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THE ENUMERATION OF GENERALIZED TAMARI INTERVALS

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Abstract. For any finite path v on the square grid consisting of north and east unit steps, starting at (0,0), we construct a poset $\mathrm{Tam}(v)$ that consists of all the paths weakly above v with the same number of north and east steps as v. For particular choices of v, we recover the traditional Tamari lattice and the m-Tamari lattice.

Let $\overleftarrow{v}$ be the path obtained from v by reading the unit steps of v in reverse order, replacing the east steps by north steps and vice versa. We show that the poset $\mathrm{Tam}(v)$ is isomorphic to the dual of the poset $\mathrm{Tam}(\overleftarrow{v})$. We do so by showing bijectively that the poset $\mathrm{Tam}(v)$ is isomorphic to the poset based on rotation of full binary trees with the fixed canopy v, from which the duality follows easily. This also shows that $\mathrm{Tam}(v)$ is a lattice for any path v. We also obtain as a corollary of this bijection that the usual Tamari lattice, based on Dyck paths of height n, can be partitioned into the (smaller) lattices $\mathrm{Tam}(v)$, where the v are all the paths on the square grid that consist of $n-1$ unit steps.

We explain possible connections between the poset $\mathrm{Tam}(v)$ and (the combinatorics of) the generalized diagonal coinvariant spaces of the symmetric group.

1. Introduction

In this article, we generalize the m-Tamari lattice to posets of arbitrary paths, as explained in Section 1.3. We prove that these posets are actually lattices, that they satisfy a duality property, and that they partition the ordinary Tamari lattice into intervals. We first introduce some basic definitions in Section 1.1 and some motivations in Section 1.2.

This article is the full version of the extended abstract [23].

1.1. **Basic definitions.** A binary tree is defined recursively as follows. A binary tree T is either the empty set or else a triple (L, r, R) where L and R are binary trees and r is the root vertex of T. The binary trees L and R are called respectively the left and the right subtrees of r. The root of L (respectively R) is called the left (respectively right) child of T. The degree of a vertex is its number of children, which is either 0, 1 or 2 for all the vertices of a binary tree. An external vertex is a vertex with degree 0, and an internal vertex is a vertex of degree 1 or 2. An external edge is an edge adjacent to an external vertex. For s a vertex in T with left and right subtrees L_s and R_s, the subtree at s in T is the (binary) tree (L_s, s, R_s). The vertices in the subtree at s are the descendants of s. A complete binary tree is a binary tree such that all the vertices have degrees 0 or 2. For more on trees and other combinatorial structures see [24].

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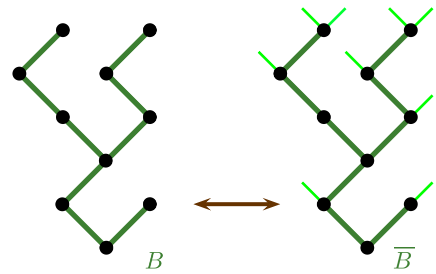

FIGURE 1. A binary tree B and its associated complete binary tree $\overline{B}$.

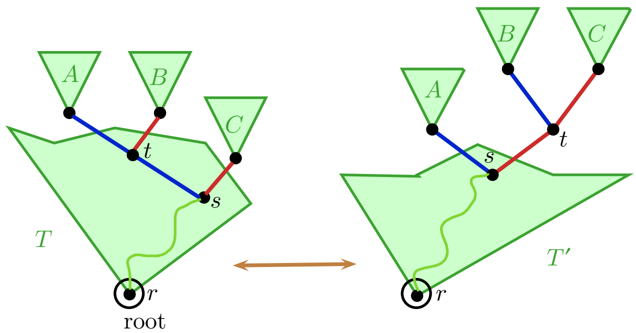

FIGURE 2. Right rotation on a complete binary tree: the covering relation in the Tamari lattice.

It is easy to establish a bijection between binary trees with n vertices and complete binary trees with n internal vertices. It consists of completing a binary tree by adding $n+1$ edges so that all the initial vertices become internal vertices and it becomes a complete binary tree. We denote by $\overline{B}$ the completion of the binary tree B. And conversely, if $\overline{B}$ is a complete binary tree, B is obtained from $\overline{B}$ by deleting its external vertices. An example of this notation is given in Figure 1. These two families of trees are enumerated by the well studied Catalan numbers $C_n = \frac{1}{2n+1}\binom{2n+1}{n}$.

We now define the Tamari lattice. The complete binary trees with n interior vertices can be equipped with a rotation. As in Figure 2, consider a complete binary tree $\overline{T}$ with an internal vertex s such that the left child of s, denoted by t, is also an internal vertex. Let A be the left subtree of t, B the right subtree of t and C the right subtree of s. Let $\overline{T'}$ be the complete binary tree constructed from $\overline{T}$ such that t becomes the right child of s, A the left subtree of s, B the left subtree of t and C the right subtree of t. This operation from $\overline{T}$ to $\overline{T'}$ is called a right rotation, and the operation from $\overline{T'}$ to $\overline{T}$ is called a left rotation. In fact, the covering relations of the well known Tamari lattice (see [14, 26]) are the relations $\overline{T} \prec \overline{T'}$.

In this article, we consider a path to be a (finite) walk on the square grid, starting at (0,0), consisting of north and east unit steps denoted by N and E respectively. The set of ballot paths of height n is the set of paths that consist of n north steps, n east steps and that lie weakly above the diagonal, that is, weakly above the path

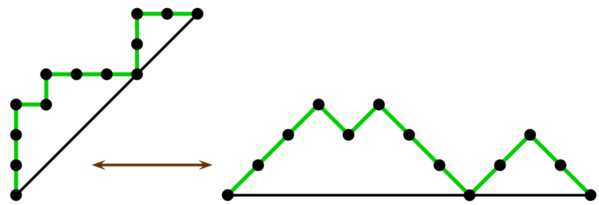

FIGURE 3. Ballot path (left) and Dyck path (right).

$(NE)^n$. They are also counted by the Catalan numbers. By applying a clockwise rotation of 45 degrees on ballot paths so that the diagonal becomes horizontal, these ballot paths become the well known Dyck paths (see Figure 3). The ballot paths can be generalized with a parameter m that is a positive integer. The m-ballot paths are the paths that consist of n north steps, mn east steps and that lie weakly above the line $y = \frac{x}{m}$, that is, weakly above the path $(NE^m)^n$.

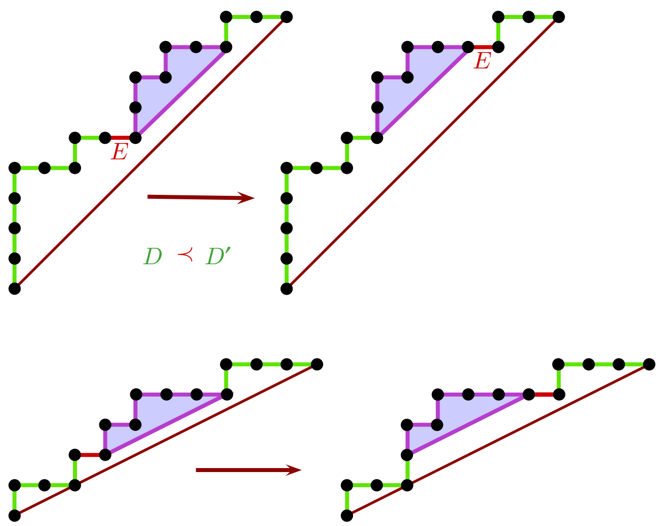

FIGURE 4. The Tamari covering relation for ballot (Dyck) paths (top figure). The covering relation in the m-Tamari lattice (m=2) (bottom figure).

Using a classical bijection between complete binary trees with n internal vertices and ballot paths of height n, the covering relations for the Tamari lattice can be translated into the following procedure on ballot paths. Let D be a ballot path of height n. Let E be an east step that precedes a north step in D. Draw a diagonal of slope 1 starting at the right extremity of E until it touches D again. Construct D' from D by switching E and the portion of the path above this diagonal. Then the covering relation in the Tamari lattice based on ballot paths becomes $D \prec D'$ (see Figure 4 for such a covering relation).

Motivated by the higher diagonal coinvariant spaces of the symmetric group, the covering relation on ballot paths is generalized in [6] to m-ballot paths by mimicking the above procedure as follows. Let D be an m-ballot path. Let E be an east step that precedes a north step in D. Draw a diagonal of slope $\frac{1}{m}$ starting at the right endpoint of E until it touches D again. Construct D' from D by switching E and the portion of the path above this diagonal. Then the covering relation in the m-Tamari lattice is given by $D \prec D'$ (see Figure 4 for an example). For more on these lattices and for enumerations of their intervals, we refer the reader to Section 6.

1.2. **Rational Catalan combinatorics** (a, b)**.** Let a and b be two relatively prime integers. We consider paths starting at $(0, 0)$ on the square grid with north and east steps and strictly above the line $y = \frac{a}{b}x$, excluding the start and endpoints (see [7]). An example is given in Figure 5. They are called (a, b)-ballot paths (or (a, b)-Dyck paths), and their study is the subject of very recent work under the term "rational Catalan combinatorics" (see [2, 3, 15, 16, 20] for more on this subject). The classical ballot paths and their extensions with any integer m are particular cases of such (a, b)-ballot paths. As shown in Figure 6, the set of m-ballot paths of height n can be seen as a particular case of these (a, b)-ballot paths, since they are obtained from the $(n, mn + 1)$-ballot paths after deleting their last east step.

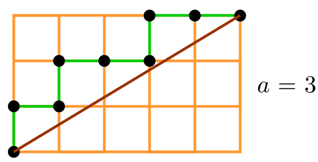

FIGURE 5. A (3,5)-ballot path.

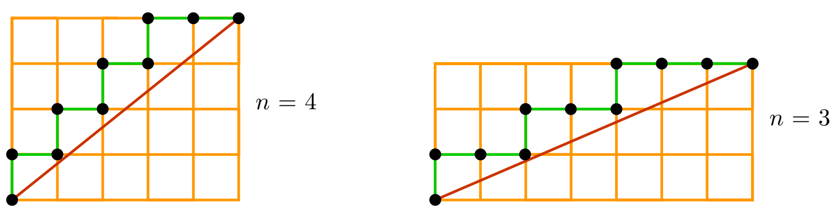

FIGURE 6. A (4,5)-ballot path (left). A (3,7)-ballot path (right).

An open question is to give an extension of the Tamari lattice, and more generally of the m-Tamari lattice, to any pair (a, b) of relatively prime integers. We propose an answer to this question by giving a far more general extension of these Tamari lattices and in particular give a construction of a rational (a, b)-Tamari lattice.

1.3. **Extension: The Tamari lattice** $\mathrm{Tam}(v)$**, where** v **is an arbitrary path.** Let v be an arbitrary path, starting at (0,0). Consider all the lattice paths weakly above v that start at (0,0) and finish at the end of v. We define the poset $\mathrm{Tam}(v)$ on this set of paths with a covering relation. Let u be such a path above v. Let p be a lattice point on u. We define the horizontal distance $\mathrm{horiz}_v(p)$ to be the maximum number of east steps that can be added to the right of p without crossing v (an

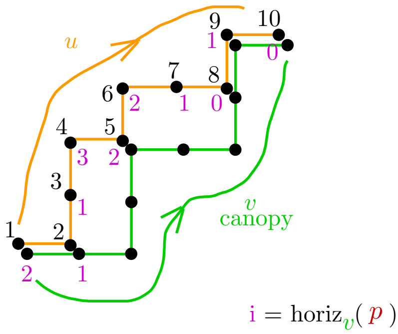

FIGURE 7. A pair (u, v) of paths with the horizontal distance $\mathrm{horiz}_v(p)$.

example of these horizontal distances is given in Figure 7). Suppose that p is preceded by an east step E and followed by a north step in u.

Let p' be the first lattice point in u that is after p and such that $\mathrm{horiz}_v(p') = \mathrm{horiz}_v(p)$. As in Figure 8, let $D_{[p,p']}$ be the subpath of u that starts at p and finishes at p'. Let u' be obtained from u by switching E and $D_{[p,p']}$. We define the covering relation to be $u \prec_v u'$ (see Figure 8 for an example). Then the poset $\mathrm{Tam}(v)$ is the transitive closure $\leq_v$ of this relation. It is easy to see that $\mathrm{Tam}((NE^m)^n)$ is the m-Tamari lattice.

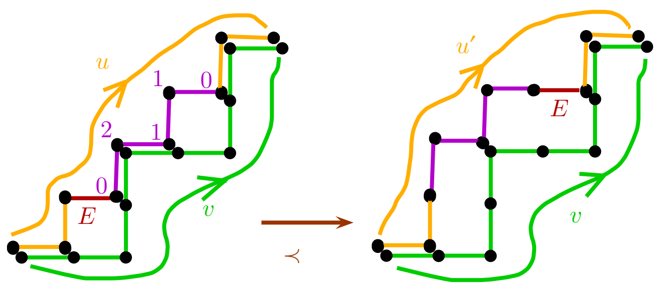

FIGURE 8. The covering relation defining the poset $\mathrm{Tam}(v)$.

For v an arbitrary path, let $\overleftarrow{v}$ be the path obtained by reading v backward and replacing the east steps by north steps and vice versa. We can now state our main results:

Theorem 1.1. *For any path v, $Tam(v)$ is a lattice.*

Recall from the first section that the usual Tamari lattice on complete binary trees with n interior vertices is isomorphic to the lattice $\mathrm{Tam}((NE)^n)$.

Theorem 1.2. *The lattice $Tam(v)$ is isomorphic to the dual of $Tam(\overleftarrow{v})$.*

For any pair (a, b) of relatively prime integers, we define the lattice $\mathrm{Tam}(a, b)$ to be the lattice $\mathrm{Tam}(v)$ where v is the closest path above the segment passing through the origin $(0, 0)$ and the point (a, b). The duality between $\mathrm{Tam}(v)$ and $\mathrm{Tam}(\overleftarrow{v})$ becomes the duality between $\mathrm{Tam}(a, b)$ and $\mathrm{Tam}(b, a)$ (see Figure 9). In the same figure, we have drawn with dotted arrows the covering relation for the

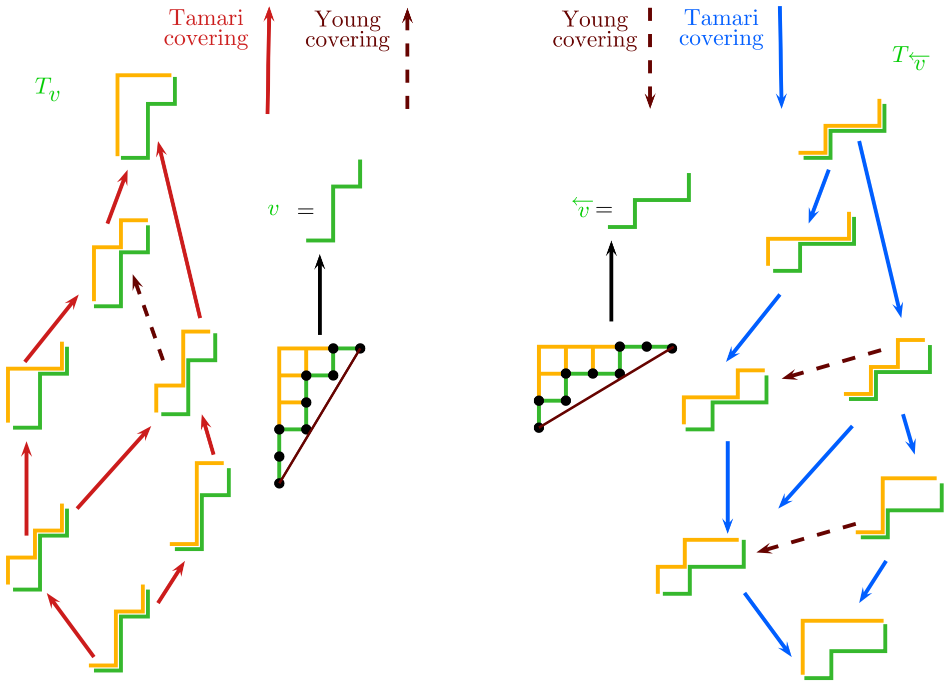

FIGURE 9. Duality and rational Catalan combinatorics: The lattices Tam(5,3) (left) and Tam(3,5) (right).

Young lattices $Y(v)$ of Ferrers diagrams included in the Ferrers diagram defined by the path v. The lattice $Y(v)$ can be seen as a refinement of the lattice Tam(v), and in that case, by the simple symmetry exchanging rows and columns, the lattice $Y(v)$ is isomorphic to $Y(\overleftarrow{v})$.

Theorem 1.3. *The usual Tamari lattice $Tam((NE)^n)$ can be partitioned into disjoint intervals $I(v)$ indexed by the lattice paths v consisting of a total of $n-1$ east and north steps, i.e.*

$$Tam((NE)^n) = \bigcup_{|v|=n-1} I(v),$$

where each $I(v) \cong Tam(v)$.

An example of Theorem 1.3 is given in Figure 10.

1.4. **Outline.** In Section 2, we define the concept of the canopy of a binary tree. We state a transformation (a bijection) from the set of binary trees with n vertices to the set of pairs of weakly non-crossing paths that contain $n-1$ edges each such that they share the same endpoints. The canopy of a binary tree is a word, which we identify with a path made from an alphabet based on two letters, and that is mainly used to partition the Tamari lattice into intervals (sublattices) that are of importance to us. These sublattices are isomorphic to the lattices Tam(v) mentioned in Section 1.3. The fact that the transformation mentioned previously is a bijection is proved in Section 3. Some simple and useful properties of the canopy of binary trees related to the Tamari lattice are given in Section 4. The

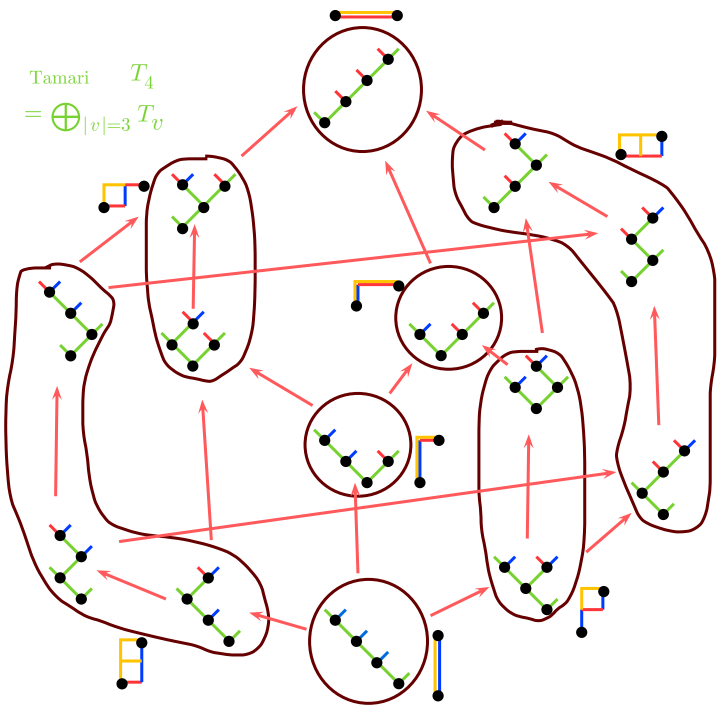

FIGURE 10. The decomposition of the Tamari lattice on complete binary trees with 4 interior vertices into the union of 8 disjoint intervals Tam(v) (Theorem 1.3).

demonstrations of the main results presented in Section 1.3 appear in Section 5. Finally, the connections of our work with the diagonal coinvariant spaces and some perspectives appear in Section 6.

2. Canopy of a binary tree

For any binary tree B, we construct a word $w(B)$ on the alphabet $\{a, \bar{a}, b, \bar{b}\}$. Walking clockwise around B and starting at the root, we write the letter a when we use a left edge for the first time and $\bar{a}$ when we use a left edge for the second time. Similarly we repeat with the letters b and $\bar{b}$ for the right edges (see Figure 11 for an example).

From $w(B)$, we construct two subwords. The first subword $u(B)$ is obtained by keeping track only of the two letters $\{\bar{a}, \bar{b}\}$ in $w(B)$. We identify a path with $u(B)$ by replacing in this sequence the letter $\bar{a}$ by a north step and $\bar{b}$ by an east step. The canopy $v(B)$ of B, which is also a subword of $w(B)$, is obtained similarly by keeping track only of the letters $\{\bar{a}, b\}$. We identify a path with $v(B)$ by replacing in this sequence the letter $\bar{a}$ by a north step and b by an east step.[1] The concept of the canopy was introduced using different terminology in [21]. For the binary tree in Figure 11, we show an example of all these words in Figure 12 and draw the paths $u(B)$ and $v(B)$ in Figure 13.

[1]So by abuse of notation, we will refer to the paths $u(B)$ and $v(B)$.

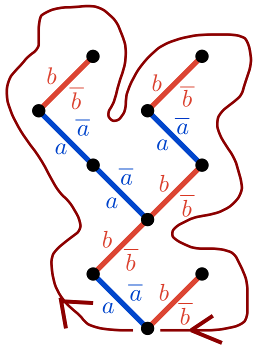

FIGURE 11. Walk around a binary tree B: the word w(B).

$$w(B) = a\ b\ a\ a\ b\ \overline{b}\ \overline{a}\ \overline{a}\ b\ a\ b\ \overline{b}\ \overline{a}\ \overline{b}\ \overline{b}\ \overline{a}\ b\ \overline{b}$$
$$u(B) = \overline{b}\ \overline{a}\ \overline{a}\ \ \overline{b}\ \overline{a}\ \overline{b}\ \overline{b}\ \overline{a}\ \ \overline{b}$$
$$v(B) = b\ \ b\ \ \overline{a}\ \overline{a}\ b\ \ b\ \ \overline{a}\ \ \overline{a}\ b$$

FIGURE 12. The words $w(B), u(B)$ and $v(B)$ associated to the binary tree in Figure 11.

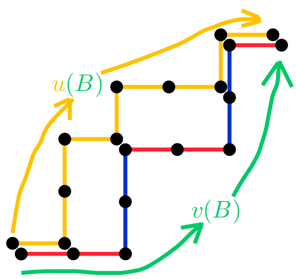

FIGURE 13. The pair (u, v) of paths associated to a binary tree in Figure 11.

It is no accident that we use the same letter v for the canopy as the letter that defines the poset in the previous section. Before explaining this, we can mention an easy property:

Lemma 2.1. *For any given binary tree B, the path $u(B)$ is weakly above the canopy (also a path) $v(B)$.*

Proof. This is straightforward since the occurrences of the letter b precede the occurrences of the letter $\bar{b}$ in $w(B)$, and these two letters determine the east steps of $v(B)$ and $u(B)$ respectively. □

Let $\overline{B}$ be a complete binary tree with n vertices. It is not difficult to prove that the canopy can also be defined using the following two equivalent definitions.

The second definition of the canopy, which is defined in [21], can be described as follows. Walking around $\overline{B}$ clockwise starting at the root, record the sequence of left and right external edges, except the first and last external edges. From this sequence, construct a path by changing the right external edges into north steps and the left external edges into east steps. The path obtained is also the canopy (see Figure 14 for an example). Because of this definition, we define the interior canopy of the complete binary tree $\overline{B}$ to be the canopy of B.

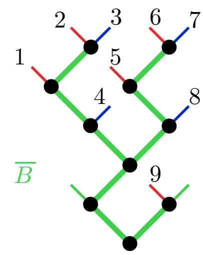

FIGURE 14. Second definition of the canopy.

For the third definition of the canopy, we first label the n vertices of the binary tree $B = (L, r, R)$ by the sequence of integers following B according to the in-order. This order is defined recursively by first visiting the vertices on the left subtree L, then the root r, then finishing with the vertices on the right subtree R (see Figure 15). Construct a path from the sequence of vertices with labels $\{1, 2, ..., n-1\}$ in B such that the i^{th} step is an east step if the vertex with label i in B has a right child, and a north step otherwise. This path is also the canopy. An example of this procedure is given in Figure 15.

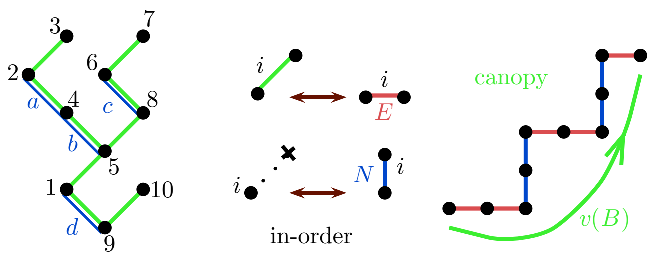

FIGURE 15. Third definition of the canopy (left and middle). The canopy (right).

Proposition 2.2. *The three previous definitions of the canopy of a binary tree are equivalent.*

Proof. In the first definition, the canopy is the word $v(B)$ formed by the letters b and $\overline{a}$ constructed by walking around the binary tree B as in Figure 11. For each letter b, that is, for each right edge of the binary tree B, one can associate an external left edge of the binary tree $\overline{B}$ by the following correspondence. This right edge connects two vertices s and t of B, where t is the right son of s. If t has no left son, then we associate the left external edge starting from t in the extended binary tree $\overline{B}$. Else, we follow the left branch starting from t until we reach a vertex t' with no left son. In that case, we associate again the corresponding left external edge starting from t' in $\overline{B}$.

Analogously, for each letter $\overline{a}$ corresponding to a left edge joining s and t in the tree B, with t the left son of s, we can draw from t the right branch until we reach a vertex with no right son and similarly associate an external right edge of $\overline{B}$. It is easy to check that this corrrespondence is a bijection between the edges of B and the external edges of $\overline{B}$ which are not the first or the last one in the order presented in the second definition of the canopy. Moreover this bijection preserves the orders defined for both definitions by walking around the binary tree B.

In order to prove the equivalence of the first and third definitions of the canopy, one can check that the labels (for the in-order, see Figure 15) of the vertices of the tree B having a right son are exactly the indices of the letters b in the word $v(B)$ of the first definition. This is a property relating the in-order and the walk around a binary tree. The list of vertices in the in-order can be seen as a certain projection on a line of the vertices of the binary tree B. In this projection on a line, the walk around B uses each edge going right or left. The edges going to the right correspond to the letters b and $\overline{a}$, while the edges going to the left correspond to the letters a and $\overline{b}$. Comparing the two projections finishes the proof. □

3. Reverse bijection: From pairs of non-crossing paths to binary trees

We will prove the following proposition at the end of this section.

Proposition 3.1. *The map defined in Section 2 associating the pair (u, v) to a binary tree is a bijection from the set of binary trees with n vertices to the set of unordered pairs of non-crossing paths, each consisting of a total of $n-1$ north and east steps, with the same endpoints.*

Before defining the inverse bijection, we need to state an equivalent definition of the pair (u, v) associated to B. For a binary tree B, let $v(B)$ be the canopy. The left edges in B can be totally ordered using the in-order of their fathers in B. Note that under this order on left edges, the i^{th} left edge corresponds to the i^{th} north step in both $u(B)$ and $v(B)$. The right height of a vertex in B is the number of right edges of the path on B from the root to the vertex. The right height of a left edge in B is the right height of either of its vertices. We obtain the following simple fact, which when combined with the definition of the canopy $v(B)$ gives an equivalent definition of the pair (u, v) associated to B.

Lemma 3.2. *For B a binary tree, the number of unit squares between the i^{th} north step of $u(B)$ and the i^{th} north step of $v(B)$ is equal to the right height of the i^{th} left edge (in the in-order) of B (see Figure 16 for an example).*

Proof. In the definition of w, we walk clockwise around the binary tree B. We record an east step in $v(B)$ when we use a right edge of B for the first time and an east step in $u(B)$ when we use a right edge of B for the second time. To count the number of unit squares between the i^{th} north steps of $u(B)$ and $v(B)$, we need to count how many right edges have been used exactly once when we walk on the i^{th} left edge of B for the second time. It is easy to show that this corresponds to the number of right edges on B from the root to the i^{th} left edge of B, which is the definition of the right height of this edge. □

For a complete binary tree $\overline{T}$, let $c(\overline{T})$ be the sequence of (external) left edges \ and right edges / recorded walking around $\overline{T}$ as in the definition of the interior

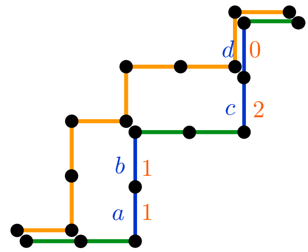

FIGURE 16. (With Figure 15 (left)) Lemma 3.2.

canopy.[2] When $\overline{T}$ has at least one internal vertex, we denote the extended interior canopy of $\overline{T}$ by the word $\overline{c}(\overline{T}) = \backslash c(\overline{T})/$. It is basically the same as the interior canopy of $\overline{T}$ except that we record also the first and last external edges. When we consider subtrees of $\overline{T}$, one special case will be important. Suppose that H is a subtree of $\overline{T}$ that consists of a single external vertex. In this case, if H is attached to a left edge $\backslash$ (respectively right edge $/$), then $\overline{c}(H) = \backslash$ (respectively $\overline{c}(H) = /$).

Lemma 3.3. *If two binary trees B and B' have the same canopy and the same sequence $(a_1, \ldots, a_k)$ of right heights of left edges (ordered by in-order), then B=B'.*

Proof. We prove this lemma by recurrence on the number of vertices of the binary trees. Let $B = (L, r, R)$ and $B' = (L', r', R')$ be the decomposition of the binary trees B and B' with root, left and right subtrees. Having the same canopy, B and B' have the same number of vertices. If every integer of the sequence $(a_1, \ldots, a_k)$ is strictly positive, then both left subtrees L and L' are empty. If there exists some integer a_i equal to zero, let m be the maximum over the indices i such that $a_i = 0$. Both binary trees B and B' will have left subtrees L and L' having $m-1$ left edges, the m^{th} left edge being the left edge going from the root to the left subtree (having right height zero). Using the first or the third definition of the canopy, we deduce that the left subtrees L and L' have the same number of vertices, and thus also the right subtrees R and R'.

It is easy to check that $\overline{c}(B) = \overline{c}(L)\overline{c}(R)$ and $\overline{c}(B') = \overline{c}(L')\overline{c}(R')$. The left subtrees L and L' having the same number of vertices, the words $c(L)$ and $c(L')$ have the same length. Therefore they are the same. The subtrees L and L' satisfy the conditions of Lemma 3.3 and thus are the same by the recurrence hypothesis. Analogously, the subtrees R and R' have the same canopy and also the same sequence of right heights of left edges (the respective heights are shifted by one), and thus by the recurrence hypothesis they are the same.

More precisely, identifying the canopy with the path v made of east and north edges, the number of elements of both subtrees L and L' is the number of edges of the path v encountered from the beginning of the path until the north edge (included) associated to a_m. For example in Figure 15 the size of L is 8, $m = 4$ from Figure 16, and the north edge related to $a_4 = 0$ (labeled d) is the 8^{th} edge of the path v. □

We now give the reverse transformation $(u, v) \to B'$ of the map defined in Section 2. Start with a pair of non-crossing paths u and v with the same endpoints such

[2]The interior canopy of a complete binary tree is defined in the same paragraph as the second definition of the canopy.

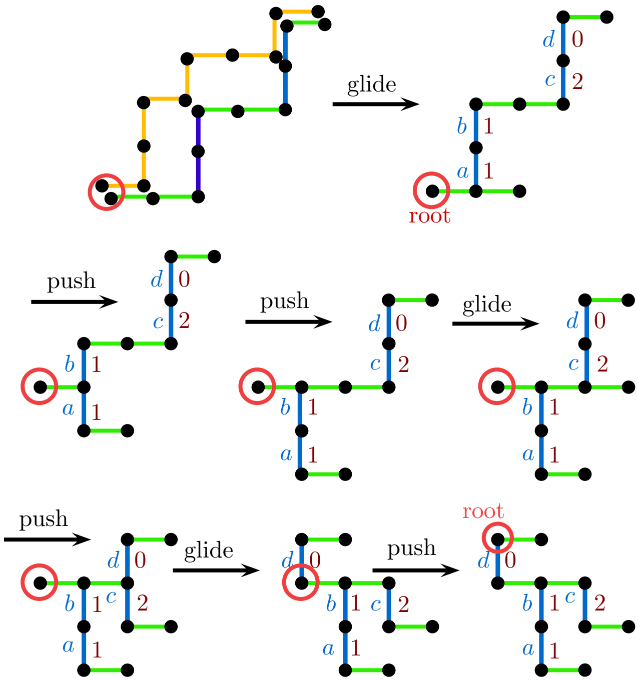

FIGURE 17. The "push-gliding" algorithm.

that u is weakly above v. Draw v and for each north step of v, assign the value a_i to be the distance between the i^{th} north step in v and the i^{th} north step in u. Now, repeat the following process for each north step in v, from bottom to top (see Figure 17 for a better understanding).

The "push-gliding" algorithm on pairs of non-crossing paths (with the same endpoints) is defined as follows. First let the initial vertex of v be the root. Slide the first north step of v horizontally with all the vertices and all the edges above it so that it is at distance a_1 from the root. Let s_1 be the vertex at the bottom of this first north step. Let e_1 be the adjacent edge of s_1 to its left (if any). This first north step pushes s_1 down and all the vertices and the edges that are connected to s_1 when e_1 is removed. If $a_1 = 0$, then the vertex at the top of the first edge becomes the root. Now redo the same procedure with the second north step and so on.[3] At the end, you will obtain a binary tree with n vertices. Now apply a reflection to this tree. This is the end of the push-gliding algorithm.

Proof of Proposition 3.1. Starting from a binary tree B, by the algorithm of Section 2 we get a pair (u, v) of paths, and after applying the push-gliding algorithm to this pair of paths we obtain a binary tree B'. We want to show that $B = B'$.

The push-gliding algorithm starts with a path v and ends with a binary tree B'. The intermediate combinatorial objects will be called Ω-trees. They are defined as being a path ω with some binary trees rooted on some vertices of the path. We define the in-order of an Ω-tree in the following way: we follow the path ω from its

[3]Note that you don't want to create cycles in this procedure, so you might have to make some edges longer to avoid them.

initial vertex to its last vertex, and each time one reaches a vertex where a binary tree is attached; then we follow the binary tree in in-order, and then continue following the path. It is easy to see that the in-order of the Ω-tree is preserved for each elementary move (push or glide) of the algorithm. At the beginning, the in-order of the path v, i.e. following the path, gives a word in two letters according to the fact that the i^{th} vertex is the starting point of an east step or not. Using the third definition of the canopy, we deduce that the canopy of the binary tree B' is the word (or path) v.

In each glide move, the i^{th} vertical edge is moved (together with a portion of the path v) to the left, gliding on a horizontal line starting from the origin, until its distance to the origin is equal to the label a_i. This distance will be kept invariant in the following glide or push steps. At the end of the algorithm, the binary tree B' is such that a_i is equal to the right height of the i^{th} left edge (in the in-order) of B'. From Lemma 3.2, we deduce that both trees B and B' have the same sequence $(a_1, \ldots, a_k)$ of right heights of left edges (ordered by in-order). Using this and the fact that both B and B' have the same canopy v, Lemma 3.3 implies that $B = B'$. This finishes the proof of Proposition 3.1. □

The bijection between binary trees and pairs of paths (u, v) was introduced in a different form by Delest and Viennot [12]. We have described here a new version of the bijection which involves a "push-gliding" algorithm and fits our purpose.

4. Canopy in the ordinary Tamari lattice

Lemma 4.1. *Let $\overline{T}$ be a complete binary tree, s and t vertices of $\overline{T}$ and A, B, C subtrees of $\overline{T}$ as in Figure 2 defining a right rotation from $\overline{T}$ to $\overline{T'}$. Let $\overline{c}(\overline{T}) = v_1\overline{c}(A)\overline{c}(B)\overline{c}(C)v_2$ be the sequence of external edges, where v_1 and v_2 are some sequences of $\backslash$ and $/$. We have the following relations:*
- if B contains more than one vertex, then $\overline{c}(\overline{T}) = \overline{c}(\overline{T'})$;
- if B is a single vertex, then $\overline{c}(\overline{T}) = v_1\overline{c}(A)/\overline{c}(C)v_2$ and $\overline{c}(\overline{T'}) = v_1\overline{c}(A)\backslash\overline{c}(C)v_2$.

Proof. This is straightforward. □

Thus the interior canopy of a complete binary tree is invariant under a rotation if and only if B contains more than one vertex. We call such a rotation a valid rotation.

We define the following order relation on the set of words in letters $\{\backslash, /\}$:

$$v_1/v_2/\ldots/v_k \leq v_1\backslash v_2\backslash\ldots\backslash v_k,$$

where the v_i are words of letters $\{\backslash, /\}$. The poset of words of length n in letters $\{\backslash, /\}$ is isomorphic to the Boolean lattice of subsets of a set of cardinality n.

Corollary 4.2. *In the Tamari lattice, if $\overline{T} \leq \overline{T'}$, then $c(\overline{T}) \leq c(\overline{T'})$.*

Proposition 4.3. *The set $I(v)$ of complete binary trees having interior canopy v is an interval of the ordinary Tamari lattice on complete binary trees with $|v| + 1$ interior vertices.*

This proposition and also Corollary 4.2 are well known; see for example Proposition 3.5 in [22] where the Boolean lattice appears as a quotient of the Tamari lattice. The intervals I(v) are the fibres over points of the usual map from the Tamari lattice to the Boolean lattice. These intervals can also be viewed as the

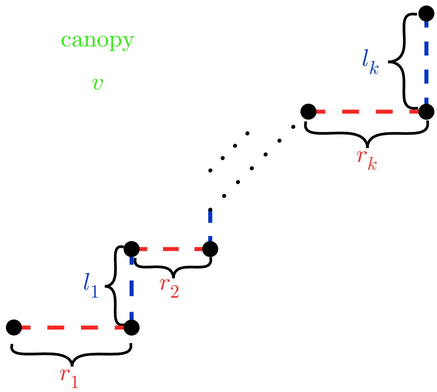

FIGURE 18. The canopy v.

images under the map from the symmetric group to the Tamari lattice of the set of permutations with a fixed descent set. Here we give a direct combinatorial proof.

Proof. Let $\overline{T}$ be a complete binary tree with interior canopy v, where v is given in Figure 18. We can always perform a valid right rotation on $\overline{T}$ if and only if there exist vertices s, t with subtrees A, B, C as in Lemma 4.1 (left part of Figure 2). In a right rotation, the length of the left branch rooted in s decreases by one. Starting from $\overline{T}$, by following a sequence of valid rotations, we will always get a tree (with interior canopy v) where no further valid right rotations are possible. In such a tree, all the left branches will be reduced to sequences of left edges, where all interior vertices have a right subtree B that consists of a single vertex. This complete binary tree is unique, and we denote it by $\overline{T_{\max}(v)}$. The binary tree $T_{\max}(v)$ is displayed in Figure 19.

Similarly, let $\overline{T'}$ be a complete binary tree with canopy v. We can always perform a valid left rotation on $\overline{T'}$ if there exist vertices s, t with subtrees A, B, C as in Lemma 4.1 (right part of Figure 2). In a left rotation, the length of the right branch rooted at s decreases by one. As in the previous case of left rotation, by following a sequence of valid left rotations, we will always get a unique complete binary tree with canopy v having all the right branches reduced to sequences of right edges, where all interior vertices have a left subtree B that consists of a single vertex. We denote this complete binary tree by $\overline{T_{\min}(v)}$. The binary tree $T_{\min}(v)$ is displayed in Figure 20.

We have proved that every complete binary tree $\overline{T}$ with interior canopy v satisfies (for the order relation of the Tamari lattice)

$$\overline{T_{\min}(v)} \leq \overline{T} \leq \overline{T_{\max}(v)}.$$

Conversely, suppose that $\overline{T}$ is a complete binary tree satisfying the above relation. From Corollary 4.2, we have

$$v = c(\overline{T_{\min}(v)}) \leq c(\overline{T}) \leq c(\overline{T_{\max}(v)}) = v.$$

Proposition 4.3 is proved. □

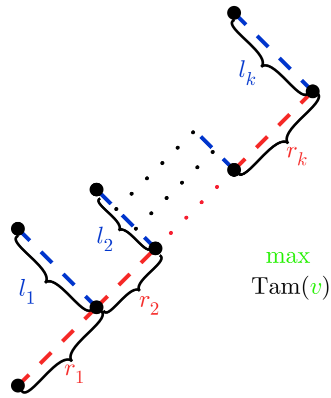

FIGURE 19. The maximum binary tree $T_{\max}(v)$ in $\mathrm{Tam}(v)$, where v is given in Figure 18.

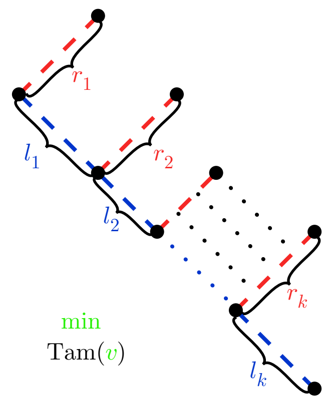

FIGURE 20. The minimum binary tree $T_{\min}(v)$ in $\mathrm{Tam}(v)$, where v is given in Figure 18.

The max and min elements of the interval can be defined easily from the "push-gliding" algorithm mentioned above. The maximum (resp. minimum) element of the interval $I(v)$ is obtained from the "push-gliding" algorithm by performing only operations "push" (resp. "gliding").

5. PROOFS OF THE MAIN THEOREMS

Using the previously defined pair (u, v) that sends a binary tree T to a pair of non-crossing paths $(u(T), v(T))$, we will show that the sequence of values $\mathrm{horiz}_v(p)$, for p being the consecutive integer points on $u(T)$ from bottom to top, can be easily read on the vertices of T. The post-order traversal on vertices of a binary tree T is the order on vertices obtained by walking clockwise around T and recording a vertex the last time we walk next to it (see Figure 21 for an example of the post-order traversal on vertices). Another equivalent way to define the post-order traversal is with the following recursive procedure: traverse the left subtree, traverse the right subtree, visit the root. Lemma 5.1 is a refinement of Lemma 3.2.

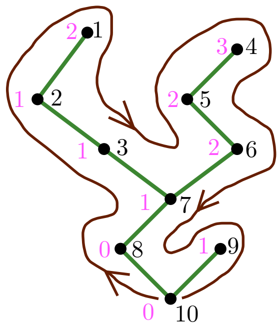

FIGURE 21. (With Figure 7) The post-order traversal in Lemma 5.1.

Lemma 5.1. *For T a binary tree, the sequence of right heights of the vertices of T in post-order traversal corresponds to the sequence of values* $\mathrm{horiz}_v(p)$*, for p being the consecutive integer points on $u(T)$ from bottom to top.*

Proof. Note that walking clockwise around T, a vertex is recorded in the post-order traversal on vertices just before an edge is visited for the second time (except for the last vertex, which is the root). This lemma can be proved using the same kind of ideas as for Lemma 3.2 (see Figure 21 for an example). □

Proposition 5.2. *For any path v, the poset $I(v)$ is isomorphic to $Tam(v)$.*

Proof. Let $\overline{T}$ be a complete binary tree with interior canopy v. We will show that the covering relations of the two posets are the same (see Figure 22 for an example). First note that a valid right rotation can be applied exactly in the following situation, and it is easy to show that this situation corresponds bijectively, under the pair $(u(T), v(T))$, to the case where you have an east step preceding a north step in the path $u(T)$, which are precisely the places where the covering relations are defined in $\mathrm{Tam}(v)$. Let s and t be vertices of the complete binary tree $\overline{T}$ such that t is the left child of s. In $\overline{T}$, let A be the left subtree of t, B the right subtree of t that contains more than once vertex, and C the right subtree of s. Apply a valid right rotation to $\overline{T}$ to obtain the complete binary tree $\overline{T'}$ (see Figure 22). The

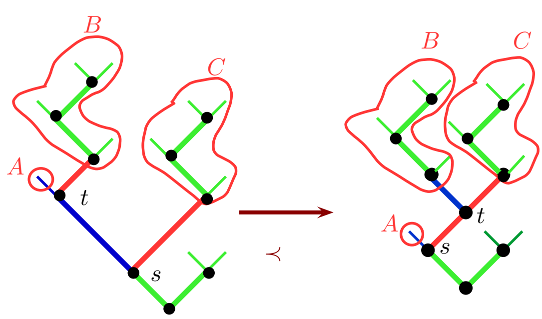

FIGURE 22. (With Figure 8) The covering relation in $\mathrm{Tam}(v)$ and the corresponding rotation in $\overline{T}$ (ordinary Tamari with complete binary trees).

sequence of right heights of the vertices of T in post-order traversal is of the form $(L, S_A, S_B, h_t, S_C, h_s, R)$, where S_A, S_B, h_t, S_C, and h_s are the sequences of right heights of the vertices in T in post-order traversal in A, B, t, C and s, respectively, and L and R are the sequences of right heights of all the vertices in T in post-order traversal that precede and succeed all the previous vertices respectively. It is clear that $h_s = h_t$. The sequence of right heights of the vertices in post-order traversal of T' is given by $(L, S_A, S_B, S_C + \bar{1}, h_s + 1, h_s, R)$, where $S_C + \bar{1}$ corresponds[4] to adding 1 to all the values in S_C. Since all the values in S_C, if any, are greater than h_s, the previous lemma and the definition of the covering relations in $\mathrm{Tam}(v)$ imply that the covering relations in $\mathrm{I}(v)$ and $\mathrm{Tam}(v)$ are the same. □

We can now prove our main theorems of Section 1.3.

Proof of Theorem 1.1. An interval of a lattice is always a lattice; therefore $\mathrm{I}(v)$ is a lattice by Proposition 4.3, and so is $\mathrm{Tam}(v)$ by Proposition 5.2. □

Proof of Theorem 1.2. After applying a reflection to a binary tree with canopy v, it is easy to see, using the second definition of canopy, that the canopy of this tree (obtained by reflection) is precisely $\overleftarrow{v}$. It is clear from the description of $\mathrm{Tam}(v)$ and $\mathrm{Tam}(\overleftarrow{v})$ as intervals of the usual Tamari lattice defined on complete binary trees that these two lattices are anti-isomorphic. □

Proof of Theorem 1.3. We partition the complete binary trees with n interior vertices into sets of trees with the same interior canopy. We then apply Proposition 5.2 to each set of trees. □

6. Connections with the diagonal coinvariant spaces and perspectives

Our work has been influenced by the combinatorics of the “generalized” diagonal coinvariant spaces of the symmetric group. We give a brief description of the subject here, and refer the reader to [5, 6, 8, 9, 17] for more details.

Let $X = (x_{i,j})_{\substack{1\le i\le k\\ 1\le j\le n}}$ be a matrix of variables. A permutation σ of the symmetric group $\mathfrak{S}_n$ permutes the variables columnwise by $\sigma(X) = (x_{i,\sigma(j)})_{\substack{1\le i\le k\\ 1\le j\le n}}$, i.e. $\sigma(x_{i,j}) = x_{i,\sigma(j)}$. This action can be directly extended to all the polynomials in $\mathbb{C}[X]$. All the variables in a same row of X are said to be contained in the same set of variables. Since X contains k rows, there are k sets of variables. Let $\mathcal{J}$ be the ideal generated by constant free invariant polynomials under this action. The diagonal coinvariant spaces of $\mathfrak{S}_n$ are defined as $\mathcal{DR}_{k,n} := \mathbb{C}[X]\big/\mathcal{J}$. They can be generalized using an additional parameter m that is a positive integer. The higher diagonal coinvariant spaces of the symmetric group are defined as $\mathcal{DR}^m_{k,n} := \varepsilon^{m-1} \otimes \mathcal{A}^{m-1}\big/\mathcal{J}\mathcal{A}^{m-1}$, where ε is the sign representation and $\mathcal{A}$ is the ideal generated by alternants, i.e. polynomials $f(X)$ such that $\sigma f(X) = \varepsilon(\sigma) f(X)$, $\forall \sigma \in \mathfrak{S}_n$. Note that $\mathcal{DR}_{k,n} = \mathcal{DR}^1_{k,n}$. The $\mathcal{DR}^m_{k,n}$ are representations of $\mathfrak{S}_n$ because the action given above can be applied to the quotient space $\mathcal{DR}^m_{k,n}$. They are graded with respect to the degree of each set of variables. We denote the subspace of alternants of $\mathcal{DR}^m_{k,n}$ by ${\mathcal{DR}^m_{k,n}}^{\varepsilon}$.

[4] Note that C might not contain a vertex of T, and therefore this sequence might be empty.

In the case $k = 1$, they are classical [25], and the dimensions of ${\mathcal{DR}_{1,n}^{m}}^{\varepsilon}$ and $\mathcal{DR}_{1,n}^{m}$ are given by 1 and $n!$, respectively.

In the case $k = 2$, they were first defined and studied by Garsia and Haiman because of their connections with the Macdonald polynomials. It was proven by Haiman [18] that the dimensions of ${\mathcal{DR}_{2,n}^{m}}^{\varepsilon}$ and $\mathcal{DR}_{2,n}^{m}$ are given by

$$\frac{1}{(m+1)n+1}\binom{(m+1)n+1}{mn} \quad \text{and} \quad (mn+1)^{n-1},$$

respectively. The first number corresponds to the number of m-ballot paths of height n and the second one to the number of m-parking functions of height n. The m-parking functions of height n are simply the m-ballot paths labelled on the north steps, with labels $\{1, 2, ..., n\}$ such that consecutive north steps are labelled increasingly. The spaces $\mathcal{DR}_{2,n}^{m}$ have been studied by many researchers for more than 20 years. Despite that, there are still some important unresolved conjectures left in the field.

For the case $k = 3$, Haiman [19] conjectured in the 1990's that the dimensions of ${\mathcal{DR}_{3,n}}^{\varepsilon}$ and $\mathcal{DR}_{3,n}$ are equal to $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ and $2^n(n+1)^{n-2}$ respectively.

Independently of this story, Chapoton [10] proved in 2006 that the number of intervals in the Tamari lattice based on complete binary trees with n interval vertices is given by $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$. In 2008, the m-Tamari lattice was introduced in [6], and it was conjectured that the number of intervals and labelled intervals in the m-Tamari lattice are given by $\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}$ and $(m+1)^n(mn+1)^{n-2}$, respectively. A labelled interval in the m-Tamari lattice is simply an interval where the top path is decorated as an m-parking function. Refinements of these two results were proven in [8, 9].

The duality that is proved in this article shows that the number of intervals in $\mathrm{Tam}((N^mE)^n)$ is the same as in the m-Tamari lattice $\mathrm{Tam}((NE^m)^n)$. Using refinements and calculations, it seems that the number of labelled intervals in $\mathrm{Tam}((NE^m)^n)$ is equal to the number of labelled intervals on east steps in $\mathrm{Tam}((N^mE)^n)$, where the labelled intervals on east steps are defined by assigning the labels $\{1, 2, ..., n\}$ on east steps of the upper path, and such that the labels on consecutive east steps are increasing. Note that for $m = 1$, this is easy to prove since you can obtain without difficulty the same functional equations for both cases from recurrences. But we have not been able to do so in the case $m > 1$. It would be interesting to see if the ideas presented in [11] could help prove this equality.

More recently, some researchers (see [1, 3, 4, 20]) have extended the combinatorics of the $\mathcal{DR}_{2,n}^{m}$ by considering paths and parking functions above the line with endpoints (0,0) and (b, a), where a, b are arbitrary positive integers.[5] They defined the combinatorial statistics area and dinv on these objects. So this rational Catalan combinatorics can be seen as the combinatorics of some possible generalizations of the spaces $\mathcal{DR}_{2,n}^{m}$. Even though these spaces have not yet been shown to exist, some preliminary calculations [4] suggest that they do. One might try now to define a dinv statistic on paths and parking functions above an arbitrary path consisting of east and north steps, even if it is not known to be possible.

[5]Note that the paths above the line with endpoints (0,0) and (mn, n) are the same as the paths above the line with endpoints (0,0) and $(mn + 1, n)$. This is why we use the term extension.

It remains to be seen if our lattices $\mathrm{Tam}(v)$, for arbitrary paths v, will give a combinatorial setup for the not yet defined generalizations[6] of the spaces $\mathcal{DR}^m_{3,n}$. It will be interesting to verify this as the theory of the "generalized" diagonal coinvariant spaces develops.

In this article, we showed that the lattices $\mathrm{Tam}(v)$ based on paths and the lattices $\mathrm{I}(v)$ based on trees with the same canopy are equivalent (i.e. isomorphic). We would like to mention that we know a third combinatorial model that is isomorphic to these two. We give only a short description here. In [9], an anonymous referee shows a combinatorial model for the m-Tamari lattice based on $(m+1)$-ary trees. The same idea can be used to define a combinatorial model isomorphic to the $\mathrm{Tam}(v)$'s. A planted rooted tree is a rooted tree such that the children of any vertex are totally ordered. Let T be a planted rooted tree. We define the prefix order sequence of T to be the sequence of degrees of the internal vertices of T in prefix order. Consider the set of planted rooted trees that have the same prefix order sequence. To define the covering relation on them, one first chooses a leaf l that is followed (in prefix order) by an internal vertex s, of degree k. Then by denoting by $T_1, T_2, ..., T_k$ the k subtrees attached to s, from left to right, we insert s with its first $k-1$ subtrees in place of the leaf l and l becomes the rightmost child of s. The rightmost subtree of s, T_k, finally takes the former place of s. This combinatorial model (a lattice) is equivalent to the $\mathrm{Tam}(v)$'s. It will be interesting to see if there are other combinatorial models equivalent to the $\mathrm{Tam}(v)$'s.

We finish this article by mentioning that in [13], it is shown that the total number of intervals in the lattices $\mathrm{Tam}(v)$, for all the paths v of length n, is given by $\frac{2(3n+3)!}{(n+2)!(2n+3)!}$. The same number also counts rooted non-separable planar maps with $n+2$ edges (sequence A000139 of OEIS).

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