

UNIMODULARITY OF INVARIANT RANDOM SUBGROUPS

IAN BIRINGER AND OMER TAMUZ

ABSTRACT. An invariant random subgroup $H \leq G$ is a random closed subgroup whose law is invariant to conjugation by all elements of G . When G is locally compact and second countable, we show that for every invariant random subgroup $H \leq G$ there almost surely exists an invariant measure on G/H . Equivalently, the modular function of H is almost surely equal to the modular function of G , restricted to H .

We use this result to construct invariant measures on orbit equivalence relations of measure preserving actions. Additionally, we prove a mass transport principle for discrete or compact invariant random subgroups.

1. INTRODUCTION

Let G be a locally compact, second countable group. We denote by Sub_G the space of closed subgroups of G , equipped with the Chabauty topology, and consider the action $G \curvearrowright \text{Sub}_G$ by conjugation.

An *invariant random subgroup* (IRS) is a Sub_G -valued random variable whose law is invariant to conjugation. Any Borel probability measure preserving action $G \curvearrowright X$ gives an IRS - the stabilizer of a random point in X ; this follows¹ from the Compact Model Theorem of Varadarajan [24, Theorem 3.2 and its corollary]. In a slight (but standard) abuse of notation we also refer to a conjugation invariant Borel probability measure on Sub_G (i.e., the law of an IRS) as an IRS.

Although measure preserving G -actions and their stabilizers have been studied for some time (e.g., [9, 23]), IRSs were first introduced by Abért, Glasner and Virág [3] and simultaneously by Vershik [26] under a different name. Since then, IRSs have appeared in a number of papers, either as direct subjects of study [12–14, 18, 26], as probabilistic limits of manifolds with increasing volume [1], or as tools to understand stationary group actions [11, 17].

The notion of an IRS is a natural weakening of that of a normal subgroup. As such, it is interesting to understand which properties of normal subgroups hold for IRSs. This is the spirit of [3] and of our main theorem.

More generally, one can consider the set of G -invariant, σ -finite Borel measures on Sub_G , which we denote by $\mathcal{M}_G(\text{Sub}_G)$. We will call an element of $\mathcal{M}_G(\text{Sub}_G)$

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¹This theorem states that given a measurable action of a locally compact second countable group on a standard probability space X , one can endow X with a precompact metrizable topology under which the action is continuous. This implies that stabilizers are closed subgroups, and that the map that assigns to each point its stabilizer is upper semicontinuous and hence measurable.

an *invariant subgroup measure*. If $(*)$ is a Borel property of subgroups of G (e.g. ‘discrete’ or ‘unimodular’), then we say that $\lambda \in \mathcal{M}_G(\text{Sub}_G)$ is $(*)$ whenever λ -a.e. $H \in \text{Sub}_G$ is $(*)$.

Our main result is the following theorem.

Theorem 1.1. *Let $\lambda \in \mathcal{M}_G(\text{Sub}_G)$ be an invariant subgroup measure. Then for all H in the topological support of λ there exists a nontrivial G -invariant measure on G/H .*

There exists a G -invariant measure on G/H if and only if the modular function μ_G of G restricts to the modular function μ_H of H (cf. [20, Theorem 1, p. 139]). Hence a corollary of Theorem 1.1 is that an invariant subgroup measure of a unimodular group G must be supported on unimodular subgroups.

When G is not unimodular, the kernel of the modular function is a unimodular, closed, normal proper subgroup of G . It can be thought of as the ‘unimodular radical’ of G : a normal unimodular subgroup that includes all other normal unimodular subgroups. Theorem 1.1 implies that any *unimodular* subgroup $H \leq G$ that lies in the support of an invariant subgroup measure must be contained in $\ker \mu_G$. That is:

Corollary 1.2. *If an invariant subgroup measure $\lambda \in \mathcal{M}_G(\text{Sub}_G)$ is unimodular, then $H \leq \ker \mu_G$ for all H in the support of λ .*

Note that, in particular, this corollary applies whenever λ is supported on discrete subgroups. It would be interesting to understand when the same result can be proved for other ‘radicals’. For example, in [5] it is shown that every amenable IRS is included in the amenable radical.

Our proof of Theorem 1.1 yields a slightly stronger theorem (Theorem 3.1), in which the G -invariant measures on the coset spaces G/H can be chosen to vary measurably with H . As an application, we construct invariant measures on orbit equivalence relations.

Suppose that $G \curvearrowright (X, \zeta)$ is a measure preserving action on a σ -finite Borel measure space. Let $E \subseteq X \times X$ denote the orbit equivalence relation

$$E = \{(x, gx) : x \in X, g \in G\},$$

which we consider with the G -action $g(x, y) = (x, gy)$. Let $p_l: E \rightarrow X$ be the projection on the left coordinate $p_l(x, y) = x$.

In many applications the interesting case is when ζ is a probability measure. More generally, we prove:

Corollary 1.3. *Let $G \curvearrowright (X, \zeta)$ be a measure preserving action on a σ -finite Borel measure space, with orbit equivalence relation E .*

Then there exists a Borel family of G -invariant, σ -finite measures ν_x on E , with ν_x supported on the fiber $p_l^{-1}(x)$, and such that

$$\nu = \int_X \nu_x d\zeta(x)$$

is a G -invariant, σ -finite measure on E .

When the action is free, each fiber $p_l^{-1}(x)$ is identified with G and one can take ν_x to be the Haar measure. When G is countable, the ν_x are counting measures and Corollary 1.3 is due to Feldman and Moore [15, Theorem 2]. Such a measure ν can

be used, for example, to define the so-called groupoid representation $G \curvearrowright L^2(\nu)$ (see [21], and [25] for a recent application).

The final goal of this paper is to formulate a version of the ‘mass transport principle’ (MTP) for invariant subgroup measures. The MTP has proved to be a useful tool in the theory of random graphs [4, 8] and in percolation theory [6, 16]. Versions of the MTP have also appeared in the theory of foliations for some time (see, e.g., the definition of an invariant transverse measure in [19, p. 82]).

A *mass transport* on a graph $\Gamma = (V, E)$ is a positive function $f : V \times V \rightarrow \mathbb{R}$, where $f(x, y)$ determines how much “mass” is transported from x to y . We assume that f is invariant to the diagonal action of the automorphism group $\text{Aut}(\Gamma)$, so that $f(x, y)$ depends only on how x and y interact with the geometry of the graph. When $\text{Aut}(\Gamma)$ is transitive and unimodular, a *mass transport principle* (MTP) applies: the total amount of mass transported from each vertex is equal to the amount transported to it. Namely, for all $v \in V$,

$$(1.1) \quad \sum_{w \in V} f(v, w) = \sum_{w \in V} f(w, v).$$

The notion of a mass transport principle can be generalized to random rooted graphs. Let \mathcal{G}_\bullet be the space of isomorphism classes of rooted graphs, equipped with the topology of convergence on finite neighborhoods. Similarly, let $\mathcal{G}_{\bullet\bullet}$ be the space of isomorphism classes of doubly rooted graphs (see [8]). A random rooted graph ‘satisfies the mass transport principle’ when its law, a measure λ on \mathcal{G}_\bullet , satisfies

$$(1.2) \quad \int_{\mathcal{G}_\bullet} \sum_{w \in \Gamma} f(\Gamma, v, w) \, d\lambda(\Gamma, v) = \int_{\mathcal{G}_\bullet} \sum_{w \in \Gamma} f(\Gamma, w, v) \, d\lambda(\Gamma, v),$$

for all positive Borel functions $f : \mathcal{G}_{\bullet\bullet} \rightarrow \mathbb{R}$. Random rooted graphs that satisfy the MTP are also called *unimodular random graphs*. Let a random rooted graph Γ be almost surely a single transitive rooted graph. Then Γ is unimodular if and only if the automorphism group $\text{Aut}(\Gamma)$ is a unimodular topological group, in which case (1.2) is exactly the MTP given in (1.1).

One should view an MTP as a replacement for group-invariance of a measure, which is especially useful when no group action is present. Our interest, however, lies with measures on Sub_G , where G acts by conjugation. We show that for measures supported on discrete or compact subgroups of G , conjugation invariance is equivalent to a suitable MTP. This generalizes the results of Abért, Glasner and Virág [3] for discrete G . We elaborate on this connection in Section 5. Discrete IRSs are particularly interesting in the case of Lie groups: the nonatomic IRSs of a connected simple Lie group are supported on discrete groups [1, Theorem 2.6].

Here, we will state our theorem in the case that G is unimodular, saving the more general statement for Section 5. So, suppose that G is a unimodular, locally compact, second countable group, and fix a Haar measure ℓ on G . If H is a compact subgroup of G , let ν_H be the pushforward of ℓ to $H \backslash G$, while if H is a discrete subgroup of G , let ν_H be the measure on $H \backslash G$ obtained by *locally pushing forward* ℓ under the covering map $G \rightarrow H \backslash G$. This is the unique measure with

$$(1.3) \quad \ell = \int_{G/H} \eta_{Hg} \, d\nu_H(Hg),$$

where η_{Hg} is the counting measure on Hg . See Section 5 for details.

Denote by Cos_G the space of cosets of closed subgroups of G :

$$\text{Cos}_G = \{Hg : H \in \text{Sub}_G, g \in G\}.$$

We discuss this space and its topology in Section 2.

Theorem 1.4 (Mass transport principle). *Let G be unimodular, and let λ be a σ -finite Borel measure on Sub_G such that λ -a.e. $H \in \text{Sub}_G$ is discrete or compact.*

Then λ is conjugation invariant (i.e., an invariant subgroup measure) if and only if for every nonnegative Borel function $f : \text{Cos}_G \rightarrow \mathbb{R}$,

$$\begin{aligned} & \int_{\text{Sub}_G} \int_{H \backslash G} f(Hg) \, d\nu_H(Hg) \, d\lambda(H) \\ &= \int_{\text{Sub}_G} \int_{H \backslash G} f(g^{-1}H) \, d\nu_H(Hg) \, d\lambda(H). \end{aligned}$$

Theorem 1.4 will be used by Abért and Biringer in [2] to show that certain invariant random subgroups of continuous groups correspond to ‘unimodular random manifolds’, i.e., measures on the space of rooted Riemannian manifolds satisfying a mass transport principle.

We should note that this is not the first version of a mass transport principle that applies in the continuous setting. In [7], Benjamini and Schramm give a version of the MTP for the hyperbolic plane.

To interpret Theorem 1.4, one should view Cos_G as foliated by the right coset spaces $H \backslash G$, where H ranges through Sub_G . The MTP says that the measure ν obtained by integrating the measures ν_H against λ is invariant under the involution $Hg \mapsto g^{-1}H$ of Cos_G .

An alternative, appealing interpretation of the MTP is the following. Call a closed normal subgroup $N \triangleleft G$ *co-unimodular* if G/N is unimodular, that is, if there exists a bi-invariant measure on G/N . Analogously, we call an IRS λ *co-unimodular* if there exists a Borel measure ν on Cos_G that projects to λ and is invariant to both the left and right G -action. In Section 5.3 we show that when G is unimodular, then there is an MTP for λ if and only if λ is co-unimodular.

A particularly aesthetic version of the MTP arises if we also define for each discrete $H \leq G$ a measure ν^H on G/H by locally pushing forward ℓ with respect to the covering map $G \rightarrow G/H$. That is,

$$(1.4) \quad \ell = \int_{H \backslash G} \eta_{gH} \, d\nu^H(gH),$$

where η_{gH} is the counting measure on Hg . It follows from (1.3) and (1.4) that the involution $Hg \mapsto g^{-1}H$ sends ν_H to ν^H . So, the MTP can be rephrased as

$$\begin{aligned} & \int_{\text{Sub}_G} \int_{H \backslash G} f(Hg) \, d\nu_H(Hg) \, d\lambda(H) \\ &= \int_{\text{Sub}_G} \int_{G/H} f(gH) \, d\nu^H(gH) \, d\lambda(H). \end{aligned}$$

In other words, the measure on Cos_G obtained by integrating the natural measures on right coset spaces $H \backslash G$ against λ is the same as the measure on Cos_G obtained by integrating the natural measures on left coset spaces G/H against λ .

2. COSETS, SUBGROUPS AND A DISINTEGRATION RESULT

We start by defining the coset space of G , Cos_G , which is the set

$$\text{Cos}_G = \{gH : g \in G, H \in \text{Sub}_G\},$$

equipped with the Fell topology of closed subsets of G . This topology is locally compact, second countable and Hausdorff, hence Polish. Note that since $gH = gHg^{-1}g = H^g$, an equivalent definition is

$$\text{Cos}_G = \{Hg : g \in G, H \in \text{Sub}_G\}.$$

The group G acts on Cos_G from the left. When viewing Cos_G as the set of left cosets, the action of $k \in G$ is given by $k(gH) = kgH$, and when considering right cosets we have $k(Hg) = (kHk^{-1})(kg) = H^k kg$.

Adopting the perspective of right cosets, consider the maps

$$\begin{array}{ccc} \text{Sub}_G \times G & \xrightarrow{\sigma_r} & \text{Cos}_G & \xrightarrow{\pi_r} & \text{Sub}_G, \\ (H, g) & \mapsto & Hg & \mapsto & H. \end{array}$$

These maps are all G -equivariant, where the actions are $(H, g) \xrightarrow{k} (kHk^{-1}, kg)$, $Hg \xrightarrow{k} kHk^{-1}(kg)$, and $H \xrightarrow{k} kHk^{-1}$. Note that $\pi_r \circ \sigma_r$ is the projection $(H, g) \mapsto H$. We will also need the maps

$$\begin{array}{ccc} \text{Sub}_G \times G & \xrightarrow{\sigma_l} & \text{Cos}_G & \xrightarrow{\pi_l} & \text{Sub}_G, \\ (H, g) & \mapsto & gH & \mapsto & H. \end{array}$$

When viewing Cos_G as right cosets, $\pi_l(Hg) = \pi_l(g \cdot g^{-1}Hg) = g^{-1}Hg$.

Suppose that G is a locally compact, second countable group. The main result of this section is the following disintegration theorem.

Proposition 2.1. *Fix a left Haar measure ℓ on G and a Borel map*

$$\text{Cos}_G \longrightarrow \mathcal{M}(G), \quad Hg \longmapsto \eta_{Hg}$$

such that each η_{Hg} is nonzero, left H -invariant and supported on Hg .

Then for each $H \in \text{Sub}_G$ there is a unique Borel measure ν_H on $H \backslash G \subset \text{Cos}_G$ such that

$$\ell = \int_{H \backslash G} \eta_{Hg} \, d\nu_H(Hg).$$

Moreover, the map $\text{Sub}_G \longrightarrow \mathcal{M}(\text{Cos}_G)$, $H \longmapsto \nu_H$ is Borel.

Succinctly, the proposition states that left H -invariant measures on cosets Hg can be realized as fiber measures in a disintegration of any left Haar measure on G , and that if the fiber measures are Borel parametrized, so are the resulting factor measures.

Proof. Since ℓ is σ -finite, the pushforward measure on $H \backslash G$ is equivalent to a σ -finite measure $\hat{\nu}_H$; for instance, one can take $\hat{\nu}_H$ to be the pushforward of any probability measure on G that is equivalent to ℓ . Note that the pushforward of ℓ is only itself σ -finite if H is compact.

Applying Rohlin’s Disintegration Theorem (see [22, Theorem 6.3]), there is a disintegration

$$(2.1) \quad \ell = \int_{H \backslash G} \hat{\eta}_{Hg} \, d\hat{\nu}_H(Hg),$$

where $Hg \mapsto \hat{\eta}_{Hg}$ is a measurable parametrization of σ -finite measures on G and $\hat{\eta}_{Hg}$ is supported on the coset Hg .

We claim that $\hat{\nu}_H$ -almost every $\hat{\eta}_{Hg}$ is left H -invariant: left multiplication by $h \in H$ on G leaves invariant each right coset Hg , and so

$$\ell = h_*(\ell) = \int_{H \backslash G} h_*(\hat{\eta}_{Hg}) \, d\hat{\nu}_H(Hg).$$

By the uniqueness of disintegrations with a given factor measure, we have $h_*(\hat{\eta}_{Hg}) = \hat{\eta}_{Hg}$ for $\hat{\nu}_H$ -a.e. coset Hg . It follows from the separability of H that $\hat{\eta}_{Hg}$ is H -invariant for $\hat{\nu}_H$ -a.e. Hg .

Equation (2.1) remains unchanged if we replace a $\hat{\nu}_H$ -negligible number of the measures $\hat{\eta}_{Hg}$ with η_{Hg} , so we may as well assume from now on that every $\hat{\eta}_{Hg}$ is nonzero and left H -invariant. By the uniqueness of the Haar measure, each $\hat{\eta}_{Hg}$ is a positive scalar multiple of η_{Hg} . Let

$$(2.2) \quad f : \text{Cos}_G \rightarrow \mathbb{R}^+, \quad f(Hg) = \frac{d\hat{\eta}_{Hg}}{d\eta_{Hg}}$$

be the function whose value gives this multiple. Define the measure ν_H on $H \backslash G$ by

$$d\nu_H(Hg) = f(Hg) \, d\hat{\nu}_H(Hg).$$

Then we have

$$(2.3) \quad \begin{aligned} \ell &= \int_{H \backslash G} \hat{\eta}_{Hg} \, d\hat{\nu}_H(Hg) \\ &= \int_{H \backslash G} \eta_{Hg} \cdot f(Hg) \, d\hat{\nu}_H(Hg) \\ &= \int_{H \backslash G} \eta_{Hg} \, d\nu_H(Hg), \end{aligned}$$

as required in the statement of the proposition. As the η_{Hg} are fixed, ν_H is the unique measure on $H \backslash G$ satisfying (2.3).

It remains to show that the ν_H are Borel parametrized when regarded as measures on the space of all cosets Cos_G . This is a consequence of the fact that the η_{Hg} are Borel parametrized, but we will nevertheless give a careful proof.

Fix a positive, continuous function $F : G \rightarrow \mathbb{R}$ such that

$$\int F \, d\eta_{Hg} < \infty \quad \forall Hg \in \text{Cos}_G.$$

One way to produce such a function is as follows. Pick a compact neighborhood B of the identity and a locally finite cover of G by open sets B_n with $B_n^{-1}B_n \subset B$. If $hg \in Hg \cap B_n^{-1}$, then

$$\begin{aligned} \eta_{Hg}(B_n) &= ((hg)_*^{-1}\eta_{Hg})(hgB_n) \\ &= \eta_{g^{-1}Hg}(hgB_n) \\ &< \eta_{g^{-1}Hg}(B). \end{aligned}$$

So, for each Hg , there is an upper bound on $\eta_{Hg}(B_n)$ that is independent of n . The function $F = \sum_n \frac{1}{2^n} \cdot \rho_n$, where ρ_n is a partition of unity subordinate to the cover B_n , then has the desired properties.

Using F , we define a positive, Borel function

$$F' : \text{Sub}_G \times G \longrightarrow \mathbb{R}, \quad F'(H, g) = \frac{F(g)}{\int F \, d\eta_{Hg}}.$$

By definition, this F' has the property that $\int F'(H, \cdot) \, d\eta_{Hg} = 1$ for all $Hg \in \text{Cos}_G$. Given a continuous function $\varphi : \text{Cos}_G \longrightarrow \mathbb{R}$, we have

$$\begin{aligned} \int \varphi(Hg) \, d\nu_H(Hg) &= \int \varphi(Hg) \left(\int_{k \in Hg} F'(H, k) \, d\eta_{Hg}(k) \right) \, d\nu_H(Hg) \\ &= \int \varphi(Hg) F'(H, g) \, d\ell(g). \end{aligned}$$

The last expression is Borel in H , so we have shown that integrating a fixed continuous function φ against the measures ν_H gives a Borel function in H , implying that the map $H \mapsto \nu_H$ is Borel. \square

Finally, let us discuss a convenient construction of such η_{Hg} . In Claim A.2, we show that there is a continuous map

$$m : \text{Sub}_G \longrightarrow \mathcal{M}(G)$$

such that each $m(H)$ is a left Haar measure on $H \subset G$. Define

$$\eta_{Hg} := g_* m(g^{-1}Hg).$$

Each η_{Hg} is a σ -finite Borel measure supported on the coset $Hg \subset G$, and if $h \in H$ we have

$$h_*(\eta_{Hg}) = h_* g_* m(g^{-1}Hg) = g_*(g^{-1}hg)_* m(g^{-1}Hg) = \eta_{Hg},$$

which shows that η_{Hg} is left H -invariant and is well defined in the sense that it depends only on the coset Hg and not on the representative g . The η_{Hg} are also permuted by the left G -action: if $k \in G$, then

$$k_*(\eta_{Hg}) = (kg)_* m(g^{-1}Hg) = \eta_{(kHk^{-1})kg} = \eta_{kHg}.$$

So, to summarize this construction:

Fact 2.2. There is a family of measures η_{Hg} as required by Proposition 2.1 such that the map $Hg \mapsto \eta_{Hg}$ is left G -equivariant:

$$k_*(\eta_{Hg}) = \eta_{kHg}, \quad \forall k \in G, \, Hg \in \text{Cos}_G.$$

Note that for some classes of subgroups H , there are ‘natural’ choices for a left Haar measure $m(H)$. For instance, if H is discrete, one can take $m(H)$ to be the counting measure, in which case the measures η_{Hg} will all be counting measures as well. When H is compact, one can take $m(H)$ to be the Haar probability measure.

As ‘discrete’ and ‘compact’ are both Borel properties of subgroups, the map $H \mapsto m(H)$ above (and therefore the measures η_{Hg}) can be adjusted to agree with these natural choices on such subgroups. This will be important in Section 5.

3. THE PROOF OF THEOREM 1.1

Our main result is the following theorem.

Theorem 3.1. *Let $\lambda \in \mathcal{M}_G(\text{Sub}_G)$ be an invariant subgroup measure. Then there exists a Borel map*

$$\text{Sub}_G \longrightarrow \mathcal{M}(\text{Cos}_G), \quad H \longmapsto \nu^H$$

such that for λ -a.e. H the measure ν^H is nonzero, G -invariant and supported on the subset $G/H \subset \text{Cos}_G$.

As we show in Lemma A.3, the set of all $H \in \text{Sub}_G$ such that G/H has an invariant measure is closed. Therefore, we will obtain as a corollary of Theorem 3.1 our main theorem from the introduction:

Theorem 1.1. *Let $\lambda \in \mathcal{M}_G(\text{Sub}_G)$ be an invariant subgroup measure. Then for all H in the topological support of λ there exists a nontrivial G -invariant measure on G/H .*

We devote the remainder of this section to proving Theorem 3.1, and so fix an invariant subgroup measure $\lambda \in \mathcal{M}_G(\text{Sub}_G)$.

Proposition 3.2. *There is a left G -invariant, σ -finite Borel measure ν on Cos_G such that $\pi_{l*}\nu \equiv \lambda \equiv \pi_{r*}\nu$.*

Here, recall that π_l and π_r are the maps $\text{Cos}_G \rightarrow \text{Sub}_G$ taking $gH \mapsto H$ and $Hg \mapsto H$, respectively.

Before proving Proposition 3.2 we show that it implies Theorem 3.1 and in fact is equivalent to it: Given Theorem 3.1, the measure ν is obtained by integrating against λ :

$$(3.1) \quad \nu = \int_{\text{Sub}_G} \nu^H \, d\lambda(H).$$

This ν is G -invariant since each ν^H is G -invariant, and $\pi_{l*}\nu \equiv \lambda$ since ν^H is nonzero and supported on $\pi_l^{-1}(H)$.

Conversely, suppose Proposition 3.2 is true. Since $\pi_{l*}\nu \equiv \lambda$, Rohlin’s Disintegration Theorem (see [22, Theorem 6.3]) implies that there is a Borel map $H \mapsto \nu^H \in \mathcal{M}(\text{Cos}_G)$, with ν^H supported on $G/H = \pi_l^{-1}(H)$, such that equation (3.1) holds.

The action of G on Cos_G leaves invariant all fibers of the map $\pi_l : \text{Cos}_G \rightarrow \text{Sub}_G$. So since ν is G -invariant, the fiber measures ν^H are G -invariant for λ -a.e. $H \in \text{Sub}_G$. In other words, for λ -a.e. H , ν^H is an invariant measure on G/H , proving Theorem 3.1.

Proof of Proposition 3.2. Fix a left Haar measure ℓ on G . Choosing a family η_{Hg} of measures as described in Fact 2.2, Proposition 2.1 gives a Borel family of measures ν_H on $H \backslash G \subset \text{Cos}_G$ with

$$\ell = \int_{H \backslash G} \eta_{Hg} \, d\nu_H(Hg)$$

for each $H \in \text{Sub}_G$, and we set

$$\nu = \int \nu_H \, d\lambda(H).$$

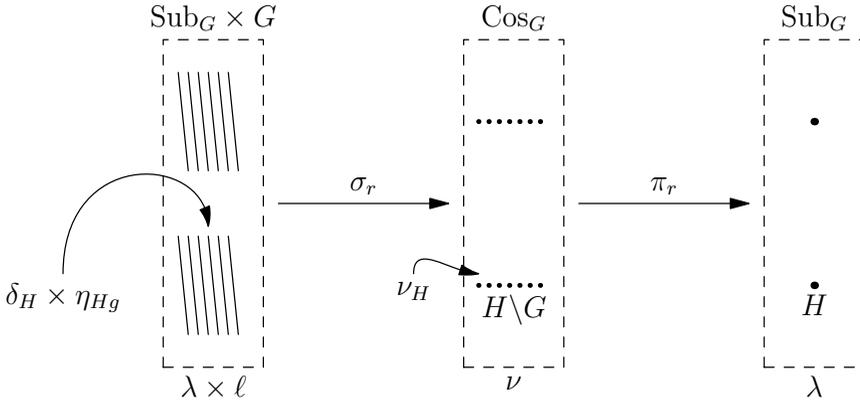


FIGURE 1. The measures involved in the proof of Proposition 3.2.

Note that $\pi_{r*}\nu \equiv \lambda$, since ν_H is supported on the fiber $\pi_r^{-1}(H)$. The measure $\lambda \times \ell$ on $\text{Sub}_G \times G$ can then be expressed as

$$\begin{aligned} \lambda \times \ell &= \int_{\text{Sub}_G} \delta_H \times \left(\int_{H \backslash G} \eta_{Hg} \, d\nu_H(Hg) \right) d\lambda(H) \\ &= \int_{\text{Cos}_G} \delta_H \times \eta_{Hg} \, d\nu(Hg), \end{aligned}$$

where δ_H is the Dirac measure at $H \in \text{Sub}_G$. This is a disintegration of $\lambda \times \ell$ with respect to $\sigma_r : \text{Sub}_G \times G \rightarrow \text{Cos}_G$, $\sigma_r(H, g) = Hg$.

The left G -action on $\text{Sub}_G \times G$ is $(H, g) \xrightarrow{k} (kHk^{-1}, kg)$. Since λ is conjugation invariant and ℓ is left G -invariant, $\lambda \times \ell$ is left G -invariant. But by the equivariance in Fact 2.2, we have that if $k \in G$, then

$$k_*(\delta_H \times \eta_{Hg}) = \delta_{kHk^{-1}} \times \eta_{(kHk^{-1})(kg)},$$

so the left G -action on $\text{Sub}_G \times G$ permutes the fiber measures $\delta_H \times \eta_{Hg}$. Therefore, as $\lambda \times \ell$ is invariant and the collection of fiber measures of its disintegration is equivariant, we have that the factor measure ν is invariant (cf. Proposition B.1).

It remains to be shown that $\pi_{l*}(\nu) \equiv \lambda$. Let

$$\psi : \text{Sub}_G \times G \rightarrow \text{Sub}_G \times G$$

be given by $\psi(H, g) = (H^{g^{-1}}, g)$. Then

$$\pi_l \circ \sigma_r = \pi_r \circ \sigma_r \circ \psi,$$

since $\pi_l \circ \sigma_r(H, g) = H^{g^{-1}}$ and $\pi_r \circ \sigma_r$ is the projection on the first coordinate. But ψ preserves $\lambda \times \ell$, since λ is conjugation invariant, so

$$\pi_{l*}(\nu) \equiv (\pi_l \circ \sigma_r)_*(\lambda \times \ell) = (\pi_r \circ \sigma_r)_*(\lambda \times \ell) \equiv \lambda. \quad \square$$

Remark 3.3. The measures ν^H in Theorem 1.1 are only determined up to a positive multiple, so the integrated measure

$$\nu = \int_{\text{Sub}_G} \nu^H \, d\lambda(H)$$

is unique up to scaling by a positive Borel function $f : \text{Cos}_G \rightarrow \mathbb{R}$ that is constant on each fiber $\pi_l^{-1}(H) = G/H$.

Sometimes, but not always, it is possible to choose the measure ν in Proposition 3.2 to be invariant under both the left and right actions of G on Cos_G . If λ is ergodic, such a ν is unique up to a global scalar. This is discussed in Section 5.3.

4. INVARIANT MEASURES ON ORBIT EQUIVALENCE RELATIONS

As in the introduction, suppose that $G \curvearrowright (X, \zeta)$ is a measure preserving action on a standard Borel probability space and let

$$E = \{(x, gx) : x \in X, g \in G\} \subset X \times X$$

be the associated orbit equivalence relation, which we consider with the G -action $g(x, y) = (x, gy)$ and the left projection $p_l : E \rightarrow X, p_l(x, y) = x$. The standard fact that E is a Borel subset of $X \times X$ follows (for example) from the Compact Model Theorem of Varadarajan [24, Theorem 3.2]; see for example [27, Corollary 2.1.20].

We aim to prove:

Corollary 1.3. *Let $G \curvearrowright (X, \zeta)$ be a measure preserving action on a σ -finite Borel measure space, with orbit equivalence relation E .*

Then there exists a Borel family of G -invariant, σ -finite measures ν_x on E , with ν_x supported on the fiber $p_l^{-1}(x)$ and such that

$$\nu = \int_X \nu_x d\zeta(x)$$

is a G -invariant, σ -finite measure on E .

To begin the proof, note that E decomposes as a union of the fibers $p_l^{-1}(x) = \{x\} \times Gx$. The fiber $\{x\} \times Gx$ is invariant under the G -action on E and is isomorphic as a G -space to the quotient G/G_x , where

$$G_x = \{g \in G : gx = x\}$$

is the stabilizer of x . Let $\text{stab} : X \rightarrow \text{Sub}_G$ be the map that assigns to each $x \in X$ its stabilizer. As stabilizers of G -actions on separable Borel spaces are closed [24], the image of stab is in Sub_G . Since stab is equivariant, $\text{stab}_*\zeta$ is an invariant subgroup measure. Theorem 3.1 then gives a Borel map

$$\text{Sub}_G \rightarrow \mathcal{M}(\text{Cos}_G), \quad H \mapsto \nu^H$$

that associates to $\text{stab}_*\zeta$ -a.e. $H \in \text{Sub}_G$ a nonzero G -invariant measure on $G/H \subset \text{Cos}_G$. So, for ζ -a.e. $x \in X$ we can define ν^x to be the measure on $\{x\} \times Gx \cong G/G_x$ corresponding to ν^{G_x} . The map

$$X \rightarrow \mathcal{M}(E), \quad x \mapsto \nu^x$$

is then Borel, so we can integrate the measures ν^x against the measure ζ on X to give a σ -finite Borel measure τ on E :

$$\tau = \int_X \nu^x d\zeta(x).$$

This τ is invariant under the G -action on the second coordinate of E , since that action preserves the p_l -fibers $\{x\} \times Gx$ and the associated measures ν^x . Therefore, Corollary 1.3 follows.

5. THE MASS TRANSPORT PRINCIPLE

We start this section in 5.1 by an additional discussion of some well known aspects of mass transport principles and their relation to invariant random subgroups (cf. [3,4]). We then prove our theorem in 5.2.

5.1. The MTP for labeled graphs and discrete groups. Fix a finite set S . An S -labeled graph is a countable directed graph with edges labeled by elements of S , such that the edges coming out from any given vertex v have labels in 1-1 correspondence with elements of S , and the same is true for the labels of edges coming into v . Let \mathcal{G}_\bullet^S and $\mathcal{G}_{\bullet\bullet}^S$ be the spaces of rooted and doubly rooted S -labeled graphs, up to (label preserving) isomorphism. The topologies on \mathcal{G}_\bullet^S and $\mathcal{G}_{\bullet\bullet}^S$ are defined so that two (doubly) rooted, S -labeled graphs are close when large finite balls around their roots are isomorphic.

A unimodular random S -labeled graph is a random S -labeled graph whose law is a probability measure λ on $\mathcal{G}_{\bullet\bullet}^S$ such that for every nonnegative Borel function $f : \mathcal{G}_{\bullet\bullet}^S \rightarrow \mathbb{R}$ we have the mass transport principle

$$(5.1) \quad \int_{\mathcal{G}_\bullet^S} \sum_{w \in G} f(G, v, w) \, d\lambda(G, v) = \int_{\mathcal{G}_\bullet^S} \sum_{w \in G} f(G, w, v) \, d\lambda(G, v).$$

In the introduction, we mentioned that unimodularity is often equivalent to invariance under a group action, when an action exists. Here, we can use the S -labels to construct an action of the free group $F(S)$ generated by S on \mathcal{G}_\bullet^S , where the action of s moves the root along the adjacent inward edge labeled ‘s’. We then have

Fact 5.1. A random S -labeled graph is unimodular if and only if it is invariant under the action of $F(S)$.

Proof. Assume first that λ is the law of a unimodular random S -labeled graph. If $s \in S$ and $E \subset \mathcal{G}_{\bullet\bullet}^S$ is Borel, define $f : \mathcal{G}_{\bullet\bullet}^S \rightarrow \mathbb{R}$ by

$$f(G, v, w) = \begin{cases} 1_E(G, v) & (G, v) = s(G, w), \\ 0 & \text{otherwise.} \end{cases}$$

Then the left side of the MTP is $\lambda(E)$, while the right-hand side is $\lambda(s(E))$. Therefore, λ is s -invariant.

For the other direction, suppose that λ is invariant under the action of each $s \in S$. By a standard reduction, it suffices to prove the MTP for functions $f : \mathcal{G}_{\bullet\bullet}^S \rightarrow \mathbb{R}$ supported on graphs (G, v, w) where there is a directed edge from w to v (compare with Proposition 2.2 in [4]). If f is such a function, the left side of the MTP becomes

$$(5.2) \quad \int_{\mathcal{G}_\bullet^S} \sum_{s \in SU_S^{-1}} \frac{f(G, v, sv)}{m(G, x, s)} \, d\lambda(G, v).$$

Here, S^{-1} is the set of formal inverses of elements of S , where the action of s^{-1} moves a vertex of a graph along the adjacent outward edge labeled ‘s’. The multiplicity function $m(G, x, s)$ records the number of elements $t \in S \cup S^{-1}$ where

$sx = tx$. Then

$$\begin{aligned}
 (5.2) &= \sum_{s \in S \cup S^{-1}} \int_{\mathcal{G}_\bullet^S} \frac{f(G, v, sv)}{m(G, v, s)} d\lambda(G, v) \\
 &= \sum_{s \in S \cup S^{-1}} \int_{\mathcal{G}_\bullet^S} \frac{f(G, s^{-1}v, v)}{m(G, s^{-1}v, s)} d\lambda(G, v) \\
 &= \int_{\mathcal{G}_\bullet^S} \sum_{s \in S \cup S^{-1}} \frac{f(G, sv, v)}{m(G, sv, s^{-1})} d\lambda(G, v) \\
 &= \int_{\mathcal{G}_\bullet^S} \sum_{w \in G} f(G, w, v) d\lambda(G, v),
 \end{aligned}$$

which proves the mass transport principle. □

In [3], Abért, Glasner and Virág show how to produce unimodular random S -labeled graphs from invariant subgroup measures. Suppose that G is a group generated by the finite set S . The *Schreier graph* of a subgroup $H \leq G$ is the graph $\text{Sch}(H \backslash G, S)$ whose vertices are right cosets of H and where each $s \in S$ contributes a directed edge labeled ‘ s ’ from every coset Hg to Hgs . We consider $\text{Sch}(H \backslash G, S)$ as an S -labeled graph rooted at the identity coset H , in which case the action of $F(S)$ described above is $Hg \mapsto Hgs^{-1}$. This defines an injection

$$\Phi : \text{Sub}_G \longrightarrow \mathcal{G}_\bullet^S, \quad H \longmapsto (\text{Sch}(H \backslash G, S), H).$$

Under Φ , the conjugation action of an element $s \in S$ on Sub_G corresponds to the natural action on \mathcal{G}_\bullet^S , since

$$(\text{Sch}(H \backslash G, S), Hs^{-1}) \cong (\text{Sch}(sHs^{-1} \backslash G, S), sHs^{-1})$$

as S -labeled rooted graphs. Therefore, invariant subgroup measures of G produce $F(S)$ -invariant measures on \mathcal{G}_\bullet^S .

We can also use the map Φ to reinterpret the MTP for S -labeled graphs (5.1) group theoretically. Let Cos_G be the space of cosets of subgroups of G , as in Section 3. There is then an injection

$$(5.3) \quad \Phi' : \text{Cos}_G \longrightarrow \mathcal{G}_{\bullet\bullet}^S, \quad Hg \longmapsto (\text{Sch}(H \backslash G, S), H, Hg).$$

To interpret the right-hand side of (5.1), note that the doubly rooted graph $(\text{Sch}(H \backslash G, S), Hg, H)$ that arises from switching the order of the roots in $\Phi(H)$ is isomorphic as an S -labeled graph to

$$(\text{Sch}(g^{-1}Hg \backslash G, S), g^{-1}Hg, g^{-1}Hgg^{-1}) = \Phi'((g^{-1}Hg)g^{-1}) = \Phi'(g^{-1}H).$$

Therefore, Fact 5.1 translates under Φ and Φ' to the following characterization of invariant subgroup measures of a discrete group G .

The discrete MTP. *Suppose that G is a finitely generated group and λ is a Borel measure on Sub_G . Then λ is conjugation invariant if and only if for every nonnegative Borel function $f : \text{Cos}_G \rightarrow \mathbb{R}$,*

$$\int_{\text{Sub}_G} \sum_{H \backslash G} f(Hg) d\lambda(H) = \int_{\text{Sub}_G} \sum_{H \backslash G} f(g^{-1}H) d\lambda(H).$$

Re-indexing, a slightly more aesthetic statement of the discrete MTP is obtained by replacing the sum on the right with $\sum_{G/H} f(gH)$.

5.2. Proof of the mass transport theorem. We now extend the Discrete MTP to general G and discrete or compact λ . As the following is intimately related to the existence of an invariant measure on Cos_G , we will frequently reference the setup of Sections 2 and 3.

Suppose that G is a locally compact, second countable topological group, and fix a left invariant Haar measure ℓ on G . If H is a compact subgroup of G , let ν_H be the pushforward of ℓ to $H \backslash G$. If H is a discrete subgroup of G , let ν_H be the measure on $H \backslash G$ obtained by *locally* pushing forward ℓ under the covering map $G \rightarrow H \backslash G$. That is, if $U \subset H \backslash G$ is an evenly covered open set with preimage $V_1 \sqcup V_2 \sqcup \dots$, then $\nu_H|_U$ is the pushforward of $\ell|_{V_i}$ for every i .

These ν_H can be understood in terms of Proposition 2.1. For discrete cosets, set η_{Hg} to be the counting measure. When H is compact, let η_{Hg} be the unique left H -invariant probability measure on Hg . Then in both cases, ν_H is characterized by the equation

$$(5.4) \quad \ell = \int_{H \backslash G} \eta_{Hg} \, d\nu_H.$$

Note that by Proposition 2.1, the map $H \mapsto \nu_H$ from Sub_G to the space of measures on Cos_G is Borel.

Theorem 5.2 (Mass transport principle). *Let λ be a σ -finite Borel measure on Sub_G such that λ -a.e. $H \in \text{Sub}_G$ is discrete or compact.*

Then λ is conjugation invariant (i.e., an invariant subgroup measure) if and only if $\mu_G|_H = 1$ for λ -a.e. $H \in \text{Sub}_G$, and for every nonnegative Borel function $f: \text{Cos}_G \rightarrow \mathbb{R}$,

$$(5.5) \quad \begin{aligned} & \int_{\text{Sub}_G} \int_{H \backslash G} f(Hg) \mu_G(Hg) \, d\nu_H(Hg) \, d\lambda(H) \\ &= \int_{\text{Sub}_G} \int_{H \backslash G} f(g^{-1}H) \, d\nu_H(Hg) \, d\lambda(H). \end{aligned}$$

Here, μ_G is the modular function of G , defined by the equation

$$\ell(S) = \mu_G(g)\ell(Sg), \quad \text{for every Borel } S \subset G.$$

The assumption $\mu_G|_H = 1$ implies that μ_G is constant over each coset Hg , and we write $\mu_G(Hg)$ for the common value on that coset. Note that the modular function of a discrete or compact group is trivial, so Theorem 1.1 implies that if λ is a discrete or compact invariant subgroup measure, then $\mu_G|_H = \mu_H = 1$ automatically for λ -a.e. H . So, in the course of the proof below, we will always assume $\mu_G|_H = \mu_H = 1$.

The MTP can be stated in a way that is more similar to our work earlier in the paper. Define an involution

$$\rho: \text{Cos}_G \rightarrow \text{Cos}_G, \quad \rho(Hg) = g^{-1}H,$$

and let ν be the measure on Cos_G defined by the integral

$$\nu := \int_{\text{Sub}_G} \nu_H \, d\lambda(H).$$

Note that the map $Hg \mapsto \eta_{Hg}$ defined above equation (5.4) is left G -equivariant, so just as in the proof of Proposition 3.2 the measure ν is left G -invariant. Now,

changing variables on the right-hand side, equation (5.5) can be rewritten as

$$(5.6) \quad \int_{\text{Cos}_G} f(Hg)\mu(Hg) \, d\nu(Hg) = \int_{\text{Cos}_G} f(Hg) \, d\rho_*\nu(Hg).$$

So, the MTP reduces to the following claim.

Claim 5.3. λ is conjugation invariant if and only if $\frac{1}{\mu_G} \cdot \nu = \rho_*\nu$.

Proof of Claim 5.3. Recall the following maps defined in Section 2:

$$\begin{array}{ccccc} \text{Sub}_G \times G & \xrightarrow{\sigma_r} & \text{Cos}_G & \xrightarrow{\pi_r} & \text{Sub}_G, \\ (H, g) & \mapsto & Hg & \mapsto & H. \end{array}$$

As in the proof of Proposition 3.2, we have a σ_r -disintegration

$$(5.7) \quad \lambda \times \ell = \int_{\text{Cos}_G} \delta_H \times \eta_{Hg} \, d\nu(Hg),$$

where ℓ is our chosen left Haar measure on G . Since $\rho(Hg) = g^{-1}H = H^{g^{-1}}g^{-1}$, there is a commutative diagram

$$\begin{array}{ccc} \text{Sub}_G \times G & \xrightarrow{\varphi} & \text{Sub}_G \times G \\ \downarrow \sigma_r & & \downarrow \sigma_r \\ \text{Cos}_G & \xrightarrow{\rho} & \text{Cos}_G, \end{array}$$

where φ is defined by $\varphi(H, g) = (H^{g^{-1}}, g^{-1})$. Note that if

$$\iota : G \longrightarrow G, \quad \iota(g) = g^{-1},$$

then $\iota_*(\ell) = \frac{1}{\mu_G} \cdot \ell$, so as φ inverts the second factor in $\text{Sub}_G \times G$ and conjugates the first, λ is conjugation invariant if and only if

$$(5.8) \quad \varphi_*(\lambda \times \ell)(H, g) = \frac{1}{\mu_G(g)} \lambda \times \ell(H, g).$$

For each $Hg \in \text{Cos}_G$, the measure η_{Hg} is either the counting measure or the unique H -invariant probability measure on Hg . In addition to being left H -invariant, in both cases η_{Hg} is in fact invariant under the right action of $H^{g^{-1}}$ on Hg . This is immediate when it is a counting measure, and when H is compact, η_{Hg} is the pushforward under left multiplication by g of the *bi-invariant* Haar probability measure on the compact group $H^{g^{-1}}$. So, the pushforward of η_{Hg} under inversion $g \mapsto g^{-1}$ is a *left* $H^{g^{-1}}$ -invariant measure on $H^{g^{-1}}g^{-1}$ and must be $\eta_{H^{g^{-1}}g^{-1}}$. Multiplying by δ_H , we then have

$$\varphi_*(\delta_H \times \eta_{Hg}) = \delta_{H^{g^{-1}}} \times \eta_{H^{g^{-1}}g^{-1}}.$$

So, the fiber measures of the disintegration in (5.7) are permuted by φ . From the commutative diagram for φ and ρ , it follows that ρ scales ν by $1/\mu_G$ if and only if φ scales $\lambda \times \ell$ by $1/\mu_G$, which we saw above was equivalent to conjugation invariance of λ . □

5.3. Bi-invariant measures and co-unimodular IRSs. As a consequence of Claim 5.3, we record the following.

Corollary 5.4. *Suppose that λ is an invariant subgroup measure in a unimodular group G such that λ -a.e. $H \in \text{Sub}_G$ is discrete or cocompact. Then there is a measure ν on Cos_G that is invariant under both the left and right actions of G and for which $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$. Moreover, if λ is ergodic, then ν is unique up to scale.*

Proof. The construction above gives a measure ν on Cos_G that is left G -invariant and also ρ -invariant. But the map $\rho(Hg) = g^{-1}H$ conjugates the left G -action on Cos_G to the right G -action, so ν is also right G -invariant. The fact that $\pi_{r*}(\nu) \equiv \lambda$ is just the definition of ν , and the fact that $\pi_{l*}(\nu) \equiv \lambda$ is the argument at the end of Proposition 3.2.

Since ν is bi-invariant, it is preserved under conjugation by $k \in G$, i.e., by the map

$$\text{Cos}_G \longrightarrow \text{Cos}_G, \quad gH \xrightarrow{k} kgHk = kgk^{-1}(kHk^{-1}).$$

This map sends G/H to $G/(kHk^{-1})$, so permutes the fibers of π_l . As it also preserves ν and its π_l -factor measure λ , the conjugation map must permute the fiber measures ν^H . Therefore, if ν and ν' are both bi-invariant and λ is ergodic, the function

$$H \longrightarrow \frac{d\nu^H}{d(\nu')^H}$$

is conjugation invariant, so is constant on a λ -full measure set. This implies that ν and ν' agree up to a scalar multiple, as desired. □

Bi-invariant ν do not always exist for general invariant subgroup measures λ , even in unimodular groups G . For instance, if λ is an atomic measure on a normal subgroup $N \triangleleft G$, then $\nu = \nu_N$ is just the left Haar measure on $G/N \subset \text{Cos}_G$, which is right invariant exactly when G/N is unimodular. Note that unimodular G may have nonunimodular quotients G/N : an example is

$$\text{Sol} = \mathbb{R}^2 \rtimes \mathbb{R}, \quad t \in \mathbb{R} \longmapsto \left(\begin{matrix} e^t & 0 \\ 0 & e^{-t} \end{matrix} \right) \circ \mathbb{R}^2,$$

where N is the x -axis in $\mathbb{R}^2 \subset \text{Sol}$. Although Sol is unimodular, the quotient Sol/N is the group of affine transformations

$$\text{Aff}(\mathbb{R}) = \{x \mapsto ax + b \mid a, b \in \mathbb{R}\},$$

which is not unimodular.

As suggested in the introduction, it is natural to call an IRS (or more generally, an invariant subgroup measure) λ co-unimodular if there exists a bi-invariant ν such that $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$. Using this terminology, Corollary 5.4 states that inside unimodular G , discrete IRSs are co-unimodular, as are compact ones. In general, co-unimodularity is equivalent to obeying a mass transport principle that is not twisted by the modular function.

Claim 5.5. Let λ be an invariant subgroup measure in a locally compact second countable group G . Then the following are equivalent.

- (1) There exists a G -left-invariant measure ν on Cos_G such that $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$ and such that $\rho_*\nu = \nu$ (i.e., an untwisted MTP holds for ν ; see (5.6) without the μ_G factor).

- (2) There exists a G -bi-invariant measure ν on Cos_G such that $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$ (that is, λ is co-unimodular).

Proof. Fix a left-invariant measure ν such that $\pi_{l*}(\nu) \equiv \pi_{r*}(\nu) \equiv \lambda$. If $\rho_*\nu = \nu$, then ν is bi-invariant, following the argument of Corollary 5.4. Conversely, if ν is bi-invariant, then $\nu + \rho_*\nu$ has the same properties as ν , but is also ρ -invariant. \square

APPENDIX A. SOME NOTES ON HAAR MEASURES

Suppose that G is a locally compact, second countable group and fix throughout this section a continuous, nonnegative function $f : G \rightarrow \mathbb{R}$ with compact support such that $f(1) = 1$.

Definition A.1. If H is a closed subgroup of G , we define $m_f(H)$ to be the unique left Haar measure on H such that $\int f dm_f(H) = 1$.

The measure $m_f(H)$ will exist and be unique as long as we have that the integral of f against any left Haar measure μ is nonzero and finite. The former follows since μ is positive on open subsets of H and f is positive on a neighborhood of the identity. The latter holds since μ is Radon and $\text{supp}(f|_H)$ is compact.

Claim A.2. The map $m_f : \text{Sub}_G \rightarrow \mathcal{M}(G)$ is continuous, where Sub_G has the Chabauty topology and $\mathcal{M}(G)$ the topology of weak convergence.

Proof. Let $H_i \rightarrow H$ in Sub_G and set $m_i = m_f(H_i)$. Fix any subsequence (m_{i_j}) of m_i . By [10, Proposition 2], the space of Haar measures on closed subgroups of G with the normalization $\int f dm_f(H) = 1$ is weak* compact. Therefore, (m_{i_j}) has a further subsequence that converges to some Haar measure m for a closed subgroup of G .

We claim that m is H -invariant. Let $h \in H$ and take a sequence of elements $h_{i_j} \in H_{i_j}$ with $h_{i_j} \rightarrow h$. Then for any continuous function $g : G \rightarrow \mathbb{R}$ with compact support, we have

$$\begin{aligned} \int g(hx) dm(x) &= \lim_j \int g(h_{i_j}x) dm_{i_j}(x) \\ &= \lim_j \int g(x) dm_{i_j}(x) \\ &= \int g(x) dm(x). \end{aligned}$$

The justification for the double limit is that $g(h_{i_j} \cdot) \rightarrow g(h \cdot)$ converge uniformly and all have support within some compact K , on which we have $m_{i_j}(K)$ bounded independently of j . It follows that m is H -invariant.

We now claim that m is supported within H . For given $g \in G \setminus H$, let U be an open set that is disjoint from some neighborhood of H . By the definition of the Chabauty topology, $U \cap H_{i_j} = \emptyset$ for large j . Therefore, $m_{i_j}(U) = 0$ for large j , implying that $m(U) = 0$.

This shows that m is a left Haar measure on H . Finally, as

$$\int f dm = \lim_j \int f dm_{i_j} = \lim_j 1 = 1,$$

it must be that $m = m_f(H)$, so the claim follows. \square

Lemma A.3. *Suppose that G is a locally compact topological group. Then the subset $\mathcal{U} \subset \text{Sub}_G$ consisting of subgroups H for which G/H has an invariant measure is closed.*

Proof. The space G/H admits a G -invariant measure if and only if the modular function of G restricts to the modular function of H : that is, $\mu_G(h) = \mu_H(h)$ for all $h \in H$ (see, e.g., [20]). So, suppose that we have a sequence of elements $H_i \in \mathcal{U}$ and that $H_i \rightarrow H$. If $h \in H$, we want to show that $\mu_G(h) = \mu_H(h)$. Let $h_i \in H_i$ with $h_i \rightarrow h$. Then

$$\begin{aligned} h_*m_f(H) &= \lim_i (h_i)_*m_f(H_i) \\ &= \lim_i \mu_{H_i}(h_i)m_f(H_i) \\ &= \lim_i \mu_G(h_i)m_f(H_i) \\ &= \mu_G(h)m_f(H), \end{aligned}$$

so $\mu_G(h) = \mu_H(h)$ and the lemma follows. □

APPENDIX B. UNIQUENESS OF FACTOR MEASURES

We give here a proof of the following standard uniqueness statement for factor measures in a disintegration with prescribed nonzero fiber measures.

Proposition B.1. *Let X and Y be Borel spaces, $p : X \rightarrow Y$ be a Borel map and*

$$Y \rightarrow \mathcal{M}(X), \quad y \mapsto \eta_y$$

be a Borel parametrization of a family of σ -finite Borel measures on X such that each η_y is nonzero and supported on $p^{-1}(y)$. If μ and μ' are σ -finite Borel measures on Y , let

$$\lambda = \int_Y \eta_y \, d\mu(y), \quad \lambda' = \int_Y \eta_y \, d\mu'(y).$$

Then $\lambda = \lambda'$ if and only if $\mu = \mu'$.

Proof. The backwards implication is immediate. So, assume $\lambda = \lambda'$. We claim that there is a Borel function $f : X \rightarrow \mathbb{R}$ such that

$$\int_X f \, d\eta_y = 1$$

for both μ -a.e. and μ' -a.e. $y \in Y$. This will finish the proof, since if $U \subset Y$ is Borel, then

$$\mu(U) = \int_X f \cdot 1_U(p(x)) \, d\lambda(x) = \int_X f \cdot 1_U(p(x)) \, d\lambda'(x) = \mu'(U).$$

The measure λ is σ -finite, so let $U_1 \subset U_2 \subset \dots$ be an increasing sequence of Borel subsets of X with $\lambda(U_i) < \infty$ and $\bigcup_i U_i = X$. For each $y \in Y$, let $n(y)$ be the minimum i such that $\eta_y(U_i) > 0$. Define

$$f : X \rightarrow \mathbb{R}, \quad f(x) = \sum_{y \in Y} \frac{1_{U_{n(y)} \cap p^{-1}(y)}(x)}{\eta_y(U_{n(y)})}.$$

Then f is Borel, and the claim will follow if we show that $\eta_y(U_{n(y)}) < \infty$ for both μ -a.e. and μ' -a.e. $y \in Y$. Assume by contradiction that there exists a $V \subseteq Y$ with

$\eta_y(U_{n(y)}) = \infty$ for all $y \in V$ and, say, $\mu(V) > 0$. It follows that there exists a $W \subseteq V$, with $\mu(W) > 0$, and an N such that $\eta_y(U_N) = \infty$ for all $y \in W$. But then

$$\lambda(U_N) = \int_Y \eta_y(U_N) d\mu \geq \int_W \eta_y(U_N) d\mu = \infty,$$

which contradicts the initial choice of the sets U_i . \square

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DEPARTMENT OF MATHEMATICS, BOSTON COLLEGE, CHESTNUT HILL, MASSACHUSETTS 02467

MICROSOFT RESEARCH NEW ENGLAND, ONE MEMORIAL DRIVE, CAMBRIDGE, MASSACHUSETTS 02142

Current address: Division of the Humanities and Social Sciences, California Institute of Technology, Pasadena, California 91125