the upper ray. By Lemma 5.4 and Lemma 5.5, there exists a minimal pair $(\alpha, \beta)$ such that
\[
(\phi_1^{-1}(\alpha), \phi_1^{-1}(\beta)) = (2n - 1, i_\gamma + 1) \text{ or } (i_\gamma + 1, 2n - 1).
\]
For the case when $(\phi_1^{-1}(\alpha), \phi_1^{-1}(\beta)) = (2n - 1, i_\gamma + 1)$, it suffices to show that there exists a surjective homomorphism
\[
V(\varpi_{i_\gamma + 1}) \otimes V(\varpi_1)(-q)^{p_\gamma - 1} \rightarrow V(\varpi_{i_\gamma})(-q)^{p_\gamma},
\]
where $p_\gamma = \phi_2^{-1}(\gamma)$.

(b-i) If $i_\gamma < n$, then $(i_\gamma + 1)^* = i_\gamma + 1$ and $((-q)^{p_\gamma - 1})^* = (-q)^{p_\gamma - 1}$. By taking the dual to (2.4), we have
\[
V(\varpi_{i_\gamma + 1}) \rightarrow V(\varpi_{i_\gamma})(-q)^{i_\gamma} \otimes V(\varpi_1)(-q)^{-i_\gamma}.
\]
By taking the left dual of $V(\varpi_1)(-q)^{-i_\gamma}$ to (5.5), we have
\[
V(\varpi_{i_\gamma + 1}) \otimes V(\varpi_1)(-q)^{2n-i_\gamma} \rightarrow V(\varpi_{i_\gamma})(-q)^{i_\gamma}.
\]
Thus we have 5.4 for $i_\gamma < n$.

(b-ii) If $i_\gamma = n$, then $(i_\gamma + 1)^* = n - 1$ and $((-q)^{p_\gamma - 1})^* = (-q)^{p_\gamma - 1}$. Then 5.4 becomes
\[
V(\varpi_{n-1})(-q)^{p_\gamma - 1} \otimes V(\varpi_1)(-q)^{p_\gamma + n - 1} \rightarrow V(\varpi_n)(-q)^{p_\gamma} \simeq V(\varpi_n)(-q)^{p_\gamma},
\]
by (5.3). The above equation follows from (2.4) directly.

The case when $(\phi_1^{-1}(\alpha), \phi_1^{-1}(\beta)) = (i_\gamma + 1, 2n - 1)$ can be proved in the same way.

Case 2: $\phi_1^{-1}(\beta_\gamma) = 1$. Note that there is no pair in the upper ray. Thus using the notation in Lemma 5.4 we have

- $N_i^l = [k]$ and $S_i^l = [\ell],$
- $[k] = N_1^l \rightarrow \cdots \rightarrow N_{b_2-1}^l \rightarrow N_{b_2}^l$ is a maximal $N$-sectional path,
- $S_{b_2}^l \rightarrow S_{b_2-1}^l \rightarrow \cdots \rightarrow S_1^l = [\ell]$ is a maximal $S$-sectional path.

Thus the situation in $\Gamma_Q$ can be drawn as follows:
for some $a < k$ and $\ell < b$. By Corollary 1.15, $\sigma_t = [c, \ell - 1]$ ($c \leq \ell - 1$) and hence $N_t = [k, \ell - 1]$. We fix a pair $(\alpha, \beta)$ as $([\ell], [k, \ell - 1])$. Then we can see that
\[ \phi_1^{-1}(\beta) = 2n + 1 - i_\alpha \text{ and } (2n - \phi_1^{-1}(\beta)) + (2n - \phi_1^{-1}(\alpha)) = (2n - \phi_1^{-1}(\gamma)) = 2n - 1. \]

(a) Assume that $i_\alpha = \phi_1^{-1}(\alpha) \geq n + 1$; then $\phi_1^{-1}(\beta) \leq n$. Thus it suffices to show that there exists a surjective homomorphism
\[ (5.6) \quad V(\omega_{2n-i_\alpha + 1})_{(-q)p_\gamma - 2n+i_\alpha} \otimes V(\omega_{2n-i_\alpha})_{(-q)p_\gamma + i_\alpha - 1} \twoheadrightarrow V(\omega_1)_{(-q)p_\gamma}. \]
By taking the dual to (2.4), we have
\[ (5.7) \quad V(\omega_{2n-i_\alpha + 1}) \hookrightarrow V(\omega_1)_{(-q)2n-i_\alpha} \otimes V(\omega_{2n-i_\alpha})_{(-q)^{-1}}. \]
By taking the left dual of $V(\omega_{2n-i_\alpha})_{(-q)^{-1}}$ to (5.7), we have
\[ V(\omega_{2n-i_\alpha + 1}) \otimes V(\omega_{2n-i_\alpha})_{(-q)^{2n-1}} \twoheadrightarrow V(\omega_1)_{(-q)2n-i_\alpha}. \]
Thus we have (5.6) for $i_\alpha \geq n + 1$.

(b) Assume that $i_\alpha \leq n$; then $\phi_1^{-1}(\beta) \geq n + 1$. Thus it suffices to show that there exists a surjective homomorphism
\[ (5.8) \quad V(\omega_{i_\alpha - 1})_{(-q)p_\gamma - 2n+i_\alpha} \otimes V(\omega_{i_\alpha})_{(-q)p_\gamma + i_\alpha - 1} \twoheadrightarrow V(\omega_1)_{(-q)p_\gamma}. \]
By taking the dual to (2.4), we have
\[ V(\omega_{i_\alpha}) \hookrightarrow V(\omega_{i_\alpha - 1})_{(-q)^{i_\alpha}} \otimes V(\omega_1)_{(-q)^{-i_\alpha + 1}}. \]
By taking the right dual of $V(\omega_{i_\alpha - 1})_{(-q)^{i_\alpha}}$, we have
\[ V(\omega_{i_\alpha - 1})_{(-q)^{-2n+1}} \otimes V(\omega_{i_\alpha}) \twoheadrightarrow V(\omega_1)_{(-q)^{-i_\alpha + 1}}. \]
Thus we have (5.8) for $i_\alpha \leq n$.

The remaining case (Case 3: $n < \phi_1^{-1}(\beta) = i_\gamma \leq 2n - 1$) can be proved by applying similar arguments as in (Case 1) and (Case 2).

**Corollary 5.6.** For every positive root $\beta \in \Phi_m^+$, there exists a sequence $(i_1, i_2, \cdots, i_{|\beta|}) \in I_{0[\beta]}$ such that
\[ \bigotimes_{t=1}^{\beta} V_Q(\alpha_{i_t}) \twoheadrightarrow V_Q(\beta). \]

From Corollary 5.6, the condition (b) in Definition 5.1 can also be restated as follows:

(b′) It contains $V_Q(\alpha_k)$ for all $\alpha_k \in \Pi_m$.

**Acknowledgements**

The author would like to express his gratitude to Professor Seok-Jin Kang, Professor Masaki Kashiwara, Professor Kyungyong Lee and Myungho Kim for many fruitful discussions. The author would like to thank the anonymous reviewers for their valuable comments and suggestions, also. The author gratefully acknowledges the hospitality of RIMS (Kyoto) during his visits in 2013 and 2014.
References


School of Mathematics, Korea Institute for Advanced Study Seoul 130-722, Korea

Current address: Department of Mathematics, Ewha Womans University, Seoul 120-750, Korea

E-mail address: sejin092@gmail.com