TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 5, May 2017, Pages 3639–3654
http://dx.doi.org/10.1090/tran/6839
Article electronically published on October 13, 2016

COSMETIC SURGERY IN L-SPACES AND NUGATORY CROSSINGS

TYE LIDMAN AND ALLISON H. MOORE

Abstract. The cosmetic crossing conjecture (also known as the "nugatory crossing conjecture") asserts that the only crossing changes that preserve the oriented isotopy class of a knot in the 3-sphere are nugatory. We use the Dehn surgery characterization of the unknot to prove this conjecture for knots in integer homology spheres whose branched double covers are L-spaces satisfying a homological condition. This includes as a special case all alternating and quasi-alternating knots with square-free determinant. As an application, we prove the cosmetic crossing conjecture holds for all knots with at most nine crossings and provide new examples of knots, including pretzel knots, non-arborescent knots and symmetric unions for which the conjecture holds.

1. Introduction

Let K be an oriented knot in S^3, and let c refer to an oriented crossing in a diagram of the knot. A fundamental question is whether a crossing change at c preserves the isotopy type of the knot. Let a crossing disk D be an embedded disk in S^3 intersecting K twice with zero algebraic intersection number. If ∂D also bounds an embedded disk in the complement of K, then the crossing is called *nugatory*, and changing this crossing does not change the isotopy type of K. A non-nugatory crossing change which preserves the oriented isotopy type of the knot is called *cosmetic*, and it is conjectured that no such crossing exists for a knot in S^3.

Conjecture 1.1 (Cosmetic crossing conjecture)**.** *If K admits a crossing change at a crossing c which preserves the oriented isotopy class of the knot, then c is nugatory.*

Remarkably, this basic question is unanswered for most classes of knots. The conjecture is attributed to X. S. Lin (see [Kir78, Problem 1.58]), and in the literature it sometimes appears as the "nugatory crossing conjecture". Scharlemann and Thompson proved that the unknot admits no cosmetic crossing changes [ST89, Theorem 1.4], and Torisu and Kalfagianni established the same for two-bridge knots and fibered knots, respectively [Tor99, Kal12]. There exist several additional obstructions amongst genus one knots and satellites [BFKP12, BK14].

Received by the editors July 19, 2015 and, in revised form, September 26, 2015.
2010 *Mathematics Subject Classification.* Primary 57M25, 57M27.
The first author was partially supported by NSF RTG grant DMS-1148490.
The second author was partially supported by NSF grant DMS-1148609.

In this paper, we prove Conjecture 1.1 for knots whose branched double covers are L-spaces that satisfy a certain homological condition. Recall that an L-space Y is a rational homology sphere with rank $\widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$, where $\widehat{HF}$ denotes the hat flavor of Heegaard Floer homology.

Theorem 1.2. *Let K be a knot in S^3 whose branched double cover $\Sigma(K)$ is an L-space. If each summand of the first homology of $\Sigma(K)$ has square-free order, then K satisfies the cosmetic crossing conjecture.*

Theorem 1.2 will be deduced from the Dehn surgery characterization of the unknot in L-spaces [Gai15, KMOS07] (see Theorem 3.1). It is interesting to juxtapose Theorem 1.2 with [BFKP12, Theorem 1.1], which implies that if a genus one knot K admits a cosmetic crossing change, then $H_1(\Sigma(K))$ is cyclic of order d^2, for some $d \in \mathbb{Z}$.

An abundant source of knots that meet the conditions of Theorem 1.2 is the *Khovanov thin* knots. These knots derive their definition from reduced Khovanov homology, which associates to an oriented link L in S^3 a bigraded vector space $\overline{Kh}^{i,j}(L)$ over $\mathbb{Z}/2\mathbb{Z}$ [Kho00]. The $\overline{Kh}$–thin links are those with their homology supported in a single diagonal $\delta = j-i$ of the bigradings. In this case, the dimension of $\overline{Kh}$ is given by the determinant of the link. By work of Manolescu and Ozsváth [MO08], all quasi-alternating links are $\overline{Kh}$–thin, and this class includes all non-split alternating links [OS05]. Of relevance here is the fact that the branched double cover of a $\overline{Kh}$–thin link is an L-space, which follows from the spectral sequence from $\overline{Kh}(L)$ to $\widehat{HF}(-\Sigma(L))$ [OS05] and the symmetry of Heegaard Floer homology under orientation reversal [OS04].

Because the determinant of a knot is equal to the order of the first homology of its branched double cover we immediately obtain the following corollary.

Corollary 1.3. *A $\overline{Kh}$–thin knot with square-free determinant satisfies the cosmetic crossing conjecture.*

We apply Theorem 1.2 (and in particular, Corollary 1.3) together with previously known obstructions for two-bridge, fibered and genus one knots to affirm the cosmetic crossing conjecture for all but ten knots in the Rolfsen tables [Rol90] of knots of ten or fewer crossings.

Theorem 1.4. *Let K be a knot of at most ten crossings not contained in the list*

$$10_{65}, 10_{66}, 10_{67}, 10_{77}, 10_{87}, 10_{98}, 10_{108}, 10_{129}, 10_{147}, 10_{164}. \tag{1.1}$$

Then K admits no cosmetic crossing changes. In particular, all knots of at most nine crossings satisfy the cosmetic crossing conjecture.

In fact, what we prove in Theorem 1.2 holds in a more general setting. Indeed, for any knot in an integer homology sphere whose branched double cover is an L-space, if each summand of the first homology of the branched double cover is square-free, then that knot does not admit cosmetic crossing changes. This statement requires an appropriate generalization of the definition of a cosmetic crossing change to an arbitrary homology sphere. (See Theorem 3.5 and the preceding remarks.)

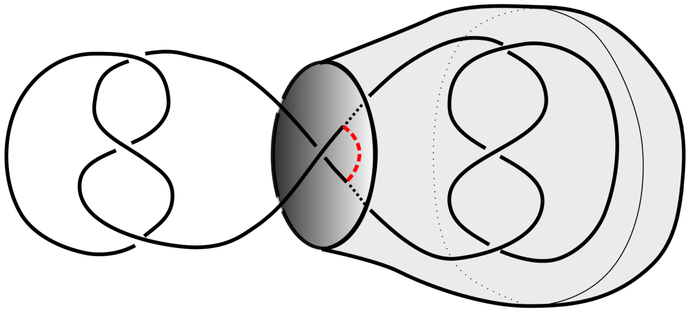

FIGURE 1. The crossing disk of a nugatory crossing.

Another interesting class of knots are those which admit a positive Dehn surgery to an L-space; such a knot is known as an *L-space knot*. Since these knots are known to be fibered by work of Ni [Ni07], it follows from Kalfagianni [Kal12] that L-space knots satisfy the cosmetic crossing conjecture. To further illustrate the techniques of the proof of the main theorem, we provide an alternate proof of this fact.

Theorem 1.5. *Let K be an L-space knot. Then K satisfies the cosmetic crossing conjecture.*

In Section 2, we establish necessary background information and prove a key homological result. In Section 3, we prove the main result Theorem 1.2 and its generalization, as well as Theorem 1.5. In Section 4, we prove Theorem 1.4 and we provide new examples of pretzel knots, non-arborescent knots and symmetric unions for which the cosmetic crossing conjecture holds.

2. Homological obstructions

In order to prove Theorem 1.2, we study the effects of a cosmetic crossing change in terms of the homology of the branched double cover. The current section is devoted to understanding this, culminating in Theorem 2.4.

2.1. **Cosmetic and nugatory crossings.** Let K be an oriented knot in S^3, and let c denote a crossing. We will abuse notation by allowing K to denote both the oriented knot and its diagram. We will write K^+ and K^- for diagrams identical to K, except possibly at the crossing c, which is positive at K^+ and negative at K^-. Without loss of generality, we assume the crossing c is positive. A *crossing disk* for K is an embedded disk D that intersects K transversely in its interior twice with zero algebraic intersection number, as in Figure 1. The boundary of the crossing disk ∂D is an unknot called the *crossing circle*. If ∂D bounds an embedded disk in the complement of K, then c is called *nugatory*. A *crossing arc* γ for the knot K is an unknotted arc with its boundary on K that may be isotoped to lie embedded in the crossing disk, again as in Figure 1. We write $\Sigma(K)$ for the double cover of S^3 branched over the knot K. If K admits a crossing change at c which preserves the isotopy type of K, then the branched double covers $\Sigma(K^+)$ and $\Sigma(K^-)$ are orientation preserving homeomorphic, and the arc γ lifts to a knot $\tilde{\gamma}$ in $\Sigma(K^+)$. The complement of a neighborhood of $\tilde{\gamma}$ in $\Sigma(K)$ is a compact, connected, oriented 3-manifold with torus boundary, which we denote by M.

There is a well-known correspondence between Dehn fillings of M and rational tangle replacements of K in S^3, commonly called "the Montesinos trick". (See, for example, [Gor09] for a standard reference.) We only state a special case here. Take a small 3-ball B containing the crossing c in K^+, so that the sphere ∂B intersects K transversely in four points. Then the double cover of $S^3 - \operatorname{int}(B)$ branched over $T = K - \operatorname{int}(B \cap K)$ is the manifold $M = \Sigma(K^+) - N(\tilde{\gamma})$. Note that $\Sigma(K^-)$ is also obtained by a Dehn filling of M. Finally, recall that the distance between any two slopes η and ξ on ∂M refers to their minimal geometric intersection number and is denoted $\Delta(\eta, \xi)$.

Lemma 2.1 (Montesinos trick). *Let α and β be the two slopes on ∂M such that $M(\alpha) = \Sigma(K^+)$ and $M(\beta) = \Sigma(K^-)$. Then $\Delta(\alpha, \beta) = 2$.*

2.2. **Rational longitude.** Now let M refer to any compact, connected, oriented 3-manifold with torus boundary and $H_1(M; \mathbb{Q}) \cong \mathbb{Q}$. For homology with $\mathbb{Z}$-coefficients, we will simply omit the coefficient group. In particular, the complement of the knot $\tilde{\gamma}$ in the rational homology sphere $\Sigma(K)$ is such a manifold. In this section we recall some facts about the rational longitude of M which we will use to study the homology of Dehn fillings of M. For more detail on the rational longitude, we refer the reader to Section 3.1 of Watson [Wat12]. We adopt the same notation as in [Wat12], and the content of our own Section 2.2 is paraphrased from this, included because it is necessary for the arguments which follow in Section 2.3.

Let us consider the long exact sequence of the pair $(M, \partial M)$,

$$\cdots \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \xrightarrow{i_*} H_1(M) \to H_1(M, \partial M) \to \cdots .$$

Via exactness and Poincaré-Lefschetz duality, it follows that the rank of i_* is one. Therefore with $\mathbb{Z}$–coefficients, i_* maps one of the $\mathbb{Z}$ summands of $H_1(\partial M) \cong \mathbb{Z} \oplus \mathbb{Z}$ injectively to $H_1(M) \cong \mathbb{Z} \oplus H$ (where H is some finite abelian group). Additionally, $\ker(i_*)$ is generated by $k\lambda_M$ for some primitive homology class $\lambda_M \in H_1(\partial M)$ and some non-negative integer k. The homology class λ_M is uniquely defined up to sign, which determines a well-defined slope in ∂M, giving rise to the following definition.

Definition 2.2. The *rational longitude* λ_M is the unique slope in ∂M such that $i_*(\lambda_M)$ is of finite order in $H_1(M)$.

We begin with the observation of [Wat12] that the order of the first homology of the manifold $M(\eta)$ obtained by Dehn filling M along some slope η is determined by $\Delta(\eta, \lambda_M)$.

Lemma 2.3. *There is a constant $c_M > 0$ (depending only on M) such that for $\eta \neq \lambda_M$,*

$$|H_1(M(\eta))| = c_M \Delta(\eta, \lambda_M).$$

We recall the definition of the constant c_M more explicitly. Fix a basis (μ, λ_M) for $H_1(\partial M)$, where $\Delta(\mu, \lambda_M) = 1$. Then under the homomorphism $i_* : H_1(\partial M) \to H_1(M) \cong \mathbb{Z} \oplus H$, we have $i_*(\mu) = (\ell, u)$ and $i_*(\lambda_M) = (0, h)$ for some $u, h \in H$ and $\ell \in \mathbb{Z}$. It turns out that $|\ell| = \operatorname{ord}_H i_*(\lambda_M)$. The constant c_M is then described as

$$c_M = \ell r_1 \cdots r_k = \operatorname{ord}_H i_*(\lambda_M)|H|, \tag{2.1}$$

where $r_1, \ldots, r_k$ are the invariant factors of the finite abelian group H. In fact, Watson gives an explicit presentation for $H_1(M(\eta))$. Writing the slope $\eta = a\mu + b\lambda_M$

on ∂M, we see that $i_*(\eta) = (a\ell, au + bh)$. Let $I_{\vec{r}}$ denote the $k \times k$ diagonal matrix with ith diagonal entry given by r_i,

$$I_{\vec{r}} = \begin{pmatrix} r_1 & & \\ & \ddots & \\ & & r_k \end{pmatrix}, \tag{2.2}$$

which is a presentation matrix for H. From this, we may identify $u \in H$ (non-uniquely) with a vector $\vec{u} = (u_1, \dots, u_k)$ and similarly for h. A presentation matrix A for $H_1(M(\eta))$ is then given in block form by

$$\begin{pmatrix} a\ell & 0 \\ a\vec{u} + b\vec{h} & I_{\vec{r}} \end{pmatrix}. \tag{2.3}$$

In particular, $|H_1(M(\eta))| = a\ell r_1 \cdots r_k$, and c_M is taken to be $\ell r_1 \cdots r_k$. Finally, note that because $\Delta(\mu, \lambda_M) = 1$, we have $a = \Delta(\eta, \lambda_M)$, and this gives the statement of Lemma 2.3.

In the following subsection, we will use (2.3) to study the homology class of the lift of a crossing arc in the presence of a cosmetic crossing change.

2.3. **The lift of the crossing arc.** We are now prepared to describe conditions which guarantee $\tilde{\gamma}$ represents a null-homologous class in $H_1(\Sigma(K))$. This will allow us to apply the Dehn surgery characterization of the unknot in the case that $\Sigma(K)$ is an L-space (see Theorem 3.1). Let us continue with the assumptions that K admits a crossing change which preserves the isotopy type of K and that α and β are the two filling slopes for the orientation preserving homeomorphic manifolds $\Sigma(K^+)$ and $\Sigma(K^-)$, respectively.

Theorem 2.4. *Suppose that $\Sigma(K)$ is the branched double cover of a knot K admitting a crossing change preserving the isotopy type of K and that*

$$H_1(\Sigma(K)) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z} \tag{2.4}$$

where each d_i is square-free. Then the lift $\tilde{\gamma}$ of the crossing arc at c is trivial in $H_1(\Sigma(K))$.

Proof. Let $M = \Sigma(K) - N(\tilde{\gamma})$. We consider the slopes α and β on M for which Dehn filling gives rise to the homeomorphic pair $M(\alpha)$ and $M(\beta)$, which are $\Sigma(K^+)$ and $\Sigma(K^-)$ respectively. Fix a basis (μ, λ_M) for $H_1(M)$, where λ_M is the rational longitude and μ is any slope with $\Delta(\mu, \lambda_M) = 1$. Write the two filling slopes α and β in terms of this basis as

$$\alpha = p\mu + q\lambda_M,$$
$$\beta = s\mu + r\lambda_M,$$

with $p, s \geq 0$. By Lemma 2.3,

$$c_M \Delta(\alpha, \lambda_M) = |H_1(M(\alpha))| = |H_1(M(\beta))| = c_M \Delta(\beta, \lambda_M),$$

which implies that $p = \Delta(\alpha, \lambda_M) = \Delta(\beta, \lambda_M) = s$. Thus, by the Montesinos trick,

$$2 = \Delta(\alpha, \beta) = |p(q - r)|,$$

and so $p = 1$ or $p = 2$. However, if $p = 2$, then $|H_1(\Sigma(K))|$ is even, and this contradicts that knots have odd determinants. Therefore $p = 1$, and we have by (2.1) that

$$|H_1(\Sigma(K))| = c_M = |H|(\operatorname{ord}_H i_*(\lambda_M)).$$

Now the distance $\Delta(\alpha, \lambda_M)$ is one, so after changing the basis (μ, λ_M) to $(\mu + q\lambda_M, \lambda_M)$, we may assume $\alpha = \mu$. Recall that we write $i_*(\mu) = (\ell, u) \in \mathbb{Z} \oplus H$. From (2.3), a presentation matrix A for $H_1(M(\alpha))$ is given by

$$A = \begin{pmatrix} \ell & & & \\ u_1 & r_1 & & \\ \vdots & & \ddots & \\ u_k & & & r_k \end{pmatrix}, \tag{2.5}$$

where $\ell = \pm \operatorname{ord}_H i_*(\lambda_M)$. After possibly multiplying the first column by -1, we may assume $\ell \geq 1$.

Now if $\ell = \operatorname{ord}_H i_*(\lambda_M) = 1$, then λ_M is integrally null-homologous, and this implies that $\tilde{\gamma}$ is in fact null-homologous in $\Sigma(K)$. To see this, consider the filling torus $N(\tilde{\gamma})$, and note that the rational longitude λ_M is homologous to the core $\tilde{\gamma}$ in $N(\tilde{\gamma})$, considered as a submanifold of $M(\alpha)$, since $\tilde{\gamma}$ has intersection number one with the meridional disk bounded by α in $N(\tilde{\gamma})$. Since λ_M is integrally null-homologous in the exterior M, then λ_M also bounds in the filled manifold $M(\alpha)$. Hence $\tilde{\gamma}$ is null-homologous in $M(\alpha)$. Thus, it is our goal to show that $i_*(\lambda_M)$ is trivial.

As an aside, we note that in the special case each d_i in (2.4) is a distinct prime (e.g. when $\det(K)$ is square-free) then it is immediate that $\ell = 1$ because $i_*(\lambda_M)$ generates a subgroup of H. In general, we will argue that $\ell = 1$ by using the Smith normal form of A.

Let Γ_i denote the greatest common divisor of the determinants of the $i \times i$ minors of A. Recall that the Smith normal form for A, since it is invertible over $\mathbb{Q}$, is given by the diagonal matrix $I_{\vec{\delta}}$ where $\delta_i = \Gamma_i / \Gamma_{i-1}$. We have that $I_{\vec{\delta}}$ presents the same group as does A. Finally, recall that $|\det(A)| = \delta_1 \cdots \delta_{k+1}$ is the order of the group being presented by A.

First, we claim that each u_i in (2.5) is a multiple of $\gcd(\ell, r_i)$, for if u_i is not a multiple of $\gcd(\ell, r_i)$, then there exists some prime p such that $p^j \mid \ell$ and $p^j \mid r_i$ but $p^j \nmid u_i$ for some j. Now let $A_{1,i+1}$ be the $k \times k$ minor obtained by deleting the first row and $(i+1)$-th column of A; we have $|\det(A_{1,i+1})| = |r_1 \cdots r_{i-1} u_i r_{i+1} \cdots r_k|$. If t is the largest power of p which divides $\det(A)$, then the largest power of p which divides $\det(A_{1,i+1})$ is at most $t-2$. Since $\delta_1 \cdots \delta_k = \Gamma_k$ divides $\det(A_{1,i+1})$, the largest power of p which divides Γ_k is at most $t-2$. Because $\det(A) = \delta_1 \cdots \delta_k \delta_{k+1}$ we see that p^2 divides δ_{k+1}. But this implies that some invariant factor in the decomposition of $H_1(\Sigma(K))$ is not square-free, a contradiction. Thus, we may now write

$$A = \begin{pmatrix} \ell & & & \\ a_1 \gcd(\ell, r_1) & r_1 & & \\ \vdots & & \ddots & \\ a_k \gcd(\ell, r_k) & & & r_k \end{pmatrix},$$

for some integers $a_1, \ldots, a_k$.

Let us now consider the cosmetic filling. By Lemma 2.1, the curve β may be written $\mu \pm 2\lambda_M$. A presentation matrix B for $M(\beta)$ is

$$B = \begin{pmatrix} & \ell & & & \\ & u_1 \pm 2h_1 & r_1 & & \\ & \vdots & & \ddots & \\ & u_k \pm 2h_k & & & r_k \end{pmatrix},$$

where $\vec{h} = (h_1, \dots, h_k)$ is identified with the image of $i_*(\lambda_M)$ in H under (2.2). The same argument as that for A now applies to B to show that $\gcd(\ell, r_i) \mid u_i \pm 2h_i$. Because $\gcd(\ell, r_i)$ divides u_i and because $\gcd(\ell, r_i)$ is odd, we know $\gcd(\ell, r_i) \mid h_i$ and we write $h_i = b_i \gcd(\ell, r_i)$ for some integers $b_1, \dots, b_k$. Now, recall that $|\ell| = \operatorname{ord}_H i_*(\lambda_M)$. This means that $\ell\vec{h}$ is in the column span of $A_{1,1}$, so for each i,

$$\ell h_i = \ell b_i \gcd(\ell, r_i) = c_i r_i$$

for some integers $c_1, \dots, c_k$. But then

$$\begin{aligned} c_i \frac{r_i}{\gcd(\ell, r_i)} = \ell b_i \quad &\Rightarrow \quad \frac{r_i}{\gcd(\ell, r_i)} \mid b_i \\ &\Rightarrow \quad b_i = g_i \frac{r_i}{\gcd(\ell, r_i)}, \text{ for some integer } g_i \\ &\Rightarrow \quad h_i = g_i \frac{r_i}{\gcd(\ell, r_i)} \gcd(\ell, r_i). \end{aligned}$$

Because each h_i is a multiple of r_i, in fact $\vec{h}$ is in the column space of $A_{1,1}$. However, $A_{1,1}$ is a presentation matrix for H, the torsion subgroup of $H_1(M)$, by (2.2). This says that h represents the trivial element in H. Hence $i_*(\lambda_M)$ is trivial in $H_1(M)$, and we conclude that $\tilde{\gamma}$ is null-homologous in $M(\alpha)$. ☐

3. Proofs of main theorems

3.1. **Proof of Theorem 1.2.** Both Theorem 1.2 and Theorem 1.5 depend on the Dehn surgery characterization of the unknot generalized to the case of null-homologous knots in L-spaces. The characterization of the unknot U in S^3 is due to Kronheimer, Mrówka, Ozsváth and Szabó [KMOS07, Theorem 1.1], and its generalization in L-spaces is due to Gainullin [Gai15, Theorem 8.2].

Theorem 3.1. *Let K be a null-homologous knot in an L-space Y, and let $Y_{p/q}(K)$ denote the result of p/q-Dehn surgery along K. If $Y_{p/q}(K)$ is orientation preserving homeomorphic to $Y_{p/q}(U)$, then K is isotopic to U.*

With this, we are now prepared to prove the main theorem.

Theorem 3.2. *Let K be a knot in S^3 with branched double cover $\Sigma(K)$ an L-space. If*

$$H_1(\Sigma(K)) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z}$$

with each d_i square-free, then the cosmetic crossing conjecture holds for K.

Proof. Suppose that the branched double cover $\Sigma(K)$ is an L-space and K admits a crossing change at a crossing c which preserves the isotopy type of K. Without loss of generality, assume c is positive and that $K^+ = K$. We will show c is nugatory.

Let the knot $\tilde{\gamma}$ in $\Sigma(K^+)$ be the lift of the crossing arc γ at c. Theorem 2.4 implies that in fact $\tilde{\gamma}$ must be null-homologous in $\Sigma(K^+)$. Since $\tilde{\gamma}$ is null-homologous and $|H_1(\Sigma(K^-))|$ is obviously equal to $|H_1(\Sigma(K^+))|$, we have that $\Sigma(K^-)$ is obtained by $\pm 1/n$-surgery on $\tilde{\gamma}$. By assumption, K^+ is isotopic to K^-; thus $\Sigma(K^+)$ is orientation preserving homeomorphic to $\Sigma(K^-)$. By doing $\pm 1/n$–framed Dehn surgery along an unknot in $\Sigma(K^+)$, we do not change the oriented homeomorphism type of $\Sigma(K^+)$ for any n. Thus applying Theorem 3.1 we deduce that $\tilde{\gamma}$ is isotopic to the unknot in $\Sigma(K^+)$.

It remains to show that this implies the crossing c is nugatory. This will follow as a special case of the equivariant Dehn's lemma of Meeks and Yau [YM84], which was instrumental in the proof of the Smith conjecture. The case for an involution is due to Kim and Tollefson [KT80] and Gordon and Litherland [GL84]. The following is well known to experts (see for instance [Tor99]). However, we include the proof for completeness.

Proposition 3.3. *If $\tilde{\gamma}$ is a null-homologous unknot in $\Sigma(K^+)$ and the arc $\gamma \in S^3$ is the image of $\tilde{\gamma}$ under the covering involution of $\Sigma(K^+)$, then the crossing associated with the arc γ must be nugatory.*

Proof. Let c be the crossing associated with γ and let M denote the exterior of $\tilde{\gamma}$. By assumption $\tilde{\gamma}$ is an unknot; therefore $M = D^2 \times S^1 \# \Sigma(K^+)$. We also will think of M as $\Sigma(B, T)$, where (B, T) is the tangle obtained by removing an open 3-ball neighborhood of the crossing arc at c. Finally, let τ denote the covering involution on $\Sigma(K^+)$.

Let $\widetilde{\Gamma} = D^2 \times \{pt\}$ in $M = D^2 \times S^1 \# \Sigma(K^+)$. Of course, $\widetilde{\Gamma}$ is a compressing disk and $\partial\widetilde{\Gamma}$ is essential in ∂M. Since $\partial\widetilde{\Gamma}$ is the unique slope on ∂M which bounds in M, we see that $\partial\widetilde{\Gamma}$ is the rational longitude λ_M. By the equivariant Dehn's lemma, we may assume that either $\tau(\widetilde{\Gamma}) \cap \widetilde{\Gamma}$ is empty or $\tau(\widetilde{\Gamma}) = \widetilde{\Gamma}$.

First, suppose that $\tau(\widetilde{\Gamma}) \cap \widetilde{\Gamma}$ is empty. This implies that $\widetilde{\Gamma}$ descends to a disk Γ in the exterior of $T \subset B$, and thus the exterior of K^+, since the lift of K^+ to $\Sigma(K^+)$ is precisely the fixed point set of τ. It remains to see that $\partial\Gamma$ is a crossing circle for the crossing c, since this would imply that c is nugatory. We recall our previous notation $M(\alpha) = \Sigma(K^+)$ and $M(\beta) = \Sigma(K^-)$. By the same arguments as in the proof of Theorem 2.4, we have that $\Delta(\alpha, \beta) = 2$ and $\Delta(\alpha, \lambda_M) = \Delta(\beta, \lambda_M) = 1$. Taking again (α, λ_M) as a basis for $H_1(\partial M)$ we further have that $\beta = \alpha \pm 2\lambda_M$ in this basis. It is straightforward to verify that for any slope η on ∂M, if $\Delta(\alpha, \eta) = \Delta(\beta, \eta) = 1$, then either $\eta = \lambda_M$ or $\eta = \pm\alpha + \lambda_M$ (where the sign is determined by $\beta = \alpha \pm 2\lambda_M$). In particular, there are precisely two slopes on ∂M which have distance one from each of α and β.

Consider the two curves w_1 and w_2 on the boundary of the neighborhood of the crossing c shown in Figure 2. These curves lift to distinct slopes on ∂M and have distance one from each of α and β. It follows that the lift of either w_1 or w_2 must be $\lambda_M = \partial\widetilde{\Gamma}$. In other words, we must have that $\widetilde{\Gamma}$ descends to a disk in the exterior of K^+ with either w_1 or w_2 as its boundary. As the crossing circle (i.e. w_2 in Figure 2) is the curve which is null-homologous in the complement of K^+, this implies that the crossing circle bounds a disk in the exterior of K^+ and thus c is nugatory, as desired.

Now, we consider the case that $\tau(\widetilde{\Gamma}) = \widetilde{\Gamma}$. Noting that the fixed point set of τ restricted to $\widetilde{\Gamma}$ is at most one-dimensional, Smith theory implies that $\widetilde{\Gamma}$ intersects

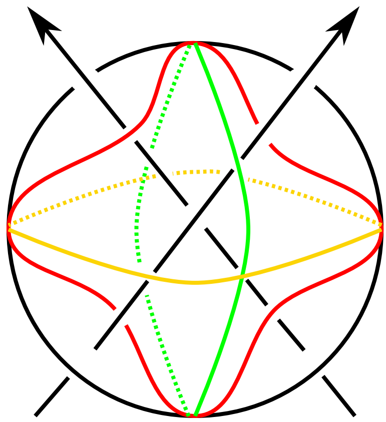

FIGURE 2. The two curves w_1 and w_2 in yellow and green, respectively, along with the quotient of α under the action of τ in red.

$\mathrm{Fix}(\tau)$ in either an arc or a single point. In the case this intersection is an arc, standard arguments allow us to perturb $\widetilde{\Gamma}$ such that $\tau(\widetilde{\Gamma}) \cap \widetilde{\Gamma}$ is empty, which was handled in the previous case. Thus, it remains to consider the case that $\widetilde{\Gamma}$ intersects the fixed point set of τ in a single point. In this case, it follows that $\widetilde{\Gamma}$ descends to a disk Γ in B which intersects $T = \mathrm{Fix}(\tau|_M)$ in a single point. Note that $\partial\Gamma$ also bounds a disk Γ' contained in ∂B which intersects ∂T in one point. But now Γ' lifts to a disk $\widetilde{\Gamma}'$ in $\Sigma(B,T) = M$ which is contained in ∂M and has $\partial\widetilde{\Gamma}' = \partial\widetilde{\Gamma}$. This contradicts that $\widetilde{\Gamma}$ is a compressing disk, since $\partial\widetilde{\Gamma}$ bounds a disk in ∂M. □

The above proposition completes the proof of Theorem 1.2. □

Remark 3.4. While we are not able to prove the cosmetic crossing conjecture in the case that $H_1(\Sigma(K))$ has summands which are not square-free, Theorem 1.2 can still be useful in the following sense. If a knot K has $\Sigma(K)$ an L-space, then if a particular crossing arc lifts to a null-homologous knot, then that crossing change must either be nugatory or change the isotopy class of the knot.

Although our main focus is on the cosmetic crossing conjecture for knots in the 3-sphere, the proof of Theorem 1.2 also applies to knots in arbitrary integer homology spheres. For a knot K in an integer homology sphere Y, we need only modify our definition of a crossing change; rather than change a crossing from positive to negative in a diagram of K, we define a *crossing change* as a ± 1–framed Dehn surgery along the crossing circle C, where C is the boundary of a crossing disk D as above. A *cosmetic crossing change* for K in Y is analogously defined as for knots in S^3.

Theorem 3.5. *Let K be a knot in an integer homology sphere Y such that the double cover of Y branched over K is an L-space. If each integer d_i in the decomposition*

$$H_1(\Sigma(Y,K)) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_k\mathbb{Z}$$

is square-free, then K admits no cosmetic crossing changes.

Proof. Note that Proposition 3.3 applies to any closed, oriented 3-manifold with a $\mathbb{Z}/2\mathbb{Z}$-action, and the Montesinos trick may be applied in any integer homology sphere. The content of Section 2.2 regarding the rational longitude is valid for any compact, oriented 3-manifold M with torus boundary and $H_1(M;\mathbb{Q}) \cong \mathbb{Q}$, and Theorem 2.4 will hold whenever $\Sigma(Y,K)$ is the double cover of an integer homology sphere Y branched over a knot K, satisfying the same homology condition as (2.4). Thus the proof of Theorem 1.2 applies mutatis mutandis. □

Boyer, Gordon and Watson conjectured that an irreducible rational homology 3-sphere is an L-space if and only if its fundamental group is not left-orderable [BGW13, Conjecture 1]. An affirmative answer to this conjecture, together with a result of Boyer, Rolfsen and Wiest [BRW05, Theorem 3.7], suggests the following.

Conjecture 3.6. *Let Y be a rational homology 3-sphere. If $\Sigma(Y,K)$ is a prime L-space, then Y is an L-space.*

Another conjecture, due to Ozsváth and Szabó (see for instance [HL14]), is that the only integer homology sphere L-spaces are the 3-sphere and connected sums of the Poincaré sphere. We remark that these conjectures conspire to limit the scope of Theorem 3.5 to the 3-sphere and connected sums of the Poincaré sphere.

3.2. **Proof of Theorem 1.5.** Next we turn our attention from knots whose branched double covers are L-spaces to knots which admit L-space surgeries. Recall that an L-space knot is a knot in S^3 such that a positive p/q–Dehn surgery along K yields an L-space $S^3_{p/q}(K)$. By work of Ni, an L-space knot K is fibered [Ni07]; thus Kalfagianni's result for fibered knots immediately implies that all L-space knots satisfy the cosmetic crossing conjecture. Thus the following theorem was previously known to be true, but the proof provided here does not appeal to Kalfagianni's work.

Theorem 3.7. *Let K be a knot in S^3 with an L-space surgery. Then K satisfies the cosmetic crossing conjecture.*

Proof. After perhaps mirroring K, we can assume that K admits a positive L-space surgery. Suppose that K admits a crossing change at c that preserves the knot type of K. Without loss of generality, assume that c is a positive crossing; the argument in the case c is negative is identical. Recall that if K is an L-space knot, then p–framed surgery along K in S^3 yields an L-space for all integers $p \geq 2g(K)-1$. Denote by $\widetilde{C}_p$ the image of the crossing circle C after p–surgery along K. Note that $\widetilde{C}_p$ is null-homologous. It follows that

$$(S^3_p(K^+))_{-1}(\widetilde{C}_p) \cong S^3_p(K^-)$$

where K^- is the image of K^+ after -1–surgery along C. By assumption, the pairs (S^3,K^+) and (S^3,K^-) are orientation preserving homeomorphic. Thus

$$(S^3_p(K^+))_{-1}(\widetilde{C}_p) \cong S^3_p(K^-) \cong S^3_p(K^+) \cong (S^3_p(K^+))_{-1}(U),$$

where U is an unknot in $S^3_p(K^+)$. As another application of Theorem 3.1, we find that $\widetilde{C}_p$ is itself an unknot for each $p \geq 2g(K)-1$. In particular, this implies that the manifold with boundary $S^3_p(K) - N(\widetilde{C}_p)$ is reducible for all $p \geq 2g(K)-1$.

It is well known that if M is a compact, orientable, irreducible 3-manifold and $F \subset \partial M$ is a toroidal boundary component, then at most finitely many slopes on F yield reducible Dehn fillings. The exterior of the two-component link $K \cup C$ in S^3 has infinitely many fillings along K yielding the reducible manifolds $S^3_p(K) - N(\widetilde{C}_p)$, so it must be the case that the exterior of $(K \cup C)$ is reducible. It follows that $K \cup C$ is split. Since C sits in an embedded 3-ball, the crossing is nugatory. □

4. Examples

4.1. **Knots of at most ten crossings.**

In this section we settle the cosmetic crossing conjecture for all but ten of the knots in the Rolfsen tables. Let us summarize the known obstructions to a knot K admitting a cosmetic crossing change. If:

- K is two-bridge [Tor99], or
- K is fibered [Kal12], or
- K is genus one and is not algebraically slice or $H_1(\Sigma(K))$ fails to be cyclic [BFKP12], or
- $\Sigma(K)$ is an L-space and $H_1(\Sigma(K))$ has only square-free summands [Theorem 1.2],

then K admits no cosmetic crossing changes. With the exception of the knots 9_{46} and 10_{128}, all of the knots mentioned in Table 1 and Table 2 are known to be alternating or quasi-alternating, hence are $\overline{Kh}$–thin and therefore have branched double covers that are L-spaces. The quasi-alternating status is the collected effort of many authors, and relevant summaries may be found in [CK09, Jab14].

Table 1. Knots of nine or fewer crossings that are non-fibered and have bridge number at least three. When the determinant is not square-free, we provide the first homology of the branched double cover.

Knot	Determinant	Genus	$H_1(\Sigma(K))$
8_{15}	33	2	
9_{16}	39	3	
9_{25}	47	2	
9_{35}	27	1	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$
9_{37}	45	2	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$
9_{38}	57	2	
9_{39}	55	2	
9_{41}	49	2	$\mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$
9_{46}	9	1	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$
9_{49}	25	2	$\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}$

Proposition 4.1. *All knots of at most nine crossings satisfy the cosmetic crossing conjecture.*

Proof. Utilizing the KnotInfo database [CL15], in Table 1 we collect the knots of nine or fewer crossings that are non-fibered and have bridge number at least three. These are listed with their determinants and genera. Where indicated, the homology groups of the branched double covers were computed using the "Cyclic Branched Cover Homology Calculator" program, available at KnotInfo.

TABLE 2. The ten-crossing knots that are non-fibered and have bridge number at least three. When the determinant is not square-free we provide the first homology of the branched double cover, to determine if Theorem 1.2 applies.

Knot	Determinant	$H_1(\Sigma(K))$	Knot	Determinant	$H_1(\Sigma(K))$
10_{49}	59		10_{95}	91	
10_{50}	53		10_{97}	87	
10_{51}	67		10_{98}	81	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$
10_{52}	59		10_{101}	85	
10_{53}	73		10_{102}	73	
10_{54}	47		10_{103}	75	$\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z}$
10_{55}	61		10_{108}	63	$\mathbb{Z}/63\mathbb{Z}$
10_{56}	65		10_{111}	77	
10_{57}	79		10_{113}	111	
10_{58}	65		10_{114}	93	
10_{61}	33		10_{117}	103	
10_{63}	57		10_{119}	101	
10_{65}	63	$\mathbb{Z}/63\mathbb{Z}$	10_{120}	105	
10_{66}	75	$\mathbb{Z}/75\mathbb{Z}$	10_{121}	115	
10_{67}	63	$\mathbb{Z}/63\mathbb{Z}$	10_{122}	105	
10_{68}	57		10_{128}	11	
10_{72}	73		10_{129}	25	$\mathbb{Z}/25\mathbb{Z}$
10_{74}	63	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/21\mathbb{Z}$	10_{130}	17	
10_{76}	57		10_{131}	31	
10_{77}	63	$\mathbb{Z}/63\mathbb{Z}$	10_{134}	23	
10_{80}	71		10_{135}	37	
10_{83}	83		10_{142}	15	
10_{84}	87		10_{144}	39	
10_{86}	85		10_{146}	33	
10_{87}	81	$\mathbb{Z}/81\mathbb{Z}$	10_{147}	27	$\mathbb{Z}/27\mathbb{Z}$
10_{90}	77		10_{162}	35	
10_{92}	89		10_{164}	45	$\mathbb{Z}/45\mathbb{Z}$
10_{93}	67		10_{165}	39	

The knots $8_{15}, 9_{16}, 9_{25}, 9_{37}, 9_{38}, 9_{39}, 9_{41}$, and 9_{49} are quasi-alternating, hence $\overline{Kh}$–thin. By the computations of the first homologies of the branched double covers in Table 1, Theorem 1.2 implies they admit no cosmetic crossing changes. The knot 9_{46} is known to be reduced Khovanov thin, but not *odd* Khovanov homology thin, due to Shumakovitch [Shu11]. Thus it is not quasi-alternating, but its branched double cover is still an L-space, and Theorem 1.2 again applies. Theorem 1.2 does not apply to the only remaining knot, 9_{35}, because $H_1(\Sigma(9_{35}))$ has a $\mathbb{Z}/9\mathbb{Z}$ summand. Since this knot has genus one and $H_1(\Sigma(K))$ is not cyclic, it admits no cosmetic crossing changes by [BFKP12]. □

Proof of Theorem 1.4. Using the same strategy, we collect the relevant data for the ten-crossings knots that are non-fibered and have bridge number at least three. All of the knots in Table 2 have genus at least two, so the obstruction of [BFKP12] will not help in the present case. Aside from the knot 10_{128}, each knot in Table 2 is $\overline{Kh}$–thin, so Theorem 1.2 applies to each knot other than those listed in (1.1). Finally, we consider the knot 10_{128}. This is the Montesinos knot $M(-2; 4/7, 1/2, 2/3)$. Using

[LM04, Theorem 1.3] and [LS07, Theorem 1.1], the branched double cover may be checked to be an L-space, and again Theorem 1.2 applies. This analysis of the ten-crossing knots together with Proposition 4.1 completes the proof of Theorem 1.4. □

Remark 4.2. The ten knots in (1.1) share the following additional properties: they are hyperbolic, non-fibered, bridge number three, and have genus either two or three.

4.2. **Further examples.** Using the obstructions of Theorem 1.2, it is not difficult to find new families of knots which satisfy the cosmetic crossing conjecture.

Example 4.3 (Pretzel knots)**.** Consider the 3-stranded pretzel knots $P(-p, q, r)$ where $p > 0$ is even and $q, r > 0$ are odd. When $q = p - 1$, we have that

$$\det(P(-p, q, r)) = | - pq - pr + qr| = p^2 - p + r; \tag{4.1}$$

thus for any odd integer n, the pretzel knots $P(-p, p-1, n+p-p^2)$ are an infinite family of knots of determinant n. By Greene [Gre10, Theorem 1.4(d)], the pretzel knots $P(-p, p-1, r)$ are quasi-alternating, but in particular r must be positive. Combining these statements, we see that for every odd integer $n \geq 3$ we have a finite set of quasi-alternating pretzel knots $P(-p, p-1, r)$ of determinant n, where $r = n + p - p^2$. By Corollary 1.3, we see that for every square-free odd integer $n \geq 3$, there exists even $p > 0$ such that the pretzel knot $P(-p, p-1, n+p-p^2)$ has determinant n and satisfies the cosmetic crossing conjecture.

Note also that when $p \geq 4$ is even and $q, r \geq 3$ are odd, then by [Gab86], $P(-p, q, r)$ is non-fibered, and the genus of $P(-p, q, r)$ is $(q+r)/2$ (see [KL07, Corollary 2.7]). Moreover, these knots are hyperbolic [KL07, Theorem 2.4] and not two-bridge, and so excluding the handful of cases where $0 < p, q, r < 4$, the previously known obstructions to admitting a cosmetic crossing change do not apply.

Example 4.4 (Branch sets of L-space surgeries)**.** Let K be a strongly invertible L-space knot. For $p/q \geq 2g(K) - 1$ with p odd, we have that $S^3_{p/q}(K)$ is an L-space which can be expressed as the branched double cover of a knot $J_{p/q}$ by work of Montesinos [Mon76]. (This is a knot and not a link, as the determinant is necessarily odd since p is.) If p is square-free, then we have by Theorem 1.2 that $J_{p/q}$ necessarily satisfies the cosmetic crossing conjecture. Further, if K is hyperbolic, then by Thurston's hyperbolic Dehn surgery theorem, all but finitely many of the surgeries on K will be hyperbolic. For such surgeries, the corresponding knot $J_{p/q}$ is not arborescent (and in particular is not a pretzel or Montesinos knot). While K is necessarily fibered, the quotient knot $J_{p/q}$ need not be fibered; for example, every two-bridge knot is the quotient of a surgery on the unknot.

Finally, we remark that conjecturally every L-space knot is strongly invertible.

In Example 4.3, we made use of finite sets of quasi-alternating knots of a fixed determinant. In [Gre10, Conjecture 3.1], Greene conjectured that this phenomenon is always the case, namely, that there exist only finitely many quasi-alternating links with a given determinant. We now describe an infinite family of knots with fixed determinant satisfying the cosmetic crossing conjecture. Though these knots will be $\overline{Kh}$–thin, presumably they are not quasi-alternating.

Example 4.5 (Symmetric unions)**.** Further examples can be generated using symmetric unions, a classical construction due to Kinoshita and Terasaka [KT57]. In

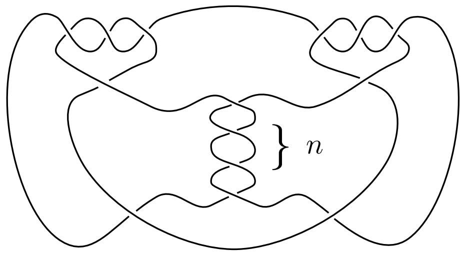

FIGURE 3. A symmetric union of the knot 5_2 with $n = 2$ full twists in the clasp.

particular this construction can be used to create infinite families of examples with a fixed determinant, unlike the two examples described above. The symmetric unions $K_n(5_2)$ of the knot 5_2 (see Figure 3 for the knot $K_2(5_2)$) are an example of such a construction.

In [Moo15], the knots $K_n = K_n(5_2)$ with $n \equiv 0 \pmod 7$ are shown to be reduced Khovanov thin, non-alternating, non-fibered, hyperbolic, of genus two, bridge number three, and have $H_1(\Sigma(K_n)) \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$. Applying Theorem 1.2, we see that this infinite family of knots with constant determinant satisfies the cosmetic crossing conjecture.

Acknowledgments

The authors would like to thank Cameron Gordon for explaining the proof of Proposition 3.3, Danny Ruberman and Steven Sivek for insightful conversations, and Effie Kalfagianni, Mark Powell, and Liam Watson for helpful correspondence.

References

[BFKP12] Cheryl Balm, Stefan Friedl, Efstratia Kalfagianni, and Mark Powell, *Cosmetic crossings and Seifert matrices*, Comm. Anal. Geom. **20** (2012), no. 2, 235–253, DOI 10.4310/CAG.2012.v20.n2.a1. MR2928712

[BGW13] Steven Boyer, Cameron McA. Gordon, and Liam Watson, *On L-spaces and left-orderable fundamental groups*, Math. Ann. **356** (2013), no. 4, 1213–1245, DOI 10.1007/s00208-012-0852-7. MR3072799

[BK14] Cheryl Jaeger Balm and Efstratia Kalfagianni, *Knots without cosmetic crossings*, Topology Appl. **207** (2016), 33–42, DOI 10.1016/j.topol.2016.04.009. MR3501264

[BRW05] Steven Boyer, Dale Rolfsen, and Bert Wiest, *Orderable 3-manifold groups* (English, with English and French summaries), Ann. Inst. Fourier (Grenoble) **55** (2005), no. 1, 243–288. MR2141698

[CK09] Abhijit Champanerkar and Ilya Kofman, *Twisting quasi-alternating links*, Proc. Amer. Math. Soc. **137** (2009), no. 7, 2451–2458, DOI 10.1090/S0002-9939-09-09876-1. MR2495282

[CL15] Jae Choon Cha and Charles Livingston, *KnotInfo: Table of knot invariants*, http://www.indiana.edu/~knotinfo, 2015.

[Gab86] David Gabai, *Detecting fibred links in S^3*, Comment. Math. Helv. **61** (1986), no. 4, 519–555, DOI 10.1007/BF02621931. MR870705

[Gai15] Fyodor Gainullin, *Heegaard Floer homology and knots determined by their complements*, preprint, 2015, arXiv:1504.06180 [math.GT].

[GL84] C. McA. Gordon and R. A. Litherland, *Incompressible surfaces in branched coverings*, The Smith conjecture (New York, 1979), Pure Appl. Math., vol. 112, Academic Press, Orlando, FL, 1984, pp. 139–152, DOI 10.1016/S0079-8169(08)61639-6. MR758466

[Gor09] Cameron Gordon, *Dehn surgery and 3-manifolds*, Low dimensional topology, IAS/Park City Math. Ser., vol. 15, Amer. Math. Soc., Providence, RI, 2009, pp. 21–71. MR2503492

[Gre10] Joshua Greene, *Homologically thin, non-quasi-alternating links*, Math. Res. Lett. **17** (2010), no. 1, 39–49, DOI 10.4310/MRL.2010.v17.n1.a4. MR2592726

[HL14] Matthew Hedden and Adam Simon Levine, *Splicing knot complements and bordered Floer homology*, to appear in J. Reine Angew. Math.

[Jab14] Slavik Jablan, *Tables of quasi-alternating knots with at most 12 crossings*, preprint, 2014, arXiv:1404.4965v2 [math.GT].

[Kal12] Efstratia Kalfagianni, *Cosmetic crossing changes of fibered knots*, J. Reine Angew. Math. **669** (2012), 151–164. MR2980586

[Kho00] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. **101** (2000), no. 3, 359–426, DOI 10.1215/S0012-7094-00-10131-7. MR1740682

[Kir78] Rob Kirby, *Problems in low dimensional manifold theory*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978, pp. 273–312. MR520548

[KL07] Dongseok Kim and Jaeun Lee, *Some invariants of pretzel links*, Bull. Austral. Math. Soc. **75** (2007), no. 2, 253–271, DOI 10.1017/S0004972700039198. MR2312569

[KMOS07] P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó, *Monopoles and lens space surgeries*, Ann. of Math. (2) **165** (2007), no. 2, 457–546, DOI 10.4007/annals.2007.165.457. MR2299739

[KT57] Shin'ichi Kinoshita and Hidetaka Terasaka, *On unions of knots*, Osaka Math. J. **9** (1957), 131–153. MR0098386

[KT80] Paik Kee Kim and Jeffrey L. Tollefson, *Splitting the PL involutions of nonprime* 3-*manifolds*, Michigan Math. J. **27** (1980), no. 3, 259–274. MR584691

[LM04] Paolo Lisca and Gordana Matić, *Transverse contact structures on Seifert 3-manifolds*, Algebr. Geom. Topol. **4** (2004), 1125–1144 (electronic), DOI 10.2140/agt.2004.4.1125. MR2113899

[LS07] Paolo Lisca and András I. Stipsicz, *Ozsváth-Szabó invariants and tight contact 3-manifolds. III*, J. Symplectic Geom. **5** (2007), no. 4, 357–384. MR2413308

[MO08] Ciprian Manolescu and Peter Ozsváth, *On the Khovanov and knot Floer homologies of quasi-alternating links*, Proceedings of Gökova Geometry-Topology Conference 2007, Gökova Geometry/Topology Conference (GGT), Gökova, 2008, pp. 60–81. MR2509750

[Mon76] José M. Montesinos, *Three-manifolds as* 3*-fold branched covers of* S^3, Quart. J. Math. Oxford Ser. (2) **27** (1976), no. 105, 85–94. MR0394630

[Moo15] Allison H. Moore, *Symmetric unions without cosmetic crossing changes*, Advances in the Mathematical Sciences: Association for Women in Mathematics Series, vol. 6, Springer International Publishing, Switzerland, 2016, pp. 103–116.

[Ni07] Yi Ni, *Knot Floer homology detects fibred knots*, Invent. Math. **170** (2007), no. 3, 577–608, DOI 10.1007/s00222-007-0075-9. MR2357503

[OS04] Peter Ozsváth and Zoltán Szabó, *Holomorphic disks and three-manifold invariants: properties and applications*, Ann. of Math. (2) **159** (2004), no. 3, 1159–1245, DOI 10.4007/annals.2004.159.1159. MR2113020

[OS05] Peter Ozsváth and Zoltán Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33, DOI 10.1016/j.aim.2004.05.008. MR2141852

[Rol90] Dale Rolfsen, *Knots and links*, Mathematics Lecture Series, vol. 7, Publish or Perish, Inc., Houston, TX, 1990. Corrected reprint of the 1976 original. MR1277811

[Shu11] Alexander N. Shumakovitch, *Patterns in odd Khovanov homology*, J. Knot Theory Ramifications **20** (2011), no. 1, 203–222, DOI 10.1142/S0218216511008802. MR2777025

[ST89] Martin Scharlemann and Abigail Thompson, *Link genus and the Conway moves*, Comment. Math. Helv. **64** (1989), no. 4, 527–535, DOI 10.1007/BF02564693. MR1022995

[Tor99] Ichiro Torisu, *On nugatory crossings for knots*, Topology Appl. **92** (1999), no. 2, 119–129, DOI 10.1016/S0166-8641(97)00238-1. MR1669827

[Wat12] Liam Watson, *Surgery obstructions from Khovanov homology*, Selecta Math. (N.S.) **18** (2012), no. 2, 417–472, DOI 10.1007/s00029-011-0070-2. MR2927239

[YM84] Shing-Tung Yau and William H. Meeks III, *The equivariant loop theorem for three-dimensional manifolds and a review of the existence theorems for minimal surfaces*, The Smith conjecture (New York, 1979), Pure Appl. Math., vol. 112, Academic Press, Orlando, FL, 1984, pp. 153–163, DOI 10.1016/S0079-8169(08)61640-2. MR758467

Institute for Advanced Study, Einstein Drive, Princeton, New Jersey 08540

Current address: Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695

E-mail address: tlid@math.ncsu.edu

Department of Mathematics, Rice University, 6100 Main Street, Houston, Texas 77005

Current address: Department of Mathematics, University of California at Davis, One Shields Avenue, Davis, California 95616

E-mail address: amoore@math.ucdavis.edu