

A BILATERAL EXTENSION OF THE q -SELBERG INTEGRAL

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ABSTRACT. A multi-dimensional bilateral q -series extending the q -Selberg integral is studied using concepts of *truncation*, *regularization* and *connection formulae*. Following Aomoto’s method, which involves regarding the q -series as a solution of a q -difference equation fixed by its asymptotic behavior, an infinite product evaluation is obtained. The q -difference equation is derived applying the *shifted symmetric polynomials* introduced by Knop and Sahi. As a special case of the infinite product formula, Askey–Evans’s q -Selberg integral evaluation and its generalization by Tarasov–Varchenko and Stokman is reclaimed, and an explanation in the context of Aomoto’s setting is thus provided.

1. INTRODUCTION

The Selberg integral [37] is a multi-dimensional generalization of the evaluation of the Euler beta integral in terms of products of gamma functions. It reads

$$(1.1) \quad \begin{aligned} & \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n z_i^{\alpha-1} (1-z_i)^{\beta-1} \prod_{1 \leq j < k \leq n} |z_j - z_k|^{2\tau} dz_1 dz_2 \cdots dz_n \\ &= \prod_{j=1}^n \frac{\Gamma(\alpha + (j-1)\tau) \Gamma(\beta + (j-1)\tau) \Gamma(j\tau)}{\Gamma(\alpha + \beta + (n+j-2)\tau) \Gamma(\tau)}. \end{aligned}$$

The Euler beta integral is fundamental to the theory of hypergeometric functions. And fundamental in the theory of hypergeometric functions is the notion of a q (or basic) generalization. From the 1980’s to 1990’s, a q -analog of the Selberg integral formula was established and proved by Askey [13], Habsieger [22], Kadell [31] and Evans [16].

Proposition 1.1. *Suppose $|q^\alpha| < 1$, $|q^\beta| < 1$ and τ is a positive integer. Let the q -integral be defined in terms of a sum as specified by (2.5) below. We have*

$$(1.2) \quad \begin{aligned} & \frac{1}{n!} \int_0^1 \cdots \int_0^1 \prod_{i=1}^n z_i^{\alpha-1} (qz_i)_{\beta-1} \prod_{\substack{1 \leq j < k \leq n \\ 1-\tau \leq l \leq \tau-1}} (z_j - q^l z_k) \prod_{1 \leq j < k \leq n} (z_j - z_k) d_q z_1 \cdots d_q z_n \\ &= q^{\alpha\tau \binom{n}{2} + 2\tau^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)\tau) \Gamma_q(\beta + (j-1)\tau) \Gamma_q(j\tau)}{\Gamma_q(\alpha + \beta + (n+j-2)\tau) \Gamma_q(\tau)}. \end{aligned}$$

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An important point for present purposes is that the above q -generalization of the Selberg integral is restricted to the case that the parameter τ is a positive integer.

On the other hand, in the 1990's, Aomoto gave a q -analog of (1.1) valid for general complex τ .

Proposition 1.2 ([8, p.121, Proposition 2]). *Let α, β and τ be complex numbers satisfying $|q^{\alpha+(i-1)\tau}| < 1$ for $i = 1, 2, \dots, n$. Then*

$$\begin{aligned}
 & \int_{z_1=0}^1 \int_{z_2=0}^{q^\tau z_1} \cdots \int_{z_n=0}^{q^\tau z_{n-1}} \prod_{i=1}^n z_i^\alpha \frac{(qz_i)_\infty}{(q^\beta z_i)_\infty} \\
 (1.3) \quad & \times \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(q^{1-\tau} z_k/z_j)_\infty}{(q^\tau z_k/z_j)_\infty} (z_j - z_k) \frac{d_q z_n}{z_n} \cdots \frac{d_q z_2}{z_2} \frac{d_q z_1}{z_1} \\
 & = q^{\alpha\tau \binom{n}{2} + 2\tau^2 \binom{n}{3}} \prod_{j=1}^n \frac{\Gamma_q(\alpha + (j-1)\tau) \Gamma_q(\beta + (j-1)\tau) \Gamma_q(j\tau)}{\Gamma_q(\alpha + \beta + (n+j-2)\tau) \Gamma_q(\tau)}.
 \end{aligned}$$

An immediate question presents itself: how are (1.2) and (1.3) related when in the latter τ is a positive integer? First, one should remark that in this case they both coincide with Selberg's formula (1.1) in the limit $q \rightarrow 1$. However, this fact aside, it soon becomes apparent that it is not possible to obtain one result from the other by analytic continuation. How to remedy this situation motivates the first study of this paper. Thus we present in Section 3 the more general theory of Aomoto relating to (1.3), which allows for the development of a theory of the *truncated Jackson integral of type A*. This theory is based on q -difference equations and asymptotic behaviors. In particular, by adding our supplementary results (see Lemmas 3.1 and 3.2), we are able to obtain Proposition 1.1 as a corollary of Proposition 1.2.

However our main purpose of this paper is not just a supplementary commentary to Aomoto's work. In the paper [13], Askey gave a conjecture for another q -extension of the Selberg integral, which is proved by Evans [17] using Anderson's method [1]. Askey–Evans's formula is written as follows:

Proposition 1.3 ([17, p.342, Theorem 1, (1.9)]). *If τ is a positive integer, then*

$$\begin{aligned}
 & \frac{1}{n!} \int_{x_1}^{x_2} \cdots \int_{x_1}^{x_2} \prod_{i=1}^n \left(\frac{qx_i}{x_1}\right)_{\alpha-1} \left(\frac{qx_i}{x_2}\right)_{\beta-1} \prod_{\substack{1 \leq j < k \leq n \\ 1-\tau \leq l \leq \tau-1}} (z_j - q^l z_k) \\
 (1.4) \quad & \times \prod_{1 \leq j < k \leq n} (z_j - z_k) d_q z_1 \cdots d_q z_n \\
 & = (-1)^{\tau \binom{n}{2}} q^{\tau^2 \binom{n}{3} - \binom{\tau}{2} \binom{n}{2}} \prod_{j=1}^n \left[\frac{\Gamma_q(\alpha + (j-1)\tau) \Gamma_q(\beta + (j-1)\tau) \Gamma_q(j\tau)}{\Gamma_q(\alpha + \beta + (n+j-2)\tau) \Gamma_q(\tau)} \right. \\
 & \quad \left. \times \frac{(x_1 x_2)^{1+(j-1)\tau}}{x_1 - x_2} \left(\frac{x_2}{x_1}\right)_{\alpha+(j-1)\tau} \left(\frac{x_1}{x_2}\right)_{\beta+(j-1)\tau} \right].
 \end{aligned}$$

This is the q -analog of the Selberg integral (1.1) whose integral area is transformed from $[0, 1]^n$ to $[x_1, x_2]^n$ by using the linear transformation $z_i \mapsto (x_2 - x_1)z_i + x_1$. This formula, like (1.2), was also formulated under the assumption the parameter τ is a positive integer. Tarasov and Varchenko [39] and Stokman [38] independently gave an extension of Askey–Evans's formula in the case of τ being an

arbitrary complex number (see (4.9) in Corollary 4.3), using a residue calculus on a certain contour integral (see also Gustafson's q -Selberg contour integral [20, 21]).

Our primary goal is to investigate the case of τ being an arbitrary complex number defining a bilateral extension of the q -Selberg integral, following Aomoto's method as presented in Section 3, i.e., the q -difference equations and its solutions fixed by the asymptotic behaviors. The main theorem of this paper, providing the solution of this problem, is Theorem 4.1. (Theorem 4.1 is equivalent to Theorem 4.12, whose expression seems to be simpler by the term *regularization*.) And as with our findings in Section 3 we can understand the formula (4.9) of Tarasov–Varchenko and Stokman as a special case (called the *truncation*) of the formula (4.6) in Theorem 4.1, and can also understand Askey–Evans's formula (1.4) as a special case of (4.12) in Corollary 4.8, i.e., the formula (4.6) with the restriction of τ being a positive integer. (See Section 4.1.) In particular, we can see that the degeneration occurs in the factors written by theta functions in the left-hand side of the formula (4.6). This means it is not so easy to guess the exact form of the formula (4.6) only from knowledge of Askey–Evans's formula (1.4) as a special case; rather the Aomoto viewpoint plays an essential role.

The method for proving the results in this paper is consistent with the concept introduced by Aomoto and Aomoto–Kato in the early 1990's in the series of papers [4–12]. Aomoto showed an isomorphism between a class of Jackson integrals of hypergeometric type, which he called the *q -analog de Rham cohomology* [4, 5], and a class of theta functions, i.e., holomorphic functions possessing a quasi-periodicity [7, Theorem 1]. This isomorphism indicates that it is essential to analyze the class of holomorphic functions as a counterpart of that of Jackson integrals in order to know the structure of q -hypergeometric functions, in particular, the meaning of known special formulae. In this paper the process to obtain the holomorphic functions through this isomorphism is called the *regularization*. When we fix a basis of the class of holomorphic functions as a linear space, an arbitrary function of the space can be expressed as a linear combination of the elements of the specific basis, which Aomoto called the *connection formula* [7, Theorem 3]. As its simplest examples, Ramanujan's ${}_1\psi_1$ summation formula and the q -Selberg integral [13, 16, 22, 31] have been explained. See the original literature [7, Examples 1, 2] and the recent review [28] for details. One way to choose a good basis is to fix it by its asymptotic behavior of a limiting process with respect to parameters included in the definition of the Jackson integral of hypergeometric type. And the asymptotic behavior can be calculated from the Jackson integrals possessing appropriate cycles which include their critical points. We call the process to fix the cycles *truncation*. (These cycles are called the *characteristic cycles* [12] or the *α -stable or α -unstable cycles* [6] by Aomoto. The meaning of " α " is mentioned in Section 3. The word *truncation* itself was first used by van Diejen in other context [14, 24].) The connection formula is also characterized as a formula showing that a multi-dimensional bilateral series giving a general solution of the q -difference equation of the Jackson integrals with respect to parameters is expressed as a linear combination of multi-dimensional unilateral series as special solutions, each fixed by their asymptotic behaviors [6, Theorem (4.2)]. (We can see different examples of q -difference equations and the connection formulae in [25, 26, 29, 30], and [29, 30] explain the Sears–Slater transformation for the very-well-poised q -hypergeometric series from the present viewpoint in the setting of BC type symmetry.)

The paper is organized as follows. After defining some basic terminology in Section 2, we first show the product expression of the q -Selberg integral along Aomoto’s setting (we called it the *Jackson integral of A-type*) using concepts of *truncation*, *regularization* and *connection formulae* in Section 3. Though the Jackson integral of A-type can be obtained from our other example, we explain it individually, because the Jackson integral of A-type has simpler structure than the other Jackson integral, it is instructive in outlining the concept of this paper, and it highlights the issue of the relationship between (1.2) and (1.3). Section 4 is devoted to explaining a bilateral extension of Askey–Evans’s q -Selberg integral. Its situation looks a little more complex than the case of the Jackson integral of A-type in their details, but still it is consistent with the outlines of the proofs for the product expressions of these sums. In the Appendix we explain the detail of the derivation of the q -difference equation. In particular, we applied the *shifted symmetric polynomials* introduced by Knop and Sahi [33] to the key lemma (Lemma A.5) for deriving the q -difference equation.

2. DEFINITION OF THE JACKSON INTEGRAL

Throughout this paper, we fix q as $0 < q < 1$ and use the symbols $(a)_\infty := \prod_{i=0}^\infty (1 - q^i a)$ and $(a)_N := (a)_\infty / (q^N a)_\infty$. We define $\theta(a)$ by $\theta(a) := (a)_\infty (q/a)_\infty$, which satisfies

$$(2.1) \quad \theta(qa) = -\theta(a)/a.$$

By repeated use of the latter, $\theta(a)$ satisfies

$$(2.2) \quad \theta(a)/\theta(q^m a) = (-a)^m q^{\binom{m}{2}} \quad \text{for } m \in \mathbb{Z}.$$

We define $\Gamma_q(x)$ by $\Gamma_q(x) := (1 - q)^{1-x} (q)_\infty / (q^x)_\infty$, which satisfies

$$\Gamma_q(x)\Gamma_q(1 - x) = \frac{(1 - q)(q)_\infty^2}{(q^x)_\infty (q^{1-x})_\infty} = (1 - q) \frac{(q)_\infty^2}{\theta(q^x)},$$

this being a q -analog of the relation $\Gamma(x)\Gamma(1 - x) = \pi / \sin \pi x$.

Let S_n be the symmetric group on $\{1, 2, \dots, n\}$. For a function $f(z) = f(z_1, z_2, \dots, z_n)$ on $(\mathbb{C}^*)^n$, we define an action of the symmetric group S_n on $f(z)$ by

$$(\sigma f)(z) := f(\sigma^{-1}(z)) = f(z_{\sigma(1)}, z_{\sigma(2)}, \dots, z_{\sigma(n)}) \quad \text{for } \sigma \in S_n.$$

We say that a function $f(z)$ on $(\mathbb{C}^*)^n$ is *symmetric* or *skew-symmetric* if $\sigma f(z) = f(z)$ or $\sigma f(z) = (\text{sgn } \sigma) f(z)$ for all $\sigma \in S_n$, respectively. We denote by $\mathcal{A}f(z)$ the alternating sum over S_n defined by

$$(2.3) \quad (\mathcal{A}f)(z) := \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma f(z),$$

which is skew-symmetric.

For $a, b \in \mathbb{C}$, we define

$$(2.4) \quad \int_a^b f(z) d_q z := \int_0^b f(z) d_q z - \int_0^a f(z) d_q z,$$

where

$$(2.5) \quad \int_0^a f(z) d_q z := (1 - q) \sum_{\nu=0}^\infty f(aq^\nu) aq^\nu,$$

which is called the *Jackson integral*. As $q \rightarrow 1$, $\int_a^b f(z)d_qz \rightarrow \int_a^b f(z)dz$ [2]. In this paper we use the Jackson integral of multiplicative measure, specified by

$$\int_0^a f(z) \frac{d_qz}{z} = (1 - q) \sum_{\nu=0}^{\infty} f(aq^\nu),$$

as is consistent with (2.5). Let \mathbb{N} be the set of non-negative integers. For a function $f(z) = f(z_1, \dots, z_n)$ on $(\mathbb{C}^*)^n$ and an arbitrary point $x = (x_1, \dots, x_n) \in (\mathbb{C}^*)^n$, we define the multiple Jackson integral as

$$(2.6) \quad \int_0^x f(z) \frac{d_qz_1}{z_1} \wedge \dots \wedge \frac{d_qz_n}{z_n} := (1 - q)^n \sum_{(\nu_1, \dots, \nu_n) \in \mathbb{N}^n} f(x_1q^{\nu_1}, \dots, x_nq^{\nu_n}).$$

In this paper we use the bilateral sum extending the Jackson integral (2.4)

$$(2.7) \quad \int_{a^\infty}^{b^\infty} f(z)d_qz := \int_0^{b^\infty} f(z)d_qz - \int_0^{a^\infty} f(z)d_qz,$$

where

$$\int_0^{a^\infty} f(z)d_qz := (1 - q) \sum_{\nu=-\infty}^{\infty} f(aq^\nu)aq^\nu, \text{ i.e., } \int_0^{a^\infty} f(z) \frac{d_qz}{z} := (1 - q) \sum_{\nu=-\infty}^{\infty} f(aq^\nu).$$

We also use the multiple bilateral sum extending the Jackson integral (2.6)

$$(2.8) \quad \int_0^{x^\infty} f(z) \frac{d_qz_1}{z_1} \wedge \dots \wedge \frac{d_qz_n}{z_n} := (1 - q)^n \sum_{(\nu_1, \dots, \nu_n) \in \mathbb{Z}^n} f(x_1q^{\nu_1}, \dots, x_nq^{\nu_n}),$$

which we also call the *Jackson integral*. By definition the Jackson integral (2.8) is invariant under the shift $x_i \rightarrow qx_i$, $1 \leq i \leq n$. While we can consider the limit $q \rightarrow 1$ for the Jackson integral (2.6) defined over \mathbb{N}^n , the Jackson integral (2.8) defined over \mathbb{Z}^n generally diverges if $q \rightarrow 1$. However, as we will see later, since the *truncation* of the Jackson integral (2.8) is corresponding to the sum (2.6) over \mathbb{N}^n , if we need to consider the limit $q \rightarrow 1$, we switch from (2.8) to (2.6) by the process of the truncation. For simplicity of notation, we use the symbol

$$\varpi_q = \frac{d_qz_1}{z_1} \wedge \dots \wedge \frac{d_qz_n}{z_n}.$$

3. AOMOTO'S q -EXTENSION OF THE SELBERG INTEGRAL

In this section we will review some known results in the context of Aomoto's q -extension of the Selberg integral.

3.1. Aomoto's setting. For $\alpha \in \mathbb{C}$, $a_1, b_1, t \in \mathbb{C}^*$, $z = (z_1, z_2, \dots, z_n) \in (\mathbb{C}^*)^n$, let $\Phi(z)$ and $\Delta(z)$ be specified by

$$(3.1) \quad \Phi(z) := \prod_{i=1}^n z_i^\alpha \frac{(qa_1^{-1}z_i)_\infty}{(b_1z_i)_\infty} \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(qt^{-1}z_k/z_j)_\infty}{(tz_k/z_j)_\infty},$$

$$(3.2) \quad \Delta(z) := \prod_{1 \leq i < j \leq n} (z_i - z_j),$$

where τ is given by $t = q^\tau$. For $x = (x_1, x_2, \dots, x_n) \in (\mathbb{C}^*)^n$ we define $I(x)$ by

$$(3.3) \quad I(x) = I(x_1, x_2, \dots, x_n) := \int_0^{x^\infty} \Phi(z)\Delta(z)\varpi_q,$$

which is called the *Jackson integral of A-type* in the context of [8]. For a general point $x \in (\mathbb{C}^*)^n$, excluding poles of $\Phi(z)$, $I(x)$ converges absolutely under the condition

$$(3.4) \quad |qa_1^{-1}b_1^{-1}| < |q^\alpha t^{2i-2}| < 1 \quad \text{for } i = 1, 2, \dots, n.$$

Let ζ be the point defined by

$$(3.5) \quad \zeta := (a_1, a_1t, a_1t^2, \dots, a_1t^{n-1}) \in (\mathbb{C}^*)^n,$$

and let Λ be the subset of \mathbb{Z}^n defined by

$$\Lambda := \{(\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{Z}^n; 0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_n\},$$

which is written as $\Lambda = \{\sum_{i=1}^n m_i \epsilon_i; m_i \in \mathbb{N}\} \cong \mathbb{N}^n$ where $\epsilon_i = \overbrace{(0, \dots, 0, 1, \dots, 1)}^{i-1} \in \mathbb{Z}^n$. Since $\Phi(\zeta q^\nu) = \Phi(a_1q^{\nu_1}, a_1tq^{\nu_2}, \dots, a_1t^{n-1}q^{\nu_n}) = 0$ if $\nu \notin \Lambda$, by definition $I(\zeta)$ is defined as the sum over the fan region $\Lambda \cong \mathbb{N}^n$. For this special point ζ , $I(\zeta)$ converges absolutely if

$$(3.6) \quad |q^\alpha t^{i-1}| < 1 \quad \text{for } i = 1, 2, \dots, n,$$

and we call $I(\zeta)$ the *truncated Jackson integral of A-type*. Note that $I(\zeta)$ is expressed as the following iterated q -integral form:

$$I(\zeta) = \int_{z_1=0}^{a_1} \int_{z_2=0}^{tz_1} \dots \int_{z_n=0}^{tz_{n-1}} \Phi(z)\Delta(z) \frac{d_q z_n}{z_n} \dots \frac{d_q z_2}{z_2} \frac{d_q z_1}{z_1}.$$

Although we regard all parameters as complex numbers throughout the paper, it is often very important to distinguish between the parameter τ being a positive integer or not, as we see in Propositions 1.1 and 1.3 for instance. We initially state a basic property of $I(x)$ under the assumption τ is not a positive integer. Let \mathbb{Z}_+ be the set of positive integers.

By definition the function $\Phi(z)$ satisfies the *quasi-symmetric property* that

$$\sigma\Phi(z) = U_\sigma(z)\Phi(z) \quad \text{for } \sigma \in S_n,$$

where

$$(3.7) \quad U_\sigma(z) := \prod_{\substack{1 \leq i < j \leq n \\ \sigma^{-1}(i) > \sigma^{-1}(j)}} \left(\frac{z_i}{z_j}\right)^{1-2\tau} \frac{\theta(q^{1-\tau} z_i/z_j)}{\theta(q^\tau z_i/z_j)},$$

which is invariant under the q -shift $z_i \rightarrow qz_i$. From (3.7) and $\sigma\Delta(z) = (\text{sgn } \sigma)\Delta(z)$, we have

$$(3.8) \quad \sigma I(x) = (\text{sgn } \sigma)U_\sigma(x)I(x).$$

Lemma 3.1. *Suppose $\tau \notin \mathbb{Z}_+$. If $x_i = x_j$ for some i and j ($1 \leq i < j \leq n$), then $I(x_1, \dots, x_n) = 0$.*

Proof. Set σ as the interchange of i and j . If we impose $x_i = x_j$, then $\sigma I(x) = I(x)$, so that we have $(1 + U_\sigma(x))I(x) = 0$ from (3.8). Since $(1 + U_\sigma(x)) \neq 0$, we obtain $I(x) = 0$. □

On the other hand, under the assumption that τ is a positive integer we generally have $I(x_1, \dots, x_n) \neq 0$ even if $x_i = x_j$ ($1 \leq i < j \leq n$). In particular, we have the following lemma.

Lemma 3.2. *Suppose that $\tau \in \mathbb{Z}_+$. For an arbitrary $x \in \mathbb{C}^*$,*

$$(3.9) \quad I(x, xt, \dots, xt^{n-1}) = \frac{I(x, x, \dots, x)}{n!}.$$

Proof. Under the condition $\tau \in \mathbb{Z}_+$, since

$$(3.10) \quad \begin{aligned} & \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(q^{1-\tau} z_k / z_j)_\infty}{(q^\tau z_k / z_j)_\infty} (z_j - z_k) \\ &= \prod_{\substack{1 \leq j < k \leq n \\ 1-\tau \leq l \leq \tau-1}} (z_j - q^l z_k) \prod_{1 \leq j < k \leq n} (z_j - z_k) \\ &= (-1)^{\tau \binom{n}{2}} q^{-\binom{\tau}{2} \binom{n}{2}} \prod_{1 \leq j < k \leq n} (z_j z_k)^\tau (z_j / z_k)_\tau (z_k / z_j)_\tau, \end{aligned}$$

the function $\Phi(z)\Delta(z)$ is symmetric. Therefore we obtain (3.9). □

Remark 3.3. As pointed out in Lemma 3.1, the right-hand side of (3.9) makes sense only when τ is a positive integer. However, as a function the left-hand side of (3.9) is defined continuously whether τ is a positive integer or not. Thus, as our basic strategy we first obtain several results for $I(x)$ under the condition $\tau \notin \mathbb{Z}_+$. Then the corresponding results in the cases $\tau \in \mathbb{Z}_+$ follow using analytic continuation. Furthermore, if necessary they will be rewritten using the relation (3.9), as we will see later.

We now state the formula corresponding to Proposition 1.2, extending τ from a positive integer to a complex number.

Proposition 3.4 (Aomoto). *Under the condition (3.6), the truncated Jackson integral $I(\zeta)$ is expressed as*

$$(3.11) \quad I(\zeta) = (1 - q)^n \prod_{j=1}^n (a_1 t^{j-1})^{\alpha+2(n-j)\tau} \frac{(q)_\infty (t)_\infty (q^\alpha a_1 b_1 t^{n+j-2})_\infty}{(t^j)_\infty (q^\alpha t^{j-1})_\infty (a_1 b_1 t^{j-1})_\infty}.$$

Proof. See the proof written after Lemma 3.14 in the next subsection. As an alternative proof, see also Corollary 4.25 below. □

Remark 3.5. If we substitute a_1 and b_1 as $a_1 \rightarrow 1$ and $b_1 \rightarrow q^\beta$, respectively, then (3.11) coincides with (1.3).

Moreover, we have the multiple bilateral sum version of the above formula as follows:

Proposition 3.6 (Aomoto). *Suppose $\tau \notin \mathbb{Z}_+$. Under the condition (3.4), $I(x)$ is expressed as a ratio of theta functions:*

$$(3.12) \quad I(x) = c_0 \prod_{i=1}^n x_i^{\alpha+2(n-i)\tau} \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(b_1 x_i)} \prod_{1 \leq j < k \leq n} \frac{\theta(x_k / x_j)}{\theta(tx_k / x_j)},$$

where c_0 is a constant independent of $x \in (\mathbb{C}^*)^n$, which is explicitly written as

$$(3.13) \quad c_0 = (1 - q)^n \prod_{j=1}^n \frac{(q)_\infty (qt^{-j})_\infty (qa_1^{-1} b_1^{-1} t^{-(j-1)})_\infty}{(qt^{-1})_\infty (q^\alpha t^{j-1})_\infty (q^{1-\alpha} a_1^{-1} b_1^{-1} t^{-(n+j-2)})_\infty}.$$

Remark 3.7. If $n = 1$, then (3.12) is equivalent to Ramanujan’s ${}_1\psi_1$ summation theorem. This is another multi-dimensional bilateral extension of Ramanujan’s ${}_1\psi_1$ summation theorem, which is different from the Milne–Gustafson summation theorem [19,35]. (See also [27] for the explanation of the Milne–Gustafson summation theorem along the context of this paper.) Another class of extension relates to the theory of Macdonald polynomials; see for example [32,36,40] as cited in [41].

Proof. Taking account of the poles of $\Phi(z)$, we have the expression

$$(3.14) \quad I(x) = f(x) \prod_{i=1}^n \frac{x_i^\alpha}{\theta(b_1 x_i)} \prod_{1 \leq j < k \leq n} \frac{x_j^{2\tau-1}}{\theta(tx_k/x_j)},$$

where $f(x)$ is some holomorphic function on $(\mathbb{C}^*)^n$. Under the condition $\tau \notin \mathbb{Z}_+$, from Lemma 3.1, $I(x)$ is divisible by $\prod_{1 \leq i < j \leq n} x_i \theta(x_j/x_i)$. This indicates that $f(x) = g(x) \prod_{1 \leq i < j \leq n} x_i \theta(x_j/x_i)$, where $g(x)$ is a holomorphic function on $(\mathbb{C}^*)^n$. Taking account of the q -periodicity of both sides of (3.14), we have

$$T_{x_i} g(x) = -\frac{g(x)}{q^\alpha b_1 t^{n-1} x_i} \quad \text{for } i = 1, 2, \dots, n,$$

where T_{x_i} means the shift operator of $x_i \rightarrow qx_i$, i.e., $T_{x_i} g(\dots, x_i, \dots) = g(\dots, qx_i, \dots)$. Then $g(x)$ is uniquely determined as $g(x) = c_0 \prod_{i=1}^n \theta(q^\alpha b_1 t^{n-1} x_i)$, where c_0 is a constant independent of x . Therefore we obtain the expression (3.12). Comparing (3.11) with (3.12) of $x = \zeta$, the explicit form of c_0 is obtained as (3.13). \square

Combining Lemma 3.2 and Proposition 3.6, we obtain a multiple bilateral summation formula extending (1.2) in Proposition 1.1.

Corollary 3.8. *Suppose $\tau \in \mathbb{Z}_+$. For an arbitrary $x \in \mathbb{C}^*$,*

$$(3.15) \quad \frac{I(x, x, \dots, x)}{n!} = c_1 \prod_{i=1}^n (xt^{i-1})^{\alpha+2(n-i)\tau} \frac{\theta(q^\alpha b_1 t^{n+i-2} x)}{\theta(b_1 t^{i-1} x)},$$

where c_1 is given by

$$(3.16) \quad c_1 = (1 - q)^n \prod_{j=1}^n \frac{(q)_\infty(t)_\infty (qa_1^{-1} b_1^{-1} t^{-(j-1)})_\infty}{(t^j)_\infty (q^\alpha t^{j-1})_\infty (q^{1-\alpha} a_1^{-1} b_1^{-1} t^{-(n+j-2)})_\infty}.$$

Proof. First we temporarily assume $\tau \notin \mathbb{Z}_+$. From Proposition 3.6, we immediately have the following for the point (x, xt, \dots, xt^{n-1}) :

$$\begin{aligned} I(x, xt, \dots, xt^{n-1}) &= c_0 \prod_{i=1}^n (xt^{i-1})^{\alpha+2(n-i)\tau} \frac{\theta(q^\alpha b_1 t^{n+i-2} x) \theta(t)}{\theta(b_1 t^{i-1} x) \theta(t^i)} \\ &= c_1 \prod_{i=1}^n (xt^{i-1})^{\alpha+2(n-i)\tau} \frac{\theta(q^\alpha b_1 t^{n+i-2} x)}{\theta(b_1 t^{i-1} x)}, \end{aligned}$$

where c_1 is written as (3.16). Then, by analytic continuation, the above formula is valid for $\tau \in \mathbb{Z}_+$. Using Lemma 3.2, we obtain (3.15). \square

Remark 3.9. (3.15) of Corollary 3.8 is also expressed as

$$\frac{I(x, x, \dots, x)}{n!} = \frac{I(a_1, a_1, \dots, a_1)}{n!} \prod_{i=1}^n \left(\frac{x}{a_1}\right)^{\alpha+(n-1)\tau} \frac{\theta(q^\alpha x b_1 t^{n+i-2}) \theta(a_1 b_1 t^{i-1})}{\theta(q^\alpha a_1 b_1 t^{n+i-2}) \theta(x b_1 t^{i-1})},$$

which is the connection between $I(x, x, \dots, x)$ and $I(a_1, a_1, \dots, a_1)$.

As a special case of Corollary 3.8, we immediately have the formula (1.2) in Proposition 1.1.

Corollary 3.10 (Askey, Habsieger, Kadell, Evans). *Suppose $\tau \in \mathbb{Z}_+$. Then*

$$\frac{I(a_1, a_1, \dots, a_1)}{n!} = (1 - q)^n \prod_{j=1}^n (a_1 t^{j-1})^{\alpha+2(n-j)\tau} \frac{(q)_\infty (t)_\infty (q^\alpha a_1 b_1 t^{n+j-2})_\infty}{(t^j)_\infty (q^\alpha t^{j-1})_\infty (a_1 b_1 t^{j-1})_\infty}.$$

Remark 3.11. If we substitute a_1 and b_1 as $a_1 \rightarrow 1$ and $b_1 \rightarrow q^\beta$, respectively, then the above formula coincides with (1.2).

3.2. q -difference equation with respect to α . In this subsection we derive the q -difference equation with respect to α satisfied by $I(x)$. We use $I(\alpha; x)$ instead of $I(x)$ to highlight the α dependence. The following lemma is known as Aomoto’s method [2, 3, 18].

Lemma 3.12 (Aomoto). *Let $e_i(z), i = 0, 1, \dots, n$, be the elementary symmetric polynomials, i.e.,*

$$e_r(z) := \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} z_{i_1} z_{i_2} \dots z_{i_r} \quad \text{for } r = 1, 2, \dots, n,$$

and $e_0(z) := 1$. Then

$$(3.17) \quad \int_0^{x\infty} e_i(z) \Phi(z) \Delta(z) \varpi_q = \frac{a_1 t^{i-1} (1 - t^{n-i+1}) (1 - q^\alpha t^{n-i})}{(1 - t^i) (1 - q^\alpha a_1 b_1 t^{2n-i-1})} \int_0^{x\infty} e_{i-1}(z) \Phi(z) \Delta(z) \varpi_q.$$

Proof. See [18, Chapter 4, Exercises 4.6, q. 2]. □

Since we have

$$I(\alpha + 1; x) = \int_0^{x\infty} z_1 z_2 \dots z_n \Phi(z) \Delta(z) \varpi_q,$$

by definition, using (3.17) repeatedly we immediately obtain the q -difference equation with respect to α .

Corollary 3.13. *The recurrence relation for $I(\alpha; x)$ is given by*

$$(3.18) \quad I(\alpha + 1; x) = I(\alpha; x) \prod_{i=1}^n \frac{a_1 t^{i-1} (1 - q^\alpha t^{i-1})}{1 - q^\alpha a_1 b_1 t^{n+i-2}}.$$

By definition $I(\zeta)$ is a sum over $\Lambda \cong \mathbb{N}^n$, while $I(x)$ is generally the sum over the lattice \mathbb{Z}^n . It has the advantage of simplifying the computation of the $\alpha \rightarrow +\infty$ asymptotic behavior, as will be seen below. (The multiplicative lattice $\{(x_1 q^{\nu_1}, \dots, x_n q^{\nu_n}) \in (\mathbb{C}^*)^n; (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n\}$ is called the q -cycle [5] of $I(x)$, while the set $\{(a_1 q^{\nu_1}, a_1 t q^{\nu_2}, \dots, a_1 t^{n-1} q^{\nu_n}) \in (\mathbb{C}^*)^n; (\nu_1, \dots, \nu_n) \in \Lambda\}$ as the support of $I(\zeta)$ is called the α -stable cycle in [6, 12].)

Lemma 3.14. *The asymptotic behavior of $I(\alpha + N; \zeta)$ as $N \rightarrow +\infty$ is given by*

$$(3.19) \quad I(\alpha + N; \zeta) \sim (1 - q)^n \prod_{i=1}^n (a_1 t^{i-1})^{\alpha+2(n-i)\tau+N} \frac{(q)_\infty (t)_\infty}{(a_1 b_1 t^{i-1})_\infty (t^i)_\infty} \quad (N \rightarrow +\infty).$$

Proof. Since $\Phi(\zeta q^\nu) = \Phi(a_1 q^{\nu_1}, a_1 t q^{\nu_2}, \dots, a_1 t^{n-1} q^{\nu_n}) = 0$ if $\nu \notin \Lambda$, by definition $I(\zeta)$ is written as

$$I(\alpha + N; \zeta) = (1 - q)^n \sum_{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_n} \prod_{i=1}^n (a_1 t^{i-1} q^{\nu_i})^{\alpha + 2(n-i)\tau + N} \frac{(t^{i-1} q^{1+\nu_i})_\infty}{(a_1 b_1 t^{i-1} q^{\nu_i})_\infty} \\ \times \prod_{1 \leq j < k \leq n} \frac{(t^{k-j-1} q^{1+\nu_k-\nu_j})_\infty}{(t^{k-j+1} q^{\nu_k-\nu_j})_\infty} (1 - t^{k-j} q^{\nu_k-\nu_j}),$$

so that the leading term of the asymptotic behavior of $I(\alpha + N; \zeta)$ as $N \rightarrow +\infty$ is given by the term corresponding to $(\nu_1, \dots, \nu_n) = (0, \dots, 0)$ in the above sum, which is (3.19). □

Proof of Proposition 3.4. By repeated use of the recurrence relation (3.18), we have

$$I(\alpha; x) = I(\alpha + N; x) \prod_{i=1}^n \frac{(q^\alpha a_1 b_1 t^{n+i-2})_N}{(a_1 t^{i-1})_N (q^\alpha t^{i-1})_N}.$$

If we put $x = \zeta$ and take $N \rightarrow +\infty$, we obtain

$$(3.20) \quad I(\alpha; \zeta) = \lim_{N \rightarrow \infty} \frac{I(\alpha + N; \zeta)}{\prod_{i=1}^n (a_1 t^{i-1})_N} \times \prod_{i=1}^n \frac{(q^\alpha a_1 b_1 t^{n+i-2})_\infty}{(q^\alpha t^{i-1})_\infty},$$

which coincides with the right-hand side of (3.11) if we use (3.19). This means that the truncated Jackson integral $I(\zeta)$ is the special solution of the q -difference equation (3.18), fixed by the asymptotic behavior (3.19) as $\alpha \rightarrow +\infty$. □

3.3. Regularization and connection formula. Let $\mathcal{I}(x)$ and $h(x)$ be the functions defined by

$$(3.21) \quad \mathcal{I}(x) = \frac{I(x)}{h(x)} \quad \text{where} \quad h(x) := \prod_{i=1}^n \frac{x_i^\alpha}{\theta(b_1 x_i)} \prod_{1 \leq j < k \leq n} x_j^{2\tau} \frac{\theta(x_k/x_j)}{\theta(tx_k/x_j)}.$$

We call $\mathcal{I}(x)$ the *regularized Jackson integral* of $I(x)$. Since the trivial poles and zeros of $I(x)$ are canceled out by multiplying together $1/h(x)$ and $I(x)$, we have the following.

Lemma 3.15. *The regularization $\mathcal{I}(x)$ is holomorphic on $(\mathbb{C}^*)^n$ and symmetric.*

Proof. From the expression (3.1) of $\Phi(z)$ as integrand of (3.3), the function $I(x)$ has the poles lying only in the set

$$\left\{ x = (x_1, x_2, \dots, x_n) \in (\mathbb{C}^*)^n; \prod_{i=1}^n \theta(b_1 x_i) \prod_{1 \leq i < j \leq n} \theta(tx_j/x_i) = 0 \right\}.$$

Moreover, from Lemma 3.1, $I(x)$ is divisible by $x_j \theta(x_i/x_j)$. We therefore obtain

$$I(x) = \mathcal{I}(x)h(x),$$

where $\mathcal{I}(x)$ is some holomorphic function on $(\mathbb{C}^*)^n$. Since $h(x)$ also satisfies $\sigma h(x) = (\text{sgn } \sigma) U_\sigma(x) h(x)$ as (3.8), $\mathcal{I}(x)$ is symmetric. □

From Proposition 3.6 the regularization $\mathcal{I}(x)$ is written as

$$(3.22) \quad \mathcal{I}(x) = c_0 \prod_{i=1}^n \theta(q^\alpha b_1 t^{n-1} x_i).$$

Lemma 3.16 (Connection formula). *For an arbitrary $x, y \in (\mathbb{C}^*)^n$, the connection formula between $\mathcal{I}(x)$ and $\mathcal{I}(y)$ is written as*

$$(3.23) \quad \mathcal{I}(x) = \mathcal{I}(y) \prod_{i=1}^n \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(q^\alpha b_1 t^{n-1} y_i)}.$$

In particular, if we set $y = \zeta \in (\mathbb{C}^*)^n$, then

$$(3.24) \quad \mathcal{I}(x) = \mathcal{I}(\zeta) \prod_{i=1}^n \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(q^\alpha a_1 b_1 t^{n+i-2})}.$$

Proof. From (3.22) we immediately have (3.23), and recalling (3.5) gives (3.24). \square

Remark 3.17. In the equation (3.24), if we switch the symbols from $\mathcal{I}(x)$ to $I(x)$ we obtain

$$(3.25) \quad I(x) = I(\zeta) \frac{h(x)}{h(\zeta)} \prod_{i=1}^n \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(q^\alpha a_1 b_1 t^{n+i-2})},$$

which is also the connection formula between a solution $I(x)$ of the q -difference equation (3.18) and the special solution $I(\zeta)$ fixed by its asymptotic behavior (3.19) as $\alpha \rightarrow +\infty$. In addition, its connection coefficient is written as a ratio of theta functions (i.e., that of q -gamma functions) and is of course invariant under the shift $\alpha \rightarrow \alpha + 1$. From the evaluation (3.11) of $I(\zeta)$, the connection formula (3.25) is another expression for the product formula (3.12) in Proposition 3.6.

If we set

$$(3.26) \quad \beta := 1 - \alpha_1 - \beta_1 - 2(n - 1)\tau - \alpha,$$

where α_1 and β_1 are given by $a_1 = q^{\alpha_1}, b_1 = q^{\beta_1}$, after rearrangement, the formula (3.12) is also expressed as the following Macdonald-type sum, whose value is given by an x -independent constant [14, 23, 34].

Proposition 3.18. *Under the condition $a_1 b_1 t^{2n-2} q^{\alpha+\beta} = q$,*

$$(3.27) \quad \begin{aligned} & \int_0^{x_\infty} \prod_{i=1}^n \frac{(qa_1^{-1} z_i)_\infty (qb_1^{-1} z_i^{-1})_\infty}{(q^\alpha b_1 t^{n-1} z_i)_\infty (q^\beta a_1 t^{n-1} z_i^{-1})_\infty} \\ & \times \prod_{1 \leq j < k \leq n} \frac{(qt^{-1} z_j / z_k)_\infty (qt^{-1} z_k / z_j)_\infty}{(qz_j / z_k)_\infty (qz_k / z_j)_\infty} \frac{d_q z_1}{z_1} \wedge \dots \wedge \frac{d_q z_n}{z_n} \\ & = (1 - q)^n \prod_{j=1}^n \frac{(q)_\infty (qt^{-j})_\infty (qa_1^{-1} b_1^{-1} t^{-(j-1)})_\infty}{(qt^{-1})_\infty (q^\alpha t^{j-1})_\infty (q^\beta t^{j-1})_\infty}. \end{aligned}$$

Proof. Since $h(x) \prod_{i=1}^n \theta(q^\alpha b_1 t^{n-1} x_i)$ is invariant under the q -shift $x_i \rightarrow qx_i$, from (3.22) we have

$$\int_0^{x_\infty} \frac{\Phi(z) \Delta(z)}{h(z) \prod_{i=1}^n \theta(q^\alpha b_1 t^{n-1} z_i)} \frac{d_q z_1}{z_1} \wedge \dots \wedge \frac{d_q z_n}{z_n} = c_0,$$

so that

$$\int_0^{x\infty} \prod_{i=1}^n \frac{(qa_1^{-1}z_i)_\infty (qb_1^{-1}z_i^{-1})_\infty}{\theta(q^\alpha b_1 t^{n-1} z_i)} \times \prod_{1 \leq j < k \leq n} \frac{(qt^{-1}z_j/z_k)_\infty (qt^{-1}z_k/z_j)_\infty}{(qz_j/z_k)_\infty (qz_k/z_j)_\infty} \frac{d_q z_1}{z_1} \wedge \dots \wedge \frac{d_q z_n}{z_n} = c_0,$$

which is rewritten as (3.27) using (3.13) under the condition (3.26). □

As a corollary, it is confirmed that the following identity for a contour integral is equivalent to the formula (3.27) of the special case $x = \zeta$.

Corollary 3.19. *Let \mathbb{T}^n be the direct product of the unit circle, that is, $\mathbb{T}^n := \{(z_1, \dots, z_n) \in \mathbb{C}^n; |z_i| = 1\}$. Suppose that $|a_1| < 1$, $|b_1| < 1$, $|t| < 1$ and $a_1 b_1 t^{2n-2} q^{\alpha+\beta} = q$. Then*

$$(3.28) \quad \left(\frac{1}{2\pi\sqrt{-1}}\right)^n \frac{1}{n!} \int_{\mathbb{T}^n} \prod_{i=1}^n \frac{(q^\alpha t^{n-1} a_1 z_i^{-1})_\infty (q^\beta t^{n-1} b_1 z_i)_\infty}{(a_1 z_i^{-1})_\infty (b_1 z_i)_\infty} \times \prod_{1 \leq j < k \leq n} \frac{(z_j/z_k)_\infty (z_k/z_j)_\infty}{(tz_j/z_k)_\infty (tz_k/z_j)_\infty} \frac{dz_1}{z_1} \dots \frac{dz_n}{z_n} = \prod_{i=1}^n \frac{(t)_\infty (q^{1-\beta} t^{-(i-1)})_\infty (q^{1-\alpha} t^{-(i-1)})_\infty}{(q)_\infty (t^i)_\infty (a_1 b_1 t^{i-1})_\infty}.$$

Proof. This follows by a residue calculation using (3.27) of the case $x = \zeta$. □

3.4. Dual expression of the Jackson integral $I(x)$. For an arbitrary $x = (x_1, x_2, \dots, x_n) \in (\mathbb{C}^*)^n$ we specify x^{-1} as

$$(3.29) \quad x^{-1} := (x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}) \in (\mathbb{C}^*)^n.$$

For the point $\bar{\zeta} = (b_1, b_1 t, \dots, b_1 t^{n-1}) \in (\mathbb{C}^*)^n$, if we set $y = \bar{\zeta}^{-1}$ in the connection formula (3.23), then we obtain the expression

$$(3.30) \quad \mathcal{I}(x) = \mathcal{I}(\bar{\zeta}^{-1}) \prod_{i=1}^n \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(q^\alpha t^{i-1})}.$$

Since $x = \bar{\zeta}^{-1} = (b_1^{-1}, b_1^{-1} t^{-1}, \dots, b_1^{-1} t^{-(n-1)})$ is a pole of the function $I(x)$ by definition, $\mathcal{I}(\bar{\zeta}^{-1})$ no longer makes sense. However, the regularization $\mathcal{I}(\bar{\zeta}^{-1})$ appearing on the right-hand side of (3.30) still has meaning as a special value of a holomorphic function. We will show a way to realize the regularization $\mathcal{I}(\bar{\zeta}^{-1})$ as a computable object by another Jackson integral. For this purpose, let $\bar{\Phi}(z)$ be the function specified by

$$(3.31) \quad \bar{\Phi}(z) := \prod_{i=1}^n z_i^{1-\alpha_1-\beta_1-2(n-1)\tau-\alpha} \frac{(qb_1^{-1}z_i)_\infty}{(a_1 z_i)_\infty} \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(qt^{-1}z_k/z_j)_\infty}{(tz_k/z_j)_\infty},$$

where α_1 and β_1 are given by $a_1 = q^{\alpha_1}$, $b_1 = q^{\beta_1}$. For $x = (x_1, x_2, \dots, x_n) \in (\mathbb{C}^*)^n$, we define the sum $\bar{I}(x)$ by

$$(3.32) \quad \bar{I}(x) := \int_0^{x\infty} \bar{\Phi}(z) \Delta(z) \varpi_q,$$

which converges absolutely under the condition (3.4). We call $\bar{I}(x)$ the *dual Jackson integral of $I(x)$* , and call $\bar{I}(\bar{\zeta})$ its *truncation*. Note that the sum $I(x)$ transforms to its dual $\bar{I}(x)$ if we interchange the parameters as

$$(3.33) \quad \alpha \leftrightarrow \beta \quad \text{and} \quad a_1 \leftrightarrow b_1,$$

where β is specified by (3.26). We also define the *regularization* $\bar{\mathcal{I}}(x)$ of $\bar{I}(x)$ as

$$(3.34) \quad \bar{\mathcal{I}}(x) = \frac{\bar{I}(x)}{\bar{h}(x)} \quad \text{where} \quad \bar{h}(x) := \prod_{i=1}^n \frac{x_i^{1-\alpha_1-\beta_1-2(n-1)\tau-\alpha}}{\theta(a_1x_i)} \prod_{1 \leq j < k \leq n} x_j^{2\tau} \frac{\theta(x_k/x_j)}{\theta(tx_k/x_j)}.$$

In the same manner as Lemma 3.15, we can confirm that the function $\bar{\mathcal{I}}(x)$ is also holomorphic and symmetric.

Lemma 3.20 (Reflective equation). *The connection between $I(x)$ and $\bar{I}(x)$ is*

$$(3.35) \quad I(x) = \frac{h(x)}{\bar{h}(x^{-1})} \bar{I}(x^{-1}),$$

where x^{-1} is specified as in (3.29) and

$$\frac{h(x)}{\bar{h}(x^{-1})} = (-1)^{\binom{n}{2}} \prod_{i=1}^n x_i^{1-\alpha_1-\beta_1} \frac{\theta(qa_1^{-1}x_i)}{\theta(b_1x_i)} \prod_{1 \leq j < k \leq n} \left(\frac{x_k}{x_j}\right)^{1-2\tau} \frac{\theta(qt^{-1}x_k/x_j)}{\theta(tx_k/x_j)}.$$

In other words, the relation between $\mathcal{I}(x)$ and $\bar{\mathcal{I}}(x)$ is

$$(3.36) \quad \mathcal{I}(x) = \bar{\mathcal{I}}(x^{-1}).$$

Proof. From the definitions (3.21) and (3.34) the ratio $h(x)/\bar{h}(x^{-1})$ is written as in (3.35). Since $\Delta(z) = (-1)^{\binom{n}{2}}(z_1z_2 \cdots z_n)^{n-1}\Delta(z^{-1})$, from (3.1), (3.2), (3.31), we have

$$(3.37) \quad \Phi(z)\Delta(z) = \frac{h(z)}{\bar{h}(z^{-1})}\bar{\Phi}(z^{-1})\Delta(z^{-1}).$$

Also since $h(z)/\bar{h}(z^{-1})$ is invariant under the shift $z_i \rightarrow qz_i$, by the definitions (3.3) and (3.32) of $I(z)$ and $\bar{I}(z)$, the connection (3.35) between $I(z)$ and its dual $\bar{I}(z)$ is derived from (3.37). □

We use $\bar{I}(\alpha; x)$ instead of $\bar{I}(x)$ to see the α dependence. From (3.35), the recurrence relation for $\bar{I}(\alpha; x)$ is completely the same as (3.18) of $I(\alpha; x)$.

Lemma 3.21. *The function $\bar{I}(\alpha; x)$ also satisfies the recurrence relation (3.18) of $I(\alpha; x)$ and is rewritten as*

$$(3.38) \quad \bar{I}(\alpha; x) = \bar{I}(\alpha - 1; x) \prod_{i=1}^n \frac{1 - q^{1-\alpha}t^{-(i-1)}}{b_1t^{i-1}(1 - q^{1-\alpha}a_1^{-1}b_1^{-1}t^{-(n+i-2)})}.$$

We saw above that although $I(\bar{\zeta}^{-1})$ no longer makes sense, its regularization $\mathcal{I}(\bar{\zeta}^{-1})$ still has meaning as a special value of a holomorphic function, and $\mathcal{I}(\bar{\zeta}^{-1})$ is evaluated by the dual integral $\bar{\mathcal{I}}(\bar{\zeta})$ via the reflective equation (3.36). Moreover, by definition the regularization $\bar{\mathcal{I}}(\bar{\zeta})$ itself is calculated using $\bar{I}(\bar{\zeta})$, which is then normally defined as a truncated Jackson integral. Though we already know the value of $\mathcal{I}(\bar{\zeta}^{-1})$ through the connection formula (3.24) with $x = \bar{\zeta}^{-1}$, the point is that we can calculate $\mathcal{I}(\bar{\zeta}^{-1})$ directly from $\bar{I}(\bar{\zeta})$. This requires the leading term of its asymptotic behavior as $\alpha \rightarrow -\infty$.

Corollary 3.22. *The asymptotic behavior of $\bar{I}(\alpha - N; \bar{\zeta})$ as $N \rightarrow +\infty$ is written as*

$$(3.39) \quad \begin{aligned} & \bar{I}(\alpha - N; \bar{\zeta}) \\ & \sim (1 - q)^n \prod_{i=1}^n (b_1 t^{i-1})^{1-\alpha_1-\beta_1-2(i-1)\tau-\alpha+N} \frac{(q)_\infty(t)_\infty}{(a_1 b_1 t^{i-1})_\infty (t^i)_\infty} \quad (N \rightarrow +\infty). \end{aligned}$$

Moreover, by repeated use of (3.38), the truncated Jackson integral $\bar{I}(\bar{\zeta})$ is written as

$$(3.40) \quad \bar{I}(\bar{\zeta}) = (1 - q)^n \prod_{i=1}^n \frac{(b_1 t^{i-1})^{1-\alpha_1-\beta_1-2(i-1)\tau-\alpha} (q)_\infty(t)_\infty (q^{1-\alpha} t^{-(i-1)})_\infty}{(t^i)_\infty (q^{1-\alpha} a_1^{-1} b_1^{-1} t^{-(n+i-2)})_\infty (a_1 b_1 t^{i-1})_\infty}.$$

Proof. Using Lemma 3.21, the arguments are completely parallel to Lemma 3.14 and (3.20). Actually, using the rule (3.33), if we substitute a_1 , b_1 and α in $\Phi(z)$ of (3.1) by b_1 , a_1 and β , respectively, then $\Phi(z)$ transforms to $\bar{\Phi}(z)$ in (3.31), so that we obtain the same result as Lemma 3.14 and Proposition 3.4 with these substitutions. □

From (3.23) and (3.36), for $x, y \in (\mathbb{C}^*)^n$ we have the connection formula between $\mathcal{I}(x)$ and $\bar{\mathcal{I}}(y)$ as

$$\mathcal{I}(x) = \bar{\mathcal{I}}(y) \prod_{i=1}^n \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(q^\alpha b_1 t^{n-1} y_i^{-1})}.$$

In particular, if $y = \bar{\zeta}$, then we have

$$\mathcal{I}(x) = \bar{\mathcal{I}}(\bar{\zeta}) \prod_{i=1}^n \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(q^\alpha t^{i-1})}.$$

If we switch the symbols from $\mathcal{I}(x)$ and $\bar{\mathcal{I}}(\bar{\zeta})$ to $I(x)$ and $\bar{I}(\bar{\zeta})$, respectively, then we obtain

$$(3.41) \quad I(x) = \bar{I}(\bar{\zeta}) \frac{h(x)}{h(\bar{\zeta})} \prod_{i=1}^n \frac{\theta(q^\alpha b_1 t^{n-1} x_i)}{\theta(q^\alpha t^{i-1})}.$$

We once again obtain the connection formula between a solution $I(x)$ of the q -difference equation (3.18) and the special solution $\bar{I}(\bar{\zeta})$ fixed by its asymptotic behavior (3.39) as $\alpha \rightarrow -\infty$ as a counterpart of the formula (3.25) of the case $\alpha \rightarrow +\infty$.

The connection formula (3.41) with (3.40) is also another expression for the product formula (3.12) in Proposition 3.6, like the formula (3.25).

Remark 3.23. The truncated Jackson integrals $I(\zeta)$ and $\bar{I}(\bar{\zeta})$ both satisfy the q -difference equation (3.18) with respect to α . $I(\zeta)$ is the special solution fixed by the asymptotic behavior (3.19) as $\alpha \rightarrow +\infty$. On the other hand, $\bar{I}(\bar{\zeta})$ is the solution fixed by the asymptotic behavior (3.39) as $\alpha \rightarrow -\infty$. The connection formula (3.41) shows that

$$(3.42) \quad I(\zeta) = \bar{I}(\bar{\zeta}) \prod_{i=1}^n \frac{(a_1 t^{i-1})^{\alpha+2(n-i)\tau} \theta(q^\alpha a_1 b_1 t^{n+i-2})}{(b_1 t^{i-1})^{1-\alpha_1-\beta_1-2(i-1)\tau-\alpha} \theta(q^\alpha t^{i-1})},$$

which connects $I(\zeta)$ and $\bar{I}(\bar{\zeta})$ by the q -periodic function of the right-hand side. This formula is explained like the formula $\Gamma(\alpha)\Gamma(1 - \alpha) = \pi/\sin \pi\alpha$, which indicates that $\Gamma(\alpha)$ and $1/\Gamma(1 - \alpha)$ are solutions of the difference equation $f(\alpha + 1) = \alpha f(\alpha)$

and they are fixed by the specific asymptotic behaviors (i.e., Stirling’s formula) as $\alpha \rightarrow +\infty$ and $-\infty$, respectively, and these solutions are connected by the periodic function $\pi/\sin \pi\alpha$.

Remark 3.24. As we have seen above, we used the integrand $\bar{\Phi}(z)$ instead of $\Phi(z)$, which coincides with $\bar{\Phi}(z)$ up to the q -periodic factor $h(z)/h(z^{-1})$, and used the set $\{(b_1q^{\nu_1}, b_1tq^{\nu_2}, \dots, b_1t^{n-1}q^{\nu_n}) \in (\mathbb{C}^*)^n; (\nu_1, \dots, \nu_n) \in \Lambda\}$ as the “ $(-\alpha)$ -stable cycle” for the dual integral $\bar{I}(x)$ when we construct a special solution $\bar{I}(\zeta)$ expressed by the (Jackson) integral representation for the q -difference equation (3.18) as $\alpha \rightarrow -\infty$. In the classical setting, this process is usually done by taking an imaginary cycle without changing the integrand $\Phi(z)$ under the ordinary integral representation. In the q -analog setting Aomoto and Aomoto–Kato used the integral representation without changing the integrand $\Phi(z)$, but instead, they adopted the residue sum on the set $\{(b_1^{-1}q^{-\nu_1}, b_1^{-1}t^{-1}q^{-\nu_2}, \dots, b_1^{-1}t^{-(n-1)}q^{-\nu_n}) \in (\mathbb{C}^*)^n; (\nu_1, \dots, \nu_n) \in \Lambda\}$ of poles of $I(x)$. They call this cycle the α -unstable cycle [6, 12] of $I(x)$ for the parameter α . To carry out this process is called *regularization* in their original paper [4]. We hope our slight changes of terminology do not create confusion for the reader.

4. FURTHER EXTENSION OF THE q -SELBERG INTEGRAL

In this section we will explain a bilateral extension of Askey–Evans’s q -Selberg integral.

4.1. Jackson integral of Selberg type. Let a_1, a_2, b_1, b_2 and t be complex numbers satisfying

$$(4.1) \quad |qt^{2i-2}| < 1 \quad \text{and} \quad q < |a_1a_2b_1b_2t^{2i-2}| \quad \text{for} \quad i = 1, 2, \dots, n.$$

Let $\Phi(z)$ be specified by

$$(4.2) \quad \Phi(z) := \prod_{i=1}^n z_i \frac{(qa_1^{-1}z_i)_\infty (qa_2^{-1}z_i)_\infty}{(b_1z_i)_\infty (b_2z_i)_\infty} \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(qt^{-1}z_k/z_j)_\infty}{(tz_k/z_j)_\infty},$$

where τ is given by $t = q^\tau$, and let $\Delta(z)$ be the difference product specified by (3.2). For $x = (x_1, x_2, \dots, x_n) \in (\mathbb{C}^*)^n$, we define the sum $J(x)$ by

$$(4.3) \quad J(x) = J(x_1, x_2, \dots, x_n) := \int_0^{x_\infty} \Phi(z)\Delta(z)\varpi_q,$$

which converges absolutely under the condition (4.1). We call $J(x)$ the *Jackson integral of Selberg type*. For arbitrary $x_1, x_2 \in \mathbb{C}^*$, we set the points $\zeta_i(x_1, x_2) \in (\mathbb{C}^*)^n$ by

$$(4.4) \quad \zeta_i(x_1, x_2) := (\underbrace{x_1, x_1t, \dots, x_1t^{i-1}}_i, \underbrace{x_2, x_2t, \dots, x_2t^{n-i-1}}_{n-i}) \in (\mathbb{C}^*)^n$$

for $i = 0, 1, \dots, n$. When we set $x_1 = a_1$ and $x_2 = a_2$ on (4.4), for the special points $\zeta_i(a_1, a_2)$, $i = 0, 1, \dots, n$, by definition $J(\zeta_i(a_1, a_2))$ is defined as the sum (4.3) over the fan region

$$(4.5) \quad \Lambda_i = \left\{ (\nu_1, \nu_2, \dots, \nu_n) \in \mathbb{Z}^n; \begin{array}{l} 0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_i \text{ and} \\ 0 \leq \nu_{i+1} \leq \nu_{i+2} \leq \dots \leq \nu_n \end{array} \right\}.$$

We call $J(\zeta_i(a_1, a_2))$ the *truncated Jackson integral of Selberg type*.

The main theorem of this paper is the following:

Theorem 4.1.

$$\begin{aligned}
 (4.6) \quad & \sum_{i=0}^n J(\zeta_i(x_1, x_2)) \prod_{j=1}^i \frac{(x_1 t^{j-1})^{-1-2(n-j)\tau}}{\theta(x_2 x_1^{-1} t^{-j+1})} \prod_{k=1}^{n-i} \frac{(x_2 t^{k-1})^{-1-2(n-i-k)\tau}}{\theta(x_1 x_2^{-1} t^{i-k+1})} \\
 & = C_0 \prod_{k=1}^n \frac{\theta(x_1 x_2 b_1 b_2 t^{n+k-2})}{\prod_{i=1}^2 \prod_{j=1}^2 \theta(x_i b_j t^{k-1})},
 \end{aligned}$$

where C_0 is a constant independent of x_1 and x_2 . The constant C_0 is explicitly expressed as

$$(4.7) \quad C_0 = (1 - q)^n \prod_{k=1}^n \frac{(q)_\infty(t)_\infty \prod_{i=1}^2 \prod_{j=1}^2 (q a_i^{-1} b_j^{-1} t^{-(k-1)})_\infty}{(t^k)_\infty (q a_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(n+k-2)})_\infty}.$$

Remark 4.2. In particular, if $\tau \in \mathbb{Z}_+$, then the formula simplifies to

$$\begin{aligned}
 (4.8) \quad & \sum_{i=0}^n (-1)^i J(\zeta_i(x_1, x_2)) \\
 & = C_0 (-1)^\tau \binom{n}{\tau} q^{-\binom{\tau}{2}} \prod_{j=1}^n \frac{(x_1 x_2 t^{j-1})^{(n-j)\tau} x_2 \theta(x_1/x_2) \theta(x_1 x_2 b_1 b_2 t^{n+j-2})}{\theta(x_1 b_1 t^{j-1}) \theta(x_1 b_2 t^{j-1}) \theta(x_2 b_1 t^{j-1}) \theta(x_2 b_2 t^{j-1})},
 \end{aligned}$$

which is equivalent to Corollary 4.8 as will be explained later.

We will give the proof of Theorem 4.1 in the next subsection. (See the proof of Theorem 4.12, which is equivalent to Theorem 4.1.) In this subsection we will explain the relation between this main theorem and other known results which are deduced from this theorem as corollaries. As a special case $x_1 = a_1, x_2 = a_2$ of Theorem 4.1, we have the formula for the truncated Jackson integrals $J(\zeta_i(a_1, a_2))$.

Corollary 4.3 (Tarasov–Varchenko, Stokman).

$$\begin{aligned}
 (4.9) \quad & \sum_{i=0}^n J(\zeta_i(a_1, a_2)) \prod_{j=1}^i \frac{(a_1 t^{j-1})^{-1-2(n-j)\tau}}{\theta(a_2 a_1^{-1} t^{-j+1})} \prod_{k=1}^{n-i} \frac{(a_2 t^{k-1})^{-1-2(n-i-k)\tau}}{\theta(a_1 a_2^{-1} t^{i-k+1})} \\
 & = (1 - q)^n \prod_{k=1}^n \frac{(q)_\infty(t)_\infty (a_1 a_2 b_1 b_2 t^{n+k-2})_\infty}{(t^k)_\infty \prod_{i=1}^2 \prod_{j=1}^2 (a_i b_j t^{k-1})_\infty}.
 \end{aligned}$$

Remark 4.4. This formula was given by Tarasov–Varchenko [39, Theorem (E.10)] and Stokman [38, Corollary 7.6] independently. The proof of [39] is by computing residues for an A-type generalization of Askey–Roy’s q -beta integral, while the proof of [38] is by computing residues for Gustafson’s q -Selberg contour integral [20, 21] and an appropriate limiting procedure. This formula extends Askey–Evans’s formula (1.4) from τ a positive integer to τ an arbitrary complex number. (Compare with Corollary 4.9.)

As we saw in the section on the Jackson integral of A-type and in keeping with the above remark, it is also very important for the Jackson integral of Selberg type to distinguish between the situations in the cases when the parameter τ is a positive integer or not.

Lemma 4.5. *Suppose $\tau \notin \mathbb{Z}_+$. If $x_i = x_j$ for some i and j ($1 \leq i < j \leq n$), then $J(x_1, \dots, x_n) = 0$.*

Proof. In the same way as Lemma 3.1. □

On the other hand, under the condition τ being a positive integer we generally have $J(x_1, \dots, x_n) \neq 0$ even if $x_i = x_j$ ($1 \leq i < j \leq n$). In particular, we have the following:

Lemma 4.6. *Suppose $\tau \in \mathbb{Z}_+$. For arbitrary $x_1, x_2 \in \mathbb{C}^*$, $J(\zeta_i(x_1, x_2))$ is expressed as*

$$(4.10) \quad J(\zeta_i(x_1, x_2)) = \frac{1}{i!(n-i)!} J(\underbrace{x_1, x_1, \dots, x_1}_i, \underbrace{x_2, x_2, \dots, x_2}_{n-i}).$$

Proof. In the same way as Lemma 3.2. □

Remark 4.7. As pointed out in Lemma 4.5, the right-hand side of (4.10) makes sense only when τ is a positive integer. However, as a function of τ the left-hand side of (4.10) is defined continuously whether τ is a positive integer or not. Thus, as our basic strategy we first obtain several results for $J(x)$ under the condition $\tau \notin \mathbb{Z}_+$. Then, using analytic continuation, the results for $J(x)$ can automatically be regarded as those for $\tau \in \mathbb{Z}_+$. And if necessary, we will rewrite them appropriately using the relation (4.10).

Recalling the binomial theorem

$$(x_2 - x_1)^n = \sum_{i=0}^n (-1)^i \binom{n}{i} x_2^{n-i} x_1^i,$$

under the condition $\tau \in \mathbb{Z}_+$ and using (2.7) and Lemma 4.6, we can deform the following iterated Jackson integral as

$$(4.11) \quad \begin{aligned} & \frac{1}{n!} \int_{x_1 \infty}^{x_2 \infty} \dots \int_{x_1 \infty}^{x_2 \infty} \Phi(z) \Delta(z) \frac{d_q z_1}{z_1} \dots \frac{d_q z_n}{z_n} \\ &= \frac{1}{n!} \sum_{i=0}^n (-1)^i \binom{n}{i} \underbrace{\int_0^{x_2 \infty} \dots \int_0^{x_2 \infty}}_{n-i} \underbrace{\int_0^{x_1 \infty} \dots \int_0^{x_1 \infty}}_i \Phi(z) \Delta(z) \frac{d_q z_1}{z_1} \dots \frac{d_q z_n}{z_n} \\ &= \sum_{i=0}^n (-1)^i \frac{1}{i!(n-i)!} J(\underbrace{x_1, x_1, \dots, x_1}_i, \underbrace{x_2, x_2, \dots, x_2}_{n-i}) \\ &= \sum_{i=0}^n (-1)^i J(\zeta_i(x_1, x_2)), \end{aligned}$$

because the integrand $\Phi(z)\Delta(z)$ is symmetric if $\tau \in \mathbb{Z}_+$.

If $\tau \in \mathbb{Z}_+$, using (2.2), the coefficient factor of $J(\zeta_i(x_1, x_2))$ appearing in the left-hand side of (4.6) is simplified as

$$\begin{aligned} & (-1)^i \prod_{j=1}^i (x_1 t^{j-1})^{1+2(n-j)\tau} \theta(x_2 x_1^{-1} t^{-j+1}) \prod_{k=1}^{n-i} (x_2 t^{k-1})^{1+2(n-i-k)\tau} \theta(x_1 x_2^{-1} t^{i-k+1}) \\ &= (-1)^\tau q^{-\binom{\tau}{2}} \binom{n}{2}^{\tau^2} \binom{n}{3}^\tau (x_1 x_2)^{\tau \binom{n}{2}} (x_2 \theta(x_1/x_2))^n, \end{aligned}$$

which is independent of the choice of indices $i = 0, 1, \dots, n$. From this, if $\tau \in \mathbb{Z}_+$, the formula (4.6) in the main theorem then shrinks to the form (4.8). From (4.11),

(4.8) is rewritten as

$$\begin{aligned} & \frac{1}{n!} \int_{x_1\infty}^{x_2\infty} \cdots \int_{x_1\infty}^{x_2\infty} \Phi(z)\Delta(z) \frac{d_q z_1}{z_1} \cdots \frac{d_q z_n}{z_n} \\ &= (-1)^\tau \binom{n}{2} q^{-\binom{\tau}{2}} \binom{n}{2} C_0 \prod_{j=1}^n \frac{(x_1 x_2 t^{j-1})^{(n-j)\tau} x_2 \theta(x_1/x_2) \theta(x_1 x_2 b_1 b_2 t^{n+j-2})}{\theta(x_1 b_1 t^{j-1}) \theta(x_1 b_2 t^{j-1}) \theta(x_2 b_1 t^{j-1}) \theta(x_2 b_2 t^{j-1})}. \end{aligned}$$

Using (3.10), we therefore obtain

Corollary 4.8. *Suppose $\tau \in \mathbb{Z}_+$. Then*

$$\begin{aligned} & \frac{1}{n!} \int_{x_1\infty}^{x_2\infty} \cdots \int_{x_1\infty}^{x_2\infty} \prod_{i=1}^n z_i^{(n-1)\tau} \frac{(qa_1^{-1}z_i)_\infty}{(b_1z_i)_\infty} \frac{(qa_2^{-1}z_i)_\infty}{(b_2z_i)_\infty} \\ & \times \prod_{1 \leq j < k \leq n} (z_j/z_k)_\tau (z_k/z_j)_\tau d_q z_1 \cdots d_q z_n \\ (4.12) \quad &= C_0 \prod_{j=1}^n \frac{(x_1 x_2 q^{(j-1)\tau})^{(n-j)\tau} x_2 \theta(x_1/x_2) \theta(x_1 x_2 b_1 b_2 q^{(n+j-2)\tau})}{\theta(x_1 b_1 q^{(j-1)\tau}) \theta(x_1 b_2 q^{(j-1)\tau}) \theta(x_2 b_1 q^{(j-1)\tau}) \theta(x_2 b_2 q^{(j-1)\tau})}, \end{aligned}$$

where C_0 is the constant given by (4.7).

In particular, putting $x_1 = a_1$ and $x_2 = a_2$ in the above equation, we obtain

Corollary 4.9 (Askey, Evans). *Suppose $\tau \in \mathbb{Z}_+$. Then*

$$\begin{aligned} & \frac{1}{n!} \int_{a_1}^{a_2} \cdots \int_{a_1}^{a_2} \prod_{i=1}^n z_i^{(n-1)\tau} \frac{(qa_1^{-1}z_i)_\infty}{(b_1z_i)_\infty} \frac{(qa_2^{-1}z_i)_\infty}{(b_2z_i)_\infty} \\ & \times \prod_{1 \leq j < k \leq n} (z_j/z_k)_\tau (z_k/z_j)_\tau d_q z_1 \cdots d_q z_n \\ (4.13) \quad &= (1-q)^n \prod_{j=1}^n \frac{(q)_\infty (t)_\infty (a_1 a_2 b_1 b_2 t^{n+j-2})_\infty (a_1 a_2 t^{j-1})^{(n-j)\tau} a_2 \theta(a_1/a_2)}{(t^j)_\infty (a_1 b_1 t^{j-1})_\infty (a_1 b_2 t^{j-1})_\infty (a_2 b_1 t^{j-1})_\infty (a_2 b_2 t^{j-1})_\infty}. \end{aligned}$$

Remark 4.10. If we substitute a_1, a_2, b_1, b_2 as $a_1 \rightarrow x_1, a_2 \rightarrow x_2, b_1 \rightarrow q^\alpha/x_1, b_2 \rightarrow q^\beta/x_2$, respectively, then (4.13) is rewritten as

$$\begin{aligned} & \frac{1}{n!} \int_{x_1}^{x_2} \cdots \int_{x_1}^{x_2} \prod_{i=1}^n z_i^{(n-1)\tau} \frac{(qz_i/x_1)_\infty}{(q^\alpha z_i/x_1)_\infty} \frac{(qz_i/x_2)_\infty}{(q^\beta z_i/x_2)_\infty} \\ & \times \prod_{1 \leq j < k \leq n} (z_j/z_k)_\tau (z_k/z_j)_\tau d_q z_1 \cdots d_q z_n \\ &= \prod_{j=1}^n \frac{(1-q)(q)_\infty (q^{\alpha+\beta+(n+j-2)\tau})_\infty (q^\tau)_\infty}{(q^{\alpha+(j-1)\tau})_\infty (q^{\beta+(j-1)\tau})_\infty (q^{j\tau})_\infty} \frac{(x_1 x_2 q^{(j-1)\tau})^{(n-j)\tau} x_2 \theta(x_1/x_2)}{(x_2 q^{\alpha+(j-1)\tau}/x_1)_\infty (x_1 q^{\beta+(j-1)\tau}/x_2)_\infty}, \end{aligned}$$

which exactly coincides with the formula (1.4) established by Askey and Evans.

4.2. Regularization and the proof of the main theorem. Let $\mathcal{J}(x)$ be the function defined by

$$(4.14) \quad \mathcal{J}(x) := \frac{J(x)}{h(x)} \quad \text{where} \quad h(x) := \prod_{i=1}^n \frac{x_i}{\theta(b_1 x_i) \theta(b_2 x_i)} \prod_{1 \leq j < k \leq n} x_j^{2\tau} \frac{\theta(x_k/x_j)}{\theta(tx_k/x_j)}.$$

Lemma 4.11. *The function $\mathcal{J}(x)$ is holomorphic on $(\mathbb{C}^*)^n$ and symmetric.*

Proof. In the same way as Lemma 3.15. □

We call $\mathcal{J}(x)$ the *regularization of $J(x)$* or the *regularized Jackson integral of $J(x)$* . For the point $\zeta_i(x_1, x_2)$ defined by (4.4), $h(\zeta_i(x_1, x_2))$ is evaluated as

$$\begin{aligned}
 (4.15) \quad h(\zeta_i(x_1, x_2)) &= \prod_{j=1}^i \frac{(x_1 t^{j-1})^{1+2(n-j)\tau}}{\theta(b_1 x_1 t^{j-1})\theta(b_2 x_1 t^{j-1})} \frac{\theta(x_2 x_1^{-1} t^{-(j-1)})}{\theta(x_2 x_1^{-1} t^{n-i-j+1})} \frac{\theta(t)}{\theta(t^j)} \\
 &\quad \times \prod_{k=1}^{n-i} \frac{(x_2 t^{k-1})^{1+2(n-i-k)\tau}}{\theta(b_2 x_1 t^{k-1})\theta(b_2 x_2 t^{k-1})} \frac{\theta(t)}{\theta(t^k)},
 \end{aligned}$$

which is used below.

Using the definition (4.14) of the regularization $\mathcal{J}(x)$, it is directly confirmed that the equation (4.6) in the main theorem (Theorem 4.1) is rewritten in the following form:

Theorem 4.12. *Suppose $\tau \notin \mathbb{Z}_+$. Let $H_i(x_1, x_2)$ be the function defined as*

$$\begin{aligned}
 H_i(x_1, x_2) &:= \prod_{j=1}^i \frac{\theta(x_2 b_1 t^{n-j})\theta(x_2 b_2 t^{n-j})}{\theta(t^j)\theta(x_2 x_1^{-1} t^{n-i-j+1})} \prod_{k=1}^{n-i} \frac{\theta(x_1 b_1 t^{n-k})\theta(x_1 b_2 t^{n-k})}{\theta(t^k)\theta(x_1 x_2^{-1} t^{i-k+1})} \\
 &\quad \times \prod_{l=1}^n \frac{\theta(t^l)}{\theta(x_1 x_2 b_1 b_2 t^{n+l-2})}.
 \end{aligned}$$

Then

$$(4.16) \quad \sum_{i=0}^n \mathcal{J}(\zeta_i(x_1, x_2)) H_i(x_1, x_2) = C_1,$$

where C_1 is a constant independent of x_1 and x_2 .

Remark 4.13. The constant C_1 is explicitly given by

$$(4.17) \quad C_1 = (1 - q)^n \prod_{k=1}^n \frac{(q)_\infty (qt^{-k})_\infty \prod_{i=1}^2 \prod_{j=1}^2 (qa_i^{-1} b_j^{-1} t^{-(k-1)})_\infty}{(qt^{-1})_\infty (qa_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(n+k-2)})_\infty},$$

which will be confirmed later.

From Theorem 4.12 we immediately have an expression for the constant C_1 .

Corollary 4.14. *The constant C_1 in (4.16) is expressed by the special values of $\mathcal{J}(x)$ as*

$$(4.18) \quad C_1 = \mathcal{J}(\zeta_i(b_1^{-1} t^{-(i-1)}, b_2^{-1} t^{-(n-i-1)})) \quad (i = 0, 1, \dots, n).$$

Proof. Since $H_i(x_1, x_2)$ has the property

$$H_i(b_1^{-1} t^{-(j-1)}, b_2^{-1} t^{-(n-j-1)}) = \delta_{ij} \quad (i, j = 0, 1, \dots, n),$$

where δ_{ij} is Kronecker's delta, using (4.16), we obtain the constant C_1 as (4.18). □

Remark 4.15. Since $x = \zeta_i(b_1^{-1} t^{-(i-1)}, b_2^{-1} t^{-(n-i-1)})$ is a pole of the function $J(x)$ by definition, the value $J(\zeta_i(b_1^{-1} t^{-(i-1)}, b_2^{-1} t^{-(n-i-1)}))$ no longer makes sense. However, the regularization $\mathcal{J}(\zeta_i(b_1^{-1} t^{-(i-1)}, b_2^{-1} t^{-(n-i-1)}))$ still has meaning as a special value of a holomorphic function. In the next subsection we will show a way to realize the regularization $\mathcal{J}(\zeta_i(b_1^{-1} t^{-(i-1)}, b_2^{-1} t^{-(n-i-1)}))$ as a computable object

by another Jackson integral, and eventually it will lead us to the explicit evaluation of the constant C_1 as in (4.17), which will be confirmed later as Lemmas 4.22 and 4.23.

The rest of this subsection is devoted to the proof of Theorem 4.12. If we set

$$(4.19) \quad F(x_1, x_2) = \sum_{i=0}^n F_i(x_1, x_2),$$

where

$$F_i(x_1, x_2) := \mathcal{J}(\zeta_i(x_1, x_2)) \prod_{j=1}^i \frac{\theta(x_2 b_1 t^{n-j}) \theta(x_2 b_2 t^{n-j})}{\theta(t^j) \theta(x_2 x_1^{-1} t^{n-i-j+1})} \\ \times \prod_{k=1}^{n-i} \frac{\theta(x_1 b_1 t^{n-k}) \theta(x_1 b_2 t^{n-k})}{\theta(t^k) \theta(x_1 x_2^{-1} t^{i-k+1})} \prod_{l=1}^n \theta(t^l),$$

then the equation (4.16) is equivalent to

$$(4.20) \quad F(x_1, x_2) = C_1 \prod_{j=1}^n \theta(x_1 x_2 b_1 b_2 t^{n+j-2}).$$

In order to prove Theorem 4.12 we will show (4.20) instead of (4.16). For this purpose, we will prove two lemmas first.

Lemma 4.16. *Suppose that $\tau \notin \mathbb{Z}_+$. Let H be the set of the holomorphic functions on $(\mathbb{C}^*)^2$ satisfying the q -difference equation*

$$(4.21) \quad f(qx_1, x_2) = f(x_1, qx_2) = (-x_1 x_2 b_1 b_2 t^{3(n-1)/2})^{-n} f(x_1, x_2).$$

The dimension of H as a linear space is equal to n , i.e., $\dim_{\mathbb{C}} H = n$. Moreover, the set $\{\Theta_i(x_1, x_2); i = 1, 2, \dots, n\}$ is a basis of H , where $\Theta_i(x_1, x_2)$ is defined by

$$(4.22) \quad \Theta_i(x_1, x_2) := \theta(x_1 x_2 b_1 b_2 t^{n-i}) \prod_{\substack{1 \leq j \leq n \\ j \neq i}} \theta(x_1 x_2 b_1 b_2 t^{2n-j}), \quad i = 1, 2, \dots, n.$$

Remark 4.17. $\Theta_1(x_1, x_2)$ coincides with the function appearing in the right-hand side of (4.20).

Proof. For an arbitrary function $f(x_1, x_2) \in H$, since $f(x_1, x_2)$ is a holomorphic function on $(\mathbb{C}^*)^2$, $f(x_1, x_2)$ is expanded as $f(x_1, x_2) = \sum_{i,j=-\infty}^{\infty} c_{ij} x_1^i x_2^j$. From $f(qx_1, x_2) = f(x_1, qx_2)$, we have $c_{ij} q^i = c_{ij} q^j$. This indicates that $c_{ij} = 0$ if $i \neq j$. Denoting c_{ii} by c_i , $f(x_1, x_2)$ is written as $f(x_1, x_2) = \sum_{i=-\infty}^{\infty} c_i (x_1 x_2)^i$. From $f(qx_1, x_2) = (-x_1 x_2 b_1 b_2 t^{3(n-1)/2})^{-n} f(x_1, x_2)$, we have $c_i q^i = c_{i+n} (-b_1 b_2 t^{3(n-1)/2})^{-n}$. This indicates that $f(x_1, x_2)$ is determined by c_0, c_1, \dots, c_{n-1} , which means that H is spanned by its n elements.

Since it is obvious that $\Theta_i(x_1, x_2) \in H$ from the explicit expression (4.22), it suffices for our purpose to show the linear independence of $\{\Theta_i(x_1, x_2)\}$. Assume

that $\sum_{i=1}^n c_i \Theta_i(x_1, x_2) = 0$. By definition

$$(4.23) \quad \Theta_i(b_1^{-1}t^{-(n-j+1)}, b_2^{-1}t^{-(n-1)}) = \delta_{ij} \theta(t^{-n}) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} \theta(t^{j-k}) \quad (i, j = 1, 2, \dots, n),$$

where δ_{ij} is Kronecker's delta. Therefore we have

$$0 = \sum_{i=1}^n c_i \Theta_i(b_1^{-1}t^{-(n-j+1)}, b_2^{-1}t^{-(n-1)}) = c_j \theta(t^{-n}) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} \theta(t^{j-k}),$$

which indicates that $c_1 = c_2 = \dots = c_n = 0$. □

Lemma 4.18. *Suppose $\tau \notin \mathbb{Z}_+$. Then $F(x_1, x_2) \in H$.*

Proof. Since it is easy to confirm that $F(x_1, x_2)$ satisfies the q -difference equations (4.21), it suffices to prove that $F(x_1, x_2)$ is holomorphic on $(\mathbb{C}^*)^2$. For this purpose, in the expression (4.19) of $F(x_1, x_2)$, we will confirm that the residues at the apparent poles vanish. Since each $F_i(x_1, x_2)$ in (4.19) has the common quasi-periodicity (4.21), it suffices to consider the residues at the poles of the cases (1) $x_1 x_2^{-1} t^{i-j+1} = 1$ ($1 \leq j \leq n-i \leq n$) or (2) $x_2 x_1^{-1} t^{n-i-j+1} = 1$ ($1 \leq j \leq i \leq n$). We will examine these poles carefully below. The function $F_0(x_1, x_2)$ has n poles of order 1 at $x_1 = x_2 t^{j-1}$ ($1 \leq j \leq n$). The function $F_n(x_1, x_2)$ also has n poles of order 1 at $x_2 = x_1 t^{j-1}$ ($1 \leq j \leq n$). If $i \neq 0, n$, then the function $F_i(x_1, x_2)$ is supposed to have n poles at $x_1 t^i = x_2 t^{j-1}$ ($1 \leq j \leq n-i$) and $x_2 t^{n-i} = x_1 t^{j-1}$ ($1 \leq j \leq i$). But when $x_1 t^i = x_2 t^{j-1}$, if $1 < j \leq n-i$, from (4.15) and Lemma 4.5, we have $\mathcal{J}(\zeta_i(x_1, x_2)) = 0$, which is a factor of $F_i(x_1, x_2)$. In the same manner, when $x_2 t^{n-i} = x_1 t^{j-1}$, if $1 < j \leq i$, we also have $\mathcal{J}(\zeta_i(x_1, x_2)) = 0$. This indicates that $F_i(x_1, x_2)$, $i \neq 0, n$, actually has only 2 poles of order 1, i.e., $x_1 t^i = x_2$ and $x_2 t^{n-i} = x_1$. Therefore the function $F(x_1, x_2)$ may have $2n - 1$ poles of order 1 at $x_1 = x_2 t^{n-i}$ ($1 \leq i \leq n$) and $x_2 = x_1 t^i$ ($0 \leq i \leq n-1$) in total. However the residues there all vanish as is confirmed by the following calculation, valid for $1 \leq i \leq n$:

$$\begin{aligned} \operatorname{Res}_{x_1=x_2 t^{n-i}} F(x_1, x_2) &= \lim_{x_1 \rightarrow x_2 t^{n-i}} (x_1 - x_2 t^{n-i}) F(x_1, x_2) \\ &= \lim_{x_1 \rightarrow x_2 t^{n-i}} (x_1 - x_2 t^{n-i}) (F_0(x_1, x_2) + F_i(x_1, x_2)) \\ &= \lim_{x_1 \rightarrow x_2 t^{n-i}} \frac{x_1 - x_2 t^{n-i}}{\theta(x_1 x_2^{-1} t^{-(n-i)})} \mathcal{J}(\zeta_0(x_1, x_2)) \frac{\prod_{j=1}^n \theta(x_1 b_1 t^{n-j}) \theta(x_1 b_2 t^{n-j})}{\prod_{\substack{1 \leq j \leq n \\ j \neq n-i+1}} \theta(x_1 x_2^{-1} t^{-j+1})} \\ &\quad + \lim_{x_1 \rightarrow x_2 t^{n-i}} \frac{x_1 - x_2 t^{n-i}}{\theta(x_2 x_1^{-1} t^{n-i})} \mathcal{J}(\zeta_i(x_1, x_2)) \prod_{j=1}^{n-i} \frac{\theta(x_1 b_1 t^{n-j}) \theta(x_1 b_2 t^{n-j})}{\theta(t^j) \theta(x_1 x_2^{-1} t^{i-j+1})} \\ &\quad \times \frac{\prod_{j=1}^i \theta(x_2 b_1 t^{n-j}) \theta(x_2 b_2 t^{n-j})}{\prod_{j=2}^i \theta(t^j) \theta(x_2 x_1^{-1} t^{n-i-j+1})} \prod_{j=2}^n \theta(t^j) \\ &= -\frac{x_2 t^{n-i}}{(q)_\infty^2} \left[\mathcal{J}(\zeta_0(x_2 t^{n-i}, x_2)) - \mathcal{J}(\zeta_i(x_2 t^{n-i}, x_2)) \right] \\ &\quad \times \frac{\prod_{j=1}^n \theta(x_2 b_1 t^{2n-i-j}) \theta(x_2 b_2 t^{2n-i-j})}{\prod_{\substack{1 \leq j \leq n \\ j \neq n-i+1}} \theta(t^{n-i-j+1})} \\ &= 0, \end{aligned}$$

because $\mathcal{J}(\zeta_0(x_2 t^{n-i}, x_2)) = \mathcal{J}(\zeta_i(x_2 t^{n-i}, x_2)) = \mathcal{J}(x_2, x_2 t, \dots, x_2 t^{n-1})$. In the same way as above it is also confirmed that

$$\operatorname{Res}_{x_2=x_1 t^i} F(x_1, x_2) = \lim_{x_2 \rightarrow x_1 t^i} (x_2 - x_1 t^i)(F_i(x_1, x_2) + F_n(x_1, x_2)) = 0$$

for $0 \leq i \leq n - 1$. Therefore $F(x_1, x_2)$ is holomorphic on $(\mathbb{C}^*)^2$. □

Proof of Theorem 4.12. We will prove (4.20). From Lemma 4.16 and Lemma 4.18, $F(x_1, x_2)$ is expressed as a linear combination of $\Theta_i(x_1, x_2)$, $i = 1, \dots, n$, i.e., $F(x_1, x_2) = \sum_{i=1}^n C_i \Theta_i(x_1, x_2)$, where C_i are some constants. By definition, it is easy to confirm that $F(b_1^{-1} t^{-(n-j+1)}, b_2^{-1} t^{-(n-1)}) = 0$ for $j = 2, 3, \dots, n$. From (4.23) we therefore obtain

$$\begin{aligned} 0 &= F(b_1^{-1} t^{-(n-j+1)}, b_2^{-1} t^{-(n-1)}) = \sum_{i=1}^n C_i \Theta_i(b_1^{-1} t^{-(n-j+1)}, b_2^{-1} t^{-(n-1)}) \\ &= C_j \theta(t^{-n}) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} \theta(t^{j-k}) \end{aligned}$$

for $j = 2, 3, \dots, n$. This indicates that $C_2 = C_3 = \dots = C_n = 0$. Thus we obtain (4.20). □

4.3. Dual expression of the Jackson integral $J(x)$. Let $\bar{\Phi}(z)$ be specified by

$$\begin{aligned} \bar{\Phi}(z) &:= \prod_{i=1}^n z_i^{1-\alpha_i-\alpha_2-\beta_1-\beta_2-2(n-1)\tau} \frac{(qb_1^{-1}z_i)_\infty}{(a_1z_i)_\infty} \frac{(qb_2^{-1}z_i)_\infty}{(a_2z_i)_\infty} \\ &\times \prod_{1 \leq j < k \leq n} z_j^{2\tau-1} \frac{(qt^{-1}z_k/z_j)_\infty}{(tz_k/z_j)_\infty}, \end{aligned} \tag{4.24}$$

where α_i and β_i are given by $a_i = q^{\alpha_i}$, $b_i = q^{\beta_i}$, and let $\Delta(z)$ be specified by (3.2). For $x = (x_1, x_2, \dots, x_n) \in (\mathbb{C}^*)^n$, we define the sum $\bar{J}(x)$ by

$$\bar{J}(x) := \int_0^{x_\infty} \bar{\Phi}(z) \Delta(z) \varpi_q, \tag{4.25}$$

which converges absolutely under the condition (4.1). We call $\bar{J}(x)$ the *dual Jackson integral of $J(x)$* . When we set $x_1 = b_1$ and $x_2 = b_2$ in (4.4), for the special points $\zeta_i(b_1, b_2)$, $i = 0, 1, \dots, n$, $\bar{J}(\zeta_i(b_1, b_2))$ is defined as the sum over the fan region Λ_i specified by (4.5). We call $\bar{J}(\zeta_i(b_1, b_2))$ the *truncation of the dual Jackson integral $\bar{J}(x)$* .

Let $\bar{\mathcal{J}}(x)$ and $\bar{h}(x)$ be the functions defined by

$$\bar{\mathcal{J}}(x) := \frac{\bar{J}(x)}{\bar{h}(x)} \tag{4.26}$$

where $\bar{h}(x) = \prod_{i=1}^n \frac{x_i^{1-\alpha_1-\alpha_2-\beta_1-\beta_2-2(n-1)\tau}}{\theta(a_1 x_i) \theta(a_2 x_i)} \prod_{1 \leq j < k \leq n} x_j^{2\tau} \frac{\theta(x_k/x_j)}{\theta(tx_k/x_j)}$.

Since the trivial poles and zeros of $\bar{J}(x)$ are canceled out by multiplying together $1/\bar{h}(x)$ and $\bar{J}(x)$, $\bar{\mathcal{J}}(x)$ is holomorphic on $x \in (\mathbb{C}^*)^n$ and symmetric. We call $\bar{\mathcal{J}}(x)$ the *regularization of $\bar{J}(x)$* .

Lemma 4.19 (Reflective equation). *For $x \in (\mathbb{C}^*)^n$, the relation between $\mathcal{J}(x)$ and $\bar{\mathcal{J}}(x)$ is given by*

$$\bar{\mathcal{J}}(x) = \mathcal{J}(x^{-1}), \tag{4.27}$$

where x^{-1} is specified as in (3.29). In particular, for $x_1, x_2 \in \mathbb{C}^*$ the following holds:

$$(4.28) \quad \bar{\mathcal{J}}(\zeta_i(x_1, x_2)) = \mathcal{J}(\zeta_i(x_1^{-1}t^{-(i-1)}, x_2^{-1}t^{-(n-i-1)})) \quad (i = 0, 1, \dots, n).$$

Proof. By definition (4.27) is equivalent to the connection between $J(x)$ and $\bar{J}(x)$, i.e.,

$$(4.29) \quad \bar{J}(x) = \frac{\bar{h}(x)}{h(x^{-1})} J(x^{-1}),$$

which we should prove. From the definitions (4.14) and (4.26) the ratio $\bar{h}(x)/h(x^{-1})$ is written as

$$(4.30) \quad \begin{aligned} \frac{\bar{h}(x)}{h(x^{-1})} &= (-1)^{\binom{n}{2}} \prod_{i=1}^n \prod_{l=1}^2 x_i^{1-\alpha_l-\beta_l} \frac{\theta(qb_l^{-1}x_i)}{\theta(a_lx_i)} \\ &\times \prod_{1 \leq j < k \leq n} \left(\frac{x_k}{x_j}\right)^{1-2\tau} \frac{\theta(qt^{-1}x_k/x_j)}{\theta(tx_k/x_j)}. \end{aligned}$$

Since $\Delta(z) = (-1)^{\binom{n}{2}} (z_1 z_2 \cdots z_n)^{n-1} \Delta(z^{-1})$, from (4.2), (4.24) and (3.2), we have

$$(4.31) \quad \bar{\Phi}(z)\Delta(z) = \frac{\bar{h}(z)}{h(z^{-1})}\Phi(z^{-1})\Delta(z^{-1}).$$

Also since $\bar{h}(z)/h(z^{-1})$ is invariant under the shift $z_i \rightarrow qz_i$, by the definitions (4.3) and (4.25) of $J(x)$ and $\bar{J}(x)$, the connection (4.29) between $J(x)$ and its dual $\bar{J}(x)$ is derived from (4.31). Since $\mathcal{J}(x)$ and $\bar{\mathcal{J}}(x)$ are symmetric, (4.28) immediately follows from (4.27). \square

We now state the q -difference equations for $\bar{J}(x)$ under the setting $x = \zeta_i(b_1, b_2)$, $i = 0, 1, \dots, n$.

Proposition 4.20. *Suppose $x = \zeta_i(b_1, b_2)$, $i = 0, 1, \dots, n$. Then the recurrence relations for $\bar{J}(x)$ are given by*

$$(4.32) \quad T_{a_j} \bar{J}(x) = (-a_j)^n \prod_{k=1}^n \frac{(1 - a_j^{-1}b_1^{-1}t^{-(k-1)})(1 - a_j^{-1}b_2^{-1}t^{-(k-1)})}{1 - a_1^{-1}a_2^{-1}b_1^{-1}b_2^{-1}t^{-(n+k-2)}} \bar{J}(x),$$

$$(4.33) \quad T_{b_j} \bar{J}(x) = (-b_j^{-1})^n \prod_{k=1}^n \frac{(1 - a_1b_jt^{k-1})(1 - a_2b_jt^{k-1})}{1 - a_1a_2b_1b_2t^{n+k-2}} \bar{J}(x),$$

for $j = 1, 2$, where T_{a_j} and T_{b_j} are the q -shift operators of $a_j \rightarrow qa_j$ and $b_j \rightarrow qb_j$, respectively. In other words, the recurrence relations for the regularization $\bar{\mathcal{J}}(x)$ are given by

$$(4.34) \quad T_{a_j} \bar{\mathcal{J}}(x) = \prod_{k=1}^n \frac{(1 - a_j^{-1}b_1^{-1}t^{-(k-1)})(1 - a_j^{-1}b_2^{-1}t^{-(k-1)})}{1 - a_1^{-1}a_2^{-1}b_1^{-1}b_2^{-1}t^{-(n+k-2)}} \bar{\mathcal{J}}(x),$$

$$(4.35) \quad T_{b_j} \bar{\mathcal{J}}(x) = \prod_{k=1}^n \frac{(1 - a_1^{-1}b_j^{-1}t^{-(k-1)})(1 - a_2^{-1}b_j^{-1}t^{-(k-1)})}{1 - a_1^{-1}a_2^{-1}b_1^{-1}b_2^{-1}t^{-(n+k-2)}} \bar{\mathcal{J}}(x),$$

for $j = 1, 2$.

Proof. The derivation of (4.32) and (4.33) will be done in the Appendix. (See Remark A.7 after Lemma A.5.) Here we just mention that (4.34) and (4.35) are derived from (4.32) and (4.33), respectively. From the expression (4.26) of $\bar{h}(x)$, under the condition $x = \zeta_i(b_1, b_2)$, $i = 0, 1, \dots, n$, the function $\bar{h}(x)$ satisfies

$$(4.36) \quad T_{a_j} \bar{h}(x) = (-a_j)^n \bar{h}(x) \text{ and } T_{b_j} \bar{h}(x) = \left(\frac{b_j}{b_1 b_2}\right)^n t^{-\binom{n}{2}} \bar{h}(x) \quad (j = 0, 1, \dots, n).$$

Since $\bar{\mathcal{J}}(x) = \bar{J}(x)/\bar{h}(x)$, from the above equations and (4.32), (4.33) we therefore obtain (4.34), (4.35). □

4.4. Evaluation of the truncated Jackson integral. The main result of this subsection is the evaluation of the regularization of the truncated Jackson integral using the q -difference equations (4.34) and (4.35) in Proposition 4.20 and its asymptotic behavior for the special direction of parameters.

Theorem 4.21. *For $x = \zeta_i(b_1, b_2)$, $i = 0, 1, \dots, n$, the regularization $\bar{\mathcal{J}}(x)$ is evaluated as*

$$\bar{\mathcal{J}}(\zeta_i(b_1, b_2)) = (1 - q)^n \prod_{k=1}^n \frac{(q)_\infty (qt^{-k})_\infty \prod_{i=1}^2 \prod_{j=1}^2 (qa_i^{-1} b_j^{-1} t^{-(k-1)})_\infty}{(qt^{-1})_\infty (qa_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(n+k-2)})_\infty}.$$

This theorem can be deduced from the specific case of $i = n$ (or $i = 0$). The reason is explained as follows. Using the reflective equation (4.28) and Corollary 4.14, we immediately have

Lemma 4.22. *The constant C_1 in (4.16) is expressed by the special values of the regularized Jackson integral as*

$$C_1 = \bar{\mathcal{J}}(\zeta_i(b_1, b_2)) \quad (i = 0, 1, \dots, n).$$

This indicates that $\bar{\mathcal{J}}(\zeta_i(b_1, b_2))$ does not depend on the choice of indices $i = 0, 1, \dots, n$. From this fact, for the proof of Theorem 4.21 it suffices to show that of the case $i = n$ only, i.e.,

Lemma 4.23. *For $x = \bar{\zeta} = (b_1, b_1 t, \dots, b_1 t^{n-1})$, the regularization $\bar{\mathcal{J}}(x)$ is evaluated as*

$$(4.37) \quad \bar{\mathcal{J}}(\bar{\zeta}) = (1 - q)^n \prod_{k=1}^n \frac{(q)_\infty (qt^{-k})_\infty \prod_{i=1}^2 \prod_{j=1}^2 (qa_i^{-1} b_j^{-1} t^{-(k-1)})_\infty}{(qt^{-1})_\infty (qa_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(n+k-2)})_\infty}.$$

Proof. We denote by C the right-hand side of (4.37). Then it is immediate to confirm that C as a function of a_j and b_j satisfies the same q -difference equations as (4.34) and (4.35) of $\bar{\mathcal{J}}(\bar{\zeta})$. Therefore the ratio $\bar{\mathcal{J}}(\bar{\zeta})/C$ is invariant under the q -shift with respect to a_j and b_j .

Next, for an integer N , let T^N be the q -shift operator for a special direction defined as

$$T^N : b_1 \rightarrow b_1 q^{2N}, b_2 \rightarrow b_2 q^{-N}, a_1 \rightarrow a_1 q^{-N}, a_2 \rightarrow a_2 q^{-N}.$$

Since we have

$$\begin{aligned} \bar{\Phi}(z) \Delta(z) &= \prod_{i=1}^n z_i^{1-\alpha_1-\alpha_2-\beta_1-\beta_2-2(i-1)\tau} \frac{(qb_1^{-1} z_i)_\infty}{(a_1 z_i)_\infty} \frac{(qb_2^{-1} z_i)_\infty}{(a_2 z_i)_\infty} \\ &\quad \times \prod_{1 \leq j < k \leq n} \frac{(qt^{-1} z_k / z_j)_\infty}{(tz_k / z_j)_\infty} (1 - z_k / z_j), \end{aligned}$$

by definition $T^N \bar{J}(\bar{\zeta})$ is written as

$$\begin{aligned}
 T^N \bar{J}(\bar{\zeta}) &= (1 - q)^n \sum_{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_n} \prod_{i=1}^n \left[(b_1 t^{i-1} q^{\nu_i + 2N})^{1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 - 2(i-1)\tau + N} \right. \\
 &\quad \times \frac{(qt^{i-1} q^{\nu_i})_\infty (qb_2^{-1} b_1 t^{i-1} q^{\nu_i + 3N})_\infty}{(a_1 b_1 t^{i-1} q^{\nu_i + N})_\infty (a_2 b_1 t^{i-1} q^{\nu_i + N})_\infty} \left. \right] \\
 &\quad \times \prod_{1 \leq j < k \leq n} \frac{(qt^{-1+k-j} q^{\nu_k - \nu_j})_\infty}{(t^{1+k-j} q^{\nu_k - \nu_j})_\infty} (1 - t^{k-j} q^{\nu_k - \nu_j}),
 \end{aligned}$$

so that the leading term of the asymptotic behavior of $T^N \bar{J}(\bar{\zeta})$ as $N \rightarrow +\infty$ is given by the term corresponding to $(\nu_1, \dots, \nu_n) = (0, \dots, 0)$ in the above sum, which is

$$\begin{aligned}
 (4.38) \quad T^N \bar{J}(\bar{\zeta}) &\sim (1 - q)^n \prod_{i=1}^n (b_1 t^{i-1} q^{2N})^{1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 - 2(i-1)\tau + N} (qt^{i-1})_\infty \\
 &\quad \times \prod_{1 \leq j < k \leq n} \frac{(qt^{-1+k-j})_\infty (t^{k-j})_\infty}{(t^{1+k-j})_\infty (qt^{k-j})_\infty} \\
 &= (1 - q)^n \prod_{i=1}^n (b_1 t^{i-1} q^{2N})^{1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 - 2(i-1)\tau + N} \frac{(q)_\infty (t)_\infty}{(t^i)_\infty} \quad (N \rightarrow +\infty).
 \end{aligned}$$

On the other hand, from (4.26), $\bar{h}(\bar{\zeta})C$ is written as

$$\begin{aligned}
 (4.39) \quad \bar{h}(\bar{\zeta})C &= C \prod_{i=1}^n \frac{(b_1 t^{i-1})^{1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 - 2(i-1)\tau} \theta(t)}{\theta(a_1 b_1 t^{i-1}) \theta(a_2 b_1 t^{i-1})} \frac{\theta(t)}{\theta(t^i)} \\
 &= (1 - q)^n \prod_{i=1}^n \left[(b_1 t^{i-1})^{1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 - 2(i-1)\tau} \right. \\
 &\quad \times \left. \frac{(q)_\infty (t)_\infty (qa_1^{-1} b_2^{-1} t^{-(i-1)})_\infty (qa_2^{-1} b_2^{-1} t^{-(i-1)})_\infty}{(t^i)_\infty (a_1 b_1 t^{i-1})_\infty (a_2 b_1 t^{i-1})_\infty (qa_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(n+i-2)})_\infty} \right],
 \end{aligned}$$

so that we have

$$\begin{aligned}
 (4.40) \quad T^N (\bar{h}(\bar{\zeta})C) &= (1 - q)^n \prod_{i=1}^n \left[(b_1 t^{i-1} q^{2N})^{1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 - 2(i-1)\tau + N} \frac{(q)_\infty (t)_\infty}{(t^i)_\infty} \right. \\
 &\quad \times \left. \frac{(a_1^{-1} b_2^{-1} t^{-(i-1)} q^{1+2N})_\infty (a_2^{-1} b_2^{-1} t^{-(i-1)} q^{1+2N})_\infty}{(a_1 b_1 t^{i-1} q^N)_\infty (a_2 b_1 t^{i-1} q^N)_\infty (a_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(n+i-2)} q^{1+N})_\infty} \right] \\
 &\sim (1 - q)^n \prod_{i=1}^n (b_1 t^{i-1} q^{2N})^{1 - \alpha_1 - \alpha_2 - \beta_1 - \beta_2 - 2(i-1)\tau + N} \frac{(q)_\infty (t)_\infty}{(t^i)_\infty} \quad (N \rightarrow +\infty).
 \end{aligned}$$

As we saw, the ratio $\bar{J}(\bar{\zeta})/C$ is invariant under the q -shift with respect to a_j and b_j . Thus $\bar{J}(\bar{\zeta})/C$ is also invariant under the q -shift T^N . Therefore, comparing (4.38) with (4.40), we obtain

$$\frac{\bar{J}(\bar{\zeta})}{C} = T^N \frac{\bar{J}(\bar{\zeta})}{C} = \frac{T^N \bar{J}(\bar{\zeta})}{T^N \bar{h}(\bar{\zeta})C} = \lim_{N \rightarrow +\infty} \frac{T^N \bar{J}(\bar{\zeta})}{T^N \bar{h}(\bar{\zeta})C} = 1,$$

and thus $\bar{J}(\bar{\zeta}) = C$. □

Corollary 4.24. *The truncated Jackson integral $\bar{J}(\bar{\zeta})$ is evaluated as*

$$(4.41) \quad \bar{J}(\bar{\zeta}) = (1 - q)^n \prod_{i=1}^n \left[(b_1 t^{i-1})^{1-\alpha_1-\alpha_2-\beta_1-\beta_2-2(i-1)\tau} \right. \\ \left. \times \frac{(q)_\infty (t)_\infty (qa_1^{-1}b_2^{-1}t^{-(i-1)})_\infty (qa_2^{-1}b_2^{-1}t^{-(i-1)})_\infty}{(t^i)_\infty (a_1b_1t^{i-1})_\infty (a_2b_1t^{i-1})_\infty (qa_1^{-1}a_2^{-1}b_1^{-1}b_2^{-1}t^{-(n+i-2)})_\infty} \right],$$

Proof. From Lemma 4.23, $\bar{J}(\bar{\zeta}) = \bar{h}(\bar{\zeta})\bar{J}(\bar{\zeta})$ is given by (4.39). □

4.5. A remark on the relation between $\bar{J}(\bar{\zeta})$ and $\bar{I}(\bar{\zeta})$ of Aomoto’s setting.

As an application of the q -difference equations (4.32) and (4.33) for $\bar{J}(x)$, we can show that the product formula (3.40) for $\bar{I}(\bar{\zeta})$ in Corollary 3.22 (or (3.11) of $I(\zeta)$ in Proposition 3.4 by the duality (3.33) of parameters) is a special case of (4.37) in Lemma 4.23. This indicates a way to prove the summation formula (3.11) from the product formula of the Jackson integral of Selberg type.

Corollary 4.25. *For the point $x = \bar{\zeta} = (b_1, b_1t, \dots, b_1t^{n-1})$, the truncated Jackson integral $\bar{J}(x)$ of Selberg type is expressed as*

$$(4.42) \quad \bar{J}(\bar{\zeta}) = \bar{I}(\bar{\zeta}) \prod_{i=1}^n \frac{(qa_1^{-1}b_2^{-1}t^{-(i-1)})_\infty}{(b_1a_2t^{i-1})_\infty},$$

where $\bar{I}(\bar{\zeta})$ is the truncated Jackson integral defined by (3.32) with the setting $\alpha = \alpha_2 + \beta_2$. In particular, $\bar{I}(\bar{\zeta})$ is expressed as (3.40).

Remark 4.26. From (4.42), $\bar{I}(\bar{\zeta})$ is a limiting case of $\bar{J}(\bar{\zeta})$ with the q -shift $a_2 \rightarrow q^N a_2$ and $b_2 \rightarrow q^{-N} b_2$ ($N \rightarrow +\infty$). Conversely, the product formula (4.41) of $\bar{J}(\bar{\zeta})$ in Corollary 4.24 is reconstructed from the product formula (3.40) of $\bar{I}(\bar{\zeta})$ via the connection (4.42).

Proof. From (4.32) and (4.33) the recurrence relation of $\bar{J}(\bar{\zeta})$ with respect to the q -shift $a_2 \rightarrow qa_2$ and $b_2 \rightarrow q^{-1}b_2$ is written as

$$\bar{J}(\bar{\zeta}) = T_{b_2}^{-1} T_{a_2} \bar{J}(\bar{\zeta}) \times \prod_{i=1}^n \frac{1 - qa_1^{-1}b_2^{-1}t^{-(i-1)}}{1 - b_1a_2t^{i-1}}.$$

By repeated use of this equation we have

$$(4.43) \quad \bar{J}(\bar{\zeta}) = T_{b_2}^{-N} T_{a_2}^N \bar{J}(\bar{\zeta}) \prod_{i=1}^n \frac{(qa_1^{-1}b_2^{-1}t^{-(i-1)})_N}{(b_1a_2t^{i-1})_N} \\ = \lim_{N \rightarrow \infty} T_{b_2}^{-N} T_{a_2}^N \bar{J}(\bar{\zeta}) \prod_{i=1}^n \frac{(qa_1^{-1}b_2^{-1}t^{-(i-1)})_\infty}{(b_1a_2t^{i-1})_\infty}.$$

Moreover, by definition $\lim_{N \rightarrow \infty} T_{b_2}^{-N} T_{a_2}^N \bar{J}(\bar{\zeta})$ is written as

$$\begin{aligned}
 & \lim_{N \rightarrow \infty} T_{b_2}^{-N} T_{a_2}^N \bar{J}(\bar{\zeta}) \\
 &= \lim_{N \rightarrow \infty} (1-q)^n \sum_{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_n} \prod_{i=1}^n \left[(b_1 t^{i-1} q^{\nu_i})^{1-\alpha_1-\alpha_2-\beta_1-\beta_2-2(i-1)\tau} \right. \\
 & \quad \left. \times \frac{(t^{i-1} q^{1+\nu_i})_\infty (b_2^{-1} b_1 t^{i-1} q^{1+\nu_i+N})_\infty}{(a_1 b_1 t^{i-1} q^{\nu_i})_\infty (a_2 b_1 t^{i-1} q^{\nu_i+N})_\infty} \right] \\
 (4.44) \quad & \quad \times \prod_{1 \leq j < k \leq n} \frac{(t^{k-j-1} q^{1+\nu_k-\nu_j})_\infty}{(t^{k-j+1} q^{\nu_k-\nu_j})_\infty} (1-t^{k-j} q^{\nu_k-\nu_j}) \\
 &= (1-q)^n \sum_{0 \leq \nu_1 \leq \nu_2 \leq \dots \leq \nu_n} \prod_{i=1}^n \left[(b_1 t^{i-1} q^{\nu_i})^{1-\alpha_1-\alpha_2-\beta_1-\beta_2-2(i-1)\tau} \right. \\
 & \quad \left. \times \frac{(t^{i-1} q^{1+\nu_i})_\infty}{(a_1 b_1 t^{i-1} q^{\nu_i})_\infty} \right] \prod_{1 \leq j < k \leq n} \frac{(t^{k-j-1} q^{1+\nu_k-\nu_j})_\infty}{(t^{k-j+1} q^{\nu_k-\nu_j})_\infty} (1-t^{k-j} q^{\nu_k-\nu_j}),
 \end{aligned}$$

which exactly coincides with $\bar{I}(\bar{\zeta})$ under the setting $\alpha = \alpha_2 + \beta_2$. From (4.43) and (4.44), we therefore obtain (4.42).

Next, using (4.42) and (4.41) of Corollary 4.24, the sum $\bar{I}(\bar{\zeta})$ is conversely calculated as

$$\begin{aligned}
 \bar{I}(\bar{\zeta}) &= \bar{J}(\bar{\zeta}) \prod_{i=1}^n \frac{(b_1 a_2 t^{i-1})_\infty}{(q a_1^{-1} b_2^{-1} t^{-(i-1)})_\infty} \\
 &= (1-q)^n \prod_{i=1}^n \frac{(b_1 t^{i-1})^{1-\alpha_1-\alpha_2-\beta_1-\beta_2-2(i-1)\tau} (q)_\infty (t)_\infty (q a_2^{-1} b_2^{-1} t^{-(i-1)})_\infty}{(t^i)_\infty (a_1 b_1 t^{i-1})_\infty (q a_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(n+i-2)})_\infty},
 \end{aligned}$$

which coincides with (3.40) of Corollary 3.22 under the setting $\alpha = \alpha_2 + \beta_2$. □

APPENDIX A. DERIVATION OF THE DIFFERENCE EQUATIONS

The aim of this section is to show a way to derive the q -difference equations (4.32), (4.33) in Proposition 4.20 for $\bar{J}(\zeta_i(b_1, b_2))$ using the shifted symmetric polynomials (the interpolation polynomials), which are defined as follows.

Lemma A.1 (Knop–Sahi). *For $a, t \in \mathbb{C}^*$, $z = (z_1, z_2, \dots, z_n) \in (\mathbb{C}^*)^n$, let $E_i(a; t; z)$ be the polynomials defined by*

$$E_r(a; t; z) := \sum_{1 \leq i_1 < \dots < i_r \leq n} \prod_{k=1}^r (z_{i_k} - a t^{i_k-k}) \quad \text{for } r = 1, 2, \dots, n,$$

which are symmetric with respect to z , where $E_0(a; t; z) = 1$. Then the polynomials $E_i(a; t; z)$ satisfy the vanishing property

$$(A.1) \quad E_i(a; t; \zeta_j) = 0 \quad \text{if } 0 \leq j < i \leq n,$$

where

$$(A.2) \quad \zeta_j := (z_1, z_2, \dots, z_j, \underbrace{a, at, \dots, at^{n-j-1}}_{n-j}) \in (\mathbb{C}^*)^n.$$

Proof. See [33, p. 476, Proposition 3.1]. □

Remark A.2. The polynomials $E_i(a; t; z)$, $i = 0, 1, \dots, n$, are called the *shifted symmetric polynomials* in the context [33]. Using the factor theorem for the vanishing property (A.1) in this lemma, we immediately have

$$(A.3) \quad E_i(a; t; \zeta_i) = \prod_{k=1}^i (z_k - at^{n-i}).$$

In particular, this is consistent with the n th symmetric polynomial having the explicit form

$$(A.4) \quad E_n(a; t; z) = \prod_{k=1}^n (z_k - a).$$

We rewrite the vanishing property (A.1) appropriately for the succeeding arguments of this section.

Lemma A.3. *Suppose that variables z_1, \dots, z_j in ζ_j are real numbers and satisfy*

$$(A.5) \quad z_1 \gg z_2 \gg \dots \gg z_j \gg 0,$$

which means $z_1/z_2 \rightarrow \infty, z_2/z_3 \rightarrow \infty, \dots, z_{j-1}/z_j \rightarrow \infty$ and $z_j \rightarrow \infty$. Then the following asymptotic behavior holds:

$$(A.6) \quad \frac{E_i(a; t; z)\Delta(z)}{z_1^n z_2^{n-1} \dots z_j^{n-j+1}} \Big|_{z=\zeta_j} \sim \delta_{ij} \Delta^{(n-i)}(a, at, \dots, at^{n-i-1}),$$

where $\Delta^{(k)}(z_1, \dots, z_k)$ denotes the difference product $\Delta(z)$ of k variables.

Proof. From (A.1), if $i > j$, then the left-hand side of (A.6) is exactly equal to 0. On the other hand, if $i < j$, then the degree of $E_i(a; t; z)\Delta(z)$ is lower than $z_1^n z_2^{n-1} \dots z_j^{n-j+1}$, so that the left-hand side of (A.6) is estimated as 0 under the condition (A.5). If $i = j$, from (A.3), (3.2) and (A.2), we have

$$E_i(a; t; \zeta_i)\Delta(\zeta_i) = \Delta^{(n-i)}(a, at, \dots, at^{n-i-1})\Delta^{(i)}(z_1, z_2, \dots, z_i) \prod_{j=1}^i \prod_{k=1}^{n-i+1} (z_j - at^{k-1}),$$

which indicates (A.6). □

We will state a key technical lemma for deriving q -difference equations. For this let $\bar{\Phi}(z)$ be the function defined by (4.24) and for a function $\varphi(z)$, define the function $\nabla_i \varphi(z)$ ($1 \leq i \leq n$) by

$$(A.7) \quad (\nabla_i \varphi)(z) := \varphi(z) - \frac{T_{z_i} \bar{\Phi}(z)}{\bar{\Phi}(z)} T_{z_i} \varphi(z),$$

where T_{z_i} means the shift operator of $z_i \rightarrow qz_i$, i.e., $T_{z_i} f(\dots, z_i, \dots) = f(\dots, qz_i, \dots)$. We then have

Lemma A.4. *For a meromorphic function $\varphi(z)$ on $(\mathbb{C}^*)^n$, if the integral*

$$\int_0^{x\infty} \varphi(z)\bar{\Phi}(z)\varpi_q$$

converges, then

$$(A.8) \quad \int_0^{x\infty} \bar{\Phi}(z)\nabla_i \varphi(z)\varpi_q = 0.$$

Moreover,

$$(A.9) \quad \int_0^{x\infty} \bar{\Phi}(z) \mathcal{A} \nabla_i \varphi(z) \varpi_q = 0,$$

where \mathcal{A} indicates the skew-symmetrization defined in (2.3).

Proof. From the definition (A.7) of ∇_i , (A.8) is equivalent to the statement

$$\int_0^{x\infty} \varphi(z) \bar{\Phi}(z) \varpi_q = \int_0^{x\infty} T_{z_i} \varphi(z) T_{z_i} \bar{\Phi}(z) \varpi_q$$

if the left-hand side converges. And this equation is confirmed from the fact that the Jackson integral is invariant under the q -shift $z_i \rightarrow qz_i$ ($1 \leq i \leq n$). Next we will confirm (A.9). Taking account of the quasi-symmetry $\sigma \bar{\Phi}(z) = U_\sigma(z) \bar{\Phi}(z)$, we have

$$\begin{aligned} \bar{\Phi}(z) \mathcal{A} \nabla_i \varphi(z) &= \bar{\Phi}(z) \sum_{\sigma \in S_n} (\text{sgn } \sigma) \sigma(\nabla_i \varphi)(z) \\ &= \sum_{\sigma \in S_n} (\text{sgn } \sigma) U_\sigma(z)^{-1} \sigma \bar{\Phi}(z) \sigma(\nabla_i \varphi)(z) \\ &= \sum_{\sigma \in S_n} (\text{sgn } \sigma) U_\sigma(z)^{-1} \sigma \left(\bar{\Phi}(z) \nabla_i \varphi(z) \right). \end{aligned}$$

Since $U_\sigma(z)$ is invariant under the q -shift $z_i \rightarrow qz_i$ ($1 \leq i \leq n$), we therefore obtain

$$\begin{aligned} \int_0^{x\infty} \bar{\Phi}(z) \mathcal{A} \nabla_i \varphi(z) \varpi_q &= \sum_{\sigma \in S} (\text{sgn } \sigma) U_\sigma(x)^{-1} \int_0^{x\infty} \sigma \left(\bar{\Phi}(z) \nabla_i \varphi(z) \right) \varpi_q \\ &= \sum_{\sigma \in S} (\text{sgn } \sigma) U_\sigma(x)^{-1} \int_0^{\sigma^{-1}x\infty} \bar{\Phi}(z) \nabla_i \varphi(z) \varpi_q \\ &= \sum_{\sigma \in S} (\text{sgn } \sigma) U_\sigma(x)^{-1} \sigma \int_0^{x\infty} \bar{\Phi}(z) \nabla_i \varphi(z) \varpi_q, \end{aligned}$$

which vanishes from (A.8). □

We set

$$(A.10) \quad e_i(a; t; z) := E_i(a; t; z^{-1}),$$

where z^{-1} is specified by (3.29). Since we have

$$\frac{T_{a_i} \bar{\Phi}(z)}{\bar{\Phi}(z)} = \prod_{j=1}^n (z_j^{-1} - a_i) = e_n(a_i; t; z), \quad \frac{T_{b_i} \bar{\Phi}(z)}{\bar{\Phi}(z)} = \prod_{j=1}^n (z_j^{-1} - b_i^{-1}) = e_n(b_i^{-1}; t^{-1}; z),$$

the q -shifts of $\bar{J}(x)$ with respect to a_i and b_i are expressed by

$$(A.11) \quad T_{a_i} \bar{J}(x) = \int_0^{x\infty} e_n(a_i; t; z) \bar{\Phi}(z) \Delta(z) \varpi_q,$$

$$(A.12) \quad T_{b_i} \bar{J}(x) = \int_0^{x\infty} e_n(b_i^{-1}; t^{-1}; z) \bar{\Phi}(z) \Delta(z) \varpi_q.$$

Lemma A.5. *Suppose that $x = \zeta_k(b_1, b_2)$ ($k = 0, 1, \dots, n$), where $\zeta_k(x_1, x_2)$ is defined by (4.4). Then the relation between $e_i(a_j; t; z)$ and $e_{i-1}(a_j; t; z)$ via the truncated Jackson integral is expressed as*

$$(A.13) \quad \int_0^x e_i(a_j; t; z) \bar{\Phi}(z) \Delta(z) \varpi_q \\ = (-a_j) \frac{(1 - t^{n-i+1})(1 - a_j^{-1} b_1^{-1} t^{-(n-i)})(1 - a_j^{-1} b_2^{-1} t^{-(n-i)})}{(1 - t^i)(1 - a_1^{-1} a_2^{-1} b_1^{-1} b_2^{-1} t^{-(2n-i-1)})} \\ \times \int_0^x e_{i-1}(a_j; t; z) \bar{\Phi}(z) \Delta(z) \varpi_q,$$

and the relation between $e_i(b_j^{-1}; t^{-1}; z)$ and $e_{i-1}(b_j^{-1}; t^{-1}; z)$ is expressed as

$$(A.14) \quad \int_0^x e_i(b_j^{-1}; t^{-1}; z) \bar{\Phi}(z) \Delta(z) \varpi_q \\ = (-b_j^{-1}) \frac{(1 - t^{-(n-i+1)})(1 - a_1 b_j t^{n-i})(1 - a_2 b_j t^{n-i})}{(1 - t^{-i})(1 - a_1 a_2 b_1 b_2 t^{2n-i-1})} \\ \times \int_0^x e_{i-1}(b_j^{-1}; t^{-1}; z) \bar{\Phi}(z) \Delta(z) \varpi_q.$$

Remark A.6. The relations (A.13) and (A.14) are identical upon the interchange of parameters $(a_1, a_2, b_1, b_2, t) \rightarrow (b_1^{-1}, b_2^{-1}, a_1^{-1}, a_2^{-1}, t^{-1})$.

Remark A.7. By repeated use of (A.13), from (A.11), we immediately obtain the q -difference equation (4.32) presented in Proposition 4.20. In the same manner, the q -difference equation (4.33) in Proposition 4.20 is deduced from (A.14) using (A.12).

The rest of this subsection is devoted to the proof of the above lemma. We will show a further lemma before proving Lemma A.5. For this purpose we abbreviate $e_i(a_1; t; z)$ and $E_i(a_1; t; z)$ by $e_i(z)$ and $E_i(z)$, respectively, and (k) of $e_i^{(k)}(z)$, $E_i^{(k)}(z)$, $\Delta^{(k)}(z)$ means that these functions are of k variables. We also use the symbol $(\widehat{z}_i) := (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_n) \in (\mathbb{C}^*)^{n-1}$ for $i = 1, \dots, n$.

Lemma A.8. *Put*

$$(A.15) \quad \phi(z) := z_1^{-1}(z_1 - b_1)(z_1 - b_2) \prod_{k=2}^n (z_1 - tz_k).$$

Then

$$(A.16) \quad (-1)^{n-1} \mathcal{A} \left[\nabla_1 \left(\phi(z) e_{i-1}^{(n-1)}(\widehat{z}_1) \Delta^{(n-1)}(\widehat{z}_1) \right) \right] = \left(c_i e_i^{(n)}(z) + c_{i-1} e_{i-1}^{(n)}(z) \right) \Delta^{(n)}(z),$$

where the coefficients c_{i-1} and c_i are given by

$$(A.17) \quad c_{i-1} = (n-1)! (-1)^{n-1} \frac{(1 - a_1 b_1 t^{n-i})(1 - a_1 b_2 t^{n-i})(1 - t^{n-i+1})}{a_1 t^{n-i} (1-t)},$$

$$(A.18) \quad c_i = (n-1)! (-1)^n \frac{(1 - a_1 a_2 b_1 b_2 t^{2n-i-1})(1 - t^i)}{a_1 a_2 t^{n-1} (1-t)}.$$

Proof. We initially compute $\nabla_1\phi(z)$. By definition $T_{z_1}\bar{\Phi}(z)/\bar{\Phi}(z)$ is written as

$$\begin{aligned} \frac{T_{z_1}\bar{\Phi}(z)}{\bar{\Phi}(z)} &= (a_1a_2b_1b_2q^{n-2})^{-1} \frac{(1-a_1z_1)(1-a_2z_1)}{(1-qb_1^{-1}z_1)(1-qb_2^{-1}z_1)} \prod_{i=2}^n \frac{1-t^{-1}z_i/z_1}{1-q^{-1}tz_i/z_1} \\ (A.19) \quad &= \frac{q^{-n}z_1^{-n}(1-a_1^{-1}z_1^{-1})(1-a_2^{-1}z_1^{-1})}{z_1^{-n}(1-q^{-1}b_1z_1^{-1})(1-q^{-1}b_2z_1^{-1})} \prod_{k=2}^n \frac{1-t^{-1}z_1^{-1}/z_k^{-1}}{1-q^{-1}tz_1^{-1}/z_k^{-1}}. \end{aligned}$$

Since the function $\phi(z)$ in (A.15) is rewritten as

$$(A.20) \quad \phi(z) = \frac{1}{z_1^{-n}}(1-b_1z_1^{-1})(1-b_2z_1^{-1}) \prod_{k=2}^n (1-tz_1^{-1}/z_k^{-1}),$$

using (A.7) and (A.19) the function $\nabla_1\phi(z)$ is computed explicitly as

$$(A.21) \quad \nabla_1\phi(z) = \phi(z) - \frac{1}{z_1^{-n}}(1-a_1^{-1}z_1^{-1})(1-a_2^{-1}z_1^{-1}) \prod_{k=2}^n (1-t^{-1}z_1^{-1}/z_k^{-1}).$$

Next, using (A.10), the equation (A.16) is transformed by $z \rightarrow z^{-1}$, i.e.,

$$\begin{aligned} &(-1)^{n-1} \mathcal{A} \left[\left(\nabla_1\phi(z^{-1}) \right) E_{i-1}^{(n-1)}(\widehat{z}_1) \Delta^{(n-1)}(\widehat{z}_1^{-1}) \right] \\ &= \left(c_i E_i^{(n)}(z) + c_{i-1} E_{i-1}^{(n)}(z) \right) \Delta^{(n)}(z^{-1}), \end{aligned}$$

which is rewritten as

$$\begin{aligned} &(-1)^{n-1} \mathcal{A} \left[\nabla_1\phi(z^{-1}) E_{i-1}^{(n-1)}(\widehat{z}_1) \frac{(-1)^{\binom{n-1}{2}} \Delta^{(n-1)}(\widehat{z}_1)}{(z_2 \cdots z_n)^{n-2}} \right] \\ &= \left(c_i E_i^{(n)}(z) + c_{i-1} E_{i-1}^{(n)}(z) \right) \frac{(-1)^{\binom{n}{2}} \Delta^{(n)}(z)}{(z_1 z_2 \cdots z_n)^{n-1}}, \end{aligned}$$

so that it suffices to prove the following instead of (A.16):

$$(A.22) \quad \begin{aligned} &\mathcal{A} \left[z_1^{n-1} z_2 \cdots z_n \nabla_1\phi(z^{-1}) E_{i-1}^{(n-1)}(\widehat{z}_1) \Delta^{(n-1)}(\widehat{z}_1) \right] \\ &= \left(c_i E_i^{(n)}(z) + c_{i-1} E_{i-1}^{(n)}(z) \right) \Delta^{(n)}(z). \end{aligned}$$

We will prove the above equation. If we put

$$\bar{\phi}_i(z) := z_1^{n-1} z_2 \cdots z_n \nabla_1\phi(z^{-1}) E_{i-1}^{(n-1)}(\widehat{z}_1) \Delta^{(n-1)}(\widehat{z}_1),$$

then, from (A.21) $\bar{\phi}_i(z)$ is computed as

$$(A.23) \quad \begin{aligned} \bar{\phi}_i(z) &= z_1^{-1} \left[(1-b_1z_1)(1-b_2z_1) \prod_{k=2}^n (z_k - tz_1) \right. \\ &\quad \left. - (1-a_1^{-1}z_1)(1-a_2^{-1}z_1) \prod_{k=2}^n (z_k - t^{-1}z_1) \right] E_{i-1}^{(n-1)}(\widehat{z}_1) \Delta^{(n-1)}(\widehat{z}_1), \end{aligned}$$

which is a polynomial of z_1, \dots, z_n . Taking account of the degree of the polynomial $\bar{\phi}_i(z)$,

$$\mathcal{A}(z_1 z_2 \cdots z_i \times z_1^{n-1} z_2^{n-2} \cdots z_{n-1})$$

is the term of highest degree in the skew-symmetrization $\mathcal{A}\bar{\phi}_i(z)$, which is thus expanded as

$$(A.24) \quad \mathcal{A}\bar{\phi}_i(z) = \sum_{j=0}^i c_j E_j^{(n)}(z) \Delta^{(n)}(z),$$

where c_j are some constants. For any $x \in \mathbb{R}$ we set

$$\xi_j := \underbrace{(x^j, x^{j-1}, \dots, x^2, x)}_j, \underbrace{(a_1, a_1 t, \dots, a_1 t^{n-j-1})}_{n-j} \in (\mathbb{C}^*)^n,$$

which is a special case of ζ_j specified by (A.2). Then, from Lemma A.3, c_j in (A.24) is written as

$$(A.25) \quad \lim_{x \rightarrow \infty} \frac{\mathcal{A}\bar{\phi}_i(z)}{z_1^n z_2^{n-1} \dots z_j^{n-j+1}} \Big|_{z=\xi_j} = c_j \lim_{x \rightarrow \infty} \frac{E_j^{(n)}(z) \Delta^{(n)}(z)}{z_1^n z_2^{n-1} \dots z_j^{n-j+1}} \Big|_{z=\xi_j} \\ = c_j \Delta^{(n-j)}(a_1, a_1 t, \dots, a_1 t^{n-j-1}).$$

On the other hand, from the explicit form (A.23) of $\bar{\phi}_i(z)$, $\mathcal{A}\bar{\phi}_i(z)$ is also expressed as

$$(A.26) \quad \mathcal{A}\bar{\phi}_i(z) = (n-1)! \sum_{j=1}^n (-1)^{j-1} (F_j(z) - G_j(z)) \Delta^{(n-1)}(\widehat{z}_j) E_{i-1}^{(n-1)}(\widehat{z}_j),$$

where

$$(A.27) \quad F_j(z) = z_j^{-1} (1 - b_1 z_j) (1 - b_2 z_j) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} (z_k - t z_j),$$

$$(A.28) \quad G_j(z) = z_j^{-1} (1 - a_1^{-1} z_j) (1 - a_2^{-1} z_j) \prod_{\substack{1 \leq k \leq n \\ k \neq j}} (z_k - t^{-1} z_j),$$

which satisfy the vanishing property

$$(A.29) \quad \begin{aligned} F_j(\xi_k) &= 0 & \text{if } k < j < n, \\ G_j(\xi_k) &= 0 & \text{if } k < j \leq n, \end{aligned}$$

and the evaluation

$$(A.30) \quad \lim_{x \rightarrow \infty} \frac{F_n(z)}{z_1 z_2 \dots z_k} \Big|_{z=\xi_k} = \frac{(1 - a_1 b_1 t^{n-k-1})(1 - a_1 b_2 t^{n-k-1})(1 - t^{n-k})}{a_1 t^{n-k-1} (1 - t)} \\ \times \prod_{j=1}^{n-k-1} (a_1 t^{j-1} - a_1 t^{n-k-1}).$$

Notice that, from Lemma A.3, $E_{i-1}^{(n-1)}(\widehat{z}_j) \Delta^{(n-1)}(\widehat{z}_j)$ satisfies the vanishing properties

$$(A.31) \quad \lim_{x \rightarrow \infty} \frac{E_{i-1}^{(n-1)}(\widehat{z}_j) \Delta^{(n-1)}(\widehat{z}_j)}{z_1^{n-1} \dots z_{j-1}^{n-j+1} z_{j+1}^{n-j} \dots z_k^{n-k+1}} \Big|_{z=\xi_k} \\ = \delta_{ik} \Delta^{(n-k)}(a_1, a_1 t, \dots, a_1 t^{n-k-1}) \quad \text{if } 1 \leq j \leq k$$

and

$$(A.32) \quad \lim_{x \rightarrow \infty} \frac{E_{i-1}^{(n-1)}(\widehat{z}_n) \Delta^{(n-1)}(\widehat{z}_n)}{z_1^{n-1} z_2^{n-2} \dots z_k^{n-k}} \Big|_{z=\xi_k} = \delta_{i-1,k} \Delta^{(n-k-1)}(a_1, a_1 t, \dots, a_1 t^{n-k-2}).$$

We now prove the expression (A.22). If $0 \leq k \leq i - 2$, then, from (A.29), (A.31) and (A.32) the equation (A.26) indicates that

$$(A.33) \quad \lim_{x \rightarrow \infty} \frac{\mathcal{A}\bar{\phi}_i(z)}{z_1^n z_2^{n-1} \dots z_k^{n-k+1}} \Big|_{z=\xi_k} = 0.$$

Comparing (A.25) with (A.33), we obtain $c_0 = c_1 = \dots = c_{i-2} = 0$, which means (A.22) holds.

Next we will evaluate c_{i-1} . From (A.29), (A.30), (A.31) and (A.32),

$$(A.34) \quad \begin{aligned} & \lim_{x \rightarrow \infty} \frac{\mathcal{A}\bar{\phi}_i(z)}{z_1^n z_2^{n-1} \dots z_{i-1}^{n-i+2}} \Big|_{z=\xi_{i-1}} \\ &= (n-1)!(-1)^{n-1} \lim_{x \rightarrow \infty} \frac{F_n(z)}{z_1 z_2 \dots z_{i-1}} \frac{E_{i-1}^{(n-1)}(\widehat{z}_n) \Delta^{(n-1)}(\widehat{z}_n)}{z_1^{n-1} z_2^{n-2} \dots z_{i-1}^{n-i+1}} \Big|_{z=\xi_{i-1}} \\ &= (n-1)!(-1)^{n-1} \frac{(1 - a_1 b_1 t^{n-i})(1 - a_1 b_2 t^{n-i})(1 - t^{n-i+1})}{a_1 t^{n-i}(1 - t)} \\ & \quad \times \left[\prod_{j=1}^{n-i} (a_1 t^{j-1} - a_1 t^{n-i}) \right] \Delta^{(n-i)}(a_1, a_1 t, \dots, a_1 t^{n-i-1}). \end{aligned}$$

Comparing (A.25) with (A.34) using the relation

$$\frac{\Delta^{(n-i+1)}(a_1, a_1 t, \dots, a_1 t^{n-i})}{\Delta^{(n-i)}(a_1, a_1 t, \dots, a_1 t^{n-i-1})} = \prod_{j=1}^{n-i} (a_1 t^{j-1} - a_1 t^{n-i}),$$

we therefore obtain the explicit expression of c_{i-1} as (A.17).

Lastly we will evaluate c_i . From the explicit forms (A.27) and (A.28) of $F_j(z)$ and $G_j(z)$, we have

$$\lim_{x \rightarrow \infty} \frac{F_j(z) - G_j(z)}{z_1 z_2 \dots z_{j-1} z_j^{n-j+1}} \Big|_{z=\xi_i} = (-1)^{n-j} (b_1 b_2 t^{n-j} - a_1^{-1} a_2^{-1} t^{-(n-j)}) \quad \text{if } 1 \leq j \leq i.$$

Using (A.29), (A.31), (A.32) and the above evaluation, we have

$$(A.35) \quad \begin{aligned} & \lim_{x \rightarrow \infty} \frac{\mathcal{A}\bar{\phi}_i(z)}{z_1^n z_2^{n-1} \dots z_i^{n-i+1}} \Big|_{z=\xi_i} \\ &= (n-1)! \sum_{j=1}^i (-1)^{j-1} \lim_{x \rightarrow \infty} \frac{F_j(z) - G_j(z)}{z_1 z_2 \dots z_{j-1} z_j^{n-j+1}} \frac{E_{i-1}^{(n-1)}(\widehat{z}_j) \Delta^{(n-1)}(\widehat{z}_j)}{z_1^{n-1} \dots z_{j-1}^{n-j+1} z_{j+1}^{n-j} \dots z_i^{n-i+1}} \Big|_{z=\xi_i} \\ &= (n-1)!(-1)^{n-1} \Delta^{(n-i)}(a_1, a_1 t, \dots, a_1 t^{n-i-1}) \sum_{j=1}^i (b_1 b_2 t^{n-j} - a_1^{-1} a_2^{-1} t^{-(n-j)}) \\ &= (n-1)!(-1)^n \frac{(1 - a_1 a_2 b_1 b_2 t^{2n-i-1})(1 - t^i)}{a_1 a_2 t^{n-1}(1 - t)} \Delta^{(n-i)}(a_1, a_1 t, \dots, a_1 t^{n-i-1}). \end{aligned}$$

Comparing (A.25) with (A.35), we therefore obtain the explicit expression of c_i as (A.18). □

Proof of Lemma A.5. We will prove (A.13) for a_j first. Without loss of generality, it suffices to show (A.13) for a_1 . Suppose that $x = \zeta_k(b_1, b_2)$, $k = 0, 1, \dots, n$. If we set $\varphi(z) = \phi(z) e_{i-1}^{(n-1)}(\widehat{z}_1) \Delta^{(n-1)}(\widehat{z}_1)$, where $\phi(z)$ is defined by (A.15), then

the truncated Jackson integral $\int_0^x \varphi(z) \bar{\Phi}(z) \varpi_q$ converges absolutely if $|a_1 a_2 b_1 b_2|$ is sufficiently large. Therefore, applying (A.9) in Lemma A.4 to the fact (A.16) in Lemma A.8, we obtain the relation

$$c_i \int_0^x e_i(a_1; t; z) \bar{\Phi}(z) \Delta(z) \varpi_q + c_{i-1} \int_0^x e_{i-1}(a_1; t; z) \bar{\Phi}(z) \Delta(z) \varpi_q = 0,$$

where c_{i-1} and c_i are given in (A.17) and (A.18), respectively. This relation coincides with (A.13).

Next we will show (A.14) for b_j of the case $j = 1$ in the same manner as above. Here if we exchange a_1, a_2 and t with b_1^{-1}, b_2^{-1} and t^{-1} , respectively, in the above proof of (A.13), including that of Lemma A.8, the argument is completely symmetric for this exchange. Therefore (A.14) for b_1 is obtained exchanging a_1, a_2 and t with b_1^{-1}, b_2^{-1} and t^{-1} , respectively, on the coefficient of (A.13). \square

Remark A.9. In the above proof, the assumption $x = \zeta_k(b_1, b_2)$, $k = 0, 1, \dots, n$, for the truncated Jackson integral $\int_0^x \varphi(z) \bar{\Phi}(z) \varpi_q$ is necessary from the technical viewpoint. In the case for any $x \neq \zeta_i(b_1, b_2)$, taking account of the influence of the terms $\varphi(xq^\nu) \bar{\Phi}(xq^\nu)$, $\nu \notin \Lambda_i$, the convergence of $\int_0^{x\infty} \varphi(z) \bar{\Phi}(z) \varpi_q$ is very subtle, and generally it is not assured. Lemma A.4 requires this convergence.

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