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COMPACT LIE GROUPS: EULER CONSTRUCTIONS AND GENERALIZED DYSON CONJECTURE

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Abstract. A generalized Euler parameterization of a compact Lie group is a way for parameterizing the group starting from a maximal Lie subgroup, which allows a simple characterization of the range of parameters. In the present paper we consider the class of all compact connected Lie groups. We present a general method for realizing their generalized Euler parameterization starting from any symmetrically embedded Lie group. Our construction is based on a detailed analysis of the geometry of these groups. As a byproduct this gives rise to an interesting connection with certain Dyson integrals. In particular, we obtain a geometry based proof of a Macdonald conjecture regarding the Dyson integrals correspondent to the root systems associated to all irreducible symmetric spaces. As an application of our general method we explicitly parameterize all groups of the class of simple, simply connected compact Lie groups. We provide a table giving all necessary ingredients for all such Euler parameterizations.

1. Introduction

A simple procedure to parameterize compact simple Lie groups is provided by the generalized Euler parameterization [2–8, 17, 18]. The role of Lie groups both in mathematics and in physics is very important and well known. In several situations their local properties are sufficient, but there are also numerous concrete applications that require explicit realizations of the group matrices including the correct range of the parameters. This is the case in lattice gauge theories or in separability criteria for entangled configurations in quantum mechanics, just to cite two relevant physical examples. For $SU(2)$ this problem has a natural solution in the Euler parameterization. Important progress was first achieved in [17] (followed by [18]), where the authors were able to define generalized Euler angles for $SU(N)$ groups. However, their construction was quite involved and does not allow an obvious generalization to other Lie groups. In [5, 7], in studying the case of G_2, a more general strategy (later reviewed in [6]), which was expected to be applicable case by case to all compact simply connected simple Lie groups, was developed. Indeed, it was first applied to the $SU(N)$ case in [8], showing the drastic simplification with respect to [17, 18], and successively to the exceptional Lie groups F_4, E_6 and E_7 in [4], [3] and [2], respectively. It is only after these several years' experience that we realized that the success of our method was due to the deep geometrical structure of the compact connected simple Lie groups and, in fact, of all compact

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connected Lie groups. In the present paper we will analyze the geometry underlying our generalized Euler parameterization. This, beyond providing us with a final strategy valid for parameterizing all compact connected Lie groups, will also provide us with a direct connection between the geometry of compact Lie groups and certain integrals known as generalized Dyson integrals. As a consequence, we will obtain a geometrical proof of a class of particular cases of a conjecture stated in [14] by Macdonald, and already proved in general by Opdam in [15].

1.1. **Geometry of the Euler parameterization.** Let us consider the geometric structure underlying the generalized Euler parameterization of a simple Lie group. Let $\mathfrak{g} \equiv \mathrm{Lie}(\boldsymbol{G})$ be the Lie algebra of a real compact Lie group $\boldsymbol{G}$. More precisely, we will assume that $\mathfrak{g}$ is some matrix realization supporting a faithful representation of $\boldsymbol{G} = \exp \mathfrak{g}$. Our strategy is to start from a maximal symmetrically embedded proper subgroup $\boldsymbol{H}$ of $\boldsymbol{G}$. Let $\boldsymbol{G}'$ be any real form of $\boldsymbol{G}$ different from the compact one. Then $\boldsymbol{G}'$ contains a maximal compact subgroup $\boldsymbol{H}$ which is invariant under the action of the Cartan involution, and $\boldsymbol{H}$ can be embedded in $\boldsymbol{G}$. For any real form this is our choice for $\boldsymbol{H}$ (in other words $\boldsymbol{G}/\boldsymbol{H}$ is the compact dual of the noncompact symmetric space $\boldsymbol{G}'/\boldsymbol{H}$). In particular when $\boldsymbol{G}'$ is the split form we will call $\boldsymbol{H}$ the "the maximal compact subgroup of a split form $\boldsymbol{G}'$ of $\boldsymbol{G}$ (MCS)". This has the property $\dim \boldsymbol{G} = 2 \dim \boldsymbol{H} + r \equiv 2h + r$, where r is the rank of $\boldsymbol{G}$. For the split case, the generalized Euler parameterization of $\boldsymbol{G}$ takes the form

$$\begin{aligned} \boldsymbol{G}[x_1, \ldots, x_h; y_1, \ldots, y_r; z_1, \ldots, z_h] =& \boldsymbol{H}[x_1, \ldots, x_h] \exp(y_1 c_1 + \ldots + y_r c_r) \cdot \\ & \cdot \boldsymbol{H}[z_1, \ldots, z_h], \end{aligned} \tag{1.1}$$

where $\boldsymbol{H}[x_1, \ldots, x_m]$ is a parameterization of the maximal subgroup and $c_1, \ldots, c_r$ is a basis for a Cartan subalgebra $\mathfrak{c}$ in the complement of $\mathrm{Lie}(\boldsymbol{H})$ in $\mathfrak{g}$. Since the group is compact, one can choose c_i so that the coordinates y_i are periodic. A parameterization obtained in this way in general is redundant for two reasons.

The first one is due to the fact that $\boldsymbol{H}$ contains a finite subgroup $\boldsymbol{\Gamma}$ of the maximal torus $\boldsymbol{T}^r = \exp \mathfrak{c}$ of $\boldsymbol{G}$. Indeed, we will see that $\boldsymbol{\Gamma}$ is isomorphic to $(\mathbb{Z}_2)^r$ if $\boldsymbol{G}$ is simply connected, otherwise to one of its proper subgroups. Thus,

$$\begin{aligned} &\boldsymbol{H}[x_1, \ldots, x_h] \exp(y_1 c_1 + \ldots + y_r c_r) \boldsymbol{H}[z_1, \ldots, z_h] = \boldsymbol{H}[x_1, \ldots, x_h] \gamma^{-1} \\ &\cdot \exp(y_1 c_1 + \ldots + y_r c_r) \gamma \boldsymbol{H}[z_1, \ldots, z_h], \end{aligned}$$

for any $\gamma \in \boldsymbol{\Gamma}$, so that we must reduce the range of the $\vec{x}$ coordinates w.r.t. the action of $\boldsymbol{\Gamma}$:

$$\boldsymbol{H}[x_1, \ldots, x_h] \gamma^{-1} = \boldsymbol{H}[\tilde{x}_1, \ldots, \tilde{x}_h].$$

This is easily accomplished by accordingly restricting the range.

The second problem is due to the fact that the Weyl group $\boldsymbol{W}$ acts nontrivially on $t \in \boldsymbol{T}^r$,

$$t \mapsto w^{-1} t w \in \boldsymbol{T}^r,$$

for any $w \in \boldsymbol{W}$, so also the range of the $\vec{y}$ coordinates has to be reduced. We will show that this problem can be completely characterized in terms of the highest root of $\boldsymbol{G}$: after a suitable linear change of variables $s_i = \sum_{j=1}^r A_{ij} y^j$ we will see that the right range of coordinates is expressed by the set of inequalities

$$0 \leq n_1 s_1 + \ldots + n_r s_r \leq \pi, \qquad 0 \leq s_i \leq \pi, \qquad i = 1, \ldots, r, \tag{1.2}$$

where $(n_1, \ldots, n_r)$ are the coefficients of the highest root $\tilde{\alpha}$ w.r.t. a basis of simple roots: $\tilde{\alpha} = n_1 \alpha_1 + \ldots + n_r \alpha_r$. The volume of the whole group $\boldsymbol{G}$ can be expressed in

terms of the volume of the MCS subgroup $\boldsymbol{H}$ times an integral directly connected to the generalized Dyson integral. On the other hand, the volumes of the compact Lie groups can be computed employing the Macdonald formula [13]. Thus, incidentally, we see that our construction will turn out to be equivalent to proving certain particular cases of a conjecture due to Macdonald, generalizing the Dyson integrals [14].

This construction is more involved, but it works as well for nonsimplyconnected compact Lie groups, as we will show in Section 2.2. Moreover, this parameterization applies to all compact connected Lie groups. Furthermore, we will extend all the results to the case in which a more general subgroup $\boldsymbol{H}$ symmetrically embedded in $\boldsymbol{G}$ is considered, in place of the MCS one. In this case the construction will turn out to be related to a version of Macdonald's conjecture for certain integrals associated to nonreduced root lattices. In fact, the more interesting point is not the proof of this conjecture, which can be obtained in a more general form using different methods ([15]; see also [16]), but its relation to the geometry of compact symmetric spaces.

We remark that this parameterization is also useful for concrete applications in physics. Indeed, one often needs to work with an explicit realization of the parameterization of a Lie group, including the right range for the parameters.

1.2. **Macdonald's conjecture.** Macdonald's conjecture [14] may be considered as a generalization of a conjecture due to Dyson [10]. Let R be a reduced root system, e^α denote the formal exponential corresponding to $\alpha \in R$ and k be a nonnegative integer. Then Macdonald conjectured (cf. [14], Conjecture 2.1) that the constant term in the polynomial

$$\prod_{\alpha\in R}(1-e^\alpha)^k \tag{1.3}$$

should be equal to $\prod_{i=1}^{l}\binom{kd_i}{k}$, where the d_i are the degrees of the fundamental invariants of the Weyl group of R and l the rank of R. Macdonald wrote this relation in an equivalent form which will turn out to be useful later. Let G be a compact connected Lie group, T a maximal torus of G, such that R is the root system of (G,T) and define

$$\Delta(t) = \prod_{\alpha\in R^+}(e^{\alpha/2}(t) - e^{-\alpha/2}(t)), \tag{1.4}$$

where $t \in T$, the exponentials are regarded as characters of T and R^+ is a choice of positive roots. Then $|\Delta(t)|^2 = \prod_{\alpha\in R}(1-e^\alpha(t))$ is a positive real-valued continuous function on T. This function enters in Weyl's integration formula

$$\int_G f(x)dx = \frac{1}{|W|}\int_T |\Delta(t)|^2 f(t)dt \tag{1.5}$$

for any continuous class function f on G. In (1.5), dx and dt are the normalized Haar measure on G and T respectively ($\int_G dx = \int_T dt = 1$). Thus, the conjecture can be rewritten as (cf. [14], Conjecture 2.1')

$$\int_T |\Delta(t)|^{2k}dt = \prod_{i=1}^{l}\binom{kd_i}{k}. \tag{1.6}$$

The equivalence of the two formulations follows from the fact that the integration over T kills all but the trivial character, or in other words, selects the constant term in $|\Delta(t)|^{2k} = \prod_{\alpha\in R}(1-e^{\alpha}(t))^k$. An observation that generalizes further the conjecture is that (1.6) makes sense if the integer k is replaced by a complex number, s, with positive real part, $\Re(s) > 0$. In this case the right hand side is replaced by

$$\prod_{i=1}^{l} \frac{\Gamma(sd_i+1)}{\Gamma(s+1)\Gamma(sd_i-s+1)}. \tag{1.7}$$

In the same paper Macdonald generalized the conjecture further (cf. [14], Conjecture 2.3). For this, let R be a root system, now not necessarily reduced, and for each $\alpha \in R$ let k_α be a nonnegative integer such that $k_\alpha = k_\beta$ if $|\alpha| = |\beta|$. Then the constant term in the Laurent polynomial

$$\prod_{\alpha\in R}(1-e^{\alpha})^{k_\alpha} \tag{1.8}$$

should be equal to the product

$$\prod_{\alpha\in R} \frac{\left(\left|\langle\rho_k,\check{\alpha}\rangle + k_\alpha + \frac{1}{2}k_{\alpha/2}\right|\right)!}{\left(\left|\langle\rho_k,\check{\alpha}\rangle + \frac{1}{2}k_{\alpha/2}\right|\right)!}, \tag{1.9}$$

where $\rho_k = \frac{1}{2}\sum_{\alpha\in R^+} k_\alpha\alpha$, $\check{\alpha} = \frac{2\alpha}{|\alpha|^2}$ is the coroot corresponding to α, $k_{\alpha/2} = 0$ if $\frac{1}{2}\alpha \notin R$ and $\langle,\rangle$ is the usual scalar product induced by the Killing form. When the k_α are all equal this reduces to the previous conjecture.

Macdonald's conjecture was proved in a slightly more general form by Opdam [15] considering k_α a complex valued Weyl invariant function with positive real part. This is the content of Theorem 4.1 of [15]:

Theorem 1 (Macdonald-Opdam)**.** *Let R be a possibly nonreduced root system, and let $k \in \mathbb{K}$ such that*[1] $\Re(k_\alpha) \geq 0$, $\forall\alpha \in R$. *Then*

$$\int_T \sigma(k,t)dt = \prod_{\alpha\in R^+} \frac{\Gamma(\langle\rho(k),\check{\alpha}\rangle + k_\alpha + \frac{1}{2}k_{\alpha/2}+1)\Gamma(\langle\rho(k),\check{\alpha}\rangle - k_\alpha - \frac{1}{2}k_{\alpha/2}+1)}{\Gamma(\langle\rho(k),\check{\alpha}\rangle + \frac{1}{2}k_{\alpha/2}+1)\Gamma(\langle\rho(k),\check{\alpha}\rangle - \frac{1}{2}k_{\alpha/2}+1)}, \tag{1.10}$$

where $\sigma(k,t) = \prod_{\alpha\in R^+}|t^{\frac{\alpha}{2}} - t^{-\frac{\alpha}{2}}|^{2k_\alpha}$ and T is the compact part in the "polar decomposition" of the maximal torus.

2. The split case

In this section we describe our general construction of the Euler parameterization for an arbitrary compact connected Lie group, $\boldsymbol{G}$, relative to a specific choice of the corresponding subgroups $\boldsymbol{H}$.

2.1. **The compact Lie groups.** Let $\boldsymbol{G}_0$ be a real compact connected semisimple Lie group (cf. the definition in [11], page 131). This means that $\boldsymbol{G}_0 \simeq \boldsymbol{G}_1 \times \boldsymbol{G}_2 \times \cdots \times \boldsymbol{G}_n$, where $\boldsymbol{G}_i$, $i = 1,\ldots,n$, are simple Lie groups uniquely determined (up to permutations). Let $\boldsymbol{H} \simeq \boldsymbol{H}_1 \times \boldsymbol{H}_2 \times \ldots \times \boldsymbol{H}_n$ such that $\boldsymbol{H}_i$ is a maximal Lie subgroup symmetrically embedded in $\boldsymbol{G}_i$. Then $\boldsymbol{H}$ is connected (see [11], Chapter VI, Theorem 1.1). In general, $\boldsymbol{H}_i$ is not simple, nor semisimple, but it has the form

[1]The vector space $\mathbb{K} \simeq \mathbb{C}^n$ is the space of all complex valued Weyl invariant functions on R, m equals the numbers of conjugacy classes of roots in R, and elements of $\mathbb{K}$ are called multiplicity functions on R. The notation k_α denotes the evaluation of $k \in \mathbb{K}$ on $\alpha \in R$.

$\boldsymbol{H}_i \simeq \boldsymbol{H}_{0,i} \times \boldsymbol{T}^{s_i}/\Delta_i$, where $\boldsymbol{H}_{0,i}$ is semisimple, $\boldsymbol{T}^{s_i}$ is an Abelian torus, and Δ_i a finite subgroup. Since our aim is to construct the Euler parameterization of $\boldsymbol{G}_0$ relative to the subgroup $\boldsymbol{H}$, and then applying the same procedure to $\boldsymbol{H}$ inductively, we are forced to consider the more general case

$$\boldsymbol{G} \simeq \boldsymbol{G}_0 \times \boldsymbol{T}^s/\Delta, \tag{2.1}$$

with $\boldsymbol{G}_0$ as before, $\boldsymbol{T}^s$ an Abelian torus, and Δ a finite subgroup.

Remark. The class of compact connected Lie groups of the form (2.1) coincides with the class of all compact connected Lie groups. We need only show that any compact connected Lie group $\boldsymbol{G}$ has the form (2.1). Let $\boldsymbol{G}_0 := \boldsymbol{G}'$ be the derived group and let the torus $T^s = \boldsymbol{Z}^0$ be the connected component of the identity of the center $\boldsymbol{Z}$ of $\boldsymbol{G}$. Then, the multiplication map $m : \boldsymbol{G}' \times \boldsymbol{Z}^0 \to \boldsymbol{G}$ is surjective. Indeed, it is a homomorphism since $\boldsymbol{Z}^0$ is central. Moreover, since $\mathrm{Lie}\,(\boldsymbol{G}^0) \oplus \mathrm{Lie}\,(\boldsymbol{Z}^0) = \mathrm{Lie}\,(\boldsymbol{G})$, the differential of m is surjective, so the image of m is open. Since $\boldsymbol{G}$ is connected, it follows that m is surjective. The kernel of m is obtained by embedding $\boldsymbol{G}' \cap \boldsymbol{Z}^0$ in $\boldsymbol{G}' \times \boldsymbol{Z}^0$ via $\gamma \to (\gamma, \gamma^{-1})$, $\gamma \in \boldsymbol{G}' \cap \boldsymbol{Z}^0$. The image of such a map is the kernel of m and is a finite group.

The parameterization of $\boldsymbol{G}$ from $\boldsymbol{G}_0$ is quite elementary, and we can concentrate here on the parameterization of $\boldsymbol{G}_0$ only. From now on we will assume that

$$\boldsymbol{G} \equiv \boldsymbol{G}_0. \tag{2.2}$$

Note that $\boldsymbol{H}$ is symmetrically embedded in $\boldsymbol{G}$, but is not maximal, unless $\boldsymbol{G}$ is simple.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras associated to $\boldsymbol{G}$ and $\boldsymbol{H}$ respectively. In this section, we will assume that $\boldsymbol{H}$ is MCS: with this, we mean that $\boldsymbol{H}_i$ is an MCS subgroup of $\boldsymbol{G}_i$. Since in this case $\mathrm{rank}\,(\boldsymbol{G}/\boldsymbol{H}) = \mathrm{rank}\,\boldsymbol{G}$, we can choose a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that $\mathfrak{c} \cap \mathfrak{h} = 0$. Thus, the generalized Euler parameterization of the group $\boldsymbol{G}$ w.r.t. $\boldsymbol{H}$ takes the form

$$\boldsymbol{G} = (\boldsymbol{H}'/\boldsymbol{\Gamma}) e^{\mathfrak{c}} \boldsymbol{H}, \tag{2.3}$$

where $\boldsymbol{H}'$ is a copy of $\boldsymbol{H}$ and $\boldsymbol{\Gamma} = \boldsymbol{H} \cap e^{\mathfrak{c}}$ is a finite subgroup of the maximal torus that will be specified later. In Section 3 we will extend the parameterization to an arbitrary symmetrically embedded subgroup. Before entering the details of the construction, we need to specify some further technical facts.

2.2. **Some technical facts and definitions.** Any finite dimensional semisimple Lie algebra $\mathfrak{g}$ admits a unique compact form. There is a unique (up to isomorphisms) simply connected, compact Lie group $\tilde{\boldsymbol{G}}$ having $\mathfrak{g}$ as the associated Lie algebra. However, more in general, there is more than one connected compact Lie group having the same Lie algebra. These are a finite number:

$$\boldsymbol{G}^k = \tilde{\boldsymbol{G}}/\Gamma^k, \qquad k = 1, 2, \ldots, m, \tag{2.4}$$

where Γ^k are finite subgroups of the center Z of $\tilde{\boldsymbol{G}}$. Notice that $\tilde{\boldsymbol{G}}$ has finite center (cf. [9], Proposition 23.11, page 200). In particular, we set $\boldsymbol{G}^1 \equiv \tilde{\boldsymbol{G}}$ and $\boldsymbol{G}^m \equiv \tilde{\boldsymbol{G}}/Z =: \boldsymbol{G}_Z$. Then each $\boldsymbol{G}^k$ is a covering of $\boldsymbol{G}_Z$ and is covered by $\tilde{\boldsymbol{G}}$. It

is known that each of such groups admits a faithful linear representation (cf. [9], Theorem 4.2, page 26). Let (R_k, V_k) be such a representation for $\boldsymbol{G}^k$ (in particular $(Ad, \mathfrak{g})$ is faithful for $\boldsymbol{G}_Z$). It induces a faithful representation (ρ_i, V_i) of $\mathfrak{g}$, so that the following diagram is commutative:

$$\begin{tikzcd} \boldsymbol{G}^k \arrow[r, "R_k"] & Aut(V_k) \\ \mathfrak{g} \arrow[u, "\exp_{\boldsymbol{G}^k}"] \arrow[r, "\rho_k"] & End(V_k) \arrow[u, "\mathrm{Exp}"'] \end{tikzcd}$$

Since $\boldsymbol{G}^k$ is compact, R_k is injective and continuous, and $\mathrm{Aut}(V_k)$ is T_2, then $\boldsymbol{G}^k$ and $R_k(\boldsymbol{G}^k)$ are homeomorphic (see [12], Theorem 8.8), and, in particular, they have the same fundamental group.[2] More generally, this means that we can construct a realization of the desired compact form simply by exponentiating the matrices associated to the Lie algebra $\mathfrak{g}$ via the representation ρ_k induced by the faithful representation R_k of $\boldsymbol{G}^k$. For this reason, we will call ρ_k a $\boldsymbol{G}^k$*-faithful representation.* Thus, we parameterize the desired compact $\boldsymbol{G}$ form by working with the right $\boldsymbol{G}$-faithful representation of the algebra.

Let $\boldsymbol{H}$ be a subgroup of $\boldsymbol{G}$ as defined previously. If we are working with a $\boldsymbol{G}$-faithful representation (ρ, V) for $\mathfrak{g}$, then ρ will decompose into a direct sum of representations of the Lie algebra $\mathfrak{h}$ of $\boldsymbol{H}$, among which at least one is surely $\boldsymbol{H}$-faithful (whereas the complementary ones will give rise to compact forms of the group covered by $\boldsymbol{H}$). As a consequence, one can construct the corresponding parameterization of $\boldsymbol{H}$ by worrying about the $\boldsymbol{H}$-faithful representation only, which is automatically present in the decomposition.

2.3. **Parameterization.** The problem of parameterizing $\boldsymbol{H}$ and $\boldsymbol{H'}$ is then the same as for $\boldsymbol{G}$ and can be obtained inductively. Thus, if we want to get an almost everywhere one to one parameterization of $\boldsymbol{G}$, the only problem is to determine the right range for the parameterization of the toric part

$$e^{\mathfrak{c}}[y_1, \ldots, y_r] = \exp(y_1 c_1 + \ldots + y_r c_r). \tag{2.5}$$

Now, we will see that the range for the y's is independent from the starting $\boldsymbol{G}$-faithful representation, depending on the adjoint representation only. This means that such determination is in a sense universal, and the details discriminating among the different compact forms of $\boldsymbol{G}$ will depend only on the periodicities of the $U(1)$ factors entering the parameterizations and the action of the finite subgroups. Furthermore, employing the isomorphism $\boldsymbol{G} \simeq \boldsymbol{G}_1 \times \boldsymbol{G}_2 \times \ldots \times \boldsymbol{G}_n$, we can parameterize each factor independently. Thus without loss of generality, we focus on the case when $\boldsymbol{G}$ is simple. In particular, then, $\boldsymbol{H}$ is maximal in $\boldsymbol{G}$.

Using the notation in Section 2.1, we write $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{c} \oplus \mathfrak{p}'$, with $\dim \mathfrak{h} = \dim \mathfrak{p}' = h$ and $\dim \mathfrak{c} = r$. Now consider the complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$. It also has the decomposition $\mathfrak{g}_{\mathbf{C}} = \boldsymbol{W}_- \oplus \mathfrak{c} \oplus \boldsymbol{W}_+$, where $\boldsymbol{W}_\pm$ is the direct sum of the root spaces $\boldsymbol{W}_{\pm\alpha}$ such that α is a positive root. We can thus pick out the following two bases for $\mathfrak{g}_{\mathbf{C}}$:

- $\{c_i\}_{i=1}^r \cup \{\lambda_{\alpha_a} \cup \lambda_{-\alpha_a}\}_{a=1}^h$, where λ_α is the eigenvector corresponding to the root α and the α_a are the positive roots;

[2]We are grateful to S. Pigola for explaining these points to us.

- $\{c_i\}_{i=1}^r \cup \{t_a\}_{a=1}^h \cup \{p_b\}_{b=1}^h$, where t_a and p_b generate $\mathfrak{h}$ and $\mathfrak{p}'$ respectively and are chosen so that ad_{t_a} and ad_{p_b} are diagonalizable, and the decomposition is Killing orthogonal.

Notice that only the second one is a basis for the compact algebra $\mathfrak{g}$. It satisfies the following relations:

$$\begin{aligned} &[t_a, t_b] \in \mathfrak{h}, \qquad [t_a, c_i] \in \mathfrak{p}', \qquad [t_a, p_b] \in \mathfrak{p}, \\ &[c_i, c_j] = 0, \qquad [c_i, p_b] \in \mathfrak{h}, \qquad [p_a, p_b] \in \mathfrak{h}. \end{aligned} \tag{2.6}$$

Indeed, the maximal symmetrically embedded compact subalgebras are in biunivocal correspondence with the real forms or, equivalently, with the Cartan decompositions of the algebra. This means that there exists an involution $\theta : \mathfrak{g} \to \mathfrak{g}$, $\theta^2 = \mathrm{id}$, such that $\mathfrak{h}$ and $\mathfrak{p}$ are the corresponding eigenspaces, with eigenvalues 1 and -1 respectively. Since θ is a homomorphism, this implies

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}, \tag{2.7}$$

and the Killing orthogonality between the two spaces. Moreover, ad-invariance of the Killing form $\langle , \rangle$ implies

$$\langle [t_a, c_i], c_j \rangle = \langle t_a, [c_i, c_j] \rangle = 0, \tag{2.8}$$

so that $[t_a, c_i] \in \mathfrak{p}'$.

The rules $[c_i, p_b] \in \mathfrak{h}$, $[t_a, c_i] \in \mathfrak{p}'$ allow us to provide a simple relation between the two bases defined above. Indeed, we can obtain from $\mathfrak{g}$ a new real form of $\mathfrak{g}_{\mathbb{C}}$ by means of Weyl's unitary trick, which consists of defining the new generators

$$\tilde{t}_a = t_a, \qquad \tilde{c}_j = ic_j, \qquad \tilde{p}_b = ip_b.$$

This is the noncompact form $\mathfrak{g}_{(r)}$, with signature r. In this case, the operators $\mathrm{ad}_{\tilde{h}_j}$ are represented by symmetric matrices since ad-invariance and symmetry of the Killing form give

$$\langle [\tilde{c}_i, \tilde{p}_a], \tilde{t}_b \rangle = -\langle \tilde{p}_a, [\tilde{c}_i, \tilde{t}_b] \rangle, \tag{2.9}$$

and the form is positive definite over the $\tilde{t}_b$ and negative over the complementary space. This means that such matrices can be diagonalized, with real eigenvalues, by means of real combinations of the vectors $\tilde{t}_a$, $\tilde{p}_b$. Then, an eigenvector corresponding to a nonzero root α will have the form $\lambda_\alpha = t_\alpha + ip_\alpha$, with $t_\alpha \in \mathfrak{h}$ and $p_\alpha \in \mathfrak{p}$. Notice that both t_α and p_α are necessarily nonvanishing. Indeed, $[c, \lambda_\alpha] = \alpha(c)\lambda_\alpha$ for all $c \in \mathfrak{c}$ implies

$$[c, t_\alpha] = i\alpha(c)p_\alpha, \quad [c, p_\alpha] = -i\alpha(c)t_\alpha, \tag{2.10}$$

and $t_\alpha = 0$ or $p_\alpha = 0$ would imply $\alpha(c) = 0$ for any $c \in \mathfrak{c}$.

In conclusion, we can choose the basis t_a, p_b, c_i so that the relation between the two bases is

$$\lambda_{\alpha_a} = t_a + ip_a, \qquad \lambda_{-\alpha_a} = t_a - ip_a, \qquad a = 1, \ldots, h. \tag{2.11}$$

Moreover, since $\langle \lambda_\alpha, \lambda_\beta \rangle \neq 0$ if and only if $\alpha + \beta = 0$, we can normalize the basis so that it becomes an orthonormal basis.

Now, we show that these known facts have interesting consequences for the Euler parameterization. We can write

$$\boldsymbol{G}[x_1, \ldots, x_h, y_1, \ldots, y_r, z_1, \ldots, z_h] = e^{\sum_{a=1}^h x_a t_a} e^{\sum_{i=1}^r y_i c_i} e^{\sum_{b=1}^h z_b t_b} \equiv (\boldsymbol{H'}/\boldsymbol{\Gamma}) e^{\mathfrak{c}} \boldsymbol{H}. \tag{2.12}$$

The invariant measure expressed in terms of this parameterization is

$$d\mu_{\boldsymbol{G}}[\vec{x};\vec{y};\vec{z}] = d\mu_{\boldsymbol{H}}[\vec{z}]d\mu_{\boldsymbol{B}}[\vec{x};\vec{y}], \tag{2.13}$$

where $d\mu_{\boldsymbol{H}}[\vec{z}]$ is the invariant measure associated to $\boldsymbol{H}$ and

$$d\mu_{\boldsymbol{B}}[\vec{x};\vec{y}] = \det J(\vec{x},\vec{y}) \prod_{a=1}^{h} dx_a \prod_{i=1}^{r} dy_i, \tag{2.14}$$

J being the $h \times h$ matrix with components

$$J^a_{\ b} := \langle e^{-\mathfrak{c}} \boldsymbol{H'}^{-1} \frac{\partial \boldsymbol{H'}}{\partial x_a} e^{\mathfrak{c}}, p_b \rangle. \tag{2.15}$$

Notice that $\boldsymbol{H'}^{-1} d\boldsymbol{H'} =: J_{\boldsymbol{H}}$ is the left invariant one form for the $\boldsymbol{H}$ subgroup in the $\boldsymbol{H'}$ parameterization, $J_{\boldsymbol{H}} = \sum_{a=1}^{h} J^a_{\boldsymbol{H}} t_a$. Thus

$$d\mu_{\boldsymbol{B}}[\vec{x};\vec{y}] = d\mu_{\boldsymbol{H}}[\vec{x}] \det M \prod_{i=1}^{r} dy_i, \qquad M^a_{\ b} := \langle e^{-\mathfrak{c}} t_a e^{\mathfrak{c}}, p_b \rangle. \tag{2.16}$$

Now $t_a = (\lambda_{\alpha_a} + \lambda_{-\alpha_a})/2$ so that

$$e^{-\mathfrak{c}} t_a e^{\mathfrak{c}} = \cosh(\alpha_a(\mathfrak{c})) t_a + i \sinh(\alpha_a(\mathfrak{c})) p_a. \tag{2.17}$$

Since the roots are real on $\tilde{c}_i$, if we define $\vec{\alpha}_a \equiv (\alpha^1_a, \ldots, \alpha^r_a)$ with $\alpha^i_a = \alpha_a(\tilde{c}_i)$, we get $\alpha_a(\mathfrak{c}) = -i \sum_{i=1}^{r} \alpha^i_a y_i \equiv -i\vec{\alpha}_a \cdot \vec{y}$. Then

$$\det M = \prod_{a=1}^{h} \sin(\vec{\alpha}_a \cdot \vec{y}). \tag{2.18}$$

Thus, the invariant measure takes the form

$$d\mu_{\boldsymbol{G}}[\vec{x};\vec{y};\vec{z}] = d\mu_{\boldsymbol{H}}[\vec{z}] d\mu_{\boldsymbol{H}}[\vec{x}] \prod_{a=1}^{h} \sin(\vec{\alpha}_a \cdot \vec{y}) \prod_{i=1}^{r} dy_i. \tag{2.19}$$

The range of the z coordinates is such as to cover the subgroup $\boldsymbol{H}$, whereas the range R_y for the y coordinates is defined by the conditions $0 \le \vec{\alpha}_a \cdot \vec{y} \le \pi$, and the range for the x coordinates is such as to cover $\boldsymbol{H'}/\boldsymbol{\Gamma}$. In particular, as a consequence of equation (2.17), the range for the y_i's depends on the adjoint representation and not on the particular $\boldsymbol{G}$-faithful representation we are considering. Notice that equation (2.19) implies the interesting relation

$$\int_{R_y} \prod_{a=1}^{h} \sin(\vec{\alpha}_a \cdot \vec{y}) \prod_{i=1}^{r} dy_i = \frac{\mathrm{Vol}(\boldsymbol{G})\ |\boldsymbol{\Gamma}|}{\mathrm{Vol}(\boldsymbol{H})^2}, \tag{2.20}$$

where the volumes can be computed by means of Macdonald's formula [13] and $|\boldsymbol{\Gamma}|$ is the cardinality of $\boldsymbol{\Gamma}$.

When $\boldsymbol{G}$ is simply connected, it is easy to see that $\boldsymbol{\Gamma} \simeq \mathbb{Z}_2^r$. Indeed, the elements of $\boldsymbol{\Gamma} = \boldsymbol{H} \cap e^{\boldsymbol{c}}$ are the elements of $e^{\boldsymbol{c}}$ whose square is the identity (see [11], Section VII, Theorem 8.5). Since the basis $c_1, \ldots, c_r$ of $\boldsymbol{c}$ can be chosen so that e^{tc_i} has period T, $\boldsymbol{\Gamma}$ is generated by $e^{\frac{T}{2}c_i}$, which proves our claim. In particular, $|\boldsymbol{\Gamma}| = 2^r$.

When $\boldsymbol{G}$ is not simply connected, this is not true in general and Γ is isomorphic to a proper subgroup of $\mathbb{Z}_2^r$. Indeed, if $\Phi : \tilde{\boldsymbol{G}} \longrightarrow \boldsymbol{G}$ is the universal covering map, then $\Gamma \simeq \Phi(\mathbb{Z}_2^r)$.

2.4. **Connections with the generalized Dyson integrals.** Let us now look closer at the integral (2.20). It is convenient to introduce the following change of variables. Let $\alpha_{a_1}, \ldots, \alpha_{a_r}$ be simple roots. Then, we define the new coordinates s_i, $i = 1, \ldots, r$, by

$$s_i := \vec{y} \cdot \vec{\alpha}_{a_i}. \tag{2.21}$$

From this we get

$$ds_1 \wedge \ldots \wedge ds_r = V_F \prod_{i=1}^{r} \frac{\|\alpha_{a_i}\|^2}{2} dy_1 \wedge \ldots \wedge dy_r, \tag{2.22}$$

where V_F is the volume of the fundamental region (parallelogram) defined by the simple coroots $\check{\alpha}_{a_i}$. Then, the integral in (2.20) takes the form

$$I = \frac{2^r}{V_F \prod_{i=1}^{r} \|\alpha_{a_i}\|^2} \int_{\tilde{R}_s} \prod_{a=1}^{h} \sin(\vec{n}_a \cdot \vec{s}) \prod_{i=1}^{r} ds_i, \tag{2.23}$$

where $\vec{n}_a$ are the coordinates of the positive roots expressed w.r.t. the simple roots and take values in $\mathbb{N}^r$ and $\tilde{R}_s$ is the range for the s coordinates. In particular, for the simple roots we have

$$\prod_{i=1}^{r} \sin(\vec{n}_{a_i} \cdot \vec{s}) = \prod_{i=1}^{r} \sin(s_i), \tag{2.24}$$

so that the range of coordinates is a subset of the cube $0 \leq s_i \leq \pi$. The remaining conditions are $0 \leq \vec{n}_a \cdot \vec{s} \leq \pi$ for all the other positive roots. The hyperplanes $\vec{n}_a \cdot \vec{s} = k\pi$, with k an integer, cut the cube into a tiling whose sectors are all equivalent, being related to each other by the Weyl reflections. We know that for a simple group the highest root (relative to the simple root system) $\alpha_{\tilde{a}} = \sum_{i=1}^{r} \tilde{n}_i \alpha_{a_i}$ has the property $\tilde{n}_i \geq n_a^i$ for all a and i (indeed, it is nothing but the highest weight of the adjoint representation). Thus, the inequalities $0 \leq \vec{n}_a \cdot \vec{s} \leq \pi$ defining the tiling reduce just to one. Indeed,

$$0 \leq \vec{\tilde{n}} \cdot \vec{s} \leq \pi \tag{2.25}$$

inside the cube implies all the remaining inequalities and then defines a fundamental region Δ. The volume of this region is

$$V = \int_{\Delta} \prod_{i=1}^{r} ds_i = \frac{1}{\prod_{i=1}^{r} \tilde{n}_i} \int_{0 \leq y_1 + \ldots + y_r \leq \pi, 0 \leq y_i \leq \pi} \prod_{i=1}^{r} dy_i = \frac{\pi^r}{r! \prod_{i=1}^{r} \tilde{n}_i}, \tag{2.26}$$

whereas the torus has volume π^r, so that the number ν of elementary cells in the cube is

$$\nu = r! \prod_{i=1}^{r} \tilde{n}_i. \tag{2.27}$$

Thus, we can write

$$I = \frac{2^r}{V_F \prod_{i=1}^{r} \|\alpha_{a_i}\|^2} \frac{1}{r! \prod_{i=1}^{r} \tilde{n}_i} \int_{Q} \prod_{a=1}^{h} |\sin(\vec{n}_a \cdot \vec{s})| \prod_{i=1}^{r} ds_i, \tag{2.28}$$

Q being the cube. By setting $2s_i = \zeta_i$ this can also be written as

$$I = \frac{(2\pi)^r}{2^h V_F \prod_{i=1}^r \|\alpha_{a_i}\|^2} \frac{1}{r! \prod_{i=1}^r \tilde{n}_i} J_{\frac{1}{2}}, \tag{2.29}$$

$$J_{\frac{1}{2}} = \frac{1}{(2\pi)^r} \int_0^{2\pi} d\zeta_1 \ldots \int_0^{2\pi} d\zeta_r \prod_{\alpha \in R} (1 - e^{\vec{n}_\alpha \cdot \vec{\zeta}})^{\frac{1}{2}}. \tag{2.30}$$

Here $J_{\frac{1}{2}}$ is a generalized Dyson integral, as conjectured by Macdonald in [14], Conjecture 2.1", for any root system.

Conjecture 2.1". *For all $s \in \mathbb{C}$ with* $\mathrm{Re}(s) > 0$,

$$J_s = \frac{1}{(2\pi)^r} \int_0^{2\pi} d\zeta_1 \ldots \int_0^{2\pi} d\zeta_r \prod_{\alpha \in R} (1 - e^{\vec{n}_\alpha \cdot \vec{\zeta}})^s = \prod_{i=1}^r \frac{\Gamma(sd_i + 1)}{\Gamma(s+1)\Gamma(sd_i - s + 1)}. \tag{2.31}$$

This formula is known as Macdonald's conjecture; in fact it has been proven for all root systems [15]. From (2.20) and (2.29) we get

$$J_{\frac{1}{2}} = \frac{2^h V_F r! \prod_{i=1}^r (n_i \|\alpha_{a_i}\|^2)}{\pi^r} \frac{\mathrm{Vol}(\boldsymbol{G})}{\mathrm{Vol}(\boldsymbol{H})^2} \frac{|\boldsymbol{\Gamma}|}{2^r}. \tag{2.32}$$

The last factor is 1 for $\boldsymbol{G}$ simply connected. This formula provides a proof of (2.31) for $s = \frac{1}{2}$ and for all the reduced simple lattices.

3. Arbitrary maximal symmetric embedded subgroups

As in the previous section we restrict our attention to the case of $\boldsymbol{G}$ simple. Here we extend previous results to the general case when $\boldsymbol{H}$ is not an MCS. In this case

$$l := \mathrm{Rank}(\boldsymbol{G}/\boldsymbol{H}) < \mathrm{Rank}(\boldsymbol{G}) = r, \tag{3.1}$$

so that the largest possible intersection between the Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ and the complement of $\mathfrak{h}$ has dimension l. We choose the Cartan subalgebra $\mathfrak{c}$ just in this way, so that

$$\mathfrak{c} = \mathfrak{c}_h \oplus \mathfrak{c}_p, \qquad \mathfrak{c}_p := \mathfrak{c} \cap \mathfrak{p}, \qquad \dim \mathfrak{c}_p = l, \quad \mathfrak{c}_h \subset \mathfrak{h}. \tag{3.2}$$

Let us fix a basis $k_1, \ldots, k_s$ for $\mathfrak{c}_h$, $h_1, \ldots, h_l$ for $\mathfrak{c}_p$, $s + l = r$. Let $\mathfrak{k}$ be the largest Lie subalgebra of $\mathfrak{h}$ such that $[\mathfrak{k}, \mathfrak{c}_p] = 0$. It is the Lie algebra of the normalizer $\boldsymbol{K}$ of $\mathfrak{c}_p$ in $\boldsymbol{H}$. Thus, we can write $\mathfrak{h} =: \mathfrak{k} \oplus \tilde{\mathfrak{h}}$ and $\mathfrak{p} =: \mathfrak{c}_p \oplus \tilde{\mathfrak{p}}$, so that

$$\mathfrak{g} = (\mathfrak{k} \oplus \tilde{\mathfrak{h}}) \oplus (\mathfrak{c}_p \oplus \tilde{\mathfrak{p}}). \tag{3.3}$$

Since $\mathfrak{h}$ is maximal, we have

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}, \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{h}, \qquad [\mathfrak{h}, \mathfrak{p}] \subseteq \mathfrak{p}. \tag{3.4}$$

Moreover, $[\mathfrak{k}, \mathfrak{c}_p] = 0$ implies

$$[\tilde{\mathfrak{h}}, \mathfrak{c}_p] \subseteq \tilde{\mathfrak{p}}, \qquad [\tilde{\mathfrak{p}}, \mathfrak{c}_p] \subseteq \tilde{\mathfrak{h}}. \tag{3.5}$$

Notice that the roots of $\mathfrak{g}$ can be divided as follows. Since $\mathfrak{c}_h$ is the Cartan subalgebra of both $\mathfrak{k}$ and $\mathfrak{h}$, $\mathrm{Rank}(\mathfrak{k}) = \mathrm{Rank}(\mathfrak{h}) = s$. We represent the roots as the simultaneous eigenvalues of the operators $(\mathrm{ad}_{k_1}, \ldots, \mathrm{ad}_{k_s}; \mathrm{ad}_{h_1}, \ldots, \mathrm{ad}_{h_l})$. The eigenvectors of the roots $\alpha_{\mathfrak{h},a}$, $a = 1, \ldots, k - s$ ($k := \dim \boldsymbol{K}$), of $\mathfrak{k}$ are in the complexification of $\mathfrak{k}$ and thus in the kernel of ad_{h_i}, $i = 1, \ldots, l$: the last l components are zero. Indeed, these are all the nonvanishing roots with this property; the remaining ones have necessarily nonvanishing elements out of the first s ones. We

will call the corresponding roots $\alpha_{\mathfrak{p},b}$, $b = 1, \dots, 2q$, where q is the number of positive roots. This root system is not reduced so that each root $\alpha_{\mathfrak{p},b}$ has multiplicity m_b, and $\sum_{b=1}^{q} m_b = h - k$. Indeed, these correspond to the nonvanishing roots of the h_i. As usual, we can divide all roots in positive and negative, $R = R^+ \oplus R^-$. This will determine a corresponding decomposition of the restricted root system: $R_{\mathfrak{p}} = R_{\mathfrak{p}}^+ \oplus R_{\mathfrak{p}}^-$. The main difference w.r.t. the case of an MCS subgroup is that now $R_{\mathfrak{p}}$ is not a reduced root lattice system and generically each root α is characterized by a multiplicity $m_\alpha \geq 1$. All such systems are classified in [1]; see also [11].

From now on, we can proceed exactly as in the previous section, by choosing an orthonormal basis of $\mathfrak{g}$, $B = B_{\boldsymbol{K}} \cup \{t_1, \dots, t_{h-k}\} \cup \{h_1, \dots, h_l\} \cup \{p_1, \dots, p_{h-k}\}$, where $B_{\boldsymbol{K}} = \{k_1, \dots, k_s, g_1, \dots, g_{k-s}\}$ is an orthonormal basis for $\mathfrak{k}$, the t_a generate $\tilde{\mathfrak{h}}$, and the p_b generate $\tilde{\mathfrak{p}}$. The Euler parameterization for $\boldsymbol{G}$ is then

$$\boldsymbol{G}[\vec{x}; \vec{y}; \vec{z}] = e^{\sum_{a=1}^{h-k} x^a t_a} e^{\sum_{i=1}^{l} y^i h_i} \boldsymbol{H}[z_1, \dots, z_h], \tag{3.6}$$

where $\boldsymbol{H}$ can be parameterized itself by means of the Euler parameterization, but it is not important here. The range of the z coordinates must be chosen in such a way as to cover the whole subgroup $\boldsymbol{H}$. The invariant measure can be computed exactly as in the previous section, giving

$$d\mu_{\boldsymbol{G}}[\vec{x}; \vec{y}; \vec{z}] = d\mu_{\boldsymbol{H}}[\vec{z}]\; d\mu_{\boldsymbol{H}/\boldsymbol{K}}[\vec{x}] \prod_{a=1}^{q} \sin^{m_a}(\vec{\alpha}_{\mathfrak{p},a} \cdot \vec{y}) \prod_{i=1}^{l} dy_i, \tag{3.7}$$

where $\vec{\alpha}_{\mathfrak{p},a} := (\alpha^1_{\mathfrak{p},a}, \dots, \alpha^l_{\mathfrak{p},a})$, $a = 1, \dots, q$, are the last l components of the positive $\alpha_{\mathfrak{p}a}$, corresponding to the eigenvalues of the ad_{h_i} only. As before, we can choose a basis of l simple roots $\vec{\alpha}_1, \dots, \vec{\alpha}_l$ in $R_{\mathfrak{p}}^+$ to prove that the range for the coordinates $\vec{y}$ is given by

$$0 \leq \vec{\alpha}_i \cdot \vec{y} \leq \pi, \qquad 0 \leq \sum_{i=1}^{l} n_i \vec{\alpha}_i \cdot \vec{y} \leq \pi, \tag{3.8}$$

where $\sum_{i=1}^{l} n_i \vec{\alpha}_i$ is the highest root of the quotient manifold.

3.1. **Further connections with the generalized Dyson integrals.** In [14] Macdonald proposed a generalization of the Dyson integrals extended to not necessarily reduced root lattices. This general conjecture has been proved by Opdam [15] in the form:

$$\begin{aligned} J^{\mathfrak{p}}_{\{k_\alpha\}} &:= \frac{1}{(2\pi)^r} \int_0^{2\pi} d\zeta_1 \dots \int_0^{2\pi} d\zeta_r \prod_{\alpha \in R_{\mathfrak{p}}} (1 - e^{\vec{n}_\alpha \cdot \vec{\zeta}})^{k_\alpha} \\ &= \prod_{\alpha \in R_{\mathfrak{p}}^+} \frac{\Gamma(\langle \rho(k), \check{\alpha} \rangle + k_\alpha + \frac{1}{2} k_{\frac{\alpha}{2}} + 1) \Gamma(\langle \rho(k), \check{\alpha} \rangle - k_\alpha - \frac{1}{2} k_{\frac{\alpha}{2}} + 1)}{\Gamma(\langle \rho(k), \check{\alpha} \rangle + \frac{1}{2} k_{\frac{\alpha}{2}} + 1) \Gamma(\langle \rho(k), \check{\alpha} \rangle - \frac{1}{2} k_{\frac{\alpha}{2}} + 1)}, \end{aligned} \tag{3.9}$$

where $R_{\mathfrak{p}}$ is a root system, $R_{\mathfrak{p}}^+$ is a choice of corresponding positive roots,

$$\rho(k) = \frac{1}{2} \sum_{R_{\mathfrak{p}}^+} k_\alpha \alpha, \tag{3.10}$$

and k is a Weyl invariant function over $R_{\mathfrak{p}}$ whose values k_α have positive real part. For example, the multiplicities m_α select such a function. Finally, $\langle \rho(k), \check{\alpha}\rangle$ indicates the invariant product with the coroot $\check{\alpha}$.

Repeating the same procedure as in Section 2.4 we get the following formula:

$$J^{\mathfrak{p}}_{\{\frac{m_\alpha}{2}\}} = \frac{2^{h-k}|\vec{\alpha}_1 \wedge \ldots \wedge \vec{\alpha}_l| l! \prod_{i=1}^{l} n_i}{\pi^l} \frac{\mathrm{Vol}(\boldsymbol{G})\mathrm{Vol}(\boldsymbol{K})}{\mathrm{Vol}(\boldsymbol{H})^2}. \tag{3.11}$$

Compared with Theorem 4.1 in [15], with the invariant functions $k_\alpha = m_\alpha/2$, this expression indeed provides the right value for the generalized Dyson integrals $J^{\mathfrak{p}}_{\{\frac{m_\alpha}{2}\}}$, thus a proof of Macdonald's conjecture ([14], Conjecture 2.3) for $k_\alpha = \frac{m_\alpha}{2}$ and for the lattices associated to all the irreducible symmetric spaces. The ingredients necessary to compute (3.11) are given in Table 1. One then easily checks, case by case, that formula (3.11) provides the same result as (3.9).

4. Euler parameterizations of the simply connected simple Lie groups

As an application of our results, we summarize how to realize the generalized Euler parameterization of any simple, simply connected, compact Lie group $\boldsymbol{G}$ w.r.t. a maximal symmetrically embedded Lie subgroup $\boldsymbol{H}$. This is given by expression (3.6), which we repeat here for convenience:

$$\boldsymbol{G}[\vec{x};\vec{y};\vec{z}] = e^{\sum_{a=1}^{h-k} x^a t_a} e^{\sum_{i=1}^{l} y^i h_i} \boldsymbol{H}[z_1, \ldots, z_h]. \tag{4.1}$$

The parameterization of $\boldsymbol{H}[z_1, \ldots, z_h]$ can be done inductively in the same way. As we have seen, this is obtained by putting the subgroup $\boldsymbol{K}$ in evidence so that $z_1, \ldots, z_h$ are chosen in such a way as to cover the whole $\boldsymbol{H}$; the coordinates $x_1, \ldots, x_{h-k}$ have the same range as $z_1, \ldots, z_{h-k}$. Finally, the range for $y_1, \ldots, y_l$ is specified by (3.8). All possible Euler parameterizations of the simple, simply connected, compact Lie groups are listed in Table 1.

From the same table one can easily verify that formula (3.11) indeed agrees with (3.9), thus providing an alternative proof of the Macdonald-Dyson conjecture for all simple groups for the case $k_\alpha = \frac{m_\alpha}{2}$. The volumes of the groups can be computed as in [13].

We point out the fact that not all subgroups $\boldsymbol{H}$ and $\boldsymbol{K}$ are semisimple but can contain $U(1)$ factors which must be discussed separately. The measure is normalized so that the volume of a $U(1)$ factor is just the length of its period. It is interesting to notice that such periods can be related to the length of the roots. We will provide a proof of this fact together with a detailed construction of Table 1, which requires much more space, in a separate publication. Here we limit ourselves to specifying the length of the period for the $U(1)$ factors appearing in the table after normalizing the long roots of $\boldsymbol{G}$ to $\sqrt{2}$. They are the following:

- in the AIII_a case there is a phase factor in $\boldsymbol{H}$ with period $T_H = 2\pi\sqrt{\frac{p+q}{pq}}$, whereas $\boldsymbol{K}$ contains p phase factors with periods $T_i = \frac{2\pi}{i}\sqrt{2i(i+1)}$, for $i = 1, \ldots, p-1$ and $T_p = 2\pi\sqrt{\frac{2p(p+q)}{q-p}}$;
- in the AIII_b case there is a phase factor in $\boldsymbol{H}$ with period $T_H = 2\pi\sqrt{\frac{2}{p}}$, and $p-1$ phase factors in $\boldsymbol{K}$ with periods $T_i = \frac{2\pi}{i}\sqrt{2i(i+1)}$, for $i = 1, \ldots, p-1$;

- in the AIV case there is a phase factor in $\boldsymbol{H}$ with period $T_H = 2\pi\sqrt{\frac{n+1}{n}}$, and a phase factor in $\boldsymbol{K}$ with period $T_K = 2\pi\sqrt{\frac{n+1}{2(n-1)}}$;
- in the CI case the phase factor in $\boldsymbol{H}$ has period $T_H = 2\pi\sqrt{2n}$;
- in the DI_b case the phase factor in $\boldsymbol{K}$ has period $T_K = 4\pi$;
- in the DIII_a case the subgroup $\boldsymbol{H}$ is $U(2n+1) \simeq SU(2n+1) \times U(1)/\mathbb{Z}_{2n+1}$, and the period of the phase factor is $T_H = 4\pi\sqrt{2n+1}$, whereas the phase factor in $\boldsymbol{K}$ has period $T_K = 4\pi$;
- in the DIII_b case the subgroup $\boldsymbol{H}$ is $U(2n) \simeq SU(2n) \times U(1)/\mathbb{Z}_{2n}$, and the period of the phase factor is $T_H = 4\pi\sqrt{2n}$;
- in the EII case the two phase factors in $\boldsymbol{K}$ have periods $T_{K_1} = 4\pi$ and $T_{K_2} = 4\pi\sqrt{3}$;
- in the EIII case the periods of the phase factors in $\boldsymbol{H}$ and in $\boldsymbol{K}$ are $T_H = 4\pi\sqrt{3}$ and $T_K = 4\pi\sqrt{3}$;
- in the EVII case the period of the phase factor in $\boldsymbol{H}$ is $T_H = 2\pi\sqrt{\frac{3}{2}}$.

Moreover, there are some particular cases that must be considered separately in the table, so that we list them apart:

- AI: for $n = 1$, $\boldsymbol{H} = SO(2)$ with period $T = 4\pi$ and obviously α_h is not defined;
  for $n = 2$, $\boldsymbol{H} = SO(3)$ which has only the short root, so that $|\alpha_G|/|\alpha_H| = 2$;
- BI_a: for $n = 2$, $\boldsymbol{H} = SO(2) \times SO(3)$ whose phase factor has period $T = 4\pi$ and $|\alpha_G|/|\alpha_H| = \sqrt{2}$;
  for $n = 3$, $\boldsymbol{H} = SO(3) \times SO(4)$ and the ratios of the root lengths are $|\alpha_G|/|\alpha_{SO(3)}| = \sqrt{2}$ and $|\alpha_G|/|\alpha_{SO(4)}| = 1$;
- BI_b: for $p = 2$ and $q > 3$, $\boldsymbol{H} = SO(2) \times SO(q)$ whose phase factor has period $T = 4\pi$, and the ratio of the root lengths is $|\alpha_G|/|\alpha_{SO(q)} = 1$;
  for $p = 3$ and $q > 5$, $\boldsymbol{H} = SO(3) \times SO(q)$, and the ratios of the root lengths are $|\alpha_G|/|\alpha_{SO(3)}| = \sqrt{2}$ and $|\alpha_G|/|\alpha_{SO(q)}| = 1$;
- BII: for $n = 1$ is the same as AI for $n = 1$;
- DI_a: for $n = 2$, $\boldsymbol{H} = SO(2) \times SO(2)$ whose phase factors have both period $T = 4\pi$;
- DI_b: for $n = 3$, $\boldsymbol{H} = SO(2) \times SO(4)$ whose phase factor has period $T = 4\pi$;
- DI_c: for $p = 2$, $q > 3$, $\boldsymbol{H} = SO(2) \times SO(q)$ whose phase factor has period $T = 4\pi$;
  for $p = 3$ and $q > 4$, $\boldsymbol{H} = SO(3) \times SO(q)$, and the ratios of the root lengths are $|\alpha_G|/|\alpha_{SO(3)}| = \sqrt{2}$ and $|\alpha_G|/|\alpha_{SO(q)}| = 1$;
- DII: for $n = 2$, $\boldsymbol{H} = SO(3)$, and the ratio of the root lengths is $|\alpha_G|/|\alpha_{SO(3)}| = \sqrt{2}$.

In this list the unspecified data can be read from Table 1. Note that in the table we are referring to the universal coverings, so that $\boldsymbol{SO}(3) \simeq \boldsymbol{SU}(2)$, $\boldsymbol{USp}(4) \simeq \boldsymbol{SO}(5)$, $\boldsymbol{USp}(2) \simeq \boldsymbol{SU}(2)$, $\boldsymbol{SO}(6) \simeq \boldsymbol{SU}(4)$, $\boldsymbol{SO}(4) \simeq \boldsymbol{SU}(2) \times \boldsymbol{SU}(2)$, and $\boldsymbol{SO}(n) \simeq \boldsymbol{Spin}(n)$; in EVII $\boldsymbol{Ss}(16) \simeq \boldsymbol{SO}(16)/\boldsymbol{\mathbb{Z}}_2$ is a semispin group. In the second column we indicate the compact form associated to the real form listed in the third column. Z indicates the center of the compact form. In particular, Z is $\mathbb{Z}_4$ if the dimension of the spin group is $4k+2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ if the dimension is $4k$.

TABLE 1. Maximal symmetrically embedded proper subgroups for the compact simple Lie algebras.

Label	$\mathbf{G}_c$	$\mathbf{G}_{nc}$	$\dim(\mathbf{G})$	$\boldsymbol{Z}$	MCS $\mathbf{H}$	$\dim(\mathbf{H})$	$\boldsymbol{\Lambda_{G/H}}$	$(\boldsymbol{n_1},\ldots,\boldsymbol{n_r})$	$\lvert\boldsymbol{\alpha_G}\rvert/\lvert\boldsymbol{\alpha_H}\rvert$	$\vec{\boldsymbol{m}}_{\boldsymbol{\lambda}}$, $\vec{\boldsymbol{m}}_{\boldsymbol{2\lambda}}$	$\lvert\boldsymbol{\alpha_G}\rvert/\lvert\boldsymbol{\alpha_{G/H}}\rvert$	$\boldsymbol{K}$	$\lvert\boldsymbol{\alpha_H}\rvert/\lvert\boldsymbol{\alpha_K}\rvert$	$\boldsymbol{\rho}$
AI	$\mathbf{SU}$(n+1)	$\boldsymbol{SL}(n+1,\mathbb{R})$	n^2+2n	$\mathbf{Z}_{n+1}$	$\mathbf{SO}$(n+1)/$\mathbf{Z}_2$	$n(n+1)/2$	$A_n\,(n\geq 1)$	(1,1,...,1)	$\sqrt{2}$	(1), (0)	1	$\mathbf{Z}_2^n$	-	V_{λ_1}
AII	$\mathbf{SU}(2n)$	$\mathbf{SU}^*(2n)$	$4n^2-1$	$\mathbf{Z}_{2n}$	$\mathbf{USp}$(2n)	$2n^2+n$	$A_{n-1}\,(n>1)$	(1,1,...,1)	1	(4), (0)	$\sqrt{2}$	$\mathbf{SU}(2)^n$	1	V_{λ_1}
AIII$_a$	$\mathbf{SU}$(p+q)	$\mathbf{SU}$(p,q)	$(p+q)^2-1$	$\mathbf{Z}_{p+q}$	$\boldsymbol{S}(\boldsymbol{U}(p)\times\boldsymbol{U}(q))$	p^2+q^2-1	$B_p\,(1<p<q)$	(1,2,...,2)	1	$2(1,q-p),(0,1)$	1	$S(\mathbf{U}(1)^p\times\mathbf{U}(q-p))$	1	V_{λ_1}
AIII$_b$	$\mathbf{SU}(2p)$	$\mathbf{SU}$(p,p)	$4p^2-1$	$\mathbf{Z}_{2p}$	$\boldsymbol{S}(\boldsymbol{U}(p)\times\boldsymbol{U}(p))$	$2p^2-1$	$C_p\,(p>1)$	(2,2,...,2,1)	1	(1,2), (0,0)	1	$\boldsymbol{S}(\boldsymbol{U}(1)^p)=\boldsymbol{SO}(2)^{p-1}\times\mathbf{Z}_2$	1	V_{λ_1}
AIV	$\mathbf{SU}(n+1)$	$\mathbf{SU}(1,n)$	n^2+2n	$\mathbf{Z}_{n+1}$	$\boldsymbol{S}(\boldsymbol{U}(1)\times\boldsymbol{U}(n))$	n^2	A_1	(2)	1	(2n-2), (1)	1	$\boldsymbol{S}(\boldsymbol{U}(n-1)\times\boldsymbol{U}(1))$	1	V_{λ_1}
BI$_a$	$\mathbf{SO}(2n+1)$	$\mathbf{SO}_0(n,n+1)$	$2n^2+n$	$\mathbf{Z}_2$	$\mathbf{SO}$(n)×$\mathbf{SO}$(n+1)	n^2	$B_n\,(n>3)$	(1,2,...,2)	1	(1,1), (0,0)	1	$\mathbf{Z}_2^n$	-	V_{λ_n}
BI$_b$	$\mathbf{SO}(p+q)=\mathbf{SO}(2n+1)$	$\mathbf{SO}_0(p,q)$	(p+q)(p+q-1)/2	$\mathbf{Z}_2$	$\mathbf{SO}$(p)×$\mathbf{SO}$(q)	$p(p-1)/2+q(q-1)/2$	$B_p\,(1<p<n)$	(1,2,...,2)	1	(1,2(n-p)+1), (0,0)	1	$\boldsymbol{SO}(q-p)\times\mathbf{Z}_2\ltimes\mathbf{Z}_2^p$	1	V_{λ_n}
BII	$\mathbf{SO}(2n+1)$	$\mathbf{SO}_0(1,2n)$	$2n^2+n$	$\mathbf{Z}_2$	$\mathbf{SO}(2n)$	n(2n-1)	A_1	(1)	1	(2n-1), (0)	$\sqrt{2}$	$\boldsymbol{SO}(2n-1)$	1	V_{λ_n}
CI	$\mathbf{USp}(2n)$	$\mathbf{Sp}(2n,\mathbb{R})$	$2n^2+n$	$\mathbf{Z}_2$	$\mathbf{U}$(n)	n^2	$C_n\,(n\geq 3)$	(2,2,...,2,1)	$\sqrt{2}$	(1,1), (0,0)	1	$\mathbf{Z}_2^n$	-	V_{λ_1}
CII_a	$\mathbf{USp}$(2p+2q)=$\mathbf{USp}$(2n)	$\mathbf{USp}$(2p,2q)	$(p+q)(2p+2q+1)$	$\mathbf{Z}_2$	$\mathbf{USp}$(2p)×$\mathbf{USp}$(2q)	$2p^2+p+2q^2+q$	$B_p\,(1\leq p\leq(n-1)/2)$	(2,2,...,2)	1	(4,4n-8p), (0,3)	1	$\boldsymbol{USp}(2q-2p)\times\boldsymbol{SU}(2)^p$	1	V_{λ_1}
CII_b	$\mathbf{USp}$(4n)	$\mathbf{USp}$(2n,2n)	$8n^2+2n$	$\mathbf{Z}_2$	$\mathbf{USp}$(2n)×$\mathbf{USp}$(2n)	$4n^2+2n$	C_n	(2,2,...,2,1)	1	(3,4), (0,0)	$\sqrt{2}$	$\boldsymbol{SU}(2)^n$	$\sqrt{2}$	V_{λ_1}
DI_a	$\mathbf{SO}(2n)$	$\mathbf{SO}$(n,n)	$n(2n-1)$	Z	$\mathbf{SO}$(n)×$\mathbf{SO}$(n)	$n(n-1)$	$D_n\,(n>1)$	(1,2,...,2,1,1)	1	(1), (0)	1	$\mathbf{Z}_2^n$	-	V_{λ_n}
DI_b	$\mathbf{SO}(2n)$	$\mathbf{SO}$(n-1,n+1)	$n(2n-1)$	Z	$\mathbf{SO}$(n-1)×$\mathbf{SO}$(n+1)	n^2-n+1	$B_{n-1}\,(n>2)$	(1,2,...,2)	1	(1,2), (0,0)	1	$\boldsymbol{U}(1)\times\mathbf{Z}_2\ltimes\mathbf{Z}_2^{n-1}$	1	V_{λ_n}
DI_c	$\mathbf{SO}$(p+q)=$\mathbf{SO}(2n)$	$\mathbf{SO}$(p,q)	(p+q)(p+q-1)/2	Z	$\mathbf{SO}$(p)×$\mathbf{SO}$(q)	$p(p-1)/2+q(q-1)/2$	$B_p\,(1<p<n-1)$	(1,2,...,2)	1	(1,2(n-p)), (0,0)	1	$\boldsymbol{SO}(q-p)\times\mathbf{Z}_2\ltimes\mathbf{Z}_2^p$	1	V_{λ_n}
DII	$\mathbf{SO}(2n)$	$\mathbf{SO}$(1,2n-1)	$n(2n-1)$	Z	$\mathbf{SO}$(2n-1)	$(2n-1)(n-1)$	A_1	(1)	1	(2n-2), (0)	1	$\mathbf{SO}$(2n-2)	1	V_{λ_n}
$DIII_a$	$\mathbf{SO}$(4n+2)	$\boldsymbol{SO}^*(4n+2)$	$(2n+1)(4n+1)$	$\mathbf{Z}_4$	$\mathbf{U}$(2n+1)	$(2n+1)^2$	$B_n\,(n\geq 2)$	(2,2,...,2)	1	(4,4), (0,1)	1	$\boldsymbol{SU}(2)^n\times\boldsymbol{SO}(2)$	2	$V_{\lambda_{2n+1}}$
$DIII_b$	$\mathbf{SO}(4n)$	$\boldsymbol{SO}^*(4n)$	$2n(4n-1)$	$\mathbf{Z}_2\times\mathbf{Z}_2$	$\mathbf{U}(2n)$	$4n^2$	$C_n\,(n\geq 2)$	(2,2,...,2,1)	1	(1,4), (0,0)	1	$\boldsymbol{SU}(2)^n$	1	$V_{\lambda_{2n}}$
EI	$\mathbf{E}_{6(-78)}$	$\mathbf{E}_{6(6)}$	78	$\mathbf{Z}_3$	$\mathbf{USp}(8)/\mathbf{Z_2}$	36	E_6	(1,2,2,3,2,1)	1	(1), (0)	1	$\mathbf{Z}_2^6$	-	V_{λ_1}
EII	$\mathbf{E}_{6(-78)}$	$\mathbf{E}_{6(2)}$	78	$\mathbf{Z}_3$	$(\mathbf{USp}(2)\times\mathbf{SU}(6))/\mathbf{Z}_2$	38	F_4	(2,3,4,2)	1	(1,2), (0,0)	1	$\boldsymbol{U}(1)^2\times\mathbf{Z}_2$	1	V_{λ_1}
EIII	$\mathbf{E}_{6(-78)}$	$\mathbf{E}_{6(-14)}$	78	$\mathbf{Z}_3$	$(\mathbf{U}(1)\times\mathbf{SO}(10))/\mathbf{Z}_4$	46	B_2	(2,2)	1	(6,8), (0,1)	1	$(\mathbf{SO}(6)\times\mathbf{U}(1))/\mathbf{Z}_2$	1	V_{λ_1}
EIV	$\mathbf{E}_{6(-78)}$	$\mathbf{E}_{6(-26)}$	78	$\mathbf{Z}_3$	$\boldsymbol{F}_4$	52	A_2	(1,1)	1	(8), (0)	$\sqrt{2}$	$\boldsymbol{SO}(8)$	1	V_{λ_1}
EV	$\mathbf{E}_{7(-133)}$	$\mathbf{E}_{7(7)}$	133	$\mathbf{Z}_2$	$\mathbf{SU}(8)/\mathbf{Z}_2$	63	E_7	(2,2,3,4,3,2,1)	1	(1), (0)	1	$\mathbf{Z}_2^7$	-	V_{λ_6}
EVI	$\mathbf{E}_{7(-133)}$	$\mathbf{E}_{7(-5)}$	133	$\mathbf{Z}_2$	$(\boldsymbol{SU}(2)\times\boldsymbol{SO}(12))/\mathbf{Z}_2$	69	F_4	(2,3,4,2)	1	(1,4), (0,0)	1	$\boldsymbol{SU}(2)^3\times\mathbf{Z}_2\times\mathbf{Z}_2$	1	V_{λ_6}
EVII	$\mathbf{E}_{7(-133)}$	$\mathbf{E}_{7(-25)}$	133	$\mathbf{Z}_2$	$(\boldsymbol{U}(1)\times E_6)/\mathbf{Z}_3$	79	C_3	(2,2,1)	1	(1,8), (0,0)	$\sqrt{2}$	$\boldsymbol{SO}(8)$	1	V_{λ_6}
EVIII	$\mathbf{E}_{8(-248)}$	$\mathbf{E}_{8(8)}$	248	$\mathbf{1}$	$\mathbf{Ss}(16)$	120	E_8	(2,3,4,6,5,4,3,2)	1	(1), (0)	1	$\mathbf{Z}_2^8$	-	Any
EIX	$\mathbf{E}_{8(-248)}$	$\mathbf{E}_{8(-24)}$	248	$\mathbf{1}$	$(\boldsymbol{SU}(2)\times E_7)/\mathbf{Z}_2$	136	F_4	(2,3,4,2)	1	(1,8), (0,0)	1	$\boldsymbol{SO}(8)\times\mathbf{Z}_2\times\mathbf{Z}_2$	1	Any
FI	$\mathbf{F}_{4(-52)}$	$\mathbf{F}_{4(4)}$	52	$\mathbf{1}$	$(\mathbf{USp}(6)\times\mathbf{USp}(2))/\mathbf{Z}_2$	24	F_4	(2,3,4,2)	1	(1,1), (0,0)	1	$\mathbf{Z}_2^4$	-	Any
FII	$\mathbf{F}_{4(-52)}$	$\mathbf{F}_{4(-20)}$	52	$\mathbf{1}$	$\mathbf{SO}(9)$	36	A_1	(2)	1	(8), (7)	$2\sqrt{2}$	$\mathbf{SO}(7)$	1	Any
G	$\mathbf{G}_{2(-14)}$	$\mathbf{G}_{2(2)}$	14	$\mathbf{1}$	$\mathbf{SO}(4)/\mathbf{Z}_2$	6	G_2	(3,2)	1	(1,1), (0,0)	1	$\mathbf{Z}_2^2$	-	Any

In the column $\Lambda_{G/H}$ we indicate the reduced form of the root system associated to the symmetric space G/H. However, these in general can also contain double roots. Notice that the rank of the reduced system gives the rank of the symmetric space. The quotients $|\alpha_G|/|\alpha_H|$, $|\alpha_G|/|\alpha_{G/H}|$ and $|\alpha_H|/|\alpha_K|$ indicate the ratio of the long roots (including eventual double roots) of the indicated root systems. $(n_1, \ldots, n_r)$ are the coefficients of the highest root of the root system for the symmetric manifold. $m_\lambda = (m_{\lambda_l}, m_{\lambda_s})$ and $m_{2\lambda} = (m_{2\lambda_l}, m_{2\lambda_s})$ indicate the multiplicities of the roots of the reduced lattice and of the double roots respectively, where l and s denote long and short respectively. In the last column ρ denotes a choice for a G-faithful representation of the algebra; V_{λ_i} means the fundamental representation associated to the i-th weight in the corresponding Dynkin diagram.

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