TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 5, May 2017, Pages 3443–3481
http://dx.doi.org/10.1090/tran/6751
Article electronically published on September 27, 2016

ALGEBRAIC SUPERGROUPS AND HARISH-CHANDRA PAIRS OVER A COMMUTATIVE RING

AKIRA MASUOKA AND TAIKI SHIBATA

Abstract. We prove a category equivalence between algebraic supergroups and Harish-Chandra pairs over a commutative ring which is 2-torsion free. The result is applied to reconstruct the Chevalley $\mathbb{Z}$-supergroups constructed by Fioresi and Gavarini (2012) and by Gavarini (2014). For a wide class of algebraic supergroups we describe their representations by using their super-hyperalgebras.

1. Introduction

Let $\Bbbk$ be a non-zero commutative ring over which we work. The word "super" is used as a synonym of "graded by $\mathbb{Z}_2 = \{0, 1\}$". Ordinary objects, such as Lie/Hopf algebras, which are defined in the tensor category of $\Bbbk$-modules, given the trivial symmetry $v \otimes w \mapsto w \otimes v$, are generalized by their super-analogues, such as Lie/Hopf superalgebras, which are defined in the tensor category of $\mathbb{Z}_2$-graded $\Bbbk$-modules, given the super-symmetry (2.1). We are mainly concerned with the super-analogues of affine/algebraic groups. By saying *affine groups* (resp., *algebraic groups*), we mean, following Jantzen [14], what are formally called affine group schemes (resp., affine algebraic group schemes), and we will use analogous simpler names for their super-analogues.

An *algebraic supergroup* (over $\Bbbk$) is thus a representable group-valued functor $\mathbf{G}$ defined on the category of commutative superalgebras over $\Bbbk$, such that the commutative Hopf superalgebra $\mathcal{O}(\mathbf{G})$ representing $\mathbf{G}$ is finitely generated; see [4, Chapter 11], for example. Associated with such $\mathbf{G}$ are a Lie superalgebra, $\mathrm{Lie}(\mathbf{G})$, and an algebraic group, $\mathbf{G}_{ev}$. The latter is the (necessarily representable) group-valued functor obtained from $\mathbf{G}$ by restricting the domain to the category of commutative algebras.

Important examples of algebraic supergroups over the complex number field $\mathbb{C}$ are *Chevalley $\mathbb{C}$-supergroups*; they are the algebraic supergroups $\mathbf{G}$ over $\mathbb{C}$ such that $\mathrm{Lie}(\mathbf{G})$ is one of the complex simple Lie superalgebras, which were classified by Kac [15]. Just as Kostant [16] once did in the classical, non-super situation, Fioresi and Gavarini constructed natural $\mathbb{Z}$-forms of the Chevalley $\mathbb{C}$-supergroups; see [8–10]. Those $\mathbb{Z}$-forms, called *Chevalley $\mathbb{Z}$-supergroups*, are important and would be useful especially to study Chevalley supergroups in positive characteristic. A motivation of this paper is to make part of Fioresi and Gavarini's construction

Received by the editors May 30, 2013 and, in revised form, March 22, 2015 and May 9, 2015.
2010 *Mathematics Subject Classification.* Primary 14M30, 16T05, 16W55.
Key words and phrases. Algebraic supergroup, Hopf superalgebra, Harish-Chandra pair, super-hyperalgebra, Chevalley supergroup.
The first author was supported by JSPS Grant-in-Aid for Scientific Research (C) 23540039.
The second author was supported by Grant-in-Aid for JSPS Fellows 26E2022.

simpler and more rigorous, and we realize it by using *Harish-Chandra pairs*, as will be explained below. Their construction is parallel to the classical one; it starts with (1) proving the existence of "Chevalley basis" for each complex simple Lie superalgebra $\mathfrak{g}$, and then turns to (2) constructing from the basis a natural $\mathbb{Z}$-form, called a *Kostant superalgebra*, of $\mathbf{U}(\mathfrak{g})$. Our construction, which will be given in Section 6, uses results from these (1) and (2), but dispenses with the following procedures, which include choosing a faithful representation of $\mathfrak{g}$ on a finite-dimensional complex super-vector space including an appropriate $\mathbb{Z}$-lattice; see Remarks 6.3 and 6.8.

In this and the following paragraphs, let us suppose that $\Bbbk$ is a field of characteristic $\neq 2$. Even in this case, algebraic supergroups have not been studied so long as Lie supergroups. Indeed, the latter has a longer history of study founded by Kostant [17], Koszul [18] and others in the 1970's. An important result from the study is the equivalence, shown by Kostant, between the category of Lie supergroups and the category of Harish-Chandra pairs; see [4, Section 7.4], [28]. The corresponding result for algebraic supergroups, that is, the equivalence

$$\mathsf{ASG} \approx \mathsf{HCP} \tag{1.1}$$

between the category ASG of algebraic supergroups and the category HCP of Harish-Chandra pairs, was only recently proved by Carmeli and Fioresi [5] when $\Bbbk = \mathbb{C}$, and then by the first-named author [20] for an arbitrary field of characteristic $\neq 2$; see [12, 20] for applications of the result. As was done for Lie supergroups, Carmeli and Fioresi define a *Harish-Chandra pair* to be a pair $(G, \mathfrak{g})$ of an algebraic group G and a finite-dimensional Lie superalgebra $\mathfrak{g}$ which satisfy some conditions (see Definition 4.4) and prove that the equivalence (1.1) is given by $\mathbf{G} \mapsto (\mathbf{G}_{ev}, \mathrm{Lie}(\mathbf{G}))$ (see the third paragraph above). In [20], the definition of Harish-Chandra pairs and the category equivalence are given by purely Hopf algebraic terms, but they will be easily seen to be essentially the same as those in [5] and in this paper; see Remarks 4.5 (1) and 4.27.

To prove the category equivalence, the articles [5] and [20] both use the following property of $\mathcal{O}(\mathbf{G})$, which was proved in [19] and will be reproduced as Theorem 2.3 below: given $\mathbf{G} \in \mathsf{ASG}$, the Hopf superalgebra $\mathcal{O}(\mathbf{G})$ is *split* in the sense that there exists a counit-preserving isomorphism

$$\mathcal{O}(\mathbf{G}) \simeq \mathcal{O}(\mathbf{G}_{ev}) \otimes \wedge(W) \tag{1.2}$$

of left $\mathcal{O}(\mathbf{G}_{ev})$-comodule superalgebras, where W is the odd component of the cotangent super-vector space of $\mathbf{G}$ at 1, and $\wedge(W)$ is the exterior algebra on it. This basic property played a role in [22] as well; see also [21]. As another application of the property we will prove a representation-theoretic result, Corollary 5.10, which generalizes results which were proved in [2, 3, 24] for some special algebraic supergroups.

Throughout the text of this paper we assume that $\Bbbk$ is a non-zero commutative ring which is 2-*torsion free* or, namely, is such that an element $a \in \Bbbk$ must be zero whenever $2a = 0$. We chose this assumption because it seems natural, in order to keep the super-symmetry (2.1) non-trivial. Our main result, Theorem 4.22, proves the category equivalence (1.1) over such $\Bbbk$ as above. We pose some assumptions to objects in the relevant categories, which are necessarily satisfied if $\Bbbk$ is a field. Indeed, an algebraic supergroup $\mathbf{G}$ in ASG is required to satisfy, in particular, the condition that $\mathcal{O}(\mathbf{G})$ is split, while an object $(G, \mathfrak{g})$ in HCP is required to satisfy, in particular, the condition that $\mathfrak{g}$ is *admissible* (see Definition 3.1), and so, given an odd element $v \in \mathfrak{g}_1$, the even component $\mathfrak{g}_0$ of $\mathfrak{g}$ must contain a unique element, $\frac{1}{2}[v,v]$, whose double equals $[v,v]$; see Section 4.3 and Definition 4.4 for the precise definitions of ASG and HCP, respectively. A novelty of our proof of the result is to construct a functor $\mathbf{G} : \mathsf{HCP} \to \mathsf{ASG}$, which will be proved an equivalence, as follows: given $(G, \mathfrak{g}) \in \mathsf{HCP}$, we realize the Hopf superalgebra $\mathcal{O}(\mathbf{G})$ corresponding to $\mathbf{G} = \mathbf{G}(G, \mathfrak{g})$ as a discrete Hopf super-subalgebra of some complete topological Hopf superalgebra, $\widehat{\mathcal{A}}$, that is simply constructed from the given pair. Indeed, this Hopf algebraic idea was used in [20], but our construction has been modified to be applicable when $\Bbbk$ is a commutative ring. Based on the proved equivalence we will reconstruct the Chevalley $\mathbb{Z}$-supergroups by giving the corresponding Harish-Chandra pairs.

The category equivalence theorem, Theorem 4.22, is proved in Section 4, while the Chevalley $\mathbb{Z}$-supergroups are reconstructed in Section 6. The contents of the remaining three sections are as follows. Section 2 is devoted to preliminaries on Hopf superalgebras and affine/algebraic supergroups. In Section 3, admissible Lie superalgebras are discussed. Especially, we prove in Corollary 3.6 that the universal envelope $\mathbf{U}(\mathfrak{g})$ of such a Lie superalgebra $\mathfrak{g}$ has the property which is dual to the splitting property (1.2); the corollary plays a role in the proof of our main result. In Section 5, we discuss supermodules over an algebraic supergroup $\mathbf{G}$ and over the super-hyperalgebra $\mathrm{hy}(\mathbf{G})$ of $\mathbf{G}$, when $\mathbf{G}_{ev}$ is a split reductive algebraic group. Let T be a split maximal torus of such $\mathbf{G}_{ev}$. Theorem 5.8 shows, roughly speaking, equivalence of $\mathbf{G}$-supermodules with $\mathrm{hy}(\mathbf{G})$-T-supermodules. When $\Bbbk$ is a field, the theorem gives Corollary 5.10 cited before.

After an earlier version of this paper was submitted, the article [11] by Gavarini was in circulation. Theorem 4.3.14 of [11] essentially proves our category equivalence theorem in the generalized situation that $\Bbbk$ is an arbitrary commutative ring. A point is to use the additional structure, called 2-*operations*, on Lie superalgebras $\mathfrak{g}$, which generalizes the map $v \mapsto \frac{1}{2}[v,v]$, $\mathfrak{g}_1 \to \mathfrak{g}_0$ given on an admissible Lie superalgebra in our situation. Given a Harish-Chandra pair, Gavarini constructs an affine supergroup in a quite different method from ours, realizing it as a group sheaf in the Zariski topology. In the appendix of this paper we will refine his category equivalence, using our construction and giving detailed arguments on 2-operations, in particular. This will not be meaningless because such detailed arguments are not given in [11]; see Remark A.11.

2. Preliminaries

2.1. We work over a non-zero commutative ring $\Bbbk$. Throughout in what follows, except in the appendix, we assume that $\Bbbk$ is 2-*torsion free*; this means that an element $a \in \Bbbk$ must be zero whenever $2a = 0$. It follows that any flat $\Bbbk$-module is 2-torsion free.

A $\Bbbk$-module is said to be $\Bbbk$-*finite* (resp., $\Bbbk$-*finite free/projective*) if it is finitely generated (resp., finitely generated and free/projective).

The unadorned $\otimes$ denotes the tensor product over $\Bbbk$. We let Hom denote the $\Bbbk$-module consisting of $\Bbbk$-linear maps. Given a $\Bbbk$-module V, we let V^* denote the dual $\Bbbk$-module $\mathrm{Hom}(V, \Bbbk)$ of V.

2.2. A *supermodule* (over $\Bbbk$) is precisely a $\Bbbk$-module $V = V_0 \oplus V_1$ graded by the group $\mathbb{Z}_2 = \{0, 1\}$ of order 2. The degree of a homogeneous element $v \in V$ is denoted by $|v|$. Such an element is said to be *even* (resp., *odd*) if $|v| = 0$ (resp., if $|v| = 1$). We say that V is *purely even* (resp., *purely odd*) if $V = V_0$ (resp., if $V = V_1$). The supermodules $V, W, \dots$ and the $\mathbb{Z}_2$-graded (or super-)linear maps naturally form a tensor category SMod; the tensor product is the $\Bbbk$-module $V \otimes W$ graded so that $(V \otimes W)_i = \bigoplus_{j+k=i} V_j \oplus W_k$, $i = 0, 1$, and the unit object is $\Bbbk$, which is supposed to be purely even. The tensor category is symmetric with respect to the so-called *super-symmetry* $c_{V,W} : V \otimes W \xrightarrow{\simeq} W \otimes V$ defined by

$$c_{V,W}(v \otimes w) = (-1)^{|v||w|} w \otimes v = \begin{cases} -w \otimes v & \text{if } v,\ w \text{ are odd,} \\ \ \ w \otimes v & \text{otherwise.} \end{cases} \tag{2.1}$$

The dual $\Bbbk$-module V^* of a supermodule V is a supermodule graded so that $(V^*)_i = (V_i)^*$, $i = 0, 1$.

Ordinary objects, such as Lie algebras or Hopf algebras, defined in the symmetric category of $\Bbbk$-modules are generalized by super-objects, such as Lie superalgebras or Hopf superalgebras, defined in SMod. The ordinary objects are regarded as purely even super-objects.

A superalgebra (resp., super-coalgebra) is said to be *commutative* (resp., *cocommutative*) if the product (resp., coproduct) is invariant, composed with the super-symmetry.

2.3. Given a Hopf superalgebra $\mathbf{A}$, we denote the coproduct, the counit and the antipode by

$$\Delta : \mathbf{A} \to \mathbf{A} \otimes \mathbf{A},\ \Delta(a) = a_{(1)} \otimes a_{(2)}, \quad \varepsilon : \mathbf{A} \to \Bbbk, \quad S : \mathbf{A} \to \mathbf{A},$$

respectively. The antipode S preserves the unit and the counit and satisfies

$$m \circ (S \otimes S) = S \circ m \circ c_{\mathbf{A},\mathbf{A}}, \quad (S \otimes S) \circ \Delta = c_{\mathbf{A},\mathbf{A}} \circ \Delta \circ S,$$

where $m : \mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$ denotes the product. We let $\mathbf{A}^+$ denote the augmentation super-ideal $\operatorname{Ker} \varepsilon$ of $\mathbf{A}$.

Let $\mathbf{H}$, $\mathbf{A}$ be Hopf superalgebras. A bilinear map $\langle\ ,\ \rangle : \mathbf{H} \times \mathbf{A} \to \Bbbk$ is called a *Hopf pairing* [20, Section 2.2] if $\langle \mathbf{H}_i, \mathbf{A}_j \rangle = 0$ whenever $i \neq j$, and if we have

$$\begin{aligned} \langle xy, a \rangle &= \langle x, a_{(1)} \rangle \, \langle y, a_{(2)} \rangle, \\ \langle x, ab \rangle &= \langle x_{(1)}, a \rangle \, \langle x_{(2)}, b \rangle, \\ \langle 1, a \rangle &= \varepsilon(a), \quad \langle x, 1 \rangle = \varepsilon(x), \end{aligned} \tag{2.2}$$

where $x, y \in \mathbf{H}$, $a, b \in \mathbf{A}$. The last conditions imply

$$\langle S(x), a \rangle = \langle x, S(a) \rangle, \quad x \in \mathbf{H},\ a \in \mathbf{A}. \tag{2.3}$$

Let V be a $\Bbbk$-module, and regard it as a purely odd supermodule. The tensor algebra $\mathbf{T}(V)$ on V uniquely turns into a Hopf superalgebra in which every element of V is an odd primitive. The *exterior algebra* $\wedge(V)$ on V is the quotient Hopf superalgebra of $\mathbf{T}(V)$ by the Hopf super-ideal generated by the even primitives v^2, where $v \in V$. Note that $\mathbf{T}(V)$ is cocommutative, while $\wedge(V)$ is commutative and cocommutative. Suppose that V is $\Bbbk$-finite free. Then $\wedge(V)$ is $\Bbbk$-finite free, so that the dual supermodule $\wedge(V)^*$, given the ordinary, dual-algebra and dual-coalgebra structures, is a Hopf superalgebra; see [20, Remark 1]. The canonical pairing $\langle\ ,\ \rangle : V \times V^* \to \Bbbk$ uniquely extends to a Hopf pairing $\langle\ ,\ \rangle : \wedge(V) \times \wedge(V^*) \to \Bbbk$;

it is determined by the property that $\langle \wedge^m(V), \wedge^n(V^*)\rangle = 0$ unless $m = n$ and by the formula

$$\langle v_1 \wedge \cdots \wedge v_n,\, w_1 \wedge \cdots \wedge w_n\rangle = \sum_{\sigma\in\mathfrak{S}_n} \operatorname{sgn}\sigma\, \langle v_1, w_{\sigma(1)}\rangle \dots \langle v_n, w_{\sigma(n)}\rangle, \tag{2.4}$$

where $v_i \in V,\ w_i \in V^*,\ n > 0$. Since this Hopf pairing is non-degenerate, it induces the isomorphism $\wedge(V^*) \xrightarrow{\simeq} \wedge(V)^*$, $a \mapsto \langle\ , a\rangle$ of Hopf superalgebras, through which we will identify

$$\wedge(V^*) = \wedge(V)^*.$$

2.4. Let $\mathbf{A}$ be a commutative Hopf superalgebra. Define

$$\overline{\mathbf{A}} := \mathbf{A}/(\mathbf{A}_1), \quad W^{\mathbf{A}} := \mathbf{A}_1/\mathbf{A}_0^+\mathbf{A}_1,$$

where $(\mathbf{A}_1)$ denotes the (Hopf super-)ideal of $\mathbf{A}$ generated by the odd component $\mathbf{A}_1$, and $\mathbf{A}_0^+ = \mathbf{A}_0 \cap \mathbf{A}^+$; see [19, Section 4]. Note that $\overline{\mathbf{A}} = \mathbf{A}_0/\mathbf{A}_1^2$ and that this is the largest purely even quotient Hopf superalgebra of $\mathbf{A}$. We denote the quotient map by

$$\mathbf{A} \to \overline{\mathbf{A}}, \quad a \mapsto \overline{a}. \tag{2.5}$$

We regard $\mathbf{A}$ as a left $\overline{\mathbf{A}}$-comodule superalgebra, naturally, by $\mathbf{A} \to \overline{\mathbf{A}} \otimes \mathbf{A}$, $a \mapsto \overline{a}_{(1)} \otimes a_{(2)}$. Similarly, $\mathbf{A}$ is regarded as a right $\overline{\mathbf{A}}$-comodule superalgebra.

Definition 2.1. $\mathbf{A}$ is said to be *split* if $W^{\mathbf{A}}$ is $\Bbbk$-free and if there exists an isomorphism $\psi : \mathbf{A} \xrightarrow{\simeq} \overline{\mathbf{A}} \otimes \wedge(W^{\mathbf{A}})$ of left $\overline{\mathbf{A}}$-comodule superalgebras.

Remark 2.2. (1) If the second condition above is satisfied, then ψ can be rechosen as *counit-preserving* in the sense that $(\varepsilon\otimes\varepsilon)\circ\psi = \varepsilon$. Indeed, if we set $\gamma := (\varepsilon\otimes\varepsilon)\circ\psi$, then $a \mapsto \psi(a_{(1)})\,\gamma\circ S(a_{(2)})$ is seen to be a counit-preserving isomorphism.

(2) The same condition as above is equivalent to the condition with the sides switched, that is, the condition that there exists a (counit-preserving) isomorphism $\mathbf{A} \xrightarrow{\simeq} \wedge(W^{\mathbf{A}}) \otimes \overline{\mathbf{A}}$ of right $\overline{\mathbf{A}}$-comodule superalgebras. Indeed, if ψ is a left- or right-sided isomorphism, then the composite $c \circ \psi \circ S$, where $c = c_{\overline{\mathbf{A}},\wedge(W^{\mathbf{A}})}$ or $c_{\wedge(W^{\mathbf{A}}),\overline{\mathbf{A}}}$, gives an opposite-sided one.

Theorem 2.3 ([19, Theorem 4.5])**.** *If $\Bbbk$ is a field of characteristic $\neq 2$, then every commutative Hopf superalgebra is split.*

A Hopf superalgebra is said to be *affine* if it is commutative and finitely generated. A split commutative Hopf superalgebra $\mathbf{A}$ is affine if and only if $\overline{\mathbf{A}}$ is affine and $W^{\mathbf{A}}$ is $\Bbbk$-finite (free).

All commutative Hopf superalgebras and all Hopf superalgebra maps form a category. The affine Hopf superalgebras form a full subcategory of the category.

2.5. The notions of *affine groups* and of *algebraic groups* (see [14, Part I, 2.1]) are directly generalized to the super-situation, as follows. A *supergroup* is a functor from the category of commutative superalgebras to the category of groups. An *affine supergroup* $\mathbf{G}$ is a representable supergroup. By Yoneda's Lemma it is represented by a uniquely determined, commutative Hopf superalgebra, which we denote by $\mathcal{O}(\mathbf{G})$. We call $\mathbf{G}$ an *algebraic supergroup* if $\mathcal{O}(\mathbf{G})$ is affine.

The category formed by all affine supergroups and all natural transformations of group-valued functors is anti-isomorphic to the category of commutative Hopf superalgebras. The full subcategory of the former category which consists of all algebraic supergroups is anti-isomorphic to the category of affine Hopf superalgebras.

Let $\mathbf{G}$ be an affine supergroup, and set $\mathbf{A} := \mathcal{O}(\mathbf{G})$. Then $\overline{\mathbf{A}}$ represents the supergroup

$$R \mapsto \mathbf{G}(R_0),$$

where R is a commutative superalgebra. This affine supergroup is denoted by $\mathbf{G}_{ev}$, so that

$$\overline{\mathbf{A}} = \mathcal{O}(\mathbf{G}_{ev}).$$

We will often regard $\mathbf{G}_{ev}$ as the affine group corresponding to the commutative Hopf algebra $\overline{\mathbf{A}}$. One sees that $W^{\mathbf{A}}$ is the odd component of the cotangent supermodule $\mathbf{A}^+/(\mathbf{A}^+)^2$ of $\mathbf{G}$ at 1.

2.6. Let $\mathbf{G}$ be an affine supergroup. Given a supermodule W, the left (resp., right) $\mathbf{G}$-supermodule structures on W correspond precisely to the right (resp., left) $\mathcal{O}(\mathbf{G})$-super-comodule structures on W.

Let $W \to W \otimes \mathcal{O}(\mathbf{G})$, $w \mapsto w^{(0)} \otimes w^{(1)}$ be a right $\mathcal{O}(\mathbf{G})$-super-comodule structure. The corresponding left $\mathbf{G}$-supermodule structure is given by the R-super-linear automorphism of $W \otimes R$ which is defined by

$${}^{\gamma}(w \otimes 1) = w^{(0)} \otimes \gamma(w^{(1)}), \quad \gamma \in \mathbf{G}(R),\ w \in W,$$

where R is an arbitrary commutative superalgebra. For simplicity this left (resp., the analogous right) $\mathbf{G}$-supermodule structure is represented as

$$(2.6) \qquad {}^{\gamma}w \quad (\text{resp., } w^{\gamma}), \quad \gamma \in \mathbf{G},\ w \in W.$$

Actually, this notational convention will be applied only when $\mathbf{G}$ is an affine group.

Given a Hopf pairing $\langle\ ,\ \rangle : \mathbf{H} \times \mathcal{O}(\mathbf{G}) \to \Bbbk$, where $\mathbf{H}$ is a Hopf superalgebra, we may induce the left $\mathbf{H}$-supermodule structure on W defined by

$$(2.7) \qquad xw := w^{(0)} \langle x, w^{(1)} \rangle, \quad x \in \mathbf{H},\ w \in W.$$

Similarly, a right $\mathbf{H}$-supermodule structure is induced from a right $\mathbf{G}$-supermodule structure.

Let G be an affine group. Note that a G-supermodule is a supermodule W given a G-module structure such that each component W_i, $i = 0, 1$, is G-stable. We let

$$(2.8) \qquad G\text{-}\mathsf{SMod} \quad (\textit{resp.}, \mathsf{SMod}\text{-}G)$$

denote the category of left (resp., right) G-supermodules. This is naturally a tensor category and is symmetric with respect to the super-symmetry.

3. Admissible Lie superalgebras

3.1. A *Lie superalgebra* is a supermodule $\mathfrak{g}$, given a super-linear map $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, called a *super-bracket*, which satisfies:

(i) $[u, u] = 0,\ u \in \mathfrak{g}_0$,
(ii) $[[v, v], v] = 0,\ v \in \mathfrak{g}_1$,
(iii) $[\ ,\ ] \circ (\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}}) = 0$, and
(iv) $[[\ ,\ ],\ ] \circ (\mathrm{id}_{\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g},\mathfrak{g}\otimes\mathfrak{g}} + c_{\mathfrak{g}\otimes\mathfrak{g},\mathfrak{g}}) = 0$.

Note that (i) implies the equation (iii) restricted to $\mathfrak{g}_0^{\otimes 2}$. If $\mathfrak{g}_1$ is 2-torsion free, then (ii) and (iii) imply the equation (iv) restricted to $\mathfrak{g}_1^{\otimes 3}$. Indeed, this follows by applying (ii) to the sum $v_1 + v_2 + v_3$ of elements $v_i \in \mathfrak{g}_1$.

A *Lie algebra* is a $\Bbbk$-module with a bracket which satisfies (i) and the Jacobi identity, that is, (iv) in the purely even situation; it is, therefore, the same as a purely even Lie superalgebra. It follows that if $\mathfrak{g}$ is a Lie superalgebra, then $\mathfrak{g}_0$ is a Lie algebra.

Definition 3.1. A Lie superalgebra $\mathfrak{g}$ is said to be *admissible* if:

(A1) $\mathfrak{g}_0$ is $\Bbbk$-flat,
(A2) $\mathfrak{g}_1$ is $\Bbbk$-free and
(A3) for every $v \in \mathfrak{g}_1$, the element $[v,v]$ in $\mathfrak{g}_0$ is 2-*divisible*; this means that there exists an element $u \in \mathfrak{g}_0$ such that $[v,v] = 2u$.

Note that (A3) is satisfied if $\mathfrak{g}_1$ has a $\Bbbk$-free basis X such that for every $x \in X$, $[x,x]$ is 2-divisible in $\mathfrak{g}_0$.

Remark 3.2. For any 2-divisible element w in a 2-torsion free $\Bbbk$-module, the element u such that $w = 2u$ is unique, and it will be denoted by $\frac{1}{2}w$. By (A1) above, this can apply to the even component of any admissible Lie superalgebra, so that we have $\frac{1}{2}[v,v]$ by (A3).

3.2. Let $\mathfrak{g}$ be an admissible Lie superalgebra. The tensor algebra $\mathbf{T}(\mathfrak{g})$ on $\mathfrak{g}$ uniquely turns into a cocommutative Hopf superalgebra in which every even (resp., odd) element of $\mathfrak{g}$ is an even (resp., odd) primitive. The *universal envelope* $\mathbf{U}(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient Hopf superalgebra of $\mathbf{T}(\mathfrak{g})$ by the Hopf super-ideal generated by the homogeneous primitives

$$zw - (-1)^{|z||w|}wz - [z,w], \quad v^2 - \frac{1}{2}[v,v], \tag{3.1}$$

where z and w are homogeneous elements in $\mathfrak{g}$, and $v \in \mathfrak{g}_1$. We remark that if 2 is invertible in $\Bbbk$, then the second elements $v^2 - \frac{1}{2}[v,v]$ in (3.1) may be removed since they are covered by the first. The universal envelope $U(\mathfrak{g}_0)$ of the Lie algebra $\mathfrak{g}_0$ is thus defined, as usual, to be the quotient algebra of the tensor algebra $T(\mathfrak{g}_0)$ by the ideal generated by $zw - wz - [z,w]$, where $z, w \in \mathfrak{g}_0$; this is a cocommutative Hopf algebra. The $\Bbbk$-flatness assumption (A1) on $\mathfrak{g}_0$ ensures the following.

Lemma 3.3 (See [13])**.** *The canonical map $\mathfrak{g}_0 \to U(\mathfrak{g}_0)$ is an injection.*

Through the injection above we will suppose $\mathfrak{g}_0 \subset U(\mathfrak{g}_0)$. The inclusion $\mathfrak{g}_0 \subset \mathfrak{g}$ induces a Hopf superalgebra map $U(\mathfrak{g}_0) \to \mathbf{U}(\mathfrak{g})$, by which we will regard $\mathbf{U}(\mathfrak{g})$ as a $U(\mathfrak{g}_0)$-ring, and in particular as a left and right $U(\mathfrak{g}_0)$-module. Recall that given an algebra R, an *R-ring* [1, p. 195] is an algebra given an algebra map from R.

Proposition 3.4. $\mathbf{U}(\mathfrak{g})$ *is free as a left as well as a right $U(\mathfrak{g}_0)$-module. In fact, if X is an arbitrary $\Bbbk$-free basis of $\mathfrak{g}_1$ given a total order, then the products*

$$x_1 \dots x_n, \quad x_i \in X, \; x_1 < \dots < x_n, \; n \geq 0,$$

in $\mathbf{U}(\mathfrak{g})$ form a $U(\mathfrak{g}_0)$-free basis, where x_i in the product denotes the image of the element under the canonical map $\mathfrak{g} \to \mathbf{U}(\mathfrak{g})$.

This is proved in [20, Lemma 11] in the generalized situation treating dual Harish-Chandra pairs, but over a field of characteristic $\neq 2$. Our proof of the proposition will confirm the proof of the cited lemma in our present situation. To use the same notation as in [20] we set

$$J := U(\mathfrak{g}_0), \quad V := \mathfrak{g}_1.$$

Then the right adjoint action

$$\mathrm{ad}_r(u)(v) = [v, u], \quad u \in \mathfrak{g}_0,\ v \in V, \tag{3.2}$$

by $\mathfrak{g}_0$ on V uniquely gives rise to a right J-module structure on V, which we denote by $v \triangleleft a$, where $v \in V$, $a \in J$. If $i : V \to \mathbf{U}(\mathfrak{g})$ denotes the canonical map, we have

$$i(v \triangleleft a) = S(a_{(1)})\, i(v)\, a_{(2)}, \quad v \in V,\ a \in J, \tag{3.3}$$

in $\mathbf{U}(\mathfrak{g})$. Indeed, this follows by induction on the largest length r when we express a as a sum of elements $u_1 \dots u_r$, where $u_i \in \mathfrak{g}_0$.

Lemma 3.5. *The right J-module structure on V and the super-bracket $[\ ,\ ] : V \otimes V \to \mathfrak{g}_0 \subset J$ restricted to V make (J, V) into a dual Harish-Chandra pair* [20, Definition 6]*, or explicitly we have:*

(a) $[u \triangleleft a_{(1)}, v \triangleleft a_{(2)}] = S(a_{(1)})[u, v]a_{(2)}$,
(b) $[u, v] = [v, u]$ *and*
(c) $v \triangleleft [v, v] = 0$

for all $u, v \in V$, $a \in J$. Properties (b), (c) *imply that*

(d) $u \triangleleft [v, w] + v \triangleleft [w, u] + w \triangleleft [u, v] = 0, \quad u, v, w \in V.$

We remark that (a) is an equation in $\mathfrak{g}_0$ and the product of the right-hand side is computed in J, which is possible since $\mathfrak{g}_0 \subset J$.

Proof of Lemma 3.5. One verifies (a), just as proving (3.3). Properties (b), (c) are those of Lie superalgebras. One sees that (b), applied to $u + v + w$ and combined with (c), implies (d). $\square$

Proof of Proposition 3.4. We will prove only the left J-freeness. The result with the antipode applied shows the right J-freeness.

Let X be a totally ordered basis of V. We confirm the proof of [20, Lemma 11] as follows. First, we introduce the same order as in the proof into all words in the letters from $X \cup \{*\}$, where $*$ stands for any element of J. Second, we see by using (3.3) that the J-ring $\mathbf{U}(\mathfrak{g})$ is generated by X and is defined by the reduction system consisting of:

(i) $xa \to a_{(1)}(x \triangleleft a_{(2)}), \quad x \in X,\ a \in J,$
(ii) $xy \to -yx + [x, y], \quad x, y \in X,\ x > y,$
(iii) $x^2 \to \frac{1}{2}[x, x], \quad x \in X,$

where we suppose that in (i), $x \triangleleft a_{(2)}$ is presented as a $\Bbbk$-linear combination of elements in X. Third, we see that the reduction system satisfies the assumptions required by Bergman's Diamond Lemma [1, Proposition 7.1], indeed its opposite-sided version.

To prove the desired result from the Diamond Lemma, it remains to verify the following by using the properties (a)–(d) in Lemma 3.5: the overlap ambiguities which may occur when we reduce the words

(iv) $xya, \quad x \geq y$ in $X,\ a \in J,$
(v) $xyz, \quad x \geq y \geq z$ in $X,$

are all resolvable. The proof of [20, Lemma 11] verifies the resolvability only when x, y and z are distinct, and the same proof works now as well.

As for the remaining cases (omitted in the cited proof), first let xya be a word from (iv) with $x = y$. This is reduced on the one hand as

$$xxa \to xa_{(1)}(x \triangleleft a_{(2)}) \to a_{(1)}(x \triangleleft a_{(2)})(x \triangleleft a_{(3)}),$$

and on the other hand as

$$\begin{aligned} xxa \to \Big(\frac{1}{2}[x,x]\Big)a &= a_{(1)}S(a_{(2)})\Big(\frac{1}{2}[x,x]\Big)a_{(3)} \\ &= a_{(1)}\Big(\frac{1}{2}[x \triangleleft a_{(2)}, x \triangleleft a_{(3)}]\Big). \end{aligned}$$

Let $b \in J$. The last equality holds since $S(b_{(1)})(\frac{1}{2}[x,x])b_{(2)}$ and $\frac{1}{2}[x \triangleleft b_{(1)}, x \triangleleft b_{(2)}]$ coincide since their doubles do by (a). For the desired resolvability it suffices to see that the two polynomials

$$(x \triangleleft b_{(1)})(x \triangleleft b_{(2)}), \quad \frac{1}{2}[x \triangleleft b_{(1)}, x \triangleleft b_{(2)}] \tag{3.4}$$

are reduced to the same one. For this, suppose

$$(x \triangleleft b_{(1)}) \otimes (x \triangleleft b_{(2)}) = \sum_{i,j=1}^{n} t_{ij}\, x_i \otimes x_j \text{ in } V \otimes V,$$

where $t_{ij} \in \Bbbk$, and $x_1 < \cdots < x_n$ in X. Note that $t_{ij} = t_{ji}$ since J is cocommutative. Then the first polynomial in (3.4) is reduced as

$$\sum_{i<j} t_{ij}(x_i x_j + x_j x_i) + \sum_i t_{ii}\, x_i x_i \to \sum_{i<j} t_{ij}[x_i, x_j] + \sum_i t_{ii}\Big(\frac{1}{2}[x_i, x_i]\Big).$$

This and the second polynomial in (3.4) coincide since by (b), their doubles do. This proves the desired result.

Next, let xyz be a word from (v), and suppose $x = y > z$. Note that if $(u, w) = ([x, z], x)$ or $(\frac{1}{2}[x, x], z)$, then u is primitive, and so we have the reduction $wu \to uw + w \triangleleft u$ given by (i). Then it follows that $xyz = xxz$ is reduced as

$$\begin{aligned} xxz \;&\to\; -xzx + x[x,z] \to zxx - [x,z]x + [x,z]x + x \triangleleft [x,z] \\ &\to\; z\Big(\frac{1}{2}[x,x]\Big) + x \triangleleft [x,z] \to \Big(\frac{1}{2}[x,x]\Big)z + z \triangleleft \Big(\frac{1}{2}[x,x]\Big) + x \triangleleft [x,z]. \end{aligned}$$

The word is alternatively reduced as

$$xxz \to \Big(\frac{1}{2}[x,x]\Big)z.$$

These two results coincide, since the element $z \triangleleft (\frac{1}{2}[x,x]) + x \triangleleft [x,z]$, whose double is zero by (d), is zero. The ambiguity for the word xyz is thus resolvable when $x = y > z$. One proves similarly the resolvability in the remaining cases, $x > y = z$ and $x = y = z$, using (d) and (c), respectively. $\square$

The proposition just proven shows the following.

Corollary 3.6. *If $\mathfrak{g}$ is an admissible Lie superalgebra, then there exists a unit-preserving, left $U(\mathfrak{g}_0)$-module super-coalgebra isomorphism*

$$U(\mathfrak{g}_0) \otimes \wedge(\mathfrak{g}_1) \xrightarrow{\simeq} \mathbf{U}(\mathfrak{g}).$$

Here, "unit-preserving" means that the isomorphism sends $1 \otimes 1$ to 1.

4. Algebraic supergroups and Harish-Chandra pairs

4.1. Let $\mathbf{G}$ be an affine supergroup. Set $\mathbf{A} := \mathcal{O}(\mathbf{G})$. Then the following is easy to see.

Lemma 4.1. *For homogeneous elements $a, b \in \mathbf{A}^+$, we have*

$$\Delta(ab) \equiv 1 \otimes ab + ab \otimes 1 + a \otimes b + (-1)^{|a||b|} b \otimes a$$

modulo $\mathbf{A}^+ \otimes (\mathbf{A}^+)^2 + (\mathbf{A}^+)^2 \otimes \mathbf{A}^+$.

Set $\mathfrak{d} := \mathbf{A}^+/(\mathbf{A}^+)^2$. This is a supermodule. The *Lie superalgebra*

$$\mathfrak{g} = \mathrm{Lie}(\mathbf{G})$$

of $\mathbf{G}$ is the dual supermodule $\mathfrak{d}^*$ of $\mathfrak{d}$. Note that $\mathbf{A}^*$ is the dual superalgebra of the super-coalgebra $\mathbf{A}$. Regard $\mathfrak{g}$ as a super-submodule of $\mathbf{A}^*$ through the natural embedding $\mathfrak{g} \subset \Bbbk \oplus \mathfrak{d}^* = (\mathbf{A}/(\mathbf{A}^+)^2)^* \subset \mathbf{A}^*$. By definition we have

$$\mathfrak{g}_1 = (W^{\mathbf{A}})^*.$$

Proposition 4.2. *The super-linear endomorphism* $\mathrm{id} - c_{\mathbf{A}^*,\mathbf{A}^*}$ *on $\mathbf{A}^* \otimes \mathbf{A}^*$, composed with the product on $\mathbf{A}^*$, restricts to a map, $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, with which $\mathfrak{g}$ is indeed a Lie superalgebra. This satisfies* (A3)*.*

Proof. By Lemma 4.1 it follows that $(\mathrm{id} - c_{\mathbf{A},\mathbf{A}}) \circ \Delta$ induces a super-linear map

$$\delta : \mathfrak{d} \to \mathfrak{d} \otimes \mathfrak{d}, \tag{4.1}$$

which is seen to satisfy

$$(\mathrm{id}_{\mathfrak{d}\otimes\mathfrak{d}} + c_{\mathfrak{d},\mathfrak{d}}) \circ \delta = 0, \quad (\mathrm{id}_{\mathfrak{d}\otimes\mathfrak{d}\otimes\mathfrak{d}} + c_{\mathfrak{d},\mathfrak{d}\otimes\mathfrak{d}} + c_{\mathfrak{d}\otimes\mathfrak{d},\mathfrak{d}}) \circ (\delta \otimes \mathrm{id}_{\mathfrak{d}}) \circ \delta = 0.$$

Therefore, δ is dualized to a map $[\ ,\ ]$ such as above, which satisfies (i), (iii) and (iv) required by super-brackets; see Section 3.1. Let $v \in \mathfrak{g}_1$. Then it follows from Lemma 4.1 that given a, b as in the lemma, we have

$$v^2(ab) = v(a)v(b) + (-1)^{|a||b|} v(b)v(a) = 0,$$

since $v(a)v(b) = 0$ unless $|a| = |b| = 1$. Therefore, $v^2 \in \mathfrak{g}_0$ and $[v, v] = 2v^2$. Thus (A3) is satisfied. The remaining (ii) is satisfied since $[[v, v], v] = 2[v^2, v] = 0$. □

Set $G := \mathbf{G}_{ev}$. Then $\overline{\mathbf{A}} = \mathcal{O}(G)$. We have the Lie algebra $\mathrm{Lie}(G) = (\overline{\mathbf{A}}^+/(\overline{\mathbf{A}}^+)^2)^*$ of G.

Lemma 4.3. *The natural embedding $\overline{\mathbf{A}}^* \subset \mathbf{A}^*$ induces an isomorphism* $\mathrm{Lie}(G) \simeq \mathfrak{g}_0$ *of Lie algebras.*

Proof. One sees that this is the dual of the canonical isomorphism

$$\mathbf{A}_0^+/((\mathbf{A}_0^+)^2 + \mathbf{A}_1^2) \simeq (\mathbf{A}_0^+/\mathbf{A}_1^2)/(((\mathbf{A}_0^+)^2 + \mathbf{A}_1^2)/\mathbf{A}_1^2).$$

□

4.2. Let G be an algebraic group. The Lie algebra $\mathrm{Lie}(G)$ of G is naturally embedded into $\mathcal{O}(G)^*$, and the embedding gives rise to an algebra map $U(\mathrm{Lie}(G)) \to \mathcal{O}(G)^*$. The associated pairing

$$\langle\ ,\ \rangle : U(\mathrm{Lie}(G)) \times \mathcal{O}(G) \to \Bbbk \tag{4.2}$$

is a Hopf pairing. Therefore, given a left G-module (resp., right) structure on a $\Bbbk$-module, there is induced a left (resp., right) $U(\mathrm{Lie}(G))$-module structure on the $\Bbbk$-module, as was seen in (2.7).

The right adjoint action by G on itself is dualized to the right co-adjoint coaction

$$\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G), \quad a \mapsto a_{(2)} \otimes S(a_{(1)})a_{(3)}. \tag{4.3}$$

This induces on $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$ a right $\mathcal{O}(G)$-comodule (or left G-module) structure. We assume

(B1) $\mathcal{O}(G)/(\mathcal{O}(G)^+)^2$ is $\Bbbk$-finite projective.

This is necessarily satisfied if $\Bbbk$ is a field. Under the assumption, the left G-module structure on $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$ just obtained is transposed to a right G-module structure on $\mathrm{Lie}(G)$. The induced right $U(\mathrm{Lie}(G))$-module structure coincides with the right adjoint action $\mathrm{ad}_r(u)(v) = [v, u]$, $u, v \in \mathrm{Lie}(G)$, as is seen by using the fact that the pairing above satisfies

$$\langle u, ab\rangle = \langle u, a\rangle\, \varepsilon(b) + \varepsilon(a)\, \langle u, b\rangle, \quad \langle u, S(a)\rangle = -\langle u, a\rangle \tag{4.4}$$

for $u \in \mathrm{Lie}(G)$, $a, b \in \mathcal{O}(G)$.

Let G be an algebraic group which satisfies (B1), and let $\mathfrak{g}$ be a Lie superalgebra such that $\mathfrak{g}_0 = \mathrm{Lie}(G)$. Note that $\mathfrak{g}_0$ is $\Bbbk$-finite projective and so $\Bbbk$-flat; it is a right G-module, as was just seen. We assume in addition that:

(B2) $\mathfrak{g}_1$ is $\Bbbk$-finite free, and $\mathfrak{g}$ is admissible, and
(B3) $\mathcal{O}(G)$ is $\Bbbk$-flat.

Assuming (B1) we see that (B2) is equivalent to $\mathfrak{g}_1$ being $\Bbbk$-finite free, and $\mathfrak{g}$ satisfies (A3).

Definition 4.4 (cf. [5, Definition 3.1])**.** (1) Suppose that the pair $(G, \mathfrak{g})$ is accompanied with a right G-module structure on $\mathfrak{g}_1$ such that the induced right $U(\mathfrak{g}_0)$-module structure coincides with the right adjoint $\mathfrak{g}_0$-action given by (3.2). Then $(G, \mathfrak{g})$ is called a *Harish-Chandra pair* if the super-bracket $[\ ,\ ] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ restricted to $\mathfrak{g}_1 \otimes \mathfrak{g}_1$ is right G-equivariant.

(2) A *morphism* $(G, \mathfrak{g}) \to (G', \mathfrak{g}')$ between Harish-Chandra pairs is a pair (α, β) of a morphism $\alpha : G \to G'$ of affine groups and a Lie superalgebra map $\beta = \beta_0 \oplus \beta_1 : \mathfrak{g} \to \mathfrak{g}'$, such that

(i) the Lie algebra map $\mathrm{Lie}(\alpha)$ induced from α coincides with β_0, and
(ii) $\beta_1(v^\gamma) = \beta_1(v)^{\alpha(\gamma)}, \quad \gamma \in G,\ v \in \mathfrak{g}_1$.

(3) The Harish-Chandra pairs and their morphisms form a category HCP.

By convention (see (2.6)) the equation (ii) of (2) above should read

$$(\beta_1 \otimes \mathrm{id}_R)((v \otimes 1)^\gamma) = ((\beta_1 \otimes \mathrm{id}_R)(v \otimes 1))^{\alpha_R(\gamma)},$$

where R is a commutative algebra, and $\gamma \in G(R)$.

Remark 4.5. (1) Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. In this situation the notion of Harish-Chandra pairs was defined by [20, Definition 7] in purely Hopf algebraic terms. It is remarked in [20, Remark 9 (2)] that if the characteristic $\mathrm{char}\,\Bbbk$

of $\Bbbk$ is zero, there is a natural category anti-isomorphism between our HCP defined above and the category of the Harish-Chandra pairs as defined by [20, Definition 7]. But this is indeed the case without the restriction on $\operatorname{char}\Bbbk$. A key fact is the following: once we are given an algebraic group G, a finite-dimensional right G-module V and a right G-equivariant linear map $[\ ,\ ] : V \otimes V \to \mathrm{Lie}(G)$, then the pair $(\mathcal{O}(G), V^*)$, accompanied with $[\ ,\ ]$, is a Harish-Chandra pair in the sense of [20] if and only if the direct sum $\mathfrak{g} := \mathrm{Lie}(G) \oplus V$ is a Lie superalgebra (in our sense), with respect to the grading $\mathfrak{g}_0 = \mathrm{Lie}(G)$, $\mathfrak{g}_1 = V$, and with respect to the super-bracket which uniquely extends (a) the bracket on $\mathrm{Lie}(G)$, (b) the map $[\ ,\ ]$, and (c) the right adjoint $\mathrm{Lie}(G)$-action on V which is induced from the right G-action on V. See [20, Remark 2 (1)], but note that in [20], the notion of Lie superalgebras is used in a restrictive sense when $\operatorname{char}\Bbbk = 3$; indeed, to define the notion, the article excludes condition (ii) from our axioms given in the beginning of Section 3.1.

(2) As the referee pointed out, our definition of Harish-Chandra pairs looks different from those definitions given in [4, Section 7.4] and [5, Section 3.1], which require that the whole super-bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ be G-equivariant. But this follows from the weaker requirement of ours that the restricted super-bracket $[\ ,\ ]|_{\mathfrak{g}_1 \otimes \mathfrak{g}_1}$ is G-equivariant, since $[\ ,\ ]|_{\mathfrak{g}_0 \otimes \mathfrak{g}_0}$ is obviously G-equivariant, and $[\ ,\ ]|_{\mathfrak{g}_1 \otimes \mathfrak{g}_0}$ is, too, as will be seen below. Let $\gamma \in G$, $u \in \mathfrak{g}_0$ and $v \in \mathfrak{g}_1$. Note that

$$\langle u, a_{(1)} \rangle\, \gamma(a_{(2)}) = \gamma(a_{(1)})\, \langle u^{\gamma}, a_{(2)} \rangle, \quad a \in \mathcal{O}(G).$$

Then the common requirement for the induced $U(\mathfrak{g}_0)$-module structure on $\mathfrak{g}_1$ shows that $[v, u]^{\gamma} = [v^{\gamma}, u^{\gamma}]$.

4.3. We define AHSA to be the full subcategory of the category of affine Hopf superalgebras which consists of the affine Hopf superalgebras $\mathbf{A}$ such that:

(C1) $\mathbf{A}$ is split (see Definition 2.1),
(C2) $\overline{\mathbf{A}}$ is $\Bbbk$-flat and
(C3) $\overline{\mathbf{A}}/(\overline{\mathbf{A}}^+)^2$ is $\Bbbk$-finite projective.

Note that the affinity and (C1) imply that $W^{\mathbf{A}}$ is $\Bbbk$-finite free. If $\Bbbk$ is a field of characteristic $\neq 2$, then AHSA is precisely the category of all affine Hopf superalgebras.

We define ASG to be the full subcategory of the category of algebraic supergroups which consists of the algebraic supergroups $\mathbf{G}$ such that $\mathcal{O}(\mathbf{G})$ is split, and $\mathbf{G}_{ev}$ satisfies (B1), (B3). This is anti-isomorphic to AHSA and is precisely the category of all algebraic supergroups if $\Bbbk$ is a field of characteristic $\neq 2$.

Let $\mathbf{G} \in \mathsf{ASG}$. Set

$$\mathbf{A} := \mathcal{O}(\mathbf{G}), \quad G := \mathbf{G}_{ev}, \quad \mathfrak{g} := \mathrm{Lie}(\mathbf{G}).$$

Then $\mathbf{A} \in \mathsf{AHSA}$, and $\mathcal{O}(G)\,(= \overline{\mathbf{A}})$ satisfies (B1), (B3). By Proposition 4.2, $\mathfrak{g}$ satisfies (B2). By Lemma 4.3 we have a natural isomorphism $\mathrm{Lie}(G) \simeq \mathfrak{g}_0$, through which we will identify the two, and suppose $\mathfrak{g}_0 = \mathrm{Lie}(G)$. Just as was seen in (4.3), the right co-adjoint $\overline{\mathbf{A}}$-coaction defined by

$$\mathbf{A} \to \mathbf{A} \otimes \overline{\mathbf{A}}, \quad a \mapsto a_{(2)} \otimes S(\overline{a}_{(1)})\overline{a}_{(3)}, \tag{4.5}$$

using the notation (2.5), induces on $\mathbf{A}^+/(\mathbf{A}^+)^2$ a right $\overline{\mathbf{A}}$-super-comodule (or left G-supermodule) structure; by (C3), it is transposed to a right G-supermodule structure on $\mathfrak{g}$, which is restricted to $\mathfrak{g}_1$.

Lemma 4.6. *Given the restricted right G-module structure on $\mathfrak{g}_1$, the pair $(G,\mathfrak{g})$ forms a Harish-Chandra pair, and so $(G,\mathfrak{g}) \in \mathsf{HCP}$.*

Proof. The right G-module structure on $\mathfrak{g}_1$ induces the right adjoint $\mathfrak{g}_0$-action, as is seen by using (4.4). Since one sees that the map δ given in (4.1) is G-equivariant, so is its dual, $[\ ,\ ]$. □

We denote this object in HCP by

$$\mathbf{P}(\mathbf{G}) = (G,\mathfrak{g}).$$

Proposition 4.7. $\mathbf{G} \mapsto \mathbf{P}(\mathbf{G})$ *gives a functor* $\mathbf{P} : \mathsf{ASG} \to \mathsf{HCP}$.

Proof. Indeed, the constructions of G and of $\mathfrak{g}$ are functorial. □

4.4. Let $(G,\mathfrak{g}) \in \mathsf{HCP}$. Modifying the construction of $A(C,W)$ given in [20], we construct an object $\mathbf{A}(G,\mathfrak{g})$ in AHSA. To be close to [20] for notation we set

$$J := U(\mathfrak{g}_0), \quad C := \mathcal{O}(G), \quad W := \mathfrak{g}_1^*.$$

Then W is $\Bbbk$-finite free. It is a right C-comodule, or a left G-module, with the right G-module structure on $\mathfrak{g}_1$ transposed to W.

Let $\mathbb{N} = \{0,1,2,\dots\}$ denote the semigroup of non-negative integers. A supermodule is said to be $\mathbb{N}$-*graded* if it is $\mathbb{N}$-graded as a $\Bbbk$-module and if the original $\mathbb{Z}_2$-grading equals the $\mathbb{N}$-grading modulo 2. A Hopf superalgebra is said to be $\mathbb{N}$-*graded* [20, Definition 1] if it is $\mathbb{N}$-graded as an algebra and coalgebra and if the original $\mathbb{Z}_2$-grading equals the $\mathbb{N}$-grading modulo 2.

Recall from Section 2.3 that the tensor algebra $\mathbf{T}(\mathfrak{g}_1) = \bigoplus_{n=0}^{\infty} \mathbf{T}^n(\mathfrak{g}_1)$ on $\mathfrak{g}_1$ is a cocommutative Hopf superalgebra; this is $\mathbb{N}$-graded. Recall that $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$ by the right adjoint; see (3.2). This uniquely extends to a right J-module-algebra structure on $\mathbf{T}(\mathfrak{g}_1)$, with which is associated the smash-product algebra [25, p. 155]

$$\mathcal{H} := J \ltimes \mathbf{T}(\mathfrak{g}_1).$$

Given the tensor-product coalgebra structure on $J \otimes \mathbf{T}(\mathfrak{g}_1)$, this $\mathcal{H}$ is a cocommutative Hopf superalgebra, which is $\mathbb{N}$-graded so that $\mathcal{H}(n) = J \otimes \mathbf{T}^n(\mathfrak{g}_1)$, $n \in \mathbb{N}$; see [20, Section 3.2]. Set

$$\mathbf{U} := \mathbf{U}(\mathfrak{g}).$$

Since we see that $\mathcal{H}$ is the quotient Hopf superalgebra of $\mathbf{T}(\mathfrak{g})$ divided by the Hopf super-ideal generated by

$$zw - wz - [z,w], \quad z \in \mathfrak{g},\ w \in \mathfrak{g}_0,$$

it follows that $\mathbf{U} = \mathcal{H}/\mathcal{I}$, where $\mathcal{I}$ is the Hopf super-ideal of $\mathcal{H}$ generated by the even primitives

$$1 \otimes (uv + vu) - [u,v] \otimes 1,\ 1 \otimes v^2 - \frac{1}{2}[v,v] \otimes 1, \tag{4.6}$$

where $u, v \in \mathfrak{g}_1$.

Let $\mathbf{T}_c(W)$ denote the *tensor coalgebra* on W, as given in [20, Section 4.1]; this is a commutative $\mathbb{N}$-graded Hopf superalgebra. In fact, this equals the tensor algebra $\mathbf{T}(W) = \bigoplus_{n=0}^{\infty} \mathbf{T}^n(W)$ as an $\mathbb{N}$-graded module and is the *graded dual* $\bigoplus_{n=0}^{\infty} \mathbf{T}^n(\mathfrak{g}_1)^*$ of $\mathbf{T}(\mathfrak{g}_1)$ (see [25, p. 231]) as an algebra and coalgebra. Suppose that $\mathbf{T}^0(W) = \Bbbk$ is the trivial right C-comodule and $\mathbf{T}^n(W)$, $n > 0$, is the n-fold tensor product of the right C-comodule W. Then $\mathbf{T}_c(W)$ turns into a right C-comodule coalgebra. The associated smash coproduct $C \blacktriangleright\!\!< \mathbf{T}_c(W)$, given the tensor-product algebra structure on $C \otimes \mathbf{T}_c(W)$, is a commutative $\mathbb{N}$-graded Hopf superalgebra. Explicitly, the coproduct is given by

$$\Delta(c \otimes d) = \big(c_{(1)} \otimes (d_{(1)})^{(0)}\big) \otimes \big((d_{(1)})^{(1)} c_{(2)} \otimes d_{(2)}\big), \tag{4.7}$$

where $c \in C$, $d \in \mathbf{T}_c(W)$, and $d \mapsto d^{(0)} \otimes d^{(1)}$ denotes the right C-comodule structure on $\mathbf{T}_c(W)$.

In general, given an $\mathbb{N}$-graded supermodule $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}(n)$, we suppose that it is given the linear topology defined by the descending chains of super-ideals

$$\bigoplus_{i>n} \mathcal{A}(n), \ n = 0, 1, \dots .$$

The completion $\widehat{\mathcal{A}}$ coincides with the direct product $\prod_{n=0}^{\infty} \mathcal{A}(n)$. This is not $\mathbb{N}$-graded anymore, but is still a supermodule. Given another $\mathbb{N}$-graded supermodule $\mathcal{B}$, the tensor product $\mathcal{A} \otimes \mathcal{B}$ is naturally an $\mathbb{N}$-graded supermodule. The complete tensor product $\widehat{\mathcal{A}} \mathbin{\widehat{\otimes}} \widehat{\mathcal{B}}$ coincides with the completion of $\mathcal{A} \otimes \mathcal{B}$. We regard $\Bbbk$ as a trivially $\mathbb{N}$-graded supermodule which is discrete. Suppose that $\mathcal{A}$ is an $\mathbb{N}$-graded Hopf superalgebra. The structure maps on $\mathcal{A}$, being $\mathbb{N}$-graded and hence continuous, are completed to

$$\widehat{\Delta} : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{A}} \mathbin{\widehat{\otimes}} \widehat{\mathcal{A}}, \quad \widehat{\varepsilon} : \widehat{\mathcal{A}} \longrightarrow \Bbbk, \quad \widehat{S} : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{A}}.$$

Satisfying the axiom of Hopf superalgebras with $\otimes$ replaced by $\widehat{\otimes}$, this $\widehat{\mathcal{A}}$ may be called a *complete topological Hopf superalgebra*. If $\mathcal{A}$ is commutative, then $\widehat{\mathcal{A}}$ is, too. See [20, Section 2.3].

Applying the construction above to $C \blacktriangleright\!\!< \mathbf{T}_c(W)$, we suppose

$$\mathcal{A} = C \blacktriangleright\!\!< \mathbf{T}_c(W), \quad \widehat{\mathcal{A}} = \prod_{n=0}^{\infty} C \otimes \mathbf{T}^n(W)$$

in what follows. We let

$$\pi : \widehat{\mathcal{A}} \to C \otimes \mathbf{T}^0(W) = C \tag{4.8}$$

denote the natural projection.

We regard C as a left J-module by

$$xc := c_{(1)} \langle x, c_{(2)} \rangle, \quad x \in J, \ c \in C,$$

where $\langle \ , \ \rangle : J \times C \to \Bbbk$ denotes the canonical Hopf pairing; see (4.2).

Let Hom_J denote the $\Bbbk$-module of left J-module maps. We regard $\mathrm{Hom}_J(\mathcal{H}, C)$ as the completion of the $\mathbb{N}$-graded supermodule $\bigoplus_{n=0}^{\infty} \mathrm{Hom}_J(J \otimes \mathbf{T}^n(\mathfrak{g}_1), C)$. The canonical isomorphisms

$$C \otimes \mathbf{T}^n(W) = \mathrm{Hom}(\mathbf{T}^n(\mathfrak{g}_1), C) \xrightarrow{\simeq} \mathrm{Hom}_J(J \otimes \mathbf{T}^n(\mathfrak{g}_1), C), \quad n = 0, 1, \dots, \tag{4.9}$$

altogether amount to a super-linear homeomorphism

$$\xi : \widehat{\mathcal{A}} \xrightarrow{\simeq} \mathrm{Hom}_J(\mathcal{H}, C). \tag{4.10}$$

Tensoring the canonical pairings $J \times C \to \Bbbk$ and $\mathbf{T}(\mathfrak{g}_1) \times \mathbf{T}_c(W) \to \Bbbk$, we define

$$\langle \ , \ \rangle : \mathcal{H} \times \mathcal{A} \to \Bbbk, \quad \langle x \otimes y, c \otimes d \rangle = \langle x, c \rangle \langle y, d \rangle, \tag{4.11}$$

where $x \in J$, $y \in \mathbf{T}(\mathfrak{g}_1)$, $c \in C$, $d \in \mathbf{T}_c(W)$. This is a Hopf pairing, as was seen in [20, Proposition 17].

Lemma 4.8. *ξ is determined by*

$$\xi(a)(x) = \pi(a_{(1)})\,\langle x, a_{(2)}\rangle, \quad a \in \mathcal{A},\ x \in \mathcal{H}. \tag{4.12}$$

Proof. Note that if $a = c \otimes d$, where $c \in C$, $d \in \mathbf{T}_c(W)$, then

$$\pi(a_{(1)}) \otimes a_{(2)} = c_{(1)} \otimes (c_{(2)} \otimes d).$$

Then the lemma follows since ξ is the completion of the $\mathbb{N}$-graded linear map

$$\mathcal{A} = C \otimes (\bigoplus_{n=0}^{\infty} \mathbf{T}^n(W)) \to \mathrm{Hom}_J(J \otimes (\bigoplus_{n=0}^{\infty} \mathbf{T}^n(\mathfrak{g}_1)), C)$$

given by $c \otimes d \mapsto \big(x \otimes y \mapsto xc\,\langle y, d\rangle\big)$. This last element equals $c_{(1)}\,\langle x \otimes y, c_{(2)} \otimes d\rangle$. $\square$

Remark 4.9. Recall that $\langle \mathcal{H}(n), \mathcal{A}(m)\rangle = 0$ unless $n = m$. Therefore, the pairing (4.11) uniquely extends to

$$\langle\ ,\ \rangle : \mathcal{H} \times \widehat{\mathcal{A}} \to \Bbbk \tag{4.13}$$

so that for each $x \in \mathcal{H}$, $\langle x,\ \rangle : \widehat{\mathcal{A}} \to \Bbbk$ is continuous. Using this pairing one sees that the value $\xi(a)$ at $a \in \widehat{\mathcal{A}}$ is given by the same formula as (4.12), with $\pi(a_{(1)}) \otimes a_{(2)}$ understood to be $(\pi \mathbin{\widehat{\otimes}} \mathrm{id}) \circ \widehat{\Delta}(a)$.

We aim to transfer the structures on $\widehat{\mathcal{A}}$ to $\mathrm{Hom}_J(\mathcal{H}, C)$ through ξ; see Proposition 4.11 below.

Recall from Section 4.3 that $\mathfrak{g}_0$ is a right G-module. Combined with the given right G-module structure on $\mathfrak{g}_1$, it results that $\mathfrak{g} \in \mathsf{SMod}\text{-}G$; see (2.8). Moreover, $\mathfrak{g}$ is a Lie-algebra object in $\mathsf{SMod}\text{-}G$, since the super-bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is G-equivariant, as was proved in Remark 4.5 (2).

We regard $\mathcal{A}$ as a right C-super-comodule, or an object in $G\text{-}\mathsf{SMod}$, with respect to the right co-adjoint coaction

$$\mathcal{A} \to \mathcal{A} \otimes C, \quad a \mapsto a_{(2)} \otimes S(\pi(a_{(1)}))\,\pi(a_{(3)}). \tag{4.14}$$

Lemma 4.10. *We have the following.*

1. *The right G-supermodule structure on $\mathfrak{g}$ uniquely extends to that on $\mathcal{H}$ so that $\mathcal{H}$ turns into an algebra object in $\mathsf{SMod}\text{-}G$. In fact, $\mathcal{H}$ turns into a Hopf-algebra object in $\mathsf{SMod}\text{-}G$.*
2. *With the structure above, $\mathcal{A}$ turns into a Hopf-algebra object in $G\text{-}\mathsf{SMod}$.*
3. *The resulting structures are dual to each other in the sense that*

$$\langle x^\gamma, a\rangle = \langle x, {}^\gamma a\rangle, \quad \gamma \in G,\ x \in \mathcal{H},\ a \in \mathcal{A}. \tag{4.15}$$

Proof. (1) The right G-supermodule structure on $\mathfrak{g}$ uniquely extends to that on $\mathbf{T}(\mathfrak{g})$ so that $\mathbf{T}(\mathfrak{g})$ turns into an algebra object in $\mathsf{SMod}\text{-}G$. The extended structure factors to $\mathcal{H}$, since we have $[z, w]^\gamma = [z^\gamma, w^\gamma]$, where $\gamma \in G$, $z \in \mathfrak{g}$ and $w \in \mathfrak{g}_0$. One sees easily that the resulting structure on $\mathcal{H}$ is such as mentioned above.

(2) This is easy to see.

(3) Let $a \in C$, and let $x = u_1 \dots u_r$ be an element of J with $u_i \in \mathfrak{g}_0$. One sees by induction on r that (4.15) holds for these x and a, using the fact that G-actions preserve the algebra structure on J and the coalgebra structure on C.

We see from (4.7) that the left G-module structure on $\mathcal{A}$, restricted to $\mathbf{T}_c(W) = \Bbbk \otimes \mathbf{T}_c(W)$, is precisely what corresponds to the original right C-comodule structure on $\mathbf{T}_c(W)$. It follows that (4.15) holds for $x \in \mathbf{T}(\mathfrak{g}_1)$, $a \in \mathbf{T}_c(W)$.

The desired equality now follows from the definition (4.11) together with the fact that the G-actions preserve the products on $\mathcal{H}$ and on $\mathcal{A}$. $\square$

For each $n \geq 0$ we have a natural linear isomorphism (see (4.9)) from

$$\bigoplus_{i+j=n} \mathrm{Hom}_J(J \otimes \mathbf{T}^i(\mathfrak{g}_1), C) \otimes \mathrm{Hom}_J(J \otimes \mathbf{T}^j(\mathfrak{g}_1), C)$$

onto the $\Bbbk$-module

$$\bigoplus_{i+j=n} \mathrm{Hom}_{J\otimes J}((J \otimes \mathbf{T}^i(\mathfrak{g}_1)) \otimes (J \otimes \mathbf{T}^j(\mathfrak{g}_1)), C \otimes C)$$

which consists of left $J \otimes J$-module maps. The direct product $\prod_{n=0}^{\infty}$ of the isomorphisms gives the super-linear homeomorphism

$$\mathrm{Hom}_J(\mathcal{H}, C) \,\widehat{\otimes}\, \mathrm{Hom}_J(\mathcal{H}, C) \xrightarrow{\approx} \mathrm{Hom}_{J\otimes J}(\mathcal{H} \otimes \mathcal{H}, C \otimes C),$$

which is indeed the completion of the continuous map $f \otimes g \mapsto (x \otimes y \mapsto f(x) \otimes g(y))$, where $f, g \in \mathrm{Hom}_J(\mathcal{H}, C)$, $x, y \in \mathcal{H}$. This homeomorphism will be used in part (2) below.

Proposition 4.11. *Suppose that $f, g \in \mathrm{Hom}_J(\mathcal{H}, C)$, $x, y \in \mathcal{H}$ and $\gamma, \delta \in G(R)$, where R is an arbitrary commutative algebra.*

(1) *The product, the identity, the counit $\widehat{\varepsilon}$ and the antipode $\widehat{S}$ on $\widehat{\mathcal{A}}$ are transferred to* $\mathrm{Hom}_J(\mathcal{H}, C)$ *through ξ so that*

$$\begin{aligned} fg(x) &= f(x_{(1)})g(x_{(2)}), \\ \xi(1)(x) &= \varepsilon(x)1, \\ \widehat{\varepsilon}(f) &= \varepsilon(f(1)), \\ \langle \gamma,\ \widehat{S}(f)(x)\rangle &= \langle \gamma^{-1},\ f(S(x)^{\gamma^{-1}})\rangle. \end{aligned}$$

(2) *Through ξ and $\xi \mathbin{\widehat{\otimes}} \xi$, the coproduct on $\widehat{\mathcal{A}}$ is translated to*

$$\widehat{\Delta} : \mathrm{Hom}_J(\mathcal{H}, C) \to \mathrm{Hom}_J(\mathcal{H}, C) \,\widehat{\otimes}\, \mathrm{Hom}_J(\mathcal{H}, C) \approx \mathrm{Hom}_{J\otimes J}(\mathcal{H} \otimes \mathcal{H}, C \otimes C)$$

so that

$$\langle (\gamma, \delta),\ \widehat{\Delta}(f)(x \otimes y)\rangle = \langle \gamma\delta,\ f(x^{\delta}\, y)\rangle.$$

Here, $\langle \gamma^{\pm 1},\ \rangle$, $\langle \gamma\delta,\ \rangle$ and $\langle (\gamma, \delta),\ \rangle$ denote the functor points in $G(R)$ and in $(G \times G)(R)$, respectively.

The formulas are essentially the same as those given in [20, Proposition 18 (2), (3)]. One will see below that the proof here, using Lemma 4.8, is simpler.

Proof. (1) Let $a \in \mathcal{A}$, and write as $\pi(a) = \overline{a}$. Then one has

$$ {}^{\gamma}a = \langle \gamma^{-1}, \overline{a}_{(1)}\rangle\, a_{(2)}\, \langle \gamma, \overline{a}_{(3)}\rangle, \quad \gamma \in G. \tag{4.16}$$

To prove the last formula we may suppose $f = \xi(a)$, since we evaluate f, $\widehat{S}(f)$ on $\mathcal{H}$. By using Lemma 4.8 we see that

$$\begin{aligned} \text{LHS} &= \langle x, S(a_{(1)})\rangle \,\langle \gamma, S(\overline{a}_{(2)})\rangle = \langle S(x), a_{(1)}\rangle\, \langle \gamma^{-1}, \overline{a}_{(2)}\rangle \\ &= \langle \gamma^{-1}, \overline{a}_{(1)}\rangle\, \langle \gamma, \overline{a}_{(2)}\rangle\, \langle S(x), a_{(3)}\rangle\, \langle \gamma^{-1}, \overline{a}_{(4)}\rangle \\ &= \langle \gamma^{-1}, \overline{a}_{(1)}\rangle\, \langle S(x), {}^{\gamma^{-1}}a_{(2)}\rangle = \text{RHS}. \end{aligned}$$

The rest is easy to see.

(2) As above we may suppose $f = \xi(a)$, $a \in \mathcal{A}$. Then

$$\begin{aligned}\text{LHS} &= \langle \gamma, \overline{a}_{(1)} \rangle \langle x, a_{(2)} \rangle \langle \delta, \overline{a}_{(3)} \rangle \langle y, a_{(4)} \rangle \\ &= \langle \gamma, \overline{a}_{(1)} \rangle \langle \delta, \overline{a}_{(2)} \rangle \langle \delta, S(\overline{a}_{(3)}) \rangle \langle x, a_{(4)} \rangle \langle \delta, \overline{a}_{(5)} \rangle \langle y, a_{(6)} \rangle \\ &= \langle \gamma, \overline{a}_{(1)} \rangle \langle \delta, \overline{a}_{(2)} \rangle \langle x, {}^{\delta}a_{(3)} \rangle \langle y, a_{(4)} \rangle = \text{RHS}.\end{aligned}$$

□

Recall from (4.6) that $\mathcal{I}$ is the Hopf super-ideal of $\mathcal{H}$ such that $\mathcal{H}/\mathcal{I} = \mathbf{U}$. Note that by the $\Bbbk$-flatness assumption (B3), the following statement makes sense.

Lemma 4.12. *$\mathcal{I}$ is G-stable or, in other words, it is C-costable. Therefore, $\mathbf{U} \in$ SMod-G.*

Proof. Since $[\ ,\ ] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ is G-equivariant, it follows that the elements $uv + vu - [u, v]$ from (4.6) generate in $\mathcal{H}$ a C-costable $\Bbbk$-submodule.

Let $\rho : \mathcal{H} \to C \otimes \mathcal{H}$ be the left C-comodule structure on $\mathcal{H}$. Let $v \in \mathfrak{g}_1$, and suppose $\rho(v) = \sum_i c_i \otimes v_i$. By (B3), $C \otimes \mathfrak{g}_0$ is 2-torsion free. Therefore, we can conclude that

$$\rho(\frac{1}{2}[v, v]) = \sum_i c_i^2 \otimes \frac{1}{2}[v_i, v_i] + \sum_{i<j} c_i c_j \otimes [v_i, v_j] \tag{4.17}$$

by seeing that the doubles of both sides coincide. It follows that

$$\rho(v^2 - \frac{1}{2}[v, v]) = \sum_i c_i^2 \otimes (v_i^2 - \frac{1}{2}[v_i, v_i]) + \sum_{i<j} c_i c_j \otimes (v_i v_j + v_j v_i - [v_i, v_j]).$$

Since this is contained in $C \otimes \mathcal{I}$, the lemma follows. □

Since $\mathfrak{g}$ is admissible, it follows by Corollary 3.6 that there is a unit-preserving left J-module super-coalgebra isomorphism

$$\phi : J \otimes \wedge(\mathfrak{g}_1) \xrightarrow{\simeq} \mathbf{U}. \tag{4.18}$$

We fix this ϕ for use in what follows.

Corollary 4.13. *$\mathrm{Hom}_J(\mathbf{U}, C)$ is a discrete super-subalgebra of $\mathrm{Hom}_J(\mathcal{H}, C)$ and is stable under $\widehat{S}$. Moreover, the map $\widehat{\Delta}$ given in Proposition 4.11 (2) sends $\mathrm{Hom}_J(\mathbf{U}, C)$ into $\mathrm{Hom}_{J\otimes J}(\mathbf{U} \otimes \mathbf{U}, C \otimes C)$.*

Proof. Since $\mathbf{U}$ is finitely generated as a left J-module by (4.18), we have $\mathrm{Hom}_J(\mathbf{U}, C) \subset \mathrm{Hom}_J(J \otimes \big(\bigoplus_{i<n} \mathbf{T}^i(\mathfrak{g}_1)\big), C)$ for n large enough. This means that $\mathrm{Hom}_J(\mathbf{U}, C)$ is discrete. The rest follows easily from Lemma 4.12. □

Given a Harish-Chandra pair $(G, \mathfrak{g})$ as above, we define

$$\mathbf{A}(G, \mathfrak{g})$$

to be the $\Bbbk$-submodule of $\widehat{\mathcal{A}}$ such that the homeomorphism ξ given in (4.10) restricts to a linear isomorphism

$$\eta : \mathbf{A}(G, \mathfrak{g}) \xrightarrow{\simeq} \mathrm{Hom}_J(\mathbf{U}, C). \tag{4.19}$$

In what follows we set $\mathbf{A} := \mathbf{A}(G, \mathfrak{g})$.

Lemma 4.14. *We have the following.*

(1) $\mathbf{A}$ *is a discrete super-subalgebra of* $\widehat{\mathcal{A}}$, *which is stable under* $\widehat{S}$.
(2) *The canonical map* $\mathbf{A}\otimes\mathbf{A} \to \widehat{\mathcal{A}}\widehat{\otimes}\widehat{\mathcal{A}}$ *is an injection. Regarding this injection as an inclusion, we have* $\widehat{\Delta}(\mathbf{A}) \subset \mathbf{A}\otimes\mathbf{A}$.
(3) $(\mathbf{A}, \widehat{\Delta}|_{\mathbf{A}}, \widehat{\varepsilon}|_{\mathbf{A}}, \widehat{S}|_{\mathbf{A}})$ *is a commutative Hopf superalgebra.*

Proof. (1) This follows from Corollary 4.13.

(2) By using η, the canonical map above is identified with the composite of the canonical map

$$\mathrm{Hom}_J(\mathbf{U}, C)\otimes \mathrm{Hom}_J(\mathbf{U}, C) \to \mathrm{Hom}_{J\otimes J}(\mathbf{U}\otimes\mathbf{U}, C\otimes C) \tag{4.20}$$

with the embedding $\mathrm{Hom}_{J\otimes J}(\mathbf{U}\otimes\mathbf{U}, C\otimes C) \subset \mathrm{Hom}_{J\otimes J}(\mathcal{H}\otimes\mathcal{H}, C\otimes C)$. By using ϕ, the map (4.20) is identified with the canonical map

$$\mathrm{Hom}(\wedge(\mathfrak{g}_1), C)\otimes \mathrm{Hom}(\wedge(\mathfrak{g}_1), C) \to \mathrm{Hom}(\wedge(\mathfrak{g}_1)\otimes\wedge(\mathfrak{g}_1), C\otimes C),$$

which is an isomorphism since $\wedge(\mathfrak{g}_1)$ is $\Bbbk$-finite free. This proves the desired injectivity. The rest follows from Corollary 4.13.

(3) Just as above the canonical map $\mathbf{A}\otimes\mathbf{A}\otimes\mathbf{A} \to \widehat{\mathcal{A}}\widehat{\otimes}\widehat{\mathcal{A}}\widehat{\otimes}\widehat{\mathcal{A}}$ is seen to be an injection. From this we see that $\widehat{\Delta}|_{\mathbf{A}}$ is coassociative. The rest is easy to see. $\square$

The restriction $\pi|_{\mathbf{A}}$ of the projection (4.8) to $\mathbf{A}$ is a Hopf superalgebra map, which we denote by

$$\mathbf{A} \to C, \quad a \mapsto \overline{a}. \tag{4.21}$$

This notation is consistent with (2.5), as will be seen from Lemma 4.16 (2). We see from Remark 4.9 that the pairing (4.13) induces

$$\langle\ ,\ \rangle : \mathbf{U}\times\mathbf{A} \to \Bbbk, \tag{4.22}$$

and the following lemma holds.

Lemma 4.15. η *is given by essentially the same formula as* (4.12) *so that*

$$\eta(a)(x) = \overline{a}_{(1)}\,\langle x, a_{(2)}\rangle, \quad a\in\mathbf{A},\ x\in\mathbf{U}.$$

Define a map ϱ to be the composite

$$\varrho : \mathbf{A} \xrightarrow{\eta} \mathrm{Hom}_J(\mathbf{U}, C) \simeq \mathrm{Hom}(\wedge(\mathfrak{g}_1), C) \xrightarrow{\varepsilon_*} \wedge(\mathfrak{g}_1)^* = \wedge(W), \tag{4.23}$$

where the second isomorphism is the one induced from the fixed ϕ (see (4.18)), and the following ε_* denotes $\mathrm{Hom}(\wedge(\mathfrak{g}_1), \varepsilon)$.

Lemma 4.16. *We have the following.*

(1) *The map*

$$\psi : \mathbf{A} \to C\otimes\wedge(W), \quad \psi(a) = \overline{a}_{(1)}\otimes\varrho(a_{(2)})$$

is a counit-preserving isomorphism of left C*-comodule superalgebras.*

(2) *We have natural isomorphisms*

$$\overline{\mathbf{A}} \simeq C, \quad W^{\mathbf{A}} \simeq W = \mathfrak{g}_1^* \tag{4.24}$$

of Hopf algebras and of $\Bbbk$*-modules, respectively.*

Proof. (1) Compose the isomorphism $\mathrm{Hom}_J(\mathbf{U}, C) \simeq \mathrm{Hom}(\wedge(\mathfrak{g}_1), C)$ in (4.23) with the canonical one $\mathrm{Hom}(\wedge(\mathfrak{g}_1), C) \simeq C \otimes \wedge(W)$. Through the composite we will identify $\mathrm{Hom}_J(\mathbf{U}, C) = C \otimes \wedge(W)$. Since $\langle x, a\rangle = \varepsilon(\eta(a)(x))$, $a \in \mathbf{A}$, $x \in \mathbf{U}$, one sees that ψ is identified with η, whence it is a bijection. The desired result follows since ϱ is a counit-preserving superalgebra map.

(2) We see from the isomorphism just obtained that the Hopf superalgebra map (4.21) induces $\overline{\mathbf{A}} \simeq C$, and the pairing (4.22), restricted to $\mathfrak{g}_1 \times \mathbf{A}$, induces $W^{\mathbf{A}} \simeq \mathfrak{g}_1^*$. □

The lemma shows the following.

Proposition 4.17. $\mathbf{A}(G, \mathfrak{g}) \in \mathsf{AHSA}$.

We let

$$\mathbf{G}(G, \mathfrak{g})$$

denote the object in ASG which corresponds to $\mathbf{A}(G, \mathfrak{g})$.

Proposition 4.18. $(G, \mathfrak{g}) \mapsto \mathbf{G}(G, \mathfrak{g})$ *gives a functor* $\mathbf{G} : \mathsf{HCP} \to \mathsf{ASG}$.

Proof. This follows since the constructions of $\widehat{\mathcal{A}}$, $\mathrm{Hom}_J(\mathcal{H}, C)$ and $\mathrm{Hom}_J(\mathbf{U}, C)$ are all functorial, and the homeomorphism ξ is natural. □

Proposition 4.19. *The Harish-Chandra pair* $\mathbf{P}(\mathbf{G}(G, \mathfrak{g}))$ *associated with* $\mathbf{G}(G, \mathfrak{g})$ *is naturally isomorphic to the original* $(G, \mathfrak{g})$.

To prove this we need a lemma. Set $\mathbf{A} := \mathbf{A}(G, \mathfrak{g})$, again. Then $\mathbf{A}$ is an object (indeed, a Hopf-algebra object) in G-SMod, being defined by the same formula as (4.16). Recall from Lemma 4.12 that $\mathbf{U} \in \mathsf{SMod}$-$G$.

Lemma 4.20. *The pairing* (4.22) *is a Hopf pairing such that*

$$\langle x^\gamma, a\rangle = \langle x, {}^\gamma a\rangle, \quad x \in \mathbf{U},\ a \in \mathbf{A}. \tag{4.25}$$

Proof. Note that the co-adjoint coaction $\mathcal{A} \to \mathcal{A} \otimes C$ given in (4.14) is completed to $\widehat{\mathcal{A}} \to \widehat{\mathcal{A}} \otimes C$, by which $\widehat{\mathcal{A}}$ is a left G-supermodule including $\mathbf{A}$ as a G-super-submodule. One sees that the pairing (4.13) satisfies the same formula as (4.15) for $a \in \widehat{\mathcal{A}}$. The resulting formula shows (4.25).

The rest follows since the pairing (4.13) satisfies the formulas (2.2) required by Hopf pairings. Here we understand that for $x, y \in \mathcal{H}$ and $a \in \widehat{\mathcal{A}}$, $\langle x, a_{(1)}\rangle \langle y, a_{(2)}\rangle$ represents $\langle x \otimes y, \widehat{\Delta}(a)\rangle$; this last denotes the pairing on $(\mathcal{H} \otimes \mathcal{H}) \times (\widehat{\mathcal{A}} \widehat{\otimes} \widehat{\mathcal{A}})$ which is obtained naturally from the pairing on $(\mathcal{H} \otimes \mathcal{H}) \times (\mathcal{A} \otimes \mathcal{A})$, just as (4.13) is obtained from (4.11). □

Proof of Proposition 4.19. We see from the definition of ψ that the pairing $\langle\ ,\ \rangle : \mathbf{U} \times \mathbf{A} \to \Bbbk$ given in (4.22) satisfies

$$\langle \phi(x \otimes y), a\rangle = \langle x, \overline{a}_{(1)}\rangle \langle y, \varrho(a_{(2)})\rangle, \quad x \in J,\ y \in \wedge(\mathfrak{g}_1),\ a \in \mathbf{A}.$$

What appear on the right-hand side are the canonical pairings on $J \times C$ and on $\wedge(\mathfrak{g}_1) \times \wedge(W)$. It follows that the pairing induces a non-degenerate pairing $\mathfrak{g} \times \mathbf{A}^+/(\mathbf{A}^+)^2 \to \Bbbk$. Lemma 4.20 shows that the last pairing induces an isomorphism $\mathrm{Lie}(\mathbf{G}) \simeq \mathfrak{g}$ of Lie superalgebras, where $\mathbf{G} := \mathbf{G}(G, \mathfrak{g})$. In addition, the isomorphism $W^{\mathbf{A}} \simeq \mathfrak{g}_1^*$ obtained in (4.24) is indeed G-equivariant. It follows that the Lie superalgebra isomorphism together with $\overline{\mathbf{A}} \simeq C$ gives the desired isomorphism of Harish-Chandra pairs. It is natural since the construction of (4.22) is functorial. □

Remark 4.21. One sees that the construction above gives an affine (not necessarily algebraic) supergroup, more generally, starting with a pair $(G, \mathfrak{g})$ such that:

(i) G is an affine group with $\mathcal{O}(G)$ $\Bbbk$-flat,
(ii) $\mathfrak{g}$ is an admissible Lie superalgebra with $\mathfrak{g}_1$ $\Bbbk$-finite (free),
(iii) $\mathfrak{g}$ is given a right G-supermodule structure such that the super-bracket on $\mathfrak{g}$ is G-equivariant, and
(iv) there is given a bilinear map $\langle\ ,\ \rangle : \mathfrak{g}_0 \times \mathcal{O}(G) \to \Bbbk$ such that
$$\langle x, ab\rangle = \langle x, a\rangle\, \varepsilon(b) + \varepsilon(a)\langle x, b\rangle,$$
$$\langle x^\gamma, a\rangle = \langle x, {}^\gamma a\rangle,$$
$$[z, x] = \langle x, z^{(-1)}\rangle\, z^{(0)},$$
where $x \in \mathfrak{g}_0$, $a, b \in \mathcal{O}(G)$, $\gamma \in G$, $z \in \mathfrak{g}$, and $z \mapsto z^{(-1)} \otimes z^{(0)}$ denotes the left $\mathcal{O}(G)$-super-comodule structure on $\mathfrak{g}$ which corresponds to the given right G-supermodule structure.

Here we do not assume (B1) or G being algebraic. Given a super Lie group, say $\mathfrak{G}$, we have in mind as G and $\mathfrak{g}$ above the universal algebraic hull of the associated Lie group $\mathfrak{G}_{red}$ and the Lie superalgebra $\mathrm{Lie}(\mathfrak{G})$ of $\mathfrak{G}$, respectively.

See [20, Remark 11] for a similar construction in an alternative situation.

4.5. The following is our main result.

Theorem 4.22. *We have a category equivalence* $\mathsf{ASG} \approx \mathsf{HCP}$. *In fact the functors* $\mathbf{P} : \mathsf{ASG} \to \mathsf{HCP}$ *and* $\mathbf{G} : \mathsf{HCP} \to \mathsf{ASG}$ *are quasi-inverse to each other.*

Since Proposition 4.19 shows that $\mathbf{P} \circ \mathbf{G}$ is naturally isomorphic to the identity functor id, it remains to prove $\mathbf{G} \circ \mathbf{P} \simeq \mathrm{id}$.

Let $\mathbf{G} \in \mathsf{ASG}$. Set
$$\mathbf{A} := \mathcal{O}(\mathbf{G}), \quad \mathfrak{g} := \mathrm{Lie}(\mathbf{G}), \quad \mathbf{U} := \mathbf{U}(\mathfrak{g}), \quad G := \mathbf{G}_{ev}.$$

Lemma 4.23. *The natural embedding* $\mathfrak{g} \subset \mathbf{A}^*$ *uniquely extends to a superalgebra map* $\mathbf{U} \to \mathbf{A}^*$. *The associated pairing* $\langle\ ,\ \rangle : \mathbf{U} \times \mathbf{A} \to \Bbbk$ *is a Hopf pairing.*

Proof. The superalgebra map $\mathbf{T}(\mathfrak{g}) \to \mathbf{A}^*$ which extends $\mathfrak{g} \subset \mathbf{A}^*$ kills the first elements in (3.1) by definition of the super-bracket. For $v \in \mathfrak{g}_1$ it kills $2v^2 - [v, v]$, whence it does $v^2 - \frac{1}{2}[v, v]$ since $\mathbf{A}^*$ is 2-torsion free. This proves the first assertion.

As for the second it is easy to see that $\langle x, 1\rangle = \varepsilon(x)$, $x \in \mathbf{U}$. It remains to prove
$$\langle x, ab\rangle = \langle x_{(1)}, a\rangle\, \langle x_{(2)}, b\rangle, \quad x \in \mathbf{U},\ a, b \in \mathbf{A}.$$
We may suppose that x is of the form $x = u_1 \dots u_r$, where u_i are homogeneous elements in $\mathfrak{g}$. Then the equation is proved by induction on the length r. $\square$

Recall $\mathbf{A} \in G\text{-}\mathsf{SMod}$, $\mathbf{U} \in \mathsf{SMod}\text{-}G$; see (4.5) or (4.16) for $\mathbf{A}$, and see Lemma 4.12 for $\mathbf{U}$. Indeed, $\mathbf{A}$ and $\mathbf{U}$ are Hopf-algebra objects in the respective categories.

Lemma 4.24. *The Hopf pairing* $\langle\ ,\ \rangle : \mathbf{U} \times \mathbf{A} \to \Bbbk$ *just obtained satisfies the same formula as* (4.25).

Proof. The G-module structure on $\mathfrak{g}$ is transposed from that on $\mathbf{A}^+/(\mathbf{A}^+)^2$. Therefore, the formula holds for every $x \in \mathfrak{g}$ and for any $a \in \mathbf{A}$. The desired formula follows by induction, as in the last proof; see also the proof of Lemma 4.10 (3). $\square$

Set

$$C := \mathcal{O}(G), \quad J := U(\mathfrak{g}_0).$$

Note $\mathbf{P}(\mathbf{G}) = (G, \mathfrak{g})$. We aim to show that the affine Hopf superalgebra $\mathbf{A}(G, \mathfrak{g})$, which is constructed from this last Harish-Chandra pair as in the previous subsection, is naturally isomorphic to the present $\mathbf{A}$. By using the Hopf pairing above and the notation (2.5), we define

$$\eta' : \mathbf{A} \to \mathrm{Hom}_J(\mathbf{U}, C), \quad \eta'(a)(x) = \overline{a}_{(1)} \langle x, a_{(2)} \rangle,$$

where $a \in \mathbf{A}$, $x \in \mathbf{U}$. Note that $\mathrm{Hom}_J(\mathbf{U}, C)$ has the Hopf superalgebra structure which is transferred from $\mathbf{A}(G, \mathfrak{g})$ through η (see (4.19)), and which is presented by the formulas given in Proposition 4.11 with the obvious modification. (To answer a question by the referee we remark here that our η' above is essentially the same, up to sign, as the existing ones, such as η^* in [4, p. 133, lines 2–3]. The authors will discuss the difference of sign somewhere else.)

Proposition 4.25. *η' is an isomorphism of Hopf superalgebras.*

Proof. Using Lemma 4.24 one computes in the same way as proving Proposition 4.11 (2) so that

$$\langle (\gamma, \delta),\ (\eta'(a_{(1)}) \otimes \eta'(a_{(2)}))(x \otimes y) \rangle = \langle \gamma\delta,\ \eta'(a)(x^{\delta}\, y) \rangle,$$

where $a \in \mathbf{A}$, $\gamma, \delta \in G$, $x, y \in \mathbf{U}$. The right-hand side equals

$$\langle (\gamma, \delta),\ \Delta(\eta'(a))(x \otimes y) \rangle,$$

by the formula giving the coproduct on $\mathrm{Hom}_J(\mathbf{U}, C)$. Therefore, η' preserves the coproduct. It is easy to see that η' preserves the remaining structure maps and is hence a Hopf superalgebra map.

Set $W := W^{\mathbf{A}}$. Choose ϕ such as in (4.18), and define $\varrho' : \mathbf{A} \to \wedge(W)$ as ϱ in (4.23), with η replaced by η'. Then as was seen for η in the proof of Lemma 4.16 (1), η' is identified with

$$\psi' : \mathbf{A} \to C \otimes \wedge(W), \quad \psi'(a) = \overline{a}_{(1)} \otimes \varrho'(a_{(2)}).$$

Since one sees that this ψ' satisfies the assumption of Lemma 4.26 below, the lemma proves that ψ' and so η' are isomorphisms. $\square$

Lemma 4.26. *In general, let $\mathbf{A}$ be a split affine Hopf superalgebra, and set $C := \overline{\mathbf{A}}$, $W := W^{\mathbf{A}}$. Let $\psi : \mathbf{A} \to C \otimes \wedge(W)$ be a counit-preserving map of left C-comodule superalgebras. Assume that the composite $(\varepsilon \otimes \varpi) \circ \psi : \mathbf{A} \to W$, where $\varpi : \wedge(W) \to W$ denotes the canonical projection, coincides with the canonical projection $\mathbf{A} \to \mathbf{A}_1/\mathbf{A}_0^+\mathbf{A}_1 = W$. Then ψ is necessarily an isomorphism.*

Proof. Let $\mathbf{B} := C \otimes \wedge(W)$. Set $\mathfrak{a} := (\mathbf{A}_1)$ and $\mathfrak{b} := (\mathbf{B}_1)\,(= C \otimes \wedge(W)^+)$ in $\mathbf{A}$ and in $\mathbf{B}$, respectively. Since $\psi(\mathfrak{a}^n) \subset \mathfrak{b}^n$ for every $n \geq 0$, there is induced a counit-preserving, left C-comodule $\mathbb{N}$-graded algebra map

$$\mathrm{gr}\,\psi : \mathrm{gr}\,\mathbf{A} = \bigoplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1} \to \mathrm{gr}\,\mathbf{B} = \bigoplus_{n=0}^{\infty} \mathfrak{b}^n/\mathfrak{b}^{n+1}.$$

One sees that $\mathrm{gr}\,\mathbf{B} = \mathbf{B} = C \otimes \wedge(W)$. Since $\mathbf{A}$ is split, we have as in [19, Proposition 4.9 (2)], a canonical isomorphism $\mathrm{gr}\,\mathbf{A} \simeq C \otimes \wedge(W)$, through which we will identify the two. Then $\mathrm{gr}\,\psi$ is a counit-preserving endomorphism of the left C-comodule

$\mathbb{N}$-graded algebra $C\otimes\wedge(W)$. Being a counit-preserving endomorphism of the left C-comodule algebra C, $\operatorname{gr}\psi(0)$ is the identity on C. This together with the assumption above implies that $\operatorname{gr}\psi(1)$ is the identity on $C\otimes W$. It follows that $\operatorname{gr}\psi$ is an isomorphism. Since the affinity assumption implies $\operatorname{gr}\mathbf{A}(n) = 0 = \operatorname{gr}\mathbf{B}(n)$ for $n \gg 0$, one sees that ψ is an isomorphism. □

Proof of Theorem 4.22. Since we see that η and η' are both natural, it follows that $\mathbf{A}(G,\mathfrak{g})$ and $\mathbf{A}$ are naturally isomorphic. This proves $\mathbf{G}\circ\mathbf{P}\simeq \mathrm{id}$, as desired. □

Remark 4.27. Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. Identify ASG with AHSA, through the obvious category anti-isomorphism. Identify our HCP defined by Definition 4.4 with that defined by [20, Definition 7], through the category anti-isomorphism given in Remark 4.5 (1). Then the category equivalences $\mathbf{P}$ and $\mathbf{G}$ given by Theorem 4.22 are easily identified with those $A\mapsto(\overline{A},W^A)$ and $(C,W)\mapsto A(C,W)$ given by [20, Theorem 29].

5. G-supermodules and hy(G)-supermodules

Throughout this section we suppose that $\Bbbk$ is an integral domain. Our assumption that $\Bbbk$ is 2-torsion free is equivalent to $2\neq 0$ in $\Bbbk$.

5.1. Let $\mathbf{G}\in\mathsf{ASG}$, and set $G:=\mathbf{G}_{ev}$. As before, we let $\mathbf{A}:=\mathcal{O}(\mathbf{G})$, whence $\overline{\mathbf{A}}=\mathcal{O}(G)$. We assume that G is *infinitesimally flat* [14, Part I, 7.4]. This means that

(D1) For every $n>0$, $\overline{\mathbf{A}}/(\overline{\mathbf{A}}^+)^n$ is $\Bbbk$-finite projective.

By (C1), it follows that for every $n>0$, $\mathbf{A}/(\mathbf{A}^+)^n$ is $\Bbbk$-finite projective.

Recall that $\mathbf{A}^*$ is the dual superalgebra of the super-coalgebra $\mathbf{A}$. We suppose $(\mathbf{A}/(\mathbf{A}^+)^n)^*\subset\mathbf{A}^*$ through the natural embedding and set

$$\operatorname{hy}(\mathbf{G}):=\bigcup_{n>0}(\mathbf{A}/(\mathbf{A}^+)^n)^*.$$

We call this the *super-hyperalgebra* of $\mathbf{G}$. This is often denoted alternatively by $\operatorname{Dist}(\mathbf{G})$, called the *super-distribution algebra* of $\mathbf{G}$.

It is easy to see that $\operatorname{hy}(\mathbf{G})$ is a super-subalgebra of $\mathbf{A}^*$. By (D1), each $(\mathbf{A}/(\mathbf{A}^+)^n)^*$ is the dual coalgebra of the algebra $\mathbf{A}/(\mathbf{A}^+)^n$. One sees that if $n<m$, then $(\mathbf{A}/(\mathbf{A}^+)^n)^*\subset(\mathbf{A}/(\mathbf{A}^+)^m)^*$ is a coalgebra embedding, so that all $(\mathbf{A}/(\mathbf{A}^+)^n)^*$, $n>0$, form an inductive system of coalgebras.

Lemma 5.1. *Given the coalgebra structure of the inductive limit, the superalgebra* $\operatorname{hy}(\mathbf{G})$ *forms a cocommutative Hopf superalgebra such that the canonical pairing* $\mathcal{O}(\mathbf{G})^*\times\mathcal{O}(\mathbf{G})\to\Bbbk$ *restricts to a Hopf pairing*

$$\langle\ ,\ \rangle:\operatorname{hy}(\mathbf{G})\times\mathcal{O}(\mathbf{G})\to\Bbbk. \tag{5.1}$$

Proof. Let $\mathbf{H}:=\operatorname{hy}(\mathbf{G})$. Since each $(\mathbf{A}/(\mathbf{A}^+)^n)^*$ is cocommutative, so is $\mathbf{H}$. The dual S^* of the antipode S of $\mathbf{A}$ stabilizes $\mathbf{H}$. Denote $S^*|_{\mathbf{H}}$ by S. Then we see that the restricted pairing satisfies (2.2), (2.3). It follows that $\mathbf{H}$ satisfies the compatibility required by super-bialgebras (see [20, Lemma 1]) and has $S=S^*|_{\mathbf{H}}$ as an antipode. □

Let $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$. Note that the primitive elements in $\mathrm{hy}(\mathbf{G})$ coincide precisely with $\mathfrak{g}$. In addition, if $\Bbbk$ is a field of characteristic zero, then we have $\mathrm{hy}(\mathbf{G}) = \mathbf{U}(\mathfrak{g})$.

The Hopf superalgebra quotient $\mathcal{O}(\mathbf{G}) \to \mathcal{O}(G)$ gives rise to a Hopf superalgebra embedding of the hyperalgebra $\mathrm{hy}(G)$ of G into $\mathrm{hy}(\mathbf{G})$. Let $W := W^{\mathbf{A}}$ $(= \mathfrak{g}_1^*)$, and choose a counit-preserving isomorphism

$$\psi : \mathcal{O}(\mathbf{G}) \xrightarrow{\simeq} \mathcal{O}(G) \otimes \wedge(W)$$

of left $\mathcal{O}(G)$-comodule superalgebras.

Lemma 5.2. *There uniquely exists a unit-preserving isomorphism*

$$\phi : \mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_1) \xrightarrow{\simeq} \mathrm{hy}(\mathbf{G})$$

of left $\mathrm{hy}(G)$*-module super-coalgebras such that*

$$\langle \phi(z), a \rangle = \langle z, \psi(a) \rangle, \quad a \in \mathcal{O}(\mathbf{G}),\ z \in \mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_1),$$

where the right-hand side gives the tensor product of the canonical pairings

$$\mathrm{hy}(G) \times \mathcal{O}(G) \to \Bbbk, \quad \wedge(\mathfrak{g}_1) \times \wedge(W) \to \Bbbk. \tag{5.2}$$

Proof. We see that ψ^* restricts to $\mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_1) \xrightarrow{\simeq} \mathrm{hy}(\mathbf{G})$, and this isomorphism is such as mentioned above. □

We will identify

$$\mathcal{O}(\mathbf{G}) = \mathcal{O}(G) \otimes \wedge(W), \quad \mathrm{hy}(G) \otimes \wedge(\mathfrak{g}_1) = \mathrm{hy}(\mathbf{G}) \tag{5.3}$$

through ψ, ϕ, respectively.

Let Q be the quotient field of $\Bbbk$, and let G_Q denote the base change of G to Q. In addition to (D1), we assume

(D2) G_Q is connected or, in other words, $\mathcal{O}(G_Q) = \mathcal{O}(G) \otimes Q$ contains no non-trivial idempotent.

This assumption ensures the following.

Lemma 5.3. *For every $r > 0$, the superalgebra map*

$$\mathcal{O}(\mathbf{G})^{\otimes r} \to (\mathrm{hy}(\mathbf{G})^{\otimes r})^*$$

which is associated with the r-fold tensor product of the Hopf pairing (5.1) *is injective.*

Proof. By Lemma 5.2 it suffices to prove that the algebra map $\mathcal{O}(G)^{\otimes r} \to (\mathrm{hy}(G)^{\otimes r})^*$ similarly given is injective. By [26, Proposition 0.3.1(g)], (D2) ensures that the Q-algebra map $\mathcal{O}(G_Q)^{\otimes r} \to (\mathrm{hy}(G_Q)^{\otimes r})^*$ for G_Q is injective. Since $\mathrm{hy}(G_Q) = \mathrm{hy}(G) \otimes Q$, we have the canonical map $(\mathrm{hy}(G)^{\otimes r})^* \otimes Q \to (\mathrm{hy}(G_Q)^{\otimes r})^*$. By (B3) we have $\mathcal{O}(G)^{\otimes r} \subset \mathcal{O}(G)^{\otimes r} \otimes Q$. The desired injectivity follows from the commutative diagram

$$\begin{array}{ccc} \mathcal{O}(G)^{\otimes r} \otimes Q & \longrightarrow & (\mathrm{hy}(G)^{\otimes r})^* \otimes Q \\ \downarrow{\simeq} & & \downarrow \\ \mathcal{O}(G_Q)^{\otimes r} & \longrightarrow & (\mathrm{hy}(G_Q)^{\otimes r})^*. \end{array}$$

□

Let M be a supermodule. Given a left $\mathbf{G}$-supermodule (resp., G-module) structure on M, one defines by the formula (2.7), using the Hopf pairing (5.1) (resp., the first one of (5.2)), a left $\mathrm{hy}(\mathbf{G})$-supermodule (resp., $\mathrm{hy}(G)$-module) structure on M. We see that in the super-situation, this indeed defines a map from

- the set of all left $\mathbf{G}$-supermodule structures on M

to

- the set of those locally finite, left $\mathrm{hy}(\mathbf{G})$-supermodule structures on M whose restricted (necessarily, locally finite) $\mathrm{hy}(G)$-module structures arise from left G-module structures.

Note that the left and the right $\mathbf{G}$-supermodule structures (resp., locally finite $\mathrm{hy}(\mathbf{G})$-supermodule structures with the property as above) on M are in one-to-one correspondence, since one can switch the sides through the inverse on $\mathbf{G}$ (resp., the antipode on $\mathrm{hy}(\mathbf{G})$). Therefore, we may replace "left" with "right" in the sets above to prove the following proposition. Indeed, we do so, to make the argument fit in with our results so far obtained.

Proposition 5.4. *If M is $\Bbbk$-projective, the map above is a bijection.*

Proof. Since M is $\Bbbk$-projective, the injection given by Lemma 5.3, tensored with M, remains injective. In addition the canonical map $(\mathrm{hy}(\mathbf{G})^{\otimes r})^* \otimes M \to \mathrm{Hom}(\mathrm{hy}(\mathbf{G})^{\otimes r}, M)$ is injective. Let

$$\mu^{(r)} : \mathcal{O}(\mathbf{G})^{\otimes r} \otimes M \to \mathrm{Hom}(\mathrm{hy}(\mathbf{G})^{\otimes r}, M)$$

denote their composite, which is an injective super-linear map. We will use only $\mu^{(1)}$, $\mu^{(2)}$.

Suppose that we are given a structure from the second set; it is a *right* $\mathrm{hy}(\mathbf{G})$-supermodule structure, in particular. We claim that the super-linear map

$$\rho : M \to \mathrm{Hom}(\mathrm{hy}(\mathbf{G}), M), \quad \rho(m)(x) = mx$$

factorizes into $\mu^{(1)}$ and a uniquely determined map, $\rho' : M \to \mathcal{O}(\mathbf{G}) \otimes M$. To show this we use the identification (5.3). Then, ρ decomposes as

$$M \xrightarrow{\rho_1} \mathrm{Hom}(\mathrm{hy}(G), M) \xrightarrow{(\rho_2)_*} \mathrm{Hom}(\mathrm{hy}(G), \mathrm{Hom}(\wedge(\mathfrak{g}_1), M)),$$

where the first map is defined, just as ρ, by $\rho_1(m)(x) = mx$, and the second $(\rho_2)_*$ denotes $\mathrm{Hom}(\mathrm{id}, \rho_2)$ induced by the map $\rho_2 : M \to \mathrm{Hom}(\wedge(\mathfrak{g}_1), M)$ similarly defined. We have the injections

$$\nu_1 : \mathcal{O}(G) \otimes M \to \mathrm{Hom}(\mathrm{hy}(G), M),$$
$$\nu_2 : \mathcal{O}(G) \otimes \mathrm{Hom}(\wedge(\mathfrak{g}_1), M) \to \mathrm{Hom}(\mathrm{hy}(G), \mathrm{Hom}(\wedge(\mathfrak{g}_1), M)),$$

which are defined in the same way as $\mu^{(1)}$. Indeed, ν_2 is identified with $\mu^{(1)}$. The condition regarding the restricted $\mathrm{hy}(G)$-module structures means that ρ_1 factorizes into ν_1 and a uniquely determined map, $\rho'' : M \to \mathcal{O}(G) \otimes M$. The composite $(\mathrm{id} \otimes \rho_2) \circ \rho''$ is identified with the desired map ρ', as is seen from the commutative diagram

$$\begin{array}{ccc}
\mathcal{O}(G)\otimes M & \xrightarrow{\mathrm{id}\otimes\rho_2} & \mathcal{O}(G)\otimes \mathrm{Hom}(\wedge(\mathfrak{g}_1), M) \\
\Big\downarrow{\scriptstyle \nu_1} & & \Big\downarrow{\scriptstyle \nu_2} \\
\mathrm{Hom}(\mathrm{hy}(G), M) & \xrightarrow{(\rho_2)_*} & \mathrm{Hom}(\mathrm{hy}(G), \mathrm{Hom}(\wedge(\mathfrak{g}_1), M)).
\end{array}$$

By using $\mu^{(2)}$, we see that the associativity of the $\mathrm{hy}(\mathbf{G})$-action on M implies that $\rho' : M \to \mathcal{O}(\mathbf{G}) \otimes M$ is coassociative. Similarly, the unitality of the action implies that ρ' is counital. Thus, ρ' is a left $\mathcal{O}(\mathbf{G})$-super-comodule structure on M. It is the unique such structure that gives rise to the originally given structure, as is easily seen. □

5.2. Let $G_{\mathbb{Z}}$ be a split reductive algebraic group over $\mathbb{Z}$; see [14, p. 153]. By saying a reductive algebraic group we assume that it is connected and smooth. Choose a split maximal torus $T_{\mathbb{Z}}$. The pair $(G_{\mathbb{Z}}, T_{\mathbb{Z}})$ naturally corresponds to a root datum $(\mathsf{X}, \mathsf{R}, \mathsf{X}^\vee, \mathsf{R}^\vee)$. In particular, X equals the character group $\mathsf{X}(T_{\mathbb{Z}})$ of $T_{\mathbb{Z}}$. It is known that $\mathcal{O}(G_{\mathbb{Z}})$ is $\mathbb{Z}$-free, and $G_{\mathbb{Z}}$ is infinitesimally flat. Moreover, for any field K, the base change $(G_{\mathbb{Z}})_K$ is a split reductive (in particular, connected) algebraic group over K, and $(T_{\mathbb{Z}})_K$ is its split maximal torus. Conversely, every split reductive algebraic group over K and its split maximal torus are obtained uniquely (up to isomorphism) in this manner.

Recall that $\Bbbk$ is supposed to be an integral domain. Let

$$G = (G_{\mathbb{Z}})_{\Bbbk}, \quad T = (T_{\mathbb{Z}})_{\Bbbk}$$

be the base changes to $\Bbbk$. Note that $\mathcal{O}(G)$ is $\Bbbk$-free. In addition, G satisfies (D1) (with $\overline{A}$ supposed to be $\mathcal{O}(G)$) and (D2).

We have the inclusion $\mathrm{hy}(G) \supset \mathrm{hy}(T)$ of hyperalgebras, which coincides with the base changes of the hyperalgebras $\mathrm{hy}(G_{\mathbb{Z}}) \supset \mathrm{hy}(T_{\mathbb{Z}})$ over $\mathbb{Z}$. Since $\Bbbk$ contains no non-trivial idempotent, the character group $\mathsf{X}(T)$ of T remains X.

Let M be a left or right $\mathrm{hy}(G)$-module. We say that M is a $\mathrm{hy}(G)$-T-*module* [14, p. 171] if the restricted $\mathrm{hy}(T)$-module structure on M arises from some T-module structure on it. This is equivalent to saying that M is a direct sum $M = \bigoplus_{\lambda \in \mathsf{X}} M_\lambda$ of $\Bbbk$-submodules M_λ, $\lambda \in \mathsf{X}$, so that

$$xm = \lambda(x)\, m, \quad x \in \mathrm{hy}(T),\ m \in M_\lambda,\ \lambda \in \mathsf{X},$$

where we have supposed that M is a *left* $\mathrm{hy}(T)$-module. One sees that the T-module structure above is uniquely determined if M is $\Bbbk$-torsion free. A $\mathrm{hy}(G)$-T-module is said to be *locally finite* if it is locally finite as a $\mathrm{hy}(G)$-module.

Let M be a $\Bbbk$-module. Given a left G-module structure on M, there arises, as before, a left $\mathrm{hy}(G)$-module structure on M; it is indeed a locally finite $\mathrm{hy}(G)$-T-module structure, as is easily seen. Thus we have a map from

- the set of all left G-module structures on M

to

- the set of all locally finite, left $\mathrm{hy}(G)$-T-module structures on M.

The structures in each set above are in one-to-one correspondence with the opposite-sided structures, as before. The following is known.

Theorem 5.5 ([14, Part II, 1.20, p. 171])**.** *If M is $\Bbbk$-projective, the map above is a bijection.*

Remark 5.6. Let $\Bbbk = \mathbb{Z}$, and suppose that $G_{\mathbb{Z}}$ is semisimple or, equivalently, $[\mathsf{X} : \mathbb{Z}\mathsf{R}] < \infty$; see [14, Part II, 1.6, p. 158]. Then it is known (see [16, 27]) that

$$\mathcal{O}(G_{\mathbb{Z}}) = \mathrm{hy}(G_{\mathbb{Z}})^{\circ}. \tag{5.4}$$

It follows that every $\mathbb{Z}$-free, locally finite $\mathrm{hy}(G_{\mathbb{Z}})$-module is necessarily a $\mathrm{hy}(G_{\mathbb{Z}})$-$T_{\mathbb{Z}}$-module.

Given a Hopf algebra H over $\mathbb{Z}$, we let H° denote, just when working over a field (see [25, Section 6.0]), the union of the $\mathbb{Z}$-submodules $(H/I)^*$ in H^*, where I runs over the ideals of H such that H/I is $\mathbb{Z}$-finite. Since the canonical map $(H/I)^* \otimes (H/I)^* \to (H/I \otimes H/I)^*$ is an isomorphism, each $(H/I)^*$ is a ($\mathbb{Z}$-finite free) coalgebra; whence H° is a coalgebra and is in fact a Hopf algebra.

Keep G, T as above. Let us consider objects $\mathbf{G} \in \mathsf{ASG}$ such that $\mathbf{G}_{ev} = G$.

Remark 5.7. (1) As will be seen in Section 6.2, if $\Bbbk = \mathbb{Z}$, the *Chevalley $\mathbb{Z}$-supergroups of classical type* which were constructed by Fioresi and Gavarini [8] and by Gavarini [9] (see also [7]) are examples of $\mathbf{G}$ as above. Therefore, their base changes are as well.

(2) Suppose that $\Bbbk$ is a field of characteristic $\neq 2$. Recall that every split reductive algebraic group is of the form G as above. Then it follows from Theorem 2.3 that the objects under consideration are precisely all algebraic supergroups $\mathbf{G}$ such that $\mathbf{G}_{ev}$ is a split reductive algebraic group.

Let $\mathbf{G} \in \mathsf{ASG}$ such that $\mathbf{G}_{ev} = G$. Let M be a left or right $\mathrm{hy}(\mathbf{G})$-supermodule. We say that M is a $\mathrm{hy}(\mathbf{G})$-T-*supermodule* if the restricted $\mathrm{hy}(T)$-module structure on M arises from some T-module structure on it; this is equivalent to saying that M is a $\mathrm{hy}(G)$-T-module, regarded as a $\mathrm{hy}(G)$-module by restriction. A $\mathrm{hy}(\mathbf{G})$-T-*supermodule* is said to be *locally finite* if it is so as a $\mathrm{hy}(\mathbf{G})$-supermodule or, equivalently, as a $\mathrm{hy}(G)$-module.

Let M be a supermodule. Given a left $\mathbf{G}$-supermodule structure on M, there arises, as before, a left $\mathrm{hy}(\mathbf{G})$-supermodule structure on M; it is indeed a locally finite $\mathrm{hy}(\mathbf{G})$-T-supermodule structure, as is easily seen. Thus we have a map from

- the set of all left $\mathbf{G}$-supermodule structures on M

to

- the set of all locally finite, left $\mathrm{hy}(\mathbf{G})$-T-supermodule structures on M.

The structures in each set above are in one-to-one correspondence with the opposite-sided structures, as before. Proposition 5.4 and Theorem 5.5 prove the following.

Theorem 5.8. *If M is $\Bbbk$-projective, the map above is a bijection.*

Remark 5.9. Let $\Bbbk = \mathbb{Z}$, and suppose that $G_{\mathbb{Z}}$ is semisimple. Then by using the same argument as in proving [20, Proposition 31], we see from (5.4) that $\mathcal{O}(\mathbf{G}) = \mathrm{hy}(\mathbf{G})^{\circ}$. It follows that every $\mathbb{Z}$-free, locally finite $\mathrm{hy}(\mathbf{G})$-supermodule is necessarily a $\mathrm{hy}(\mathbf{G})$-$T_{\mathbb{Z}}$-supermodule.

Theorem 5.8 can be reformulated as an isomorphism between the category of $\Bbbk$-projective, left $\mathbf{G}$-supermodules and the category of $\Bbbk$-projective, locally finite left $\mathrm{hy}(\mathbf{G})$-T-supermodules. When $\Bbbk$ is a field of characteristic $\neq 2$, the result is formulated as follows, in view of Remark 5.7 (2).

Corollary 5.10. *Suppose that $\Bbbk$ is a field of characteristic $\neq 2$, and let $\mathbf{G}$ be an algebraic supergroup over $\Bbbk$ such that $\mathbf{G}_{ev}$ is a split reductive algebraic group. Choose a split maximal torus T of $\mathbf{G}_{ev}$. Then there is a natural isomorphism between the category of left $\mathbf{G}$-supermodules and the category of locally finite, left* $\mathrm{hy}(\mathbf{G})$*-T-supermodules.*

This has been known only for some special algebraic supergroups with the property as above; see Brundan and Kleshchev [2, Corollary 5.7], Brundan and Kujawa [3, Corollary 3.5], and Shu and Wang [24, Theorem 2.8].

6. Harish-Chandra pairs corresponding to Chevalley supergroups over $\mathbb{Z}$

6.1. Those finite-dimensional simple Lie superalgebras over the complex number field $\mathbb{C}$ which are not purely even were classified by Kac [15]. They are divided into classical type and Cartan type. A *Chevalley $\mathbb{C}$-supergroup of classical/Cartan type* is a connected algebraic supergroup $\mathbf{G}$ over $\mathbb{C}$ such that $\mathrm{Lie}(\mathbf{G})$ is a simple Lie superalgebra of classical/Cartan type. As was mentioned in Remark 5.7 (1), Fioresi and Gavarini [8,9] constructed natural $\mathbb{Z}$-forms of Chevalley $\mathbb{C}$-supergroups of classical type. Gavarini [10] accomplished the same construction for Cartan type. The resulting $\mathbb{Z}$-forms are called *Chevalley $\mathbb{Z}$-supergroups of classical/Cartan type*; they are indeed objects in our category ASG defined over $\mathbb{Z}$.

Based on our Theorem 4.22, we will reconstruct the Chevalley $\mathbb{Z}$-supergroups by giving the corresponding Harish-Chandra pairs. Indeed, our construction depends on part of Fioresi and Gavarini's, but simplifies the rest; see Remarks 6.3 and 6.8.

6.2. Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra over $\mathbb{C}$ which is of classical type. Then $\mathfrak{g}_0$ is a reductive Lie algebra, and $\mathfrak{g}_1$, with respect to the right adjoint $\mathfrak{g}_0$-action, decomposes as the direct sum of weight spaces for a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$. Let Δ_0 (resp., Δ_1) denote the set of the even (resp., odd) roots, that is, the weights with respect to the adjoint $\mathfrak{h}$-action on $\mathfrak{g}_0$ (resp., on $\mathfrak{g}_1$).

Let

$$(\mathsf{X}, \mathsf{R}, \mathsf{X}^\vee, \mathsf{R}^\vee), \quad G_{\mathbb{Z}} \supset T_{\mathbb{Z}} \tag{6.1}$$

be a root datum and the corresponding split reductive algebraic $\mathbb{Z}$-group and split maximal torus. Suppose that $\mathfrak{g}_0 \supset \mathfrak{h}$ coincide with the complexifications of $\mathrm{Lie}(G_{\mathbb{Z}}) \supset \mathrm{Lie}(T_{\mathbb{Z}})$. Then one has

$$\mathsf{R} = \Delta_0, \quad \mathsf{X}^\vee \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}, \quad \mathrm{hy}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C} = U(\mathfrak{g}_0).$$

Recall that $\mathrm{hy}(G_{\mathbb{Z}})$ is called a *Kostant form* of $U(\mathfrak{g}_0)$. We assume

$$\Delta_1 \subset \mathsf{X}. \tag{6.2}$$

Theorem 6.1 (Fioresi, Gavarini)**.** *There exists a $\mathbb{Z}$-lattice $V_{\mathbb{Z}}$ of $\mathfrak{g}_1$ such that:*

(i) $\mathfrak{g}_{\mathbb{Z}} := \mathrm{Lie}(G_{\mathbb{Z}}) \oplus V_{\mathbb{Z}}$ *is a Lie-superalgebra $\mathbb{Z}$-form of $\mathfrak{g}$.*
(ii) *This Lie superalgebra $\mathfrak{g}_{\mathbb{Z}}$ over $\mathbb{Z}$ is admissible.*
(iii) *$V_{\mathbb{Z}}$ is $\mathrm{hy}(G_{\mathbb{Z}})$-stable in the right $U(\mathfrak{g}_0)$-module $\mathfrak{g}_1$.*

Fioresi and Gavarini [8] and Gavarini [9] introduced the notion of *Chevalley bases*, gave an explicit example of such a basis for each $\mathfrak{g}$, and constructed from the basis a natural Hopf-superalgebra $\mathbb{Z}$-form, called a *Kostant superalgebra*, of $\mathbf{U}(\mathfrak{g})$; the even basis elements coincide with the classical Chevalley basis for $\mathfrak{g}_0$. They do not refer to root data. But, once an explicit Chevalley basis is given as in [8, 9], one can rechoose the basis so that it includes a $\mathbb{Z}$-free basis of $\mathsf{X}^\vee$ by replacing part of the original basis, $H_1, \ldots, H_\ell$, with a desired $\mathbb{Z}$-free basis; this replacement is possible, since it affects only the adjoint action on the basis elements X_α, and the new basis elements still act via the roots α. (The method of [8, Remark 3.8] attributed to the referee gives an alternative construction of the desired basis from scratch.) One sees that the odd elements in the Chevalley basis generate the desired $\mathbb{Z}$-lattice $V_\mathbb{Z}$ as above; see [8, Sections 4.2, 6.1] and [9, Section 3.4] to verify condition (ii), in particular.

Set $\mathfrak{g}_\mathbb{Z} := \mathrm{Lie}(G_\mathbb{Z}) \oplus V_\mathbb{Z}$ in $\mathfrak{g}$, as above. One sees from (iii) and (6.2) that $V_\mathbb{Z}$ is a right $\mathrm{hy}(G_\mathbb{Z})$-$T_\mathbb{Z}$-module, whence it is a right $G_\mathbb{Z}$-module by Theorem 5.5. The restricted super-bracket $[\ ,\ ] : V_\mathbb{Z} \times V_\mathbb{Z} \to \mathrm{Lie}(G_\mathbb{Z})$, being $\mathrm{hy}(G_\mathbb{Z})$-linear, is $G_\mathbb{Z}$-equivariant. This proves the following.

Proposition 6.2. *$(G_\mathbb{Z}, \mathfrak{g}_\mathbb{Z})$ is a Harish-Chandra pair.*

We let

$$\mathbf{G}_\mathbb{Z} = \mathbf{G}(G_\mathbb{Z}, \mathfrak{g}_\mathbb{Z})$$

denote the algebraic $\mathbb{Z}$-supergroup in ASG which is associated with the Harish-Chandra pair just obtained. Since one sees that the category equivalences in Theorem 4.22 are compatible with base extensions, it follows that $\mathbf{G}_\mathbb{Z}$ is a $\mathbb{Z}$-form of the algebraic $\mathbb{C}$-supergroup associated with the Harish-Chandra pair $(G, \mathfrak{g})$, where G denotes the base change of $G_\mathbb{Z}$ to $\mathbb{C}$. Recall from Section 6.1 the definition of Chevalley $\mathbb{C}$-supergroups of classical type, and note that every such $\mathbb{C}$-supergroup is associated with some Harish-Chandra pair of the last form. We have thus constructed a natural $\mathbb{Z}$-form of every Chevalley $\mathbb{C}$-supergroup of classical type.

Remark 6.3. (1) After constructing Kostant superalgebras, Fioresi and Gavarini's construction, which is parallel to the classical construction of Chevalley $\mathbb{Z}$-groups, continues as follows: (a) Choose a faithful rational representation $\mathfrak{g} \to \mathfrak{gl}_\mathbb{C}(M)$ on a finite-dimensional super-vector space M over $\mathbb{C}$, (b) choose a $\mathbb{Z}$-lattice $M_\mathbb{Z}$ in M which is stable under the action of the Kostant superalgebra, (c) construct a natural group-valued functor which is realized as subgroups of $\mathbf{GL}_R(M_\mathbb{Z} \otimes_\mathbb{Z} R)$, where R runs over the commutative superalgebras over $\mathbb{Z}$, and (d) prove that the sheafification, say $\mathbf{G}_\mathbb{Z}^{FG}$, of the constructed group-valued functor is representable and has desired properties, which include the property that $\mathcal{O}(\mathbf{G}_\mathbb{Z}^{FG})$ is split; see [8, Corollary 5.20] and [9, Corollary 4.22] for the last property.

Our method of construction dispenses with these procedures.

(2) The algebraic group $(\mathbf{G}_\mathbb{Z}^{FG})_{ev}$ associated with Fioresi and Gavarini's $\mathbf{G}_\mathbb{Z}^{FG}$ is a split reductive algebraic $\mathbb{Z}$-group. As was noted in an earlier version of the present paper, it was not clear to the authors whether the split reductive algebraic $\mathbb{Z}$-groups which correspond to all *possible* root data (namely, all relevant root data satisfying (6.2)) can be realized as $(\mathbf{G}_\mathbb{Z}^{FG})_{ev}$; note that by definition, those algebraic $\mathbb{Z}$-groups are realized as our $(\mathbf{G}_\mathbb{Z})_{ev} = G_\mathbb{Z}$. Later, Gavarini kindly showed the first-named author that they are indeed realized; essentially the same argument of his proof is contained in the erratum added to a new version of [10].

6.3. Let $\mathfrak{g}$ be a finite-dimensional simple Lie superalgebra over $\mathbb{C}$ which is of Cartan type. Then $\mathfrak{g}_0$ is a direct sum $\mathfrak{g}_0^r \ltimes \mathfrak{g}_0^n$ of a reductive Lie algebra $\mathfrak{g}_0^r$ with a nilpotent Lie algebra $\mathfrak{g}_0^n$. With respect to the right adjoint $\mathfrak{g}_0^r$-action, $\mathfrak{g}_0^n$ and $\mathfrak{g}_1$ decompose as direct sums of weight spaces for a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0^r$; we let Δ_0^r, Δ_0^n and Δ_1 denote the sets of the roots for $\mathfrak{g}_0^r$, $\mathfrak{g}_0^n$ and $\mathfrak{g}_1$, respectively. The nilpotent Lie algebra $\mathfrak{g}_0^n$ acts on $\mathfrak{g}_1$ nilpotently.

This time we assume that the root datum and the corresponding algebraic $\mathbb{Z}$-groups given in (6.1) are as follows: $\mathfrak{g}_0^r \supset \mathfrak{h}$ coincide with the complexifications of $\mathrm{Lie}(G_{\mathbb{Z}}) \supset \mathrm{Lie}(T_{\mathbb{Z}})$, and $\Delta_0^n \subset \mathsf{X} \supset \Delta_1$.

Theorem 6.4 (Gavarini)**.** *There exist $\mathbb{Z}$-lattices $N_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$ of $\mathfrak{g}_0^n$ and $\mathfrak{g}_1$, respectively, such that:*

(i) $\mathfrak{g}_{\mathbb{Z}} := \mathrm{Lie}(G_{\mathbb{Z}}) \oplus N_{\mathbb{Z}} \oplus V_{\mathbb{Z}}$ *is a Lie-superalgebra $\mathbb{Z}$-form of* $\mathfrak{g}$.

(ii) *This Lie superalgebra $\mathfrak{g}_{\mathbb{Z}}$ over $\mathbb{Z}$ is admissible.*

(iii) $V_{\mathbb{Z}}$ *is* $\mathrm{hy}(G_{\mathbb{Z}})$*-stable in the right $U(\mathfrak{g}_0^r)$-module* $\mathfrak{g}_1$.

(iv) $N_{\mathbb{Z}}$ *contains a $\mathbb{Z}$-free basis $x_1, \dots, x_s$ such that*

(iv-1) *the $\mathbb{Z}$-submodule $H_{\mathbb{Z}}$ of $U(\mathfrak{g}_0^n)$ which is (freely) generated by*

$$\frac{x_1^{n_1}}{n_1!} \cdots \frac{x_s^{n_s}}{n_s!}, \quad n_1 \geq 0, \dots, n_s \geq 0$$

is a $\mathbb{Z}$-subalgebra,

(iv-2) $V_{\mathbb{Z}}$ *is $H_{\mathbb{Z}}$-stable in the right $U(\mathfrak{g}_0^n)$-module $\mathfrak{g}_1$, and*

(iv-3) $H_{\mathbb{Z}}$ *is* $\mathrm{hy}(G_{\mathbb{Z}})$*-stable in the right $U(\mathfrak{g}_0^r)$-module $U(\mathfrak{g}_0^n)$.*

Gavarini's construction in [10] is parallel to those in [8,9]. One sees that among Gavarini's Chevalley basis elements, the elements contained in $\mathfrak{g}_0^n$ and the odd elements generate the desired $\mathbb{Z}$-lattices $N_{\mathbb{Z}}$ and $V_{\mathbb{Z}}$, respectively; the former are precisely the desired elements for (iv). See [10, Section 3.1] for (ii), and see [10, Section 3.3] for (iii), (iv). Note that the $\mathbb{Z}$-algebra $H_{\mathbb{Z}}$ given in (iv-1) is indeed a Hopf-algebra $\mathbb{Z}$-form of $U(\mathfrak{g}_0^n)$.

Recall from [6, IV, Sect. 2, 4.5] that there uniquely exists a unipotent algebraic group F over $\mathbb{C}$ such that $\mathrm{Lie}(F) = \mathfrak{g}_0^n$. The corresponding Hopf algebra $\mathcal{O}(F)$ is the polynomial algebra $\mathbb{C}[t_1, \dots, t_s]$ such that

$$(6.3) \quad \langle\ ,\ \rangle : U(\mathfrak{g}_0^n) \times \mathcal{O}(F) \to \mathbb{C}, \quad \langle \frac{x_1^{n_1}}{n_1!} \cdots \frac{x_s^{n_s}}{n_s!},\ t_1^{m_1} \cdots t_s^{m_s} \rangle = \delta_{n_1,m_1} \cdots \delta_{n_s,m_s}$$

is a Hopf pairing. This induces a Hopf algebra isomorphism

$$(6.4) \qquad \mathcal{O}(F) \xrightarrow{\simeq} U(\mathfrak{g}_0^n)'.$$

Here and in what follows, given a finitely generated Hopf algebra B over a field or $\mathbb{Z}$, we define

$$B' := \bigcup_{n>0} (B/(B^+)^n)^*,$$

as in [23, Section 9.2]. This is a Hopf subalgebra of B°. If B is the commutative Hopf algebra corresponding to an algebraic group, then B' is the hyperalgebra of the algebraic group.

Lemma 6.5. *$\mathbb{Z}[t_1, \dots, t_s]$ is a Hopf-algebra $\mathbb{Z}$-form of $\mathcal{O}(F) = \mathbb{C}[t_1, \dots, t_s]$. The Hopf pairing (6.3) over $\mathbb{C}$ restricts to a Hopf pairing $\langle\ ,\ \rangle : H_{\mathbb{Z}} \times \mathbb{Z}[t_1, \dots, t_s] \to \mathbb{Z}$ over $\mathbb{Z}$, and it induces an isomorphism*

$$\mathbb{Z}[t_1, \dots, t_s] \xrightarrow{\simeq} H'_{\mathbb{Z}}$$

of $\mathbb{Z}$-Hopf algebras.

Proof. It is easy to see that the Hopf algebra isomorphism (6.4) restricts to a $\mathbb{Z}$-algebra map $\mathbb{Z}[t_1, \ldots, t_n] \to H'_{\mathbb{Z}}$. We have the following commutative diagram, which contains the isomorphism and the restricted algebra map:

$$\begin{tikzcd}
\mathbb{Z}[t_1, \ldots, t_s] \arrow[r, hook] \arrow[d, hook] & \mathcal{O}(F) = \mathbb{C}[t_1, \ldots, t_s] \arrow[d, "\simeq"] \\
H'_{\mathbb{Z}} \arrow[r, hook] \arrow[d, hook] & U(\mathfrak{g}_0^n)' \arrow[d, hook] \\
H^*_{\mathbb{Z}} \arrow[r, hook] & U(\mathfrak{g}_0^n)^*
\end{tikzcd}$$

Since $H^*_{\mathbb{Z}} \simeq \mathbb{Z}[\![t_1, \ldots, t_n]\!]$, $U(\mathfrak{g}_0^n)^* \simeq \mathbb{C}[\![t_1, \ldots, t_n]\!]$, we see that the outer big square is a pull-back. The lower square is a pull-back, too, as is easily seen. It follows that the upper square is a pull-back, whence $\mathbb{Z}[t_1, \ldots, t_n] \to H'_{\mathbb{Z}}$ is an isomorphism. This implies that $\mathbb{Z}[t_1, \ldots, t_n]$ is a Hopf-algebra $\mathbb{Z}$-form of $\mathcal{O}(F)$. The rest is now easy to see. □

Let $F_{\mathbb{Z}}$ denote the algebraic $\mathbb{Z}$-group corresponding to the $\mathbb{Z}$-Hopf algebra $\mathbb{Z}[t_1, \ldots, t_s]$. Then

$$\mathcal{O}(F_{\mathbb{Z}}) = \mathbb{Z}[t_1, \ldots, t_s], \quad \mathrm{hy}(F_{\mathbb{Z}}) = H_{\mathbb{Z}}, \quad \mathrm{Lie}(F_{\mathbb{Z}}) = N_{\mathbb{Z}}.$$

Note from (i) of Theorem 6.4 that $N_{\mathbb{Z}}$ is a Lie-algebra $\mathbb{Z}$-form of $\mathfrak{g}_0^n$. From the first two equalities above or from Gavarini's original construction one sees that the construction of $H_{\mathbb{Z}}$ does not depend on the order of the basis elements.

Let $G \supset T$ denote the base changes of $G_{\mathbb{Z}} \supset T_{\mathbb{Z}}$ to $\mathbb{C}$. The right $U(\mathfrak{g}_0^r)$-module structure on $\mathfrak{g}_0^n$, which arises from the right adjoint action, is indeed a $U(\mathfrak{g}_0^r)$-T-module structure. Hence it gives rise to a right G-module structure, by which $\mathfrak{g}_0^n$ is a Lie-algebra object in the symmetric tensor category Mod-G of right G-modules. The structure uniquely extends to $U(\mathfrak{g}_0^n)$ so that $U(\mathfrak{g}_0^n)$ turns into a Hopf-algebra object in Mod-G. One sees that the structure just obtained is transposed through (6.3) to $\mathcal{O}(F)$, so that $\mathcal{O}(F)$ is a Hopf-algebra object in the symmetric category G-Mod of left G-modules. Thus, F turns into a right G-equivariant algebraic group. The associated semi-direct product $G \ltimes F$ of algebraic groups has $\mathfrak{g}_0 = \mathfrak{g}_0^r \ltimes \mathfrak{g}_0^n$ as its Lie algebra, as is easily seen. Note that $\mathfrak{g}_1$ is a right $U(\mathfrak{g}_0^r)$-T-module and is such a right $U(\mathfrak{g}_0^n)$-module that is annihilated by $(U(\mathfrak{g}_0^n)^+)^m$ for some m. Then it follows that $\mathfrak{g}_1$ turns into a right G-module and F-module. Moreover, it is a right $G \ltimes F$-module, as is seen by using (1) $\mathrm{Lie}(G \ltimes F) = \mathfrak{g}_0^r \ltimes \mathfrak{g}_0^n$, (2) $G \ltimes F$ is connected, and (3) $\mathfrak{g}_1$ is a right $U(\mathfrak{g}_0)$-module.

What were constructed in the last paragraph are all defined over $\mathbb{Z}$, as is seen from the following lemma.

Lemma 6.6. *Keep the notation as above.*

1. *The right $\mathcal{O}(G)$-comodule structure $\mathcal{O}(F) \to \mathcal{O}(F) \otimes_{\mathbb{C}} \mathcal{O}(G)$ on $\mathcal{O}(F)$ restricts to $\mathcal{O}(F_{\mathbb{Z}}) \to \mathcal{O}(F_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{O}(G_{\mathbb{Z}})$, by which $F_{\mathbb{Z}}$ turns into a right $G_{\mathbb{Z}}$-equivariant algebraic group. Therefore, we have the associated semi-direct product $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$ of algebraic groups.*
2. *$V_{\mathbb{Z}}$ is naturally a right $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$-module.*

Proof. (1) One sees that the right $\mathrm{hy}(G_{\mathbb{Z}})$-module structure on $H_{\mathbb{Z}}$ which is given by (iv-3) of Theorem 6.4 is indeed a $\mathrm{hy}(G_{\mathbb{Z}})$-$T_{\mathbb{Z}}$-module structure. Hence it gives rise to a right $G_{\mathbb{Z}}$-module structure on $H_{\mathbb{Z}}$, by which $H_{\mathbb{Z}}$ turns into a Hopf-algebra object in Mod-$G_{\mathbb{Z}}$. Since the isomorphism given in Lemma 6.5 is compatible with base extension, it follows that the last structure is transposed to a left $G_{\mathbb{Z}}$-module structure on $\mathcal{O}(F_{\mathbb{Z}})$, so that $\mathcal{O}(F_{\mathbb{Z}})$ is a Hopf-algebra object in $G_{\mathbb{Z}}$-Mod. By construction the corresponding right $\mathcal{O}(G_{\mathbb{Z}})$-comodule structure on $\mathcal{O}(F_{\mathbb{Z}})$ is the restriction of the right $\mathcal{O}(G)$-comodule structure on $\mathcal{O}(F)$. This proves the first assertion. The rest is easy to see.

(2) Just as for $H_{\mathbb{Z}}$, we see from (iii) of the theorem that $V_{\mathbb{Z}}$ is a right $\mathrm{hy}(G_{\mathbb{Z}})$-$T_{\mathbb{Z}}$-module, whence it is a right $G_{\mathbb{Z}}$-module. We see from (iv-2) that $V_{\mathbb{Z}}$ is a right $H_{\mathbb{Z}}$-module, and it is indeed a right $H_{\mathbb{Z}}/(H_{\mathbb{Z}}^{+})^{m}$-module for the same m as before. It follows by Lemma 6.5 that $V_{\mathbb{Z}}$ is a right $F_{\mathbb{Z}}$-module.

It remains to prove that

$$(vf)g = (vg)f^{g}, \quad v \in V_{\mathbb{Z}},\ f \in F_{\mathbb{Z}},\ g \in G_{\mathbb{Z}}.$$

Let R be a commutative ring. The equality in $R \otimes_{\mathbb{Z}} \mathbb{C}$-points follows from the analogous equality for $\mathfrak{g}_1$, since $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{g}_1 \otimes_{\mathbb{C}} (R \otimes_{\mathbb{Z}} \mathbb{C})$. To prove the equality in R-points, we may suppose $R = \mathcal{O}(F_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathcal{O}(G_{\mathbb{Z}})$, and so that R is $\mathbb{Z}$-flat. In this case the equality follows from the previous result since we then have $V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \subset V_{\mathbb{Z}} \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} \mathbb{C}$. ☐

Recall that $\mathfrak{g}_{\mathbb{Z}}$ is a Lie-superalgebra $\mathbb{Z}$-form as given in (i) of Theorem 6.4. Its odd component $V_{\mathbb{Z}}$ is a right $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$-module by Lemma 6.6.

Proposition 6.7. *$(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ is a Harish-Chandra pair.*

Proof. As is easily seen, $\mathrm{Lie}(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}})$ coincides with the even component $\mathrm{Lie}(G_{\mathbb{Z}}) \ltimes N_{\mathbb{Z}}$ of $\mathfrak{g}_{\mathbb{Z}}$. The restricted super-bracket $[\ ,\ ] : V_{\mathbb{Z}} \times V_{\mathbb{Z}} \to \mathrm{Lie}(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}})$, being $\mathrm{hy}(G_{\mathbb{Z}})$- and $H_{\mathbb{Z}}$-linear, is $G_{\mathbb{Z}}$- and $F_{\mathbb{Z}}$-equivariant. It is necessarily $G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}$-equivariant. ☐

We thus have the algebraic $\mathbb{Z}$-supergroup $\mathbf{G}(G_{\mathbb{Z}} \ltimes F_{\mathbb{Z}}, \mathfrak{g}_{\mathbb{Z}})$ in ASG which is associated with the Harish-Chandra pair just obtained. It is a $\mathbb{Z}$-form of the algebraic $\mathbb{C}$-supergroup which is associated with the Harish-Chandra pair $(G \ltimes F, \mathfrak{g})$. Since every Chevalley $\mathbb{C}$-supergroup of Cartan type (see Section 6.1) is associated with some Harish-Chandra pair of the last form, we have constructed a natural $\mathbb{Z}$-form of every such $\mathbb{C}$-supergroup.

Remark 6.8. Just as in the classical-type case (see Remark 6.3 (1)), Gavarini's construction requires faithful representations of $\mathfrak{g}$, which, however, must satisfy more involved conditions as given in [10, Definition 3.14]; Proposition 3.16 of [10] proves that part of the conditions is satisfied if the representation is completely reducible. The required representations look thus rather restrictive. On the other hand, Theorem 4.42 of [10] implies that the required representations are numerous enough to ensure that our $\mathbb{Z}$-forms all are realized by Gavarini's construction. But the proof of the theorem is wrong, as was pointed out in an earlier version of this paper. After the publication of [10], a corrected proof of the theorem, which uses the category equivalence [11, Theorem 4.3.14] (= Theorem A.10 below), was given in an erratum added to a new version of [10]. As far as the authors see, the proof is correct if the same argument for proving our Lemma 6.6 is added.

Appendix A. Generalization using 2-operations

In this appendix we work over an arbitrary non-zero commutative ring $\Bbbk$. As was announced in the last paragraph of the Introduction we will refine Gavarini's category equivalence; see Theorem A.10.

A.1. Let $\mathfrak{g}$ be a *Lie superalgebra*; see Section 3.1.

Definition A.1 ([11, Definition 2.2.1])**.** A 2-*operation* on $\mathfrak{g}$ is a map $(\ )^{\langle 2\rangle} : \mathfrak{g}_1 \to \mathfrak{g}_0$ such that:

(i) $(cv)^{\langle 2\rangle} = c^2 v^{\langle 2\rangle}$,
(ii) $(v+w)^{\langle 2\rangle} = v^{\langle 2\rangle} + [v,w] + w^{\langle 2\rangle}$ and
(iii) $[v^{\langle 2\rangle}, z] = [v,[v,z]]$,

where $c \in \Bbbk$, $v, w \in \mathfrak{g}_1$, $z \in \mathfrak{g}$.

This is related to the admissibility defined by Definition 3.1 as follows.

Lemma A.2. *Assume that $\Bbbk$ is 2-torsion free. If $\mathfrak{g}$ is admissible, then*

$$v^{\langle 2\rangle} := \frac{1}{2}[v,v],\ \ v \in \mathfrak{g}_1,$$

gives the unique 2-operation on $\mathfrak{g}$, and this is indeed the unique map $\mathfrak{g}_1 \to \mathfrak{g}_0$ that satisfies (i), (ii) *above.*

Proof. The left- and right-hand sides of (i)–(iii) coincide since their doubles are seen to coincide. The uniqueness follows, since we see from (i), (ii) that $4v^{\langle 2\rangle} = (2v)^{\langle 2\rangle} = 2v^{\langle 2\rangle} + [v,v]$, and so $2v^{\langle 2\rangle} = [v,v]$. □

If $\Bbbk$ is 2-torsion free, an admissible Lie superalgebra is thus the same as a Lie superalgebra $\mathfrak{g}$ given a (unique) 2-operation, such that $\mathfrak{g}_0$ is $\Bbbk$-flat and $\mathfrak{g}_1$ is $\Bbbk$-free.

Let us return to the situation that $\Bbbk$ is arbitrary. Let $\mathfrak{g}$ be a Lie superalgebra given a 2-operation. One directly verifies the following.

Proposition A.3. *Suppose that the odd component $\mathfrak{g}_1$ is $\Bbbk$-free, and choose a totally ordered basis X arbitrarily. Given a commutative algebra R, define a map*

$$(\ )_R^{\langle 2\rangle} : \mathfrak{g}_1 \otimes R \to \mathfrak{g}_0 \otimes R$$

by

$$\Big(\sum_{i=1}^n x_i \otimes c_i\Big)_R^{\langle 2\rangle} := \sum_{i=1}^n x_i^{\langle 2\rangle} \otimes c_i^2 + \sum_{i<j}[x_i,x_j] \otimes c_i c_j,$$

where $x_1 < \cdots < x_n$ in X, and $c_i \in R$. This definition is independent of choice of ordered bases, and the map gives a 2-operation on the R-Lie superalgebra $\mathfrak{g}\otimes R$. For arbitrary elements $v_i \in \mathfrak{g}_1$, $c_i \in R$, $1 \le i \le m$, we have

$$\Big(\sum_{i=1}^m v_i \otimes c_i\Big)_R^{\langle 2\rangle} = \sum_{i=1}^m v_i^{\langle 2\rangle} \otimes c_i^2 + \sum_{i<j}[v_i,v_j] \otimes c_i c_j.$$

In this appendix we let $\mathbf{U}(\mathfrak{g})$ denote the cocommutative Hopf superalgebra which is defined as in [11, Section 4.3.4]. This is the quotient Hopf superalgebra of the tensor algebra $\mathbf{T}(\mathfrak{g})$ divided by the super-ideal generated by the homogeneous primitives

$$zw - (-1)^{|z||w|}wz - [z,w], \quad v^2 - v^{\langle 2\rangle},$$

where z and w are homogeneous elements in $\mathfrak{g}$, and $v \in \mathfrak{g}_1$. The only difference from the definition given in Section 3.2 is that the second generators $v^2 - \frac{1}{2}[v, v]$ in (3.1) are here replaced (indeed, generalized) by $v^2 - v^{\langle 2 \rangle}$.

Lemma A.4. *Suppose that the homogeneous components $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are both $\Bbbk$-free, and choose their totally ordered bases X_0 and X_1. Then $\mathbf{U}(\mathfrak{g})$ has the following monomials as a $\Bbbk$-free basis:*

$$a_1^{r_1} \dots a_m^{r_m} x_1 \dots x_n,$$

where $a_1 < \cdots < a_m$ in X_0, $r_i > 0$, $m \geq 0$, and $x_1 < \cdots < x_n$ in X_1, $n \geq 0$.

Proof. To prove Proposition 3.4 we used the Diamond Lemma [1, Proposition 7.1] for R-rings. But here we use the Diamond Lemma [1, Theorem 1.2] for $\Bbbk$-algebras. We suppose that $X_0 \cup X_1$ is the set of generators and extend the total orders on X_i, $i = 0, 1$, to the set so that $a < x$ whenever $a \in X_0$, $x \in X_1$. The reduction system consists of the obvious reductions arising from the super-bracket and

$$x^2 \to x^{\langle 2 \rangle}, \quad x \in X_1,$$

where the last $x^{\langle 2 \rangle}$ is supposed to be presented as a linear combination of elements in X_0. It is essential to prove that the overlap ambiguities which may occur when we reduce the words

- xxa, $\quad x \in X_1$, $a \in X_0$,
- xyz, $\quad x = y \geq z$ or $x \geq y = z$ in X_1

are resolvable. This is easily proved (indeed, more easily than in the proof of Proposition 3.4) by using condition (iii) in Definition A.1. For example, the word xxa is reduced on the one hand as

$$xxa \to x[x, a] + xax \to x[x, a] + [x, a]x + ax^{\langle 2 \rangle} \to [x, [x, a]] + ax^{\langle 2 \rangle},$$

and on the other hand as

$$xxa \to x^{\langle 2 \rangle} a.$$

The two results coincide by (iii). $\square$

Remark A.5. To use condition (iii) as above, we cannot treat $\mathbf{U}(\mathfrak{g})$ as a $J = U(\mathfrak{g}_0)$-ring as in the proof of Proposition 3.4. Indeed, to reduce the word xxa with $a \in J$ in the proof, we are not allowed to present a as (a linear combination of) bc with $b \in \mathfrak{g}_0$, $c \in J$, and to reduce as

$$xxa \to xxbc \to x[x, b]c + xbxc,$$

because by the first step, the lengths of words increase: $\mathrm{length}(xx*) < \mathrm{length}(xx{*}{*})$; see the proof of [20, Lemma 11].

Corollary A.6 (cf. [11, (4.7)]). *If $\mathfrak{g}_0$ is $\Bbbk$-finite projective and $\mathfrak{g}_1$ is $\Bbbk$-free, then the same result as Corollary* 3.6 *holds; that is, there exists a unit-preserving, left $U(\mathfrak{g}_0)$-module super-coalgebra isomorphism $U(\mathfrak{g}_0) \otimes \wedge(\mathfrak{g}_1) \xrightarrow{\simeq} \mathbf{U}(\mathfrak{g})$.*

Proof. Choose a totally ordered basis X of $\mathfrak{g}_1$, and define a left $U(\mathfrak{g}_0)$-module (super-coalgebra) map $\phi : U(\mathfrak{g}_0) \otimes \wedge(\mathfrak{g}_1) \to \mathbf{U}(\mathfrak{g})$ by

$$\phi(1 \otimes (x_1 \wedge \cdots \wedge x_n)) = x_1 \dots x_n,$$

where $x_1 < \cdots < x_n$ in X, $n \geq 0$. To prove that this is bijective, it suffices to prove that the localization $\phi_{\mathfrak{m}}$ at each maximal ideal $\mathfrak{m}$ of $\Bbbk$ is bijective. Note that $\mathfrak{g}_{\mathfrak{m}}$ is a $\Bbbk_{\mathfrak{m}}$-Lie superalgebra given a 2-operation by Proposition A.3 and

$$U(\mathfrak{g}_0)_{\mathfrak{m}} = U((\mathfrak{g}_0)_{\mathfrak{m}}), \quad (\wedge(\mathfrak{g}_1))_{\mathfrak{m}} = \wedge((\mathfrak{g}_1)_{\mathfrak{m}}), \quad \mathbf{U}(\mathfrak{g})_{\mathfrak{m}} = \mathbf{U}(\mathfrak{g}_{\mathfrak{m}}).$$

Since $(\mathfrak{g}_0)_{\mathfrak{m}}$ is $\Bbbk_{\mathfrak{m}}$-free under the assumption above, Lemma A.4 shows that $\phi_{\mathfrak{m}}$ is bijective. □

Let $\mathbf{G}$ be an *affine supergroup*; see Section 2.5. Recall from Section 4.1 that

$$\mathrm{Lie}(\mathbf{G}) := (\mathcal{O}(\mathbf{G})^+/(\mathcal{O}(\mathbf{G})^+)^2)^*.$$

Note that the proof of Proposition 4.2 does not use the assumption that $\Bbbk$ is 2-torsion free. From the proposition and the proof one sees the following.

Proposition A.7. *Let* $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$.

(1) $\mathfrak{g}$ *is naturally a Lie superalgebra.*
(2) *Given* $v \in \mathfrak{g}_1$*, the square* v^2 *in* $\mathcal{O}(\mathbf{G})^*$ *is contained in* $\mathfrak{g}_0$*. Moreover, the square map* $(\ )^2 : \mathfrak{g}_1 \to \mathfrak{g}_0$ *gives a* 2*-operation on* $\mathfrak{g}$.

We will suppose that $\mathrm{Lie}(\mathbf{G})$ is given this specific 2-operation.

A.2. Recall from [11, Definitions 3.2.6 and 4.1.2] the following definitions of two categories: $(\text{gss-fsgroups})_{\Bbbk}$, $(\text{sHCP})_{\Bbbk}$.

Let $(\text{gss-fsgroups})_{\Bbbk}$ denote the category of the affine supergroups $\mathbf{G}$ such that when we set $\mathbf{A} := \mathcal{O}(\mathbf{G})$,

(E1) $\mathbf{A}$ is *split* (Definition 2.1),
(E2) $\overline{\mathbf{A}}/(\overline{\mathbf{A}}^+)^2$ is $\Bbbk$-finite projective, and
(E3) $W^{\mathbf{A}} = \mathbf{A}_1/\mathbf{A}_0^+\mathbf{A}_1$ is $\Bbbk$-finite (free).

The morphisms in $(\text{gss-fsgroups})_{\Bbbk}$ are the natural transformations of group-valued functors.

Let $(G, \mathfrak{g})$ be a pair of an affine group G and a Lie superalgebra $\mathfrak{g}$ given a 2-operation, such that $\mathfrak{g}_1$ is $\Bbbk$-finite free and is given a right G-module structure. Suppose that this pair satisfies:

(F1) $\mathfrak{g}_0 = \mathrm{Lie}(G)$,
(F2) $\mathcal{O}(G)/(\mathcal{O}(G)^+)^2$ is $\Bbbk$-finite projective, so that $\mathfrak{g}_0 = \mathrm{Lie}(G)$ is necessarily $\Bbbk$-finite projective, and it is naturally a right G-module (recall from Section 4.2 that the corresponding left $\mathcal{O}(G)$-comodule structure on $\mathrm{Lie}(G)$ is transposed from the right co-adjoint $\mathcal{O}(G)$-coaction on $\mathcal{O}(G)^+/(\mathcal{O}(G)^+)^2$),
(F3) the right $U(\mathfrak{g}_0)$-module structure on $\mathfrak{g}_1$ induced from the given right G-module structure coincides with the right adjoint $\mathfrak{g}_0$-action on $\mathfrak{g}_1$,
(F4) the restricted super-bracket $[\ ,\ ] : \mathfrak{g}_1 \otimes \mathfrak{g}_1 \to \mathfrak{g}_0$ is G-equivariant, and
(F5) the diagram

$$\begin{tikzcd} \mathfrak{g}_1 \arrow[r, "(\ )^{\langle 2 \rangle}"] \arrow[d] & \mathfrak{g}_0 \arrow[d] \\ \mathcal{O}(G) \otimes \mathfrak{g}_1 \arrow[r, "(\ )^{\langle 2 \rangle}_{\mathcal{O}(G)}"] & \mathcal{O}(G) \otimes \mathfrak{g}_0 \end{tikzcd}$$

commutes, where the vertical arrows are the left $\mathcal{O}(G)$-comodule structures.

One sees that under (F4), condition (F5) is equivalent to

$$(v_R^{\langle 2\rangle})^{\gamma} = (v^{\gamma})_R^{\langle 2\rangle}, \quad v \in \mathfrak{g}_1 \otimes R,\ \gamma \in G(R),$$

where R is an arbitrary commutative algebra.

Let $(\mathrm{sHCP})_{\Bbbk}$ denote the category of all those pairs $(G, \mathfrak{g})$ which satisfy conditions (F1)–(F5) above. A morphism $(G, \mathfrak{g}) \to (G', \mathfrak{g}')$ in $(\mathrm{sHCP})_{\Bbbk}$ is a pair (α, β) of a morphism $\alpha : G \to G'$ of affine groups and a Lie superalgebra map $\beta = \beta_0 \oplus \beta_1 : \mathfrak{g} \to \mathfrak{g}'$, which satisfies conditions (i), (ii) in Definition 4.4, and

(iii) $\beta_0(v^{\langle 2\rangle}) = \beta_1(v)^{\langle 2\rangle}, \quad v \in \mathfrak{g}_1.$

Remark A.8. One sees from Lemma A.2 that if $\Bbbk$ is 2-torsion free, then our HCP and ASG (see Definition 4.4 and Section 4.3), roughly speaking, coincide with $(\mathrm{sHCP})_{\Bbbk}$ and $(\text{gss-fsgroups})_{\Bbbk}$, respectively. To be precise, ours are more restrictive in that for objects $(G, \mathfrak{g}) \in \mathsf{HCP}$, $\mathbf{G} \in \mathsf{ASG}$, the commutative Hopf algebras $\mathcal{O}(G)$ and $\mathcal{O}(\mathbf{G}_{ev})$ are assumed to be affine and $\Bbbk$-flat.

We may remove the affinity assumption so long as (B1) and (C3) are assumed. But the assumption seems natural, since if $\Bbbk$ is a field of characteristic $\neq 2$, it ensures that (B1) and (C3) are satisfied, so that our Theorem 4.22 then coincides with the known category equivalence between all algebraic supergroups and the Harish-Chandra pairs; see Remark 4.27.

Note from (4.17) that under the $\Bbbk$-flatness assumption above, $\mathcal{O}(G) \otimes \mathfrak{g}_1$ is 2-torsion free, and condition (F5) for $v^{\langle 2\rangle} = \frac{1}{2}[v, v]$ is necessarily satisfied. Recall that the condition is not contained in the axioms for objects in HCP.

A.3. Our category equivalences between $(\text{gss-fsgroups})_{\Bbbk}$ and $(\mathrm{sHCP})_{\Bbbk}$ will be presented differently from Gavarini's Φ_g, Ψ_g; see Remark A.11. So, we will use different symbols, $\mathbf{P}'$, $\mathbf{G}'$, to denote them.

Let us construct a functor $\mathbf{P}' : (\text{gss-fsgroups})_{\Bbbk} \to (\mathrm{sHCP})_{\Bbbk}$. Given $\mathbf{G} \in (\text{gss-fsgroups})_{\Bbbk}$, set $G := \mathbf{G}_{ev}$, $\mathfrak{g} := \mathrm{Lie}(\mathbf{G})$. Recall from Proposition A.7 and the following remark that $\mathfrak{g}$ is a Lie superalgebra given the square map as a 2-operation. As in Lemma 4.3 one has $\mathfrak{g}_0 \simeq \mathrm{Lie}(G)$, through which we will identify the two, and we suppose $\mathfrak{g}_0 = \mathrm{Lie}(G)$. Since $\mathfrak{g}$ is $\Bbbk$-finite projective by (E2), (E3), the co-adjoint $\mathcal{O}(G)$-coaction on $\mathcal{O}(\mathbf{G})^+/(\mathcal{O}(\mathbf{G})^+)^2$ (see (4.5)) is transposed to $\mathfrak{g}$, so that $\mathfrak{g}$ is a right G-supermodule. The restricted right G-module structure on $\mathfrak{g}_1$ satisfies (F3), (F4), as was seen in the proof of Lemma 4.6. To conclude $(G, \mathfrak{g}) \in (\mathrm{sHCP})_{\Bbbk}$, it remains to prove the following.

Lemma A.9. (F5) *is satisfied.*

Proof. Let $v \mapsto \sum_i c_i \otimes v_i$ denote the left $\mathcal{O}(G)$-comodule structure $\mathfrak{g}_1 \to \mathcal{O}(G) \otimes \mathfrak{g}_1$ on $\mathfrak{g}_1$. Let $a \mapsto a^{(0)} \otimes a^{(1)}$ denote the right co-adjoint $\mathcal{O}(G)$-coaction $\mathcal{O}(\mathbf{G}) \to \mathcal{O}(\mathbf{G}) \otimes \mathcal{O}(G)$ on $\mathcal{O}(\mathbf{G})$. Since $\mathfrak{g}$ is $\Bbbk$-finite projective, we have the canonical injection $\mathcal{O}(G) \otimes \mathfrak{g} = \mathrm{Hom}(\mathfrak{g}^*, \mathcal{O}(G)) \to \mathrm{Hom}(\mathcal{O}(\mathbf{G}), \mathcal{O}(G))$. Therefore, it suffices to prove

$$\langle v^2, a^{(0)}\rangle\, a^{(1)} = \sum_i c_i^2\, \langle v_i^2, a\rangle + \sum_{i<j} c_i c_j\, \langle [v_i, v_j], a\rangle$$

for $v \in \mathfrak{g}_1$, $a \in \mathcal{O}(\mathbf{G})$, where $\langle\ ,\ \rangle$ denotes the canonical pairing $\mathcal{O}(\mathbf{G})^* \times \mathcal{O}(\mathbf{G}) \to \Bbbk$. This is proved as follows:

$$\begin{aligned} \text{LHS} &= \langle v, (a_{(1)})^{(0)}\rangle \langle v, (a_{(2)})^{(0)}\rangle (a_{(1)})^{(1)}(a_{(2)})^{(1)} \\ &= \sum_{i,j} c_i c_j \langle v_i, a_{(1)}\rangle \langle v_j, a_{(2)}\rangle = \sum_{i,j} c_i c_j \langle v_i v_j, a\rangle = \text{RHS}. \end{aligned}$$

□

Let $\mathbf{P}'(\mathbf{G})$ denote the thus obtained object $(G, \mathfrak{g})$ in $(\text{sHCP})_\Bbbk$. As in Proposition 4.7, we see that $\mathbf{P}' : (\text{gss-fsgroups})_\Bbbk \to (\text{sHCP})_\Bbbk$ gives the desired functor, since the Lie superalgebra map induced from a morphism of affine supergroups obviously preserves the 2-operation.

Let us construct a functor $\mathbf{G}' : (\text{sHCP})_\Bbbk \to (\text{gss-fsgroups})_\Bbbk$. Let $(G, \mathfrak{g}) \in (\text{sHCP})_\Bbbk$. Then the natural right G-module structure on $\mathfrak{g}_0 = \text{Lie}(G)$ and the given right G-module structure on $\mathfrak{g}_1$ amount to a right G-supermodule structure on $\mathfrak{g}$, by which the super-bracket on $\mathfrak{g}$ is G-equivariant, as is seen in Remark 4.5 (2) by using (F3), (F4). (According to the original definition [11, Definition 4.1.2], the proved G-equivariance is assumed as an axiom for objects in $(\text{sHCP})_\Bbbk$. But it can be weakened to (F4), as was just seen.) Using (F5), one sees as in Lemma 4.12 (indeed, more easily) that the right G-supermodule structure on $\mathfrak{g}$ uniquely extends to $\mathbf{U}(\mathfrak{g})$, so that $\mathbf{U}(\mathfrak{g})$ turns into a Hopf-algebra object in $\mathsf{SMod}\text{-}G$. By using an isomorphism $U(\mathfrak{g}_0) \otimes \wedge(\mathfrak{g}_1) \simeq \mathbf{U}(\mathfrak{g})$ such as given by Corollary A.6, we can trace the argument in Section 4.4 to construct a split commutative Hopf superalgebra, $\mathbf{A} = \mathbf{A}(G, \mathfrak{g})$, such that

$$\mathbf{A} \simeq \text{Hom}_{U(\mathfrak{g}_0)}(\mathbf{U}(\mathfrak{g}), \mathcal{O}(G)), \quad \overline{\mathbf{A}} \simeq \mathcal{O}(G), \quad W^{\mathbf{A}} \simeq \mathfrak{g}_1^*.$$

It follows that this $\mathbf{A}$ satisfies (E1)–(E3). We let $\mathbf{G}'(G, \mathfrak{g})$ denote the affine supergroup corresponding to $\mathbf{A}$. Then one sees that $\mathbf{G}'(G, \mathfrak{g}) \in (\text{gss-fsgroups})_\Bbbk$, and $(G, \mathfrak{g}) \mapsto \mathbf{G}'(G, \mathfrak{g})$ gives the desired functor. As for the functoriality, note that condition (iii) given just above Remark A.8 is used to see that a morphism (α, β) in $(\text{sHCP})_\Bbbk$ induces, in particular, a Hopf superalgebra map $\mathbf{U}(\mathfrak{g}) \to \mathbf{U}(\mathfrak{g}')$; see the proof of Proposition 4.18.

Theorem A.10 ([11, Theorem 4.3.14]). *We have a category equivalence*

$$(\text{gss-fsgroups})_\Bbbk \approx (\text{sHCP})_\Bbbk.$$

In fact the functors $\mathbf{P}'$ and $\mathbf{G}'$ constructed above are quasi-inverse to each other.

Proof. To prove $\mathbf{P}' \circ \mathbf{G}' \simeq \text{id}$, $\mathbf{G}' \circ \mathbf{P}' \simeq \text{id}$, we can trace the argument of Section 4.5 proving $\mathbf{P} \circ \mathbf{G} \simeq \text{id}$, $\mathbf{G} \circ \mathbf{P} \simeq \text{id}$, except in two points.

First, to prove $\mathbf{P}' \circ \mathbf{G}' \simeq \text{id}$, we have to show that if $(G, \mathfrak{g}) \in (\text{sHCP})_\Bbbk$, and setting $\mathbf{G} := \mathbf{G}'(G, \mathfrak{g})$, then the natural Lie superalgebra isomorphism $\text{Lie}(\mathbf{G}) \simeq \mathfrak{g}$ as given in the proof of Proposition 4.19 preserves the 2-operation. Note that we have a Hopf pairing $\mathbf{U}(\mathfrak{g}) \times \mathcal{O}(\mathbf{G}) \to \Bbbk$ as given in (4.22), and it restricts to a non-degenerate pairing $\mathfrak{g} \times \mathcal{O}(\mathbf{G})^+/(\mathcal{O}(\mathbf{G})^+)^2 \to \Bbbk$, which induces the isomorphism above. Therefore, we have the commutative diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{Lie}(\mathbf{G}) \arrow[d, hook] \\
\mathbf{U}(\mathfrak{g}) \arrow[r] & \mathcal{O}(\mathbf{G})^*,
\end{tikzcd}$$

where the arrow in the bottom is the map induced from the Hopf pairing above. Given $v \in \mathfrak{g}_1$, the composite $\mathfrak{g} \xrightarrow{\simeq} \mathrm{Lie}(\mathbf{G}) \hookrightarrow \mathcal{O}(\mathbf{G})^*$, which factors through $\mathbf{U}(\mathfrak{g})$ as above, sends $v^{\langle 2\rangle}$ to v^2. This proves the desired result.

Second, to prove $\mathbf{G}' \circ \mathbf{P}' \simeq \mathrm{id}$, we should remark that Lemma 4.26 can apply, since the conclusion of the lemma holds so long as $W^{\mathbf{A}}$ is $\Bbbk$-finite, even if the split commutative Hopf superalgebra $\mathbf{A}$ is not finitely generated. □

Remark A.11. In [11], details are not given for the following two.

(1) 2-*operations*. Condition (F5) is not explicitly given in [11]. The functor Φ_g : (gss-fsgroups)$_\Bbbk \to$ (sHCP)$_\Bbbk$ in [11] is almost the same as our $\mathbf{P}'$, but it does not specify the associated 2-operation; see [11, Proposition 4.1.3]. Accordingly, it is not proved that $\Phi_g(G_{\mathcal{P}}) \xrightarrow{\simeq} \mathcal{P}$ preserves the 2-operation on the associated Lie superalgebras; see the first paragraph of the proof of [11, Theorem 4.3.14].

(2) *Proof of* $U(\mathfrak{g}_0) \otimes \wedge(\mathfrak{g}_1) \simeq \mathbf{U}(\mathfrak{g})$. This isomorphism is what was proved by our Corollary A.6. The proof of [11] given in the three lines above (4.7) is rather sketchy, and it might overlook the localization argument used in our proof. Note that the argument uses Proposition A.3; this last result or any equivalent one is not given in [11].

Acknowledgment

The first author thanks Fabio Gavarini, who kindly answered his questions about results in [10, 11], sending errata.

Note added in proof

After the present paper was accepted for publication the authors proved in Theorem 5.7 of the preprint “On functor points of affine supergroups”, arXiv: 1505.06558v2, the following: in the situation of Section A.2 above, conditions (E2) and (E3) imply (E1), provided $\overline{\mathbf{A}}$ is $\Bbbk$-flat. As is remarked by Remark 5.8(2) of the preprint, it follows that to define the category AHSA in Section 4.3 above, we may weaken condition (C1) to (C1′): $W^{\mathbf{A}}$ is $\Bbbk$-free. For, this weakened condition (C1′) together with (C2) and (C3) turns out to ensure the existence of an isomorphism $\mathbf{A} \xrightarrow{\simeq} \overline{\mathbf{A}} \otimes \wedge(W^{\mathbf{A}})$ of left $\overline{\mathbf{A}}$-comodule superalgebras; see Definition 2.1 above.

References

[1] George M. Bergman, *The diamond lemma for ring theory*, Adv. in Math. **29** (1978), no. 2, 178–218, DOI 10.1016/0001-8708(78)90010-5. MR506890 (81b:16001)

[2] Jonathan Brundan and Alexander Kleshchev, *Modular representations of the supergroup $Q(n)$. I*, J. Algebra **260** (2003), no. 1, 64–98, DOI 10.1016/S0021-8693(02)00620-8. Special issue celebrating the 80th birthday of Robert Steinberg. MR1973576 (2004f:20081)

[3] Jonathan Brundan and Jonathan Kujawa, *A new proof of the Mullineux conjecture*, J. Algebraic Combin. **18** (2003), no. 1, 13–39, DOI 10.1023/A:1025113308552. MR2002217 (2004j:20017)

[4] Claudio Carmeli, Lauren Caston, and Rita Fioresi, *Mathematical foundations of supersymmetry*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2011. MR2840967 (2012h:58010)

[5] Claudio Carmeli and Rita Fioresi, *Superdistributions, analytic and algebraic super Harish-Chandra pairs*, Pacific J. Math. **263** (2013), no. 1, 29–51, DOI 10.2140/pjm.2013.263.29. MR3069075

[6] Michel Demazure and Pierre Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs* (French), Masson & Cie, Éditeur, Paris; North-Holland Publishing Co., Amsterdam, 1970. Avec un appendice *Corps de classes local* par Michiel Hazewinkel. MR0302656

[7] R. Fioresi and F. Gavarini, *On the construction of Chevalley supergroups*, Supersymmetry in mathematics and physics, Lecture Notes in Math., vol. 2027, Springer, Heidelberg, 2011, pp. 101–123, DOI 10.1007/978-3-642-21744-9_5. MR2906339

[8] R. Fioresi and F. Gavarini, *Chevalley supergroups*, Mem. Amer. Math. Soc. **215** (2012), no. 1014, vi+64, DOI 10.1090/S0065-9266-2011-00633-7. MR2918543

[9] F. Gavarini, *Chevalley supergroups of type* $D(2,1;a)$, Proc. Edinb. Math. Soc. (2) **57** (2014), no. 2, 465–491, DOI 10.1017/S0013091513000503. MR3200319

[10] Fabio Gavarini, *Algebraic supergroups of Cartan type*, Forum Math. **26** (2014), no. 5, 1473–1564, DOI 10.1515/forum-2011-0144. MR3334037

[11] Fabio Gavarini, *Global splittings and super Harish-Chandra pairs for affine supergroups*, Trans. Amer. Math. Soc. **368** (2016), no. 6, 3973–4026, DOI 10.1090/tran/6456. MR3453363

[12] A. N. Grishkov and A. N. Zubkov, *Solvable, reductive and quasireductive supergroups*, J. Algebra **452** (2016), 448–473, DOI 10.1016/j.jalgebra.2015.11.013. MR3461076

[13] P. J. Higgins, *Baer invariants and the Birkhoff-Witt theorem*, J. Algebra **11** (1969), 469–482. MR0238913 (39 #273)

[14] Jens Carsten Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR2015057 (2004h:20061)

[15] V. G. Kac, *Lie superalgebras*, Advances in Math. **26** (1977), no. 1, 8–96. MR0486011 (58 #5803)

[16] Bertram Kostant, *Groups over Z*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 90–98. MR0207713 (34 #7528)

[17] Bertram Kostant, *Graded manifolds, graded Lie theory, and prequantization*, Differential geometrical methods in mathematical physics (Proc. Sympos., Univ. Bonn, Bonn, 1975), Lecture Notes in Math., vol. 570, Springer, Berlin, 1977, pp. 177–306. MR0580292 (58 #28326)

[18] J.-L. Koszul, *Graded manifolds and graded Lie algebras*, Proceedings of the international meeting on geometry and physics (Florence, 1982), Pitagora, Bologna, 1983, pp. 71–84. MR760837 (85m:58019)

[19] Akira Masuoka, *The fundamental correspondences in super affine groups and super formal groups*, J. Pure Appl. Algebra **202** (2005), no. 1-3, 284–312, DOI 10.1016/j.jpaa.2005.02.010. MR2163412 (2006e:16066)

[20] Akira Masuoka, *Harish-Chandra pairs for algebraic affine supergroup schemes over an arbitrary field*, Transform. Groups **17** (2012), no. 4, 1085–1121, DOI 10.1007/s00031-012-9203-8. MR3000482

[21] A. Masuoka, *Hopf algebraic techniques applied to super algebraic groups*, Proceedings of Algebra Symposium (Hiroshima, 2013), pp. 48–66, Math. Soc. Japan, 2013; available at arXiv:1311.1261.

[22] Akira Masuoka and Alexandr N. Zubkov, *Quotient sheaves of algebraic supergroups are superschemes*, J. Algebra **348** (2011), 135–170, DOI 10.1016/j.jalgebra.2011.08.038. MR2852235

[23] Susan Montgomery, *Hopf algebras and their actions on rings*, CBMS Regional Conference Series in Mathematics, vol. 82, published for the Conference Board of the Mathematical Sciences, Washington, DC, by the American Mathematical Society, Providence, RI, 1993. MR1243637 (94i:16019)

[24] Bin Shu and Weiqiang Wang, *Modular representations of the ortho-symplectic supergroups*, Proc. Lond. Math. Soc. (3) **96** (2008), no. 1, 251–271, DOI 10.1112/plms/pdm040. MR2392322 (2009d:20108)

[25] Moss E. Sweedler, *Hopf algebras*, Mathematics Lecture Note Series, W. A. Benjamin, Inc., New York, 1969. MR0252485 (40 #5705)

[26] Mitsuhiro Takeuchi, *On coverings and hyperalgebras of affine algebraic groups*, Trans. Amer. Math. Soc. **211** (1975), 249–275. MR0429928 (55 #2937a)

[27] M. Takeuchi, *Hyperalgebraic construction of Chevalley group schemes* (in Japanese), RIMS Kokyuroku **473** (1982), 57–70.

[28] E. G. Vishnyakova, *On complex Lie supergroups and split homogeneous supermanifolds*, Transform. Groups **16** (2011), no. 1, 265–285, DOI 10.1007/s00031-010-9114-5. MR2785503 (2012b:58010)

Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan
E-mail address: akira@math.tsukuba.ac.jp

Graduate School of Pure and Applied Sciences, University of Tsukuba, Ibaraki 305-8571, Japan
Current address: Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada
E-mail address: shibata@ualberta.ca