TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 4, April 2017, Pages 2651–2673
http://dx.doi.org/10.1090/tran/6728
Article electronically published on April 15, 2016

RATIONALITY OF HOMOGENEOUS VARIETIES

CHEEWHYE CHIN AND DE-QI ZHANG

Abstract. Let G be a connected linear algebraic group over an algebraically closed field k, and let H be a connected closed subgroup of G. We prove that the homogeneous variety G/H is a rational variety over k whenever H is solvable or when $\dim(G/H) \leqslant 10$ and $\mathrm{char}(k) = 0$. When H is of maximal rank in G, we also prove that G/H is rational if the maximal semisimple quotient of G is isogenous to a product of almost-simple groups of type A, type C (when $\mathrm{char}(k) \neq 2$), or type B_3 or G_2 (when $\mathrm{char}(k) = 0$).

Contents

1. Introduction

Let k be an algebraically closed field of arbitrary characteristic $p \geq 0$. Throughout this paper, we work only with the Zariski topology on varieties or algebraic groups over k. By a result that goes back to Chevalley when $\mathrm{char}(k) = 0$ (cf. [Ch54, §2, Cor. 2 to Th. 1]) and to Rosenlicht for arbitrary $\mathrm{char}(k)$ (cf. [Ro57, end of §3]), a connected linear algebraic group G over k is a rational variety: the field $k(G)$ of rational functions on G is a purely transcendental extension of k. If $H \subseteq G$ is any closed subgroup, the homogeneous variety G/H is thus unirational: its field $k(G/H)$ of rational functions is contained in a purely transcendental extension of k (namely, $k(G)$). It is thus natural to consider the following:

Problem 1.1. Let G be a connected linear algebraic group over the algebraically closed field k, and let $H \subseteq G$ be a closed subgroup. Is the homogeneous variety G/H rational? Equivalently, is the field $k(G/H)$ of rational functions on G/H a purely transcendental extension of k?

Received by the editors March 9, 2015 and, in revised form, April 15, 2015.
2010 *Mathematics Subject Classification.* Primary 14E08 14M17, 14M20.
Key words and phrases. Homogeneous variety, rationality.

This is a long-standing question in algebraic geometry, known as the *rationality problem*. When H is not assumed to be connected, the answer is negative in general; see Remark 1.3 below. When H is connected, the rationality problem in the generality of Problem 1.1 is open even when $\mathrm{char}(k) = 0$. The problem was mentioned (possibly for the first time) in [Ha84], in which it was suggested that the answer is negative in general even when H is connected, although no counterexample is known to date. However, affirmative answers have been established in several cases; for instance, the rationality of G/H when $\dim(G/H) \leqslant 4$ and $\mathrm{char}(k) = 0$ was proved in [MU83, Lemma 1.15].

A more general form of the rationality problem is concerned with the rationality of the field $k(X)^H$ when X itself is a rational variety on which a linear group H acts. Two typical versions of this problem have been considered in the literature: namely, when X is the underlying vector space of a finite dimensional representation of H, and when X is the underlying variety of a connected group H acting on itself by conjugation; see [CTS07] for a survey of the former and [CTKPR11] for works on the latter. The variant 1.1 of the rationality problem we consider in this paper amounts to the case when X is the underlying variety of a connected group G on which a subgroup H acts by right multiplication. Our goal is to give several more criteria under which one can establish an affirmative answer to this variant of the rationality problem and, to the extent possible, in a characteristic-free manner.

1.2. Here, we give an overview of the main results of this paper. We refer the reader to Theorems 2.9, 3.1, 4.4 and 5.9 for the precise statements.

In Theorem 2.9, we show that G/H is rational for any closed subgroup H contained in a Borel subgroup B of G; in particular, this is so whenever G or H is connected and solvable (for then H is necessarily contained in some Borel subgroup of G). This result holds for any characteristic of k and is probably folklore to the experts, although it seems not to have appeared in the literature. It is analogous to a theorem of Miyata (cf. [Mi71]) asserting that the field $k(V)^H$ is a purely transcendental extension of k when H is a linear group and V is a finite dimensional representation of H which is triangularizable. We establish Theorem 2.9 via a certain splitting principle for the quotient map $G \dashrightarrow G/B$ (cf. Corollary 2.5) which was "morally speculated" by Prof. V. Popov in a conversation with the second author. The main tools needed are some classical results of Rosenlicht (cf. Lemmas 2.1 and 2.4).

In Theorem 3.1, we show that G/H is rational for any connected closed subgroup H of maximal rank in G if the maximal semisimple quotient of G is isogenous to a product of almost-simple groups of type A, type C (when $\mathrm{char}(k) \neq 2$), or type B_3 or G_2 (when $\mathrm{char}(k) = 0$). Here, the main ingredient is the Borel-de Siebenthal algorithm (cf. [BdS49, Th. 6] and the algebraic group version in [Le, Prop. 6.6]) classifying maximal connected reductive subgroups of maximal rank in a given connected semisimple group. We also employ general structural results of algebraic groups such as the Bruhat decomposition (cf. Lemma 3.3), the theorem of Borel-Tits (cf. Lemma 3.5), as well as properties of special groups (in the sense of Serre). Our Theorem 3.1 may be compared with [CTKPR11, Th. 0.2], where the authors prove, among other things, that for an almost-simple group G of type A or C, acting on itself by conjugation, the field extension $k(G)/k(G)^G$ is purely transcendental.

In sections 4 and 5, we assume $\mathrm{char}(k) = 0$ and use geometric arguments (cf. Lemma 4.1) to show the rationality of certain low dimensional homogeneous

varieties G/H, our starting point being the classical theorems of Lüroth and Castelnuovo (cf. Lemma 4.2) asserting the rationality of unirational varieties of dimension $\leqslant 2$ (over our algebraically closed field k of characteristic zero). We show in Theorem 4.4 that G/H is rational whenever $\dim(G/H) \leqslant 5$, and in Theorem 5.9 that if H is connected, G/H is rational whenever $\dim(G/H) \leqslant 10$. We follow the approach in [MU83, Lemma 1.15] of reducing to the case when G is semisimple, but we utilize new ingredients such as the geometry of the big cell and the structure of the centralizers of subtori in a connected reductive group (cf. [Hu75, §28.5, §22.4, §26.2]).

Remark 1.3. Our results provide affirmative answers to the rationality problem 1.1 in the respective situations, but except for low dimensional homogeneous varieties (i.e. when $\dim(G/H) \leqslant 5$ as in Theorem 4.4), we require that the closed subgroup $H \subseteq G$ be connected. This is not surprising, since there are examples of finite subgroups H in a product G of general linear groups for which G/H is not rational. Indeed, by the work of Saltman [Sa84, Th. 3.6] in the context of the Noether problem, one knows that for each prime p, there exist finite p-groups H with the following property: for any algebraically closed field k with $\mathrm{char}(k) \neq p$, if V denotes the vector space of the regular representation of H over k, and if $k(V)^H$ denotes the subfield of H-invariants in the field $k(V)$ of rational functions on V over k, then $k(V)^H$ is not a purely transcendental extension of k, so the variety V/H is not rational. If we decompose $V \cong \bigoplus_i V_i^{\oplus m_i}$ into irreducible representations V_i of H, then $m_i = \dim_k V_i$, and the linear representation of H on the isotypic component $V_i^{\oplus m_i}$ yields an action of H by right multiplication on $G_i := \mathrm{GL}_{m_i}(k)$. Thus V contains an H-stable open subvariety isomorphic to $G := \prod_i G_i$. Since V/H is not rational, G/H is not rational either.

1.4. As we are mainly interested in the birational properties of homogeneous varieties, we consider only rational actions and rational quotients (unless explicitly stated otherwise). A (right) *rational action* of a linear algebraic group H on an irreducible variety X (cf. [BSU13, §2.3]) is a group homomorphism $H \to \mathrm{Aut}(k(X))$ from H to the group $\mathrm{Aut}(k(X))$ of birational automorphisms of X, such that the resulting rational map $a : X \times H \dashrightarrow X$ is (defined and) regular on an open dense subset of $X \times H$. Given such a rational action, the *rational quotient of X by H* (in the sense of Rosenlicht; cf. [Ro56, Th. 2]) is any irreducible variety whose field of rational functions is identified with $k(X)^H$; i.e. it is a k-variety X/H characterized up to birational equivalence by the equality $k(X/H) = k(X)^H$ of rational function fields.* The k-inclusion $k(X/H) \subseteq k(X)$ of fields induces a dominant rational map $X \dashrightarrow X/H$, called the *rational quotient map*, which is H-equivariant with respect to the trivial action on X/H and has the universal property that any dominant rational map from X which is constant on general H-orbits in X factors through it. Thus, if G is a connected linear algebraic group and $H \subseteq G$ is a closed subgroup, the homogeneous space G/H of H-cosets in G is regarded as the rational quotient of the variety G by the right multiplication action of H. Similarly, if $K \subseteq G$ is another closed subgroup, the space of double cosets $K\backslash G/H$ is both the rational

*Given a rational action of H on X, we may replace X by another birational model and assume that H acts on X regularly (cf. [Ro56, Th. 1]), and it then follows (essentially by [Ro56, Th. 2]) that there exists a geometric quotient of (an open subset of) X by H (in the sense of GIT). The rational quotient X/H can thus be regarded as the birational equivalence class of the moduli space of "general H-orbits in X".

quotient of the variety $K\backslash G$ by the right multiplication action of H and the rational quotient of the variety G/H by the left multiplication action of K.

2. Quotients by subgroups contained in a Borel

In this section, we work over an algebraically closed base field k of arbitrary characteristic, and we consider the rationality problem 1.1 for G/H when H is contained in a Borel subgroup of G. Our goal is to establish Theorem 2.9. We first review some classical results and arguments.

Lemma 2.1 (Rosenlicht)**.** *Let H be a linear algebraic group acting rationally on a variety X, and let $K \lhd H$ be a closed normal subgroup. Then $H' := H/K$ acts rationally on the quotient variety $X' := X/K$, and X'/H' is (naturally) birational to X/H.*

See [Ro56, Th. 5]. In particular, if $K \lhd H$ lies in the kernel of the action of H on the field $k(X)$ (i.e. K acts trivially on an open dense subset of X), then $X' = X/K$ is (birational to) X itself, and hence X/H is birational to X/H'. In other words, in forming the rational quotient X/H, we may replace H by its image in the birational automorphism group $\mathrm{Aut}(k(X))$ of X. This is also clear from the characterization of the rational quotient by its rational function field.

Let H be a linear algebraic group acting rationally on a variety X. The action is called *generically free* iff there exists an open dense subset $U \subseteq X$ such that for every $x \in U$, the stabilizer subgroup ${}_xH := \{h \in H \,|\, x\cdot h = x\}$ is trivial. An easy example of a generically free action is the right multiplication action of H on a connected linear algebraic group G which contains H as a closed subgroup. A generically free action is necessarily *generically faithful*, i.e. the homomorphism $H \to \mathrm{Aut}(k(X))$ is injective, but the converse does not hold in general. However, one has the following:

Lemma 2.2. *Let H be a connected commutative group acting rationally on a variety X. If the action is generically faithful, then it is generically free. In particular, one has* $\dim X = \dim(X/H) + \dim H$.

Proof. We proceed as in the beginning of the proof of [Ro56, Th. 10]: replacing X by another birational model, we may assume that, over the function field $K := k(Y)$ of the rational quotient $Y := X/H$, the variety $X \otimes_k K$ is a homogeneous space for a regular action of $H \otimes_k K$. By [De70, §1, Prop. 7(b)], one knows that the intersection of the stabilizer subgroups over the geometric points of $X \otimes_k K$ is the kernel of the rational action of $H \otimes_k K$ on $X \otimes_k K$; this kernel is trivial by our assumption. But since $H \otimes_k K$ is commutative, these stabilizer subgroups are all equal, and hence they are all trivial. It follows that over the base field k, the action of H on X is generically free. □

Lemma 2.3. † *Let M be a linear algebraic group acting rationally on a variety X. Suppose that the action of M on X is generically free and that the rational quotient map $\pi : X \dashrightarrow X/M$ admits a rational section $s : X/M \dashrightarrow X$. Then there exists a birational map*

$$f \;:\; (X/M) \times M \dashrightarrow X \qquad \textit{which is } M\textit{-equivariant}$$

†After the second author gave a talk at Fudan University on a preliminary version of this paper, the PhD student JinSong Xu of the host Professor Meng Chen informed us of an independent proof of this result in the case when X is a homogeneous variety.

with respect to the natural right action of M on the domain $\mathrm{Dom}(f)$ *of f via multiplication on the second factor M. In particular, for any closed subgroup $H \subseteq M$, X/H is birational to $(X/M) \times (M/H)$.*

This is the birational analogue of the classical statement that a principal bundle with a section is trivial.

Proof. We assume that M acts on X from the right. Replacing X by another birational model, we may assume (cf. [Ro56, Th. 1]) that the action is biregular, so that one has a quotient morphism $\pi : X \to X/M$. Since the singular locus of X is stable under the action of M and the action is generically free, we may replace X by an open dense subset and assume that X is non-singular and that the action is in fact free; i.e. the stabilizer subgroup ${}_xM$ is trivial for every $x \in X$. Likewise, shrinking X/M (and X correspondingly) if necessary, we also assume that X/M is non-singular. Further replacing X/M by the domain $\mathrm{Dom}(s)$ of the rational section s and X by $\pi^{-1}(\mathrm{Dom}(s))$, we may assume that $s : X/M \to X$ is a regular section, i.e. a morphism such that $\pi \circ s = \mathrm{id}_{X/M}$. Thus we have a bijective morphism

$$f \ : \ (X/M) \times M \longrightarrow X, \qquad (y, m) \mapsto s(y) \cdot m$$

which is clearly M-equivariant: $f(y, m) \cdot m' = f(y, m \cdot m')$. Since X/M and X are normal varieties, we may apply [Bo91, Lemma 6.14(ii)] to infer that f is in fact an isomorphism. This shows the main assertion of the lemma. The final claim follows by passing to the quotient by H. □

Lemma 2.4 (Rosenlicht's generic section theorem)**.** *Let M be a connected solvable group acting rationally on a variety X. Then the rational quotient map $\pi : X \dashrightarrow X/M$ admits a rational section $s : X/M \dashrightarrow X$.*

See [Ro56, Th. 10].

Corollary 2.5. *Let G be a connected linear algebraic group, and let $H \subseteq G$ be any closed subgroup contained in a Borel subgroup B of G. Then G/H is birational to $(G/B) \times (B/H)$.*

Proof. Since B is connected and solvable, the quotient map $G \dashrightarrow G/B$ has a rational section by Lemma 2.4. The right-multiplication action of B on G is generically free, so Lemma 2.3 is applicable and yields the corollary. □

Lemma 2.6. *Let M be a connected solvable group acting rationally on a variety X. Then X is birational to $(X/M) \times \mathbb{P}^d$ for some $d \leqslant \dim M$. In particular, if X/M is rational, then so is X.*

Proof. Since M is connected solvable, we can find a sequence of connected normal closed subgroups $M = M_r \rhd M_{r-1} \rhd \cdots \rhd M_1 \rhd M_0 = \{1\}$ such that the subquotients M_{i+1}/M_i are 1-dimensional groups, isomorphic to either $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{G}_{\mathrm{a}}$. For each $i \in \{0, \ldots, r-1\}$, we will show that X/M_i is birational to X/M_{i+1} or $(X/M_{i+1}) \times \mathbb{P}^1$. By descending induction on i, it would then follow that $X = X/M_0$ is birational to $(X/M_r) \times \mathbb{P}^d$ for some $d \leqslant r = \dim M$, whence the lemma.

Consider the rational action of $H_i := M_{i+1}/M_i$ on the variety $X_i := X/M_i$, and let H_i' denote the image of H_i in $\mathrm{Aut}(k(X_i))$; thus H_i' is either trivial or isomorphic to $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{G}_{\mathrm{a}}$. By Lemma 2.1, X/M_{i+1} is naturally birational to X_i/H_i', and by Lemma 2.2, the action of H_i' on X_i is generically free. By Lemma 2.4, the rational

quotient map $X_i \dashrightarrow X_i/H_i'$ admits a rational section $X_i/H_i' \dashrightarrow X_i$. Hence by Lemma 2.3, X_i is birational to $(X_i/H_i') \times H_i'$, which proves what we want. □

Corollary 2.7. *Let M be a connected solvable group. Then for any closed subgroup $H \subseteq M$, the quotient variety M/H is rational.*

Proof. Apply Lemma 2.6 to the natural left action of M on $X := M/H$, and note that the rational quotient $M\backslash X$ of X by M is a point. □

Remark 2.8. In Lemma 2.6, the connected solvable group M acts on a variety X which is not necessarily a group; this mildly generalizes [Ro56, Cor. 1 to Th. 10], and will be very convenient for us later on. However, Corollary 2.7, which is deduced from Lemma 2.6, does not give the best result: the quotient variety M/H there is in fact isomorphic to a product of copies of $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$; see [Ro63, Theorem 5]. Both results, as well as Lemma 2.4, hold for split connected solvable linear algebraic groups over an arbitrary base field by essentially the same argument.

Theorem 2.9. *Let G be a connected linear algebraic group, and let $H \subseteq G$ be any closed subgroup contained in a Borel subgroup of G. Then G/H is a rational variety.*

Proof. Let B be a Borel subgroup of G containing H. Then G/H is birational to $(G/B) \times (B/H)$ by Corollary 2.5. The quotient flag variety G/B is rational (well-known; see also Lemma 3.3 below), and B/H is rational by Corollary 2.7. Hence G/H is rational. □

We record below some reduction arguments which will be useful later on.

Lemma 2.10. *Let G be a connected linear algebraic group and let $H \subseteq G$ be any closed subgroup. Let $R := R(G)$ be the solvable radical of G, let $G' := G/R$ be the maximal semisimple quotient of G, and let $H' := H/(H \cap R)$ be the image of H in G'. Then G/H is birational to $G'/H' \times \mathbb{P}^s$ for some $s \leqslant \dim R$. In particular, if G'/H' is rational, then so is G/H.*

Proof. By Lemma 2.6 applied to the natural left action of R on G/H, we see that G/H is birational to $(G/HR) \times \mathbb{P}^s$ for some $s \leqslant \dim R$. The result follows from the observation that G/HR is isomorphic to $(G/R)/(H/H \cap R) = G'/H'$. □

Lemma 2.11. *For $i = 1, 2$, let M_i be a connected solvable group acting rationally on a variety X_i. Assume that X_1/M_1 is birational to X_2/M_2 and that $\dim X_1 \leqslant \dim X_2$. Then X_2 is birational to $X_1 \times \mathbb{P}^d$ where $d := \dim X_2 - \dim X_1$. In particular, if X_1 is rational, then so is X_2.*

Proof. By Lemma 2.6, X_i is birational to $(X_i/M_i) \times \mathbb{P}^{d_i}$ for some $d_i \geqslant 0$. Since X_1/M_1 is birational to X_2/M_2 by assumption, it follows that $d = d_2 - d_1$ and that X_2 is birational to $X_1 \times \mathbb{P}^d$. □

3. Quotients by connected subgroups of maximal rank

We continue to work over an algebraically closed base field k, of arbitrary characteristic unless otherwise stated. Let G be a connected linear algebraic group. The maximal semisimple adjoint quotient $\overline{G}$ of G decomposes as a direct product $\overline{G} = \overline{G}_1 \times \cdots \times \overline{G}_\ell$ of connected adjoint almost-simple groups; we refer to these $\overline{G}_i$'s as the *adjoint factors of G*. Our main result in this section is the following theorem.

Theorem 3.1. *Let G be a connected linear algebraic group, and let $H \subseteq G$ be a connected closed subgroup of maximal rank in G. Assume that each adjoint factor of G is of type A, type C (when $\mathrm{char}(k) \neq 2$), or type B_3 or G_2 (when $\mathrm{char}(k) = 0$). Then G/H is a rational variety.*

We prove this in section 3.7; the essential case is when G is simply-connected and almost-simple. To reduce to this case, let G_i denote the simply-connected cover of the adjoint factor $\overline{G}_i$, let $\overline{H}_i$ denote the image of H in $\overline{G}_i$, and let H_i denote the preimage of $\overline{H}_i$ in G_i.

Lemma 3.2. *With the above notation, each H_i is a subgroup of maximal rank in G_i, and G/H is birational to $(G_1/H_1) \times \cdots \times (G_\ell/H_\ell) \times \mathbb{P}^s$ for some $s \leqslant \dim R$, where $R := R(G)$ is the solvable radical of G. In particular, if each G_i/H_i is rational, then so is G/H.*

Proof. Let $G' := G/R$ be the maximal semisimple quotient of G. The image H' of H in G' is a subgroup of maximal rank in G'. By Lemma 2.10, G/H is birational to $(G'/H') \times \mathbb{P}^s$ for some $s \leqslant \dim R$.

Next, let $Z(G')$ be the center of G', let $\overline{G} := G'/Z(G')$ be the adjoint quotient of G', and consider the quotient isogeny $q : G' \to \overline{G}$. The map $\overline{H} \mapsto H' := q^{-1}(\overline{H})$ is a bijection between the collection of connected subgroups of maximal rank of $\overline{G}$ and the collection of connected subgroups of maximal rank of G' (cf. [SGA3-II, Exp. XII, Cor. 7.12]). Since H' contains $Z(G')$, one obtains an isomorphism between G'/H' with $\overline{G}/\overline{H}$. Applying the same argument to the quotient isogeny $\tilde{G} \to \overline{G}$ from the simply-connected cover $\tilde{G}$ of $\overline{G}$, one obtains an isomorphism between $\tilde{G}/\tilde{H}$ with $\overline{G}/\overline{H}$, where $\tilde{H}$ is the preimage of $\overline{H}$ in $\tilde{G}$.

In our notation above, we can thus write $\tilde{G}$ as the direct product $\tilde{G} = G_1 \times \cdots \times G_\ell$; the subgroup $\tilde{H}$, which is of maximal rank in $\tilde{G}$, is then of the form $\tilde{H} = H_1 \times \cdots \times H_\ell$, and each H_i is a subgroup of maximal rank in G_i (cf. [BdS49, §3], or the algebraic group version in [Le, Prop. 4.1]). Thus $\tilde{G}/\tilde{H}$ is isomorphic to $(G_1/H_1) \times \cdots \times (G_\ell/H_\ell)$. The lemma follows. □

To proceed further, let us first review some preliminary results pertaining to the rationality of G/H in the greater generality when G is a connected reductive group and $H \subseteq G$ is any closed (connected) subgroup.

Lemma 3.3. *Let G be a connected reductive group, and let $P \subseteq G$ be a parabolic subgroup. Then G/P is rational, and the rational quotient map $G \dashrightarrow G/P$ admits a rational section.*

Proof. This is a standard consequence of the Bruhat decomposition; we give a detailed proof here for the sake of clarity.

Let $B = T \cdot U \subseteq G$ be a Borel subgroup contained in P, with maximal torus T and unipotent radical U; write $B^- = U^- \cdot T$ for the opposite Borel subgroup. The parabolic subgroup P is then the standard parabolic subgroup P_I associated to a subset I of the set of simple roots of G relative to T. By the Bruhat decomposition, G contains the Zariski-dense open subset (big cell) $U^- \cdot T \cdot U$, and P contains the Zariski-dense open subset $U_I^- \cdot T \cdot U$, where U_I^- denotes the subgroup of U^- generated by the 1-dimensional unipotent subgroups associated to the negative

roots belonging to the subroot lattice spanned by I (cf. [Hu75, §30.1]). Write V_I^- for the product variety of the 1-dimensional unipotent subgroups associated to the negative roots in the complement of the subroot lattice spanned by I; thus V_I^- is a rational variety, and one has $U^- \cong V_I^- \times U_I^-$ as varieties. Then the Bruhat decomposition above shows that the natural multiplication map

$$f \ : \ V_I^- \times P \dashrightarrow G, \qquad (v,p) \mapsto v \cdot p$$

is a birational map. With respect to the trivial action on V_I^- and the natural right action of P on P and on G, the map f is P-equivariant. Thus it induces the birational map $V_I^- \dashrightarrow G/P$, showing that G/P is rational. The inverse birational map yields the desired rational section of $G \dashrightarrow G/P$. □

Corollary 3.4. *Let G be a connected reductive group, and let $H \subseteq G$ be any closed subgroup contained in a parabolic subgroup P of G. Then G/H is birational to $(G/P) \times (P/H)$. In particular, if P/H is rational, then so is G/H.*

Proof. The first assertion is deduced from Lemma 2.3 by the same argument as in the proof of Corollary 2.5, using Lemma 3.3 above instead of Lemma 2.4. The second assertion then follows by another use of Lemma 3.3. □

Lemma 3.5 (Borel-Tits)**.** *Let G be a connected reductive (resp. connected semisimple) group. Suppose $H \subseteq G$ is a connected closed subgroup which is not contained in any proper parabolic subgroup of G. Then H is reductive (resp. semisimple).*

Proof. If H is not reductive, its unipotent radical $U := R_u(H)$ is a non-trivial normal subgroup of H. Since G is connected reductive, there exists (cf. [Hu75, §30.3, Cor. A]) a parabolic subgroup P of G with $N_G(U) \subseteq P$ and $U \subseteq R_u(P)$. The first inclusion gives $H \subseteq P$; the second inclusion forces P to be a proper parabolic subgroup, contradicting our hypothesis on H. Hence H is reductive.

Now suppose G is connected semisimple. If H is not semisimple, its center $Z(H)$ is of positive dimension, and we may choose a non-trivial 1-parameter subgroup $\lambda : \mathbb{G}_{\mathrm{m}} \to G$ of G with image in $Z(H)$. By [GIT, Def. 2.3/Prop. 2.6], there is a unique closed subgroup $P(\lambda) \subseteq G$ characterized by the property that

$$\gamma \in P(\lambda) \iff \lambda(t)\,\gamma\,\lambda(t^{-1}) \quad \begin{array}{l}\text{has a specialization in } G \\ \text{when } t \in \mathbb{G}_{\mathrm{m}} \text{ specializes to } 0.\end{array}$$

Moreover, one knows that $P(\lambda)$ is a parabolic subgroup of G and that the image of λ is contained in the solvable radical of $P(\lambda)$. As G is semisimple, this last fact forces $P(\lambda)$ to be a proper parabolic subgroup of G. But since H centralizes λ by construction, the characterizing property of $P(\lambda)$ shows that $H \subseteq P(\lambda)$, contradicting our hypothesis on H. Thus H is semisimple. □

Recall that an algebraic group M over k is called *special* (in the sense of Serre) iff $H^1(K, M) = \{1\}$ for every field K containing k. By the classification theorem of Serre and Grothendieck for special groups (see for instance [Re00, Th. 5.4]), one knows that a connected semisimple group M is special if and only if it is a direct product of simply-connected almost-simple groups of type A_n or C_n (i.e. SL_{n+1} or Sp_{2n}). Special groups enjoy the following important property: if X is an irreducible variety on which a special group M acts generically freely, the rational quotient map $X \dashrightarrow X/M$ admits a rational section (cf. [Re00, Lemma 5.2 and Prop. 5.3]).

Proposition 3.6. *Let G be isomorphic to Sp_{2n} for some $n \geqslant 2$, and let $M \subsetneq G$ be a maximal connected proper subgroup which is semisimple and of maximal rank. Then:*

(a) *M is a product of two simply-connected almost-simple groups of type C (i.e. M is G-conjugate to $\mathrm{Sp}_{2m} \times \mathrm{Sp}_{2n-m}$ for some $0 < m < n$), and hence it is a special group;*
(b) *the rational quotient map $G \dashrightarrow G/M$ admits a rational section;*
(c) *for any closed subgroup $H \subseteq G$ contained in M, G/H is birational to $(G/M) \times (M/H)$;*
(d) *if $\mathrm{char}(k) \neq 2$, then G/M is a rational variety.*

Proof. By the Borel-de Siebenthal algorithm (cf. [BdS49, Th. 6] and the algebraic group version in [Le, Prop. 6.6]), one knows that the Dynkin diagram of M is obtained from the extended Dynkin diagram of G by removing a vertex corresponding to a simple root α of G whose corresponding coefficient n_α for the longest root α_0 of G is a prime number. Moreover, one has the exact sequence

$$1 \to \mu(n'_\alpha) \to Z(\tilde{M}) \to Z(M) \to 1,$$

where $\tilde{M}$ is the simply-connected cover of M, $n'_\alpha := \frac{|\alpha|^2}{|\alpha_0|^2} \cdot n_\alpha$, and $\mu(n'_\alpha)$ is the group of roots of unity of order n'_α. Since G is simply-connected of type C, one has $n_\alpha = 2$ and $n'_\alpha = 1$, and the Dynkin diagram of M consists of two connected components both of type C. Hence $M = \tilde{M}$ is simply-connected and is a product of two simply-connected almost-simple groups of type C. Thus M is a special group. The right-multiplication action of M on G is generically free, and hence by the property of M being a special group (cf. [Re00, Lemma 5.2 and Prop. 5.3]), the rational quotient map $G \dashrightarrow G/M$ has a rational section. Hence by Lemma 2.3, for any closed subgroup H contained in M, G/H is birational to $(G/M) \times (M/H)$. This proves parts (a), (b) and (c).

For part (d), using [Le, Prop. 6.6] again, we see that M is the centralizer of an element of order 2 (an involution) in G; therefore, G/M is a symmetric variety (which makes sense since $\mathrm{char}(k) \neq 2$ by assumption). It is well-known (cf. [Sp85, Th. 4.2, Cor. 4.3]) that a symmetric variety is a spherical variety: the natural left action on G/M by a Borel subgroup B of G gives rise to a Zariski-dense open orbit of G/M. Consequently, G/M is birational to a quotient variety of B, and hence by Corollary 2.7, it is a rational variety. □

3.7. *Proof of Theorem* 3.1. We can now establish our main theorem in this section. By Lemma 3.2, we are reduced to showing that when G is a connected simply-connected and almost-simple group of type A, type C (when $\mathrm{char}(k) \neq 2$), or type B_3 or G_2 (when $\mathrm{char}(k) = 0$), and $H \subsetneq G$ is a connected proper closed subgroup of maximal rank in G, then G/H is rational. We proceed by induction on the common rank n of G and H, the case of $n = 0$ being trivial. Henceforth assume that $n \geqslant 1$ and that our conclusion holds for groups of the stated types of lower ranks.

Suppose H is contained in some proper parabolic subgroup $P \subsetneq G$. By Corollary 3.4, G/H is birational to $(G/P) \times (P/H)$, and by Lemma 3.3, G/P is rational. If G is of type A, C or G_2 (resp. type B_3), the adjoint factors of P are all of type A (resp. type A_1 or C_2), and the ranks of these factors are strictly lower than that

of G. By Lemma 3.2 applied to $H \subseteq P$ and our induction hypothesis, we see that P/H is rational, and hence G/H is rational.

If G is of type A_n for $n \geqslant 1$, the Borel-de Siebenthal algorithm shows that every connected proper subgroup $H \subsetneq G$ of maximal rank in G is contained in some proper parabolic subgroup of G. Our proof of Theorem 3.1 is therefore complete in this case.

If G is of type C_n for $n \geqslant 2$ and $\mathrm{char}(k) \neq 2$, we are reduced to the case when $H \subsetneq G$ is not contained in any proper parabolic subgroup of G and is therefore semisimple by Lemma 3.5. We let $M \subsetneq G$ be a maximal connected proper subgroup containing H; thus M is also of maximal rank in G and is not contained in any proper parabolic subgroup of G, and by Lemma 3.5, M is semisimple. By Proposition 3.6, G/H is birational to $(G/M) \times (M/H)$, and G/M is rational (because $\mathrm{char}(k) \neq 2$); moreover, the adjoint factors of M are all of type C, and the ranks of these factors are strictly lower than that of G. By Lemma 3.2 applied to $H \subseteq M$ and our induction hypothesis, we see that M/H is rational, and hence G/H is rational.

In the remaining cases, G is of type B_3 or G_2 with $\mathrm{char}(k) = 0$, and $H \subsetneq G$ is not contained in any proper parabolic subgroup of G; again, H is semisimple by Lemma 3.5. The rationality of G/H is then established directly using results in section 5, and we defer the proof of these cases to Corollary 5.12. The proof of Theorem 3.1 is thus completed — modulo the use of Corollary 5.12 for the low dimensional cases. □

Remark 3.8. It is possible that the assumptions in Theorem 3.1 on the adjoint factors of G can be removed altogether. This would be the case if the assertion of Proposition 3.6(b) can be established for almost-simple groups of any type; our induction argument in 3.7 would then yield the stable-rationality of G/H in general. In turn, the rationality of G/H in general would be reduced to the assertions of Proposition 3.6(d) for almost-simple groups of any type, i.e. to the rationality of G/M when G is almost-simple (of any type) and $M \subsetneq G$ is a maximal connected proper subgroup of maximal rank in G, but which is not contained in any proper parabolic subgroup of G. As explained in 3.7, such an M is semisimple, and its (finitely many) possibilities are determined by the Borel-de Siebenthal algorithm.

4. Low dimensional homogeneous varieties

From now on, we work over an algebraically closed base field k of characteristic 0. In this and the next section, we apply geometric methods to study the rationality problem 1.1. Our goal is to establish Theorems 4.4 and 5.9 asserting the rationality of all homogeneous varieties G/H of sufficiently low dimensions, thereby answering the rationality problem 1.1 affirmatively in these cases. In this section, we place no restriction on the connectedness of H, while in section 5, we extend our rationality results further when H is assumed to be connected. The following argument will be used several times in both sections.

Lemma 4.1. *Let $H_1 \subseteq H_2$ be two connected algebraic groups such that H_2 (and hence H_1) act rationally on an algebraic variety X, and let $f : X/H_1 \dashrightarrow X/H_2$ be the dominant rational quotient map. The following are equivalent:*

(a) *f is birational;*
(b) *f is generically injective;*
(c) *f is generically finite;*

(d) $\dim(X/H_1) = \dim(X/H_2)$;
(e) *for all points $x \in X$ in general position, its orbits $x\cdot H_1$ and $x\cdot H_2$ under the action of H_1 and H_2 have the same Zariski closure.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are clear. To show the other implications, we replace X by another birational model and assume that both $H_1 \subseteq H_2$ act regularly on X (cf. [Ro56, Th. 1]). For a point $x \in X$ in general position, its orbit $x\cdot H_i$ under the action of H_i is an irreducible locally closed subvariety in X of dimension $\dim(x\cdot H_i) = \dim X - \dim(X/H_i)$. If $\dim(X/H_1) = \dim(X/H_2)$, these orbits are of the same dimension, and since we have the inclusion $x\cdot H_1 \subseteq x\cdot H_2$, these orbits have the same Zariski closure in X; hence (d) $\Rightarrow$ (e). The fiber of f over the point $x\cdot H_2$ in X/H_2 consists of those points $x'\cdot H_1$ in X/H_1 which, when regarded as orbits in X, belong to the same Zariski closure as $x\cdot H_2$ in X; hence (e) $\Rightarrow$ (b). Finally, recall that (cf. [Hu75, §4.6, Th.]) a dominant injective morphism between irreducible varieties induces a finite purely inseparable extension of their function fields. As our base field k is of characteristic 0, we infer that when f is generically injective, it induces a trivial extension of function fields and f is therefore birational; hence (b) $\Rightarrow$ (a). $\square$

Recall that a unirational curve over any field is rational by Lüroth's theorem and a unirational surface over an algebraically closed field of characteristic 0 is rational by Castelnuovo's rationality criterion. Hence over our algebraically closed base field k of characteristic 0, a unirational variety is rational if its dimension is $\leqslant 2$.

Lemma 4.2. *Let G be a connected linear algebraic group, and let $B \subseteq G$ be a Borel subgroup of G. For any closed subgroup $H \subseteq G$, if $\dim(B\backslash G/H) \leqslant 2$, then G/H is rational.*

Proof. The underlying variety of G is rational (cf. [Ch54]); the space of double cosets $B\backslash G/H$, being dominated by G, is therefore a unirational variety. Hence our hypothesis on its dimension implies that $B\backslash G/H$ is rational. By Lemma 2.6 applied to the left action of B on G/H, it now follows that G/H is rational. $\square$

Lemma 4.3. *Let G be a connected semisimple group with maximal torus T, and let $B \subseteq G$ be a Borel subgroup of G containing T. For any closed subgroup $H \subseteq G$, one has $\dim(B\backslash G/H) \leqslant \dim(T\backslash G/H)$; and if equality holds, then G/H is rational.*

Proof. Let U be the unipotent radical of B. The inclusion of T in $B = T \ltimes U$ induces a dominant rational map $T\backslash G/H \dashrightarrow B\backslash G/H$, so one always has $\dim(B\backslash G/H) \leqslant \dim(T\backslash G/H)$. Assume that equality holds. By Lemma 4.1 applied to the left action of T and B on G/H, we see that for a point $x \in G/H$ in general position, its orbits $T\cdot x$ and $B\cdot x$ under the action of T and B have the same Zariski closure in G/H. Since $B = U\cdot T$ and since G contains the Zariski-dense open subset (big cell) $U^-\cdot T\cdot U$ (cf. [Hu75, §28.5]),

$$G/H = G\cdot x = \overline{U^-B\cdot x} = \overline{U^-T\cdot x} = \overline{B^-\cdot x}.$$

Hence G/H is birational to B^-/B^-_x where B^-_x is the stabilizer of x in B^-; and since B^- is connected solvable, it follows from Corollary 2.7 that this is rational. $\square$

Theorem 4.4. *Let G be a connected linear algebraic group, and let $H \subseteq G$ be any closed subgroup. If $\dim(G/H) \leqslant 5$, then G/H is a rational variety.*

Proof. By Lemma 2.10, we may replace G by its maximal semisimple quotient G' and H by its image H' in G'. The rationality of G'/H' implies that of G/H, but the dimension of G'/H' can only be at most that of G/H. Henceforth, we assume G is semisimple.

Let T be a maximal torus of G, and let B be a Borel subgroup of G containing T. If $\dim T \leqslant 1$, the semisimple group G is either trivial or of rank 1, isogenous to SL_2. In either case, we see that

$$\dim(B\backslash G/H) \;\leqslant\; \dim(B\backslash G) \;\leqslant\; 1,$$

and the rationality of G/H follows from Lemma 4.2. Henceforth we assume $\dim T \geqslant 2$.

Replacing G further by its image in $\mathrm{Aut}(G/H)$ if necessary, we may also assume that the natural left action of G on G/H is generically faithful. Applying Lemma 2.2 to the left action of T on G/H, we have $\dim(G/H) = \dim(T\backslash G/H) + \dim T$. This is $\leqslant 5$ by hypothesis, so $\dim(T\backslash G/H) \leqslant 3$. But as one always has $\dim(B\backslash G/H) \leqslant \dim(T\backslash G/H)$, this means that

$$\text{either} \qquad \dim(B\backslash G/H) \leqslant 2 \qquad \text{or} \qquad \dim(B\backslash G/H) = \dim(T\backslash G/H) = 3,$$

and the rationality of G/H follows from Lemma 4.2 and Lemma 4.3 respectively. □

5. Low dimensional homogeneous varieties, continued

In this section, we consider the rationality of G/H when $H \subseteq G$ is a *connected* closed subgroup. We still work over an algebraically closed base field k of characteristic 0. In the series of lemmas below leading up to the main theorem, 5.9, of this section, we adopt the following hypotheses and notation.

5.1. Let G be a connected semisimple group, and let $H \subseteq G$ be a connected closed subgroup. We fix once and for all:

$U(H)$	the unipotent radical of H,
S	a (reductive) Levi subgroup of H so that $H = S \ltimes U(H)$;
T_H	a maximal torus of S (and hence of H),
$B_S^{\pm}$	a pair of opposite Borel subgroups of S containing T_H,
$U_S^{\pm}$	the unipotent radical of the corresponding $B_S^{\pm}$ so that $B_S^{\pm} = T_H \ltimes U_S^{\pm}$,
$B_H^{\pm}$	the preimage in H of $B_S^{\pm}$ so that $B_H^{\pm} = B_S^{\pm} \ltimes U(H)$,
$U_H^{\pm}$	the preimage in H of $U_S^{\pm}$ so that $U_H^{\pm} = U_S^{\pm} \ltimes U(H)$.

Here, $B_H^{\pm}$ are Borel subgroups of H with unipotent radicals $U_H^{\pm}$, and T_H is a maximal torus of H contained in B_H. Having fixed these, we choose:

$B = B^+$	a Borel subgroup of G containing B_H;
T	a maximal torus of B (and hence of G) containing T_H;
B^-	the opposite Borel subgroup of G containing T, such that $B^- \cap B = T$;
$U^{\pm}$	the unipotent radical of the corresponding $B^{\pm}$ so that $B^{\pm} = T \ltimes U^{\pm}$.

We also set

$$\begin{aligned} u(H) &:= \dim U(H), \\ u_G &:= \dim U = \dim U^-, \qquad & t_G &:= \dim T \quad \text{(the rank of } G\text{)}, \\ u_H &:= \dim U_H = \dim U_H^-, \qquad & t_H &:= \dim T_H \quad \text{(the rank of } H\text{)}. \end{aligned}$$

Thus:

$$\begin{aligned} \dim S &= \dim T_H + 2\dim U_S &&= t_H + 2(u_H - u(H)), \\ \dim H &= \dim S + \dim U(H) &&= t_H + 2u_H - u(H), \\ \dim G &= \dim T + 2\dim U &&= t_G + 2u_G, \end{aligned}$$

and hence

$$(*) \qquad \dim G/H \;=\; \dim G - \dim H \;=\; (t_G - t_H) + 2(u_G - u_H) + u(H).$$

The subgroup inclusion maps $U_H \subseteq B_H$, $B_H \subseteq H$ and $U^- \subseteq B^-$ induce the dominant rational maps α, γ and φ between the respective spaces of double cosets:

$$(**) \qquad \begin{tikzcd} B^-\backslash G/U_H \arrow[r, dashed, "\alpha"] & B^-\backslash G/B_H \arrow[d, "\gamma"] & U^-\backslash G/B_H \arrow[l, dashed, "\varphi"'] \\ & B^-\backslash G/H & \end{tikzcd}$$

We will consider these rational maps in the series of lemmas below leading up to Theorem 5.9.

Lemma 5.2. *In the situation of section* 5.1*:*

(a) *$B^-\backslash G/U_H$ is rational, of dimension $u_G - u_H$.*
(b) *One has $\dim(B^-\backslash G/H) \leqslant \dim(B^-\backslash G/U_H)$; and if equality holds, then G/H is rational.*
(c) *If $u_G - u_H \leqslant 3$, then G/H is rational.*

Proof. The Bruhat (big cell) decomposition of G shows that $B^-\backslash G/U_H$ contains a Zariski-dense constructible subset $B^-\backslash B^-U/U_H$ which is birational to U/U_H; in turn, this is rational by Corollary 2.7. Since $\dim(U/U_H) = u_G - u_H$, we see that $B^-\backslash G/U_H$ is rational and of that dimension; this shows part (a).

The asserted inequality of part (b) follows from the existence of the dominant rational map $\gamma \circ \alpha$ in the diagram $(**)$ of section 5.1. If equality holds, Lemma 4.1 applied to the right action of U_H and H on $B^-\backslash G$ shows that $\gamma \circ \alpha$ is a birational map, and hence $B^-\backslash G/H$ is rational. By Lemma 2.6 applied to the left action of B^- on G/H, it then follows that G/H is rational.

For part (c), if $\dim(B^-\backslash G/H) \leqslant 2$, the rationality of G/H follows from Lemma 4.2; henceforth, assume that $\dim(B^-\backslash G/H) \geqslant 3$. Our hypothesis together with parts (a) and (b) then yields

$$3 \;\leqslant\; \dim(B^-\backslash G/H) \;\leqslant\; \dim(B^-\backslash G/U_H) \;\leqslant\; 3,$$

whence equality holds throughout, and the rationality of G/H follows from part (b) again. □

Remark 5.3. Although we do not need it below, it is of interest to note that $B^-\backslash G/B_H$ is in fact rational. Indeed, $B^-\backslash G/U_H$ contains the Zariski-open subset $B^-\backslash B^-U/U_H \cong U/U_H$, and with respect to the (regular) action of T_H on U/U_H by conjugation, the isomorphism is T_H-equivariant. Thus the quotient $B^-\backslash G/B_H$ of $B^-\backslash G/U_H$ by T_H is birational to the quotient of U/U_H by T_H. Moreover, U/U_H is T_H-equivariantly isomorphic to the quotient $V := \mathrm{Lie}(U)/\mathrm{Lie}(U_H)$ of Lie algebras, on which T_H acts linearly via its adjoint actions on $\mathrm{Lie}(U)$ and $\mathrm{Lie}(U_H)$. So $B^-\backslash G/B_H$ is birational to the quotient V/T_H. Now choose a basis of V for which the action of T_H is diagonal, and let $V_0 \subseteq V$ denote the open subset on which all coordinates are non-zero. Then V_0 is isomorphic to a torus, on which T_H acts by

multiplication; thus V_0/T_H is a torus as well. This shows that V_0/T_H and hence V/T_H and $B^-\backslash G/B_H$ are all rational.

Lemma 5.4. *In the situation of* 5.1*, one has* $\dim(B^-\backslash G/H) \leqslant \dim(B^-\backslash G/B_H)$*; and if equality holds, then* G/H *is rational.*

Proof. The asserted inequality follows from the existence of the dominant rational map γ in the diagram $(**)$ of 5.1. If equality holds, then by Lemma 4.1 applied to the right action of B_H and H on $B^-\backslash G$, we see that $\gamma : B^-\backslash G/B_H \dashrightarrow B^-\backslash G/H$ is birational. By Lemma 5.2, the variety $X_1 := B^-\backslash G/U_H$ is rational, of dimension $u_G - u_H$. The torus $M_1 := B_H/U_H$ acts by right multiplication on X_1 with quotient $X_1/M_1 = B^-\backslash G/B_H$. On the other hand, the variety $X_2 := G/H$ is of dimension $(t_G - t_H) + 2(u_G - u_H) + u(H)$ by the formula $(*)$ in 5.1. The solvable group $M_2 := B^-$ acts by left multiplication on X_2 with quotient $M_2\backslash X_2 = B^-\backslash G/H$. Thus X_1/M_1 is birational to $M_2\backslash X_2$ via γ; since $\dim X_1 \leqslant \dim X_2$, Lemma 2.11 is applicable and shows that the rationality of X_1 implies that of $X_2 = G/H$. This completes the proof of the lemma. □

Lemma 5.5. *In the situation of* 5.1*, set*

$$d \;:=\; \dim(U^-\backslash G/B_H) - \dim(B^-\backslash G/B_H).$$

(a) *Let* L *denote the identity component of the kernel* L' *of the natural left action of* T *on* $U^-\backslash G/B_H$*. Then* $d = t_G - t_L$*, where* $t_L := \dim L$*.*
(b) *If* $d \leqslant 1$*, then* G/H *is rational.*

Proof. First note that $d \geqslant 0$ by the existence of the dominant rational map φ in the diagram $(**)$ of 5.1. The torus L is of finite index in the diagonalizable group L' contained in T; hence $\dim(T/L') = \dim(T/L) = t_G - t_L$. By construction, the induced left action of T/L' on $U^-\backslash G/B_H$ is generically faithful, and its quotient is $TU^-\backslash G/B_H = B^-\backslash G/B_H$. Hence by Lemma 2.2, we have

$$\dim(U^-\backslash G/B_H) \;=\; \dim(B^-\backslash G/B_H) + \dim(T/L'),$$

from which it follows that $d = \dim(T/L') = t_G - t_L$. This shows part (a) of the lemma.

Let $D := C_G(L)$ denote the centralizer of L in G; it is a connected reductive subgroup of G (cf. [Hu75, §26.2, Cor. A; §22.3, Th.]) containing the maximal torus T, and a Borel subgroup is given by $B \cap D$ (cf. [Hu75, §22.4, Cor.]), whose unipotent radical is $U_D := U \cap D$. We set $u_D := \dim U_D$.

We claim that U is contained in the image $U_D \cdot U_H$ of multiplying U_D and U_H in G. Assuming this for the moment, we infer that $u_G \leqslant u_D + u_H$, and part (b) of the lemma can be deduced from this as follows. If $d = 0$, then $t_L = t_G$, so $L = T$ is the maximal torus of G, and it is self-centralizing in G (cf. [Hu75, §26.2, Cor. A]), whence $D = T$, and we have $u_D = 0$. If $d = 1$, then L is a subtorus of codimension 1 in T; if L is a regular subtorus, then $D = T$ and we have $u_D = 0$ as before; if L is a singular subtorus, then D is isogenous to $L \times \mathrm{SL}_2$ (cf. [Hu75, §26.2, Cor. B]), and we have $u_D = 1$. In any case, we see that $d \leqslant 1$ implies $u_G - u_H \leqslant u_D \leqslant 1 \leqslant 3$, and the rationality of G/H follows from Lemma 5.2.

We now proceed to prove our claim that $U \subseteq U_D \cdot U_H$. The torus T normalizes U^-, and so it acts regularly from the left on $U^-\backslash G/B_H$, which contains $U^-\backslash U^-B/B_H$ (isomorphic to B/B_H) as a Zariski-dense T-stable open subset, by the Bruhat (big cell) decomposition of G. Hence L acts trivially from the

left on $U^-\backslash U^-B/B_H$. This means that for any $b \in B$ and any $\ell \in L$, one has $U^-\cdot \ell\cdot b\cdot B_H = U^-\cdot b\cdot B_H$, whence

$$\ell\cdot b \;=\; v_1\cdot b\cdot b_1 \qquad \text{for some} \quad v_1 \in U^-,\ b_1 \in B_H.$$

Thus $v_1 \in U^- \cap B = \{1\}$, and if we write $b_1 = t_1\cdot u_1$ (with $t_1 \in T_H$, $u_1 \in U_H$) and $b = t\cdot u$ (with $t \in T$, $u \in U$), then reducing modulo U shows that $\ell = t_1$ in T. Hence, for any $b \in B$ and $\ell \in L$, there exists $u_1 \in U_H$ such that

$$b^{-1}\cdot \ell\cdot b \;=\; \ell\cdot u_1.$$

Specializing this relation to the case when b lies in U_H, we see that L normalizes U_H, and hence $L\cdot U_H$ is a connected subgroup of B containing L as a maximal torus. Specializing the relation to the case when b equals $u \in U$, we see that $u^{-1}\cdot L\cdot u$ is also a maximal torus in $L\cdot U_H$, so it is U_H-conjugate to L: there exists $u_2 \in U_H$ such that $(u\cdot u_2)^{-1}\cdot L\cdot (u\cdot u_2) = L$ or, equivalently, $u\cdot u_2 \in U \cap N_G(L)$. But since U is connected and solvable, by [Hu75, §19.4, Prop.]), $U \cap N_G(L) = N_U(L)$ is equal to $C_U(L) = U \cap C_G(L) = U \cap D = U_D$. We have thus shown that for any $u \in U$, there exists $u_2 \in U_H$ (depending on u) such that $u\cdot u_2 \in U_D$. Hence U is contained in $U_D\cdot U_H$, and our claim follows. □

Lemma 5.6. *In the situation of* 5.1*:*

(a) *$U^-\backslash G/B_H$ is rational, of dimension $(t_G - t_H) + (u_G - u_H)$.*
(b) *If $(t_G - t_H) + (u_G - u_H) \leqslant 5$, then G/H is rational.*

Proof. The Bruhat (big cell) decomposition of G shows that $U^-\backslash G/B_H$ contains a Zariski-dense constructible subset $U^-\backslash U^-B/B_H$ which is birational to B/B_H; in turn, this is rational by Corollary 2.7. Since $\dim(B/B_H) = (t_G - t_H) + (u_G - u_H)$, we see that $U^-\backslash G/B_H$ is rational and of that dimension; this shows part (a).

For part (b), consider the dominant rational maps γ and φ in the diagram $(**)$ of 5.1, which give the inequalities

$$\dim(B^-\backslash G/H) \;\leqslant\; \dim(B^-\backslash G/B_H) \;\leqslant\; \dim(U^-\backslash G/B_H).$$

If $\dim(B^-\backslash G/H) \leqslant 2$, the rationality of G/H follows from Lemma 4.2, while if one has $\dim(B^-\backslash G/H) = \dim(B^-\backslash G/B_H)$, the rationality of G/H follows from Lemma 5.4. In the remaining cases, our hypothesis yields

$$4 \;\leqslant\; \dim(B^-\backslash G/H) + 1 \;\leqslant\; \dim(B^-\backslash G/B_H) \;\leqslant\; \dim(U^-\backslash G/B_H) \;\leqslant\; 5,$$

whence $d := \dim(U^-\backslash G/B_H) - \dim(B^-\backslash G/B_H)$ is $\leqslant 1$; the rationality of G/H now follows from Lemma 5.5. □

Lemma 5.7. *In the situation of* 5.1*, suppose H is reductive; set*

$$e \;:=\; \dim(B^-\backslash G/U_H) - \dim(B^-\backslash G/B_H).$$

(a) *Let K denote the identity component of the kernel K' of the natural right action of T_H on $B^-\backslash G/U_H$. Then $e = t_H - t_K$, where $t_K := \dim K$.*
(b) *If the natural left action of G on G/H has zero-dimensional kernel, then $t_K = 0$ and hence $e = t_H$.*
(c) *If $e = 0$, which is to say $\dim(B^-\backslash G/U_H) = \dim(B^-\backslash G/B_H)$, then G/H is rational.*

Proof. Part (a) of the lemma is established along the same lines as part (a) of Lemma 5.5, using the generically faithful right action of T_H/K' on $B^-\backslash G/U_H$, passing to the quotient and applying Lemma 2.2 to get

$$\dim(B^-\backslash G/U_H) \;=\; \dim(B^-\backslash G/B_H) + \dim(T_H/K').$$

Let $E := C_G(K)$ denote the centralizer of K in G; it is a connected reductive subgroup of G (cf. [Hu75, §26.2, Cor. A; §22.3, Th.]) containing the maximal torus T, and a Borel subgroup is given by $B\cap E$ (cf. [Hu75, §22.4, Cor.]), whose unipotent radical is $U_E := U \cap E$. Let $E_H := C_H(K) = E \cap H$ denote the centralizer of K in H; it is a connected reductive subgroup of H containing the maximal torus T_H, and a Borel subgroup is given by $B_H \cap E_H = B \cap E \cap H$, whose unipotent radical is $U_{E_H} := U_H \cap E_H = U \cap E \cap H$. We set $u_E := \dim U_E$ and $u_{E_H} := \dim U_{E_H}$; hence $\dim E = t_G + 2u_E$ and $\dim E_H = t_H + 2u_{E_H}$.

We claim that U is equal to the Zariski closure $\overline{U_E\cdot U_H}$ of the image of multiplying U_E and U_H in G. Assuming this for the moment, we infer that $u_G \leqslant u_E + u_H$. More precisely, since the multiplication map $U_E \times U_H \to \overline{U_E\cdot U_H} = U$ has a general fiber isomorphic to $U_{E_H} = U_E \cap U_H$, we see that $u_G + u_{E_H} = u_E + u_H$. We have the natural inclusion map

$$E/E_H \;=\; E/(E\cap H) \;\hookrightarrow\; G/H.$$

Here, since H is reductive by hypothesis, one has $u(H) = 0$, and so by the dimension formula $(*)$ in 5.1, we have

$$\dim(G/H) \;=\; (t_G - t_H) + 2(u_G - u_H).$$

On the other hand,

$$\dim(E/E_H) \;=\; (t_G + 2u_E) - (t_H + 2u_{E_H}) \;=\; (t_G - t_H) + 2(u_G - u_H).$$

Thus $\dim E/E_H = \dim G/H$, which shows that the locally closed subvariety E/E_H of G/H is in fact a Zariski-open subset, whence G/H and E/E_H are birational to each other. This is the key fact needed for showing parts (b) and (c) of the lemma. For part (b), we note that K is a normal subgroup of both E and E_H; if we let $\overline{E} := E/K$ and $\overline{E_H} := E_H/K$ denote the respective quotient groups, then E/E_H is naturally isomorphic to $\overline{E}/\overline{E_H}$, and the morphisms

$$\overline{E}/\overline{E_H} \;\cong\; E/E_H \;\hookrightarrow\; G/H$$

are E-equivariant with respect to the natural left actions of E. But K acts trivially on $\overline{E}/\overline{E_H}$ by construction, so it also acts trivially on G/H. Hence K is contained in the kernel of the natural left action of G on G/H; if this kernel is zero-dimensional, so is K, which is to say $t_K = 0$. For part (c), if we have $e = 0$, then $t_K = t_H$, so $K = T_H$ is the maximal torus of H, and it is therefore self-centralizing in H (cf. [Hu75, §26.2, Cor. A]), whence $E_H = E \cap H = C_H(K)$ is equal to T_H. Then $E/E_H = E/T_H$ is rational by Theorem 2.9, and the rationality of G/H follows.

To prove our claim, it suffices to show that a point in general position in U belongs to (the Zariski closure of) $U_E\cdot U_H$, since the reverse inclusion is clear. The torus T_H normalizes U_H, and so it acts regularly from the right on $B^-\backslash G/U_H$, which contains $B^-\backslash B^-U/U_H$ (isomorphic to U/U_H) as a Zariski-dense open subset, by the Bruhat (big cell) decomposition of G. Hence K acts trivially from the right on $B^-\backslash B^-U/U_H$. This means that for a point $u \in U$ in general position and for any

$k \in K$, one has $B^- \cdot u \cdot k \cdot U_H = B^- \cdot u \cdot U_H$, whence

$$u \cdot k \;=\; v_1 \cdot t_1 \cdot u \cdot u_1 \qquad \text{for some} \quad v_1 \in U^-,\ t_1 \in T,\ u_1 \in U_H.$$

Thus $v_1 \in U^- \cap B = \{1\}$, and reducing modulo U shows that $k = t_1$ in T; hence $u^{-1} \cdot k \cdot u = k \cdot u_1^{-1}$ lies in $K \cdot U_H$. Note that $K \cdot U_H$ is a connected subgroup of B_H containing K as a maximal torus; the above discussion shows that $u^{-1} \cdot K \cdot u$ is also a maximal torus in $K \cdot U_H$, so it is U_H-conjugate to K: there exists $u_2 \in U_H$ such that $(u \cdot u_2)^{-1} \cdot K \cdot (u \cdot u_2) = K$ or, equivalently, $u \cdot u_2 \in U \cap N_G(K)$. But since U is connected and solvable, by [Hu75, §19.4, Prop.]), $U \cap N_G(K) = N_U(K)$ is equal to $C_U(K) = U \cap C_G(K) = U \cap E = U_E$. We have thus shown that for a point $u \in U$ in general position, there exists $u_2 \in U_H$ (depending on u) such that $u \cdot u_2 \in U_E$. This establishes our claim and hence completes the proof of the lemma as well. □

Lemma 5.8. *In the situation of* 5.1*, suppose H is reductive. If $u_G - u_H \leqslant 4$, then G/H is rational.*

Proof. Consider the dominant rational maps γ and α in the diagram $(**)$ of 5.1, which give the inequalities

$$\dim(B^- \backslash G/H) \;\leqslant\; \dim(B^- \backslash G/B_H) \;\leqslant\; \dim(B^- \backslash G/U_H).$$

If $\dim(B^- \backslash G/H) \leqslant 2$, the rationality of G/H follows from Lemma 4.2; henceforth, assume that $\dim(B^- \backslash G/H) \geqslant 3$. We have $\dim(B^- \backslash G/U_H) \leqslant 4$ by our hypothesis and Lemma 5.2. Hence among the two inequalities in the above display, equality holds for at least one of them. The rationality of G/H then follows from Lemma 5.4 or Lemma 5.7 respectively. □

We are now in a position to show our main result of this section.

Theorem 5.9. *Let G be a connected linear algebraic group, and let $H \subseteq G$ be a connected closed subgroup. If $\dim(G/H) \leqslant 10$, then G/H is a rational variety.*

Proof. By Lemma 2.10, we may replace G by its maximal semisimple quotient; henceforth we assume that G is semisimple and adopt the notation of 5.1. By the dimension formula $(*)$ in 5.1, we have

$$\dim(G/H) \;=\; (t_G - t_H) + 2(u_G - u_H) + u(H).$$

If $(t_G - t_H) + (u_G - u_H) \leqslant 5$, the rationality of G/H follows from Lemma 5.6, while if $u_G - u_H \leqslant 3$, the rationality of G/H follows from Lemma 5.2. In the remaining cases, our hypothesis $\dim(G/H) \leqslant 10$ together with the above dimension formula implies that

$$u(H) = 0, \quad u_G - u_H = 4, \quad \text{and} \quad t_G - t_H = 2.$$

Hence H is reductive, and the rationality of G/H now follows from Lemma 5.8. □

With a bit more work, we can establish a slightly technical but also more applicable result in Theorem 5.11. First, we note that the dimension formula

$$\dim(U^- \backslash G/B_H) \;=\; (t_G + u_G) - (t_H + u_H), \qquad \dim(B^- \backslash G/U_H) \;=\; u_G - u_H$$

of Lemmas 5.6 and 5.2 yields:

Lemma 5.10. *In the situation of* 5.1*, the following are equivalent:*

(a) $\dim(B^- \backslash G/B_H) = u_G - u_H - t_H$.
(b) $\dim(U^- \backslash G/B_H) - \dim(B^- \backslash G/B_H) = t_G$ *(i.e. $d = t_G$ in Lemma* 5.5*).*
(c) $\dim(B^- \backslash G/U_H) - \dim(B^- \backslash G/B_H) = t_H$ *(i.e. $e = t_H$ in Lemma* 5.7*).*

Theorem 5.11. *Let G be a connected semisimple group, and let $H \subseteq G$ be a connected reductive closed subgroup. Suppose the natural left action of G on G/H has zero-dimensional kernel. If $\dim(G/H) < t_G + t_H + 8$, then G/H is a rational variety.*

Proof. We adopt the notation of 5.1. Our hypothesis on the action of G on G/H together with Lemma 5.7 gives $e = t_H$, which by Lemma 5.10 means that $\dim(B^-\backslash G/B_H) = u_G - u_H - t_H$. Since H is reductive by hypothesis, one has $u(H) = 0$, and so by the dimension formula $(*)$ in 5.1, we have

$$\dim(G/H) \;=\; (t_G - t_H) + 2(u_G - u_H).$$

Hence our assumption that this is $< t_G + t_H + 8$ amounts to the inequality

$$\dim(B^-\backslash G/B_H) \;<\; 4.$$

If $\dim(B^-\backslash G/H) \leqslant 2$, the rationality of G/H follows from Lemma 4.2. In the remaining case, by the inequality in Lemma 5.4, we must have

$$3 \;\leqslant\; \dim(B^-\backslash G/H) \;\leqslant\; \dim(B^-\backslash G/B_H) \;\leqslant\; 3,$$

whence equality holds throughout, and the rationality of G/H follows from Lemma 5.4 again. □

Corollary 5.12. *Let G be a connected group which is almost-simple of type B_3 or G_2, and let $H \subseteq G$ be a connected semisimple subgroup of maximal rank in G. Then G/H is rational.*

Proof. Since the case when $H = G$ is trivial, we shall assume that the connected semisimple closed subgroup $H \subsetneq G$ is properly contained in G. The natural left action of G on G/H is thus non-trivial, and since G is almost-simple by hypothesis, the kernel of the action is zero-dimensional. Hence Theorem 5.11 is applicable whenever the required bound on $\dim(G/H)$ holds. As H is semisimple, we have the crude lower bound $\dim H \geqslant 3n$ where n denotes the common rank of G and H. From the following table of values:

G of type	B_3	G_2
$\dim G =$	21	14
$\dim H \geqslant$	9	6
$\dim(G/H) \leqslant$	12	8
$n + n + 8 =$	14	12

we see that $\dim(G/H) < n + n + 8$ in each of the cases considered, whence Theorem 5.11 yields the rationality of G/H. □

We note that Corollary 5.12 completes the proof of Theorem 3.1 in section 3.7.

6. Concluding remarks

In this final section, we indicate a few other cases in which our results yield an affirmative answer to the rationality problem 1.1. We still work over an algebraically closed base field k of characteristic 0.

Proposition 6.1. *Let G be a connected linear algebraic group with $\dim G \leqslant 13$. Then for any connected closed subgroup $H \subseteq G$, G/H is a rational variety.*

Proof. If $\dim(G/H) \leqslant 10$, the rationality of G/H follows from Theorem 5.9. The remaining cases are when $11 \leqslant \dim(G/H) \leqslant 13$, but this means $\dim H \leqslant 2$ and thus H is solvable, whence the rationality of G/H follows from Theorem 2.9. □

Proposition 6.2. *Let G be a connected linear algebraic group with $\dim G = 14$. Then for any connected closed subgroup $H \subseteq G$, G/H is a rational variety, except possibly when G is the simple group of type G_2 and $H \subseteq G$ is semisimple of type A_1 (in which case $\dim(G/H) = 11$).*

Proof. As in the proof of the previous result, when $\dim H \geqslant 4$ or when $\dim H \leqslant 2$, the rationality of G/H follows from Theorem 5.9 or Theorem 2.9 respectively. Henceforth we assume that $\dim H = 3$ and that H is not solvable. This means that H is semisimple of type A_1; in particular, $\dim(G/H) = 11$.

Let $R = R(G)$ be the solvable radical of G. If $\dim(G/HR) < \dim(G/H) = 11$, then G/HR is rational by Theorem 5.9, and so G/H is rational by Lemma 2.10. Hence we may assume that $\dim(G/HR) = \dim(G/H) = 11$, which means $H = HR$ is a connected closed subgroup of G containing R in its radical. Since H is semisimple, this forces R to be trivial, and hence G is also semisimple. By the classification of semisimple groups, $\dim G = 14$ implies that G is either of type $A_2 + 2A_1$ or of type G_2. In the former case, we have $u_G - u_H = (3+2) - 1 = 4$, and the rationality of G/H follows from Lemma 5.8. In the latter case, we are in the possible exceptional situation of the proposition. □

When the homogeneous variety G/H is of dimension $\leqslant 10$, the rationality problem 1.1 has been answered affirmatively in Theorem 5.9. We consider the cases when G/H is of dimension 11 and 12 in Propositions 6.3 and 6.4 below. To put the homogeneous variety G/H in a somewhat "reduced form", we impose the hypothesis that G acts on G/H with a zero-dimensional kernel. By Lemma 2.1, this can always be achieved without changing the birational type of G/H by replacing G and H by their images in $\mathrm{Aut}(k(G/H))$.

Proposition 6.3. *Let G be a connected semisimple group, and let $H \subseteq G$ be a connected closed subgroup. Suppose the natural left action of G on G/H has zero-dimensional kernel. If $\dim(G/H) = 11$, then G/H is a rational variety, except possibly when G is semisimple of type G_2 and H is semisimple of type A_1.*

Proof. We adopt the notation of 5.1. If H is contained in some proper parabolic subgroup P of G, then by Corollary 3.4, G/H is birational to $(G/P)\times(P/H)$, and by Lemma 3.3, G/P is rational. Since $\dim(P/H) \leqslant \dim(G/H) - 1 = 10$, Theorem 5.9 shows that P/H is rational, and the rationality of G/H follows. Henceforth we assume H is not contained in any proper parabolic subgroup P of G; thus by Lemma 3.5, H is semisimple.

Proceeding as in the proof of Theorem 5.9, we see that G/H is rational if $(t_G - t_H) + (u_G - u_H) \leqslant 5$ or $u_G - u_H \leqslant 3$ or even $u_G - u_H = 4$ (by Lemma 5.8). In the remaining cases, our hypothesis $\dim(G/H) = 11$ together with the dimension formula $(*)$ in 5.1 implies that

$$(\, u_G - u_H \,,\, t_G - t_H \,) \quad \text{is equal to} \quad (5, 1).$$

By Lemma 5.2, we have $\dim(B^- \backslash G/U_H) = u_G - u_H$ which is $= 5$ here, so we may argue as in the proof of Lemma 5.8 to see that G/H is rational, except possibly

when

$$\dim(B^-\backslash G/U_H) = 5, \quad \dim(B^-\backslash G/B_H) = 4, \quad \dim(B^-\backslash G/H) = 3.$$

In this case, our hypothesis on the action of G on G/H together with Lemma 5.7 gives $e = t_H$ in the notation there, which by Lemma 5.10 means that $\dim(B^-\backslash G/B_H) = u_G - u_H - t_H$. Thus H is a semisimple group of rank $t_H = 1$ and hence of type A_1, and G is a semisimple group of rank $t_G = t_H + 1 = 2$, with $\dim G = \dim(G/H) + \dim H = 14$. By the classification of semisimple groups, this implies G is of type G_2, and we are in the possible exceptional situation of the proposition. □

Proposition 6.4. *Let G be a connected semisimple group, and let $H \subseteq G$ be a connected reductive closed subgroup. Suppose the natural left action of G on G/H has zero-dimensional kernel. If $\dim(G/H) = 12$, then G/H is a rational variety, except possibly when G is semisimple of type A_3 and H is semisimple of type A_1.*

Proof. Again we adopt the notation of 5.1, but note that $u(H) = 0$ by hypothesis here. As in the proof of the previous result, we see that G/H is rational if $(t_G - t_H) + (u_G - u_H) \leqslant 5$ or $u_G - u_H \leqslant 4$. In the remaining cases, our hypothesis $\dim(G/H) = 12$ together with the dimension formula $(*)$ in 5.1 implies that

$$(\, u_G - u_H \,,\, t_G - t_H \,) \quad \text{is equal to} \quad (6,0) \;\text{ or }\; (5,2).$$

Our hypothesis on the action of G on G/H together with Theorem 5.11 shows that G/H is rational if $12 = \dim(G/H) < t_G + t_H + 8$; henceforth we assume that $t_G + t_H \leqslant 4$.

In the first case above, we have $t_G = t_H \leqslant 2$. By the classification of semisimple groups, this implies that $\dim G \leqslant 14$, and hence $\dim H = \dim G - \dim(G/H) \leqslant 2$. This means that H is solvable, and so G/H is rational by Theorem 2.9.

In the second case above, we must have $t_H = 1$ and $t_G = t_H + 2 = 3$. Again, if H is solvable, then G/H is rational by Theorem 2.9; henceforth we assume that H is not solvable. This means that H is semisimple of rank $t_H = 1$ and hence of type A_1, and G is a semisimple group of rank $t_G = 3$, with $\dim G = \dim(G/H) + \dim H = 15$. By the classification of semisimple groups, this implies G is of type A_3, and we are in the possible exceptional situation of the proposition. □

In view of the possible exceptional situations in Propositions 6.2 and 6.3, our rationality results do not entirely cover the case when G is the simple group of type G_2, and we are thus led to pose the following:

Question 6.5. Let G be the 14-dimensional connected semisimple group of type G_2. Let $H \subseteq G$ be a connected semisimple closed subgroup of type A_1 which is not contained in a proper parabolic subgroup of G. Is the 11-dimensional homogeneous variety G/H a rational variety?

The question can be made a bit more precise. By the Jacobson-Morozov theorem, the semisimple closed subgroups of type A_1 in a semisimple group G are in bijection with the non-trivial unipotent elements in G. For $G \cong G_2$, there are four such unipotent conjugacy classes, of which the regular and the subregular unipotent classes are the ones corresponding to (G-conjugacy classes of) subgroups H of type A_1 which are not contained in any proper parabolic subgroup of G. If H corresponds to the subregular unipotent class, one knows that $H \cong \mathrm{PGL}_2 \cong \mathrm{SO}_3$ is contained in a maximal connected semisimple subgroup M of type A_2 in $G \cong G_2$ and that $M \cong \mathrm{SL}_3$, which is a special group (in the sense of Serre). By Lemma

2.3, this implies that G/H is birational to $(G/M) \times (M/H)$, and since both G/M and M/H are of dimension $\leqslant 10$ and hence rational by Theorem 5.9, it follows that G/H is also rational in this case. Thus the only outstanding case of the question, yielding the possible exceptional situation in Propositions 6.2 and 6.3, is when H corresponds to the regular unipotent class of $G \cong G_2$, i.e. when $H \cong \mathrm{PGL}_2$ arises as the image of SL_2 under its irreducible 7-dimensional representation to $G_2 \subseteq \mathrm{SO}_7$.[‡]

We conclude by the following result indicating the nature of a potential "minimal counter-example" for the rationality of homogeneous variety G/H.

Proposition 6.6. *Let G be a connected linear algebraic group, and let $H \subseteq G$ be a connected closed subgroup such that the natural left action of G on G/H is faithful. Suppose that the homogeneous variety G/H is not rational and that $\dim(G/H)$ is minimal among these non-rational varieties. Then:*

(a) *G is semisimple; H is semisimple and is not contained in any proper parabolic subgroup of G.*
(b) *H is of finite index in its normalizer $N_G(H)$ in G; consequently, the Tits fibration $G/H \to G/N_G(H)$ is a finite morphism.*
(c) *Let X be a smooth projective G-equivariant compactification of G/H with $D := X \smallsetminus (G/H)$ a simple normal crossing divisor. Then the rational map $\Phi_{|-(K_X+D)|}$ given by the complete linear system $|-(K_X + D)|$ is a generically finite map. In particular, $-(K_X + D)$ is a big divisor.*

Proof. Let $R := R(G)$ be the radical of G. By Lemma 2.10, G/H is birational to $G'/H' \times \mathbb{P}^s$ for some $s \leqslant \dim R$, where $G' = G/R$ is the maximal semisimple quotient of G, and $H' = H/(H \cap R)$ is the image of H in G'. Since G/H is non-rational, so is G'/H', whence the minimality of $\dim(G/H)$ implies that $s = 0$; i.e. G/H is birational to G'/H'. Since G acts faithfully on G/H by assumption, while R acts trivially on G'/H', it follows that R is trivial and G is semisimple.

Suppose H is contained in a proper parabolic subgroup P of G. Since $\dim(P/H)$ is strictly smaller than $\dim(G/H)$, the minimality of $\dim(G/H)$ forces P/H to be rational. But G/P is rational by Lemma 3.3, while by Corollary 3.4, G/H is birational to $(G/P) \times (P/H)$ and hence is rational, contradicting our assumption of the non-rationality of G/H. Hence H is not contained in any proper parabolic subgroup of G. By Lemma 3.5, it follows that H is semisimple. This proves part (a).

Suppose the closed subvariety $N_G(H)/H$ of G/H has positive dimension. Choose a connected 1-dimensional closed subgroup $\overline{M}$ in $N_G(H)/H$, and let $M \subseteq N_G(H)$ be its preimage in $N_G(H)$. By Lemma 2.6 applied to the natural right action of $\overline{M}$ on G/H, we see that G/H is birational to G/M or to $(G/M) \times \mathbb{P}^1$. Since $\dim(G/M)$ is strictly smaller than $\dim(G/H)$, the minimality of $\dim(G/H)$ forces G/M to be rational, but this implies that G/H is also rational, contradicting our assumption. Hence $N_G(H)/H$ is 0-dimensional, which proves part (b).

Let (X, D) be a smooth projective G-equivariant compactification of the positive-dimensional homogeneous variety G/H, and let $D = X \smallsetminus (G/H)$. As observed in [Br07, §2.1, Proof of Prop. 3.3.5 (iii)], the Tits fibration $G/H \to G/N_G(H)$ (as a

[‡]Likewise, a similar analysis shows that the only possible exceptional situation in Proposition 6.4 is when the semisimple group H of type A_1 corresponds via the Jacobson-Morozov theorem to the regular unipotent class of G of type A_3; i.e. H is the isomorphic image of SL_2 under its irreducible 4-dimensional representation to SL_4. In the "adjoint" case, the homogeneous space $\mathrm{PGL}_4/\mathrm{PGL}_2$ is known (cf. [PS85]) to be rational.

rational map from $X \supseteq G/H$) is given by $\Phi_{|V|}$, where

$$V := \mathrm{Im}(\wedge^{\dim G}(\mathrm{Lie}G) \longrightarrow H^0(X, -(K_X + D)))$$

is a non-zero vector space over k. The generical finiteness of $\Phi_{|V|}$ then implies the same for $\Phi_{|-(K_X+D)|}$, which proves part (c). □

Acknowledgements

The authors heartily thank their friends and colleagues, and especially the referee of an earlier version of this paper, for the Remarks 1.3 and 5.3, and also Proposition 3.6(d), which allows the case of type C in Theorem 3.1 to be included. The second author would like to offer his thanks to M. Brion for valuable discussions about the rationality of G/H when G is solvable and for reminding him of the reference to [GIT] in the argument for Lemma 3.5. He also thanks M. Chen for the warm hospitality during his visit to Fudan University while working on this paper. He is partially supported by an ARF of NUS.

References

[Bo91] Armand Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR1102012 (92d:20001)

[BdS49] A. Borel and J. De Siebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos* (French), Comment. Math. Helv. **23** (1949), 200–221. MR0032659 (11,326d)

[Br07] Michel Brion, *Log homogeneous varieties*, Proceedings of the XVIth Latin American Algebra Colloquium (Spanish), Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2007, pp. 1–39. MR2500349 (2010m:14063)

[BSU13] Michel Brion, Preena Samuel, and V. Uma, *Lectures on the structure of algebraic groups and geometric applications*, CMI Lecture Series in Mathematics, vol. 1, Hindustan Book Agency, New Delhi; Chennai Mathematical Institute (CMI), Chennai, 2013. MR3088271

[Ch54] C. Chevalley, *On algebraic group varieties*, J. Math. Soc. Japan **6** (1954), 303–324. MR0067122 (16,672g)

[CTS07] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *The rationality problem for fields of invariants under linear algebraic groups (with special regards to the Brauer group)*, Algebraic groups and homogeneous spaces, Tata Inst. Fund. Res. Stud. Math., Tata Inst. Fund. Res., Mumbai, 2007, pp. 113–186. MR2348904 (2008m:13010)

[CTKPR11] Jean-Louis Colliot-Thélène, Boris Kunyavskiĭ, Vladimir L. Popov, and Zinovy Reichstein, *Is the function field of a reductive Lie algebra purely transcendental over the field of invariants for the adjoint action?*, Compos. Math. **147** (2011), no. 2, 428–466, DOI 10.1112/S0010437X10005002. MR2776610 (2012d:14084)

[De70] Michel Demazure, *Sous-groupes algébriques de rang maximum du groupe de Cremona* (French), Ann. Sci. École Norm. Sup. (4) **3** (1970), 507–588. MR0284446 (44 #1672)

[GIT] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. MR1304906 (95m:14012)

[Ha84] W. Haboush, *Brauer groups of homogeneous spaces. I*, Methods in ring theory (Antwerp, 1983), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 129, Reidel, Dordrecht, 1984, pp. 111–144. MR770586 (86j:14047)

[Hu75] James E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Springer-Verlag, New York-Heidelberg, 1975. MR0396773 (53 #633)

[Le] S. P. Lehalleur, *Subgroups of maximal rank of reductive groups*, retrieved from: `http://www.math.ens.fr/~gille/pepin.pdf`, to appear in "Autour des schémas en groupes", Panoramas et Synthèses.

[Mi71] Takehiko Miyata, *Invariants of certain groups. I*, Nagoya Math. J. **41** (1971), 69–73. MR0272923 (42 #7804)

[MU83] Shigeru Mukai and Hiroshi Umemura, *Minimal rational threefolds*, Algebraic geometry (Tokyo/Kyoto, 1982), Lecture Notes in Math., vol. 1016, Springer, Berlin, 1983, pp. 490–518, DOI 10.1007/BFb0099976. MR726439 (85c:14027)

[PS85] Ragni Piene and Michael Schlessinger, *On the Hilbert scheme compactification of the space of twisted cubics*, Amer. J. Math. **107** (1985), no. 4, 761–774, DOI 10.2307/2374355. MR796901 (86m:14004)

[Re00] Z. Reichstein, *On the notion of essential dimension for algebraic groups*, Transform. Groups **5** (2000), no. 3, 265–304, DOI 10.1007/BF01679716. MR1780933 (2001j:20073)

[Ro56] Maxwell Rosenlicht, *Some basic theorems on algebraic groups*, Amer. J. Math. **78** (1956), 401–443. MR0082183 (18,514a)

[Ro57] Maxwell Rosenlicht, *Some rationality questions on algebraic groups*, Ann. Mat. Pura Appl. (4) **43** (1957), 25–50. MR0090101 (19,767h)

[Ro63] Maxwell Rosenlicht, *Questions of rationality for solvable algebraic groups over nonperfect fields* (English, with Italian summary), Ann. Mat. Pura Appl. (4) **61** (1963), 97–120. MR0158891 (28 #2113)

[Sa84] David J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), no. 1, 71–84, DOI 10.1007/BF01389135. MR751131 (85m:13006)

[SGA3-II] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux* (French), Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Dirigé par M. Demazure et A. Grothendieck. Lecture Notes in Mathematics, Vol. 152, Springer-Verlag, Berlin-New York, 1970. MR0274459 (43 #223b)

[Sp85] T. A. Springer, *Some results on algebraic groups with involutions*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 525–543. MR803346 (86m:20050)

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Singapore

E-mail address: cheewhye@nus.edu.sg

Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076, Singapore

E-mail address: matzdq@nus.edu.sg