TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 7, July 2017, Pages 5137–5155
http://dx.doi.org/10.1090/tran/7070
Article electronically published on March 17, 2017

TORUS INVARIANT TRANSVERSE KÄHLER FOLIATIONS

HIROAKI ISHIDA

Abstract. In this paper, we show the convexity of the image of a moment map on a transverse symplectic manifold equipped with a torus action under a certain condition. We also study properties of moment maps in the case of transverse Kähler manifolds. As an application, we give a positive answer to the conjecture posed by Cupit-Foutou and Zaffran.

1. Introduction

In [11], a family of complex manifolds (say LV manifolds here) which includes classical Hopf manifolds and Calabi-Eckmann manifolds are constructed by López de Medrano and Verjovsky. Most of LV manifolds are non-Kähler; it is shown that an LV manifold admits a Kähler form if and only if it is a compact complex torus of complex dimension 1. In [10], as a contrast, it is shown by Loeb and Nicolau that each LV manifold carries a transverse Kähler vector field. In [12], Meersseman generalizes the construction of LV manifolds and gives a new family of complex manifolds which are known as LVM manifolds. As well as LV manifolds, an LVM manifold admits a Kähler form if and only if it is a compact complex torus. He also constructs a foliation $\mathcal{F}$ on each LVM manifold and shows that $\mathcal{F}$ is transverse Kähler. In [13], the condition (K)[1] is introduced for LVM manifolds by Meersseman and Varjovsky. They also show that if an LVM manifold M satisfies the condition (K), then the leaf space $M/\mathcal{F}$ is a projective toric orbifold. In [4], Bosio generalizes LVM manifolds and now they are known as LVMB manifolds. As well as LVM manifolds, if an LVMB manifold is Kähler, then it is a compact complex torus. In [3], Battisti gives an explanation of the difference between LVM manifolds and LVMB manifolds in terms of toric geometry.

In [6], Cupit-Foutou and Zaffran construct a foliation $\mathcal{F}$ on each LVMB manifold as a natural generalization of the case of LVM manifolds. They also generalize the condition (K) to LVMB manifolds. They show that the condition (K) is invariant under biholomorphic equivalence and if the foliation $\mathcal{F}$ on an LVMB manifold M satisfying the condition (K) is transverse Kähler, then M is an LVM manifold. Using this, they produce examples of LVMB manifolds that are not biholomorphic

Received by the editors October 12, 2015 and, in revised form, May 17, 2016.
2010 *Mathematics Subject Classification.* Primary 53D20; Secondary 14M25, 32M05, 57S25.
Key words and phrases. Torus action, complex manifold, toric variety, moment-angle manifold, LVM manifold, LVMB manifold, non-Kähler manifold, transverse Kähler form, moment map.
The author was supported by JSPS Research Fellowships for Young Scientists.

[1]The condition (K) is defined for *admissible configurations* (respectively, LVMB datum) that are ingredients of LVM (respectively, LVMB) manifolds. For short, we say that an LVM (respectively, LVMB) manifold satisfies the condition (K) if the corresponding admissible configuration (respectively, LVMB datum) satisfies the condition (K).

to LVM manifolds. In particular, the family of LVMB manifolds properly contains the family of LVM manifolds.

From this point of view, they give the following conjecture which is the motivation of this paper:

Conjecture 1.1. *An LVMB manifold is an LVM manifold if and only if the foliation $\mathcal{F}$ is transverse Kähler.*

Our approach to Conjecture 1.1 uses techniques of Hamiltonian torus actions on symplectic manifolds. Especially, (an analogue of) the convexity theorem plays an important role. The convexity theorem shown by Atiyah, Guillemin and Sternberg in [1] and [7] states that if a compact torus G acts on a compact connected symplectic manifold M in Hamiltonian fashion, then the image of a corresponding moment map is a convex polytope.

A closed 2-form on a smooth manifold is called a *presymplectic form* (see [9]). A smooth manifold M equipped with a presymplectic form ω is called a *presymplectic manifold*. Let $\mathcal{F}$ be a smooth foliation on M. A *transverse symplectic form* ω with respect to $\mathcal{F}$ is a presymplectic form on M whose kernel coincides with $T\mathcal{F}$. As well as the symplectic case, we can consider a *moment map* for an action of a Lie group G on a presymplectic manifold.

In this paper, first we show the following:

Theorem 1.2 (See also Theorem 2.7)**.** *Let M be a compact connected manifold equipped with an action of a compact torus G. Let $\mathfrak{g}'$ be a subspace of the Lie algebra $\mathfrak{g}$ of G such that the action of $\mathfrak{g}'$ is locally free. Let $\mathcal{F}_{\mathfrak{g}'}$ be the foliation on M whose leaves are $\mathfrak{g}'$-orbits. Let ω be a G-invariant transverse symplectic form on M with respect to $\mathcal{F}_{\mathfrak{g}'}$. If there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to ω, then the image of M by Φ is the convex hull of the image of common critical points of h_v for $v \in \mathfrak{g}$, where $h_v : M \to \mathbb{R}$ is given by $h_v(x) = \langle \Phi(x), v\rangle$.*

Example 1.3. The setting of Theorem 1.2 naturally appears in symplectic manifolds. Let N be a compact connected symplectic manifold equipped with an effective Hamiltonian action of a compact torus G. Let $\mathfrak{g}'$ be any subspace of $\mathfrak{g}$ and let $i : \mathfrak{g}' \to \mathfrak{g}$ denote the inclusion. Let $c \in (\mathfrak{g}')^*$ be a regular value of $i^* \circ \Phi : N \to (\mathfrak{g}')^*$ and let $M := (i^* \circ \Phi)^{-1}(c)$. Then, $\mathfrak{g}'$ acts on M locally freely and the restriction $\omega|_M$ of the symplectic form on N is a transverse symplectic form on M with respect to $\mathcal{F}_{\mathfrak{g}'}$. The image $\Phi(M)$ coincides with $(i^*)^{-1}(c) \cap \Phi(N)$, and it is a convex polytope.

For a connected complex manifold M equipped with an effective action of a compact torus G which preserves the complex structure J on M, the subspace

$$\mathfrak{g}_J := \{v \in \mathfrak{g} \mid X_v = -JX_{v'}, {}^{\exists} v' \in \mathfrak{g}\}$$

of $\mathfrak{g}$ acts on M locally freely (see Proposition 3.3). The foliation $\mathcal{F}_{\mathfrak{g}_J}$ whose leaves are $\mathfrak{g}_J$-orbits coincides with the foliation which has been considered in [6] in the case of LVMB manifolds and the foliation which has been considered in [15] in the case of moment-angle manifolds equipped with complex-analytic structures. $\mathcal{F}_{\mathfrak{g}_J}$ gives a lower bound of G-invariant foliations that admit G-invariant transverse Kähler forms such that there exist moment maps with respect to the forms (see Proposition 4.2).

An effective action of a compact torus G on a connected manifold M is said to be *maximal* if there exists a point $x \in M$ such that $\dim G + \dim G_x = \dim M$ (see

[8] for details). If M is a compact connected complex manifold equipped with a maximal action of a compact torus G preserving the complex structure J on M, one can associate a complete fan $q(\Delta)$ in $\mathfrak{g}/\mathfrak{g}_J$ with M. On the other hand, if there exists a G-invariant transverse Kähler form on M with respect to $\mathcal{F}_{\mathfrak{g}_J}$ and if there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to the form, then one can find a lift $\widetilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ of Φ. The following is the main theorem in this paper:

Theorem 1.4 (See also Theorem 5.8)**.** *Let M be a compact connected complex manifold equipped with a maximal action of a compact torus G which preserves the complex structure J on M. If $\mathcal{F}_{\mathfrak{g}_J}$ is transverse Kähler, then a moment map Φ with respect to a G-invariant transverse Kähler form exists and $\widetilde{\Phi}(M)$ is a convex polytope normal to $q(\Delta)$. Conversely, if $q(\Delta)$ is polytopal, then $\mathcal{F}_{\mathfrak{g}_J}$ is transverse Kähler.*

As an application of Theorem 1.4, we show that Conjecture 1.1 is true.

Remark 1.5. It is shown that if the fan $q(\Delta)$ is weakly normal, then there exists a closed 2-form on M whose restriction to an open dense subset is transverse Kähler; see [16].

This paper is organized as follows. In Section 2, we investigate Hamiltonian functions for almost periodic vector fields on transverse symplectic manifolds and show the convexity of the image of a moment map with almost the same argument as Atiyah. In Section 3, we construct a G-invariant foliation $\mathcal{F}_{\mathfrak{g}_J}$ on a complex manifold equipped with an action of a compact torus G. In Section 4, we show that the foliation $\mathcal{F}_{\mathfrak{g}_J}$ is a lower bound of G-invariant foliations that admit moment maps. In Section 5, we consider the case of maximal torus action and give a proof of the conjecture posed by Cupit-Foutou and Zaffran.

Conventions and notation. For a smooth manifold M and a smooth foliation $\mathcal{F}$ on M, we denote by $T\mathcal{F}$ the subbundle of the tangent bundle TM consisting of vectors tangent to leaves of $\mathcal{F}$. Let G be a compact torus acting on M smoothly. We say that $\mathcal{F}$ is *G-invariant* if $T\mathcal{F}$ is a G-equivariant subbundle of TM. We denote by $\mathfrak{g}$ the Lie algebra of G. Through the exponential map, $\mathfrak{g}$ also acts on M. For a subspace $\mathfrak{g}'$ that acts on M locally freely, we denote by $\mathcal{F}_{\mathfrak{g}'}$ the foliation on M whose leaves are $\mathfrak{g}'$-orbits. For a point $x \in M$, we denote by G_x the isotropy subgroup of G at x. We also denote by $\mathfrak{g}_x$ the Lie algebra of G_x. Remark that $\mathfrak{g}_x$ is not the isotropy subgroup of $\mathfrak{g}$ at x. For $v \in \mathfrak{g}$, we denote by X_v the fundamental vector field generated by v on M. For a vector field X on M, we denote by X_x the value of X at x. For the fundamental vector field, we denote by $(X_v)_x$ the value of X_v at x. For a differential form ω, we denote by ω_x the value of ω at x. We denote by $\mathcal{L}_X\omega$ the Lie derivative for ω along X and by $\iota_X\omega$ the interior product of X and ω. We identify $\mathbb{R}$ with the Lie algebra of S^1 by the differential of the map $t \mapsto e^{2\pi\sqrt{-1}t}$. For $\alpha : G \to S^1$, we denote by $d\alpha$ the differential at the unit of G and $d\alpha$ is regarded as an element in $\mathfrak{g}^*$. We denote by $H^1_{\mathcal{F}}(M)$ the first basic cohomology group with coefficients in $\mathbb{R}$.

2. The convexity theorem

Let ω be a transverse symplectic form on a smooth manifold M with respect to a smooth foliation $\mathcal{F}$ on M. Let a compact torus G act on M effectively. We assume that the action of G preserves ω (and hence, $\mathcal{F}$ is G-invariant). In this case, by the Cartan formula we have that

$$0 = \mathcal{L}_{X_v}\omega = d\iota_{X_v}\omega + \iota_{X_v} d\omega = d\iota_{X_v}\omega$$

for $v \in \mathfrak{g}$. We say that a smooth map $\Phi : M \to \mathfrak{g}^*$ is a *moment map* if the function $h_v : M \to \mathbb{R}$ given by $\langle \Phi(x), v\rangle = h_v(x)$ satisfies that $dh_v = -\iota_{X_v}\omega$. A moment map Φ with respect to ω exists if and only if $\iota_{X_v}\omega$ is exact for any v. In particular, the obstruction for the existence of a moment map sits in $H^1_{\mathcal{F}}(M)$. The purpose of this section is to show the convexity of the image of a moment map under certain conditions by almost the same argument as Atiyah (see [1]).

Let $\{\psi_\alpha : U_\alpha \to V_\alpha\}_\alpha$ be foliation charts of $(M, \mathcal{F})$. The local leaf space $U_\alpha/\mathcal{F}$ is diffeomorphic to an open subset of $\mathbb{R}^{\dim M - \dim \mathcal{F}}$. The quotient map $\pi_\alpha : U_\alpha \to U_\alpha/\mathcal{F}$ is a fiber bundle whose fibers are diffeomorphic to open balls of dimension $\dim \mathcal{F}$. The transition functions $\psi_{\alpha\beta} : \psi_\beta(U_\alpha \cap U_\beta) \to \psi_\alpha(U_\alpha \cap U_\beta)$ can be written as

$$\psi_{\alpha\beta}(x, z) = (\psi^T_{\alpha\beta}(x), \psi^{\mathcal{F}}_{\alpha\beta}(x, z)) \in \psi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^{\dim M - \dim \mathcal{F}} \times \mathbb{R}^{\dim \mathcal{F}}$$

for $(x, z) \in \psi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^{\dim M - \dim \mathcal{F}} \times \mathbb{R}^{\dim \mathcal{F}}$.

Let $x \in M$ and let G_x denote the isotropy subgroup at x of G. Let $\psi : U \to V$ be a local foliation chart on an open neighborhood at x. The local leaf space $U/\mathcal{F}$ is diffeomorphic to an open subset of $\mathbb{R}^{\dim M - \dim \mathcal{F}}$. Let $\pi : U \to U/\mathcal{F}$ be the quotient map. Since G_x is compact, the intersection $U' := \bigcap_{g \in G_x} g(U)$ is a G_x-invariant open neighborhood at x. Since $\mathcal{F}$ is G-invariant, $\pi(U')$ is a G_x-manifold of dimension $\dim M - \dim \mathcal{F}$. Let ω be a G-invariant transverse symplectic form on M with respect to $\mathcal{F}$ and suppose that $\dim \mathcal{F} = \ell$ and $\dim M = 2n + \ell$. ω descends to a symplectic form $\underline{\omega}$ on $\pi(U')$. By the equivariant Darboux theorem, there exist G_x-invariant open subsets U_x of $\pi(U')$ and V_x of $T_xM/T_x\mathcal{F}$, a G_x-equivariant diffeomorphism $\varphi : U_x \to V_x$ and local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on $T_xM/T_x\mathcal{F}$ such that

$$(\varphi^{-1})^*\underline{\omega} = \sum_{i=1}^n dx_i \wedge dy_i$$

and

$$X_v = 2\pi \sum_{i=1}^n \langle d\alpha_i, v\rangle \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)$$

for $v \in \mathfrak{g}_x$, where $\alpha_1, \dots, \alpha_n \in \mathrm{Hom}(G_x, S^1)$ are weights at $0 \in T_xM/T_x\mathcal{F}$. Remark that $\psi|_{\pi^{-1}(U_x)} : \pi^{-1}(U_x) \to \psi(\pi^{-1}(U_x))$ is a local foliation chart near x. We state this fact as a lemma for later use.

Lemma 2.1. *Let M be a smooth manifold of dimension $2n + \ell$ equipped with an action of a compact torus G. Let $\mathcal{F}$ be a G-invariant smooth foliation on M of dimension ℓ and let ω be a G-invariant transverse symplectic form on M with respect to $\mathcal{F}$. Then, for any $x \in M$, there exist*

- *a local foliation chart $\psi_x : \widetilde{U_x} \to \widetilde{V_x}$ on an open neighborhood $\widetilde{U_x}$ at x such that $\widetilde{U_x}/\mathcal{F}$ carries the action of G_x,*
- *a G_x-invariant open neighborhood V_x at 0 of $T_xM/T_x\mathcal{F}$,*
- *a G_x-equivariant diffeomorphism $\varphi_x : \widetilde{U_x}/\mathcal{F} \to V_x$, and*
- *local coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ on $T_xM/T_x\mathcal{F}$*

such that $(\varphi_x^{-1})^*\underline{\omega} = \sum_{i=1}^n dx_i \wedge dy_i$ *and*

$$X_v = 2\pi \sum_{i=1}^n \langle d\alpha_i, v\rangle \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)$$

for $v \in \mathfrak{g}_x$, *where* $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}(G_x, S^1)$ *are weights at* $0 \in T_xM/T_x\mathcal{F}$.

Let $M, G, \mathcal{F}, \omega$ be as Lemma 2.1. Let $v \in \mathfrak{g}$ and suppose that there exists a smooth function $h_v : M \to \mathbb{R}$ such that $-\iota_{X_v}\omega = dh_v$. Since ω is transverse symplectic with respect to $\mathcal{F}$, a point $x \in M$ is a critical point of h_v if and only if $(X_v)_x \in T_x\mathcal{F}$. Because of this, unfortunately, we cannot deduce the property that h_v is nondegenerate[2] for a general foliation, unlike the symplectic case. Let $\mathfrak{g}'$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g}'$ acts on M locally freely. Since G is abelian, $\mathcal{F}_{\mathfrak{g}'}$ is a G-invariant foliation.

Lemma 2.2. *Let* M *be a smooth manifold equipped with an action of a compact torus* G. *Let* $\mathfrak{g}'$ *be a subspace of* $\mathfrak{g}$ *such that* $\mathfrak{g}'$ *acts on* M *locally freely. Let* ω *be a* G*-invariant transverse symplectic form on* M *with respect to* $\mathcal{F}_{\mathfrak{g}'}$. *Let* $v \in \mathfrak{g}$ *and suppose that there exists a smooth function* $h_v : M \to \mathbb{R}$ *such that* $dh_v = -\iota_{X_v}\omega$. *Then,* h_v *is a nondegenerate function and the index of each critical submanifold is even.*

Proof. Let $x \in M$ be a critical point of h_v. Then, $(X_v)_x \in T_x\mathcal{F}_{\mathfrak{g}'}$ implies that there exist $v_x \in \mathfrak{g}_x$ and $v' \in \mathfrak{g}'$ such that $v = v_x + v'$. Since $\iota_{X_{v'}}\omega = 0$, we have that $\iota_{X_v}\omega = \iota_{X_{v_x}}\omega$. Since $\iota_{X_v}\omega$ is basic for $\mathcal{F}_{\mathfrak{g}'}$, so is h_v. Let $\psi_x : \widetilde{U_x} \to \widetilde{V_x}$, V_x, φ_x, $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ be as in Lemma 2.1. Since h_v is basic for $\mathcal{F}_{\mathfrak{g}'}$, h_v descends to a smooth function $\underline{h_v} : \widetilde{U_x}/\mathcal{F}_{\mathfrak{g}'} \to \mathbb{R}$ such that $\pi^*\underline{h_v} = h_v$, where $\pi : \widetilde{U_x} \to \widetilde{U_x}/\mathcal{F}_{\mathfrak{g}'}$ denotes the quotient map. By definition of $\mathcal{F}_{\mathfrak{g}'}$, π sends the fundamental vector field X_v generated by v on $\widetilde{U_x}$ to the fundamental vector field X_{v_x} generated by v_x on $\widetilde{U_x}/\mathcal{F}_{\mathfrak{g}'}$. Also, since φ_x is G_x-equivariant, φ_x sends X_{v_x} on $\widetilde{U_x}/\mathcal{F}_{\mathfrak{g}'}$ to X_{v_x} on $T_xM/T_x\mathcal{F}_{\mathfrak{g}'}$.

Therefore

$$dh_v = -\iota_{X_v}\omega = -\iota_{X_{v_x}}\omega = \pi^*(-\iota_{X_{v_x}}\underline{\omega}) = \pi^* \circ \varphi_x^*(-\iota_{X_{v_x}}(\varphi_x^{-1})^*\underline{\omega}).$$

On the other hand,

$$dh_v = d(\pi^*\underline{h_v}) = \pi^*(d\underline{h_v}) = \pi^* \circ (\varphi_x^{-1})^*(d((\varphi_x^{-1})^*\underline{h_v})).$$

Since π^* is injective and φ_x is a diffeomorphism, we have that $d((\varphi_x^{-1})^*\underline{h_v}) = -\iota_{X_{v_x}}(\varphi_x^{-1})^*\underline{\omega}$. Let $\alpha_1, \ldots, \alpha_n \in \mathrm{Hom}(G_x, S^1)$ be the weights at the origin in $T_xM/T_x\mathcal{F}_{\mathfrak{g}'}$. Then, X_{v_x} on $T_xM/T_x\mathcal{F}_{\mathfrak{g}'}$ can be represented as

$$X_{v_x} = 2\pi \sum_{i=1}^n \langle d\alpha_i, v_x\rangle \left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right)$$

with the coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$. Therefore

$$-\iota_{X_{v_x}}(\varphi_x^{-1})^*\underline{\omega} = 2\pi \sum_{i=1}^n \langle d\alpha_i, v_x\rangle (x_i dx_i + y_i dy_i)$$

[2]A smooth function $f : M \to \mathbb{R}$ is said to be *non-degenerate (in the sense of Bott)* if its critical set is a submanifold and the second derivative is a nondegenerate quadric form in the transverse direction.

and hence

$$(\varphi_x^{-1})^*\underline{h_v} = (\varphi_x^{-1})^*\underline{h_v}(0) + \pi \sum_{i=1}^{n} \langle d\alpha_i, v_x \rangle (x_i^2 + y_i^2). \tag{2.1}$$

Therefore $(\varphi_x^{-1})^*\underline{h_v}$ is nondegenerate at 0 and the index at 0 is twice as many as the number of α_i such that $\langle d\alpha_i, v_x \rangle < 0$. Since $\varphi_x \circ \pi : \widetilde{U_x} \to V_x$ is a fiber bundle, h_v is nondegenerate at x and the index at x is twice as many as the number of α_i such that $\langle d\alpha_i, v_x \rangle < 0$, proving the lemma. □

Remark 2.3. In the proof of Lemma 2.2, it follows from (2.1) that h_v attains a local minimum at x if and only if $\langle d\alpha_i, v_x \rangle \geq 0$ for all α_i.

Remark 2.4. We are not sure whether Lemma 2.2 holds even if we replace $\mathcal{F}_{\mathfrak{g}'}$ in any G-invariant foliation $\mathcal{F}$ or not.

The following is the key to the convexity theorem.

Lemma 2.5 ([1, Lemma 2.1]). *Let $\phi : N \to \mathbb{R}$ be a non-degenerate function (in the sense of Bott) on the compact connected manifold N, and assume that neither ϕ nor $-\phi$ has a critical manifold of index 1. Then $\phi^{-1}(c)$ is connected (or empty) for every $c \in \mathbb{R}$.*

By Lemmas 2.2 and 2.5, if M is compact, then the level set $h_v^{-1}(c)$ is connected unless empty. Moreover, if c is a regular value, then $h_v^{-1}(c)$ is a connected submanifold of M. Since G is abelian, $dh_v = -\iota_{X_v}\omega$ is G-invariant. Therefore h_v is also G-invariant. Therefore $h_v^{-1}(c)$ is G-invariant for $c \in \mathbb{R}$.

Lemma 2.6. *Let M be a compact connected manifold equipped with an action of a compact torus G. Let $\mathfrak{g}'$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g}'$ acts on M locally freely. Let ω be a G-invariant transverse symplectic form on M with respect to $\mathcal{F}_{\mathfrak{g}'}$. Let $v_1, \dots, v_k \in \mathfrak{g}$ and suppose that there exists a smooth function h_{v_i} such that $dh_{v_i} = -\iota_{X_{v_i}}\omega$ for $i = 1, \dots, k$. Let $c \in \mathbb{R}^k$ be a regular value of $h = (h_{v_1}, \dots, h_{v_k}) : M \to \mathbb{R}^k$. Then, $\mathfrak{g}'' = \mathfrak{g}' + \mathbb{R}v_1 + \cdots + \mathbb{R}v_k$ acts on $h^{-1}(c)$ locally freely and $\omega|_{h^{-1}(c)}$ is a transverse symplectic form with respect to the foliation $\mathcal{F}_{\mathfrak{g}''}$.*

Proof. Let $x \in h^{-1}(c)$. Since c is a regular value, $(-\iota_{X_{v_i}}\omega)_x$ are linearly independent. This together with the fact that the action of $\mathfrak{g}'$ is locally free yields that $(X_{v''})_x = 0$ if and only if $v'' = 0$ for $v'' \in \mathfrak{g}''$. Therefore the action of $\mathfrak{g}''$ is locally free.

$T_x(h^{-1}(c))$ is given by $\ker(dh)_x = (T_x\mathcal{F}_{\mathfrak{g}''})^{\perp}$, where $(T_x\mathcal{F}_{\mathfrak{g}''})^{\perp}$ denotes the annihilator of $T_x\mathcal{F}_{\mathfrak{g}''}$ with respect to ω. Therefore ω_x descends to a symplectic form on $T_x(h^{-1}(c))/T_x\mathcal{F}_{\mathfrak{g}''}$. It turns out that $Y_x \in \ker(\omega|_{h^{-1}(c)})_x$ if and only if $Y_x \in T_x\mathcal{F}_{\mathfrak{g}''}$. Therefore $\omega|_{h^{-1}(c)}$ is a transverse symplectic form with respect to $\mathcal{F}_{\mathfrak{g}''}$, proving the lemma. □

Now we are in a position to prove the convexity theorem.

Theorem 2.7. *Let M be a compact connected manifold equipped with an action of a compact torus G. Let $\mathfrak{g}'$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g}'$ acts on M locally freely. Let ω be a G-invariant transverse symplectic form on M with respect to $\mathcal{F}_{\mathfrak{g}'}$. Let $v_1, \dots, v_k \in \mathfrak{g}$ and suppose that there exists a smooth function h_{v_i} such* *that* $dh_{v_i} = -\iota_{X_{v_i}}\omega$ *for* $i = 1, \dots, k$. *Put* $h = (h_1, \dots, h_k) : M \to \mathbb{R}^k$. *Then the following hold:*

(A_k) *For* $c \in \mathbb{R}^k$, *the fiber* $h^{-1}(c)$ *is connected unless empty.*

(B_k) $h(M)$ *is convex.*

(C_k) *If* $Z_1, \dots, Z_N$ *are the connected components of the set of common critical points of* h_{v_i}, *then* $h(Z_j)$ *is a point* c_j *and* $h(M)$ *is a convex hull of* $c_1, \dots, c_N$.

Proof. The proof consists of the following steps.

Step 1. (A_k) implies (B_{k+1}).

Step 2. (A_k) holds by induction on k.

Step 3. (B_k) implies (C_k).

Remark that it follows from the connectedness of M that (B_1) holds because $h(M)$ is a closed interval in $\mathbb{R}$.

Step 1. Assume that (A_k) holds. Let $\pi : \mathbb{R}^{k+1} \to \mathbb{R}^k$ be any linear projection given by $\pi(e_i) = \sum_{j=1}^k a_{ij} e_j$ for $i = 1, \dots, k+1$. The composition $h' := \pi \circ h : M \to \mathbb{R}^k$ satisfies the assumption of the theorem. Namely, the j-th component of h' is a smooth function $\sum_{i=1}^{k+1} a_{ij} h_{v_i}$, but

$$d\left(\sum_{i=1}^{k+1} a_{ij} h_{v_i}\right) = -\iota_{X_{\left(\sum_{i=1}^{k+1} a_{ij} v_i\right)}}\omega.$$

Therefore each fiber of h' is connected unless empty. Let $x, y \in h(M)$ and assume that π is surjective and $\pi(x) = \pi(y) = c$. Since the fiber $\pi^{-1}(c)$ is a line in $\mathbb{R}^{k+1}$, it suffices to see that $h(M) \cap \pi^{-1}(c)$ is connected. Since $h' = \pi \circ h$, we have that $h(M) \cap \pi^{-1}(c) = h(h'^{-1}(c))$. Since h is continuous and $h'^{-1}(c)$ is connected, $h(M) \cap \pi^{-1}(c)$ is connected, proving that (A_k) implies (B_{k+1}).

Step 2. It follows from Lemmas 2.2 and 2.5 that (A_1) holds. Assume that (A_k) holds. Let $v_1, \dots, v_{k+1} \in \mathfrak{g}$ and assume that there exists a smooth function $h_{v_i} : M \to \mathbb{R}$ such that $dh_{v_i} = -\iota_{X_{v_i}}\omega$ for $i = 1, \dots, k+1$. Let $h = (h_{v_1}, \dots, h_{v_{k+1}})$ and let $c = (c_1, \dots, c_{k+1})$ be a point in $\mathbb{R}^{k+1}$. We want to show that $h^{-1}(c) = h_{v_1}^{-1}(c_1) \cap \dots \cap h_{v_{k+1}}^{-1}(c_{k+1})$ is connected unless empty. If h has no regular value, then one of dh_{v_i} is a linear combination of the others. By the assumption that (A_k) holds, we are done. Assume that h has a regular value. Then, the set of regular values is dense in $h(M)$. By continuity, we only need to show that $h^{-1}(c)$ is connected for any regular value c. Then, $N := h_{v_1}^{-1}(c_1) \cap \dots \cap h_{v_k}^{-1}(c_k)$ is a connected submanifold by (A_k). Moreover, it follows from Lemma 2.6 that $\omega|_N$ is a transverse symplectic form on N with respect to $\mathcal{F}_{\mathfrak{g}''}$ on N, where $\mathfrak{g}'' = \mathfrak{g} + \mathbb{R}v_1 + \dots + \mathbb{R}v_k$. The function $h_{v_{k+1}}|_N$ satisfies that $dh_{v_{k+1}}|_N = -\iota_{X_{v_{k+1}}}\omega|_N$. Therefore by Lemmas 2.2 and 2.5, $(h_{v_{k+1}}|_N)^{-1}(c_{k+1})$ is connected. Therefore $h^{-1}(c) = h_{v_1}^{-1}(c_1) \cap \dots \cap h_{v_{k+1}}^{-1}(c_{k+1})$ is connected, proving that (A_k) holds for all k.

Step 3. The former assertion that states that $h(Z_j)$ is a point c_j is obvious. Let H be the closure of $\exp(\mathfrak{g}'')$ in G. Let x be a common critical point of $h_{v_1}, \dots, h_{v_k}$. Since $(dh_{v_i})_x = (-\iota_{X_{v_i}}\omega)_x = 0$, we have that $(X_{v_i})_x \in T_x\mathcal{F}_{\mathfrak{g}'}$ for $i = 1, \dots, k$. Therefore there exist $v_{i,x} \in \mathfrak{h}_x$ and $v_i' \in \mathfrak{g}'$ such that $v_i = v_{i,x} + v_i'$ for $i = 1, \dots, k$. Let H_x^0 denote the identity component of H_x. Then, $\{\exp(t_1 v_{1,x}) \cdot \dots \cdot \exp(t_k v_{k,x}) \mid t_i \in \mathbb{R}\}$ is dense in H_x^0 and $\exp(\mathfrak{h}_x + \mathfrak{g}')$ is dense in H. Conversely, for a subtorus H' of H, if

$\exp(\mathfrak{h}'+\mathfrak{g}')$ is dense in H, then each fixed point $x \in M^{H'}$ is a common critical point of $h_{v_1}, \dots, h_{v_k}$. Let $u' \in \mathfrak{g}'$ and $v := \sum_{i=1}^k a_i v_i + u' \in \mathfrak{g}''$ such that $\{\exp(tv) \mid t \in \mathbb{R}\}$ is dense in H. Put $h_v := \sum_{i=1}^k a_i h_{v_i}$. We claim that each critical point of h_v is a common critical point of $h_{v_1}, \dots, h_{v_k}$. Let x be a critical point of h_v. Since $(dh_v)_x = (-\iota_{X_v}\omega)_x = 0$, there exists $v_x \in \mathfrak{h}_x$ and $v' \in \mathfrak{g}'$ such that $v = v_x + v'$. Since $\{\exp(tv) \mid t \in \mathbb{R}\}$ is dense in H, we have that $\{\exp(tv_x) \mid t \in \mathbb{R}\}$ is also dense in H_x^0. By definition of v and v_x, the closure of $\exp(\mathfrak{h}_x + \mathfrak{g}')$ is H. Therefore the critical point x of h_v is a common critical point of $h_{v_1}, \dots, h_{v_k}$. In particular, h_v takes the minimum value in a common critical point of $h_{v_1}, \dots, h_{v_k}$. It turns out that the linear form $\alpha := \sum_{i=1}^k a_i e_i^*$ restricted to $h(M)$ takes the minimum value at one of the c_j's. Therefore

(2.2)
$$h(M) \subseteq \bigcap_{(a_1,\dots,a_k)\in A} \{y = (y_1, \dots, y_k) \in \mathbb{R}^k \mid \langle \alpha, y\rangle \geq \min(\langle \alpha, c_j\rangle \mid j = 1, \dots, N)\},$$

where

$$A := \{(a_1, \dots, a_k) \mid \{\exp(tv) \mid t \in \mathbb{R}\} \text{ is dense in } H\}.$$

Since A is dense in $\mathbb{R}^k$, the right hand side of (2.2) is the convex hull of c_j's. It follows from (B_k) and $c_j \in h(M)$ for all j that $h(M)$ is the convex full of c_j's, proving the theorem. $\square$

3. Holomorphic foliations from torus actions

Let M be a complex manifold and let G be a compact torus acting on M as holomorphic transformations. In this section, we define a subspace $\mathfrak{g}_J$ of $\mathfrak{g}$ which acts on M locally freely by using the complex structure J on M and the action of G. We begin with the following lemmas.

Lemma 3.1. *Let M be a complex manifold equipped with an action of a compact torus G which acts by holomorphic transformations. For $x \in M$, there exist G_x-invariant open neighborhoods U at $x \in M$ and V at $0 \in T_xM$ such that U and V are G_x-equivariantly biholomorphic.*

Proof. Let U_0 be an open neighborhood at x and let $\varphi : U_0 \to V_0$ be a local holomorphic coordinate centered at x, where V_0 is an open subset of $\mathbb{C}^n$. Since G_x is compact, the intersection $\bigcap_{g\in G_x} g(U_0)$ is a G_x-invariant open neighborhood at x. By restricting the domain of definition, we may assume that U_0 is G_x-invariant. Through the differential $(d\varphi)_x : T_xM \to T_0\mathbb{C}^n = \mathbb{C}^n$, we identify $\mathbb{C}^n$ with T_xM. Then we have a biholomorphism $(d\varphi)_x^{-1} \circ \varphi : U_0 \to (d\varphi)_x^{-1}(V_0) \subseteq T_xM$. By averaging on G_x, we have a G_x-equivariant holomorphic map

$$\varphi' := \int_{g\in G_x} (dg)_x \circ ((d\varphi)_x^{-1} \circ \varphi) \circ g^{-1} dg : U_0 \to T_xM.$$

φ' is no longer injective, but $(d\varphi')_x = \mathrm{id}_{T_xM}$. Therefore, it follows from the implicit function theorem that there exists an open subset U of U_0 such that $\varphi'|_U : U \to \varphi'(U)$ is biholomorphic. As before, we may assume that U is G_x-invariant and then $V := \varphi'(U)$ is also G_x-invariant. Therefore there exists a G_x-equivariant biholomorphism $\varphi' : U \to V \subseteq T_xM$, proving the lemma. $\square$

Lemma 3.2. *Let M be a connected complex manifold with the complex structure J. Let X be a nonzero almost periodic vector field on M whose flows preserve J. Assume that JX is complete. If X vanishes at a point $x \in M$, then JX is not almost periodic.*

Proof. Assume that JX is almost periodic. Since X is an infinitesimal automorphism of J, $[X, JX] = 0$. Therefore we may assume that a compact torus G acts on M effectively and as holomorphic transformations and there exist $v, v' \in \mathfrak{g}$ such that $X = X_v$, $JX = X_{v'}$ and the subgroup $\{\exp(sv)\exp(tv') \mid s, t \in \mathbb{R}\}$ is dense in G. Since $X_x = (JX)_x = 0$, x is a G-fixed point. Let $\alpha_1, \dots, \alpha_n \in \mathrm{Hom}(G, S^1)$ be the weights of the G-representation T_xM. By Lemma 3.1, there exists an equivariant biholomorphic map $U \to V \subseteq T_xM$, where U is an open neighborhood at x. Combining with the decomposition of T_xM into 1-dimensional representations of weights $\alpha_1, \dots, \alpha_n$, we have a local coordinate $(z_1, \dots, z_n) : U \to \mathbb{C}^n$ such that $z_i(g \cdot p) = \alpha_i(g) z_i(p)$ for $p \in U$. Let x_i and y_i denote the real and imaginary parts of z_i, respectively. Then, through the local coordinate $(z_1, \dots, z_n)$ we can represent X and JX as

$$X = 2\pi \sum_{i=1}^{n} \langle d\alpha_i, v\rangle \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right)$$

and

$$JX = 2\pi \sum_{i=1}^{n} \langle d\alpha_i, v'\rangle \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right).$$

On the other hand, J is represented as

$$J = \sum_{i=1}^{n} \left(\frac{\partial}{\partial y_i} \otimes dx_i - \frac{\partial}{\partial x_i} \otimes dy_i\right).$$

Therefore

$$\begin{aligned} 0 &= X + J^2 X \\ &= 2\pi \sum_{i=1}^{n} \left(\langle d\alpha_i, v\rangle \left(-y_i \frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial y_i}\right) + \langle d\alpha_i, v'\rangle \left(-x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i}\right)\right) \\ &= 2\pi \sum_{i=1}^{n} \left(\langle d\alpha_i, -y_i v - x_i v'\rangle \frac{\partial}{\partial x_i} + \langle d\alpha_i, x_i v - y_i v'\rangle \frac{\partial}{\partial y_i}\right). \end{aligned}$$

Therefore, by substituting ϵ, $0 < |\epsilon| \ll 1$ for x_i and y_i, we have that

$$0 = \langle d\alpha_i, -\epsilon v - \epsilon v'\rangle = -\epsilon \langle d\alpha_i, v + v'\rangle$$

and

$$0 = \langle d\alpha_i, \epsilon v - \epsilon v'\rangle = \epsilon \langle d\alpha_i, v - v'\rangle$$

for all $i = 1, \dots, n$. Thus we have $\langle d\alpha_i, v\rangle = 0$ for all $i = 1, \dots, n$. Since the action of G on M is effective and M is connected, $d\alpha_i \in \mathfrak{g}^*$ for $i = 1, \dots, n$ spans $\mathfrak{g}^*$. This together with the fact that $\langle d\alpha_i, v\rangle = 0$ for all i shows that $v = 0$. This contradicts the assumption that $X = X_v$ is nonzero and hence JX is not almost periodic, as required. $\square$

Proposition 3.3. *Let M be a connected complex manifold with the complex structure J. Let G be a compact torus acting on M effectively and as holomorphic transformations. Define*

$$\mathfrak{g}_J := \{v \in \mathfrak{g} \mid \text{there exists } v' \in \mathfrak{g} \text{ such that } X_v = -JX_{v'}\}.$$

Then:

(1) $\mathfrak{g}_J$ *is a Lie subalgebra of* $\mathfrak{g}$.
(2) $\mathfrak{g}_J$ *has the complex structure* J_0 *which satisfies* $X_{J_0(v)} = JX_v$.
(3) $\mathfrak{g}_J$ *acts on* M *holomorphically and locally freely.*

Proof. Part (1) follows from the fact that G is commutative. For part (2), let $v \in \mathfrak{g}_J$. Assume that $v', v'' \in \mathfrak{g}_J$ satisfy that $X_v = -JX_{v'} = -JX_{v''}$. Then, $X_{v'} = X_{v''}$. It follows from the effectiveness of the G-action that $v' = v''$. Therefore for $v \in \mathfrak{g}_J$, there exists unique $J_0(v) \in \mathfrak{g}_J$ such that $X_{J_0(v)} = JX_v$. The map $J_0 : \mathfrak{g}_J \to \mathfrak{g}_J$ is linear and $J_0^2 = -1$, proving part (2). Part (3) follows from part (2) and Lemma 3.2. The proposition is proved. □

Let M be a connected complex manifold with the complex structure J and let a compact torus G act on M effectively and as holomorphic transformations. By Proposition 3.3, $\mathfrak{g}_J$ acts on M holomorphically and locally freely. Therefore we have a holomorphic foliation $\mathcal{F}_{\mathfrak{g}_J}$ whose leaves are $\mathfrak{g}_J$-orbits.

4. Torus invariant transverse Kähler foliations

A transverse Kähler form is a special kind of transverse symplectic form. Let M be a complex manifold with the complex structure J. Let $\mathcal{F}$ be a holomorphic foliation on M. A real 2-form ω on M is called *transverse Kähler* with respect to $\mathcal{F}$ if the following conditions are satisfied:

(1) ω is transverse symplectic with respect to $\mathcal{F}$.
(2) ω is of type $(1,1)$. Namely, for $Y_x, Z_x \in T_xM$, $\omega_x(JY_x, JZ_x) = \omega_x(Y_x, Z_x)$.
(3) ω is positive. Namely, $\omega_x(Y_x, JY_x) \geq 0$ for all $Y_x \in T_xM$.

The conditions (1) and (3) imply that $\omega_x(Y_x, JY_x) = 0$ if and only if $Y_x \in T_x\mathcal{F}$. For a holomorphic foliation $\mathcal{F}$ on M, if a transverse Kähler form ω exists, we say that $\mathcal{F}$ is *transverse Kähler*.

Proposition 4.1. *Let M be a complex manifold with the complex structure J. Let G be a compact torus acting on M as holomorphic transformations. Let $\mathcal{F}$ be a G-invariant foliation and let ω be a transverse Kähler form with respect to $\mathcal{F}$. Then,*

$$\int_{g \in G} g^*\omega dg$$

is a transverse Kähler form with respect to $\mathcal{F}$ and invariant under the G-action on M.

Proof. For short, denote

$$\omega' = \int_{g \in G} g^*\omega dg.$$

Since ω is closed, so is ω'. Since G acts on M preserving the complex structure J, ω' is a positive $(1,1)$-form. It remains to show that $\ker \omega'_x = T_x\mathcal{F}$ for all $x \in M$.

By definition, for $Y_x \in T_xM$,

$$\begin{aligned}\omega'_x(Y_x, JY_x) &= \int_{g\in G} (g^*\omega)_x(Y_x, JY_x)dg \\ &= \int_{g\in G} \omega_{g\cdot x}((dg)_x(Y_x), (dg)_x(JY_x))dg \\ &= \int_{g\in G} \omega_{g\cdot x}((dg)_x(Y_x), J((dg)_x(Y_x)))dg\end{aligned}$$

because J is G-invariant. Since $\omega_{g\cdot x}((dg)_x(Y_x), J((dg)_x(Y_x))) \geq 0$ and the equality holds if and only if $(dg)_x(Y_x) \in T_{g\cdot x}\mathcal{F}$, it follows from the G-invariance of $\mathcal{F}$ that $\omega'_x(Y_x, JY_x) = 0$ if and only if $Y_x \in T_x\mathcal{F}$, proving the proposition. □

Thanks to Proposition 4.1, if a G-invariant foliation $\mathcal{F}$ is transverse Kähler, we may always assume that the transverse Kähler form with respect to $\mathcal{F}$ is G-invariant without loss of generality.

For foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ on a smooth manifold M, we denote by $\mathcal{F}_1 \subseteq \mathcal{F}_2$ if $T\mathcal{F}_1 \subseteq T\mathcal{F}_2$. Our next purpose is to give a lower bound of G-invariant transverse Kähler foliations that admit moment maps.

Proposition 4.2. *Let M be a connected complex manifold with the complex structure J. Let a compact torus G act on M effectively and as holomorphic transformations. Let $\mathcal{F}$ be a G-invariant holomophic foliation and let ω be a G-invariant transverse Kähler form with respect to $\mathcal{F}$. If there exists a moment map with respect to ω, then $\mathcal{F}_{\mathfrak{g}_J} \subseteq \mathcal{F}$.*

Proof. Let $\Phi : M \to \mathfrak{g}^*$ be a moment map. We denote by h_v the smooth function given by $h_v(x) = \langle \Phi(x), v\rangle$ for $v \in \mathfrak{g}$ and $x \in M$. Since h_v is G-invariant for $v \in \mathfrak{g}$, we have that

$$0 = \mathcal{L}_{X_{v_1}} h_{v_2} = \iota_{X_{v_1}} dh_{v_2} = -\iota_{X_{v_1}}\iota_{X_{v_2}}\omega = 2\omega(X_{v_1}, X_{v_2}) \tag{4.1}$$

for any $v_1, v_2 \in \mathfrak{g}$. Assume that $v \in \mathfrak{g}_J$. Then,

$$0 \leq \omega(X_v, JX_v) = \omega(X_v, X_{J_0(v)}) = 0$$

by (4.1). Therefore $(X_v)_x \in T_x\mathcal{F}$ for all x and hence $\mathcal{F}_{\mathfrak{g}_J} \subseteq \mathcal{F}$, as required. □

Theorem 4.3. *Let M, J, G, $\mathcal{F}$, ω be as in Proposition 4.2. Let $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_J$ be the quotient map. Assume that there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to ω. Then, there exist $c \in \mathfrak{g}^*$ and a smooth map $\widetilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ such that $\Phi + c = q^* \circ \widetilde{\Phi}$.*

Proof. For $v \in \mathfrak{g}$, h_v denotes the smooth function given by $h_v(x) = \langle \Phi(x), v\rangle$. It follows from Proposition 4.2 that $dh_v = -\iota_{X_v}\omega = 0$ for $v \in \mathfrak{g}_J$. It turns out that h_v is constant on M. Let $i : \mathfrak{g}_J \to \mathfrak{g}$ denote the inclusion. Then, there exists $\overline{c} \in \mathfrak{g}_J^*$ such that $i^* \circ \Phi(x) = \overline{c}$ for any $x \in M$. The sequences

$$0 \longrightarrow \mathfrak{g}_J \xrightarrow{\ i\ } \mathfrak{g} \xrightarrow{\ q\ } \mathfrak{g}/\mathfrak{g}_J \longrightarrow 0$$

and

$$0 \longleftarrow \mathfrak{g}_J^* \xleftarrow[i^*]{} \mathfrak{g}^* \xleftarrow[q^*]{} (\mathfrak{g}/\mathfrak{g}_J)^* \longleftarrow 0$$

are exact. Since i^* is surjective, there exists $c \in \mathfrak{g}^*$ such that $i^*(c) = \overline{c}$. In particular, $i^*(\Phi(x) - c) = 0$ for all $x \in M$. Therefore, there uniquely exists $\widetilde{\Phi}(x) \in (\mathfrak{g}/\mathfrak{g}_J)^*$

such that $q^*(\widetilde{\Phi}(x)) = \Phi(x) - c$ for all $x \in M$. The smoothness is obvious. The theorem is proved. □

We call $\widetilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ an *induced moment map.*

Remark 4.4. (1) Assume that $\mathfrak{g}_J$ is a Lie algebra of a subtorus (that is, closed) in G. Let G_J be the subtorus of G corresponding to $\mathfrak{g}_J$. Then the leaf space $M/\mathcal{F}_{\mathfrak{g}_J}$ is an orbifold M/G_J equipped with an action of the compact torus G/G_J. A transverse Kähler form on M with respect to $\mathcal{F}_{\mathfrak{g}_J}$ descends to a Kähler form on M/G_J. Conversely, the pullback of a Kähler form on M/G_J is a transverse Kähler form on M. An induced moment map $\widetilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ is a composition of the quotient map $M \to M/G_J$ and a moment map $\overline{\Phi} : M/G_J \to (\mathfrak{g}/\mathfrak{g}_J)^*$.

(2) In the case when M is an LVM manifold satisfying the condition (K) with a specific transverse Kähler form which is the so-called *canonical Euler form* on M, it has been shown that a moment map $\Phi : M \to \mathfrak{g}^*$ descends to a moment map $\overline{\Phi} : M/G_J \to (\mathfrak{g}/\mathfrak{g}_J)^*$. Moreover the images of moment maps are described; see [13, Theorem E, (ii)]. The context of Theorem 4.3 is more general than the LVM case, but it says nothing about the image of a moment map. In the next section, we will see the image of an LVMB manifold M by an induced moment map, without the assumption that M satisfies the condition (K).

As a corollary of Theorems 2.7 and 4.3, we have the following.

Corollary 4.5. *Let M be a compact connected complex manifold. Let a compact torus G act on M effectively and preserving the complex structure J on M. Assume that $\mathcal{F}_{\mathfrak{g}_J}$ is transverse Kähler and there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to a G-invariant transverse Kähler form. Then, the image of M by an induced moment map $\widetilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ is a convex polytope in $(\mathfrak{g}/\mathfrak{g}_J)^*$.*

5. The extreme case

In this section, we consider the extreme case. First we recall the notion of maximal torus action introduced in [8]. Let M be a connected smooth manifold equipped with an effective action of a compact torus G. Then, for any point x, we have that $\dim G_x + \dim G \leq \dim M$. The G-action on M is *maximal* if there exists a point $x \in M$ such that

$$\dim G + \dim G_x = \dim M.$$

Any compact connected complex manifold M equipped with a maximal action of a compact torus G which preserves the complex structure can be described with a fan Δ in $\mathfrak{g}$ and a complex subspace $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}}$.

Theorem 5.1 (See [8]). *Let M be a compact connected complex manifold M equipped with a maximal action of a compact torus G which preserves the complex structure J. Then, there exists a nonsingular fan Δ in $\mathfrak{g}$ and a complex subspace $\mathfrak{h}$ such that M is G-equivariantly biholomorphic to $X(\Delta)/H$, where $X(\Delta)$ denotes the toric variety associated with Δ and $H := \exp(\mathfrak{h}) \subseteq G^{\mathbb{C}} \curvearrowright X(\Delta)$.*

We shall recall how to deduce Δ and $\mathfrak{h}$ from M briefly. Each connected component of the set of fixed points of a circle subgroup of G is a closed complex submanifold of M. If such a submanifold has complex codimension one, then we call it a *characteristic submanifold* of M. The number of characteristic submanifolds is at most finite. Let $N_1, \dots, N_k$ be characteristic submanifolds of M. Each characteristic submanifold N_i is fixed by a circle subgroup G_i of G by definition. To each characteristic submanifold N_i, we assign a group isomorphism $\lambda_i : S^1 \to G_i \subseteq G$ such that

$$(\lambda_i(g))_*(\xi) = g\xi \quad \text{for all } g \in S^1 \text{ and } \xi \in TM|_{N_i}/TN_i.$$

We can think of $\lambda \in \mathrm{Hom}(S^1, G)$ as a vector in $\mathfrak{g}$ by $d\lambda(1) \in \mathfrak{g}$. We have a collection Δ of cones

$$\Delta := \left\{ \mathrm{pos}(\lambda_i \mid i \in I) \mid \bigcap_{i \in I} N_i \neq \emptyset \right\},$$

where $\mathrm{pos}(\lambda_i \mid i \in I)$ is the cone spanned by λ_i for $i \in I$. It has been shown that Δ is a nonsingular fan in $\mathfrak{g}$ with respect to the lattice $\mathrm{Hom}(S^1, G)$. Since the action of G preserves the complex structure J on M, it extends to a holomorphic action of $G^{\mathbb{C}}$ on M. Then the complex subspace $\mathfrak{h}$ of $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g} \otimes 1 + \mathfrak{g} \otimes \sqrt{-1}$ is defined to be the Lie algebra of global stabilizers of the $G^{\mathbb{C}}$-action on M. Namely,

$$\mathfrak{h} = \{u \otimes 1 + v \otimes \sqrt{-1} \in \mathfrak{g}^{\mathbb{C}} \mid X_u + JX_v = 0\}.$$

The pair of Δ and $\mathfrak{h}$ satisfies the following.

(1) The restriction $p|_{\mathfrak{h}}$ of the projection $p : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g} \otimes 1 \cong \mathfrak{g}$ is injective.
(2) The quotient map $q : \mathfrak{g} \to \mathfrak{g}/p(\mathfrak{h})$ sends Δ to a complete fan $q(\Delta)$ in $\mathfrak{g}/p(\mathfrak{h})$.

Conversely, if Δ and $\mathfrak{h}$ satisfy the conditions (1) and (2), then the quotient $X(\Delta)/H$ is a compact connected complex manifold and the action of G on $X(\Delta)$ descends to a maximal action on $X(\Delta)/H$.

Proposition 5.2. *Let M be a compact connected complex manifold M equipped with an action of a compact torus G which preserves the complex structure J. Let $\mathfrak{h}$ be the Lie algebra of global stabilizers of the $G^{\mathbb{C}}$-action on M. Then, $p(\mathfrak{h}) = \mathfrak{g}_J$.*

Proof. This follows from the definitions of $\mathfrak{h}$ and $\mathfrak{g}_J$ immediately. □

Lemma 5.3. *Let $\Delta, \mathfrak{h}, q$ be as above and let J denote the complex structure on $X(\Delta)/H$. Assume that $\mathcal{F}_{\mathfrak{g}_J}$ is a transverse Kähler foliation on $X(\Delta)/H$ and let ω be a G-invariant transverse Kähler form with respect to $\mathcal{F}_{\mathfrak{g}_J}$. In addition, assume that there exists a moment map $\Phi : X(\Delta)/H \to \mathfrak{g}^*$ with respect to ω. Then, the image of $X(\Delta)/H$ by an induced moment map $\widetilde{\Phi}$ is a convex polytope and $\widetilde{\Phi}(X(\Delta)/H)$ is a normal polytope of $q(\Delta)$.*

Before the proof of Lemma 5.3, we shall recall notions of normal fan and normal polytope. Let P be an n-dimensional polytope in a vector space V^* of dimension n. For a vector $v \in V$, we put

$$F_v := \{\alpha \in P \mid \langle \alpha, v \rangle \leq \langle \alpha', v \rangle \text{ for all } \alpha' \in P\}.$$

Then F_v is a face of P, and it is the locus where $\langle \bullet, v \rangle|_P$ attains its minimum. For a face F of P, the *(inner) normal cone* σ_F of F is given by

$$\sigma_F := \{v \in V \mid F_v \subseteq F\}.$$

Its relative interior is given by $\{v \in V \mid F_v = F\}$. The *(inner) normal fan* of P is the collection $\Delta_P := \{\sigma_F\}_F$ of cones σ_F for the faces F of P. Conversely, for given fan Δ in V, a polytope P in V^* whose (inner) normal fan coincides with Δ

is called an *(inner) normal polytope* of Δ. If such a polytope P exists for Δ, then Δ is said to be *polytopal*.

Now assume that Δ is a nonsingular fan in $\mathfrak{g}$. We also prepare some notation of submanifolds. For a cone $\sigma \in \Delta$, we denote by X_σ the closed toric subvariety of $X(\Delta)$ corresponding to σ. If $\lambda_1, \dots, \lambda_k \in \mathrm{Hom}(S^1, G)$ are the primitive generators of 1-cones of Δ, then σ can be written as $\sigma = \mathrm{pos}(\lambda_i \mid i \in I)$ for some $I \subseteq \{1, \dots, k\}$ and $(\lambda_i)_{i \in I}$ is a part of the $\mathbb{Z}$-basis of $\mathrm{Hom}(S^1, G)$. More precisely, each point of X_σ is fixed by a subtorus G_σ of G, and $(\lambda_i)_{i\in I}$ is a $\mathbb{Z}$-basis of $\mathrm{Hom}(S^1, G_\sigma)$. The image Y_σ of X_σ by the quotient map $X(\Delta) \to X(\Delta)/H$ is a closed submanifold of $X(\Delta)/H$ because X_σ and Y_σ are both connected components of the set of fixed points by the G_σ-actions. If we denote by $(\alpha_i^I)_{i\in I}$ the dual basis of $(\lambda_i)_{i\in I}$, the set of nonzero weights of $T_x Y_\sigma$ coincides with $(\alpha_i^I)_{i\in I}$ for all $x \in Y_\sigma$.

Proof of Lemma 5.3. We may assume that $q^* \circ \widetilde{\Phi} = \Phi$ without loss of generality. For $v \in \mathfrak{g}$, define $h_v : X(\Delta)/H \to \mathfrak{g}^*$ by $h_v(x) = \langle \Phi(x), v\rangle$. Then, $h_v(x) = \langle \widetilde{\Phi}(x), q(v)\rangle$. We shall see that each connected component of the set of critical points of h_v is one of Y_σ for some $\sigma \in \Delta$. Let $x \in X(\Delta)/H$. Since $dh_v = -\iota_{X_v}\omega$, x is a critical point of h_v if and only if $(X_v)_x \in T_x\mathcal{F}_{\mathfrak{g}_J}$, in particular, $v \in \mathfrak{g}_x + \mathfrak{g}_J$. Therefore, the set of critical points is

$$\bigcup_{\sigma; v\in\mathfrak{g}_\sigma + \mathfrak{g}_J} Y_\sigma.$$

Assume that h_v takes the minimum value a_v on Y_σ. Let $v_\sigma \in \mathfrak{g}_\sigma$ such that $q(v) = q(v_\sigma)$. Then, $\langle d\alpha_i^I, v_\sigma\rangle > 0$ for all $i \in I$, where $(\lambda_i)_{i\in I}$ is the set of primitive generators of σ (see Remark 2.3). Therefore v_σ sits in the relative interior of σ. In particular, $q(v)$ sits in the relative interior of $q(\sigma)$. The converse is also true: if $q(v)$ sits in the relative interior of $q(\sigma)$, then h_v takes the minimum value a_v on Y_σ.

By Corollary 4.5, $\widetilde{\Phi}(X(\Delta)/H)$ is a convex polytope P in $(\mathfrak{g}/\mathfrak{g}_J)^*$. We claim that, for each σ, the image of Y_σ by $\widetilde{\Phi}$ is a face of P. Let $v \in \mathfrak{g}$ such that $q(v)$ sits in the relative interior of σ. Then, h_v attains the minimum value a_v on Y_σ. But $h_v(x) = \langle\widetilde{\Phi}(x), q(v)\rangle$ implies that h_v attains its minimum value a_v at x if and only if $q(v)|_P$ attains its minimum value a_v at $\widetilde{\Phi}(x)$. Since $h_v^{-1}(a_v) = Y_\sigma$, we have that $(q(v)|_P)^{-1}(a_v) = \widetilde{\Phi}(Y_\sigma)$. Since $(q(v)|_P)(\alpha) \geq a_v$ for all $\alpha \in P$, $\widetilde{\Phi}(Y_\sigma)$ is a face of P that is given by $P \cap H_{q(v),a_v}$, where $H_{q(v),a_v}$ is the hyperplane in $(\mathfrak{g}/\mathfrak{g}_J)^*$ defined as $H_{q(v),a_v} := \{\alpha \in (\mathfrak{g}/\mathfrak{g}_J)^* \mid \langle\alpha, q(v)\rangle = a_v\}$.

Conversely, if a face F of P is given by $P \cap H_{q(v),a_v}$, then F is the image of Y_σ by $\widetilde{\Phi}$, where σ is the cone such that $q(v)$ sits in the relative interior of $q(\sigma)$. It turns out that for each face F of P there exists a cone σ such that the inner normal cone of F coincides with $q(\sigma)$. Hence P is a normal polytope of $q(\Delta)$, as required. $\square$

Now we consider the obstruction for the existence of a moment map in the case of an LVMB manifold with at least one indispensable index. Let Σ be an abstract simplicial complex on $\{0, 1, \dots, m\}$ (a singleton $\{i\}$ does not need to be a member of Σ). If an index i satisfies that $\{i\} \notin \Sigma$, we say that i is *indispensable*, according to the literature of LVMB manifolds (see [4], [12] and [13]). Let $G = (S^1)^m$. Then $\mathfrak{g} = \mathbb{R}^m$ and $\mathbb{Z}^m$ is identified with $\mathrm{Hom}(S^1, G)$. G acts on $\mathbb{C}P^m$ via $(g_1, \dots, g_m)\cdot[z_0, z_1, \dots, z_m] := [z_0, g_1z_1, \dots, g_mz_m]$ for $(g_1, \dots, g_m) \in G$ and $[z_0, z_1, \dots, z_m] \in \mathbb{C}P^m$. Put $e_0 := -e_1 - \cdots - e_m$ and define

$$\Delta := \{\mathrm{pos}(e_i \mid i \in I) \mid I \in \Sigma\}.$$

Δ is a nonsingular fan in $\mathbb{R}^m$ and the toric variety $X(\Delta)$ associated with Δ_Σ is given by

$$X(\Delta_\Sigma) = \bigcup_{I\in\Sigma} U_I,$$

where

$$U_I = \{[z] = [z_0, \ldots, z_m] \in \mathbb{C}P^m \mid z_j \neq 0 \text{ if } j \notin I\}.$$

Let $\mathfrak{h} \subseteq \mathbb{C}^m$ such that $\Delta := \Delta_\Sigma$ and $\mathfrak{h}$ satisfy the conditions (1) and (2). We call the manifold $X(\Delta_\Sigma)/H$ an *LVMB manifold*. Moreover, if $q(\Delta_\Sigma)$ is polytopal, we call it an *LVM manifold*, due to [3, Theorems 2.2 and 3.10].

In the case when 0 is indispensable, $[z] \in X(\Delta_\Sigma)$ implies that $z_0 \neq 0$. Therefore we can think of $X(\Delta_\Sigma)$ as an open subset of $\mathbb{C}^m$ via the map

$$[z_0, \ldots, z_m] \mapsto \left(\frac{z_1}{z_0}, \ldots, \frac{z_m}{z_0}\right).$$

Namely,

$$X(\Delta_\Sigma) = \bigcup_{I\in\Sigma} U'_I,$$

where

$$U'_I = \{z = (z_1, \ldots, z_m) \in \mathbb{C}^m \mid z_j \neq 0 \text{ if } j \notin I\}.$$

Remark 5.4. By comparing with [14, Section 3], we can see that if 0 is an indispensable index, then the LVMB manifold $X(\Delta_\Sigma)/H$ is equivariantly biholomorphic to a *moment-angle manifold* with a complex structure constructed by Panov and Ustinovsky.

In the case when there is an indispensable index, we call $X(\Delta_\Sigma)/H$ an *LVMB manifold with an indispensable index*.

It has been shown in [2] that the odd degree basic cohomology groups of $X(\Delta_\Sigma)/H$ with respect to $\mathcal{F}_{\mathfrak{g}_J}$ vanish for shellable Σ. Therefore for shellable Σ, there exists a moment map for any transverse Kähler form on $X(\Delta_\Sigma)/H$ with respect to $\mathcal{F}_{\mathfrak{g}_J}$. We can avoid the assumption on Σ for the vanishing of first basic cohomology groups with a straightforward computation.

Lemma 5.5. *Let $X(\Delta_\Sigma)/H$ be an LVMB manifold with at least one indispensable index. Let J be the complex structure on $X(\Delta_\Sigma)/H$. Then, $H^1_{\mathcal{F}_{\mathfrak{g}_J}}(X(\Delta_\Sigma)/H) = 0$.*

Proof. By renumbering, we may assume that $\{i\}$ is a member of Σ for $i = 1, \ldots, r$ but not for $i = 0, r+1, r+2, \ldots, m$ (r could be m). Then, $X(\Delta_\Sigma) = X(\Delta_{\Sigma'}) \times (\mathbb{C}\setminus\{0\})^{m-r}$, where Σ' is an abstract simplicial complex on $\{0, 1, \ldots, r\}$ such that if $I \in \Sigma$, then $I \in \Sigma'$. $X(\Delta_{\Sigma'})$ is a complement of coordinate subspaces of real codimension ≥ 4 in $\mathbb{C}^r$. Thus, $X(\Delta_{\Sigma'})$ is simply connected. Let $c_i : S^1 \to X(\Delta_\Sigma)$ be the curve defined by

$$c_i(t) = (\underbrace{1, \ldots, 1}_{i-1}, t, \underbrace{1, \ldots, 1}_{m-i}) \in X(\Delta_\Sigma) \subseteq \mathbb{C}^m$$

for $i = 1, \ldots, m$. c_i is null-homologous for $i = 1, \ldots, r$ and the homology classes $[c_i]$ determined by c_i for $i = r+1, \ldots, m$ form a basis of $H_1(X(\Delta_\Sigma))$.

Let β be a 1-form on $X(\Delta_\Sigma)/H$. β is closed and basic for $\mathcal{F}_{\mathfrak{g}_J}$ if and only if $d\beta = 0$ and $\iota_{X_v}\beta = 0$ for any $v \in \mathfrak{g}_J$. Let $\pi : X(\Delta_\Sigma) \to X(\Delta_\Sigma)/H$ be the quotient map. $\pi^*\beta$ is a 1-form basic for $\mathcal{F}_{\mathfrak{h}}$. That is, $\iota_{X_u}\pi^*\beta = 0$ and $\iota_{X_u}d\pi^*\beta = 0$ for

$u \in \mathfrak{h}$. Therefore we need to show that a closed 1-form γ on $X(\Delta_\Sigma)$ satisfying

- $\iota_{X_u}\gamma = 0$ for $u \in \mathfrak{h}$,
- $\iota_{X_v}\gamma = 0$ for $v \in \mathfrak{g}_J$

is exact. Let γ be such a 1-form on $X(\Delta_\Sigma)$. By averaging γ with the action of G, we may assume that γ is G-invariant without loss of generality. Since γ is real, we can represent

$$\gamma = \sum_{i=1}^{m} f_i dz_i + \overline{f_i} d\overline{z_i}$$

with smooth functions $f_i : X(\Delta_\Sigma) \to \mathbb{C}$. Let $v = (v_1, \dots, v_m) \in \mathfrak{g} = \mathbb{R}^m$. Then X_v can be represented as

$$X_v = 2\pi \sum_{i=1}^{m} \sqrt{-1} v_i \left(z_i \frac{\partial}{\partial z_i} - \overline{z_i} \frac{\partial}{\partial \overline{z_i}} \right). \tag{5.1}$$

Since γ is G-invariant, we have that there exists $a_i \in \mathbb{R}$ such that

$$2\pi\sqrt{-1}(z_i f_i - \overline{z_i}\overline{f_i}) = a_i \quad \text{for } i = 1, \dots, m.$$

Since $H_1(X(\Delta_\Sigma))$ is generated by c_i for $i = r+1, \dots, m$ and the Kronecker pairing is given by $\langle [c_i], [\gamma] \rangle = a_i$, it suffices to show that $a_i = 0$ for $i = r+1, \dots, m$.

If $u = (u_{1,1} + \sqrt{-1}u_{1,\sqrt{-1}}, \dots, u_{m,1} + \sqrt{-1}u_{m,\sqrt{-1}}) \in \mathbb{C}^m$, X_u can be represented as

$$X_u = 2\pi \sum_{i=1}^{m} \left(\sqrt{-1}u_{i,1} \left(z_i \frac{\partial}{\partial z_i} - \overline{z_i} \frac{\partial}{\partial \overline{z_i}} \right) - u_{i,\sqrt{-1}} \left(z_i \frac{\partial}{\partial z_i} + \overline{z_i} \frac{\partial}{\partial \overline{z_i}} \right) \right). \tag{5.2}$$

Since $p(\mathfrak{h}) = \mathfrak{g}_J$ by Proposition 5.2, it follows from (5.1) and (5.2) that the conditions $\iota_{X_v}\gamma = 0$ for $v \in \mathfrak{g}_J$ and $\iota_{X_u}\gamma = 0$ for $u \in \mathfrak{h}$ are equivalent to

$$\sum_{i=1}^{m} v_i z_i f_i = 0 \quad \text{for all } v = (v_1, \dots, v_m) \in \mathfrak{g}_J. \tag{5.3}$$

Assume that $\{1, \dots, n\} \in \Sigma$ is a maximal simplex. Then, $q(e_1), \dots, q(e_n)$ form a basis of $\mathfrak{g}/\mathfrak{g}_J$. Let $\alpha_1, \dots, \alpha_n$ be the dual basis of $q(e_1), \dots, q(e_n)$. Then, we have a basis

$$e_j - \sum_{i=1}^{n} \langle \alpha_i, q(e_j) \rangle e_i \quad \text{for } j = n+1, \dots, m$$

of $\mathfrak{g}_J = \ker q$. Therefore we have that (5.3) is equivalent to

$$z_j f_j - \sum_{i=1}^{n} \langle \alpha_i, q(e_j) \rangle z_i f_i = 0 \quad \text{for } j = n+1, \dots, m. \tag{5.4}$$

The Kronecker pairing $\langle [c_i], [\gamma] \rangle = 0$ for $i = 1, \dots, r$ because c_i for $i = 1, \dots, r$ is null-homologous. Therefore $a_i = 2\pi\sqrt{-1}(z_i f_i - \overline{z_i}\overline{f_i}) = 0$ for $i = 1, \dots, n$. This together with (5.4) yields that $a_j = 0$ for all $j = n+1, \dots, m$. Therefore γ is exact, proving the lemma. □

Corollary 5.6. *Let $X(\Delta_\Sigma)/H$ be an LVMB manifold with at least one indispensable index. Assume that $\mathcal{F}_{\mathfrak{g}_J}$ on $X(\Delta_\Sigma)/H$ is transverse Kähler. Then, the complete fan $q(\Delta)$ in $\mathfrak{g}/\mathfrak{g}_J$ is polytopal. Namely, $X(\Delta_\Sigma)/H$ is an LVM manifold.*

Proof. Let ω be a transverse Kähler form on $X(\Delta_\Sigma)/H$ with respect to $\mathcal{F}_{\mathfrak{g}_J}$. Since $\mathcal{F}_{\mathfrak{g}_J}$ is G-invariant, we may assume that ω is G-invariant by Proposition 4.1. The closed 1-form $-\iota_{X_v}\omega$ is exact for all $v \in \mathfrak{g}$ by Lemma 5.5. Therefore there exists a moment map on $X(\Delta_\Sigma)/H$ with respect to ω. Let $\widetilde{\Phi} : X(\Delta_\Sigma)/H \to (\mathfrak{g}/\mathfrak{g}_J)^*$ be an induced moment map. By Lemma 5.3, the image of $X(\Delta_\Sigma)/H$ by $\widetilde{\Phi}$ is a normal polytope of $q(\Delta)$. Therefore $q(\Delta)$ is polytopal, as required. □

Conversely, we can construct a transverse Kähler form on $X(\Delta_\Sigma)/H$ with respect to $\mathcal{F}_{\mathfrak{g}_J}$ from a normal polytope P of $q(\Delta)$. Essentially, this fact has been shown in [10] and [12]. But, the "language" in this paper is slightly different from them. For the reader's convenience, we give a brief explanation of the construction of a transverse Kähler form without a proof. Let P be a normal polytope of $q(\Delta_\Sigma)$ represented as

$$P = \{\alpha \in (\mathfrak{g}/\mathfrak{g}_J)^* \mid \langle \alpha, q(e_i)\rangle \geq a_i\}.$$

The map $q^* : (\mathfrak{g}/\mathfrak{g}_J)^* \to \mathfrak{g}^*$ is an injective map. We consider the embedding $P \to \mathfrak{g}^*$ given by

$$\alpha \mapsto \sum_{i=1}^m (\langle \alpha, q(e_i)\rangle - a_i)e_i^* = q^*(\alpha) - \sum_{i=1}^m a_i e_i^*,$$

where e_i^* denotes the i-th dual basis vector of the standard basis $e_1, \dots, e_m$ of $\mathfrak{g} = \mathbb{R}^m$. Let $i : \mathfrak{g}_J \to \mathfrak{g}$ be the inclusion and consider the dual map $i^* : \mathfrak{g}^* \to \mathfrak{g}_J^*$. The image of embedded P is the point $i^*(\sum_{i=1}^m -a_i e_i^*) =: \beta$. $X(\Delta_\Sigma)$ is an open subset of $\mathbb{C}^m$. So $X(\Delta_\Sigma)$ has the standard Kähler form

$$\omega_{\mathrm{st}} = \frac{\sqrt{-1}}{2}\sum_{i=1}^m dz_i \wedge d\overline{z_i}.$$

G acts on $X(\Delta_\Sigma)$ preserving ω_{st}. The map $\Phi : X(\Delta_\Sigma) \to \mathfrak{g}^*$ given by

$$\Phi(z_1, \dots, z_m) = \pi \sum_{i=1}^m |z_i|^2 e_i^*$$

is a moment map with respect to ω_{st}. For the composition $i^* \circ \Phi : X(\Delta_\Sigma) \to \mathfrak{g}_J^*$, the value $\beta \in \mathfrak{g}_J^*$ is a regular value and $(i^* \circ \Phi)^{-1}(\beta) =: \mathcal{Z}_P$ is a smooth manifold equipped with an action of G and the G-invariant transverse symplectic form $\omega := \omega_{\mathrm{st}}|_{\mathcal{Z}_P}$ with respect to $\mathcal{F}_{\mathfrak{g}_J}$. Each orbit of H intersects with $\mathcal{Z}_P$ at exactly one point in $\mathcal{Z}_P$, and hence the inclusion $\mathcal{Z}_P \to X(\Delta_\Sigma)$ induces an equivariant diffeomorphism $\varphi : \mathcal{Z}_P \to X(\Delta_\Sigma)/H$. The form $(\varphi^{-1})^*\omega$ on $X(\Delta_\Sigma)/H$ is what we wanted. The image of an induced moment map is nothing but P up to translations.

The construction above and Corollary 5.6 yield the following.

Theorem 5.7. *The holomorphic foliation $\mathcal{F}_{\mathfrak{g}_J}$ on an LVMB manifold $X(\Delta_\Sigma)/H$ with at least one indispensable index is transverse Kähler if and only if $X(\Delta_\Sigma)/H$ is an LVM manifold with at least one indispensable index.*

We give remarks on the foliation $\mathcal{F}_{\mathfrak{g}_J}$ and equivariant holomorphic principal bundles. Let M_1 and M_2 be complex manifolds with the complex structures J_1 and J_2, respectively. Assume that compact tori G_1 and G_2 act on M_1 and M_2 respectively. If we have an equivariant principal holomorphic bundle $\pi : M_1 \to M_2$, it is easy to see that $T_x\mathcal{F}_{{\mathfrak{g}_1}_{J_1}} = (d\pi)_x^{-1}(T_{\pi(x)}\mathcal{F}_{{\mathfrak{g}_2}_{J_2}})$ for all $x \in M_1$. Therefore we can obtain every basic form for $\mathcal{F}_{{\mathfrak{g}_1}_{J_1}}$ from a basic form for $\mathcal{F}_{{\mathfrak{g}_2}_{J_2}}$ by the pull-back operator π^*. Moreover, every basic form for $\mathcal{F}_{{\mathfrak{g}_1}_{J_1}}$ is also basic for the action of

$\ker\alpha$ (that is, invariant under the action of $\ker\alpha$ and the interior product with fundamental vector fields generated by the action of $\ker\alpha$ vanishes). Therefore there exists the inverse operator $(\pi^*)^{-1}$ of π^* defined for basic forms for $\mathcal{F}_{\mathfrak{g}_{1J_1}}$. In particular, $\mathcal{F}_{\mathfrak{g}_{1J_1}}$ on M_1 is transverse Kähler if and only if so is $\mathcal{F}_{\mathfrak{g}_{2J_2}}$ on M_2. Also, there exists a moment map $\Phi_1 : M_1 \to \mathfrak{g}_1^*$ with respect to a G_1 invariant transverse Kähler form ω_1 if and only if there exists a moment map $\Phi_2 : M_2 \to \mathfrak{g}_2^*$ with respect to $(\pi^*)^{-1}\omega_1$.

It has been shown in [8] that a compact connected complex manifold M equipped with a maximal action of a compact torus G is obtained as a quotient of an LVMB manifold with at least one indispensable index. Therefore, we can characterize the manifold with a maximal torus action which admits a transverse Kähler form with respect to $\mathcal{F}_{\mathfrak{g}_J}$.

Theorem 5.8. *Let M, G, J, Δ, $\mathfrak{h}$ be as in Theorem 5.1. Then, the following are equivalent:*

1. *$\mathcal{F}_{\mathfrak{g}_J}$ on M is transverse Kähler.*
2. *$q(\Delta)$ is polytopal.*

In this case, for any G-invariant transverse Kähler form ω, there exists a moment map $\Phi : M \to \mathfrak{g}^*$ with respect to ω and the image of M by an induced moment map $\widetilde{\Phi} : M \to (\mathfrak{g}/\mathfrak{g}_J)^*$ is an inner normal polytope of $q(\Delta)$.

As a corollary, we show that the conjecture posed in [6] holds.

Corollary 5.9. *For an LVMB manifold M, the holomorphic foliation $\mathcal{F}_{\mathfrak{g}_J}$ is transverse Kähler if and only if M is an LVM manifold.*

References

[1] M. F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc. **14** (1982), no. 1, 1–15. MR642416

[2] Fiammetta Battaglia and Dan Zaffran, *Foliations modeling nonrational simplicial toric varieties*, Int. Math. Res. Not. IMRN **22** (2015), 11785–11815. MR3456702

[3] L. Battisti, *LVMB manifolds and quotients of toric varieties*, Math. Z. **275** (2013), no. 1-2, 549–568. MR3101820

[4] Frédéric Bosio, *Variétés complexes compactes: une généralisation de la construction de Meersseman et López de Medrano-Verjovsky* (French, with English and French summaries), Ann. Inst. Fourier (Grenoble) **51** (2001), no. 5, 1259–1297. MR1860666

[5] Frédéric Bosio and Laurent Meersseman, *Real quadrics in $\mathbf{C}^n$, complex manifolds and convex polytopes*, Acta Math. **197** (2006), no. 1, 53–127. MR2285318

[6] Stéphanie Cupit-Foutou and Dan Zaffran, *Non-Kähler manifolds and GIT-quotients*, Math. Z. **257** (2007), no. 4, 783–797. MR2342553

[7] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), no. 3, 491–513. MR664117

[8] Hiroaki Ishida, *Complex manifolds with maximal torus actions*, J. Reine Angew. Math., published ahead of print, DOI 10.1515/crelle-2016-0023.

[9] Paulette Libermann and Charles-Michel Marle, *Symplectic geometry and analytical mechanics*, Mathematics and its Applications, vol. 35, D. Reidel Publishing Co., Dordrecht, 1987. Translated from the French by Bertram Eugene Schwarzbach. MR882548

[10] J. J. Loeb and M. Nicolau, *On the complex geometry of a class of non-Kählerian manifolds*, Israel J. Math. **110** (1999), 371–379. MR1750427

[11] Santiago López de Medrano and Alberto Verjovsky, *A new family of complex, compact, non-symplectic manifolds*, Bol. Soc. Brasil. Mat. (N.S.) **28** (1997), no. 2, 253–269. MR1479504

[12] Laurent Meersseman, *A new geometric construction of compact complex manifolds in any dimension*, Math. Ann. **317** (2000), no. 1, 79–115. MR1760670

[13] Laurent Meersseman and Alberto Verjovsky, *Holomorphic principal bundles over projective toric varieties*, J. Reine Angew. Math. **572** (2004), 57–96. MR2076120

[14] Taras Panov and Yuri Ustinovsky, *Complex-analytic structures on moment-angle manifolds* (English, with English and Russian summaries), Mosc. Math. J. **12** (2012), no. 1, 149–172, 216. MR2952429

[15] Taras Panov, Yury Ustinovskiy, and Misha Verbitsky, *Complex geometry of moment-angle manifolds*, Math. Z. **284** (2016), no. 1-2, 309–333. MR3545497

[16] Yu. M. Ustinovsky, *Geometry of compact complex manifolds with maximal torus action*, Proc. Steklov Inst. Math. **286** (2014), no. 1, 198–208. MR3482597

Department of Mathematics and Computer Science, Graduate School of Science and Engineering, Kagoshima University, Korimoto 1-21-40, Kagoshima 890-0065, Japan

E-mail address: ishida@sci.kagoshima-u.ac.jp