

DERIVATIVE BOUNDS FOR FRACTIONAL MAXIMAL FUNCTIONS

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ABSTRACT. In this paper we study the regularity properties of fractional maximal operators acting on BV -functions. We establish new bounds for the derivative of the fractional maximal function, both in the continuous and in the discrete settings.

1. INTRODUCTION

1.1. Historical overview. Let M denote the centered Hardy-Littlewood maximal operator on \mathbb{R}^d , i.e. for $f \in L^1_{loc}(\mathbb{R}^d)$,

$$(1.1) \quad Mf(x) = \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y)| \, dy,$$

where $B_r(x)$ is the ball centered at x with radius r and $m(B_r(x))$ is its d -dimensional Lebesgue measure. One of the cornerstones of harmonic analysis is the celebrated theorem of Hardy-Littlewood-Wiener that asserts that $M : L^p(\mathbb{R}^d) \rightarrow L^p(\mathbb{R}^d)$ is bounded for $1 < p \leq \infty$. For $p = 1$ we have $M : L^1(\mathbb{R}^d) \rightarrow L^{1,\infty}(\mathbb{R}^d)$ bounded. The process of averaging a function is, in essence, a variation-diminishing process (e.g. for a fixed radius r in (1.1)), so it is natural to wonder if such behavior is preserved when taking a pointwise supremum over averages instead. This leads one to consider maximal operators acting on functions of bounded variation and Sobolev functions. The first result in this direction is due to Kinunen [11], who showed that $M : W^{1,p}(\mathbb{R}^d) \rightarrow W^{1,p}(\mathbb{R}^d)$ is bounded for $1 < p \leq \infty$, elegantly combining basic tools of functional analysis. This paradigm has been extended to multilinear, local and fractional contexts in [7, 12, 13]. Due to the lack of reflexivity of L^1 , results for $p = 1$ are subtler. Examples of such results were motivated by the following question posed in [10]:

Question A (Hajlasz and Onninen [10]). Is the operator $f \mapsto |\nabla Mf|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$?

A standard dilation argument reveals the true nature of this question: whether the variation of the maximal function is controlled by the variation of the original function, i.e. if we have

$$\|\nabla Mf\|_{L^1(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

Progress on this problem has been restricted to dimension $d = 1$. For the uncentered maximal operator (defined similarly as in (1.1), with the supremum taken over all

Received by the editors June 4, 2015.

2010 *Mathematics Subject Classification.* Primary 26A45, 42B25, 39A12, 46E35, 46E39.

Key words and phrases. Fractional maximal operator, Sobolev spaces, discrete maximal operators, bounded variation.

balls containing the point x in its closure), which we denote here by \widetilde{M} , Tanaka [22] showed that if $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{M}f$ is weakly differentiable and

$$(1.2) \quad \|(\widetilde{M}f)'\|_{L^1(\mathbb{R})} \leq 2 \|f'\|_{L^1(\mathbb{R})}.$$

This result was later refined by Aldaz and Pérez Lázaro [2], who showed that if f is of bounded variation, then $\widetilde{M}f$ is in fact absolutely continuous and

$$(1.3) \quad \text{Var}(\widetilde{M}f) \leq \text{Var}(f),$$

where $\text{Var}(f)$ denotes the total variation of f . Observe that inequality (1.3) is sharp. More recently, in the remarkable paper [14], Kurka considered the centered maximal operator in dimension $d = 1$ and proved that

$$(1.4) \quad \text{Var}(Mf) \leq 240,004 \text{Var}(f).$$

It is also shown in [14] that if $f \in W^{1,1}(\mathbb{R})$, then Mf is weakly differentiable and (1.2) also holds with constant $C = 240,004$. It is currently unknown if one can bring down the value of such constant to $C = 1$ in the centered case. The status of Question A in the general case $d > 1$ is wide open, even in establishing the weak differentiability of Mf or $\widetilde{M}f$ (related issues were considered by Hajlasz and Maly in [9]). In [8], Carneiro and Svaiter considered maximal operators of convolution type associated to smooth kernels (namely, the Gauss kernel and the Poisson kernel) and obtained the inequalities (1.2) and (1.3) with the sharp constant $C = 1$ by exploring the connections with the underlying partial differential equations. Other interesting works related to this theory are [1, 16, 17, 21].

For $0 \leq \beta < d$, we define the centered fractional maximal operator as

$$M_\beta f(x) = \sup_{r>0} \frac{1}{m(B_r(x))^{1-\frac{\beta}{d}}} \int_{B_r(x)} |f(y)| \, dy.$$

When $\beta = 0$ we plainly recover (1.1). Such fractional maximal operators have applications in potential theory and partial differential equations. By comparison with an appropriate Riesz potential, one can show that if $1 < p < \infty$, $0 < \beta < d/p$ and $q = dp/(d - \beta p)$, then $M_\beta : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$ is bounded. When $p = 1$ we have again a weak-type bound (for details, see [20, Chapter V, Theorem 1]). In [13], Kinnunen and Saksman studied the regularity properties of such fractional maximal operators. One of the results they proved [13, Theorem 2.1] is that $M_\beta : W^{1,p}(\mathbb{R}^d) \rightarrow W^{1,q}(\mathbb{R}^d)$ is bounded for p, q, β, d as described above, extending Kinnunen’s original result [11] for the case $\beta = 0$. It is then natural to consider the extension of Question A to the fractional case at the endpoint $p = 1$:

Question B. Let $0 \leq \beta < d$ and $q = d/(d - \beta)$. Is the operator $f \mapsto |\nabla M_\beta f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$?

In the case $1 \leq \beta < d$, Question B admits a positive answer, which follows from the main result of Kinnunen and Saksman in their aforementioned work [13]. In fact, [13, Theorem 3.1] states the following regularizing effect: if $f \in L^r(\mathbb{R}^d)$ with $1 < r < d$ and $1 \leq \beta < d/r$, then $M_\beta f$ is weakly differentiable and

$$(1.5) \quad |\nabla M_\beta f(x)| \leq C M_{\beta-1} f(x)$$

holds for a.e. $x \in \mathbb{R}^d$, where $C = C(d, \beta)$ is a universal constant. In our case, given $1 \leq \beta < d$ and $f \in W^{1,1}(\mathbb{R}^d)$, by the Sobolev embedding we have $f \in L^{p^*}(\mathbb{R}^d)$,

where $p^* = d/(d - 1)$, and hence $f \in L^r(\mathbb{R}^d)$ for any $1 \leq r \leq p^*$. We may choose r with $1 < r < d$ such that $1 \leq \beta < d/r$ and hence (1.5) holds. Then

$$\|\nabla M_\beta f\|_{L^q(\mathbb{R}^d)} \leq C \|M_{\beta-1} f\|_{L^q(\mathbb{R}^d)} \leq C' \|f\|_{L^{p^*}(\mathbb{R}^d)} \leq C'' \|\nabla f\|_{L^1(\mathbb{R}^d)}.$$

Question B (in the case $0 \leq \beta < 1$) is the main motivation for this work. The presence of the fractional part introduces additional difficulties as we shall see in the course of the paper (e.g. one does not necessarily have $M_\beta(f)(x) \geq |f(x)|$ a.e.). Here we give a positive answer to Question B in dimension $d = 1$ for the uncentered fractional maximal operator (which we denote by \widetilde{M}_β), both in the continuous and discrete settings. For general $d \geq 1$, in the discrete setting, we also obtain an interesting family of inequalities that approximate the conjectured bounds, for both the centered and uncentered versions (in a more general framework, where the balls are replaced by dilations of a given convex set). We now briefly state these results.

1.2. Main results.

1.2.1. *Continuous setting.* In dimension $d = 1$, for $0 \leq \beta < 1$, the uncentered fractional maximal operator is

$$(1.6) \quad \widetilde{M}_\beta f(x) = \sup_{\substack{r,s \geq 0 \\ r+s > 0}} \frac{1}{(r+s)^{1-\beta}} \int_{x-r}^{x+s} |f(y)| \, dy.$$

For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and $1 \leq q < \infty$, motivated by the Riemann sums of a Riemann integrable function, we define its q -variation as

$$(1.7) \quad \text{Var}_q(f) := \sup_{\mathcal{P}} \left(\sum_{n=1}^{N-1} \frac{|f(x_{n+1}) - f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \right)^{1/q},$$

where the supremum is taken over all finite partitions $\mathcal{P} = \{x_1 < x_2 < \dots < x_N\}$. This is also known as the *Riesz q -variation* of f (see, for instance, the discussion in [3] for this object and its generalizations). Our first result is the extension of (1.2) and (1.3) for the uncentered fractional maximal operator, which comes with an interesting regularizing effect.

Theorem 1. *Let $0 \leq \beta < 1$ and $q = 1/(1 - \beta)$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function of bounded variation such that $\widetilde{M}_\beta f \not\equiv \infty$. Then $\widetilde{M}_\beta f$ is absolutely continuous and its derivative satisfies*

$$(1.8) \quad \left\| (\widetilde{M}_\beta f)' \right\|_{L^q(\mathbb{R})} = \text{Var}_q(\widetilde{M}_\beta f) \leq 8^{1/q} \text{Var}(f).$$

The constant $C = 8^{1/q}$ appearing in (1.8) is likely not sharp. The problem of finding the sharp constant in this inequality is certainly an interesting one. Another inviting possibility is the investigation of the validity of Theorem 1 for the centered fractional maximal function, which, if confirmed, would be an extension of Kurka’s work [14]. It is worth mentioning that our strategy to approach the fractional case is very different from that of Tanaka [22] and Aldaz and Pérez Lázaro [2] for the case $\beta = 0$. In those papers the essential idea is to prove that the maximal function does not have any local maxima in the set where it disconnects from the original function. In the fractional case $\beta > 0$, the mere notion of the disconnecting set is ill-posed, since one does not necessarily have $M_\beta(f)(x) \geq |f(x)|$ a.e. anymore.

1.2.2. *Discrete setting.* We shall denote a vector $\vec{n} \in \mathbb{Z}^d$ by $\vec{n} = (n_1, n_2, \dots, n_d)$. For a function $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ (or, in general, for a vector-valued function $f : \mathbb{Z}^d \rightarrow \mathbb{R}^m$) we define its ℓ^p -norm as usual:

$$(1.9) \quad \|f\|_{\ell^p(\mathbb{Z}^d)} = \left(\sum_{\vec{n} \in \mathbb{Z}^d} |f(\vec{n})|^p \right)^{1/p},$$

if $1 \leq p < \infty$, and

$$\|f\|_{\ell^\infty(\mathbb{Z}^d)} = \sup_{\vec{n} \in \mathbb{Z}^d} |f(\vec{n})|.$$

The gradient ∇f of a discrete function $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ is the vector

$$\nabla f(\vec{n}) = \left(\frac{\partial f}{\partial x_1}(\vec{n}), \frac{\partial f}{\partial x_2}(\vec{n}), \dots, \frac{\partial f}{\partial x_d}(\vec{n}) \right),$$

where

$$\frac{\partial f}{\partial x_i}(\vec{n}) = f(\vec{n} + \vec{e}_i) - f(\vec{n}),$$

and $\vec{e}_i = (0, 0, \dots, 1, \dots, 0)$ is the canonical i -th base vector. For $f : \mathbb{Z} \rightarrow \mathbb{R}$ and $1 \leq q < \infty$, the discrete analogue of (1.7) is the q -variation defined by

$$\text{Var}_q(f) = \left(\sum_{n=-\infty}^{\infty} |f(n+1) - f(n)|^q \right)^{1/q} = \|f'\|_{\ell^q(\mathbb{Z})}.$$

For $0 \leq \beta < 1$ and $f : \mathbb{Z} \rightarrow \mathbb{R}$, we define the one-dimensional discrete uncentered fractional maximal operator by

$$(1.10) \quad \widetilde{M}_\beta f(n) = \sup_{r,s \geq 0} \frac{1}{(r+s+1)^{1-\beta}} \sum_{k=-r}^s |f(n+k)|.$$

Our next result is the discrete analogue of Theorem 1.

Theorem 2. *Let $0 \leq \beta < 1$ and $q = 1/(1 - \beta)$. Let $f : \mathbb{Z} \rightarrow \mathbb{R}$ be a function of bounded variation such that $\widetilde{M}_\beta f \not\equiv \infty$. Then*

$$\|(\widetilde{M}_\beta f)'\|_{\ell^q(\mathbb{Z})} \leq 4^{1/q} \|f'\|_{\ell^1(\mathbb{Z})}.$$

In the case of the discrete uncentered Hardy-Littlewood maximal function ($\beta = 0$ in the setting above), Theorem 2 was proved by Bober, Carneiro, Hughes and Pierce in [4], with the sharp constant $C = 1$. The analogue of Kurka’s inequality (1.4) for the discrete centered Hardy-Littlewood maximal function was established by Temur in [23] (with constant $C = 294,912,004$). As in the continuous case, the investigation of the validity of Theorem 2 for the discrete centered operator is also a very interesting problem.

1.2.3. *Operators associated to convex sets.* We now report progress related to Question B in the multidimensional discrete setting. We do this for a more general family of fractional maximal operators defined as follows. Let $\Omega \subset \mathbb{R}^d$ be a bounded open convex set with Lipschitz boundary. Let us assume that $\vec{0} \in \text{int}(\Omega)$ and that $\pm \vec{e}_i \in \overline{\Omega}$ for $1 \leq i \leq d$ (by renormalizing Ω if necessary).¹ For $r > 0$ we write

$$\overline{\Omega}_r(\vec{x}_0) = \{ \vec{x} \in \mathbb{R}^d; r^{-1}(\vec{x} - \vec{x}_0) \in \overline{\Omega} \},$$

¹This renormalization is merely aesthetic. For general Ω , one has to dilate the constants appearing in our results accordingly.

and for $r = 0$ we consider

$$\overline{\Omega}_0(\vec{x}_0) = \{\vec{x}_0\}.$$

Whenever $\vec{x}_0 = \vec{0}$ we shall write $\overline{\Omega}_r = \overline{\Omega}_r(\vec{0})$ for simplicity. This object plays the role of the “ball of center \vec{x}_0 and radius r ” in our maximal operators below. For instance, to work with regular ℓ^p -balls, one should consider $\Omega = \{\vec{x} \in \mathbb{R}^d; \|\vec{x}\|_p < 1\}$, where $\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_d|^p)^{\frac{1}{p}}$ for $\vec{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$.

Given $0 \leq \beta < d$ and $f : \mathbb{Z}^d \rightarrow \mathbb{R}$, we denote by $M_{\Omega, \beta}$ the discrete centered fractional maximal operator² associated to Ω , i.e.

$$(1.11) \quad M_{\Omega, \beta} f(\vec{n}) = \sup_{r \geq 0} \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r} |f(\vec{n} + \vec{m})|,$$

and we denote by $\widetilde{M}_{\Omega, \beta}$ its uncentered version

$$(1.12) \quad \widetilde{M}_{\Omega, \beta} f(\vec{n}) = \sup_{\overline{\Omega}_r(\vec{x}_0) \ni \vec{n}} \frac{1}{N(\vec{x}_0, r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r(\vec{x}_0)} |f(\vec{m})|,$$

where $N(\vec{x}, r)$ is the number of the lattice points in the set $\overline{\Omega}_r(\vec{x})$ (and $N(r) := N(\vec{0}, r)$). It should be understood throughout the rest of the paper that we always consider Ω -balls with at least one lattice point.

These convex Ω -balls have roughly the same behavior as the regular Euclidean balls from the geometric and arithmetic points of view. For instance, we have the following asymptotics [15, Chapter VI §2, Theorem 2] for the number of lattice points

$$(1.13) \quad N(\vec{x}, r) = C_{\Omega} r^d + O(r^{d-1})$$

as $r \rightarrow \infty$, where $C_{\Omega} = m(\Omega)$ is the d -dimensional volume of Ω , and the constant implicit in the big O notation depends only on the dimension d and on the set Ω (e.g. if Ω is the ℓ^{∞} -ball we have the exact expression $N(r) = (2\lfloor r \rfloor + 1)^d$). From (1.13) we can find a constant c_1 depending only on the dimension d and on the set Ω such that

$$(1.14) \quad N(\vec{x}, r) \leq C_{\Omega}(r + c_1)^d$$

and

$$(1.15) \quad N(\vec{x}, r) \geq \max\{C_{\Omega}(\max\{r - c_1, 0\})^d, 1\} =: C_{\Omega}(r - c_1)_+^d.$$

We define $c_2 > c_1$ as the constant such that

$$(1.16) \quad C_{\Omega}(c_2 - c_1)^d = 1.$$

Since Ω is bounded, there exists $\lambda > 0$ (depending only on Ω) such that $\overline{\Omega} \subset \overline{B}_{\lambda} = \overline{B}_{\lambda}(\vec{0})$ (note that $\lambda \geq 1$, since we assume $\pm e_i \in \overline{\Omega}$ for $1 \leq i \leq d$). This means that if $\vec{p} \in \overline{\Omega}_r(\vec{x}_0)$, then

$$(1.17) \quad |\vec{p} - \vec{x}_0| \leq \lambda r.$$

If $0 < \beta < d$ and $f : \mathbb{Z}^d \rightarrow \mathbb{R}$, we consider the discrete fractional integral operator

$$I_{\beta} f(\vec{n}) = \sum_{\vec{m} \in \mathbb{Z}^d \setminus \{\vec{0}\}} \frac{f(\vec{n} - \vec{m})}{|\vec{m}|^{d-\beta}}.$$

²We remark that the results in this section remain valid if we choose $N(r)^{-1} r^{\beta}$ instead of $N(r)^{-1+\frac{\beta}{d}}$ in the definition of our discrete fractional maximal functions.

One can show that

$$M_{\Omega,\beta}f(\vec{n}) \leq C I_\beta |f|(\vec{n}) + |f(\vec{n})|$$

for all $\vec{n} \in \mathbb{Z}^d$, where $C = C(d, \Omega, \beta)$. It is known that if $1 < p < \infty$, $0 < \beta < d/p$ and $q = dp/(d - \beta p)$, then $I_\beta : \ell^p(\mathbb{Z}^d) \rightarrow \ell^q(\mathbb{Z}^d)$ is bounded (see, for instance, Pierce’s thesis [19, Proposition 2.4]). Observe also that if $p \leq q$, then

$$(1.18) \quad \|f\|_{\ell^q(\mathbb{Z}^d)} \leq \|f\|_{\ell^p(\mathbb{Z}^d)}.$$

This plainly implies that $M_{\Omega,\beta} : \ell^p(\mathbb{Z}^d) \rightarrow \ell^q(\mathbb{Z}^d)$ is bounded for p, q, β, d as above. One can also verify the pointwise inequality

$$\left| \frac{\partial}{\partial x_i} (M_{\Omega,\beta}f)(\vec{n}) \right| \leq M_{\Omega,\beta} \left(\frac{\partial f}{\partial x_i} \right) (\vec{n})$$

for all $\vec{n} \in \mathbb{Z}^d$. Hence, for $f \in \ell^p(\mathbb{Z}^d)$, we have³

$$(1.19) \quad \|\nabla M_{\Omega,\beta}f\|_{\ell^q(\mathbb{Z}^d)} \leq C \|M_{\Omega,\beta}|\nabla f|\|_{\ell^q(\mathbb{Z}^d)} \leq C' \|\nabla f\|_{\ell^p(\mathbb{Z}^d)}.$$

Moreover, if $f, g \in \ell^p(\mathbb{Z}^d)$, we have (recall that derivation is a bounded operator in $\ell^q(\mathbb{Z}^d)$, by the triangle inequality)

$$\begin{aligned} \|\nabla M_{\Omega,\beta}f - \nabla M_{\Omega,\beta}g\|_{\ell^q(\mathbb{Z}^d)} &\leq C \|M_{\Omega,\beta}f - M_{\Omega,\beta}g\|_{\ell^q(\mathbb{Z}^d)} \\ &\leq C \|M_{\Omega,\beta}|f - g|\|_{\ell^q(\mathbb{Z}^d)} \leq C' \|f - g\|_{\ell^p(\mathbb{Z}^d)}, \end{aligned}$$

and we see that the operator $f \mapsto \nabla M_{\Omega,\beta}f$ is continuous from $\ell^p(\mathbb{Z}^d)$ to $\ell^q(\mathbb{Z}^d)$. Similar remarks apply to the uncentered version $\widetilde{M}_{\Omega,\beta}$. In relation to (1.19), the conjectured bound suggested by Question B in the discrete endpoint case $p = 1$ is the following:

Question C. Let $0 \leq \beta < d$ and $q = d/(d - \beta)$. For a discrete function $f : \mathbb{Z}^d \rightarrow \mathbb{R}$ do we have $\|\nabla M_{\Omega,\beta}f\|_{\ell^q(\mathbb{Z}^d)} \leq C(d, \Omega, \beta) \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}$?

Our next result is related to this question. In the case $0 \leq \beta < 1$ we present a family of estimates that approximate the conjectured bounds, whereas in the case $1 \leq \beta < d$ we give a positive answer (under the assumption that $f \in \ell^1(\mathbb{Z}^d)$) by adapting the methods of Kinnunen and Saksman [13] to the discrete setting. We complement these results with their corresponding continuity statements.

Theorem 3.

(i) Let $0 \leq \beta < d$ and $0 \leq \alpha \leq 1$. Let $q \geq 1$ be such that

$$(1.20) \quad q > \frac{d}{d - \beta + \alpha}.$$

Then there exists a constant $C = C(d, \Omega, \alpha, \beta, q) > 0$ such that

$$(1.21) \quad \|\nabla M_{\Omega,\beta}f\|_{\ell^q(\mathbb{Z}^d)} \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{1-\alpha} \|f\|_{\ell^1(\mathbb{Z}^d)}^\alpha \quad \forall f \in \ell^1(\mathbb{Z}^d).$$

Moreover, the operator $f \mapsto \nabla M_{\Omega,\beta}f$ is continuous from $\ell^1(\mathbb{Z}^d)$ to $\ell^q(\mathbb{Z}^d)$.

(ii) Let $1 \leq \beta < d$ and $0 \leq \alpha < 1$. Let

$$(1.22) \quad q = \frac{d}{d - \beta + \alpha}.$$

³Throughout this paper our constants may vary from line to line.

Then there exists a constant $C = C(d, \Omega, \alpha, \beta) > 0$ such that

$$(1.23) \quad \|\nabla M_{\Omega, \beta} f\|_{\ell^q(\mathbb{Z}^d)} \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{1-\alpha} \|f\|_{\ell^1(\mathbb{Z}^d)}^\alpha \quad \forall f \in \ell^1(\mathbb{Z}^d).$$

Moreover, the operator $f \mapsto \nabla M_{\Omega, \beta} f$ is continuous from $\ell^1(\mathbb{Z}^d)$ to $\ell^q(\mathbb{Z}^d)$.

The same results hold for the discrete uncentered fractional maximal operator $\widetilde{M}_{\Omega, \beta}$.

Theorem 3 extends the result of Carneiro and Hughes in [6, Theorem 1], which corresponds to the case $\beta = 0$, $\alpha = 1$ and $q = 1$. Our approach here is different and simpler than that of [6]. The boundedness part of Theorem 3 (i) for the classical discrete Hardy-Littlewood maximal operator ($\beta = 0$) and $q = 1$ was first established by Carneiro and Rogers (unpublished manuscript). It is important to observe that inequality (1.21) (and its analogue for the uncentered case) can only hold if

$$(1.24) \quad q \geq \frac{d}{d - \beta + \alpha}.$$

This is due, essentially, to a dilation argument. To see this, let us consider, for instance, the uncentered case where $\Omega = (-1, 1)^d$ is the unit open cube. Let $k \in \mathbb{N}$ and consider the cube $Q_k = [-k, k]^d$ and its characteristic function $f_k := \chi_{Q_k}$. One has $\|f_k\|_{\ell^1(\mathbb{Z}^d)} \sim_d k^d$, $\|\nabla f_k\|_{\ell^1(\mathbb{Z}^d)} \sim_d k^{d-1}$ and $\|\nabla \widetilde{M}_{\Omega, \beta} f_k\|_{\ell^q(\mathbb{Z}^d)} \gg_{\Omega, \beta, d} k^{\frac{d}{q}-1+\beta}$. One can see this last estimate by considering the region $H = \{\vec{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d; n_1 \geq 4dk; |n_i| \leq k, \text{ for } i = 2, 3, \dots, d\}$ and showing that the maximal function at $\vec{n} \in H$ is realized by the cube of side $n_1 + k$ that contains the cube Q_k . Then we sum $|\widetilde{M}_{\Omega, \beta} f_k(\vec{n} + \vec{e}_1) - \widetilde{M}_{\Omega, \beta} f_k(\vec{n})|^q$ from $n_1 = 4dk$ to ∞ , and then sum these contributions over the $\sim k^{d-1}$ possibilities for (n_2, \dots, n_d) . Letting $k \rightarrow \infty$ we obtain the necessary condition (1.24).

We may collect the cases left open in Theorem 3 in our final question:

Question D. Does the inequality (1.21) (and its analogue for the uncentered case) hold for all $\alpha \leq \beta$ and $q = d/(d - \beta + \alpha)$?

We now proceed to the proofs of these results. In doing so, we opt to consider the discrete cases first, since they describe the essence of the main ideas with fewer technicalities than the continuous cases. In Section 2 we prove the boundedness part of Theorem 3. In Section 3 we prove the continuity part of Theorem 3, a nontrivial statement that does not follow directly from the boundedness, as the maximal operators are no longer sublinear at the derivative level. In Section 4 we prove Theorem 2, and, finally, in Section 5 we adapt some of our ideas used in the discrete setting to the continuous setting and conclude by proving Theorem 1.

2. PROOF OF THEOREM 3 — BOUNDEDNESS

Throughout this section we work with the discrete maximal operators (1.11) and (1.12) associated to a convex set Ω as described in §1.2.3, and we remove the subscript Ω in some passages for simplicity. Given such a convex set Ω , let us fix the constants $C_\Omega = m(\Omega)$, c_1 , c_2 and λ as defined in (1.13)–(1.17). In proving the boundedness statements of Theorem 3 we may assume, without loss of generality, that f is nonnegative since $|\nabla|f|(\vec{n})| \leq |\nabla f(\vec{n})|$ and $M_\beta|f| = M_\beta f$ (resp. $\widetilde{M}_\beta|f| = \widetilde{M}_\beta f$).

2.1. **Centered case — part (i).** To prove (1.21) it is sufficient to show that

$$(2.1) \quad \left\| \frac{\partial M_\beta f}{\partial x_i} \right\|_{\ell^q(\mathbb{Z}^d)}^q \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(1-\alpha)q} \|f\|_{\ell^1(\mathbb{Z}^d)}^{\alpha q}$$

for each $i = 1, 2, \dots, d$. We will work with $i = d$, and the other cases are analogous. Since $f \in \ell^1(\mathbb{Z}^d)$, for each $\vec{n} \in \mathbb{Z}^d$ there exists $r \geq 0$ such that

$$(2.2) \quad M_\beta f(\vec{n}) = A_r f(\vec{n}) := \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r} f(\vec{n} + \vec{m}).$$

Let $\ell(\vec{n})$ be the minimum $\ell \in \mathbb{Z}^+$ such that there exists $r \geq 0$ which satisfies (2.2) and $\lceil r \rceil = \ell(\vec{n})$. Also, let $r(\vec{n}) \geq 0$ be such that $r(\vec{n})$ satisfies (2.2) and $\lceil r(\vec{n}) \rceil = \ell(\vec{n})$.

For all $k \in \mathbb{Z}^+$ we define the set X_k^+ by

$$(2.3) \quad X_k^+ := \{ \vec{n} \in \mathbb{Z}^d : M_\beta f(\vec{n} + \vec{e}_d) \geq M_\beta f(\vec{n}) \text{ and } \ell(\vec{n} + \vec{e}_d) = k \}$$

and the set X_k^- by

$$(2.4) \quad X_k^- := \{ \vec{n} \in \mathbb{Z}^d : M_\beta f(\vec{n} + \vec{e}_d) < M_\beta f(\vec{n}) \text{ and } \ell(\vec{n}) = k \}.$$

Hence, we may write

$$\begin{aligned} \sum_{\vec{n} \in \mathbb{Z}^d} \left| \frac{\partial}{\partial x_d} M_\beta f(\vec{n}) \right|^q &= \sum_{k \geq 0} \sum_{\vec{n} \in X_k^+} (M_\beta f(\vec{n} + \vec{e}_d) - M_\beta f(\vec{n}))^q \\ &\quad + \sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} (M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d))^q. \end{aligned}$$

We will prove that

$$(2.5) \quad \sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} (M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d))^q \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha},$$

and, analogously, we will have

$$(2.6) \quad \sum_{k \geq 0} \sum_{\vec{n} \in X_k^+} (M_\beta f(\vec{n} + \vec{e}_d) - M_\beta f(\vec{n}))^q \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha}.$$

Inequalities (2.5) and (2.6) then imply (2.1) for $i = d$.⁴ To show (2.5), first we note that for $\vec{n} \in X_k^-$ and $r = r(\vec{n})$ we have

$$\begin{aligned} (2.7) \quad M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d) &\leq A_r f(\vec{n}) - A_r f(\vec{n} + \vec{e}_d) \\ &= \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r} f(\vec{n} + \vec{m}) - \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r} f(\vec{n} + \vec{e}_d + \vec{m}) \\ &\leq \frac{1}{N+(k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_k} |f(\vec{n} + \vec{m}) - f(\vec{n} + \vec{e}_d + \vec{m})|, \end{aligned}$$

⁴Recall that our constants may vary from line to line.

where $N^+(\vec{x}, \tau) := \max\{N(\vec{x}, \tau), 1\}$ and $N^+(\tau) = N^+(\vec{0}, \tau)$.⁵ On the other hand,

$$\begin{aligned}
 M_\beta f(\vec{n} + \vec{e}_d) &\geq \frac{1}{N(k+1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_{k+1}} f(\vec{n} + \vec{e}_d + \vec{m}) \\
 (2.8) \qquad &\geq \frac{N(k+1)^{\frac{\beta}{d}}}{N(k+1)} \sum_{\vec{m} \in \overline{\Omega}_k} f(\vec{n} + \vec{m}) \\
 &\geq \frac{N(r)}{N(k+1)} M_\beta f(\vec{n}).
 \end{aligned}$$

In the second inequality above we have used the convexity of Ω and the fact that $-\vec{e}_d \in \overline{\Omega}$ to conclude that $\overline{\Omega}_k(\vec{n}) \subset \overline{\Omega}_{k+1}(\vec{n} + \vec{e}_d)$. Using (2.8) we get

$$\begin{aligned}
 (2.9) \qquad M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d) &\leq \left(1 - \frac{N(r)}{N(k+1)}\right) M_\beta f(\vec{n}) \\
 &\leq \frac{N(k+1) - N^+(k-1)}{N(k+1)} \frac{1}{N^+(k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_k} f(\vec{n} + \vec{m}).
 \end{aligned}$$

Putting together the estimates (2.7) and (2.9), we see that $(M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d))^q$ is bounded above by the product⁶

$$\begin{aligned}
 (2.10) \qquad &\left(\frac{1}{N^+(k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_k} |f(\vec{n} + \vec{m}) - f(\vec{n} + \vec{e}_d + \vec{m})| \right)^{q(1-\alpha)} \\
 &\times \left(\frac{N(k+1) - N^+(k-1)}{N(k+1)} \frac{1}{N^+(k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_k} f(\vec{n} + \vec{m}) \right)^{q\alpha}.
 \end{aligned}$$

By (1.14) and (1.15), there is a positive constant C such that

$$(2.11) \qquad \frac{N(\vec{x}, k+1) - N^+(\vec{x}, k-1)}{N(\vec{x}, k+1)} \leq \frac{C}{N^+(\vec{x}, k-1)^{\frac{1}{d}}} \quad \forall k \in \mathbb{Z}^+; \forall \vec{x} \in \mathbb{R}^d.$$

Let

$$(2.12) \qquad \left(1 - \frac{\beta}{d}\right) q + \frac{\alpha q}{d} = 1 + \gamma.$$

⁵Here we formally include the possibility of having $\tau < 0$, with the understanding that, in this case, $N^+(\vec{x}, \tau) = 1$.

⁶If $\alpha = 0$ or 1 , it is understood that we have only one term in this product. The modifications for the rest of the proof are standard.

From inequality (1.20) we have $\gamma > 0$. Then it follows from (2.10) and (2.11) that $(M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d))^q$ is bounded above by the product

$$\left(\frac{1}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{\Omega}_k} |f(\vec{n} + \vec{m}) - f(\vec{n} + \vec{e}_d + \vec{m})| \right)^q \right)^{(1-\alpha)} \times \left(\frac{C}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{\Omega}_k} f(\vec{n} + \vec{m}) \right)^q \right)^\alpha.$$

Thus, by Hölder’s inequality with exponents $p = \frac{1}{1-\alpha}$ and $p' = \frac{1}{\alpha}$, we see that the left-hand side of (2.5) is bounded by

$$\left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{1}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{\Omega}_k} |f(\vec{n} + \vec{m}) - f(\vec{n} + \vec{e}_d + \vec{m})| \right)^q \right]^{1-\alpha} \times \left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{C}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{\Omega}_k} f(\vec{n} + \vec{m}) \right)^q \right]^\alpha.$$

Since $q \geq 1$, this last product is bounded by

$$\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{1}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{\Omega}_k} |f(\vec{n} + \vec{m}) - f(\vec{n} + \vec{e}_d + \vec{m})| \right)^q \right]^{1-\alpha} \times \|f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{C}{N^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{\Omega}_k} f(\vec{n} + \vec{m}) \right)^q \right]^\alpha.$$

By Fubini’s theorem and (1.17), this is bounded by

$$\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \sum_{k \geq \frac{|\vec{m}|}{\lambda}} \frac{1}{N^+(k-1)^{1+\gamma}} \sum_{\vec{n} \in X_k^-} |f(\vec{n} + \vec{m}) - f(\vec{n} + \vec{e}_d + \vec{m})| \right]^{1-\alpha} \times \|f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \sum_{k \geq \frac{|\vec{m}|}{\lambda}} \frac{C}{N^+(k-1)^{1+\gamma}} \sum_{\vec{n} \in X_k^-} f(\vec{n} + \vec{m}) \right]^\alpha,$$

which turns out to be bounded by

(2.13)

$$\|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \frac{1}{N^+(\frac{|\vec{m}|}{\lambda} - 1)^{1+\gamma}} \sum_{k \geq \frac{|\vec{m}|}{\lambda}} \sum_{\vec{n} \in X_k^-} |f(\vec{n} + \vec{m}) - f(\vec{n} + \vec{e}_d + \vec{m})| \right]^{1-\alpha} \times \|f\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \frac{C}{N^+(\frac{|\vec{m}|}{\lambda} - 1)^{1+\gamma}} \sum_{k \geq \frac{|\vec{m}|}{\lambda}} \sum_{\vec{n} \in X_k^-} f(\vec{n} + \vec{m}) \right]^\alpha.$$

Since the sets X_k^- are pairwise disjoint, this is bounded above by

$$(2.14) \quad \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \frac{1}{N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right)^{1+\gamma}} \right]^{1-\alpha} \times \|f\|_{\ell^1(\mathbb{Z}^d)}^{\alpha} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \frac{C}{N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right)^{1+\gamma}} \right]^{\alpha}.$$

By (1.15) we know that

$$N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right) \geq \max \left\{ C_{\Omega} \left(\max \left\{ \frac{|\vec{m}|}{\lambda} - 1 - c_1, 0 \right\} \right)^d, 1 \right\} \quad \forall \vec{m} \in \mathbb{Z}^d.$$

This implies that both sums in brackets in (2.14) are finite, and we arrive at the desired estimate (2.5). Using (1.14) we see that

$$\sum_{\vec{m} \in \mathbb{Z}^d} \frac{1}{N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right)^{1+\gamma}} \rightarrow \infty \quad \text{as } \gamma \rightarrow 0^+,$$

which implies that our constant C in (1.21) blows up when we approach the case of equality in (1.24).

2.2. Uncentered case — part (i). Here it suffices to show that

$$(2.15) \quad \left\| \frac{\partial \widetilde{M}_{\beta} f}{\partial x_i} \right\|_{\ell^q(\mathbb{Z}^d)}^q \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{(1-\alpha)q} \|f\|_{\ell^1(\mathbb{Z}^d)}^{\alpha q}$$

for each $i = 1, 2, \dots, d$. We work again with $i = d$, and the other cases are analogous. Since $f \in \ell^1(\mathbb{Z}^d)$, for each $\vec{n} \in \mathbb{Z}^d$ we can take a point $\vec{x} = \vec{x}(\vec{n}) \in \mathbb{R}^d$ and a radius $r = r(\vec{n})$ such that $\vec{n} \in \overline{\Omega}_r(\vec{x})$, and the fractional average over the set $\overline{\Omega}_r(\vec{x})$ realizes the supremum in the maximal function, i.e.

$$(2.16) \quad \widetilde{M}_{\beta} f(\vec{n}) = A_{(\vec{x}, r)} f(\vec{n}) := \frac{1}{N(\vec{x}, r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r(\vec{x})} f(\vec{m}).$$

Let $\tilde{\ell}(\vec{n})$ be the minimum $\tilde{\ell} \in \mathbb{Z}^+$ such that there is a pair (\vec{x}, r) that verifies the equality (2.16) with $\lceil r \rceil = \tilde{\ell}(\vec{n})$. Let $r(\vec{n})$ be such a radius and $\vec{x}(\vec{n})$ be such a center.

For $k \in \mathbb{Z}^+$ we now define the set \widetilde{X}_k^+ by

$$\widetilde{X}_k^+ := \{ \vec{n} \in \mathbb{Z}^d : \widetilde{M}_{\beta} f(\vec{n} + \vec{e}_d) \geq \widetilde{M}_{\beta} f(\vec{n}) \text{ and } \tilde{\ell}(\vec{n} + \vec{e}_d) = k \}$$

and the set \widetilde{X}_k^- by

$$\widetilde{X}_k^- := \{ \vec{n} \in \mathbb{Z}^d : \widetilde{M}_{\beta} f(\vec{n} + \vec{e}_d) < \widetilde{M}_{\beta} f(\vec{n}) \text{ and } \tilde{\ell}(\vec{n}) = k \}.$$

Hence,

$$\begin{aligned} \sum_{\vec{n} \in \mathbb{Z}^d} \left| \frac{\partial}{\partial x_d} \widetilde{M}_{\beta} f(\vec{n}) \right|^q &= \sum_{k \geq 0} \sum_{\vec{n} \in X_k^+} (\widetilde{M}_{\beta} f(\vec{n} + \vec{e}_d) - \widetilde{M}_{\beta} f(\vec{n}))^q \\ &\quad + \sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} (\widetilde{M}_{\beta} f(\vec{n}) - \widetilde{M}_{\beta} f(\vec{n} + \vec{e}_d))^q. \end{aligned}$$

We will prove that

$$(2.17) \quad \sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} (\widetilde{M}_\beta f(\vec{n}) - \widetilde{M}_\beta f(\vec{n} + \vec{e}_d))^q \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha},$$

and, analogously, we will have

$$(2.18) \quad \sum_{k \geq 0} \sum_{\vec{n} \in X_k^+} (\widetilde{M}_\beta f(\vec{n} + \vec{e}_d) - \widetilde{M}_\beta f(\vec{n}))^q \leq C \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{q(1-\alpha)} \|f\|_{\ell^1(\mathbb{Z}^d)}^{q\alpha}.$$

As a consequence of (2.17) and (2.18) we will obtain (2.15) for $i = d$, as desired.

For $\vec{n} \in X_k^-$ let $r = r(\vec{n})$ and $\vec{x} = \vec{x}(\vec{n})$. Then

$$\begin{aligned} \widetilde{M}_\beta f(\vec{n}) - \widetilde{M}_\beta f(\vec{n} + \vec{e}_d) &\leq A_{(\vec{x}, r)} f(\vec{n}) - A_{(\vec{x} + \vec{e}_d, r)} f(\vec{n} + \vec{e}_d) \\ &= \frac{1}{N(\vec{x}, r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r(\vec{x})} f(\vec{m}) - \frac{1}{N(\vec{x} + \vec{e}_d, r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r(\vec{x} + \vec{e}_d)} f(\vec{m}) \\ &\leq \frac{1}{N(\vec{x}, r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r(\vec{x})} |f(\vec{m}) - f(\vec{m} + \vec{e}_d)| \\ &\leq \frac{1}{N^+(\vec{x}, k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_k(\vec{x})} |f(\vec{m}) - f(\vec{m} + \vec{e}_d)|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \widetilde{M}_\beta f(\vec{n} + \vec{e}_d) &\geq \frac{1}{N(\vec{x} + \vec{e}_d, k+1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_{k+1}(\vec{x} + \vec{e}_d)} f(\vec{m}) \\ &\geq \frac{N(\vec{x}, k+1)^{\frac{\beta}{d}}}{N(\vec{x}, k+1)} \sum_{\vec{m} \in \Omega_k(\vec{x})} f(\vec{m}) \\ &\geq \frac{N(\vec{x}, r)}{N(\vec{x}, k+1)} \widetilde{M}_\beta f(\vec{n}). \end{aligned}$$

In the second inequality above we have used the convexity of Ω and the fact that $-\vec{e}_d \in \overline{\Omega}$ to conclude that $\overline{\Omega}_k(\vec{x}) \subset \overline{\Omega}_{k+1}(\vec{x} + \vec{e}_d)$. Therefore,

$$\begin{aligned} \widetilde{M}_\beta f(\vec{n}) - \widetilde{M}_\beta f(\vec{n} + \vec{e}_d) &\leq \left(1 - \frac{N(\vec{x}, r)}{N(\vec{x}, k+1)}\right) \widetilde{M}_\beta f(\vec{n}) \\ &\leq \frac{N(\vec{x}, k+1) - N^+(\vec{x}, k-1)}{N(\vec{x}, k+1)} \frac{1}{N^+(\vec{x}, k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_k(\vec{x})} f(\vec{m}). \end{aligned}$$

With these two inequalities, we can see that $(\widetilde{M}_\beta f(\vec{n}) - \widetilde{M}_\beta f(\vec{n} + \vec{e}_d))^q$ is bounded above by the product

$$(2.19) \quad \left(\frac{1}{N^+(\vec{x}, k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_k(\vec{x})} |f(\vec{m}) - f(\vec{m} + \vec{e}_d)| \right)^{q(1-\alpha)} \times \left(\frac{N(\vec{x}, k+1) - N^+(\vec{x}, k-1)}{N(\vec{x}, k+1)} \frac{1}{N^+(\vec{x}, k-1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_k(\vec{x})} f(\vec{m}) \right)^{q\alpha}.$$

Let $\gamma > 0$ be defined as in (2.12). Using (2.11), we conclude that (2.19) is bounded above by the product

$$\left(\frac{1}{N^+(\vec{x}, k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \Omega_k(\vec{x})} |f(\vec{m}) - f(\vec{m} + \vec{e}_d)| \right)^q \right)^{1-\alpha} \times \left(\frac{C}{N^+(\vec{x}, k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \Omega_k(\vec{x})} f(\vec{m}) \right)^q \right)^\alpha.$$

For each $r \in \mathbb{R}$, let $\tilde{N}^+(r) := \min \{N^+(\vec{x}, r); \vec{x} \in \mathbb{R}^d\}$. Thus, by Hölder’s inequality with exponents $p = \frac{1}{1-\alpha}$ and $p' = \frac{1}{\alpha}$ we see that the left-hand side of (2.17) is bounded by

$$\left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{1}{\tilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \Omega_k(\vec{x}(\vec{n}))} |f(\vec{m}) - f(\vec{m} + \vec{e}_d)| \right)^q \right]^{1-\alpha} \times \left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{C}{\tilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \Omega_k(\vec{x}(\vec{n}))} f(\vec{m}) \right)^q \right]^\alpha.$$

In turn, this is bounded by

$$\left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{1}{\tilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{B}_{2k\lambda}(\vec{n})} |f(\vec{m}) - f(\vec{m} + \vec{e}_d)| \right)^q \right]^{1-\alpha} \times \left[\sum_{k \geq 0} \sum_{\vec{n} \in X_k^-} \frac{C}{\tilde{N}^+(k-1)^{1+\gamma}} \left(\sum_{\vec{m} \in \overline{B}_{2k\lambda}(\vec{n})} f(\vec{m}) \right)^q \right]^\alpha,$$

since $\overline{\Omega} \subset \overline{B}_\lambda$ and thus $\overline{\Omega}_k(\vec{x}(\vec{n})) \subset \overline{B}_{k\lambda}(\vec{x}(\vec{n})) \subset \overline{B}_{2k\lambda}(\vec{n})$. The remaining steps of the proof are analogous to the centered case in §2.1.

2.3. Centered case — part (ii). In the case $1 \leq \beta < d$, the operator M_β has a certain regularizing effect. This was observed in [13, Theorem 3.1] in the continuous setting. In what follows we adapt their argument to the discrete setting. Let $\vec{n} \in \mathbb{Z}^d$ and assume that $M_\beta f(\vec{n}) \geq M_\beta f(\vec{n} + \vec{e}_d)$. Since we are assuming that $f \in \ell^1(\mathbb{Z}^d)$, there exists $r \geq 0$ such that

$$M_\beta f(\vec{n}) = A_r f(\vec{n}) = \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \Omega_r} f(\vec{n} + \vec{m}).$$

Proceeding as in (2.8) we find that

$$\begin{aligned} M_\beta f(\vec{n} + \vec{e}_d) &\geq \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_{r+1}} f(\vec{n} + \vec{e}_d + \vec{m}) \\ &\geq \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r} f(\vec{n} + \vec{m}). \end{aligned}$$

Hence

$$(2.20) \quad M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d) \leq \left(\frac{1}{N(r)^{1-\frac{\beta}{d}}} - \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \right) \sum_{\vec{m} \in \bar{\Omega}_r} f(\vec{n} + \vec{m}).$$

We claim that

$$(2.21) \quad \left(\frac{1}{N(r)^{1-\frac{\beta}{d}}} - \frac{1}{N(r+1)^{1-\frac{\beta}{d}}} \right) \leq C \frac{1}{N(r)^{1-\frac{\beta-1}{d}}}.$$

This is certainly true for small r , whereas for large r we may take c_1 in (1.14) and (1.15) to be strictly smaller than $\frac{1}{2}$ and use the mean value theorem (after clearing the denominators). From (2.20) and (2.21) we find that

$$M_\beta f(\vec{n}) - M_\beta f(\vec{n} + \vec{e}_d) \leq C M_{\beta-1} f(\vec{n}).$$

If $M_\beta f(\vec{n}) < M_\beta f(\vec{n} + \vec{e}_d)$ an analogous argument leads to

$$M_\beta f(\vec{n} + \vec{e}_d) - M_\beta f(\vec{n}) \leq C M_{\beta-1} f(\vec{n} + \vec{e}_d).$$

Moreover, we may replace \vec{e}_d by any other basis vector \vec{e}_j . This leads to the pointwise bound

$$(2.22) \quad |\nabla M_\beta f(\vec{n})| \leq C \left\{ M_{\beta-1} f(\vec{n}) + \sum_{j=1}^d M_{\beta-1} f(\vec{n} + \vec{e}_j) \right\}.$$

Finally, letting $r = d/(d + \alpha - 1)$ and $p^* = d/(d - 1)$, we use the boundedness of $M_{\beta-1}$, interpolation and the Sobolev embedding (for which the proof in the discrete setting is analogous to the proof in the continuous setting as in [20, Chapter V, §2.5]) to obtain

$$\begin{aligned} \|\nabla M_\beta f\|_{\ell^q(\mathbb{Z}^d)} &\leq C \|M_{\beta-1} f\|_{\ell^q(\mathbb{Z}^d)} \leq C' \|f\|_{\ell^r(\mathbb{Z}^d)} \\ &\leq C' \|f\|_{\ell^{p^*}(\mathbb{Z}^d)}^{1-\alpha} \|f\|_{\ell^1(\mathbb{Z}^d)}^\alpha \leq C'' \|\nabla f\|_{\ell^1(\mathbb{Z}^d)}^{1-\alpha} \|f\|_{\ell^1(\mathbb{Z}^d)}^\alpha. \end{aligned}$$

This concludes the proof of this part.

2.4. Uncentered case — part (ii). The proof in this case is analogous to §2.3, establishing the pointwise bound (2.22) for the uncentered operator \widetilde{M}_β . We leave the details to the interested reader.

3. PROOF OF THEOREM 3 — CONTINUITY

In this section we keep working with the discrete maximal operators $M_\beta = M_{\Omega, \beta}$ (resp. $\widetilde{M}_\beta = \widetilde{M}_{\Omega, \beta}$). Given our convex set Ω as described in §1.2.3, we fix the constants $C_\Omega = m(\Omega)$, c_1 , c_2 and λ as defined in (1.13) - (1.17).

3.1. Centered case — part (i). We want to show that if $f_j \rightarrow f$ in $\ell^1(\mathbb{Z}^d)$, then $\nabla M_\beta f_j \rightarrow \nabla M_\beta f$ in $\ell^q(\mathbb{Z}^d)$. From the fact that $||f_j| - |f|| \leq |f_j - f|$, we may assume without loss of generality that $f_j \geq 0$ for all j and that $f \geq 0$. It suffices to show that

$$(3.1) \quad \left\| \frac{\partial}{\partial x_i} M_\beta f_j - \frac{\partial}{\partial x_i} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)} \rightarrow 0$$

as $j \rightarrow \infty$, for each $i = 1, 2, \dots, d$. As before, we will prove it for $i = d$, and the other cases are analogous.

3.1.1. *Convergence of radii.* Given a function $g \in \ell^1(\mathbb{Z}^d)$ and a point $\vec{n} \in \mathbb{Z}^d$, we first study the set of radii that realize the supremum in the fractional maximal function at the point \vec{n} . Let us define

$$\mathcal{R}g(\vec{n}) = \left\{ r \in [0, \infty); M_\beta g(\vec{n}) = A_r |g|(\vec{n}) = \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r} |g(\vec{n} + \vec{m})| \right\}.$$

We prove the following auxiliary lemma, which can be seen as a discrete fractional analogue of a lemma of Luiro [16, Lemma 2.2].

Lemma 4. *Let $f_j \rightarrow f$ in $\ell^1(\mathbb{Z}^d)$ and let $R > 0$. There exists j_0 such that for $j \geq j_0$ we have $\mathcal{R}f_j(\vec{n}) \subset \mathcal{R}f(\vec{n})$ for each $\vec{n} \in \overline{B}_R$.*

Proof. Fix $\vec{n} \in \overline{B}_R$ and consider the map $r \mapsto A_r f(\vec{n})$. Since $f \in \ell^1(\mathbb{Z}^d)$, there is only a finite number of values in the image set $\{A_r f(\vec{n}); r \geq 0\}$ such that $A_r f(\vec{n}) \geq \frac{1}{2} M_\beta f(\vec{n})$. Therefore, there exists $\varepsilon(\vec{n}) > 0$ such that if $A_r f(\vec{n}) > M_\beta f(\vec{n}) - \varepsilon(\vec{n})$, then $A_r f(\vec{n}) = M_\beta f(\vec{n})$ and $r \in \mathcal{R}f(\vec{n})$. Let

$$\varepsilon = \frac{1}{3} \min \{ \varepsilon(\vec{n}); \vec{n} \in \overline{B}_R \}.$$

From (1.18) we see that $\|f_j - f\|_{\ell^1(\mathbb{Z}^d)} \rightarrow 0$ implies that $\|f_j - f\|_{\ell^a(\mathbb{Z}^d)} \rightarrow 0$ for each $a \geq 1$. In particular, there exists j_0 such that $\|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} < \varepsilon$ for all $j \geq j_0$ (if $\beta = 0$, then $d/\beta = \infty$). By Hölder’s inequality,

(3.2)

$$\begin{aligned} |A_r f_j(\vec{n}) - A_r f(\vec{n})| &\leq \frac{1}{N(r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \overline{\Omega}_r} |f_j(\vec{n} + \vec{m}) - f(\vec{n} + \vec{m})| \\ &\leq \frac{1}{N(r)^{1-\frac{\beta}{d}}} \left(\sum_{\vec{m} \in \overline{\Omega}_r} |f_j(\vec{n} + \vec{m}) - f(\vec{n} + \vec{m})|^{\frac{d}{\beta}} \right)^{\frac{\beta}{d}} N(r)^{1-\frac{\beta}{d}} \\ &\leq \|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} \\ &\leq \varepsilon \end{aligned}$$

for all $r \geq 0$ and $j \geq j_0$. Hence, for any $\vec{n} \in \overline{B}_R$ and $j \geq j_0$, if we take $s \in \mathcal{R}f(\vec{n})$ we get

$$(3.3) \quad M_\beta f(\vec{n}) = A_s f(\vec{n}) = A_s f_j(\vec{n}) + A_s(f - f_j)(\vec{n}) \leq M_\beta f_j(\vec{n}) + \varepsilon.$$

For any $\vec{n} \in \overline{B}_R$, $j \geq j_0$ and $r_j \in \mathcal{R}f_j(\vec{n})$, we use (3.2) and (3.3) to obtain

$$A_{r_j} f(\vec{n}) = A_{r_j}(f - f_j)(\vec{n}) + A_{r_j} f_j(\vec{n}) \geq M_\beta f(\vec{n}) - 2\varepsilon.$$

From the definition of ε we must have $r_j \in \mathcal{R}f(\vec{n})$, which completes the proof of the lemma. □

3.1.2. *Brezis-Lieb reduction.* For a fixed $\vec{n} \in \mathbb{Z}^d$, given $\varepsilon > 0$, we use Lemma 4 to find j_0 such that if $j \geq j_0$, we have $\mathcal{R}f_j(\vec{n}) \subset \mathcal{R}f(\vec{n})$ and $\|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} < \varepsilon$.

Taking $r_j \in \mathcal{R}f_j(\vec{n})$ and using (3.2) we get

$$|M_\beta f_j(\vec{n}) - M_\beta f(\vec{n})| = |A_{r_j} f_j(\vec{n}) - A_{r_j} f(\vec{n})| \leq \varepsilon$$

for $j \geq j_0$, and therefore $M_\beta f_j(\vec{n}) \rightarrow M_\beta f(\vec{n})$ as $j \rightarrow \infty$. Hence it follows that

$$(3.4) \quad \frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) \rightarrow \frac{\partial}{\partial x_d} M_\beta f(\vec{n})$$

as $j \rightarrow \infty$. From the classical Brezis-Lieb lemma [5], in order to prove (3.1) it suffices to prove the convergence of the norms, i.e.

$$\lim_{j \rightarrow \infty} \left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)} = \left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)}.$$

From Fatou’s lemma and the pointwise convergence in (3.4) we have

$$\left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)} \leq \liminf_{j \rightarrow \infty} \left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)}.$$

We prove the opposite inequality in the next subsection.

3.1.3. *Upper bound.* Let $\varepsilon_0 > 0$ be given. Our goal now is to find j_0 such that, for $j \geq j_0$,

$$(3.5) \quad \left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)} \leq \left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(\mathbb{Z}^d)} + \varepsilon_0.$$

Let R be a sufficiently large radius (to be properly chosen later) and let $Q_{2R} = \{\vec{x} \in \mathbb{R}^d : \|\vec{x}\|_\infty \leq 2R\}$ be the cube of side $4R$. We write

$$(3.6) \quad \begin{aligned} \left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)}^q &= \sum_{\|\vec{n}\|_\infty \leq 2R} \left| \frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) \right|^q + \sum_{\|\vec{n}\|_\infty > 2R} \left| \frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) \right|^q \\ &=: S_1 + S_2. \end{aligned}$$

We bound S_1 and S_2 separately.

Let $\varepsilon_1 > 0$ (to be properly chosen later). By Lemma 4, there exists j_1 such that if $j \geq j_1$, we have $\mathcal{R}f_j(\vec{n}) \subset \mathcal{R}f(\vec{n})$ for each \vec{n} with $\|\vec{n}\|_\infty \leq 2R + 1$ and

$$\|f_j - f\|_{\ell^{\frac{d}{\beta}}(\mathbb{Z}^d)} \leq \varepsilon_1.$$

By (3.2) we obtain that

$$\left| \frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) - \frac{\partial}{\partial x_d} M_\beta f(\vec{n}) \right| \leq 2\varepsilon_1,$$

for any $\vec{n} \in Q_{2R}$. Hence, by the triangle inequality,

$$(3.7) \quad S_1 \leq \left(\left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(Q_{2R})} + 2\varepsilon_1(4R + 1)^{\frac{d}{q}} \right)^q.$$

In order to bound S_2 , we recall the definition of the sets X_k^+ and X_k^- in (2.3) and (2.4). We then write

$$\begin{aligned} S_2 &= \sum_{k \geq 0} \sum_{\substack{\|\vec{n}\|_\infty > 2R \\ \vec{n} \in X_k^+}} (M_\beta f_j(\vec{n} + \vec{e}_d) - M_\beta f_j(\vec{n}))^q \\ &\quad + \sum_{k \geq 0} \sum_{\substack{\|\vec{n}\|_\infty > 2R \\ \vec{n} \in X_k^-}} (M_\beta f_j(\vec{n}) - M_\beta f_j(\vec{n} + \vec{e}_d))^q \\ &=: S_2^+ + S_2^-. \end{aligned}$$

From the calculations leading to (2.13) we have

$$S_2^- \leq \|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \frac{1}{N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right)^{1+\gamma}} \right. \\ \left. \times \sum_{k \geq \frac{|\vec{m}|}{\lambda}} \sum_{\substack{\|\vec{n}\|_\infty > 2R \\ \vec{n} \in X_k^-}} |f_j(\vec{n} + \vec{m}) - f_j(\vec{n} + \vec{e}_d + \vec{m})| \right]^{1-\alpha} \\ \times \|f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left[\sum_{\vec{m} \in \mathbb{Z}^d} \frac{C}{N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right)^{1+\gamma}} \sum_{k \geq \frac{|\vec{m}|}{\lambda}} \sum_{\vec{n} \in X_k^-} f_j(\vec{n} + \vec{m}) \right]^\alpha.$$

We have already noted after (2.14) that the sum $\sum_{\vec{m} \in \mathbb{Z}^d} \frac{1}{N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right)^{1+\gamma}}$ is finite.

We now consider two cases: (i) when $\|\vec{m}\|_\infty < R$, which implies that $\vec{n} + \vec{m} \in Q_R^c$; (ii) $\|\vec{m}\|_\infty \geq R$. We define the function

$$h(R) = \sum_{\|\vec{m}\|_\infty \geq R} \frac{1}{N^+ \left(\frac{|\vec{m}|}{\lambda} - 1\right)^{1+\gamma}}$$

and obtain the following upper bound (recall that the sets X_k^- are pairwise disjoint):

$$(3.8) \quad S_2^- \leq \|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left(C \|\nabla f_j\|_{\ell^1(Q_R^c)} + h(R) \|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)} \right)^{(1-\alpha)} \\ \times \|f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left(C \|f_j\|_{\ell^1(Q_R^c)} + h(R) \|f_j\|_{\ell^1(\mathbb{Z}^d)} \right)^\alpha.$$

The same bound as in (3.8) holds for S_2^+ . Putting together (3.6), (3.7) and (3.8) (this last one duplicated, for S_2^- and for S_2^+) we obtain

$$\left\| \frac{\partial}{\partial x_d} M_\beta f_j \right\|_{\ell^q(\mathbb{Z}^d)}^q \leq \left(\left\| \frac{\partial}{\partial x_d} M_\beta f \right\|_{\ell^q(Q_{2R})} + 2\varepsilon_1 (4R + 1)^{\frac{d}{q}} \right)^q \\ + 2 \left\{ \|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)(1-\alpha)} \left(C \|\nabla f_j\|_{\ell^1(Q_R^c)} + h(R) \|\nabla f_j\|_{\ell^1(\mathbb{Z}^d)} \right)^{(1-\alpha)} \right. \\ \left. \times \|f_j\|_{\ell^1(\mathbb{Z}^d)}^{(q-1)\alpha} \left(C \|f_j\|_{\ell^1(Q_R^c)} + h(R) \|f_j\|_{\ell^1(\mathbb{Z}^d)} \right)^\alpha \right\}.$$

The crucial point in our argument is the fact that $h(R) \rightarrow 0$ as $R \rightarrow \infty$. It is now clear that we can choose R sufficiently large to make the second term above very small (for j large), and then we choose ε_1 sufficiently small to arrive at (3.5). This completes the proof in the centered case.

3.2. Uncentered case — part (i). Minor modifications are needed in comparison to the previous argument. For a function $g \in \ell^1(\mathbb{Z}^d)$ and a point $\vec{n} \in \mathbb{Z}^d$, we now define

$$\tilde{\mathcal{R}}g(\vec{n}) = \left\{ (\vec{x}, r) \in \mathbb{R}^d \times \mathbb{R}^+; \tilde{M}_\beta g(\vec{n}) = A_{(\vec{x}, r)} |g|(\vec{n}) = \frac{1}{N(\vec{x}, r)^{1-\frac{\beta}{d}}} \sum_{\vec{m} \in \tilde{\Omega}_r(\vec{x})} |g(\vec{m})| \right\}.$$

The following lemma can be proved with the same ideas used in the proof of Lemma 4.

Lemma 5. *Let $f_j \rightarrow f$ in $\ell^1(\mathbb{Z}^d)$ and let $R > 0$. There exists j_0 such that for $j \geq j_0$ we have $\tilde{\mathcal{R}}f_j(\vec{n}) \subset \tilde{\mathcal{R}}f(\vec{n})$ for each $\vec{n} \in \overline{B}_R$.*

Once we have adjusted this auxiliary lemma, the remaining steps of the proof are analogous to the centered case.

3.3. Centered case — part (ii). The proof follows along similar lines to §3.1, and we only present the minor modifications needed. Observe that Lemma 4 continues to hold,⁷ and we may arrive at (3.6) in the same way. We bound S_1 just as we did in (3.7). To bound S_2 we need a different argument, since γ would be zero in this case. In fact, we use the pointwise estimate (2.22) to get directly

$$(3.9) \quad S_2 = \sum_{\|\vec{n}\|_\infty > 2R} \left| \frac{\partial}{\partial x_d} M_\beta f_j(\vec{n}) \right|^q \leq C \sum_{\|\vec{n}\|_\infty \geq 2R} |M_{\beta-1} f_j(\vec{n})|^q.$$

Since $f_j \rightarrow f$ in $\ell^1(\mathbb{Z}^d)$ we have $f_j \rightarrow f$ in $\ell^r(\mathbb{Z}^d)$ for $r = d/(d + \alpha - 1)$. Since $M_{\beta-1} : \ell^r(\mathbb{Z}^d) \rightarrow \ell^q(\mathbb{Z}^d)$ is bounded and continuous we have $M_{\beta-1} f_j \rightarrow M_{\beta-1} f$ in $\ell^q(\mathbb{Z}^d)$. We can then choose R sufficiently large to make the right-hand side of (3.9) very small (for j large). The rest of the proof is analogous to §3.1.

3.4. Uncentered case — part (ii). The proof is essentially analogous to §3.1, §3.2 and §3.3. One just has to use the analogue of (2.22) to the uncentered operator to bound S_2 as in (3.9). We omit the details.

4. PROOF OF THEOREM 2

In this section we work with the one-dimensional discrete uncentered maximal operator defined in (1.10). To prove Theorem 2 we may assume without loss of generality that $f \geq 0$. For $n \in \mathbb{Z}$ and $r, s \in \mathbb{Z}^+ \times \mathbb{Z}^+$ we define the fractional average

$$A_{r,s} f(n) = \frac{1}{(r+s+1)^{1-\beta}} \sum_{k=-r}^s f(n+k).$$

We start with the following preliminary lemma.

Lemma 6. *Let $f : \mathbb{Z} \rightarrow \mathbb{R}^+$ be a bounded function such that $\widetilde{M}_\beta f \not\equiv \infty$.*

- (i) *We have $\widetilde{M}_\beta f(n) < \infty$ for all $n \in \mathbb{Z}$.*
- (ii) *If $\widetilde{M}_\beta f(n)$ is not attained by any average $A_{r,s} f(n)$ with $r, s \in \mathbb{Z}^+ \times \mathbb{Z}^+$, then*

$$(4.1) \quad \widetilde{M}_\beta f(m) \geq \widetilde{M}_\beta f(n)$$

for all $m \in \mathbb{Z}$.

⁷In fact, note that all we need to establish Lemma 4 is that $f \in \ell^s(\mathbb{Z}^d)$ and $f_j \rightarrow f$ in $\ell^s(\mathbb{Z}^d)$ for some $1 \leq s < d/\beta$.

Proof. (i) If there is $n \in \mathbb{Z}$ such that $\widetilde{M}_\beta f(n) = \infty$, there exists a sequence $\{r_j, s_j\}$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$, with $r_j + s_j \rightarrow \infty$ such that $A_{r_j, s_j} f(n) \rightarrow \infty$ as $j \rightarrow \infty$. For each $m \in \mathbb{Z}$, letting $C = \|f\|_{\ell^\infty(\mathbb{Z})}$ we have

$$(4.2) \quad A_{r_j, s_j} f(m) \geq A_{r_j, s_j} f(n) - \frac{2C|m - n|}{(r_j + s_j + 1)^{1-\beta}},$$

which implies that $\widetilde{M}_\beta f(m) = \infty$, a contradiction.

(ii) If $\widetilde{M}_\beta f(n)$ is not attained, there exists a sequence $\{r_j, s_j\}$ in $\mathbb{Z}^+ \times \mathbb{Z}^+$ with $r_j + s_j \rightarrow \infty$ such that $A_{r_j, s_j} f(n) \rightarrow \widetilde{M}_\beta f(n)$ as $j \rightarrow \infty$. The inequality (4.1) plainly follows from (4.2). \square

The next lemma is the heart of the proof. It bounds the q -variation of $\widetilde{M}_\beta f$ in a monotone interval by the variation of f in a comparable interval.

Lemma 7. *Let $f : \mathbb{Z} \rightarrow \mathbb{R}^+$ be a function of bounded variation such that $\widetilde{M}_\beta f$ is nonconstant (in particular, $\widetilde{M}_\beta f \neq \infty$).*

(i) *Let $a < b$ be integers such that $\widetilde{M}_\beta f$ is nonincreasing in $[a, b]$, with $\widetilde{M}_\beta f(a) > \widetilde{M}_\beta f(a+1)$. Let r be the smallest nonnegative integer such that $\widetilde{M}_\beta f(a) = A_{r,0} f(a)$. Then we have*

$$(4.3) \quad \sum_{n=a}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \leq 2 \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|.$$

(ii) *Let $a < b$ be integers such that $\widetilde{M}_\beta f$ is nondecreasing in $[a, b]$, with $\widetilde{M}_\beta f(b-1) < \widetilde{M}_\beta f(b)$. Let s be the smallest nonnegative integer such that $\widetilde{M}_\beta f(b) = A_{0,s} f(b)$. Then we have*

$$(4.4) \quad \sum_{n=a}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \leq 2 \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a}^{b+s-1} |f(n) - f(n+1)|.$$

Proof. We prove (i), and (ii) is analogous. Observe first that the existence of such minimal r is guaranteed by Lemma 6. Then note that

$$(4.5) \quad \begin{aligned} |\widetilde{M}_\beta f(a) - \widetilde{M}_\beta f(b)| &\leq |A_{r,0} f(a) - A_{r,0} f(b)| \\ &= \left| \sum_{k=0}^{b-a-1} A_{r,0} f(a+k) - A_{r,0} f(a+k+1) \right| \\ &\leq \frac{1}{(r+1)^{1-\beta}} \sum_{n=a-r}^{b-1} (r+1) |f(n) - f(n+1)| \\ &= (r+1)^\beta \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|. \end{aligned}$$

Let m be the smallest integer in $[a, b-1]$ such that

$$\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1) = \max \{ \widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1); n \in [a, b-1] \} > 0,$$

and let $t = \min\{t \in \mathbb{Z}^+; \widetilde{M}_\beta f(m) = A_{t,0} f(m)\}$. The existence of such t is guaranteed by Lemma 6.

Step 1. Let us first consider the situation when $t \geq r$. In this case, using (4.5) we obtain

$$\begin{aligned}
 & \sum_{n=a}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \\
 & \leq |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^{q-1} \sum_{n=a}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)| \\
 & = |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^{q-1} |\widetilde{M}_\beta f(a) - \widetilde{M}_\beta f(b)| \\
 & \leq |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^{q-1} (r+1)^\beta \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
 (4.6) \quad & = |(r+1)^{1-\beta} (\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1))|^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
 & \leq |(r+1)^{1-\beta} (A_{t,0} f(m) - A_{t,0} f(m+1))|^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
 & \leq \left| \frac{(r+1)^{1-\beta}}{(t+1)^{1-\beta}} (f(m-t) - f(m+1)) \right|^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)| \\
 & \leq \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|.
 \end{aligned}$$

This establishes (4.3) in this case.

Step 2. Now assume that $t < r$ and $m - t \leq a$. Note that in this case we have $a - r < m - t$. We may proceed as in (4.5) and (4.6) to obtain

$$\begin{aligned}
 (4.7) \quad & \sum_{n=a}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \leq (t+1) |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^q \\
 & \quad + |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^{q-1} \sum_{n=m}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)| \\
 & \leq |f(m-t) - f(m+1)|^q + |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^{q-1} |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(b)| \\
 & \leq |f(m-t) - f(m+1)|^q \\
 & \quad + |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^{q-1} (t+1)^\beta \sum_{n=m-t}^{b-1} |f(n) - f(n+1)| \\
 & \leq |f(m-t) - f(m+1)|^{q-1} \left(|f(m-t) - f(m+1)| + \sum_{n=m-t}^{b-1} |f(n) - f(n+1)| \right) \\
 & \leq 2 \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=a-r}^{b-1} |f(n) - f(n+1)|,
 \end{aligned}$$

and we have again established (4.3).

Step 3. Finally, we consider the case $t < r$ and $a < m - t$. Reasoning as in (4.7) we obtain

$$\begin{aligned}
 & \sum_{n=m-t}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \\
 & \leq (t+1) |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^q \\
 (4.8) \quad & + |\widetilde{M}_\beta f(m) - \widetilde{M}_\beta f(m+1)|^{q-1} \sum_{n=m}^{b-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)| \\
 & \leq 2 \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=m-t}^{b-1} |f(n) - f(n+1)|.
 \end{aligned}$$

We then proceed inductively. Let $(m_1, t_1) = (m, t)$. Having defined $(m_1, t_1), (m_2, t_2), \dots, (m_{l-1}, t_{l-1})$, if $t_{l-1} < r$ and $a < m_{l-1} - t_{l-1}$ we define m_l as the smallest integer in the interval $[a, m_{l-1} - t_{l-1} - 1]$ such that

$$\begin{aligned}
 \widetilde{M}_\beta f(m_l) - \widetilde{M}_\beta f(m_l+1) &= \max \{ \widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1); n \in [a, m_{l-1} - t_{l-1} - 1] \} > 0, \\
 \text{and let } t_l &= \min \{ t \in \mathbb{Z}^+; \widetilde{M}_\beta f(m_l) = A_{t,0} f(m_l) \}. \text{ If } t_l < r \text{ and } a < m_l - t_l \text{ we} \\
 &\text{reboot Step 3 to obtain}
 \end{aligned}$$

$$(4.9) \quad \sum_{n=m_l-t_l}^{m_{l-1}-t_{l-1}-1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \leq 2 \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_{n=m_l-t_l}^{m_{l-1}-t_{l-1}-1} |f(n) - f(n+1)|.$$

This process must terminate; i.e. there exists a smallest N such that either (i) $t_N \geq r$ or (ii) $t_N < r$ and $m_N - t_N \leq a$. In the first case we use Step 1 to bound the q -variation of $\widetilde{M}_\beta f$ on the interval $[a, m_{N-1} - t_{N-1}]$, and in the second case we use Step 2 to bound the q -variation of $\widetilde{M}_\beta f$ on the interval $[a, m_{N-1} - t_{N-1}]$. We then sum with all the previous inequalities (4.8) and (4.9) of the inductive process to arrive at the desired result. \square

We now introduce the local maxima and minima of a discrete function $g : \mathbb{Z} \rightarrow \mathbb{R}$.⁸ We say that an interval $[n, m]$ is a *string of local maxima* of g if

$$g(n-1) < g(n) = \dots = g(m) > g(m+1).$$

If $n = -\infty$ or $m = \infty$ (but not both simultaneously) we modify the definition accordingly, eliminating one of the inequalities. The rightmost point m of such a string is a *right local maximum* of g , while the leftmost point n is a *left local maximum* of g . We define *string of local minima*, *right local minimum* and *left local minimum* analogously.

Proof of Theorem 2. Assume that $\widetilde{M}_\beta f$ is not constant (in case $\widetilde{M}_\beta f$ is constant the result is obviously true). Let $\{[a_j^-, a_j^+]\}_{j \in \mathbb{Z}}$ and $\{[b_j^-, b_j^+]\}_{j \in \mathbb{Z}}$ be the ordered strings of local maxima and local minima of $\widetilde{M}_\beta f$ (we allow the possibilities of a_j^- or $b_j^- = -\infty$ and $a_j^+ = \infty$ or $b_j^+ = \infty$), i.e.

$$(4.10) \quad \dots < a_{-1}^- \leq a_{-1}^+ < b_{-1}^- \leq b_{-1}^+ < a_0^- \leq a_0^+ < b_0^- \leq b_0^+ < a_1^- \leq a_1^+ < b_1^- \leq b_1^+ < \dots$$

⁸The local extrema are defined slightly differently in [4, 6], but used with the meaning stated here.

This sequence may terminate in one or both sides (in principle, we are not even ruling out the possibility of this sequence being empty, i.e. of $\widetilde{M}_\beta f$ being monotone), and we adjust the notation accordingly.

Let a_j^\pm be one of the right local maxima and r_j^\pm be the smallest integer radius such that $\widetilde{M}_\beta f(a_j^\pm) = A_{r_j^\pm, 0} f(a_j^\pm)$. The crucial observation is that $a_j^- \leq a_j^+ - r_j^+$, and Lemma 7 yields

$$\sum_j \sum_{n=a_j^+}^{b_j^- - 1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \leq 2 \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_j \sum_{n=a_j^-}^{b_j^- - 1} |f(n) - f(n+1)|.$$

Analogously,

$$\sum_j \sum_{n=b_j^+}^{a_{j+1}^- - 1} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \leq 2 \|f'\|_{\ell^1(\mathbb{Z})}^{q-1} \sum_j \sum_{n=b_j^+}^{a_{j+1}^+ - 1} |f(n) - f(n+1)|.$$

Adding up these inequalities we obtain the desired result (note the potential overlap in each $[a_j^-, a_j^+]$):

$$\sum_{n=-\infty}^{\infty} |\widetilde{M}_\beta f(n) - \widetilde{M}_\beta f(n+1)|^q \leq 4 \|f'\|_{\ell^1(\mathbb{Z})}^q.$$

Note that in the exceptional cases when the sequence (4.10) terminates to one or both sides, or even when the sequence (4.10) is empty (if $\widetilde{M}_\beta f$ is monotone), the q -variation of $\widetilde{M}_\beta f$ on the intervals of length infinity where it is monotone can be bounded directly using Lemma 7. □

5. PROOF OF THEOREM 1

We now move the discussion to the continuous setting. In this case, the one-dimensional uncentered fractional maximal operator \widetilde{M}_β is defined as in (1.6). The ideas presented in the previous section (discrete setting) also play a relevant role here, while new technical details arise. When proving Theorem 1 we may assume without loss of generality that $f \geq 0$ since $\text{Var}(f) \leq \text{Var}(|f|)$. The case $\beta = 0$ of Theorem 1 was proved by Aldaz and Pérez Lázaro [2] (with the sharp constant $C = 1$), so throughout this section we restrict ourselves to the case $0 < \beta < 1$. For $r, s \geq 0$ we keep denoting the fractional averages by⁹

$$A_{r,s} f(x) = \frac{1}{(r+s)^{1-\beta}} \int_{x-r}^{x+s} f(y) dy.$$

We start with the following preliminary lemma.

Lemma 8. *Let $0 < \beta < 1$. Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be a bounded function such that $\widetilde{M}_\beta f \not\equiv \infty$.*

- (i) *We have $\widetilde{M}_\beta f(x) < \infty$ for all $x \in \mathbb{R}$.*

⁹We define $A_{0,0}(f)(x) = \limsup_{r,s \rightarrow 0^+} \frac{1}{(r+s)^{1-\beta}} \int_{x-r}^{x+s} f(y) dy$. If f is locally bounded and $\beta > 0$ we have $A_{0,0}(f)(x) = 0$.

(ii) If $\widetilde{M}_\beta f(x)$ is not attained by any average $A_{r,s}f(x)$ with $r, s \geq 0$, then

$$(5.1) \quad \widetilde{M}_\beta f(y) \geq \widetilde{M}_\beta f(x)$$

for all $y \in \mathbb{R}$.

Proof. (i) If $\widetilde{M}_\beta f(x) = \infty$ for some $x \in \mathbb{R}$, since f is bounded there exists a sequence $(r_j, s_j) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $(r_j + s_j) \rightarrow \infty$ such that $A_{r_j, s_j} f(x) \rightarrow \infty$ as $j \rightarrow \infty$. Letting $C = \|f\|_{L^\infty(\mathbb{R})}$, for any $y \in \mathbb{R}$ we have

$$(5.2) \quad A_{r_j, s_j} f(y) \geq A_{r_j, s_j} f(x) - \frac{2C|x - y|}{(r_j + s_j)^{1-\beta}},$$

which implies that $\widetilde{M}_\beta f(y) = \infty$, a contradiction.

(ii) If $\widetilde{M}_\beta f(x)$ is not attained, there exists a sequence $(r_j, s_j) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $A_{r_j, s_j} f(x) \rightarrow \widetilde{M}_\beta f(x)$ as $j \rightarrow \infty$. If $(r_j + s_j)$ is bounded, passing to a subsequence if necessary, we can find $(r, s) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $r_j \rightarrow r$ and $s_j \rightarrow s$, which implies that $A_{r,s} f(x) = \widetilde{M}_\beta f(x)$, a contradiction. Hence we have $(r_j + s_j) \rightarrow \infty$, and inequality (5.1) plainly follows from (5.2). \square

Our next proposition establishes the result for Lipschitz functions.

Proposition 9. *Let $0 < \beta < 1$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a Lipschitz function such that $\widetilde{M}_\beta f \neq \infty$. Then*

$$(5.3) \quad \text{Var}_q(\widetilde{M}_\beta f) \leq 8^{1/q} \text{Var}(f).$$

We postpone the proof of Proposition 9 until §5.2. For now, let us assume the validity of this result to conclude the proof of Theorem 1. In order to do so, we also need the following classical result of F. Riesz (see [18, Chapter IX, §4, Theorem 7]).

Proposition 10 (F. Riesz). *Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a given function and $1 < q < \infty$. Then $\text{Var}_q(g) < \infty$ if and only if g is absolutely continuous and its derivative g' belongs to $L^q(\mathbb{R})$. Moreover, in this case, we have that*

$$(5.4) \quad \|g'\|_{L^q(\mathbb{R})} = \text{Var}_q(g).$$

5.1. Proof of Theorem 1. We start by showing the validity of the inequality in (1.8) for a function $f : \mathbb{R} \rightarrow \mathbb{R}^+$ of bounded variation such that $\widetilde{M}_\beta f \neq \infty$. Let $\varphi \in C_c^\infty(\mathbb{R})$ be a nonnegative smooth function with support in $[-1, 1]$ and $\|\varphi\|_{L^1(\mathbb{R})} = 1$. For $\varepsilon > 0$ we define $\varphi_\varepsilon(x) = \frac{1}{\varepsilon} \varphi(\frac{x}{\varepsilon})$ and write $f_\varepsilon = f * \varphi_\varepsilon$. Note that f_ε is Lipschitz continuous and $\text{Var}(f_\varepsilon) \leq \text{Var}(f)$ for all $\varepsilon > 0$.

Fix a partition $\mathcal{P} = \{x_1 < x_2 < \dots < x_N\}$. By Proposition 9 we have

$$(5.5) \quad \left(\sum_{n=1}^{N-1} \frac{|\widetilde{M}_\beta f_\varepsilon(x_{n+1}) - \widetilde{M}_\beta f_\varepsilon(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \right)^{1/q} \leq 8^{1/q} \text{Var}(f_\varepsilon) \leq 8^{1/q} \text{Var}(f)$$

for all $\varepsilon > 0$. We claim that

$$(5.6) \quad \lim_{\varepsilon \rightarrow 0} \widetilde{M}_\beta f_\varepsilon(x) = \widetilde{M}_\beta f(x)$$

for all $x \in \mathbb{R}$. It then follows from (5.5) and (5.6) that

$$\left(\sum_{n=1}^{N-1} \frac{|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \right)^{1/q} \leq 8^{1/q} \text{Var}(f).$$

Since the original partition \mathcal{P} was arbitrary, we arrive at (1.8).

We now prove (5.6). Recall that $\lim_{\varepsilon \rightarrow 0} f_\varepsilon(x) = f(x)$ at every point of continuity of f (in particular almost everywhere). By Fatou’s lemma, for any nontrivial interval $[x - s, x + r]$ containing x we have

$$A_{r,s}f(x) \leq \liminf_{\varepsilon \rightarrow 0} A_{r,s}f_\varepsilon(x) \leq \liminf_{\varepsilon \rightarrow 0} \widetilde{M}_\beta f_\varepsilon(x),$$

from which we conclude that

$$(5.7) \quad \widetilde{M}_\beta f(x) \leq \liminf_{\varepsilon \rightarrow 0} \widetilde{M}_\beta f_\varepsilon(x).$$

We prove the opposite inequality by contradiction. Given $x \in \mathbb{R}$, assume that for some $\eta > 0$ we have

$$(5.8) \quad \limsup_{\varepsilon \rightarrow 0} \widetilde{M}_\beta f_\varepsilon(x) > (1 + 2\eta) \widetilde{M}_\beta f(x)$$

(in particular, $f \not\equiv 0$ and $\widetilde{M}_\beta f(x) > 0$). Then, for a certain sequence of $\varepsilon \rightarrow 0$, there exist $y = y_\varepsilon$ and $r = r_\varepsilon > 0$ such that $x \in [y - r, y + r]$ and

$$(5.9) \quad \frac{1}{(2r)^{1-\beta}} \int_{y-r}^{y+r} f_\varepsilon(t) dt > (1 + \eta) \widetilde{M}_\beta f(x).$$

Since $\|f_\varepsilon\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})}$ and we are assuming that $0 < \beta < 1$, we note that we must have $r_\varepsilon > c > 0$ for this sequence of $\varepsilon \rightarrow 0$. On the other hand, observe that

$$\begin{aligned} \frac{1}{(2r)^{1-\beta}} \int_{y-r}^{y+r} f_\varepsilon(t) dt &= \frac{1}{(2r)^{1-\beta}} \int_{y-r}^{y+r} \int_{-\varepsilon}^\varepsilon f(t-u) \varphi_\varepsilon(u) du dt \\ &= \frac{1}{(2r)^{1-\beta}} \int_{-\varepsilon}^\varepsilon \left(\int_{y-r}^{y+r} f(t-u) dt \right) \varphi_\varepsilon(u) du \\ (5.10) \quad &\leq \frac{1}{(2r)^{1-\beta}} \int_{-\varepsilon}^\varepsilon \left(\int_{y-r-\varepsilon}^{y+r+\varepsilon} f(t) dt \right) \varphi_\varepsilon(u) du \\ &\leq \frac{(2r + 2\varepsilon)^{1-\beta}}{(2r)^{1-\beta}} \int_{-\varepsilon}^\varepsilon \widetilde{M}_\beta f(x) \varphi_\varepsilon(u) du \\ &\leq \frac{(2r + 2\varepsilon)^{1-\beta}}{(2r)^{1-\beta}} \widetilde{M}_\beta f(x). \end{aligned}$$

From (5.9) and (5.10) we conclude that

$$\frac{(2r + 2\varepsilon)^{1-\beta}}{(2r)^{1-\beta}} > 1 + \eta.$$

If we restrict ourselves to the range $\varepsilon \leq 1$, the inequality above implies that $r \leq N$ for some large $N = N(\beta, \eta)$ and then $|y| \leq |x| + N$. Therefore, there exists a subsequence $\{y_{\varepsilon_k}, r_{\varepsilon_k}\} \subset \{y_\varepsilon, r_\varepsilon\}$ and a pair (y_0, r_0) with $r_0 > 0$ such that $y_{\varepsilon_k} \rightarrow y_0$ and $r_{\varepsilon_k} \rightarrow r_0$ as $\varepsilon_k \rightarrow 0$. Then $x \in [y_0 - r_0, y_0 + r_0]$ and

$$\begin{aligned} \widetilde{M}_\beta f(x) &\geq \frac{1}{(2r_0)^{1-\beta}} \int_{y_0-r_0}^{y_0+r_0} f(t) dt \\ &= \lim_{\varepsilon_k \rightarrow 0} \frac{1}{(2r_{\varepsilon_k})^{1-\beta}} \int_{y_{\varepsilon_k}-r_{\varepsilon_k}}^{y_{\varepsilon_k}+r_{\varepsilon_k}} f_\varepsilon(t) dt \\ &\geq (1 + \eta) \widetilde{M}_\beta f(x), \end{aligned}$$

which is a contradiction. Hence (5.8) cannot occur, and we must have

$$\limsup_{\varepsilon \rightarrow 0} \widetilde{M}_\beta f_\varepsilon(x) \leq \widetilde{M}_\beta f(x),$$

which, together with (5.7), establishes the pointwise convergence (5.6) and concludes the proof of the inequality in (1.8).

Finally, the fact that $\widetilde{M}_\beta f$ is absolutely continuous with

$$\|(\widetilde{M}_\beta f)'\|_{L^q(\mathbb{R})} = \text{Var}_q(\widetilde{M}_\beta f)$$

now follows from Proposition 10. This concludes the proof of Theorem 1.

5.2. Proof of Proposition 9. Recall that we are assuming that $f \geq 0$. If $\text{Var}(f) = \infty$, we understand that inequality (5.3) is true, so let us also assume that f is of bounded variation. The next lemma is the continuous analogue of Lemma 7 and is the core of this proof.

Lemma 11. *Let $0 \leq \beta < 1$. Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be a Lipschitz function of bounded variation such that $\widetilde{M}_\beta f$ is nonconstant (in particular, $\widetilde{M}_\beta f \not\equiv \infty$).*

(i) *Let $x_1 < x_2 < \dots < x_N$ be a sequence of real numbers such that*

$$\widetilde{M}_\beta f(x_1) > \widetilde{M}_\beta f(x_2) \geq \dots \geq \widetilde{M}_\beta f(x_{N-1}) \geq \widetilde{M}_\beta f(x_N).$$

Let $r, s \geq 0$ be such that $\widetilde{M}_\beta f(x_1) = A_{r,s}f(x_1)$, with $r + s$ minimal (and then with r minimal, if necessary). Then

$$(5.11) \quad \sum_{n=1}^{N-1} \frac{|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \leq 4 \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1-r}^{x_N} |f'(x)| \, dx.$$

(ii) *Let $x_1 < x_2 < \dots < x_N$ be a sequence of real numbers such that*

$$\widetilde{M}_\beta f(x_1) \leq \widetilde{M}_\beta f(x_2) \leq \dots \leq \widetilde{M}_\beta f(x_{N-1}) < \widetilde{M}_\beta f(x_N).$$

Let $r, s \geq 0$ be such that $\widetilde{M}_\beta f(x_N) = A_{r,s}f(x_N)$, with $r + s$ minimal (and then with s minimal, if necessary). Then

$$(5.12) \quad \sum_{n=1}^{N-1} \frac{|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \leq 4 \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1}^{x_N+s} |f'(x)| \, dx.$$

Proof. We prove (i), and (ii) is analogous. Note that the existence of such a pair (r, s) is guaranteed by Lemma 8 and the continuity properties of the averages. For such a pair (r, s) , since $\widetilde{M}_\beta f(x_1) > \widetilde{M}_\beta f(x_N)$, we must have $s < x_N - x_1$. Let $a \in [x_1 - r, x_1 + s]$ be a point such that

$$\frac{1}{(r + s)} \int_{x_1-r}^{x_1+s} f(x) \, dx = f(a),$$

and let $b \in [x_N - r - s, x_N] \subset [x_1 - r, x_N]$ be a point such that

$$\frac{1}{r + s} \int_{x_N-r-s}^{x_N} f(x) \, dx = f(b).$$

We then obtain

$$\begin{aligned}
 (5.13) \quad & \left| \widetilde{M}_\beta f(x_1) - \widetilde{M}_\beta f(x_N) \right| \leq \left| A_{r,s} f(x_1) - A_{r+s,0} f(x_N) \right| \\
 & = (r+s)^\beta |f(a) - f(b)| \\
 & \leq (r+s)^\beta \int_{x_1-r}^{x_N} |f'(x)| \, dx.
 \end{aligned}$$

Now let m be the smallest integer with $1 \leq m \leq N - 1$ such that

$$\frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{|x_m - x_{m+1}|} = \max \left\{ \frac{\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})}{|x_n - x_{n+1}|}; 1 \leq n \leq N - 1 \right\} > 0.$$

Let $t, u \geq 0$ be such that $\widetilde{M}_\beta f(x_m) = A_{t,u} f(x_m)$, with $t + u$ minimal (and then with t minimal, if necessary). The existence of such a pair (t, u) is guaranteed by Lemma 8, and we have $0 \leq u < x_{m+1} - x_m$.

Step 1. Let us first consider the case when $t + u \geq r + s$. Using (5.13) we have

$$\begin{aligned}
 & \sum_{n=1}^{N-1} \frac{|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \\
 & \leq \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^{q-1} \sum_{n=1}^{N-1} |\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})| \\
 & = \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^{q-1} |\widetilde{M}_\beta f(x_1) - \widetilde{M}_\beta f(x_N)| \\
 & \leq \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^{q-1} (r+s)^\beta \int_{x_1-r}^{x_N} |f'(x)| \, dx \\
 & = \left| (r+s)^{1-\beta} \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^{q-1} \int_{x_1-r}^{x_N} |f'(x)| \, dx \\
 & \leq \left| \left(\frac{r+s}{t+u} \right)^{1-\beta} \int_{x_m-t}^{x_m+u} \frac{f(y) - f(y + x_{m+1} - x_m - u)}{x_{m+1} - x_m} \, dy \right|^{q-1} \int_{x_1-r}^{x_N} |f'(x)| \, dx \\
 & \leq \left| \left(\frac{r+s}{t+u} \right)^{1-\beta} \frac{1}{x_{m+1} - x_m} \int_{x_m-t}^{x_m+u} \int_0^{x_{m+1}-x_m-u} |f'(y+z)| \, dz \, dy \right|^{q-1} \\
 & \qquad \qquad \qquad \times \int_{x_1-r}^{x_N} |f'(x)| \, dx \\
 & \leq \left| \left(\frac{r+s}{t+u} \right)^{1-\beta} \frac{1}{x_{m+1} - x_m} \int_0^{x_{m+1}-x_m-u} \int_{x_m-t}^{x_m+u} |f'(y+z)| \, dy \, dz \right|^{q-1} \\
 & \qquad \qquad \qquad \times \int_{x_1-r}^{x_N} |f'(x)| \, dx \\
 & \leq \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1-r}^{x_N} |f'(x)| \, dx,
 \end{aligned}$$

which establishes (5.11) in this case.

Step 2. We now consider the case when $t + u < r + s$ and $x_m - t \leq x_1$. In this case note that $x_1 - r \leq x_m - t$ (which is clear if $m = 1$, and if $m > 1$ we note that $s < x_m - x_1$). Reasoning as before we have

$$\begin{aligned}
 & \sum_{n=1}^{N-1} \frac{|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \\
 & \leq \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^q \sum_{n=1}^{m-1} |x_n - x_{n+1}| \\
 & \quad + \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^{q-1} \sum_{n=m}^{N-1} |\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})| \\
 & \leq \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^q |x_1 - x_m| \\
 & \quad + \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^{q-1} |\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_N)| \\
 & \leq t \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^q \\
 & \quad + \left| \frac{\widetilde{M}_\beta f(x_m) - \widetilde{M}_\beta f(x_{m+1})}{x_m - x_{m+1}} \right|^{q-1} (t + u)^\beta \int_{x_m-t}^{x_N} |f'(x)| dx \\
 & \leq \left| \left(\frac{t}{t+u} \right)^{1-\beta} \int_{x_m-t}^{x_m+u} \frac{f(y) - f(y + x_{m+1} - x_m - u)}{x_{m+1} - x_m} dy \right|^q \\
 & \quad + \left| \int_{x_m-t}^{x_m+u} \frac{f(y) - f(y + x_{m+1} - x_m - u)}{x_{m+1} - x_m} dy \right|^{q-1} \int_{x_m-t}^{x_N} |f'(x)| dx \\
 & \leq \left| \frac{1}{x_{m+1} - x_m} \int_{x_m-t}^{x_m+u} \int_0^{x_{m+1}-x_m-u} |f'(y+z)| dz dy \right|^q \\
 & \quad + \left| \frac{1}{x_{m+1} - x_m} \int_{x_m-t}^{x_m+u} \int_0^{x_{m+1}-x_m-u} |f'(y+z)| dz dy \right|^{q-1} \int_{x_m-t}^{x_N} |f'(x)| dx \\
 & = \left| \frac{1}{x_{m+1} - x_m} \int_0^{x_{m+1}-x_m-u} \int_{x_m-t}^{x_m+u} |f'(y+z)| dz dy \right|^q \\
 & \quad + \left| \frac{1}{x_{m+1} - x_m} \int_0^{x_{m+1}-x_m-u} \int_{x_m-t}^{x_m+u} |f'(y+z)| dz dy \right|^{q-1} \int_{x_m-t}^{x_N} |f'(x)| dx \\
 & \leq \|f'\|_{L^1(\mathbb{R})}^{q-1} \left\{ \frac{1}{x_{m+1} - x_m} \int_0^{x_{m+1}-x_m-u} \int_{x_m-t}^{x_N} |f'(x)| dx dy + \int_{x_m-t}^{x_N} |f'(x)| dx \right\} \\
 & \leq 2 \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_m-t}^{x_N} |f'(x)| dx \\
 & \leq 2 \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_1-r}^{x_N} |f'(x)| dx,
 \end{aligned}$$

which establishes (5.11).

Step 3. Finally, we consider the case when $t + u < r + s$ and $x_1 < x_m - t$. We let k be the unique integer such that $x_{k-1} < x_m - t \leq x_k$. Arguing as in Step 2 we get

$$(5.14) \quad \sum_{n=k}^{N-1} \frac{|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \leq 2 \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_{k-1}}^{x_N} |f'(x)| dx.$$

We then proceed inductively. Let $(m_1, t_1, u_1, k_1) = (m, t, u, k)$. Having defined $(m_1, t_1, u_1, k_1), (m_2, t_2, u_2, k_2), \dots, (m_{l-1}, t_{l-1}, u_{l-1}, k_{l-1})$, if $t_{l-1} + u_{l-1} < r + s$ and $x_1 < x_{m_{l-1}} - t_{l-1}$, then k_{l-1} is the unique integer such that $x_{k_{l-1}-1} < x_{m_{l-1}} - t_{l-1} \leq x_{k_{l-1}}$. We then let m_l be the smallest integer with $1 \leq m_l \leq k_{l-1} - 1$ such that

$$\begin{aligned} & \frac{\widetilde{M}_\beta f(x_{m_l}) - \widetilde{M}_\beta f(x_{m_l+1})}{|x_{m_l} - x_{m_l+1}|} \\ & = \max \left\{ \frac{\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})}{|x_n - x_{n+1}|}; 1 \leq n \leq k_{l-1} - 1 \right\} > 0. \end{aligned}$$

Let $t_l, u_l \geq 0$ be such that $\widetilde{M}_\beta f(x_{m_l}) = A_{t_l, u_l} f(x_{m_l})$, with $t_l + u_l$ minimal (and then with t_l minimal, if necessary). The existence of such a pair (t_l, u_l) is guaranteed by Lemma 8. If $t_l + u_l < r + s$ and $x_1 < x_{m_l} - t_l$ we let k_l be the unique integer such that $x_{k_l-1} < x_{m_l} - t_l \leq x_{k_l}$. We then reboot Step 3 to obtain

$$(5.15) \quad \sum_{n=k_l}^{k_{l-1}-1} \frac{|\widetilde{M}_\beta f(x_n) - \widetilde{M}_\beta f(x_{n+1})|^q}{|x_n - x_{n+1}|^{q-1}} \leq 2 \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_{k_l-1}}^{x_{k_{l-1}}} |f'(x)| dx.$$

This process must terminate; i.e. there is a smallest integer L such that either (i) $t_L + u_L \geq r + s$ or (ii) $t_L + u_L < r + s$ and $x_{m_L} - t_L \leq x_1$. In the first case (resp. second case) we use Step 1 (resp. Step 2) to bound the q -variation of $\widetilde{M}_\beta f$ (over the partition $\{x_n\}$) from x_1 to $x_{k_{L-1}}$. We then sum all the previous inequalities (5.14) and (5.15) to arrive at the desired conclusion (note that the sum of the integrals on the right-hand sides of (5.14) and (5.15) has a two-fold overlap over each interval $[x_{k_{l-1}}, x_{k_l}]$). \square

Proof of Proposition 9. Fix a partition $\mathcal{P} = \{x_1 < x_2 < \dots < x_N\}$. For a generic function $g : \mathbb{R} \rightarrow \mathbb{R}$, we say that an interval $[x_n, x_m]$ is a *string of local maxima* of g (relative to the partition \mathcal{P}) if

$$(5.16) \quad g(x_{n-1}) < g(x_n) = \dots = g(x_m) > g(x_{m+1}),$$

provided $n \neq 1$ and $m \neq N$. In case $n = 1$ (resp. $m = N$) we disregard the leftmost (resp. rightmost) inequality in (5.16). When $m \neq N$, the rightmost point x_m of such a string is a *right local maximum* of g , and when $n \neq 1$ the leftmost point x_n is a *left local maximum* of g . We define *string of local minima*, *right local minimum* and *left local minimum* (relative to the partition \mathcal{P}) analogously.

Assume that $\widetilde{M}_\beta f$ is not constant (in case $\widetilde{M}_\beta f$ is constant the result is obviously true). The strategy is the same as in the discrete case, to bound the q -variation of $\widetilde{M}_\beta f$ between a right local maximum and the next left local minimum. The q -variation of $\widetilde{M}_\beta f$ between a right local minimum and the next left local maximum is treated analogously.

Let x_m be a right local maximum and x_l be the next left local minimum of $\widetilde{M}_\beta f$ (i.e. with the smallest $l > m$). Let $r, s \geq 0$ be such that $\widetilde{M}_\beta f(x_m) = A_{r,s} f(x_m)$,

with $r + s$ minimal (and then with r minimal, if necessary). By Lemma 11 we have

$$(5.17) \quad \sum_{n=m}^{l-1} \frac{|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \leq 4 \|f'\|_{L^1(\mathbb{R})}^{q-1} \int_{x_m-r}^{x_l} |f'(x)| \, dx.$$

In case there exists a right local minimum $x_{l'}$ before x_m (which is the case if x_m is not the first right local maximum), note that we have $x_{l'} < x_m - r$. We may therefore sum the inequalities (5.17) over all pairs of consecutive local extrema (the nonincreasing and nondecreasing pieces). Noting the additional two-fold overlap on the right-hand side we arrive at

$$\sum_{n=1}^{N-1} \frac{|\widetilde{M}_\beta f(x_{n+1}) - \widetilde{M}_\beta f(x_n)|^q}{|x_{n+1} - x_n|^{q-1}} \leq 8 \|f'\|_{L^1(\mathbb{R})}^q.$$

Since the right-hand side is now independent of the partition \mathcal{P} , we may take the supremum over all such partitions to obtain the desired result. \square

ACKNOWLEDGMENTS

The first author acknowledges support from CNPq-Brazil grants 305612/2014–0 and 477218/2013–0, and FAPERJ grant $E-26/103.010/2012$. The second author acknowledges support from CAPES-Brazil. The authors credit the main idea for the proof of the boundedness part of Theorem 3 (i) to Prof. Keith Rogers (ICMAT - Madrid), to whom they are greatly thankful. The authors are also greatly thankful to Prof. Martin Lind for bringing Proposition 10 and references [3, 18] to their attention.

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