

COMPLETE BOUNDEDNESS OF HEAT SEMIGROUPS ON THE VON NEUMANN ALGEBRA OF HYPERBOLIC GROUPS

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ABSTRACT. We prove that $\lambda_g \mapsto e^{-t|g|^r} \lambda_g$ defines a multiplier on the von Neumann algebra of hyperbolic groups with a complete bound $\simeq r$, for any $0 < t < \infty, 1 < r < \infty$. In the proof we observe that a construction of Ozawa allows us to characterize the radial multipliers that are bounded on every hyperbolic graph, partially generalizing results of Haagerup–Steenstrup–Szwarc and Wysoczański. Our argument is also based on the work of Peller.

1. INTRODUCTION

Let $\Delta = \partial^2\theta$ be the Laplacian operator on the unit circle \mathbf{T} . The maps

$$T_t^r = e^{-t(-\Delta)^{\frac{r}{2}}} : e^{i2\pi k\theta} \rightarrow e^{-4\pi^2 t|k|^r} e^{i2\pi k\theta}$$

define a semigroup of uniformly bounded operators on $L^\infty(\mathbf{T})$ for any $0 < r < \infty$. Moreover, for $0 < r \leq 2$, T_t^r are positivity preserving contractions, which can be easily seen by their integral representations. For $r = 1, 2$, the semigroups T_t^r are called Poisson semigroup and heat semigroup. They both play important roles, and very often complementary roles, in harmonic analysis. It is easier to work with heat semigroups, for some problems, because of the general Gaussian upper estimate of the heat kernels.

Let \mathbb{F}_n , $2 \leq n \leq \infty$ be the free group on n generators. Let λ_g be the left regular representations of $g \in \mathbb{F}_n$. One may consider the analogue of the classical Poisson or heat semigroups on free group von Neumann algebra $\mathcal{L}(\mathbb{F}_n)$,

$$S_t^r : \lambda_g \rightarrow e^{-t|g|^r} \lambda_g,$$

with $|g|$ the reduced word length of g . U. Haagerup proved (see [9]) that for $r = 1$, $(S_t^1)_{t \geq 0}$ is a semigroup of completely positive operators on $\mathcal{L}(\mathbb{F}_n)$. For $0 < r \leq 1$, the maps S_t^r are therefore still unital completely positive (u.c.p.) by the theory of subordinated semigroups (see e.g. [26]). For $r > 1$, S_t^r cannot be completely positive for all t , as the function $\phi_t(g) = e^{-t|g|^r}$ is in $\ell_1(\mathbb{F}_2)$ for each t , and the positive definiteness of ϕ_t would imply the amenability of the free group \mathbb{F}_2 (see the proof of [3, Theorem 2.6.8]).

This discussion settles the question of the complete positivity of the semigroup S_t^r . How about the complete boundedness (c.b.) of S_t^r ? For $r \leq 1$, S_t^r has of

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course completely bounded norm 1, since u.c.p. implies c.b. with norm 1 on C^* algebras. For $r > 1$, it is not hard to see that S_t^r is bounded with an upper bound $1 + ct^{-\frac{2}{r}}$ for each $t > 0$ because of the rapid decay property of the free group. The question is then whether S_t^r is (completely) bounded uniformly in t for any (all) $r > 1$. By duality, the principle of uniform boundness and the pointwise convergence of S_t^r , asking whether $\sup_t \|S_t^r\| < \infty$ is equivalent to asking whether S_t^r , as an operator on the predual $L_1(\widehat{\mathbb{F}}_n)$ of $\mathcal{L}(\mathbb{F}_n)$, converges to the identity in the strong (or weak) operator topology when $t \rightarrow 0$. Similarly, $\sup_t \|S_t^r\|_{cb} < \infty$ if and only if S_t^r converges to the identity in the stable point-norm (or stable point-weak) topology.

A characterization of the completely bounded radial multipliers on free groups was given by Haagerup and Szwarz in terms of some Hankel matrix being of trace class (this was published in [11]; see also [24]). On the other hand a famous result of V. V. Peller ([20]) states that a Hankel matrix belongs to the trace class iff the symbol function belongs to the Besov space B_1^1 (see Section 3). These results together provide a precise method for estimating the complete bounds of radial maps on free groups. However, the answer to the uniform complete boundedness of S_t^r (for $r > 1$) remained open, to the best knowledge of the authors, before the writing of this article. Besides the possible technicality of the corresponding estimation, another reason may be the general belief of a negative answer to the question. For example Knudby proved in [16, Theorem 1.6] that the symbol of a completely contractive semigroup of radial multipliers on \mathbb{F}_n ($n \geq 2$) grows linearly. This implies that for each $r > 1$ there does not exist a constant C such that $\|S_t^r\|_{cb} \leq e^{tC}$ for all $t > 0$.

In the present work we record a proof that S_t^r is a weak $*$ continuous analytic semigroup of completely bounded maps on the free group von Neumann algebras for any $r > 0$, by a careful estimation of the corresponding Besov norm. We remark that by [6, 10, 18, 24], this also proves that the analogous semigroups on free products of groups, on free products of operator algebras, and on amalgamated free products of finite von Neumann algebras, are completely bounded with bound independent on t . We do not elaborate on this and refer to [6, 10, 18, 24] for the definitions of radial multipliers on free products of groups and (amalgamated) free products of operator algebras, and for details.

The authors hope that the study of S_t^r would benefit the recent research in harmonic analysis on noncommutative L_p spaces (see [13–15]), as the classical case suggests.

The main ingredient that we introduce is a result on the trace class norm of Hankel matrices with smooth symbol. A particular case of our result is the following (see Theorem 3.1 for a more precise statement).

Theorem 1.1. *Let $f: [0, \infty) \rightarrow \mathbf{R}$ be a bounded continuous function of class C^2 on $(0, \infty)$, and $\frac{1}{2} \geq \alpha > 0$. Then the trace class norm of the matrix $(f(j+k) - f(j+k+1))_{j,k \geq 0}$ is less than $\frac{C}{\sqrt{\alpha}} \sqrt{\|x^{\frac{3}{2}-\alpha} f''\|_{L^2(\mathbf{R}_+)} \|x^{\frac{3}{2}+\alpha} f''\|_{L^2(\mathbf{R}_+)}}$ for some universal constant C .*

After we communicated to him the aforementioned result, Narutaka Ozawa asked us whether the same holds for other length functions on \mathbb{F}_n , or more generally for an arbitrary finitely generated hyperbolic group. To our surprise, the answer is yes.

Indeed we can extend to hyperbolic graphs the sufficient condition from [11, 24] for a radial multiplier to be bounded. This is a consequence of [19].

Theorem 1.2. *Let Γ be a hyperbolic graph with bounded degree. Then there is a constant C such that for every $\dot{\phi}: \mathbf{N} \rightarrow \mathbf{C}$, if the matrix*

$$H = (\dot{\phi}(j+k) - \dot{\phi}(j+k+1))_{j,k \geq 0}$$

belongs to the trace class S^1 , then $\lim_n \dot{\phi}(n)$ exists and the map

$$A = (a_{x,y}) \in B(\ell^2(\Gamma)) \mapsto (\dot{\phi}(d(x,y))a_{x,y}) \in B(\ell^2(\Gamma))$$

is bounded with norm less than $C\|H\|_{S^1} + \lim_n |\dot{\phi}(n)|$.

In view of this theorem it is natural to wonder whether there are nonhyperbolic groups also satisfying the same criterion. It turns out that there are, namely the groups \mathbf{Z}^d ($d \geq 1$) for their standard generating set. This follows from some delicate estimates of $L^1(\mathbf{R}^d/\mathbf{Z}^d)$ -norms that we prove in Section 4.

The main result of this paper can be summarized as

Theorem 1.3. *Let Γ be a finitely generated hyperbolic group, and $|\cdot|$ the word length associated to a fixed finite generating set. Then there is a constant C such that for every $r > 0$, and every $z \in \mathbf{C}$ with positive real part, the multiplier $S_z^r: \lambda_g \mapsto e^{-z|g|^r} \lambda_g$ is completely bounded on $\mathcal{L}(\Gamma)$ with norm less than*

$$C(1 + |\tan(\arg z)|)^{3/2}(1 + r).$$

The dependence on the constant on the argument of z is probably not optimal, but the order r is sharp as $r \rightarrow \infty$ (see Example 3.5).

Our method applies to other radial multipliers. For example in Example 3.4 we show that the Bochner-Riesz multipliers $\lambda_g \mapsto (1 - \frac{|g|^2}{N^2})^z \chi_{\{|g| \leq N\}} \lambda_g$ are completely bounded on the noncommutative L^p spaces associated with $\mathcal{L}(\Gamma)$ uniformly in N if $|\frac{2}{p} - 1| < \text{Re}(z)$, for any $1 \leq p \leq \infty$ and any finitely generated hyperbolic group Γ .

This article is organized as follows. In Section 2 we recall facts on Schur multipliers and prove Theorem 1.2. Section 3 contains estimates for the Schatten 1-norm of Hankel matrices and the proof of Theorem 1.3. Section 4 contains the proof that \mathbf{Z}^d satisfies the conclusion of Theorem 1.2. In Section 5, we explain a motivation in studying S_t^r and prove an endpoint result on H_Σ^∞ -calculus.

Notation. We denote by \mathbf{N} the set of nonnegative integers including 0 : $\mathbf{N} = \{0, 1, 2, \dots\}$. We denote by S^1 the Banach space of trace class operators on $\ell^2(\mathbf{N})$. For a discrete group Γ we denote by $L^p(\hat{\Gamma})$ the noncommutative L^p space on the von Neumann algebra of Γ .

2. RADIAL MULTIPLIERS ON HYPERBOLIC GRAPHS

2.1. Reminders on Schur and Fourier multipliers. We start with some reminders. If X is a set, a function $\varphi: X \times X \rightarrow \mathbf{C}$ is a *Schur multiplier* if the map $(a_{s,t})_{s,t \in X} \in B(\ell^2(X)) \mapsto (\varphi(s,t)a_{s,t})_{s,t \in X}$, denoted M_φ , is bounded on $B(\ell^2(X))$. The following proposition, which is essentially due to Grothendieck ([22, Theorem 5.1]), characterizes the Schur multipliers.

Proposition 2.1. *Let X be a nonempty set and assume that $\varphi: X \times X \mapsto \mathbf{C}$ and $C \geq 0$ are given. Then the following are equivalent:*

- (i) $\varphi(x,y)$ extends to a Schur multiplier on $B(\ell^2(X))$ with norm $\leq C$.

(ii) *There exists a Hilbert space H and two bounded maps $P, Q : \Gamma \mapsto H$ such that*

$$\varphi(x, y) = \langle P(x), Q(y) \rangle$$

and $\|P\|_\infty \|Q\|_\infty \leq C$, where

$$\|P\|_\infty = \sup_{x \in X} \|P(x)\|, \|Q\|_\infty = \sup_{x \in X} \|Q(x)\|.$$

Let Γ be a discrete group and $\phi : \Gamma \rightarrow \mathbf{C}$. The *Fourier multiplier* $\lambda_g \mapsto \phi(g)\lambda_g$ is completely bounded on the group von Neumann algebra $\mathcal{L}(\Gamma)$ if and only if the associated function $\varphi(g, h) = \phi(h^{-1}g)$ is a Schur multiplier. In this case the c.b. norm of the Fourier multiplier is equal to the norm (and also the c.b. norm) of the Schur multiplier (see [2]).

Let us state an immediate consequence of Proposition 2.1 that will be used later (see [22, Theorem 6.1] for a complete characterization of Hankelian Schur multipliers).

Lemma 2.2. *Let $(a_n)_{n \in \mathbf{Z}}$ be a finitely supported sequence of complex numbers. Then for each matrix $B = (b_{j,k})_{j,k \in \mathbf{N}} \in S^1$,*

$$\|(a_{j+k}b_{j,k})_{j,k \in \mathbf{N}}\|_{S^1} \leq \left(\int_0^1 \left| \sum_{n \in \mathbf{Z}} a_n e^{2i\pi n\theta} \right| d\theta \right) \|B\|_{S^1}.$$

Proof. Write $f(\theta) = \sum_{n \in \mathbf{Z}} a_n e^{2i\pi n\theta}$ and decompose $f = gh$ a product of two L^2 functions with $\|g\|_{L^2} \|h\|_{L^2} = \|f\|_{L^1}$. We can write $a_{j+k} = \int f(\theta) e^{-2i\pi(j+k)\theta} = \int g(\theta) e^{-2i\pi j\theta} h(\theta) e^{-2i\pi k\theta} = \langle g e^{-2i\pi j\theta}, \bar{h} e^{2i\pi k\theta} \rangle$. Proposition 2.1 implies that $(j, k) \mapsto a_{j+k}$ is a Schur multiplier with norm less than $\|g\|_{L^2} \|h\|_{L^2} = \|f\|_{L^1}$ on $B(\ell^2)$. By duality $M_{a_{j+k}}$ is bounded on S^1 with norm $\leq \|f\|_{L^1}$, which is the content of the lemma. \square

2.2. Radial multipliers on trees. Let Γ be a group with a fixed finite generating set, and denote by d the associated left-invariant distance on Γ , or equivalently the distance on the Cayley graph of Γ . Recall that $d(x, y) = |x^{-1}y|$ where $|\cdot|$ is the word length with respect to the fixed generating set of Γ .

Following [11], we say a Schur multiplier φ on $\Gamma \times \Gamma$ is *radial* if $\varphi(x, y) = \dot{\phi}(d(x, y))$ for some function $\dot{\phi} : \mathbf{N} \mapsto \mathbf{C}$. We say a Fourier multiplier $\lambda_x \mapsto \phi(x)\lambda_x$ is a *radial Fourier multiplier* if $\phi(x) = \dot{\phi}(|x|)$ for some $\dot{\phi} : \mathbf{N} \rightarrow \mathbf{C}$. By [2], the completely bounded norm of a radial Fourier multiplier associated with ϕ equals the norm of the Schur multiplier $\dot{\phi}(d(x, y))$.

An exact formula for the norm of radial Schur multipliers on free groups was given by Haagerup and Szwarc in 1987. The result was published in [11] with Steenstrup, where they extended the study to homogenous trees. A similar characterization for radial Fourier multipliers with respect to the block length on free products of groups was proved by Wysocański in 1995 ([24]). These results were recently extended to (amalgamated) free products of operator algebras in [6, 10, 18]. We rewrite a special version of the results from [11] below.

Theorem 2.3 (Haagerup, Steenstrup, Szwarc). *Let X be a homogenous tree with degree ≥ 3 and $\dot{\phi} : \mathbf{N} \rightarrow \mathbf{C}$. Then the function $\varphi(x, y) = \dot{\phi}(d(x, y))$ is a Schur multiplier if and only if the matrix*

$$H = (\dot{\phi}(j+k) - \dot{\phi}(j+k+2))_{j,k \geq 0}$$

belongs to the trace class. In that case $\lim_n \dot{\phi}(2n)$ and $\lim_n \dot{\phi}(2n + 1)$ exist, and

$$\|M_{\dot{\phi}}\| \leq \lim_{n \rightarrow \infty} (|\dot{\phi}(2n)| + |\dot{\phi}(2n + 1)|) + \|H\|_{S^1}.$$

In this section, we remark that a similar result to the “if” part of Theorem 2.3 holds on hyperbolic graphs. We first recall a proof of the “if” part of Theorem 2.3, valid for all trees, and that we will later adapt for general hyperbolic graphs. Fix an infinite geodesic path p . For every $x \in X$, there is a unique infinite geodesic path p_x which starts at x and flows into p . Remark that for every $x, y \in X$ the geodesics p_x and p_y first meet on a point of the geodesic segment between x and y , and then coincide forever. In formulas, the set $\{(i, j) \in \mathbf{N} \times \mathbf{N}, p_x(i) = p_x(j)\}$ is of the form $\{(i_0 + k, j_0 + k), k \in \mathbf{N}\}$, for i_0, j_0 satisfying $i_0 + j_0 = d(x, y)$.

Since $H = (\dot{\phi}(i + j) - \dot{\phi}(i + j + 2))_{0 \leq i, j < \infty}$ belongs to the trace class, we can write $H = A^*B$ where A, B are Hilbert-Schmidt operators satisfying $\|H\|_{S^1} = \|A\|_{S_2}\|B\|_{S_2}$. For $x, y \in X$, set

$$P(x) = \sum_{i \geq 0} B(e_i) \otimes \delta_{p_x(i)} \in \ell^2(\mathbf{N}) \otimes \ell^2(X)$$

and

$$Q(y) = \sum_{j \geq 0} A(e_j) \otimes \delta_{p_y(j)} \in \ell^2(\mathbf{N}) \otimes \ell^2(X)$$

where $(e_i)_{i \geq 0}$ and δ_x are the coordinate orthonormal basis of $\ell^2(\mathbf{N})$ and $\ell^2(X)$. We see that $\|P(x)\|^2 = \sum_{i \geq 0} \|B(e_i)\|^2 = \|B\|_{S_2}^2$ and $\|Q(y)\|^2 = \|A\|_{S_2}^2$. Using that $\langle Be_i, Ae_j \rangle = \dot{\phi}(i + j) - \dot{\phi}(i + j + 2)$ we can write

$$\begin{aligned} \langle P(x), Q(y) \rangle &= \sum_{i, j, p_x(i) = p_y(j)} \dot{\phi}(i + j) - \dot{\phi}(i + j + 2) \\ &= \sum_{k=0}^{\infty} \dot{\phi}(d(x, y) + 2k) - \dot{\phi}(d(x, y) + 2k + 2) \end{aligned}$$

which equals $\dot{\phi}(d(x, y)) - \lim_{n \rightarrow \infty} \dot{\phi}(2n)$ for $d(x, y)$ even and equals $\dot{\phi}(d(x, y)) - \lim_{n \rightarrow \infty} \dot{\phi}(2n + 1)$ for $d(x, y)$ odd. Therefore,

$$\dot{\phi}(d(x, y)) = \langle P(x), Q(y) \rangle + \frac{1 + (-1)^{d(x, y)}}{2} \lim_n \dot{\phi}(2n) + \frac{1 - (-1)^{d(x, y)}}{2} \lim_n \dot{\phi}(2n + 1).$$

Fix a distinguished point $e \in X$. Note $(-1)^{d(x, y)} = (-1)^{d(x, e)}(-1)^{d(y, e)}$ since $d(x, e) + d(y, e) - d(x, y)$ is even. So $1 \pm (-1)^{d(x, y)}$ is a Schur multiplier with norm ≤ 2 . We then obtain by Proposition 2.1 that

$$\|M_{\dot{\phi}}\| \leq \lim_{n \rightarrow \infty} (|\dot{\phi}(2n)| + |\dot{\phi}(2n + 1)|) + \|H\|_{S^1}.$$

2.3. Generalization to hyperbolic graphs. Identify a connected graph Γ with its vertex set and equip it with the graph distance d . A geodesic path p is a finite or infinite sequence of points $p(0), \dots, p(k) \cdots \in \Gamma$ such that $d(p(m), p(n)) = |m - n|$ for every m, n . A connected graph Γ is *hyperbolic* if there exists a constant $\delta \geq 0$ such that for every geodesic triangle each edge is contained in the δ -neighborhood of the union of the other two. A finitely generated group Γ is *hyperbolic* if its Cayley graph is hyperbolic. The hyperbolicity property of a group is independent of the choice of generating set. A tree is a hyperbolic graph for $\delta = 0$. See e.g. [3], section 5.3 for more information on hyperbolic groups.

To obtain an extension on general hyperbolic graphs, we need the following result of Ozawa [19, Proposition 10].

Proposition 2.4 (Ozawa). *Let Γ be a hyperbolic graph with bounded degree. There is $C_0 \in \mathbf{R}$, a Hilbert space \mathcal{H} and maps $\eta_i^\pm: \Gamma \rightarrow \mathcal{H}$ (for $i \in \mathbf{N}$) such that:*

- (1) $\eta_i^\pm(x) \perp \eta_j^\pm(x)$ for all $x \in \Gamma$ and $|i - j| \geq 2$.
- (2) $\|\eta_i^\pm(x)\| \leq \sqrt{C_0}$ for all $i \in \mathbf{N}$ and $x \in \Gamma$.
- (3) $\sum_{i+j=n} \langle \eta_i^+(x), \eta_j^-(y) \rangle = \begin{cases} 1 & \text{if } d(x, y) \leq n, \\ 0 & \text{otherwise.} \end{cases}$

The $\eta_i^\pm(x)$ provided by this proposition will play the role of the vectors $\delta_{p_x(i)} \in \ell^2(X)$ in the preceding proof.

Assume $H = (\dot{\phi}(i + j) - \dot{\phi}(i + j + 1))_{0 \leq i, j < \infty} \in S^1$. Then the diagonal of H belongs to ℓ^1 , and hence $\lim_n \dot{\phi}(n)$ exists. We proceed similarly. Write $H = A^*B$ with $\|H\|_{S^1} = \|A\|_{S_2} \|B\|_{S_2}$. Set

$$P(x) = \sum_{i \geq 0} B(e_i) \otimes \eta_i^+(x) \in \ell^2 \otimes \mathcal{H}$$

and

$$Q(y) = \sum_{j \geq 0} A(e_j) \otimes \eta_j^-(x) \in \ell^2 \otimes \mathcal{H},$$

for $x, y \in \Gamma$. From conditions (1) and (2) in Proposition 2.4 we see that if $|i - j| \geq 2$, $|\langle B(e_i) \otimes \eta_i^+(x), B(e_j) \otimes \eta_j^+(x) \rangle|$ is zero and always less than $C_0 \frac{\|B(e_i)\|^2 + \|B(e_j)\|^2}{2}$. Hence

$$\|P(x)\|^2 = \sum_{i, j \geq 0} \langle B(e_i) \otimes \eta_i^+(x), B(e_j) \otimes \eta_j^+(x) \rangle \leq 3C_0 \|B\|_{S_2}^2.$$

Similarly, $\sup_y \|Q(y)\|^3 \leq 3C_0 \|A\|_{S_2}^2$. We claim that $\langle P(x), Q(y) \rangle = \dot{\phi}(d(x, y))$ for all $x, y \in \Gamma$. Indeed,

$$\langle P(x), Q(y) \rangle = \sum_{i, j} \langle B e_i \otimes \eta_i^+(x), A e_j \otimes \eta_j^-(y) \rangle.$$

Using that $\langle B e_i, A e_j \rangle = \dot{\phi}(i + j) - \dot{\phi}(i + j - 1)$ we can write

$$\langle P(x), Q(y) \rangle = \sum_{n \geq 0} \left((\dot{\phi}(n) - \dot{\phi}(n + 1)) \sum_{i+j=n} \langle \eta_i^+(x), \eta_j^-(y) \rangle \right),$$

which equals $\dot{\phi}(d(x, y)) - \lim_n \dot{\phi}(n)$ from assumption 3 in Proposition 2.4. By Propostion 2.1, we obtain Theorem 1.2.

Remark 2.5. In Theorem 1.2, we can take $C = 1$ if Γ is a tree. Otherwise, C may depend on Γ .

Remark 2.6. The condition that $(\dot{\phi}(j + k) - \dot{\phi}(j + k + 1))_{j, k \geq 0}$ is of trace class is stronger than the condition in Theorem 2.3 for $(\dot{\phi}(j + k) - \dot{\phi}(j + k + 2))_{j, k \geq 0}$, but is necessary for general hyperbolic graphs. Indeed, if Γ_0 is the Cayley graph of $(\mathbf{Z}/3\mathbf{Z}) * (\mathbf{Z}/3\mathbf{Z}) * (\mathbf{Z}/3\mathbf{Z})$ with generating set the union of the 3 copies of $\mathbf{Z}/3\mathbf{Z}$, then by [24, Theorem 6.1] the Schur multiplier with symbol $\phi(d(x, y))$ is bounded on $B(\ell^2(\Gamma_0))$ if and only if $(\dot{\phi}(j + k) - \dot{\phi}(j + k + 1))_{j, k \geq 0}$ belongs to the trace class. Theorem 1.2 can therefore be read as “a function $\phi: \mathbf{N} \rightarrow \mathbf{C}$ defines a bounded

radial multiplier on every hyperbolic graph Γ with bounded degree if and only if it defines a bounded radial multiplier on Γ_0 ”.

Remark 2.7. The fact that it is the matrix $(\dot{\phi}(j+k) - \dot{\phi}(j+k+2))_{j,k \geq 0}$ that appears in Theorem 2.3 is related to the fact that trees are bipartite (a graph is bipartite if it does not contain any odd length cycle). A modification (left to the reader) of the proofs in [19] actually shows that when the hyperbolic graph Γ is bipartite, then Theorem 1.2 holds also for H replaced by $(\dot{\phi}(j+k) - \dot{\phi}(j+k+2))_{j,k \geq 0}$. In that case the statement becomes “a function $\phi: \mathbf{N} \rightarrow \mathbf{C}$ defines a bounded radial multiplier on every bipartite hyperbolic graph Γ if and only if it defines a bounded radial multiplier on \mathbb{F}_2 with standard generating set”.

In particular, if we apply the preceding Theorem 1.2 to a Cayley graph of a finitely generated hyperbolic group Γ and recall that the word length of $y^{-1}x$ equals $d(y^{-1}x, e) = d(x, y)$ on its Cayley graph, we get

Corollary 2.8. *Let Γ be a finitely generated hyperbolic group, and $|\cdot|$ the length function on Γ associated to a finite generating set of Γ . Then there is a constant $C \in \mathbf{R}$ such that if $\phi: \mathbf{N} \rightarrow \mathbf{C}$ is a function such that the infinite Hankel matrix*

$$H = (\dot{\phi}(k+j) - \dot{\phi}(k+j+1))_{0 \leq k, j < \infty}$$

belongs to the trace class S^1 , then $\lambda_g \mapsto \dot{\phi}(|g|)\lambda_g$ extends to a completely bounded map on $\mathcal{L}(\Gamma)$ with norm $\leq C\|H\|_{S^1} + \lim_{n \rightarrow \infty} |\dot{\phi}(n)|$.

2.4. Weighted length functions on \mathbb{F}_n . Corollary 2.8 in particular applies to \mathbb{F}_n equipped with the length function associated to other finite generating sets than the standard generating set.

One may also consider weighted lengths on a free group \mathbb{F}_n , $n \in \mathbf{N} \cup \{\infty\}$. Denote the free generators by $g_1, g_2, \dots, g_k, \dots$. Fix a sequence of positive real numbers $a = (a_k)_k$, for $g = g_{i_1}^{k_1} g_{i_2}^{k_2} \dots g_{i_m}^{k_m} \in \mathbb{F}_n$, $i_j \neq i_{j+1}$, let

$$|g|_a = \sum_{j=1}^m a_{i_j} |k_j|.$$

Theorem 1.3 also extends to the weighted lengths $|g|_a$ on free products groups \mathbb{F}_n , $2 \leq n \leq \infty$, because of the following lemma.

For $\phi: \mathbf{R}_+ \mapsto \mathbf{C}$, we denote by $\phi_t(\cdot) = \phi(t \cdot)$ and m_{ϕ^a} the Fourier multiplier sending λ_g to $\phi(|g|_a)\lambda_g$. We ignore the “ a ” when $a_k = 1$ for all k .

Proposition 2.9. *Given a continuous function ϕ , suppose $\|m_{\phi_t}\|_{c.b} < C$ on \mathbb{F}_∞ for all $0 < t < 1$. Then $\|m_{\phi_t^a}\|_{c.b} < C$ for the same constant C for any sequence a and any $t > 0$.*

Proof. Assume first that $a_k \in \mathbf{N}^* = \{1, 2, \dots\}$. Let T_a be the trace preserving *-homomorphism sending λ_{g_j} to $\lambda_{g_j^{a_j}}$. Then

$$T_a \circ m_{\phi_t^a} = m_{\phi_t} \circ T_a,$$

which shows that

$$\|m_{\phi_t^a}\|_{c.b} = \|T_a \circ m_{\phi_t^a}\|_{cb} = \|m_{\phi_t} \circ T_a\|_{cb} \leq \|m_{\phi_t}\|_{c.b}$$

since T_a is completely isometric.

Next we assume $a_k \in \mathbf{Q}_+$ with a common denominator N . Then

$$T_{Na} \circ m_{\phi_t^a} = m_{\phi_{\frac{t}{N}}} T_{Na}.$$

Therefore, $m_{\phi_t^a}$ is completely bounded with upper bound $\sup_{0 < t < 1} \|m_{\phi_t}\|_{c.b.}$. The general case follows by approximation. \square

3. COMPLETE BOUNDEDNESS OF S_t^r

Theorem 2.3 (respectively Corollary 2.8) states that the completely bounded norm of the map S_t^r on $\mathcal{L}(\mathbb{F}_n)$ (respectively $\mathcal{L}(\Gamma)$ for a hyperbolic group Γ) is equivalent to (respectively dominated by) the trace class norm of the corresponding Hankel matrix.

In this section we give an upper bound on the trace class norm of Hankel matrices with smooth symbol, that we apply to several explicit examples. In particular Theorem 1.3 is an immediate consequence of Corollary 2.8 and Examples 3.6 and 3.5.

We start by stating a more precise version of Theorem 1.1.

Theorem 3.1. *Let $f: [0, \infty) \rightarrow \mathbf{R}$ be a bounded continuous function of class C^2 on $(0, \infty)$, and $\frac{1}{2} \geq \alpha > 0$. Then, for any $t > 0$, the trace class norm of the matrix $f(t(j+k)) - f(t(j+k+1))$ satisfies the inequality*

$$(1) \quad \left\| (f(t(j+k)) - f(t(j+k+1)))_{j,k \geq 0} \right\|_{S_1} \leq \frac{C}{\sqrt{\alpha}} \sqrt{\tilde{A}\tilde{B}} \leq \frac{2C}{\sqrt{\alpha}} \tilde{B}$$

for some universal constant C , where

$$\begin{aligned} \tilde{A} &= \sqrt{\|x^{\frac{1}{2}-\alpha} f'\|_{L^2(\mathbf{R}_+)} \|x^{\frac{1}{2}+\alpha} f'\|_{L^2(\mathbf{R}_+)}, \\ \tilde{B} &= \sqrt{\|x^{\frac{3}{2}-\alpha} f''\|_{L^2(\mathbf{R}_+)} \|x^{\frac{3}{2}+\alpha} f''\|_{L^2(\mathbf{R}_+)}. \end{aligned}$$

We postpone the proof of Theorem 3.1 to the end of this section.

3.1. Examples of applications. We give several applications of Theorem 3.1.

Example 3.2 (The semigroup $\lambda_g \mapsto (1+|g|)^{-z} \lambda_g$ on free groups). For every $z \in \mathbf{C}$ with positive real part, the formula

$$(1+n)^{-z} = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} e^{-t} e^{-tn} dt,$$

(where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the Gamma function) together with the fact that $S_t^1: \lambda_g \mapsto e^{-t|g|} \lambda_g$ is unital completely positive for every $t > 0$ implies that $\lambda_g \mapsto (1+|g|)^{-z} \lambda_g$ is unital completely positive if $z \in \mathbf{R}_+$, and unital completely bounded with completely bounded norm less than $\Gamma(\operatorname{Re}(z))/|\Gamma(z)|$ otherwise. This estimate is far from optimal: if $\operatorname{Re}(z) \geq 3$, the rapid decay property implies that this c.b. norm is less than $\sum_{n \geq 0} (1+n)^{-2}$, whereas $\Gamma(\operatorname{Re}(z))/|\Gamma(z)|$ is unbounded. Theorem 3.1 gives complementary estimates for this norm in the regime $\operatorname{Re}(z) \leq 3$. Namely let $z = a + ib$ with $0 < a \leq 3$. Taking $f(x) = (1+x)^{-z}$ and $\alpha = \min(1, a)/2$ then

$$\|x^{\frac{1}{2} \pm \alpha} f'\|_{L^2(0, \infty)} \leq \|(1+x)^{\frac{1}{2} \pm \alpha} f'\|_{L^2(1, \infty)} = \frac{|z|}{\sqrt{2a \pm 2\alpha}} \leq \frac{|z|}{\sqrt{a}}$$

and similarly

$$\|x^{\frac{3}{2} \pm \alpha} f''\|_{L^2(0, \infty)} \leq \frac{|z(1+z)|}{\sqrt{2a \pm 2\alpha}} \leq \frac{|z(1+z)|}{\sqrt{a}}.$$

Hence Theorem 2.3 and Theorem 3.1 imply that the completely bounded norm of $\lambda_g \mapsto (1 + |g|)^{-z} \lambda_g$ is less than $C \frac{|z|}{\operatorname{Re}(z)} \sqrt{|1 + z|}$. This implies that for $\omega \in [0, \pi/2)$, the c.b. norm of $\lambda_g \mapsto (1 + |g|)^{-z} \lambda_g$ is bounded by $C(1 + \tan \omega)^{\frac{3}{2}}$ on $\{z \in \mathbf{C}, |\arg z| \leq \omega\}$.

The same estimates also hold for the multiplier $\lambda_g \mapsto \max(1, |g|)^{-z} \lambda_g$. This follows from Theorem 2.3 and from the following general inequality applied to $a_n = \max(1, n)^{-z} - (n + 1)^{-z}$. If a_n is any sequence of complex numbers,

$$\|(a_{j+k+1})_{j,k \geq 0}\|_{S^1} \leq \|(a_{j+k})_{j,k \geq 0}\|_{S^1} \leq |a_0| + 2\|(a_{j+k+1})_{j,k \geq 0}\|_{S^1}$$

the first inequality is obvious, whereas the second is the triangle inequality for the trace norm $\|\cdot\|_{S^1}$ applied to the decomposition

$$a_{j+k} = a_0 1_{j=k=0} + a_k 1_{j=0, k>0} + a_{j+k} 1_{j>0}.$$

Note also that by Corollary 2.8, the same results hold for the semigroups $\lambda_g \mapsto (1 + |g|)^{-z} \lambda_g$ and $\lambda_g \mapsto \max(1, |g|)^{-z} \lambda_g$ on every hyperbolic group, up to some multiplicative constant depending on the group.

Example 3.3 (Fejér kernel). Given $N \in \mathbf{N}$, let $F_N(k) = (1 - \frac{k}{N}) \chi_{[0, N]}(k)$. Define the Fejér multiplier as

$$m_{F_N} : \lambda_g \mapsto F_N(|g|) \lambda_g.$$

Then $\|m_{F_N}\|_{cb} \simeq \log N$ on free group von Neumann algebras. In fact,

$$(2) \quad \|(F_N(k + j) - F_N(k + j + 1))_{k,j}\|_{S^1} = \|(\frac{1}{N})_{k+j \leq N-1}\|_{S^1} \simeq \log N.$$

If one applies Theorem 3.1 for the function $f(x) = (1 - x)^\delta \chi_{[0, 1]}(x)$, we see that

$$\sup_N \|(F_N(k + j)^\delta - F_N(k + j + 2)^\delta)_{k,j}\|_{S^1} < \infty$$

if $\delta > \frac{3}{2}$, because $f'' \in L^2$ if and only if $\delta > \frac{3}{2}$. We will see in Example 3.4 that the previous inequality actually holds for all $\delta > 1$.

Example 3.4 (Bochner-Riesz mean). The Bochner-Riesz mean is a “smoothed” Fejér multiplier in modern harmonic analysis. Given $N \in \mathbf{N}$, let $B_N^\delta(k) = (1 - \frac{k^2}{N^2})^\delta \chi_{[0, N]}(k)$ for $\delta \in \mathbf{C}$. Define the Bochner-Riesz multiplier as

$$m_{B_N}^\delta : \lambda_g \mapsto B_N^\delta(|g|) \lambda_g.$$

As for the Fejér kernel, a direct application of Theorem 3.1 would yield that the Bochner-Riesz kernels are completely bounded on von Neumann algebras of hyperbolic groups if $\operatorname{Re}(\delta) > \frac{3}{2}$. However, using the known boundedness properties of Bochner-Riesz multipliers on $\mathcal{L}(\mathbf{Z})$, one can decrease this to $\operatorname{Re}(\delta) > 1$.

Before that, we observe that, as in $\mathcal{L}(\mathbf{Z}^n)$, the problem of complete boundedness of the Bochner-Riesz and Fejér multipliers on hyperbolic groups are equivalent, in the sense that for all $\operatorname{Re}(\delta) \geq 0$ and all hyperbolic groups, there is a constant C such that for all N ,

$$(3) \quad \frac{1}{C} \leq \frac{\|m_{B_N}^\delta\|_{cb(\mathcal{L}\Gamma)}}{\|m_{F_N}^\delta\|_{cb(\mathcal{L}\Gamma)}} \leq C.$$

Indeed, given $\operatorname{Re}(\delta) \geq 0$, let f_δ and g_δ be compactly supported C^2 functions on $[0, \infty)$ satisfying $f_\delta(x) = 1/g_\delta(x) = (1+x)^\delta$ for all $x \in [0, 1]$, so that $B_N^\delta(k) = f_\delta(k/N)F_N^\delta(k)$ and $F_N^\delta(k) = g_\delta(k/N)F_N^\delta(k)$. By Theorem 3.1

$$\sup_N \|(f_\delta(\frac{i+j}{N}) - f_\delta(\frac{i+j+1}{N}))_{j,k}\|_{S^1} < \infty$$

and the same for g_δ . By Corollary 2.8, the multipliers corresponding to $f_\delta(|g|/N)$ and $g_\delta(|g|/N)$ are therefore bounded uniformly in N on every hyperbolic group. This proves (3).

We can now prove that Bochner-Riesz multipliers (and hence the Fejér multipliers by (3)) are completely bounded on every hyperbolic group, and in particular on all free groups, if $\operatorname{Re}\delta > 1$. This follows from Corollary 2.8 and the estimate

$$(4) \quad \forall \operatorname{Re}(\delta) > 1, \sup_N \|(B_N^\delta(j+k) - B_N^\delta(j+k+1))_{j,k \geq 0}\|_{S^1} < \infty.$$

Let us prove (4). By derivating we can write

$$B_N^\delta(k) - B_N^\delta(k+1) = \int_0^1 2\delta(1 - \frac{(k+t)^2}{N^2})^{\delta-1} \chi_{[0,N]}(k+t) \frac{k+t}{N^2} dt.$$

Denote by $B_{N,t}^{\delta-1}(k) = (1 - \frac{(k+t)^2}{N^2})^{\delta-1} \chi_{[-N,N]}(k+t)$; for $t = 0$ and $k \geq 0$ this is $B_N^{\delta-1}$. Let f_1 be a compactly supported C^2 function on $[0, \infty)$ such that $f(x) = -x^2/2$ on $[0, 1]$. The previous equality becomes

$$B_N^\delta(k) - B_N^\delta(k+1) = \int_0^1 2\delta B_{N,t}^{\delta-1}(k) (\frac{t-1/2}{N^2} + f_1(\frac{k}{N}) - f_1(\frac{k+1}{N})) dt.$$

The trace norm of the matrix $(B_{N,t}^{\delta-1}(j+k) \frac{t-1/2}{N^2})_{j,k \geq 0}$ is less than the sum of the absolute values of its entries, which is less than $\frac{1}{2}$. Moreover by Theorem 3.1 the trace norm of the matrix $(f_1(\frac{j+k}{N}) - f_1(\frac{j+k+1}{N}))$ is bounded uniformly in N , by some constant C . Therefore Lemma 2.2 and the previous equality imply that the trace norm of $(B_N^\delta(j+k) - B_N^\delta(j+k+1))_{j,k \geq 0}$ is less than

$$\begin{aligned} & |\delta| + 2C|\delta| \sup_{t \in [0,1]} \|\sum_{n \in \mathbf{Z}} B_{N,t}^{\delta-1}(n) e^{2i\pi n\theta}\|_{L^1(\mathbf{R}/\mathbf{Z})} \\ & \leq |\delta| + 2C|\delta| \|m_{B_N^{\delta-1}}\|_{cb(L^\infty(\mathbf{R}))} \leq |\delta| + 2C|\delta| e^{C|Im\delta|^2}. \end{aligned}$$

The first inequality follows by embedding $L^\infty(\mathbf{R}/\mathbf{Z})$ into $L^\infty(\mathbf{R})$ via $\sum a_n e^{2i\pi n\theta} \mapsto \sum a_n e^{2i\pi n(x-t)}$. The second inequality is quoted from [8, Prop. 10.2.2], and the constant C depends only on $\operatorname{Re}(\delta)$. This proves (4) and

$$(5) \quad \|m_{B_N^\delta}\|_{cb(\mathcal{L}(\Gamma))} \leq e^{C+C|Im\delta|^2}$$

for $\operatorname{Re}(\delta) > 1$ with C only depends on $\operatorname{Re}(\delta)$ and Γ . We can observe that the assumption $\operatorname{Re}(\delta) > 1$ is needed. Indeed, for $\delta = 1$, (4) does not hold because of (2) and (3), so $m_{B_N^1}$ is not c.b. on $\mathcal{L}(\mathbb{F}_2)$ uniformly in N by Theorem 2.3.

Fix $0 < \varepsilon < 1$, let C be the constant in (5) such that the multiplier $F(z) = m_{B_N^\delta}^z e^{Cz^2-5C}$ is c.b. on $\mathcal{L}(\Gamma)$ uniformly on the complex line $\{z; \operatorname{Re}(z) = 1 + \varepsilon\}$. Note that $F(z)$ is c.b. on $L^2(\hat{\Gamma})$ (the noncommutative L^2 space of $\mathcal{L}(\Gamma)$), uniformly in N on the imaginary line $\{z; \operatorname{Re}(z) = 0\}$. By complex interpolation and duality, we get $m_{B_N^\delta}^z$ is c.b. on $L^p(\hat{\Gamma})$ (the noncommutative L^p space of $\mathcal{L}(\Gamma)$) for any $|\frac{2}{p} - 1| < \operatorname{Re}(\delta)$ for any $1 \leq p \leq \infty$.

Example 3.5. Given $r > 0$, let $\alpha = \frac{\min\{r, 1\}}{2}$ and $f(x) = e^{-x^r}$. We then have,

$$\|x^{\frac{1}{2} \pm \alpha} f'\|_{L^2(0, \infty)} \leq c\sqrt{r}, \|x^{\frac{3}{2} \pm \alpha} f''\|_{L^2(0, \infty)} \leq c(1+r)\sqrt{r}.$$

Applying Theorem 3.1, we get

$$\sup_{t \geq 0} \|(e^{-t(j+k)^r} - e^{-t(j+k+1)^r})_{j, k \geq 0}\|_{S^1} \leq c(1+r).$$

Moreover the order r as r goes to ∞ is optimal. Indeed, if n is the integer part of r and $t = n^{-r}$ using the inequality $\|A\|_{S^1} \geq \sum_{j=0}^n |A_{j, n-j}|$ we have

$$\|(e^{-t(j+k)^r} - e^{-t(j+k+1)^r})_{j, k \geq 0}\|_{S^1} \geq (n+1) \left(e^{-1} - e^{-(1+1/n)^r} \right) \sim r(e^{-1} - e^{-e})$$

as $r \rightarrow \infty$.

Example 3.6. For every $z = a + bi \in \mathbf{C}$ with $|\arg z| \leq \omega < \frac{\pi}{2}$, let $f(x) = e^{-zx^r}$. Denote $K = (1 + \tan^2 \omega)$. Then $|z|^2 \leq Ka^2$ and

$$\begin{aligned} |f'|^2 &= |zrx^{r-1}e^{-zx^r}|^2 \leq Ka^2r^2x^{2r-2}e^{-2ax^r}, \\ |f''|^2 &= |-z^2r^2x^{2r-2}e^{-zx^r} + zr(r-1)x^{r-2}e^{-zx^r}|^2 \\ &\leq 2K^2|a^2r^2x^{2r-2}e^{-ax^r}|^2 + 2K|ar(r-1)x^{r-2}e^{-ax^r}|^2. \end{aligned}$$

Setting $\alpha = \min\{\frac{r}{2}, 1\}$ we then have using the change of variable $s = ax^r$,

$$\begin{aligned} \|x^{\frac{1}{2} \pm \alpha} f'\|_{L^2(\mathbf{R}_+)}^2 &\leq K \int_{\mathbf{R}_+} a^2r^2x^{2r \pm 2\alpha - 1} e^{-2ax^r} dx \\ &= K \int_{\mathbf{R}_+} ra^{\mp \frac{2\alpha}{r}} s^{1 \pm \frac{2\alpha}{r}} e^{-2s} ds \simeq Ka^{\mp \frac{2\alpha}{r}} r, \\ \|x^{\frac{3}{2} \pm \alpha} f''\|_{L^2(\mathbf{R}_+)}^2 &\leq 2K^2 \int_{\mathbf{R}_+} a^4r^4x^{4r \pm 2\alpha - 1} e^{-2ax^r} dx \\ &\quad + 2K \int_{\mathbf{R}_+} a^2r^2(r-1)^2x^{2r \pm 2\alpha - 1} e^{-2ax^r} dx \\ &= 2K^2 \int_{\mathbf{R}_+} a^{\mp \frac{2\alpha}{r}} r^3 s^{3 \pm \frac{2\alpha}{r}} e^{-2s} ds \\ &\quad + 2K \int_{\mathbf{R}_+} a^{\mp \frac{2\alpha}{r}} r(r-1)^2 s^{1 \pm \frac{2\alpha}{r}} e^{-2s} ds \\ &\simeq 2K^2 a^{\mp \frac{2\alpha}{r}} r^3 + 2Ka^{\mp \frac{2\alpha}{r}} r(r-1)^2. \end{aligned}$$

Theorem 3.1 yields that

$$\sup_{t \geq 0} \|(e^{-zt(j+k)^r} - e^{-zt(j+k+1)^r})_{j, k \geq 0}\|_{S^1} < c(1 + (\tan \omega)^{\frac{3}{2}})(1+r).$$

3.2. The proof. In the sequel we consider the unit circle $\mathbf{T} = \{e^{2i\pi t}, t \in \mathbf{R}/\mathbf{Z}\}$ equipped with the Lebesgue probability measure, and the unit disk $\mathbf{D} = \{z \in \mathbf{C}, |z| < 1\}$ equipped with the Lebesgue probability measure $\frac{dz}{\pi}$.

We now turn to the proof of Theorem 3.1. The proof relies on Peller’s characterization of trace class Hankel matrices [20] that we now recall. With the formulation given in [11, Theorem 3.1] (which gives very good constants), Peller’s theorem states that a Hankel matrix $(a_{j+k})_{j, k \geq 0}$ belongs to the trace class if and only if the function $g(z) = \sum_{n \geq 0} (n+1)(n+2)a_n z^n$ belongs to $L^1(\mathbf{D}, \frac{dz}{\pi})$, and

$$\frac{\pi}{8} \|g\|_{L^1} \leq \|(a_{j+k})_{j, k \geq 0}\|_{S^1} \leq \|g\|_{L^1}.$$

The condition $\sum_{n \geq 0} (n+1)(n+2)a_n z^n \in L^1(\mathbf{D})$ is one of the equivalent conditions for the series $\sum_{n \geq 0} a_n z^n$ to belong to the so-called Besov space of analytic functions B_1^1 . In the sequence we will work with another classical condition, that is more suited for our proof.

Consider the classical de la Vallée Poussin kernels $(W_n)_{n \geq 0}$. The W_n 's are functions on \mathbf{T} given by their Fourier coefficients. $W_0(z) = 1 + z$ and for $n > 0$

$$\widehat{W_n}(k) = \begin{cases} 2^{-n+1}(k - 2^{n-1}) & \text{if } 2^{n-1} \leq k \leq 2^n, \\ 2^{-n}(2^{n+1} - k) & \text{if } 2^n \leq k \leq 2^{n+1}, \\ 0 & \text{otherwise.} \end{cases}$$

The Besov space B_1^1 of analytic functions is the Banach space of series $\varphi(z) = \sum_{n \geq 0} a_n z^n$ with $a_n \in \mathbf{C}$ such that

$$(6) \quad \|\varphi\|_{B_1^1} = \sum_{n \geq 0} 2^n \|W_n * \varphi\|_{L^1(\mathbf{T})} < \infty.$$

We refer to [21] for the equivalence of these definitions of B_1^1 , or for the following formulation of Peller's theorem: there is a constant $C > 0$ such that for every Hankel matrix $A = (a_{j+k})_{j,k \geq 0}$,

$$(7) \quad C^{-1} \left\| \sum_{n \geq 0} a_n z^n \right\|_{B_1^1} \leq \|A\|_{S^1} \leq C \left\| \sum_{n \geq 0} a_n z^n \right\|_{B_1^1}.$$

For a function $f: [0, \infty) \rightarrow \mathbf{R}$ and a subinterval I of $[0, \infty)$ we adopt the following notation:

$$(8) \quad \|f\|_{L^2(I)} = \left(\int_I |f(x)|^2 dx \right)^{\frac{1}{2}}, \quad \|f\|_{\ell^2(I)} = \left(\sum_{k \in I \cap \mathbf{N}} |f(k)|^2 \right)^{\frac{1}{2}}.$$

We will prove the following upper estimate on the B_1^1 -norm of a function with smooth symbol.

Proposition 3.7. *Let $f: [0, \infty) \rightarrow \mathbf{R}$ be a continuous function of class C^1 on $(1, \infty)$, and $\frac{1}{2} \geq \alpha > 0$. Then*

$$\left\| \sum_{n \geq 0} f(n) z^n \right\|_{B_1^1} \leq C \left(|f(0)| + \frac{1}{\sqrt{\alpha}} (A + \sqrt{AB}) \right),$$

for a universal constant C , where

$$A = \sqrt{\|x^{\frac{1}{2}-\alpha} f\|_{\ell^2(1, \infty)} \|x^{\frac{1}{2}+\alpha} f\|_{\ell^2(1, \infty)}},$$

$$B = \sqrt{\|x^{\frac{3}{2}-\alpha} f'\|_{L^2(1, \infty)} \|x^{\frac{3}{2}+\alpha} f'\|_{L^2(1, \infty)}}.$$

Before we prove the proposition, we explain how it implies Theorem 3.1.

Proof of Theorem 3.1. Let f be as in Theorem 3.1. Note that both \tilde{A} and \tilde{B} are unchanged if the function f is replaced by $x \mapsto f(tx)$. We can therefore restrict ourselves to the case $t = 1$.

We first prove the inequalities

$$(9) \quad \left(\sum_{n \geq 1} n^{2\beta} |f(n+1) - f(n)|^2 \right)^{\frac{1}{2}} \leq \|x^\beta f'\|_{L^2(1, \infty)}$$

for $\beta = \frac{1}{2} + \alpha$ and $\beta = \frac{1}{2} - \alpha$, and

$$(10) \quad \|x^\beta(f'(x+1) - f'(x))\|_{L^2(1,\infty)} \leq \|x^\beta f''\|_{L^2(1,\infty)}$$

for $\beta = \frac{3}{2} + \alpha$ and $\beta = \frac{3}{2} - \alpha$. Together with (7) and Proposition 3.7, they will imply that

$$(11) \quad \left\| (f(j+k) - f(j+k+1))_{j,k \geq 0} \right\|_{S^1} \leq C \left(|f(0) - f(1)| + \frac{1}{\sqrt{\alpha}}(\tilde{A} + \sqrt{\tilde{A}\tilde{B}}) \right).$$

For (9), note that $n^\beta \leq x^\beta$ for every integer n and $x \in [n, n+1]$. By the Cauchy-Schwarz inequality we have

$$n^\beta |f(n+1) - f(n)| \leq n^\beta \|f'\|_{L^2(n,n+1)} \leq \|x^\beta f'\|_{L^2(n,n+1)}.$$

By taking the square and summing for $n \geq 1$ we get (9). For (10) write $f'(x+1) - f'(x) = \int_0^1 f''(x+s)ds$ and use the triangle inequality to get

$$\|x^\beta(f'(x+1) - f'(x))\|_{L^2(1,\infty)} \leq \int_0^1 \|x^\beta f''(x+s)\|_{L^2(1,\infty)} ds,$$

from which (10) follows because $\|x^\beta f''(x+s)\|_{L^2(1,\infty)} \leq \|x^\beta f''(x)\|_{L^2(1,\infty)}$ for all $0 < s < 1$.

We now prove the theorem. If $\tilde{B} = \infty$ there is nothing to prove. So let us assume that $t = 1$ and $\tilde{B} < \infty$. Theorem 3.1 follows from (11) and the inequalities

$$|f(0) - f(1)| \leq \|f'\|_{L^1(\mathbf{R}_+)} \leq \frac{\sqrt{2}}{\sqrt{\alpha}} \tilde{A}$$

and

$$\tilde{A} \leq \frac{1}{\sqrt{1-\alpha^2}} \tilde{B}.$$

To prove the first inequality, decompose the integral and use the Cauchy-Schwarz inequality

$$\begin{aligned} \|f'\|_{L^1} &= \int_0^s \frac{x^{\frac{1}{2}-\alpha} f'(x)}{x^{\frac{1}{2}-\alpha}} dx + \int_s^\infty \frac{f'(x)x^{\frac{1}{2}+\alpha}}{x^{\frac{1}{2}+\alpha}} dx \\ &\leq \frac{s^\alpha}{\sqrt{2\alpha}} \|x^{\frac{1}{2}-\alpha} f'\|_{L^2(\mathbf{R}_+)} + \frac{s^{-\alpha}}{\sqrt{2\alpha}} \|x^{\frac{1}{2}-\alpha} f'\|_{L^2(\mathbf{R}_+)}. \end{aligned}$$

Taking the infimum over $s > 0$ we get $\|f'\|_{L^1(\mathbf{R}_+)} \leq \frac{\sqrt{2}}{\sqrt{\alpha}} \tilde{A}$ as claimed.

Let us move to the inequality $\tilde{A} \leq \frac{1}{\sqrt{1-\alpha^2}} \tilde{B}$. By the assumption $\tilde{B} < \infty$, we have that $f'' \in L^1([1, \infty))$ and hence $\lim_{x \rightarrow \infty} f'(x)$ exists. Since f is bounded, this limit is 0, and we can write $f'(x) = \int_1^\infty x g_r(x) dr$ where $g_r(x) = f''(rx)$. By the triangle inequality

$$\|x^{\frac{1}{2} \pm \alpha} f'\|_{L^2(\mathbf{R}_+)} \leq \int_1^\infty \|x^{\frac{3}{2} \pm \alpha} g_r\|_{L^2(\mathbf{R}_+)} dr.$$

By a change of variable

$$\|x^{\frac{3}{2} \pm \alpha} g_r\|_{L^2(\mathbf{R}_+)} = \frac{1}{r^{2 \pm \alpha}} \|x^{\frac{3}{2} \pm \alpha} f''\|_{L^2(\mathbf{R}_+)},$$

and hence using that $\int_1^\infty \frac{dr}{r^{2\pm\alpha}} = \frac{1}{1\pm\alpha}$, we get

$$\|x^{\frac{1}{2}\pm\alpha} f'\|_{L^2(\mathbf{R}_+)} \leq \frac{1}{1\pm\alpha} \|x^{\frac{3}{2}\pm\alpha} f''\|_{L^2(\mathbf{R}_+)}.$$

The inequality $\tilde{A} \leq \frac{1}{\sqrt{1-\alpha^2}} \tilde{B}$ follows. □

We give now the proof of Proposition 3.7. We start with a classical elementary lemma.

Lemma 3.8. *If $\varphi \in L^2(\mathbf{T})$, then*

$$\|\varphi\|_{L^1(\mathbf{T})} \leq \frac{2}{\sqrt{\pi}} \sqrt{\|\varphi\|_{L^2(\mathbf{T})} \|(1-z)\varphi\|_{L^2(\mathbf{T})}}.$$

Proof. Denote $g(z) = (1-z)\varphi(z)$. For any $0 < s < 1/2$:

$$\begin{aligned} \|\varphi\|_{L^1} &= \int_0^1 |\varphi(e^{2i\pi t})| dt \\ &= \int_{-s}^s |\varphi(e^{2i\pi t})| dt + \int_s^{1-s} \frac{1}{|1-e^{2i\pi t}|} |(1-e^{2i\pi t})\varphi(e^{2i\pi t})| dt \\ &\leq \sqrt{2s} \|\varphi\|_2 + \sqrt{\int_s^{1-s} \frac{1}{|1-e^{2i\pi t}|^2} dt} \|g\|_2 \end{aligned}$$

by the Cauchy-Schwarz inequality. The remaining integral can be computed:

$$\begin{aligned} \int_s^{1-s} \frac{1}{|1-e^{2i\pi t}|^2} dt &= 2 \int_s^{1/2} \frac{1}{4 \sin^2(\pi t)} dt \\ &= \frac{1}{2} \left[\frac{-\cos(\pi t)}{\pi \sin(\pi t)} \right]_s^{1/2} = \frac{1}{2\pi \tan(\pi s)} \leq \frac{1}{2\pi^2 s} \end{aligned}$$

where we used that $\tan x \geq x$ for all $0 \leq x \leq \pi/2$. Taking $s = \|g\|_2 / 2\pi \|\varphi\|_2 \leq 1/2$ we get the desired inequality. □

Let f be as in Proposition 3.7. Recall the notation introduced in (8). We prove the following.

Lemma 3.9. *Let $I_n = (2^{n-1}, 2^{n+1}]$. Denote $\varphi(z) = \sum_{n \geq 0} f(n)z^n$. Then for $n \geq 1$*

$$2^n \|W_n * \varphi\|_{L^1(\mathbf{T})} \leq \frac{4}{\sqrt{\pi}} (\|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} + \sqrt{\|x^{\frac{3}{2}} f'\|_{L^2(I_n)} \|x^{\frac{1}{2}} f\|_{\ell^2(I_n)}}).$$

Proof. The inequality $\|W_n * \varphi\|_{L^2(\mathbf{T})} \leq \|f\|_{\ell^2(I_n)}$ is clear. Writing $(1-z)(W_n * \varphi)(z)$ as

$$\sum_{2^{n-1}+1}^{2^{n+1}} (\widehat{W}_n(k) - \widehat{W}_n(k-1)) f(k) z^k + \widehat{W}_n(k-1) (f(k) - f(k-1)) z^k,$$

and noting that $|\widehat{W}_n(k) - \widehat{W}_n(k-1)| \leq 2^{1-n}$ and $|\widehat{W}_n(k-1)(f(k) - f(k-1))| \leq \|f'\|_{L^2(k-1,k)}$ for $k \in I_n$, we get

$$\|(1-z)(W_n * \varphi)\|_{L^2(\mathbf{T})} \leq 2^{1-n} \|f\|_{\ell^2(I_n)} + \|f'\|_{L^2(I_n)}.$$

By Lemma 3.8 and the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ we get

$$\|W_n * \varphi\|_{L^1(\mathbf{T})} \leq \frac{2^{\frac{3-n}{2}}}{\sqrt{\pi}} \|f\|_{\ell^2(I_n)} + \frac{2}{\sqrt{\pi}} \sqrt{\|f\|_{\ell^2(I_n)} \|f'\|_{L^2(I_n)}}.$$

Multiplying by 2^n and using that $x \geq 2^{n-1}$ on I_n we get

$$2^n \|W_n * \varphi\|_{L^1(\mathbf{T})} \leq \frac{4}{\sqrt{\pi}} \left(\|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} + \sqrt{\|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} \|x^{\frac{3}{2}} f'\|_{L^2(I_n)}} \right),$$

which concludes the proof. □

Proof of Proposition 3.7. By Lemma 3.9 and Peller’s characterization, there is a universal constant C such that we have

$$\|A\|_1 \leq C \left(|f(0)| + |f(1)| + \sum_{n \geq 1} \|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} + \sqrt{\|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} \|x^{\frac{3}{2}} f'\|_{L^2(I_n)}} \right).$$

Denote here $I_0 = \{1\}$, so that $|f(1)| = \|f\|_{\ell^2(I_0)}$. Then by the Cauchy-Schwarz inequality the previous inequality becomes

$$(12) \quad \|A\|_1 \leq C \left(|f(0)| + \sum_{n \geq 0} \|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} + \sqrt{\sum_{n \geq 1} \|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} \sum_{n \geq 1} \|x^{\frac{3}{2}} f'\|_{L^2(I_n)}} \right).$$

Let $N \geq 1$. If $n \leq N$, $x^{1/2}$ is dominated by $x^{\frac{1}{2}-\alpha} 2^{\alpha(n+1)}$ on I_n , and hence

$$\begin{aligned} \sum_{n=0}^N \|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} &\leq \left(\sum_{n=0}^N 2^{2\alpha(n+1)} \right)^{\frac{1}{2}} \left(\sum_{n=0}^N \|x^{\frac{1}{2}-\alpha} f\|_{\ell^2(I_n)}^2 \right)^{\frac{1}{2}} \\ &\leq \frac{2^{\alpha(N+2)}}{\sqrt{2^{2\alpha}-1}} \sqrt{2} \|x^{\frac{1}{2}-\alpha} f\|_{\ell^2([1, 2^{N+1}])} \\ &\leq \frac{C}{\sqrt{\alpha}} 2^{\alpha N} \|x^{\frac{1}{2}-\alpha} f\|_{\ell^2([1, \infty))}. \end{aligned}$$

The $\sqrt{2}$ is because every point in $[1, \infty)$ belongs to at most two intervals I_n for $n \in [1, N]$. For $n > N$ use that $x \geq 2^{n-1}$ on I_n to dominate

$$\begin{aligned} \sum_{n > N} \|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} &\leq \sum_{n > N} 2^{-\alpha(n-1)} \|x^{\frac{1}{2}+\alpha} f\|_{\ell^2(I_n)} \\ &\leq \left(\sum_{n > N} 2^{-2\alpha(n-1)} \right)^{\frac{1}{2}} \sqrt{2} \|x^{\frac{1}{2}+\alpha} f\|_{\ell^2([1, \infty))} \\ &\leq \frac{C}{\sqrt{\alpha}} 2^{-\alpha N} \|x^{\frac{1}{2}+\alpha} f\|_{\ell^2([1, \infty))}. \end{aligned}$$

Let $a = \|x^{\frac{1}{2}-\alpha} f\|_{\ell^2([1, \infty))}$ and $b = \|x^{\frac{1}{2}+\alpha} f\|_{\ell^2([1, \infty))}$. Since $a \leq b$ we have $\inf_{N \geq 1} 2^{\alpha N} a + 2^{-\alpha N} b \leq 2^{1+\alpha} \sqrt{ab}$. This implies that there is a constant C such that

$$\sum_{n \geq 0} \|x^{\frac{1}{2}} f\|_{\ell^2(I_n)} \leq \frac{C}{\sqrt{\alpha}} \sqrt{\|x^{\frac{1}{2}-\alpha} f\|_{\ell^2([1, \infty))} \|x^{\frac{1}{2}+\alpha} f\|_{\ell^2([1, \infty))}}.$$

The same argument implies a similar inequality with f replaced by xf' and the norm ℓ^2 replaced by the norm L^2 :

$$\sum_{n \geq 1} \|x^{\frac{3}{2}} f'\|_{L^2(I_n)} \leq \frac{C}{\sqrt{\alpha}} \sqrt{\|x^{\frac{3}{2}-\alpha} f'\|_{L^2(1,\infty)} \|x^{\frac{3}{2}+\alpha} f\|_{L^2(1,\infty)}}.$$

If we remember (12) we get the inequality in the proposition, which concludes the proof. □

4. THE CASE OF \mathbf{Z}^d

In this section we prove that \mathbf{Z}^d equipped with its standard generating set satisfies the conclusion of Corollary 2.8 for all $d \geq 1$. For the standard generating set, the word length of $n = (n_1, \dots, n_d) \in \mathbf{Z}^d$ is the ℓ^1 -length $|n| = |n_1| + \dots + |n_d|$.

As \mathbf{Z}^d is commutative, the von Neumann algebra of \mathbf{Z}^d is $L^\infty(\mathbf{R}^d/\mathbf{Z}^d)$, and for a function $\phi: \mathbf{Z}^d \rightarrow \mathbf{C}$ with finite support, the Fourier multiplier m_ϕ with symbol ϕ is the convolution by the function $x \in \mathbf{R}^d/\mathbf{Z}^d \mapsto \sum_{n \in \mathbf{Z}^d} \phi(n) e^{2i\pi n \cdot x}$ on $L^\infty(\mathbf{R}^d/\mathbf{Z}^d)$. The norm and c.b. norm of m_ϕ both coincide with the L^1 -norm of the function $\sum_{n \in \mathbf{Z}^d} \phi(n) e^{2i\pi n \cdot x}$. Taking into account Peller's Theorem [20] (see Section 3.2), we see that the validity of Corollary 2.8 for \mathbf{Z}^d is equivalent to the existence of C_d such that

$$(13) \quad \left\| \sum_{n \in \mathbf{Z}^d} a_{|n|} e^{in \cdot x} \right\|_{L^1([-\pi, \pi]^d)} \leq C_d \left\| \sum_{m \geq 0} (a_{m+1} - a_m) e^{imx} \right\|_{B_1^1}$$

for all finitely supported sequence $(a_m)_{m \geq 0}$. But it is easy to see from (6) that for $\varphi(z) = \sum_{m \geq 0} b_m z^m$,

$$C^{-1} \|\varphi\|_{B_1^1} \leq \sum_{n \geq 0} \|W_n * \psi\|_{L^1(\mathbf{T})} \leq C \|\varphi\|_{B_1^1}$$

for some universal constant C , where $\psi(z) = \sum_{m \geq 0} (m+1) b_m z^m$.

Inequality (13) therefore follows from

Proposition 4.1. *Let d be an integer. There is a constant C_d such that for every finitely supported sequence $(a_n)_{n \geq 0}$,*

$$(14) \quad \left\| \sum_{n \in \mathbf{Z}^d} a_{|n|} e^{in \cdot x} \right\|_{L^1([-\pi, \pi]^d)} \leq C_d \left\| \sum_{m \geq 0} (m+1) (a_{m+1} - a_m) e^{imx} \right\|_{L^1([-\pi, \pi])}.$$

Proof. We can assume that d is even, because (14) for d implies (14) for $d-1$ by taking the average with respect to x_d .

We can rewrite

$$\sum_{n \in \mathbf{Z}^d} a_{|n|} e^{in \cdot x} = \sum_{m \geq 0} (a_m - a_{m+1}) D_m(x)$$

where $D_m(x) = \sum_{|n| \leq m} e^{in \cdot x}$. The exact value for $D_m(x)$ was computed in [25, Theorem 4.2.3] and is equal to

$$D_m(x_1, \dots, x_d) = [\cos x_1, \dots, \cos x_d] G_m$$

where $G_m: [-1, 1] \rightarrow \mathbf{R}$ is given by

$$G_m(\cos x) = (-1)^{\frac{d}{2}-1} \sin(x)^{d-2} (\cos(mx) + \cos((m+1)x))$$

and for a function $f: [-1, 1] \rightarrow \mathbf{C}$ and d distinct numbers $t_1, \dots, t_d \in [-1, 1]$ we use the following notation of divided difference:

$$[t_1, \dots, t_d]f = \sum_{j=1}^d \frac{f(t_j)}{\prod_{k \neq j} (t_j - t_k)}.$$

Here we use that d is even, otherwise in the formula for $G_m(\cos x)$ the terms \cos have to be replaced by \sin .

Denote $f_1(x) = \sum_{m \geq 0} (a_m - a_{m+1})e^{imx}$ and $f_2(x) = e^{ix}f(x)$. By the preceding we can write

$$\sum_{n \in \mathbf{Z}^d} a_{|n|} e^{in \cdot x} = \frac{(-1)^{\frac{d}{2}-1}}{2} [\cos x_1, \dots, \cos x_d] (H_{f_1} + H_{f_2})$$

where for a function $f: [-\pi, \pi] \rightarrow \mathbf{C}$ we denote $H_f: [-1, 1] \rightarrow \mathbf{C}$ by

$$H_f(\cos x) = \sin(x)^{d-2} (f(x) + f(-x)).$$

Using that H_1 is a polynomial of degree $d - 2$ (d is even) and that $[t_1, \dots, t_d]f = 0$ for all polynomials of degree $d - 2$ we observe for further use that

$$(15) \quad [\cos x_1, \dots, \cos x_d]H_1 = 0.$$

We claim that

$$(16) \quad \sup_{-\pi < t < \pi} \|[\cos x_1, \dots, \cos x_d]H_{\chi_{[t, \pi]}}\|_{L^1([-\pi, \pi]^d)} < \infty.$$

This will imply the proposition. Indeed, if C_d is the sup in the previous inequality, and $f: [-\pi, \pi] \rightarrow \mathbf{C}$ is any C^1 function, writing $f(x) = f(-\pi) + \int_{-\pi}^x f'(t)\chi_{[t, \pi]}(x)dt$ for all $x \in [-\pi/2, \pi/2]$ and using (15), we get

$$\begin{aligned} & \|[\cos x_1, \dots, \cos x_d]H_f\|_{L^1([-\pi, \pi]^d)} \\ & \leq \int_{-\pi}^{\pi} |f'(t)| \|[\cos x_1, \dots, \cos x_d]H_{\chi_{[t, \pi]}}\|_{L^1([-\pi, \pi]^d)} dt \leq C_d \|f'\|_{L^1([-\pi, \pi])}. \end{aligned}$$

Applying this inequality to f_1 and f_2 yields (14).

We now move to (16). We start by some easy reduction. If $t > 0$ we have the relations

$$H_{\chi_{[t, \pi]}} = H_1 - H_{\chi_{(-t, \pi]}} = H_{\frac{1}{2}} - H_{\chi_{[0, t]}}$$

(except at the point 1). By (15) we can therefore assume that $t \geq 0$, in which case we have

$$\|[\cos x_1, \dots, \cos x_d]H_{\chi_{[t, \pi]}}\|_{L^1([-\pi, \pi]^d)} = \|[\cos x_1, \dots, \cos x_d]H_{\chi_{[0, t]}}\|_{L^1([-\pi, \pi]^d)}.$$

Also by symmetry we have

$$\|[\cos x_1, \dots, \cos x_d]H_{\chi_{[t, \pi]}}\|_{L^1([-\pi, \pi]^d)} = 2^d \|[\cos x_1, \dots, \cos x_d]H_{\chi_{[t, \pi]}}\|_{L^1([0, \pi]^d)}.$$

Finally, by the change of variables $x_i \mapsto \pi - x_i$,

$$\|[\cos x_1, \dots, \cos x_d]H_{\chi_{[t, \pi]}}\|_{L^1([0, \pi]^d)} = \|[\cos x_1, \dots, \cos x_d]H_{\chi_{[0, \pi-t]}}\|_{L^1([0, \pi]^d)},$$

so are left to prove

$$\sup_{0 \leq t \leq \pi/2} \|[\cos x_1, \dots, \cos x_d]H_{\chi_{[0, t]}}\|_{L^1([0, \pi]^d)} < \infty.$$

This will follow from the estimate in Lemma 4.2 below. If $d = 2$ this is exactly the lemma by letting $s \rightarrow 0$. If $d > 2$, fix $0 < t \leq \frac{\pi}{2}$, and for $x \in [0, \pi]$ write

$$H_{\chi_{[0,t]}}(\cos x) = \int_0^t (d-2)(\sin u)^{d-3} \cos u \chi_{[u,t]}(x) du.$$

With the notation of Lemma 4.2 we have for all $t \in (0, \pi/2)$,

$$[\cos x_1, \dots, \cos x_d] H_{\chi_{[0,t]}} \|_{L^1([0,\pi]^d)} \leq \int_0^t (d-2)(\sin u)^{d-3} \cos u \|A_{u,t}^d\|_{L^1} du,$$

which, by Lemma 4.2 and the inequality $(d-2)(\sin u)^{d-3} \cos u \leq (d-2)u^{d-3}$, is less than

$$(d-2)C_d \int_0^t \frac{(t-u)}{u^2} |\log(1 - \frac{u}{t})|^{d-1} du.$$

By the change of variable $u \rightarrow tu$ this is equal to

$$(d-2)C_d \int_0^1 \frac{(1-u)}{u^2} |\log(1-u)|^{d-1} du.$$

This last quantity is finite and independant of t . This concludes the proof of (16) and of the proposition. □

The previous proof used the following lemma.

Lemma 4.2. *Let $d \geq 1$ be an integer. For every $0 < s < t \leq \pi$, define a function $A_{s,t}^d: [0, \pi]^d \rightarrow \mathbf{R}$ by*

$$\begin{aligned} A_{s,t}^d(x_1, \dots, x_d) &= [\cos x_1, \dots, \cos x_d](\cos x \mapsto \chi_{[s,t]}(x)) \\ &= \sum_{i=1}^d \frac{\chi_{[s,t]}(x_i)}{\prod_{j \neq i} (\cos x_i - \cos x_j)}. \end{aligned}$$

Then there is a constant C_d such that

$$\|A_{s,t}^d\|_{L^1([0,\pi]^d)} \leq C_d(t-s) \left(\frac{1}{s} |\log(1 - \frac{s}{t})| \right)^{d-1}$$

for all $0 < s < t \leq \frac{\pi}{2}$.

Proof. We proceed by induction on d . The case $d = 1$ is obvious because

$$\|A_{s,t}^1\|_{L^1([0,\pi])} = \int_0^\pi \chi_{[s,t]}(x) dx = t - s.$$

Assume that the lemma holds for $d \geq 1$, and let $0 < s < t \leq \frac{\pi}{2}$. If $x_1, \dots, x_{d+1} \in [s, t]$, then

$$A_{s,t}^{d+1}(x_1, \dots, x_{d+1}) = [\cos x_1, \dots, \cos x_{d+1}] \mathbf{1} = 0.$$

By symmetry we therefore have

$$\|A_{s,t}^{d+1}\|_{L^1([0,\pi]^{d+1})} \leq (d+1) \left(\int_0^s + \int_t^\pi \right) \|A_{s,t}^{d+1}(\cdot, y)\|_{L^1([0,\pi]^d)} dy.$$

If $y \notin [s, t]$ we can write

$$\begin{aligned} A_{s,t}^{d+1}(x_1, \dots, x_d, y) &= \sum_{i=1}^d \frac{\chi_{[s,t]}(x_i)}{(\cos x_i - \cos y) \prod_{j \neq i} (\cos x_i - \cos x_j)} \\ &= [\cos x_1, \dots, \cos x_d] (\cos x \mapsto h_y(x) \chi_{[s,t]}(x)), \end{aligned}$$

where we denote $h_y(x) = \frac{1}{\cos x - \cos y}$.

At this point we have to distinguish the cases $y < s$ and $y > t$. Let us first consider the case $0 \leq y < s$. Then for $x \in [s, t]$ we write $h_y(x) = h_y(t) - \int_s^t h'_y(u) \chi_{[s,u]}(x) du$, so that by the triangle inequality we get

$$|A_{s,t}^{d+1}(x_1, \dots, x_d, y)| \leq |h_y(t) A_{s,t}(x_1, \dots, x_d)| + \int_s^t |h'_y(u) A_{s,u}^d(x_1, \dots, x_d)| du.$$

Integrating we obtain

$$\|A_{s,t}^{d+1}(\cdot, y)\|_{L^1([0, \pi]^d)} \leq |h_y(t)| \|A_{s,t}^d\|_{L^1} + \int_s^t |h'_y(u)| \|A_{s,u}^d\|_{L^1} du.$$

If we use the elementary inequalities $|h_y(x)| \leq \frac{C}{x(x-y)}$ and $|h'_y(x)| \leq \frac{Cx}{x^2(x-y)^2}$ valid for all $0 \leq y < x \leq \frac{\pi}{2}$ for a universal constant C , we have $\int_0^s |h_y(t)| dy \lesssim \frac{1}{s} |\log(1 - \frac{s}{t})|$ and $\int_0^s |h'_y(t)| dy \lesssim \frac{s}{u^2(u-s)}$ and the previous inequality together with the induction hypothesis yields after integration

$$\int_0^s \|A_{s,t}^{d+1}(\cdot, y)\|_{L^1([0, \pi]^d)} dy \lesssim_d \frac{|\log(1 - \frac{s}{t})|^d}{s^{d-1}t} + \int_s^t \frac{s^d}{u^2} (-\log(1 - \frac{s}{u}))^{d-1} du.$$

With the change of variable $v = 1 - \frac{s}{u}$ the last integral becomes

$$\int_0^{1-\frac{s}{t}} s^{d-1} |\log v|^{d-1} dv \lesssim_d s^{d-1} (1 - \frac{s}{t}) \log(1 - \frac{s}{t})^{d-1},$$

from which the inequality

$$\int_0^s \|A_{s,t}^{d+1}(\cdot, y)\|_{L^1([0, \pi]^d)} dy \lesssim_d \left(\frac{1}{s} |\log(1 - \frac{s}{t})| \right)^d$$

easily follows.

When $y \geq t$, we write $h_y(x) = h_y(s) + \int_s^t h'_y(t) \chi_{[u,t]}(x) du$ and with similar computations we get

$$\int_t^s \|A_{s,t}^{d+1}(\cdot, y)\|_{L^1([0, \pi]^d)} dy \lesssim_d \frac{|\log(1 - \frac{s}{t})|}{s} \|A_{s,t}^d\|_{L^1} + \int_s^t \frac{u}{t^2(t-u)} \|A_{u,t}^d\|_{L^1}.$$

By the induction hypothesis the first term is $\lesssim_d (\frac{1}{s} |\log(1 - s/t)|)^d$, and the second one is less than

$$\int_s^t \frac{u^{2-d}}{t^2} |\log(1 - s/t)|^{d-1} \leq \int_0^{1-\frac{s}{t}} \frac{s^{d-2}}{t} |\log x|^{d-1} dx.$$

The inequality

$$\int_t^\pi \|A_{s,t}^{d+1}(\cdot, y)\|_{L^1([0, \pi]^d)} dy \lesssim_d \left(\frac{1}{s} |\log(1 - \frac{s}{t})| \right)^d$$

follows easily, which completes the proof of the lemma. □

5. BMO AND H^∞ -CALCULUS

A motivation in studying S_t^r comes from harmonic analysis on free groups. We will briefly explain it in this section. We will also show a related result on bounded H^∞ -calculus.

Following [17] and [14], we may consider BMO spaces associated with the semi-groups S_t^r on the free group von Neumann algebras. For $f \in L^2(\hat{\mathbb{F}}_n)$, let

$$\|f\|_{BMO^r} = \sup_{t \geq 0} \|S_t^r |f - S_t^r f|^2\|^{\frac{1}{2}}.$$

Set

$$BMO^r(\hat{\mathbb{F}}_n) = \{f \in L^2, \max\{\|f\|_{BMO^r}, \|f^*\|_{BMO^r}\} < \infty\}.$$

Theorem 5.2 of [14] says that the complex interpolation space between BMO^r and $L^2(\hat{\mathbb{F}}_n)$ is $L^p(\hat{\mathbb{F}}_n)$, for all $2 < p < \infty$ and $0 < r \leq 1$. Thus, for any $0 < r \leq 1$, BMO^r serves as an endpoint for $L^p(\hat{\mathbb{F}}_n)$ corresponds to $p = \infty$. What will be an endpoint space which could substitute $L^1(\hat{\mathbb{F}}_n)$? A natural candidate would be the H^1 space defined by the Littlewood-Paley G -function

$$G(f) = \left(\int_0^\infty |\partial_t S_t^1 f|^2 t dt\right)^{\frac{1}{2}}$$

for $f \in L^1$ and

$$\|f\|_{H^1} = \tau G(f) < \infty.$$

In fact, for $n = 1$, we have Fefferman–Stein’s famous duality $(H^1)^* = BMO^1$ and the corresponding interpolation result. There has not been a satisfactory H^1 -BMO duality theory associated with semigroups on free group von Neumann algebra for $n > 1$. A main obstacle is due to the missing of geometric/metric tools in the non-commutative setting. For example, when $n = 1$, all the H^1 -BMO duality arguments (to the best knowledge of the authors) rely on an equivalent characterization of H^1 by the Lusin area function, the definition of which is similar to the Littlewood-Paley G -function but uses an integration on cones instead of the radial integration. The concept of “cones” on $\hat{\mathbb{F}}_n$ is a big mystery for $n > 1$. However, there is a semigroup representation of Lusin area integrations as follows:

$$Af = \left(\int_0^\infty S_{t^2}^2 |\partial_t S_t^1 f|^2 t dt\right)^{\frac{1}{2}},$$

which uses the semigroup S_t^2 to compensate the “integration on cones” and

$$\|f\|_{H^1} \simeq \|A(f)\|_{L^1}$$

for $n = 1$ (see [17] for an explanation). We should point out that the equivalence $\|f\|_{H^1} \simeq \|A(f)\|_{L^1}$ fails if we replace the extra S_t^2 in the definition of Af by S_t^1 . The complete boundedness of S_t^r , especially for $r = 2$, then draws our attention and is proved in Section 3. We still do not know whether a semigroup H^1 -BMO duality holds on $\mathcal{L}(\mathbb{F}_n)$ for $n > 1$ and leave the question for later.

Junge–Le Merdy–Xu ([13]) studied H^∞ -calculus in the noncommutative setting (see [4]). In particular, they obtain a bounded H^∞ -calculus property of $\mathcal{L}^r : \lambda_g \mapsto |g|^r \lambda_g$ on $L^p(\hat{\mathbb{F}}_n)$ and consequently a Littlewood-Paley theory for the corresponding semigroup S_t^r for all $1 < p < \infty, 0 < r \leq 1$. The endpoint cases ($p = 1, \infty$) are more subtle and \mathcal{L}^r has no bounded H^∞ -calculus on the group von Neuman algebra $\mathcal{L}\mathbb{F}_n$. In the rest of this section, we will show that $S_t^r, 0 < r < 1$ has a bounded H^∞ -calculus on $BMO^{\frac{1}{2}}(\mathbb{F}_n)$.

Proposition 5.1. *Suppose T is a sectorial operator on a Banach space X . Assume $\int_0^\infty T e^{-tT} a(t) dt$ is bounded on X with norm smaller than C for any choice $a(t) = \pm 1$. Then T has a bounded $H^\infty(S_\eta^0)$ calculus for any $\eta > \pi/2$.*

Proof. This is a consequence of Example 4.8 of [4] by setting $a(t)$ to be the sign of $\langle T e^{-tT} u, v \rangle$ for any pair $(u, v) \in (X, X^*)$. \square

Proposition 5.2. *Suppose $a(t)$ is a function on $(0, \infty)$ satisfying*

$$(17) \quad s \int_s^\infty \frac{|a(t-s)|^2}{t^2} dt \leq c_a^2$$

for any $s > 0$. Then $\int_0^\infty \mathcal{L}^{\frac{1}{2}} e^{-t\mathcal{L}^{\frac{1}{2}}} a(t) dt$ is completely bounded on $BMO^{\frac{1}{2}}(\mathbb{F}_n)$ with upper bound $\lesssim c_a$.

Proof. We apply Corollary 3.4 of [14] to S_t^1 . Note that the subordinated Poisson semigroup of S_t^1 is $S_t^{\frac{1}{2}}$. So the space $BMO(\mathcal{P})$ associated with S_t^1 as defined in [14] is the space $BMO^{\frac{1}{2}}$ defined in this section. Corollary 3.4 of [14] then implies Proposition 5.1. Because the required $\Gamma^2 \geq 0$ condition associated with S_t^1 is actually the positive definiteness of the kernel $K(g, h) = \left(\frac{|g|+|h|-|g^{-1}h|}{2}\right)^2$ on $\mathbb{F}_n \times \mathbb{F}_n$, which easily follows from the negative definiteness of the length function $|\cdot|$. \square

Remark 5.3. There are a few misprints in [14]. The condition of $a(t)$ on page 710 is incorrectly written. The correct one is (17) in this article. In Theorem 3.3 of [14], n must be a strictly positive integer.

Theorem 5.4. *For $0 < r < 1$, $\mathcal{L}^r : \lambda_g \mapsto |g|^r \lambda_g$ has a bounded $H^\infty(S_\eta^0)$ calculus on $BMO^{\frac{1}{2}}(\hat{\mathbb{F}}_n)$ for any $\eta > r\pi$.*

Proof. It is easy to see that S_t^1 is a bounded semigroup on $BMO^{\frac{1}{2}}$. So \mathcal{L}^r is a sectorial operator on $BMO^{\frac{1}{2}}$ of type $\frac{r\pi}{2}$ for $0 < r < 1$. Applying Proposition 5.2 to Proposition 5.1, for $|a(t)| = 1$ and $T = \mathcal{L}^{\frac{1}{2}}$ we conclude that $\mathcal{L}^{\frac{1}{2}}$ has a bounded $H^\infty(S_\eta^0)$ calculus on $BMO^{\frac{1}{2}}$ for any $\eta > \pi/2$. Therefore, \mathcal{L}^r has a bounded $H^\infty(S_\eta^0)$ calculus on $BMO^{\frac{1}{2}}$ for any $\eta > r\pi$. \square

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