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DENSITY CHARACTER OF SUBGROUPS OF TOPOLOGICAL GROUPS

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ABSTRACT. We give a complete characterization of subgroups of separable topological groups. Then we show that the following conditions are equivalent for an ω-narrow topological group G: (i) G is homeomorphic to a subspace of a separable regular space; (ii) G is topologically isomorphic to a subgroup of a separable topological group; (iii) G is topologically isomorphic to a closed subgroup of a separable path-connected, locally path-connected topological group.

A *pro-Lie group* is a projective limit of finite-dimensional Lie groups. We prove here that an almost connected pro-Lie group is separable if and only if its weight is not greater than the cardinality $\mathfrak{c}$ of the continuum. It is deduced from this that an almost connected pro-Lie group is separable if and only if it is homeomorphic to a subspace of a separable Hausdorff space. It is also proved that a locally compact (even feathered) topological group G which is a subgroup of a separable Hausdorff topological group is separable, but the conclusion is false if it is assumed only that G is homeomorphic to a subspace of a separable Tychonoff space.

We show that every precompact (abelian) topological group of weight less than or equal to $\mathfrak{c}$ is topologically isomorphic to a closed subgroup of a separable pseudocompact (abelian) group of weight $\mathfrak{c}$. This result implies that there is a wealth of closed non-separable subgroups of separable pseudocompact groups. An example is also presented under the Continuum Hypothesis of a separable countably compact abelian group which contains a non-separable closed subgroup.

1. Introduction

All topological spaces and topological groups are assumed to be Hausdorff. The *weight* $w(X)$ of a topological space X is defined as the smallest cardinal number of the form $|\mathcal{B}|$, where $\mathcal{B}$ is a base of topology in X. The *density character* $d(X)$ of a topological space X is $\min\{|A| : A \text{ is dense in } X\}$. If $d(X) \leq \aleph_0$, then we say that the space X is *separable*.

It is well known that a subspace S of a separable metrizable space X is separable, but a closed subspace S of a separable Hausdorff topological space X is not necessarily separable. Even a closed linear subspace S of a separable Hausdorff locally convex space X can fail to be separable [17].

In several classes of topological groups, the situation improves notably. It has been proved by W. Comfort and G. Itzkowitz [3] that a closed subgroup S of a

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separable locally compact topological group G is separable. It is known also that a metrizable subgroup of a separable topological group is separable [25, p. 89] (see also [17]).

In Section 3 we compare embeddings of topological groups in separable regular spaces and in separable topological groups. First, in Theorem 3.1, we characterize subgroups of separable topological groups as ω-narrow groups of weight at most $\mathfrak{c} = 2^\omega$. It turns out that in the class of ω-narrow topological groups, the difference between the two types of embeddings disappears, even if we require an embedding to be closed. Our Theorem 3.2 states that if an ω-narrow topological group G is *homeomorphic* to a subspace of a separable regular space, then G is topologically isomorphic to a *closed* subgroup of a separable path-connected, locally path-connected topological group.

Early this century K. H. Hofmann and S. A. Morris introduced the class of *pro-Lie groups* which consists of projective limits of finite-dimensional Lie groups and proved that it contains all compact groups, all locally compact abelian groups, and all connected locally compact groups and is closed under the formation of products and closed subgroups. They defined a topological group G to be *almost connected* if the quotient group of G by the connected component of its identity is compact.

In Section 4 we look at conditions on the topological group G which are sufficient to guarantee its separability if G is a subspace of a separable Hausdorff or regular space X. En route we show in Theorem 4.2 that an almost connected pro-Lie group (for example, a connected locally compact group or a compact group) is separable if and only if its weight is not greater than $\mathfrak{c}$. It is proved in Theorem 4.10 that if an almost connected pro-Lie group G is a subspace of a separable Hausdorff space, then G is also separable. Theorem 4.12 extends the latter result to a wider class of topological groups which contains both almost connected pro-Lie groups and locally compact σ-compact groups.

It turns out that the difference between the classes of (ω-narrow) topological groups admitting a homeomorphic embedding into a separable *Hausdorff* or *regular* space can be very big. It is clear that every *subspace* of a separable regular space has weight at most $\mathfrak{c}$, while Proposition 4.7 of [20] provides an example of a countably compact (hence ω-narrow) topological abelian group G homeomorphic to a subspace of a separable Hausdorff space which satisfies $d(G) = 2^{\mathfrak{c}}$ and $w(G) = 2^{2^{\mathfrak{c}}}$. This means, by Theorem 4.10, that almost connected pro-Lie groups are considerably more sensitive than countably compact groups with respect to homeomorphic embeddings into separable Hausdorff spaces.

In the rest of Section 4 we consider various types of subgroups of separable topological groups. For example, it is shown in Theorem 4.13 that every feathered subgroup of a separable group is separable. Since the class of feathered groups includes both locally compact and metrizable groups, we thus obtain a generalization of the aforementioned results of Comfort and Itzkowitz ([3] and [25], [17]).

A complete topological group which has a local base at the identity element consisting of open subgroups is said to be *prodiscrete*. It is shown in Proposition 4.16 that closed subgroups of separable prodiscrete abelian groups can fail to be separable. Since prodiscrete abelian groups are pro-Lie, we see that closed subgroups of separable pro-Lie groups need not be separable either; hence one cannot drop the "almost connected" assumption in Theorem 4.2.

In Section 5 we consider closed isomorphic embeddings into separable groups. We prove in Theorem 5.2 that a precompact topological group of weight $\leq \mathfrak{c}$ is topologically isomorphic to a closed subgroup of a separable pseudocompact group H of weight $\leq \mathfrak{c}$. Since there is a wealth of non-separable precompact groups of weight $\mathfrak{c}$, we conclude that closed subgroups of separable pseudocompact groups can fail to be separable. Corollary 5.3 is an abelian version of Theorem 5.2.

We also present in Proposition 5.5, under the Continuum Hypothesis, an example of a separable countably compact abelian group G which contains a non-separable closed subgroup. We do not know if such an example exists in ZFC alone (see Question 5.6).

2. Background results

In this section we collect several important well-known facts that will be frequently applied in the sequel.

Theorem 2.1 (De Groot, [11, Theorem 3.3]). *If X is a separable regular space, then $w(X) \leq \mathfrak{c}$. More generally, every regular space X satisfies $w(X) \leq 2^{d(X)}$.*

Compact dyadic spaces are defined to be continuous images of generalized Cantor cubes $\{0,1\}^\kappa$, where κ is an arbitrary cardinal number.

Proposition 2.2 (Engelking, [5, Theorem 10]). *Let κ be an infinite cardinal. A compact dyadic space K with $w(K) \leq 2^\kappa$ satisfies $d(K) \leq \kappa$. In particular, if $w(K) \leq \mathfrak{c}$, then K is separable.*

Theorem 2.3 (Comfort, [2, Theorem 3.1]). *Every infinite compact topological group G satisfies $|G| = 2^{w(G)}$ and $d(G) = \log w(G)$. In particular, if a compact group G satisfies $w(G) \leq \mathfrak{c}$, then it is separable.*

Theorem 2.3 is deduced in [2] from the fact that compact topological groups are dyadic. In fact as Comfort observes, if G is a compact group of weight $\alpha \geq \aleph_0$, then there are continuous surjections as indicated below:

$$\{0,1\}^\alpha \to G \to [0,1]^\alpha$$

and $|\{0,1\}^\alpha| = |[0,1]^\alpha| = 2^\alpha$, and so $|G| = 2^\alpha$.

A key result we shall need is the following one:

Theorem 2.4 (Hewitt–Marczewski–Pondiczery, [11, Theorem 11.2]). *Let $\{X_i : i \in I\}$ be a family of topological spaces and $X = \prod_{i\in I} X_i$, where $|I| \leq 2^\kappa$ for some cardinal number $\kappa \geq \omega$. If $d(X_i) \leq \kappa$ for each $i \in I$, then $d(X) \leq \kappa$. In particular, the product of no more than $\mathfrak{c}$ separable spaces is separable.*

In the following two definitions we introduce the main concepts of our study in Section 4.

Definition 2.5 (Hofmann–Morris, [12]). A Hausdorff topological group G is said to be *almost connected* if the quotient group G/G_0 is compact, where G_0 is the connected component of the identity in G.

Clearly the class of almost connected topological groups includes all compact groups, all connected topological groups, and their products.

Definition 2.6 (Hofmann–Morris, [12–14]). A topological group is called a *pro-Lie group* if it is a projective limit of finite-dimensional Lie groups.

As shown in [12] the class of pro-Lie groups includes all locally compact abelian topological groups, all compact groups, all connected locally compact topological groups, and all almost connected locally compact topological groups. Further, every closed subgroup of a pro-Lie group is again a pro-Lie group, and any finite or infinite product of pro-Lie groups is a pro-Lie group.

Theorem 2.7 (Hofmann–Morris, [12, Theorem 12.81])**.** *Let G be a connected pro-Lie group. Then G contains a maximal compact connected subgroup C such that G is homeomorphic to $C \times \mathbb{R}^\kappa$, for some cardinal κ.*

Almost connected pro-Lie groups also admit a topological characterization similar to the one in Theorem 2.7.

Theorem 2.8 (Hofmann–Morris, [13, Corollary 8.9])**.** *Every almost connected pro-Lie group G is homeomorphic to the product $\mathbb{R}^\kappa \times \{0,1\}^\lambda \times B$, where $\{0,1\}$ is the discrete two-element group, B is a compact connected group, and κ, λ are cardinals.*

Remark 2.9. It is known that a topological group G is separable if it contains a closed subgroup H such that both H and the quotient space G/H are separable. In particular, separability is a three space property [1, Theorem 1.5.23].

A topological group is said to be *ω-narrow* [1, Section 3.4] if it can be covered by countably many translations of every neighborhood of the identity element. It is known that every separable topological group is ω-narrow (see [1, Corollary 3.4.8]). The class of ω-narrow groups is productive and hereditary with respect to taking arbitrary subgroups [1, Section 3.4], so ω-narrow groups need not be separable. In fact, ω-narrow groups can have uncountable cellularity (see [21] or [1, Example 5.4.13]).

The following theorem characterizes the class of ω-narrow topological groups.

Theorem 2.10 (Guran, [7])**.** *A topological group G is ω-narrow if and only if G is topologically isomorphic to a subgroup of a product of second countable topological groups.*

3. Isomorphic embedding of ω-narrow topological groups

In this section we characterize the subgroups of separable topological groups as ω-narrow groups of weight at most $\mathfrak{c}$ (Theorem 3.1). Further, in Theorem 3.2, we show that for an ω-narrow group G, being a subgroup of a separable group is equivalent to admitting a closed topological embedding into a separable regular space which is also equivalent to admitting a closed topological embedding into a separable path-connected, locally path-connected topological group.

Let us recall that a topological group which has a local base at the identity element consisting of open subgroups is called *protodiscrete.* A complete protodiscrete group is said to be *prodiscrete.* Protodiscrete topological groups are exactly the totally disconnected pro-Lie groups [12, Proposition 3.30].

Theorem 3.1. *A (protodiscrete abelian) topological group H is topologically isomorphic to a subgroup of a separable (prodiscrete abelian) topological group if and only if H is ω-narrow and satisfies $w(H) \le \mathfrak{c}$.*

Proof. Assume that a topological group H is a subgroup of a separable topological group G. Then G is ω-narrow [1, Corollary 3.4.8] and satisfies $w(G) \le \mathfrak{c}$, by

Theorem 2.1. Since subgroups of ω-narrow topological groups are ω-narrow [1, Theorem 3.4.4], we see that H is also ω-narrow and satisfies $w(H) \leq \mathfrak{c}$.

Conversely, assume that an ω-narrow group H satisfies $w(H) \leq \mathfrak{c}$. It follows from Theorem 2.10 (see also [1, Theorem 3.4.23]) that H is topologically isomorphic to a subgroup of a topological product $\Pi = \prod_{i \in I} G_i$, where the index set I has the cardinality at most $\mathfrak{c}$ and each factor G_i is a second countable topological group. The group Π is separable by the Hewitt–Marczewski–Pondiczery theorem.

Further, if the group H is protodiscrete, then all factors G_i can be chosen countable and discrete. Indeed, let $\mathcal{N}(e)$ be a local base at the identity element e of H consisting of open subgroups and satisfying $|\mathcal{N}(e)| \leq \mathfrak{c}$. For every $N \in \mathcal{N}(e)$, denote by π_N the canonical homomorphism of H onto the discrete quotient group H/N. Then the diagonal product of the family $\{\pi_N : N \in \mathcal{N}(e)\}$ is a topological isomorphism of H onto a subgroup of the product group $P = \prod_{N \in \mathcal{N}(e)} H/N$. Since the group H is ω-narrow, each quotient group H/N is countable. Thus H is a topological subgroup of P, a product of countable discrete groups. As $|\mathcal{N}(e)| \leq \mathfrak{c}$, the group P is separable. Evidently, the group P is prodiscrete. This completes our argument. □

It is natural to compare the restrictions on a given topological group G imposed by the existence of either a topological embedding of G into a separable regular space or a topological isomorphism of G onto a subgroup of a separable topological group.

Let us note that the first of the two classes of topological groups is strictly wider than the second one. In order to show this, consider an arbitrary discrete group G satisfying $\omega < |G| \leq \mathfrak{c}$. Then G embeds as a *closed subspace* into the separable space $\mathbb{N}^{\mathfrak{c}}$ [4], where $\mathbb{N}$ is the set of positive integers endowed with the discrete topology. However, G does not admit a topological isomorphism onto a subgroup of a separable topological group. Indeed, every subgroup of a separable topological group is ω-narrow by Theorem 3.1. Since the discrete group G is uncountable, it fails to be ω-narrow.

The above observation makes it natural to restrict our attention to ω-narrow topological groups when considering embeddings into separable topological groups. It turns out that in the class of ω-narrow topological groups, the difference between the two types of embeddings disappears, even if we require an embedding to be closed.

In the next result, which complements Theorem 3.1, we identify a large class of topological groups with the class of closed subgroups of separable path-connected, locally path-connected topological groups. It is not surprising therefore that the Hartman–Mycielski construction [9] comes into play.

Theorem 3.2. *The following are equivalent for an arbitrary ω-narrow topological group G:*

(a) *G is homeomorphic to a subspace of a separable regular space;*
(b) *G is topologically isomorphic to a subgroup of a separable topological group;*
(c) *G is topologically isomorphic to a closed subgroup of a separable path-connected, locally path-connected topological group.*

Proof. Since Hausdorff topological groups are regular, it is clear that (c) implies (b) and (b) implies (a). Hence it suffices to show that (a) implies (c).

Assume that an ω-narrow topological group G is homeomorphic to a subspace of a separable regular space X. Then $w(X) \le \mathfrak{c}$ by Theorem 2.1 and hence $w(G) \le w(X) \le \mathfrak{c}$. Applying Theorem 3.1, we find a separable topological group H containing G as a topological subgroup. Clearly G can fail to be closed in H, so our next step is to construct another separable topological group containing G as a closed subgroup. To this end we will use the path-connected, locally path-connected group $H^\bullet$ corresponding to H and consisting of *step functions* from the semi-open interval $J = [0, 1)$ to the group H (for a description of $H^\bullet$, see Hartman and Mycielski [9] or [1, Construction 3.8.1]).

The group H is canonically isomorphic to a closed subgroup of $H^\bullet$. The corresponding monomorphism $i\colon H \to H^\bullet$ assigns to each element $h \in H$ the constant function $i(h) = h^\bullet \in H^\bullet$ defined by $h^\bullet(x) = h$ for all $x \in J$.

Let e be the identity of H. Denote by E the set of all step functions f from J to H satisfying the following condition:

(i) there exists $b \in [0, 1)$ such that $f(x) = e$ for each x with $b \le x < 1$.

It is clear that E is a subgroup of $H^\bullet$. Let D be a countable dense subgroup of H. Denote by E' the subgroup of E consisting of all $f \in E$ satisfying the following condition:

(ii) there exist rational numbers $0 = b_0 < b_1 < \cdots < b_{m-1} < b_m = 1$ such that f is constant on each subinterval $[b_k, b_{k+1})$ and $f(b_k) = g_k \in D$ for $k = 0, 1, \ldots, m-1$.

Notice that E' is countable. The argument given in the proof of [1, Theorem 3.8.8, item e)] shows that E' is dense in $H^\bullet$.

Denote by G_0 the subgroup of $H^\bullet$ generated by the set $i(G) \cup E$. Since $i\colon H \to H^\bullet$ is a topological monomorphism, the group G is topologically isomorphic to the subgroup $i(G)$ of $H^\bullet$. It also follows from $E' \subset E \subset G_0$ that the group G_0 is separable. Let us verify that $i(G)$ is closed in G_0.

First we note that $i(H)$ is closed in $H^\bullet$ according to [1, Theorem 3.8.3]. Hence the required conclusion about $i(G)$ will follow if we show that $G_0 \cap i(H) = i(G)$. Assume that $f \in G_0 \cap i(H)$. Then f is a constant function on J with a single value $h_0 \in H$. We have to show that $h_0 \in G$, i.e. $f \in i(G)$. As $f \in G_0$, we can write f in the form

$$f = i(g_1)^{m_1} t_1^{n_1} \cdots i(g_k)^{m_k} t_k^{n_k} i(g_{k+1})^{m_{k+1}},$$

where $g_1, \ldots, g_k, g_{k+1} \in G$, $t_1, \ldots, t_k \in E$, and $m_i, n_i \in \mathbb{Z}$. Item (i) of our definition of the group E implies that there exists $b < 1$ such that $t_i(b) = e$ for each $i = 1, \ldots, k$. Hence $h_0 = f(b) = g_1^{m_1} \cdots g_k^{m_k} g_{k+1}^{m_{k+1}} \in G$. Since f is a constant function, we see that $f \in i(G)$. This implies the inclusion $G_0 \cap i(H) \subset i(G)$. The inverse inclusion is evident. Therefore, $G \cong i(G)$ is closed in G_0.

Now we have to check that the group G_0 is path-connected and locally path-connected. It is worth mentioning that G_0 is a proper dense subgroup of $H^\bullet$, but not *every* dense subgroup of $H^\bullet$ inherits these properties from $H^\bullet$. We start with the path-connectedness of G_0.

Since G_0 is generated by the set $i(G) \cup E$, it suffices to verify that for every element $f \in i(G) \cup E$, there exists a path in G_0 connecting the identity $e^\bullet$ of $H^\bullet$ with f. Indeed, every element $f \in G_0$ is a product of finitely many elements $f_1, \ldots, f_n$ of $i(G) \cup E$ and, multiplying the paths connecting $f_1, \ldots, f_n$ with $e^\bullet$, we obtain a path in G_0 connecting $e^\bullet$ and f. So take an arbitrary element $f \in i(G) \cup E$.

Case 1 ($f \in i(G)$). Then $f = g^\bullet$ for some $g \in G$. For every $r \in [0,1]$, let f_r be a step function from J to H defined by $f_r(x) = g$ if $x < r$ and $f_r(x) = e$ if $r \le x < 1$. It is clear that $f_r \in H^\bullet$ for each $r \in [0,1]$ and that the mapping $\varphi\colon [0,1] \to H^\bullet$, $\varphi(r) = f_r$, is continuous. Since $\varphi(0) = e^\bullet$, $\varphi(1) = f \in i(G)$, and $f_r \in E \subset G_0$ for each $r \in [0,1)$, this proves that φ is a path in G_0 connecting $e^\bullet$ and f.

Case 2 ($f \in E$). Choose a partition $0 = b_0 < b_1 < \cdots < b_m = 1$ of J such that f is constant on each interval $[b_i, b_{i+1})$, where $0 \le i < m$, and $f(b_{m-1}) = e$. Let us define a path $\varphi\colon [0,1] \to G_0$ connecting $e^\bullet$ with f as follows. First we put $l_k = b_{k+1} - b_k$ for $k = 0, \dots, m-1$. For every $r \in [0,1]$ and every $x \in J$, let

$$f_r(x) = \begin{cases} f(x), & \text{if } b_k \le x < b_k + r\cdot l_k \text{ for some } k \text{ with } 0 \le k < m;\\ e, & \text{if } b_k + r\cdot l_k \le x < b_{k+1} \text{ for some } k \text{ with } 0 \le k < m. \end{cases}$$

It is easy to verify that $f_0 = e^\bullet$, $f_1 = f$, and $f_r(x) = e$ if $b_{m-1} \le x < 1$. Hence $f_r \in E \subset G_0$ for each $r \in [0,1]$. Again, the mapping $\varphi\colon [0,1] \to H^\bullet$ defined by $\varphi(r) = f_r$ for each $r \in [0,1]$ is continuous, so φ is a path in G_0 connecting $e^\bullet$ and f.

Summing up, the group G_0 is path-connected.

Finally, we check that G_0 is locally path-connected. Every neighborhood of $e^\bullet$ in $H^\bullet$ contains an open neighborhood of the form

$$O(U,\varepsilon) = \{f \in H^\bullet : \mu(\{x \in J : f(x) \notin U\}) < \varepsilon\},$$

where U is an open neighborhood of the identity e in H, $\varepsilon > 0$, and μ is the Lebesgue measure on J. Therefore, by the homogeneity of G_0, it suffices to verify that the intersections $G_0 \cap O(U,\varepsilon)$ are path-connected.

Take an arbitrary element $f \in G_0 \cap O(U,\varepsilon)$, where U is an open neighborhood of e in H and $\varepsilon > 0$. Then $f = i(g_1)^{m_1} t_1^{n_1} \cdots i(g_k)^{m_k} t_k^{n_k} i(g_{k+1})^{m_{k+1}}$, where $g_1, \dots, g_k, g_{k+1} \in G$, $t_1, \dots, t_k \in E$, and $m_i, n_i \in \mathbb{Z}$. Our aim is to define a path $\Phi\colon [0,1] \to G_0 \cap O(U,\varepsilon)$ connecting $e^\bullet$ with f. We cannot apply directly the formula from the above Case 1 since otherwise we lose control over the measure of the set $\{x \in J : f_r(x) \notin U\}$ for some $r \in (0,1)$, where f_r is assumed to be $\Phi(r)$. Instead, we adjust the speed of changes of the elements $i(g_1), \dots, i(g_k), i(g_{k+1})$ and $t_1, \dots, t_k$ on the appropriately chosen subintervals of J.

First we choose a partition $0 = b_0 < b_1 < \cdots < b_{m-1} < b_m = 1$ of J such that t_i is constant on $[b_j, b_{j+1})$ for all integers $i \le k$ and $j < m$. Let also $l_j = b_{j+1} - b_j$, where $j = 0, \dots, m-1$. For every $i = 1, \dots, k, k+1$ we define a path $\varphi_i\colon [0,1] \to H^\bullet$ by

$$\varphi_i(r,x) = \begin{cases} g_i, & \text{if } b_j \le x < b_j + r\cdot l_k \text{ for some } j \text{ with } 0 \le j < m;\\ e, & \text{if } b_j + r\cdot l_j \le x < b_{j+1} \text{ for some } j \text{ with } 0 \le j < m. \end{cases}$$

Then $\varphi_i(0,x) = e$, $\varphi_i(1,x) = g_i$ for each $x \in J$ and $\varphi_i(r,\cdot) \in G_0$ for each $r \in [0,1]$. The path φ_i is continuous and connects $e^\bullet$ with $g_i^\bullet$ in G_0.

Similarly, we define a path $\psi_i\colon [0,1] \to H^\bullet$ for each $i = 1, \dots, k$ by

$$\psi_i(r,x) = \begin{cases} t_i(x), & \text{if } b_j \le x < b_j + r\cdot l_k \text{ for some } j \text{ with } 0 \le j < m;\\ e, & \text{if } b_j + r\cdot l_j \le x < b_{j+1} \text{ for some } j \text{ with } 0 \le j < m. \end{cases}$$

It is clear that $\psi_i(0,x) = e$, $\psi_i(1,x) = t_i(x)$ for each $x \in J$ and $\psi_i(r,\cdot) \in G_0$ for each $r \in [0,1]$. The path ψ_i is continuous and connects $e^\bullet$ with t_i in G_0.

Finally we define a path Φ in G_0 connecting $e^\bullet$ with f by letting

$$\Phi(r,x) = \varphi_1(r,x)^{m_1} \cdot \psi_1(r,x)^{n_1} \cdots \varphi_k(r,x)^{m_k} \cdot \psi_k(r,x)^{n_k} \cdot \varphi_{k+1}(r,x)^{n_{k+1}},$$

where $r \in [0,1]$ and $x \in J$. The path Φ is continuous being a product of continuous paths φ_i and ψ_i. The following Claim describes a basic property of the path Φ:

Claim. For all $r \in [0,1]$ and $x \in J$, either $\Phi(r,x) = f(x)$ or $\Phi(r,x) = e$.

Indeed, let $r \in [0,1]$ and $x \in J$ be arbitrary. Choose an integer $j < m$ such that $b_j \le x < b_{j+1}$. If $b_j \le x < b_j + rl_j$, then $\varphi_i(r,x) = g_i$ and $\psi_i(r,x) = t_i(x)$ for all i, whence it follows that $\Phi(r,x) = f(x)$. If $b_j + r \cdot l_j \le x < b_{j+1}$, then $\varphi_i(r,x) = e$ and $\psi_i(r,x) = e$ for all i, so $\Phi(r,x) = e$. This proves our Claim.

Applying the Claim we see that

$$\{x \in J : \Phi(r,x) \notin U\} \subset \{x \in J : f(x) \notin U\},$$

for every $r \in [0,1]$. Hence $\mu(\{x \in J : \Phi(r,x) \notin U\}) < \epsilon$ for each $r \in [0,1]$. In other words, the path Φ lies in $O(U,\varepsilon)$, so the set $O(U,\varepsilon)$ is path-connected. Since the sets of the form $G_0 \cap O(U,\varepsilon)$ constitute a base for G_0 at the identity, this completes the proof of the theorem. □

4. Separability of pro-Lie groups

We prove in this section that an almost connected pro-Lie group G is separable if and only if $w(G) \le \mathfrak{c}$; i.e. almost connected pro-Lie groups are close to compact groups in this respect (see Theorem 2.3). This result is then used in Theorem 4.10 to show that if an almost connected pro-Lie group G is homeomorphic to a subspace of a separable Hausdorff space, then G is separable as well. We also prove a result stating that a locally compact subgroup of a separable topological group is separable (see Corollary 4.14).

First, we establish a simple relationship between almost connected pro-Lie groups and ω-narrow groups.

Proposition 4.1. *Every almost connected pro-Lie group has countable cellularity and hence is ω-narrow.*

Proof. By Theorem 2.8, every almost connected pro-Lie group G is homeomorphic to the product $\mathbb{R}^\kappa \times \{0,1\}^\lambda \times B$, where B is a compact connected group and κ, λ are cardinals. The space $\mathbb{R}^\kappa \times \{0,1\}^\lambda$ has countable cellularity as a product of separable spaces [4, Theorem 2.3.17]. The compact group B also has countable cellularity by [1, Corollary 4.1.8]. Hence the cellularity of the space $\mathbb{R}^\kappa \times \{0,1\}^\lambda \times B$ is countable (see Corollary 5.4.9 of [1]). Finally, every topological group of countable cellularity is ω-narrow according to [1, Theorem 3.4.7]. □

It turns out that the separability of an almost connected pro-Lie group G is equivalent to the inequality $w(G) \le \mathfrak{c}$ as is shown in the next theorem.

Theorem 4.2. *An almost connected pro-Lie group G is separable if and only if $w(G) \le \mathfrak{c}$. In particular this is the case if G is a connected locally compact group.*

Proof. If G is separable, then $w(G) \le \mathfrak{c}$ by Theorem 2.1, since Hausdorff topological groups are regular.

Conversely, assume that $w(G) \le \mathfrak{c}$. By Theorem 2.8, the group G is homeomorphic to the product $\mathbb{R}^\kappa \times \{0,1\}^\lambda \times B$, where B is a compact connected topological group and κ, λ are cardinals. It is clear that $w(G) = \kappa \cdot \lambda \cdot \mu \le \mathfrak{c}$, where $\mu = w(B)$.

Hence the spaces $\mathbb{R}^\kappa$ and $\{0,1\}^\lambda$ are separable by Theorem 2.4, while the separability of B follows from Theorem 2.3. Hence G is also separable as the product of three separable spaces. $\square$

Since every connected locally compact group is σ-compact, the last part of Theorem 4.2 admits the following slightly more general form which will be applied in the proof of Theorem 4.12:

Lemma 4.3. *Let N be a closed subgroup of an ω-narrow topological group G. If the quotient space G/N is locally compact, then the inequality $w(G/N) \leq \mathfrak{c}$ is equivalent to the separability of G/N.*

Proof. Let $\pi \colon G \to G/N$ be the quotient mapping of G onto the locally compact left coset space G/N. If G/N is separable, then Theorem 2.1 implies that $w(G/N) \leq \mathfrak{c}$. Assume therefore that $w(G/N) \leq \mathfrak{c}$.

Take an open neighborhood U of the identity element in G such that the closure of the open set $\pi(U)$ in G/N is compact. Since G is ω-narrow, there exists a countable set $C \subset G$ such that $G = CU$. Hence the compact sets $\overline{\pi(xU)}$, with $x \in C$, cover the space G/N. This proves that G/N is σ-compact.

Let $\{K_n : n \in \omega\}$ be a countable family of compact sets that covers G/N. Making use of the local compactness of G/N we can find, for every $n \in \omega$, an open set O_n with compact closure in G/N such that $K_n \subset O_n$. Since the space G/N is normal, there exists a closed G_δ-set F_n in G/N such that $K_n \subset F_n \subset O_n$. It is clear that F_n is compact for each $n \in \omega$. Summing up, each F_n is a compact G_δ-set in the quotient space G/N of the ω-narrow topological group G, so Theorem 2 of [22] implies that each F_n is a dyadic compact space. As $w(F_n) \leq w(G/N) \leq \mathfrak{c}$ for each $n \in \omega$, it follows from Proposition 2.2 that each F_n is separable. The inclusions $K_n \subset F_n$ with $n \in \omega$ imply that $G/N = \bigcup_{n\in\omega} F_n$, so the space G/N is separable. $\square$

Remark 4.4. Theorem 4.2 would not be valid in ZFC if one replaced the condition $w(G) \leq \mathfrak{c}$ by the weaker one, $|G| \leq 2^{\mathfrak{c}}$. Indeed, the compact topological group $G = \{0,1\}^\kappa$ with $\kappa = \mathfrak{c}^+$ satisfies $w(G) = \kappa$, so G is not separable by Theorem 2.3. However, it is consistent with ZFC that $|G| = 2^\kappa = 2^{\mathfrak{c}}$ (see [16, Chap. VIII, Sect. 4]).

Theorem 4.2 generalizes the second part of Theorem 2.3. It is also clear that Theorem 4.2 is not valid for arbitrary locally compact groups or arbitrary pro-Lie groups — it suffices to take a discrete group of cardinality $\mathfrak{c}$.

A family $\mathcal{N}$ of subsets of a topological space Y is called a *network* for Y if for every point $y \in Y$ and any neighbourhood U of y there exists a set $F \in \mathcal{N}$ such that $y \in F \subset U$. The *network weight* $nw(Y)$ of a space Y is defined as the smallest cardinal number of the form $|\mathcal{N}|$, where $\mathcal{N}$ is a network for Y.

Lemma 4.5. *If L is a Lindelöf subspace of a separable Hausdorff space X, then $nw(L) \leq \mathfrak{c}$. Hence every compact subspace K of a separable Hausdorff space satisfies $w(K) \leq \mathfrak{c}$.*

Proof. Denote by D a countable dense subset of X. Let

$$\mathcal{D} = \{\overline{C} : C \subseteq D\} \text{ and } \mathcal{N} = \{\bigcap \gamma : \gamma \subseteq \mathcal{D},\ |\gamma| \leq \omega\}.$$

Then $|\mathcal{D}| \leq \mathfrak{c}$, $|\mathcal{N}| \leq \mathfrak{c}^\omega = \mathfrak{c}$, and we claim that the family $\{N \cap L : N \in \mathcal{N}\}$ is a network for L. First we note that the family $\mathcal{D}$ separates points of X. In other words, for every distinct element $x, y \in X$, there exists $C \in \mathcal{D}$ such that $x \in C \not\ni y$. This is clear since the space X is Hausdorff. Take a point $x \in L$ and an arbitrary open neighborhood U of x in X. Denote by $\mathcal{D}_x$ the family of all $C \in \mathcal{D}$ with $x \in C$. Since $\mathcal{D}$ separates points of X, we see that $\bigcap \mathcal{D}_x = \{x\}$. Using the Lindelöf property of L, we can find a countable subfamily γ of $\mathcal{D}_x$ such that $L \cap \bigcap \gamma \subseteq L \cap U$. Then $F = \bigcap \gamma \in \mathcal{N}$ and $x \in L \cap F \subseteq L \cap U$. This proves that $\mathcal{N}$ is a network for L.

If K is a compact subset of X, then $w(K) = nw(K)$. Since K is obviously Lindelöf, we conclude that $w(K) \leq \mathfrak{c}$. $\square$

Combining Lemma 4.5 and Theorem 2.3, we deduce the following fact:

Corollary 4.6. *If a compact group G is a subspace of a separable Hausdorff space, then G is separable.*

It turns out that the compactness of G in Corollary 4.6 cannot be weakened to σ-compactness:

Example 4.7 (See [3, Lemma 3.1])**.** Let X be any separable compact space which contains a closed non-separable subspace Y. The free abelian topological group $A(Y)$ naturally embeds into $A(X)$ as a closed subgroup. Then $A(X)$ is a separable σ-compact group, while $A(Y)$ is not separable; otherwise Y would be separable.

The conclusion of Corollary 4.6 remains valid for connected pro-Lie groups. Later, in Theorem 4.10, we will show that "connected" can be weakened to "almost connected".

In what follows we consider topological groups homeomorphic to a subspace of a separable *Hausdorff* space. According to Theorem 2.1, a regular separable space has weight at most $\mathfrak{c}$. However, it is known that the weight of a separable Hausdorff space can be as big as $2^{2^{\mathfrak{c}}}$; see [15]. Furthermore, there exists a countably compact topological abelian group G homeomorphic to a subspace of a separable Hausdorff space which satisfies $d(G) = 2^{\mathfrak{c}}$ and $w(G) = 2^{2^{\mathfrak{c}}}$ (see [20, Proposition 4.7]). This crucial difference between (embeddings into) Hausdorff and regular separable spaces explains why Theorem 3.2 is not applicable to topological groups which are topologically embeddable into separable Hausdorff spaces. Instead, we rely on the rich structure theory of pro-Lie groups.

Proposition 4.8. *If a connected pro-Lie group G is a subspace of a separable Hausdorff space X, then G is separable.*

Proof. By Theorem 2.7, the connected pro-Lie group G is homeomorphic to the product $C \times \mathbb{R}^\kappa$, where C is a compact connected group and κ is a cardinal. Since C can be identified with a subspace of G, Lemma 4.5 implies that $w(C) \leq \mathfrak{c}$. Further, $\mathbb{R}^\kappa$ contains a compact subspace homeomorphic with $\{0,1\}^\kappa$. Since the latter space has weight κ, we apply Lemma 4.5 once again to conclude that $\kappa \leq \mathfrak{c}$. Hence $w(G) \leq \mathfrak{c}$. Therefore G is separable, by Theorem 4.2. $\square$

Since the class of connected pro-Lie groups is productive and contains connected locally compact groups [12], the next fact is immediate from Proposition 4.8.

Corollary 4.9. *Let G be a product of connected locally compact groups. If G is a subspace of a separable Hausdorff space X, then G is separable.*

The next result, one of the main results in this section, extends Corollary 4.6 and Proposition 4.8 to almost connected pro-Lie groups.

Theorem 4.10. *Let G be an almost connected pro-Lie group. If G is homeomorphic to a subspace of a separable Hausdorff space, then it is separable as well.*

Proof. By Theorem 2.8, the group G is homeomorphic to $\mathbb{R}^\kappa \times \{0,1\}^\lambda \times B$, where $\{0,1\}$ is the discrete two-element group, B is a compact connected group, and κ, λ are cardinals. Assume that G is homeomorphic to a subspace of a separable Hausdorff space. The connected pro-Lie group $\mathbb{R}^\kappa$ is separable by Proposition 4.8. The compact group $K = \{0,1\}^\lambda \times B$ is separable according to Corollary 4.6. Therefore G is separable as the product of two separable spaces, $\mathbb{R}^\kappa$ and K. (It is worth noting that $\kappa \cdot \lambda \leq \mathfrak{c}$.) □

It turns out that Theorem 4.10 can be extended to a wider class of topological groups which contains both almost connected pro-Lie groups and locally compact σ-compact groups. We have already mentioned that Theorem 3.2 is not applicable to the study of topological groups homeomorphic to subspaces of separable Hausdorff spaces. Unfortunately, no structure theory is available for the groups which are extensions of locally compact groups by almost connected pro-Lie groups. For these two principal reasons, our proof of Theorem 4.12 is unavoidably indirect and considerably longer than the argument in the proof of Theorem 4.10.

For the proof of Theorem 4.12 we need a simple auxiliary result which complements Pontryagin's open homomorphism theorem (see [18, Theorem 3]).

Lemma 4.11. *Let X and Y be locally compact and σ-compact topological spaces and let X be homogeneous. If $f\colon X \to Y$ is a continuous bijection, then f is a homeomorphism.*

Proof. Let $U_0 \subset X$ be a non-empty open set with compact closure. Take a non-empty open set U in X such that $\overline{U} \subset U_0$. Denote by $Homeo(X)$ the family of all homeomorphisms of X onto itself. Since X is homogeneous and σ-compact, there exists a countable subfamily $\mathcal{A} \subset Homeo(X)$ such that $X = \bigcup\{\alpha(U) : \alpha \in \mathcal{A}\}$. Note that the closure of $\alpha(U)$ is compact, for each $\alpha \in \mathcal{A}$.

The family $\{f(\alpha(U)) : \alpha \in \mathcal{A}\}$ is a countable cover of Y. Since Y is locally compact it has the Baire property. Hence there exists $\alpha \in \mathcal{A}$ such that the closure of $f(\alpha(U))$ has a non-empty interior in Y. Let V be a non-empty open set in Y contained in $\overline{f(\alpha(U))} = f(\alpha(\overline{U}))$. Since f is one-to-one, we see that $f^{-1}(V) \subset \alpha(\overline{U}) \subset \alpha(U_0)$, so $W = f^{-1}(V)$ is an open subset of $\alpha(U_0)$. The closure of $\alpha(U_0)$ in X is the compact set $\alpha(\overline{U_0})$, so the restriction of f to the open subset W of $\alpha(\overline{U_0})$ is a homeomorphism of W onto its image $V = f(W)$. Therefore, by the homogeneity argument, f is a homeomorphism. □

Theorem 4.12. *Let G be an ω-narrow topological group which contains a closed subgroup N such that N is an almost connected pro-Lie group and the quotient space G/N is locally compact. If G is homeomorphic to a subspace of a separable Hausdorff space, then it is separable as well.*

Proof. Assume that G is a subspace of a separable Hausdorff space X. Since $N \subset G \subset X$, it follows from Theorem 4.10 that the group N is separable and, hence, $w(N) \leq \mathfrak{c}$ by Theorem 2.1.

Let τ be the topology of X. The family of all regular open sets in X constitutes a base for a weaker topology on X, say σ. Since the topology τ is separable, the space $Y = (X, \sigma)$ has a base of the cardinality at most $\mathfrak{c}$. Indeed, let S be a countable dense subset of X. Then the family

$$\mathcal{B} = \{\mathrm{Int}_X \overline{D} : D \subseteq S\} \setminus \{\emptyset\}$$

is a base for Y and, clearly, $|\mathcal{B}| \leq \mathfrak{c}$. It is also clear that the space Y is Hausdorff. We see in particular that the pseudocharacter of Y is at most $\mathfrak{c}$, i.e. $\psi(Y) \leq \mathfrak{c}$. Since the identity mapping of X onto Y is continuous, it follows that $\psi(X) \leq \psi(Y) \leq \mathfrak{c}$. Hence the subspace G of X satisfies $\psi(G) \leq \mathfrak{c}$ as well. This is the first important property of the group G.

We claim that there exists a continuous isomorphism (not necessarily a homeomorphism) π of G onto a Hausdorff topological group H with the following properties:

(i) $w(H) \leq \mathfrak{c}$;
(ii) the restriction of π to N is a topological isomorphism of N onto the closed subgroup $K = \pi(N)$ of H;
(iii) the quotient space H/K is locally compact.

Indeed, it follows from [1, Corollary 3.4.19] that for every neighborhood U of the identity e in G, there exists a continuous homomorphism π_U of G onto a second-countable Hausdorff topological group H_U such that $\pi_U^{-1}(V) \subseteq U$, for some open neighborhood V of the identity in H_U. Let $\mathcal{P}$ be a family of open neighborhoods of e in G such that $\{e\} = \bigcap \mathcal{P}$ and $|\mathcal{P}| \leq \mathfrak{c}$ (we use the fact that $\psi(G) \leq \mathfrak{c}$). Let also $\mathcal{Q}$ be a family of open neighborhoods of e in G such that $|\mathcal{Q}| \leq \mathfrak{c}$ and $\{W \cap N : W \in \mathcal{Q}\}$ is a local base for N at e (here we use the inequality $w(N) \leq \mathfrak{c}$). Then the cardinality of the family $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$ is not greater than $\mathfrak{c}$. For every element $U \in \mathcal{R}$, we take a continuous homomorphism π_U of G onto a second countable topological group H_U as above. Further, since the space G/H is locally compact, there exists an open neighborhood U_0 of e in G such that the closure of $\varphi_G(U_0)$ in G/H is compact, where $\varphi_G \colon G \to G/H$ is the quotient mapping. Take a continuous homomorphism $p \colon G \to H_0$ to a second countable topological group H_0 such that $p^{-1}(V_0) \subset U_0$, for some open neighborhood V_0 of $p(e)$ in H_0.

Let π be the diagonal product of the family $\{\pi_U : U \in \mathcal{R}\} \cup \{p\}$. Then π is a continuous homomorphism of G to the product $P = H_0 \times \prod_{U \in \mathcal{R}} H_U$ of second countable Hausdorff topological groups. It follows from our choice of $\mathcal{P}$ and the inclusion $\mathcal{P} \subseteq \mathcal{R}$ that π is a monomorphism. Since $|\mathcal{R}| \leq \mathfrak{c}$, the group P and its subgroup $H = \pi(G)$ have weight at most $\mathfrak{c}$. Denote by p_0 the projection of P onto the factor H_0 and let $W_0 = H \cap p_0^{-1}(V_0)$. Then W_0 is an open neighborhood of the identity in H, and since $p_0 \circ \pi = p$, we see that $\pi^{-1}(W_0) = p^{-1}(V_0) \subset U_0$.

Let us verify that $\pi(N)$ is closed in H. It follows from our choice of the family $\mathcal{Q}$ and the inclusion $\mathcal{Q} \subseteq \mathcal{R}$ that the restriction of π to N is a topological isomorphism of N onto its image $\pi(N)$. Since the pro-Lie group N is complete, so is the subgroup $K = \pi(N)$ of H. Hence K is closed in H. In particular, the quotient space H/K is Hausdorff. This proves our claim.

Let $\varphi_G \colon G \to G/N$ and let $\varphi_H \colon H \to H/K$ be the canonical quotient mappings onto the left coset spaces G/N and H/K, respectively. We define a mapping $i \colon G/N \to H/K$ by letting $i(\varphi_G(x)) = \varphi_H(\pi(x))$, for each $x \in G$. Since $K = \pi(N)$, this definition is correct. It follows from our definition of i that the diagram below commutes:

$$\begin{tikzcd} G \arrow[r, "\varphi_G"] \arrow[d, "\pi"'] & G/N \arrow[d, "i"] \\ H \arrow[r, "\varphi_H"] & H/K \end{tikzcd}$$

Since π is algebraically an isomorphism and $K = \pi(N)$, we see that i is a bijection. The continuity of the mapping i follows from the facts that π and φ_H are continuous, while φ_G is open and continuous.

The set $\varphi_H(W_0)$ is an open neighborhood of the identity in H/K and the closure of $\varphi_H(W_0)$ is compact; i.e. the space H/K is locally compact. Indeed, it follows from $\pi^{-1}(W_0) \subset U_0$ that $\overline{\varphi_H(W_0)} \subset i(\overline{\varphi_G(U_0)}) \subset i(\overline{\varphi_G(U_0)})$. Since $\overline{\varphi_G(U_0)}$ is compact, so are the sets $i(\overline{\varphi_G(U_0)})$ and $\overline{\varphi_H(W_0)}$. We have thus proved that the homomorphism $\pi\colon G \to G/H$ satisfies (i)–(iii).

The group H is ω-narrow as a continuous homomorphic image of the ω-narrow group G. Hence H can be covered by countably many translations of the open set W_0. Since the set $\overline{\varphi_H(W_0)}$ is compact, it follows that the space H/K is σ-compact. Similarly, since the group G is ω-narrow and the set $\overline{\varphi_G(U_0)}$ is compact, the space G/N is also σ-compact. Therefore, both spaces G/N and H/K are locally compact, σ-compact, and homogeneous.

Finally, Lemma 4.11 implies that the bijection $i\colon G/N \to H/K$ is a homeomorphism. It is clear that $w(H/K) \le w(H) \le \mathfrak{c}$, so we conclude that $w(G/N) = w(H/K) \le \mathfrak{c}$. Hence Lemma 4.3 implies that the space G/N is separable. Since the subgroup N of G is also separable, the separability of G follows from Remark 2.9. □

In the sequel we consider embeddings into separable topological groups. As one can expect, the situation improves notably when compared to embeddings into separable Hausdorff spaces.

Let us recall that a topological group G is called *feathered* if it contains a non-empty compact subset with a countable neighborhood base in G. Equivalently, G is feathered if it contains a compact subgroup K such that the quotient space G/K is metrizable (see [1, Section 4.3]). All metrizable groups and all locally compact groups are feathered. Notice that the class of feathered groups is countably productive according to [1, Proposition 4.3.13].

Theorem 4.13. *Let a feathered topological group G be a subgroup of a separable topological group. Then G is separable.*

Proof. Assume that G is a subgroup of a separable topological group X. By [1, Corollary 3.4.8], the group X is ω-narrow. Hence, according to [1, Theorem 3.4.4], the subgroup G of X is also ω-narrow. Applying [1, 4.3.A], we deduce that G is Lindelöf. Take a compact subgroup K of G such that the quotient space G/K is metrizable. Note that the space G/K is Lindelöf as a continuous image of the Lindelöf space G. Hence G/K is separable.

Finally, the compact group K is separable by Lemma 4.5. Hence the separability of G follows from Remark 2.9. □

Since all locally compact groups and all metrizable groups are feathered, the following two corollaries are immediate from Theorem 4.13; the second of them is well-known.

Corollary 4.14. *If a locally compact topological group G is a subgroup of a separable topological group, then G is separable.*

Corollary 4.15. *If a metrizable group G is a subgroup of a separable topological group, then G is separable.*

In fact, the conclusion of Corollary 4.15 remains valid if G is a subgroup of a topological group X with countable cellularity. Indeed, the group X is ω-narrow by [1, Theorem 3.4.7], and so is its subgroup G. Since G is first countable, it follows from [1, Proposition 3.4.5] that G has a countable base and hence is separable.

Unlike the case of almost connected pro-Lie groups, closed subgroups of separable prodiscrete abelian groups can fail to be separable.

Proposition 4.16. *Closed subgroups of separable prodiscrete abelian groups need not be separable.*

Proof. Let D be a discrete space of the cardinality $\aleph_1$. Denote by $L = D \cup \{x_0\}$ the space which contains D as a dense open subspace and in which the sets of the form $L \setminus C$, where C is an arbitrary countable subset of D, constitute a local base at x_0 in L. The space L is known as a *one-point Lindelöfication* of D.

Denote by H the free abelian topological group over L. Let us note that L is a Lindelöf P*-space*; i.e. every G_δ-set in L is open. Since all finite powers of L are Lindelöf, H is Lindelöf as well. According to [1, Proposition 7.4.7], H is also a P-space. Hence H is protodiscrete by [1, Lemma 4.4.1]. Notice that H cannot be separable as a non-discrete Hausdorff P-space. Every Lindelöf P-group is complete (see [19, Proposition 2.3]), so the group H is prodiscrete.

Clearly, the Lindelöf topological group H is ω-narrow. It is not difficult to verify that the topological character of H (the minimum cardinality of a local base at the identity of H) equals $\aleph_1$; this follows, for example, from the proof of [6, Lemma 5.1]. Hence $w(H) = \chi(H) = \aleph_1 \leq \mathfrak{c}$ (see [1, Lemma 5.1.5]).

We now apply Theorem 3.1 to conclude that H is topologically isomorphic to a subgroup of a separable prodiscrete abelian group G. Since H is complete, it is closed in G. Thus G contains a closed non-separable subgroup. □

Remark 4.17. 1) A discrete (hence locally compact and metrizable) group G homeomorphic to a closed subspace of a separable Tychonoff space is not necessarily separable. Indeed, it suffices to consider the Niemytzki plane which contains a discrete copy of the real numbers, the x-axis. Therefore, Theorem 4.13 and Corollaries 4.14 and 4.15 would not be valid if the group G were assumed to be a subspace of a separable Hausdorff (or even Tychonoff) space. Neither is Corollary 4.14 valid if G is assumed only to be a pro-Lie group. Indeed, Proposition 4.16 provides an example of a separable prodiscrete abelian group which contains a closed non-separable subgroup.

2) The separable connected pro-Lie group $G = \mathbb{R}^{\mathfrak{c}}$ contains a closed non-separable subgroup. To see this, we consider the closed subgroup $\mathbb{Z}^{\mathfrak{c}}$ of G. By a theorem of Uspenskij [23], the group $\mathbb{Z}^{\mathfrak{c}}$ contains a subgroup H of uncountable cellularity. The closure of H in G, say K, is a closed non-separable subgroup of G. By Proposition 4.8, the group K cannot be connected.

3) A natural question, after Proposition 4.8 and Corollary 4.15, is whether a connected metrizable group must be separable if it is a subspace of a separable Hausdorff (or regular) space. Again the answer is 'No'. Indeed, consider an arbitrary connected metrizable group G of weight $\mathfrak{c}$. For example, one can take $G = C(X)$, the Banach space of continuous real-valued functions on a compact space X satisfying $w(X) = \mathfrak{c}$, endowed with the sup-norm topology. Since $w(G) = \mathfrak{c}$, the space G is homeomorphic to a subspace of the Tychonoff cube $I^{\mathfrak{c}}$, where $I = [0, 1]$ is the closed unit interval. Thus G embeds as a subspace in a separable regular space, but both the density and weight of G are equal to $\mathfrak{c}$.

5. Embedding theorem

Theorem 5.2 and Corollary 5.3 given below show that there is a wealth of separable pseudocompact topological (abelian) groups with closed non-separable subgroups. In Proposition 5.5, assuming the Continuum Hypothesis, we present an example of a countably compact abelian group with a closed non-separable subgroup. It is not clear, however, whether a similar example can be constructed without extra set-theoretic assumptions.

The proof of Theorem 5.2 makes use of the following auxiliary lemma.

Lemma 5.1. *There exists a sequence $\{\varphi_m : m \in \omega\}$ of mappings of ω to ω satisfying the following condition: For every integer $k \geq 1$ and every vector triple $(\overline{m}, \overline{\jmath}_1, \overline{\jmath}_2)$, where $\overline{m} = (m_1, \ldots, m_k)$, $\overline{\jmath}_1 = (j_{1,1}, \ldots, j_{k,1})$, and $\overline{\jmath}_2 = (j_{1,2}, \ldots, j_{k,2})$ are elements of ω^k and $m_1, \ldots, m_k$ are pairwise distinct, there exists $n \in \omega$ such that $\varphi_{m_i}(n) = j_{i,1}$ and $\varphi_{m_i}(n+1) = j_{i,2}$ for each i with $1 \leq i \leq k$.*

Proof. Let us enumerate the family of all triples $(\overline{m}, \overline{\jmath}_1, \overline{\jmath}_2)$ as in the lemma in such a way that if a triple has number n and its first entry is $\overline{m} = (m_1, \ldots, m_k)$, then $m_1 \ldots, m_k$ are less than or equal to n. Assume that at a stage n of our inductive construction we have defined the values $\varphi_m(j)$ for all $m < n$ and $j \leq 2n$. We now consider the triple $(\overline{m}, \overline{\jmath}_1, \overline{\jmath}_2)$ which has number n in our enumeration and note that the coordinates $m_1, \ldots, m_k$ of $\overline{m}$ are less than or equal to n. According to the conclusion of the lemma, we have to put $\varphi_{m_i}(2n+1) = j_{i,1}$ and $\varphi_{m_i}(2n+2) = j_{i,2}$ for each i with $1 \leq i \leq k$. The rest of the values $\varphi_m(j)$ with $m \leq n$ and $j \leq 2n+2$ can be chosen arbitrarily.

Continuing this way, we obtain the sequence $\{\varphi_m : m \in \omega\}$ satisfying the condition of the lemma. □

Theorem 5.2. *Every precompact topological group of weight $\leq \mathfrak{c}$ is topologically isomorphic to a closed subgroup of a separable, connected, pseudocompact group H of weight $\leq \mathfrak{c}$.*

Proof. It is known that every second countable compact topological group is topologically isomorphic to a subgroup of the group $\mathbb{U} = \prod_{n \in \mathbb{N}} U(n)$, where $U(n)$ is the group of unitary $n \times n$ matrices with complex entries for each $n \in \mathbb{N}$. Hence every compact topological group is topologically isomorphic to a subgroup of some power of the group $\mathbb{U}$. In particular, if G is a precompact topological group and $w(G) \leq \mathfrak{c}$, then the Raĭkov completion of G, ϱG, is a compact topological group of weight $\leq \mathfrak{c}$, so ϱG and G are topologically isomorphic to subgroups of the compact group $\Pi = \mathbb{U}^{\mathfrak{c}}$. As $\mathbb{U}$ is connected, so is Π.

Let us identify G with a subgroup of Π. Since $\mathbb{U}$ is second countable, the group Π is separable by the Hewitt–Marczewski–Pondiczery theorem. Let $S = \{s_n : n \in \omega\}$

be a dense subset of Π, where each s_n is distinct from the identity element e_Π of Π. First we are going to define a special countable dense subset of $\Pi^{\mathfrak{c}}$ modifying the original construction of Hewitt–Marczewski–Pondiczery. To this end, we replace the index set $\mathfrak{c}$ with $\mathbb{R}$, so we will work with $\Pi^{\mathbb{R}}$ in place of $\Pi^{\mathfrak{c}}$.

Let $\mathcal{B}$ be the base for the usual topology on $\mathbb{R}$ which consists of the intervals (a, b) with rational endpoints a, b.

We are now in the position to define a special countable dense subset D of $\Pi^{\mathbb{R}}$. Let

$$\mathcal{A} = \{(U_1, \dots, U_k, s_{i_1}, \dots, s_{i_k}) : k \geq 1,\ U_1, \dots, U_k \text{ are pairwise disjoint elements of } \mathcal{B},\ \text{and } i_1, \dots, i_k \in \omega\}.$$

It is clear that $\mathcal{A}$ is countable. Let us enumerate this family as $\mathcal{A} = \{L_n : n \in \omega\}$. We also choose a pairwise disjoint sequence $\{V_m : m \in \omega\}$ of open unbounded subsets of $\mathbb{R}$. Let $\{\varphi_m : m \in \omega\}$ be a sequence of mappings of ω to ω satisfying the conclusion of Lemma 5.1. Given $n \in \omega$ and $L_n = (U_1, \dots, U_k, s_{i_1}, \dots, s_{i_k})$ in $\mathcal{A}$, we define an element $a_n \in \Pi^{\mathbb{R}}$ as follows:

$$a_n(r) = \begin{cases} s_{i_j} & \text{if } r \in U_j \text{ for some } j \leq k; \\ s_{\varphi_n(j)} & \text{if } r \in V_j \setminus \bigcup_{i \leq k} U_k \text{ for some } j \in \omega; \\ e_\Pi & \text{otherwise,} \end{cases}$$

where $r \in \mathbb{R}$. A standard argument shows that the countable set $D = \{a_n : n \in \omega\}$ is dense in $\Pi^{\mathfrak{c}}$.

Let also Σ be the Σ-product of continuum many copies of the group Π considered as a subgroup of $\Pi^{\mathbb{R}}$. Then Σ is a dense pseudocompact subgroup of $\Pi^{\mathbb{R}}$.

Denote by Δ the diagonal subgroup of $\Pi^{\mathbb{R}}$:

$$\Delta = \{x \in \Pi^{\mathbb{R}} : x(r) = x(s) \text{ for all } r, s \in \mathbb{R}\}.$$

It is clear that

$$G_0 = \{x \in \Delta : x(0) \in G\}$$

is an isomorphic topological copy of G.

Denote by E the subgroup of $\Pi^{\mathbb{R}}$ generated by the set $G_0 \cup D$. Finally we put $H = E \cdot \Sigma$. Then H is a subgroup of $\Pi^{\mathbb{R}}$ since Σ is an invariant subgroup of $\Pi^{\mathbb{R}}$. It is easy to see that H is a separable, dense, pseudocompact subgroup of $\Pi^{\mathbb{R}} \cong \mathbb{U}^{\mathbb{R} \times \mathfrak{c}}$. Since the group $\mathbb{U}$ is compact and metrizable, the subgroup H of the connected group $\Pi^{\mathbb{R}}$ is also connected according to [1, Theorem 2.4.15].

The main difficulty here is to verify the equality $H \cap \Delta = G_0$. Once this is done, we will immediately conclude that $G_0 \cong G$ is closed in H.

Assume that $d = w \cdot b \in \Delta$, where $w \in E$ and $b \in \Sigma$. We have to show that $d \in G_0$. Let $B = \operatorname{supp}(b)$. Then $w(x) = w(y)$ for all $x, y \in \mathbb{R} \setminus B$. Further, the element $w \in E$ has the form $w = g_1 d_1^{\epsilon_1} \cdots g_n d_n^{\epsilon_n} g_{n+1}$, where $g_1, \dots, g_n, g_{n+1} \in G_0$, $d_1, \dots, d_n \in D$, and $\epsilon_1, \dots, \epsilon_n \in \{1, -1\}$. Here some (or even all) of the elements $g_1, \dots, g_n, g_{n+1}$ can be equal to the identity element e of $\Pi^{\mathbb{R}}$, and some of d_i can coincide.

The word w has the form $w = g_1 d_1^{\epsilon_1} \cdots g_n d_n^{\epsilon_n} g_{n+1}$, where $n \geq 1$. Substituting d_i with variables z_j and assigning the same variable to d_{i_1} and d_{i_2} whenever $d_{i_1} = d_{i_2}$, we obtain the word $w[\overline{z}] = g_1 z_1^{\epsilon_1} \cdots g_n z_p^{\epsilon_n} g_{n+1}$, where $\overline{z} = (z_1, \dots, z_k)$, $k \leq n$, and $1 \leq p \leq k$. For example, $k = 1$ if and only if all the d_i are equal, and $k = n$ if all the d_i are pairwise distinct. Let

$$R = \{w[\overline{z}] : \overline{z} = (z_1, \dots, z_k) \in (\Pi^{\mathbb{R}})^k\}$$

be the range of $w[\cdot]$. We consider two cases.

Case 1. All values of $w(\overline{z})$ coincide, i.e. $|R| = 1$. Let us take $\overline{z}_0 = (e, \dots, e)$, i.e. $z_1 = \dots = z_k = e$. Then $w = w[\overline{z}_0] = g_1 \cdots g_n \cdot g_{n+1} \in G_0$. It now follows from $d = w \cdot b \in \Delta$ that $b = w^{-1} d \in \Delta \cap \Sigma$, and we conclude that $b = e$ and $d = w \in G_0$.

Case 2 ($|R| \geq 2$). Then we choose $\overline{z}_1 = (z_{1,1}, \dots, z_{k,1})$ and $\overline{z}_2 = (z_{1,2}, \dots, z_{k,2})$ in $(\Pi^{\mathbb{R}})^k$ such that $w[\overline{z}_1] \neq w[\overline{z}_2]$. We claim that there exist $x, y \in \mathbb{R} \setminus B$ such that $w(x) \neq w(y)$, which is a contradiction.

Indeed, take $r \in \mathbb{R}$ such that $w[\overline{z}_1](r) \neq w[\overline{z}_2](r)$. Hence

$$\begin{aligned} &g_1(r) z_{1,1}(r)^{\epsilon_1} \cdots z_{p,1}(r)^{\epsilon_n} g_{n+1}(r) \\ \neq\ &g_1(r) z_{1,2}(r)^{\epsilon_1} \cdots z_{p,2}(r)^{\epsilon_n} g_{n+1}(r). \end{aligned}$$

Since multiplication and inversion in Π are continuous, we can find an open symmetric neighborhood W of e_Π in Π such that the sets

$$O_\delta = g_1(r)\Big(z_{1,i}(r)^{\epsilon_1} W\Big) \cdots g_n(r)\Big(z_{n,i}(r)^{\epsilon_n} W\Big) g_{n+1}(r)$$

with $\delta = 1, 2$ are disjoint. Let us put $t_{i,\delta} = z_{i,\delta}(r)$ for all $i = 1, \dots, k$ and $\delta = 1, 2$. Since the set $S = \{s_j : j \in \omega\}$ is dense in Π, we can choose, for every $i \leq n$ and $\delta = 1, 2$, an element $s_{i,\delta} \in S \cap t_{i,\delta} W \cap W t_{i,\delta}$. Then $s_{i,\delta}^\epsilon \in t_{i,\delta}^\epsilon W$ for each $\epsilon = \pm 1$. Furthermore, according to our choice of the variables z_i, the elements $s_{i,\delta}$ can be chosen to satisfy $s_{i,\delta} = s_{l,\delta}$ whenever $d_i = d_l$, where $i, l \leq n$ and $\delta = 1, 2$.

It now follows from $O_1 \cap O_2 = \emptyset$ that the elements

$$h_1 = g_1(r) s_{1,1}^{\epsilon_1} \cdots g_i(r) s_{i,1}^{\epsilon_i} \cdots g_n(r) s_{n,1}^{\epsilon_n} g_{n+1}(r) \in O_1 \tag{5.1}$$

and

$$h_2 = g_1(r) s_{1,2}^{\epsilon_1} \cdots g_i(r) s_{i,2}^{\epsilon_i} \cdots g_n(r) s_{n,2}^{\epsilon_n} g_{n+1}(r) \in O_2 \tag{5.2}$$

are distinct. Let us recall that each d_i is in $D = \{a_m : m \in \omega\}$, so for every $i \leq n$ there exists $m \in \omega$ such that $d_i = a_m$. Since the set $\{d_i : 1 \leq i \leq n\}$ contains exactly k pairwise distinct elements, there are pairwise distinct non-negative integers $m_1, \dots, m_k$ such that $\{d_i : 1 \leq i \leq n\} = \{a_{m_1}, \dots, a_{m_k}\}$ and a_{m_j} corresponds to the variable z_j for $j = 1, \dots, k$. Similarly, choose integers $j_{i,\delta}$ for $1 \leq i \leq k$ and $\delta = 1, 2$ such that $\{s_{i,\delta} : 1 \leq i \leq n\} = \{s_{j_{1,\delta}}, \dots, s_{j_{k,\delta}}\}$ for each $\delta = 1, 2$, where both $s_{j_{i,1}}$ and $s_{j_{i,2}}$ correspond to the variable z_i, $1 \leq i \leq k$.

Consider the k-tuples $(m_1, \dots, m_k)$, $(j_{1,1}, \dots, j_{k,1})$, and $(j_{1,2}, \dots, j_{k,2})$.

By Lemma 5.1, there exists $n_0 \in \omega$ such that $\varphi_{m_i}(n_0) = j_{i,1}$ and $\varphi_{m_i}(n_0 + 1) = j_{i,2}$ for all $i = 1, \dots, k$. Let $L_{n_0} = (U_1, \dots, U_l, s_{i_1}, \dots, s_{i_l})$. Take $x \in V_p \setminus (B \cup \bigcup_{i \leq l} U_i)$ and $y \in V_{p+1} \setminus (B \cup \bigcup_{i \leq l} U_i)$, where $B = \operatorname{supp}(b)$. This choice of x and y is possible since the sets V_p and V_{p+1} are unbounded in $\mathbb{R}$. Then our definition of the elements a_m implies that $a_{m_i}(x) = s_{j_{i,1}}$ and $a_{m_i}(y) = s_{j_{i,2}}$ for each $i = 1, \dots, k$. Therefore the elements $w(x) = h_1 \in O_1$ and $w(y) = h_2 \in O_2$ of Π are distinct (see the equalities (5.1) and (5.2)). This contradiction completes the proof of the theorem. $\square$

Replacing the group $\mathbb{U}$ by the circle group $\mathbb{T}$ in the proof of Theorem 5.2 we obtain, after several simplifications in the corresponding argument, the abelian version of it:

Corollary 5.3. *Every precompact abelian group of weight $\leq \mathfrak{c}$ is topologically isomorphic to a closed subgroup of a separable, connected, pseudocompact abelian group H of weight $\leq \mathfrak{c}$.*

Our next aim is to present an example of a countably compact separable abelian group with a closed non-separable subgroup. Our argument makes use of an ω-*hereditarily finally dense* subgroup of $\mathbb{Z}(2)^{\omega_1}$ constructed in [8] by A. Hajnal and I. Juhász under the assumption of the Continuum Hypothesis. Hence our example here also requires CH.

The following lemma is almost evident.

Lemma 5.4. *Let K and L be subgroups of a topological abelian group G. If K is countably compact and L is ω-bounded, then $K+L$ is a countably compact subgroup of G.*

Proof. It is known that the product of a countably compact space and an ω-bounded space is countably compact — this follows, for example, from [24, Theorem 3.3]. Since $K + L$ is a continuous image of the countably compact space $K \times L$, the required conclusion is immediate. $\square$

Proposition 5.5. *Under CH, there exists a separable countably compact topological abelian group G which contains a closed non-separable subgroup.*

Proof. Let $P = \mathbb{Z}(2)^{\omega_1}$ and $\Pi = P^{\omega_1}$, where both groups carry the usual Tychonoff product topology, so P and Π are compact topological abelian groups. Since $\Pi = (\mathbb{Z}(2)^{\omega_1})^{\omega_1} \cong \mathbb{Z}(2)^{\omega_1 \times \omega_1} \cong \mathbb{Z}(2)^{\omega_1}$, it follows from [8, Theorem 2.2] that under CH, Π contains a dense countably compact ω-*HFD* subgroup H of the cardinality $\mathfrak{c}$. The latter means that for every infinite subset S of H, there exists a countable set $C \subset \omega_1$ such that $\pi_{\omega_1 \setminus C}(S)$ is dense in $P^{\omega_1 \setminus C}$, where π_J denotes the projection of P^{ω_1} onto P^J for each non-empty set $J \subset \omega_1$. (More precisely, the fact that H is ω-HFD appears on page 202 in the proof of Theorem 2.2 in [8].) Further, according to [8, Theorem 2.2], the group H is hereditarily separable; i.e. every subspace of H is separable. In particular, H is separable.

Denote by Δ the diagonal subgroup of Π; i.e. let

$$\Delta = \{x \in \Pi : \pi_\alpha(x) = \pi_\beta(x) \text{ for all } \alpha, \beta \in \omega_1\},$$

where $\pi_\alpha \colon \Pi \to P_{(\alpha)}$ is the projection of Π to the αth factor. Then Δ is a closed subgroup of Π topologically isomorphic to P — the projection π_0 of Δ to $P_{(0)}$ is a topological isomorphism.

Let Σ_P be the Σ-product of ω_1 copies of the group $\mathbb{Z}(2)$ which is identified with the corresponding subgroup of $P = \mathbb{Z}(2)^{\omega_1}$. Then Σ_P is a proper, dense, countably compact subgroup of P. In fact, Σ_P is ω-*bounded*; i.e. the closure in Σ_P of every countable subset of Σ_P is compact [1, Corollary 1.6.34]. Further, since Σ_P is not compact, it cannot be separable.

Let $G = H + \Sigma_0$, where

$$\Sigma_0 = \{x \in \Delta : \pi_0(x) \in \Sigma_P\}$$

is the 'diagonal' copy of the group Σ_P. Again, the groups Σ_0 and Σ_P are topologically isomorphic. Let us note that by Lemma 5.4, G is a countably compact subgroup of Π. It is clear that G is separable since it contains a dense separable subgroup H.

We claim that the intersection $K = G \cap \Delta$ satisfies $|K : \Sigma_0| < \omega$. It is clear that $\Sigma_0 \subset K$, so it suffices to verify that $|H \cap \Delta| < \omega$. Indeed, since the projection of Δ to an arbitrary subproduct P^J, with $|J| \geq \omega$, is nowhere dense in P^J, we see that $H \cap \Delta$ is not finally dense in Π. Hence $H \cap \Delta$ is finite. We have thus proved that $|K : \Sigma_0| < \omega$.

It is clear that K is a closed subgroup of G. It remains to show that the group K is not separable. Since $|K : \Sigma_0| < \omega$ and $|\Delta : \Sigma_0| > \omega$, we conclude that K is a proper dense subgroup of Δ. Further, there exists a finite subset F of K such that $K = \Sigma_0 + F$. Since Σ_0 is ω-bounded, so is K. Thus, if K were separable it would be compact, contradicting the fact that K is a proper dense subgroup of Δ. □

Since we have used CH in Proposition 5.5, it is natural to ask whether a similar construction is possible in ZFC alone:

Question 5.6. Does there exist in ZFC a countably compact separable topological group which contains a non-separable closed subgroup?

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