TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 369, Number 8, August 2017, Pages 5425–5466
http://dx.doi.org/10.1090/tran/6827
Article electronically published on February 13, 2017

REDUCTION MODULO p OF CERTAIN SEMI-STABLE REPRESENTATIONS

CHOL PARK

ABSTRACT. Let $p > 3$ be a prime number and let $G_{\mathbb{Q}_p}$ be the absolute Galois group of $\mathbb{Q}_p$. In this paper, we find Galois stable lattices in the 3-dimensional irreducible semi-stable non-crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, 1, 2)$ by constructing the corresponding strongly divisible modules. We also compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules and determine which of the original representations has an absolutely irreducible mod p reduction.

CONTENTS

1. INTRODUCTION

Let $p > 3$ be a prime number and let E be a finite extension of $\mathbb{Q}_p$. We write $G_{\mathbb{Q}_p}$ for the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and $I_{\mathbb{Q}_p}$ for the inertia subgroup of $G_{\mathbb{Q}_p}$. In this paper, we construct strongly divisible modules for the admissible filtered (ϕ, N)-modules that correspond to the 3-dimensional irreducible semi-stable non-crystalline E-representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, 1, 2)$. By a result of Liu [Liu], this is equivalent to constructing Galois stable lattices in the semi-stable representations. We also compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules to determine which of the semi-stable representations has an absolutely irreducible mod p reduction (cf. Theorems 7.1 and 7.2). As a consequence, if $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ is an irreducible mod p reduction of a semi-stable non-crystalline representation with Hodge–Tate weights $(0, 1, 2)$, then $\bar{\rho}|_{I_{\mathbb{Q}_p}}$ is isomorphic to either

$$\omega_3^{2p+1} \oplus \omega_3^{2p^2+p} \oplus \omega_3^{2+p^2} \quad \text{or} \quad \omega_3^{p+2} \oplus \omega_3^{p^2+2p} \oplus \omega_3^{1+2p^2}$$

where ω_3 is a fundamental character of level 3 (cf. Theorem 7.3).

Received by the editors July 20, 2014 and, in revised form, August 27, 2015.
2010 *Mathematics Subject Classification.* Primary 11F80.

In the paper [Par], all the 3-dimensional semi-stable E-representations of $G_{\mathbb{Q}_p}$ with regular Hodge–Tate weights have been classified by determining the admissible filtered (ϕ, N)-modules of Hodge–Tate weights $(0, r, s)$ for $0 < r < s$. There are 49 families of admissible filtered (ϕ, N)-modules of dimension 3. Among them, there are 26 families with $N = 0$ (i.e., the crystalline case), there are 20 families with $\mathrm{rank}(N) = 1$, and there are 3 families with $\mathrm{rank}(N) = 2$. However, if we restrict our attention to those families which contain representations that are irreducible and of Hodge–Tate weights $(0, 1, 2)$, there are only 11 families for $N = 0$ and 7 families for $\mathrm{rank}(N) = 1$; there are none for $\mathrm{rank}(N) = 2$. Since we are concerned only with the case of absolutely irreducible residual representations and since the crystalline deformation rings are already determined (cf. [CHT], Corollary 2.4.3), the 7 families of $\mathrm{rank}(N) = 1$,

$$D^4_{\mathrm{rk}N=1}, D^6_{\mathrm{rk}N=1}, D^{10}_{\mathrm{rk}N=1}, D^{12}_{\mathrm{rk}N=1}, D^{17}_{\mathrm{rk}N=1}, D^{18}_{\mathrm{rk}N=1}, D^{20}_{\mathrm{rk}N=1} \tag{1.1}$$

(defined in [Par]), are the ones we will consider in this paper.

Finding a strongly divisible module for a given admissible filtered (ϕ, N)-module is in general very subtle and difficult even when Hodge–Tate weights are small. An iterative process for the construction of strongly divisible modules is given in [Bre99], but it is rather elaborate to execute in practice (and much more so in dimension 3 than in dimension 2). Some of the families listed above can be expected to be more difficult than others and to exhibit new features that do not occur in the GL_2-setting. For instance, there are families with two $\mathfrak{L}$-invariants in the filtration, and our construction will produce strongly divisible modules that have coefficients defined as limits of sequences in E, which depend on the values of the parameters in the families of admissible filtered (ϕ, N)-modules.

This paper is organized as follows. In the remainder of the introduction, we give a brief review of p-adic Hodge theory (filtered (ϕ, N)-modules, strongly divisible modules, and Breuil modules) and introduce notation that will be used throughout the paper. In Section 2, we study some examples of Breuil modules which occur as mod p reductions of semi-stable representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, 1, 2)$. In Section 3, we glue the seven families of admissible filtered (ϕ, N)-modules of $\mathrm{rank}(N) = 1$ together so that, as a consequence, there are two families $D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$ that parameterize all the irreducible semi-stable non-crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, 1, 2)$. In Section 4, we define various sequences and study their properties. The limits of these sequences will appear in the coefficients of our strongly divisible modules as we will see in Sections 5 and 6. In Section 5, we construct strongly divisible modules of $D_{[0,\frac{1}{2}]}$ and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules. We first divide the area in which the parameters of $D_{[0,\frac{1}{2}]}$ are defined into three pieces, and then for each case we perform the tasks we described in the previous sentence. We do similar things for $D_{[\frac{1}{2},1]}$ in Section 6. In Section 7, we state and prove the main results in this paper. We determine which admissible filtered (ϕ, N)-modules correspond to the representations whose mod p reductions are absolutely irreducible, and we also determine the irreducible mod p reductions of semi-stable representations with Hodge–Tate weights $(0, 1, 2)$.

1.1. **Review of p-adic Hodge theory.** In this subsection, we quickly review filtered (ϕ, N)-modules, strongly divisible modules, and Breuil modules. We closely follow the exposition of [BM, EGH] and we refer the reader to those papers for more

detail. Let K and E be finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ and K_0 the maximal absolutely unramified subextension of K. We also let k be the residue field of K and write $G_K = \mathrm{Gal}(\overline{\mathbb{Q}}_p/K)$ for the absolute Galois group of K.

1.1.1. *Filtered (ϕ, N)-modules.* We fix the uniformizer $p \in \mathbb{Q}_p$, thereby fixing an inclusion $\mathbf{B}_{\mathrm{st}} \to \mathbf{B}_{\mathrm{dR}}$. (See [Fon] for details.) A *filtered (ϕ, N)-module* (strictly speaking, a *filtered (ϕ, N, K, E)-module*) is a free $K_0 \otimes_{\mathbb{Q}_p} E$-module D of finite rank together with a triple $(\phi, N, \{\mathrm{Fil}^i D_K\}_{i\in\mathbb{Z}})$ where

- the *Frobenius map* $\phi : D \to D$ is a Frobenius-semilinear and E-linear automorphism;
- the *monodromy operator* $N : D \to D$ is a (nilpotent) $K_0 \otimes_{\mathbb{Q}_p} E$-linear endomorphism such that $N\phi = p\phi N$;
- the *Hodge filtration* $\{\mathrm{Fil}^i D_K\}_{i\in\mathbb{Z}}$ is a decreasing filtration on $D_K := K \otimes_{K_0} D$ such that a $K \otimes_{\mathbb{Q}_p} E$-submodule $\mathrm{Fil}^i D_K$ is D_K if $i \ll 0$ and 0 if $i \gg 0$.

The morphisms of filtered (ϕ, N)-modules are $K_0 \otimes_{\mathbb{Q}_p} E$-module homomorphisms that commute with ϕ and N and that preserve the filtration.

A filtered (ϕ, N)-module D is said to be *admissible* if it is in the sense of [BM]. The *Hodge–Tate weights* of a filtered (ϕ, N)-module D are the integers r such that $\mathrm{Fil}^r D_K \neq \mathrm{Fil}^{r+1} D_K$, each counted with multiplicity $\dim_E(\mathrm{Fil}^r D_K/\mathrm{Fil}^{r+1} D_K)$. We say that a filtered (ϕ, N)-module is *positive* if the lowest Hodge–Tate weight is greater than or equal to 0.

Let V be a finite-dimensional E-vector space equipped with continuous action of G_K, and define

$$\mathrm{D}_{\mathrm{st}}(V) := (\mathbf{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

Then $\dim_{K_0} \mathrm{D}_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p} V$. If the equality holds, then we say that V is *semi-stable*; in that case $\mathrm{D}_{\mathrm{st}}(V)$ inherits from $\mathbf{B}_{\mathrm{st}}$ the structure of an admissible filtered (ϕ, N)-module. We say that V is *crystalline* if V is semi-stable and the monodromy operator N on $\mathrm{D}_{\mathrm{st}}(V)$ is 0. Following Colmez and Fontaine [CF], the functor D_{st} provides an equivalence between the category of semi-stable E-representations of G_K and the category of admissible filtered (ϕ, N, K, E)-modules.

If V is a finite-dimensional vector space over E equipped with a continuous action of G_K, we let V^* be the dual representation of G_K. V is semi-stable (resp., crystalline) if and only if so is V^*. If we denote $\mathrm{D}^*_{\mathrm{st}}(V) := \mathrm{D}_{\mathrm{st}}(V^*)$, then the functor $\mathrm{D}^*_{\mathrm{st}}$ gives rise to an anti-equivalence between the category of semi-stable E-representations of G_K and the category of admissible filtered (ϕ, N, K, E)-modules. The quasi-inverse to $\mathrm{D}^*_{\mathrm{st}}$ is given by

$$\mathrm{V}^*_{\mathrm{st}}(D) := \mathrm{Hom}_{\phi,N}(D, \mathbf{B}_{\mathrm{st}}) \cap \mathrm{Hom}_{\mathrm{Fil}^*}(D_K, K \otimes_{K_0} \mathbf{B}_{\mathrm{st}})$$

(that is, the homomorphisms of $K_0 \otimes E$-modules that commute with ϕ and N and that preserve filtration). $\mathrm{V}^*_{\mathrm{st}}(D)$ inherits an E-module structure from the E-module structure on D and an action of G_K from the action of G_K on $\mathbf{B}_{\mathrm{st}}$.

If V is semi-stable, then when we refer to the Hodge–Tate weights of V, we mean those of $\mathrm{D}^*_{\mathrm{st}}(V)$. Our normalization implies that the cyclotomic character $\varepsilon : G_{\mathbb{Q}_p} \to E^\times$ has Hodge–Tate weight 1. Twisting V by a power ε^n of the cyclotomic character has the effect of shifting all the Hodge–Tate weights of V by n; after a suitable twist, we are therefore free to assume that the lowest Hodge–Tate weight is 0.

1.1.2. *Strongly divisible modules.* We fix a uniformizer π_K in K and let $W(k)$ be the ring of Witt vectors over k so that $K_0 = W(k)[\frac{1}{p}]$. Let $E(u) \in W(k)[u]$ be the minimal polynomial of π_K over K_0 and let S be the p-adic completion of $W(k)[u, \frac{u^{ie}}{i!}]_{i\in\mathbb{N}}$, where e is the absolute ramification index of K. We endow S with the following structure:

- a continuous Frobenius-semilinear map $\phi : S \to S$ with $\phi(u) = u^p$;
- a continuous $W(k)$-linear derivation $N : S \to S$ with $N(u) = -u$ and $N(u^{ie}/i!) = -ieu^{ie}/i!$;
- a decreasing filtration $\{\mathrm{Fil}^i S\}_{i\in\mathbb{N}_0}$ where $\mathrm{Fil}^i S$ is the p-adic completion of $\sum_{j\geq i} \frac{E(u)^j}{j!} S$.

Note that $N\phi = p\phi N$ and $\phi(\mathrm{Fil}^i S) \subset p^i S$ for $0 \leq i \leq p-1$.

Let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$. We also let $S_{\mathcal{O}_E} := S \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ and $S_E := S_{\mathcal{O}_E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and extend the definitions of Fil, ϕ, and N to $S_{\mathcal{O}_E}$ and S_E by $\mathcal{O}_E$-linearly and E-linearly, respectively. Let $\mathcal{MF}(\phi, N, K, E)$ be the category whose objects are finite free S_E-modules $\mathfrak{D}$ with

- a ϕ-semilinear and E-linear morphism $\phi : \mathfrak{D} \to \mathfrak{D}$ such that the determinant of ϕ with respect to some choice of $S_{\mathbb{Q}_p}$-basis is invertible in $S_{\mathbb{Q}_p}$ (which does not depend on the choice of basis);
- a decreasing filtration of $\mathfrak{D}$ by S_E-submodules $\mathrm{Fil}^i\mathfrak{D}$, $i \in \mathbb{Z}$, with $\mathrm{Fil}^i\mathfrak{D} = \mathfrak{D}$ for $i \leq 0$ and $\mathrm{Fil}^i S_E \cdot \mathrm{Fil}^j\mathfrak{D} \subset \mathrm{Fil}^{i+j}\mathfrak{D}$ for all j and all $i \geq 0$;
- a $K_0 \otimes E$-linear map $N : \mathfrak{D} \to \mathfrak{D}$ such that
  - $N(sx) = N(s)x + sN(x)$ for all $s \in S_E$ and $x \in \mathfrak{D}$,
  - $N\phi = p\phi N$,
  - $N(\mathrm{Fil}^i\mathfrak{D}) \subset \mathrm{Fil}^{i-1}\mathfrak{D}$ for all i.

For a filtered (ϕ, N)-module D with positive Hodge–Tate weights, one can associate an object $\mathfrak{D} \in \mathcal{MF}(\phi, N, K, E)$ by the following:

- $\mathfrak{D} := S \otimes_{W(k)} D$;
- $\phi := \phi \otimes \phi : \mathfrak{D} \to \mathfrak{D}$;
- $N := N \otimes \mathrm{Id} + \mathrm{Id} \otimes N : \mathfrak{D} \to \mathfrak{D}$;
- $\mathrm{Fil}^0\mathfrak{D} := \mathfrak{D}$ and, by induction,

$$\mathrm{Fil}^{i+1}\mathfrak{D} := \{x \in \mathfrak{D} | N(x) \in \mathrm{Fil}^i\mathfrak{D} \text{ and } f_{\pi_K}(x) \in \mathrm{Fil}^{i+1}D_K\}$$

  where $f_{\pi_K} : \mathfrak{D} \twoheadrightarrow D_K$ is defined by $s(u) \otimes x \mapsto s(\pi_K)x$.

The functor $\mathfrak{D} : D \mapsto S \otimes_{W(k)} D$ gives rise to an equivalence between the category of positive filtered (ϕ, N)-modules and $\mathcal{MF}(\phi, N, K, E)$, by a result of Breuil [Bre97].

Fix a positive integer $r \leq p-2$. The category $\mathfrak{MD}^r_{\mathcal{O}_E}$ of *strongly divisible modules of weight* r is defined to be the category of free $S_{\mathcal{O}_E}$-modules $\mathfrak{M}$ of finite rank with an $S_{\mathcal{O}_E}$-submodule $\mathrm{Fil}^r\mathfrak{M}$ and additive maps $\phi, N : \mathfrak{M} \to \mathfrak{M}$ such that the following properties hold:

- $\mathrm{Fil}^r S_{\mathcal{O}_E} \cdot \mathfrak{M} \subset \mathrm{Fil}^r\mathfrak{M}$;
- $\mathrm{Fil}^r\mathfrak{M} \cap I\mathfrak{M} = I\mathrm{Fil}^r\mathfrak{M}$ for all ideals I in $\mathcal{O}_E$;
- $\phi(sx) = \phi(s)\phi(x)$ for all $s \in S_{\mathcal{O}_E}$ and for all $x \in \mathfrak{M}$;
- $\phi(\mathrm{Fil}^r\mathfrak{M})$ is contained in $p^r\mathfrak{M}$ and generates it over $S_{\mathcal{O}_E}$;
- $N(sx) = N(s)x + sN(x)$ for all $s \in S_{\mathcal{O}_E}$ and for all $x \in \mathfrak{M}$;
- $N\phi = p\phi N$;
- $E(u)N(\mathrm{Fil}^r\mathfrak{M}) \subset \mathrm{Fil}^r\mathfrak{M}$.

The morphisms are $S_{\mathcal{O}_E}$-linear maps that preserve Fil^r and commute with ϕ and N. For a strongly divisible module $\mathfrak{M}$ of weight r, there exists a unique admissible filtered (ϕ, N)-module D with Hodge–Tate weights lying in $[0, r]$ such that $\mathfrak{M}[\frac{1}{p}] \simeq S \otimes_{W(k)} D$, so one has the following equivalent definition: let D be an admissible filtered (ϕ, N)-module such that $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^{r+1} D_K = 0$. *A strongly divisible module* in $\mathfrak{D} := \mathfrak{D}(D)$ is an $S_{\mathcal{O}_E}$-submodule $\mathfrak{M}$ of $\mathfrak{D}$ such that

- $\mathfrak{M}$ is a free $S_{\mathcal{O}_E}$-module of finite rank such that $\mathfrak{M}[\frac{1}{p}] \simeq \mathfrak{D}$;
- $\mathfrak{M}$ is stable under ϕ and N;
- $\phi(\mathrm{Fil}^r \mathfrak{M}) \subset p^r \mathfrak{M}$ where $\mathrm{Fil}^r \mathfrak{M} := \mathfrak{M} \cap \mathrm{Fil}^r \mathfrak{D}$.

For a strongly divisible module $\mathfrak{M}$, we define an $\mathcal{O}_E[G_K]$-module $\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M})$ as follows:

$$\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M}) := \mathrm{Hom}_{S, \mathrm{Fil}^r, \phi, N}(\mathfrak{M}, \widehat{\mathbf{A}}_{\mathrm{st}}).$$

(See [Bre99] for details.) The functor $\mathrm{T}^*_{\mathrm{st}}$ provides an anti-equivalence of categories between the category $\mathfrak{MD}^r_{\mathcal{O}_E}$ of strongly divisible modules of weight r and the category of G_K-stable $\mathcal{O}_E$-lattices in semi-stable E-representations of G_K with Hodge–Tate weights lying in $[0, r]$, provided that $0 \leq r \leq p-2$. Moreover, there is a compatibility: if $\mathfrak{M}$ is a strongly divisible module in $\mathfrak{D} := \mathfrak{D}(D)$ for an admissible filtered (ϕ, N)-module D, then $\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M})$ is a Galois stable $\mathcal{O}_E$-lattice in $\mathrm{V}^*_{\mathrm{st}}(D)$. This was conjectured by Breuil and proved by Liu [Liu] in the case $E = \mathbb{Q}_p$; Emerton–Gee–Herzig [EGH] gave the (essentially formal) generalization to the case of E-coefficients.

1.1.3. *Breuil modules.* Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$, k an algebraic extension of $\mathbb{F}_p$, and $e \in \mathbb{N}$. The category $\mathrm{BrMod}^r_{\mathbb{F}}$ of *Breuil modules of weight* r consists of quadruples $(\mathcal{M}, \mathrm{Fil}^r \mathcal{M}, \phi_r, N)$ where

- $\mathcal{M}$ is a finitely generated $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$-module, free over $k[u]/u^{ep}$ (which implies that $\mathcal{M}$ is in fact a free $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$-module of finite rank);
- $\mathrm{Fil}^r \mathcal{M}$ is a $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$-submodule of $\mathcal{M}$ containing $u^{er}\mathcal{M}$;
- $\phi_r : \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$ is $\mathbb{F}$-linear and ϕ-semilinear (where $\phi : k[u]/u^{ep} \to k[u]/u^{ep}$ is the p-th power map) with image generating $\mathcal{M}$ as a $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$-module;
- $N : \mathcal{M} \to \mathcal{M}$ is $k \otimes_{\mathbb{F}_p} \mathbb{F}$-linear and satisfies
  - $N(ux) = uN(x) - ux$ for all $x \in \mathcal{M}$,
  - $u^e N(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^r \mathcal{M}$, and
  - $\phi_r(u^e N(x)) = cN(\phi_r(x))$ for all $x \in \mathrm{Fil}^r \mathcal{M}$, where $c \in (k[u]/u^{ep})^\times$ is the image of $\frac{1}{p}\phi(E(u))$ under the natural map $S \to k[u]/u^{ep}$.

The morphisms are $(k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$-module homomorphisms that preserve $\mathrm{Fil}^r \mathcal{M}$ and commute with ϕ_r and N.

Suppose that k (resp., $\mathbb{F}$) is the residue field of K (resp., of E). We also assume $e = [K : K_0]$. If $\mathfrak{M}$ is an object of $\mathfrak{MD}^r_{\mathcal{O}_E}$, then $\mathcal{M} := \mathfrak{M}/(\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$ is naturally an object of $\mathrm{BrMod}^r_{\mathbb{F}}$. More precisely,

$$\mathcal{M} = (\mathfrak{M}/\mathfrak{m}_E \mathfrak{M}) \otimes_{S_{\mathcal{O}_E}/\mathfrak{m}_E S_{\mathcal{O}_E}} (k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}, \tag{1.2}$$

where

- $\mathrm{Fil}^r \mathcal{M}$ is the image of $\mathrm{Fil}^r \mathfrak{M}$ in $\mathcal{M}$;
- the map ϕ_r is induced by $\frac{1}{p^r}\phi|_{\mathrm{Fil}^r \mathfrak{M}}$;
- N is induced by the one on $\mathfrak{M}$.

Note that this association gives rise to a functor from the category $\mathfrak{MD}^r_{\mathcal{O}_E}$ to $\mathrm{BrMod}^r_{\mathbb{F}}$.

For a Breuil module $\mathcal{M}$, we define a $\mathbb{F}[G_K]$-module $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ as follows:

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{k[u]/u^{ep},\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, \widehat{\mathbf{A}}).$$

(See [EGH] for details.) $\mathrm{T}^*_{\mathrm{st}}$ gives rise to a faithful contravariant functor from the category $\mathrm{BrMod}^r_{\mathbb{F}}$ to the category of finite-dimensional $\mathbb{F}$-representations of G_K with $\dim_{\mathbb{F}} \mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) = \mathrm{rank}_{(k\otimes_{\mathbb{F}_p}\mathbb{F})[u]/u^{ep}}\mathcal{M}$. The functor is full as well if $er < p-1$. Moreover, there is a compatibility, that is, if $\mathfrak{M} \in \mathfrak{MD}^r_{\mathcal{O}_E}$ and $\mathcal{M} := (\mathfrak{M}/\mathfrak{m}_E\mathfrak{M}) \otimes_{S_{\mathcal{O}_E}/\mathfrak{m}_E S_{\mathcal{O}_E}} (k \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^{ep}$ denotes the Breuil module corresponding to the reduction of $\mathfrak{M}$, then $\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M}) \otimes_{\mathcal{O}_E} \mathbb{F} \simeq \mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$.

We say a morphism of Breuil modules $f : \mathcal{M} \to \mathcal{M}'$ is a *quotient map* if $f(\mathrm{Fil}^r\mathcal{M}) = \mathrm{Fil}^r\mathcal{M}'$. If $f : \mathcal{M} \twoheadrightarrow \mathcal{M}'$ is a quotient map of Breuil modules, then it is clear that $\mathrm{T}^*_{\mathrm{st}}(f) : \mathrm{T}^*_{\mathrm{st}}(\mathcal{M}') \hookrightarrow \mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$, i.e., $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}')$ is a subrepresentation of $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$. Moreover, the converse is also true, due to Proposition 3.2.6 in [EGH]: if $\mathcal{M} \in \mathrm{BrMod}^r_{\mathbb{F}}$ and if T' is a subrepresentation of $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$, then there is a unique quotient map $f : \mathcal{M} \twoheadrightarrow \mathcal{M}'$ in $\mathrm{BrMod}^r_{\mathbb{F}}$ such that $\mathrm{T}^*_{\mathrm{st}}(f)$ is identified with the inclusion $T' \hookrightarrow \mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$.

1.2. **Notation.** We let S be the p-adic completion of $\mathbb{Z}_p[\frac{u^i}{i!}]_{i\in\mathbb{N}}$ since we are concerned only with representations of $G_{\mathbb{Q}_p}$ in this paper. We fix a prime number p so that we fix an embedding $\mathbf{B}_{\mathrm{st}} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$, and we let $E(u) := u - p \in S$. We assume that $p > 3$ since we are concerned with strongly divisible modules of weight 2.

We let E be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_E$, maximal ideal $\mathfrak{m}_E$, and residue field $\mathbb{F}$: the field E is the coefficients of our semi-stable representations, $S_{\mathcal{O}_E} := S \otimes \mathcal{O}_E$ is the coefficient of our strongly divisible modules, and $\overline{S} := \mathbb{F}[u]/u^p$ is the coefficient of our Breuil modules. We also let $S_E := S_{\mathcal{O}_E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

We write $\overline{a} \in \mathbb{F}$ for the image of $a \in \mathcal{O}_E$ under the fixed quotient map $\mathcal{O}_E \to \mathbb{F}$, and let v_p be the valuation on $\overline{\mathbb{Q}}_p$ with $v_p(p) = 1$. We let

$$\gamma := \frac{(u-p)^p}{p} \in S.$$

It is easy to check that $\phi(\gamma) \in p^{p-1}S$ and $\phi(\frac{u-p}{p}) = \frac{u^p-p}{p} \equiv \gamma - 1$ modulo pS. It is also straightforward to check that $N(\gamma) = -p[\gamma + (u-p)^{p-1}]$. γ will appear in the coefficients of our strongly divisible modules.

Let R be a commutative ring with unity. (For example, $R = E$, $R = S_{\mathcal{O}_E}$, or $R = \overline{S}$.) By $M = R(E_1, \ldots, E_n)$, we mean that M is a free module over R of rank n with a basis $\underline{e} := (E_1, \ldots, E_n)$. If $f : M \to M$ is an R-module homomorphism, then we define an $n \times n$-matrix $\mathrm{Mat}_{\underline{e}}(f)$ by the following equation:

$$(f(E_1), \ldots, f(E_n)) = (E_1, \ldots, E_n) \cdot \mathrm{Mat}_{\underline{e}}(f).$$

Let $\mathcal{M}$ be a Breuil module of weight r over $\mathbb{F}[u]/u^p$ with a basis $\underline{e} = (E_1, \ldots, E_n)$ and let $\underline{f} := (f_1, \ldots, f_n)$ be a system of generators for $\mathrm{Fil}^r\mathcal{M}$ modulo $\mathrm{Fil}^r S \cdot \mathcal{M}$. We define an $n \times n$-matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M})$ by the equation

$$(f_1, \ldots, f_n) = (E_1, \ldots, E_n) \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M}).$$

Similarly, for Frobenius morphism $\phi_r : \mathrm{Fil}^r\mathcal{M} \to \mathcal{M}$, we define an $n \times n$-matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r)$ by the equation

$$(\phi_r(f_1), \ldots, \phi_r(f_n)) = (E_1, \ldots, E_n) \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r).$$

2. Examples of Breuil modules of weight 2

In this section, we provide some examples of Breuil modules which (as we will see later) occur as mod p reductions of semi-stable representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0,1,2)$.

2.1. **Simple Breuil modules.** The Breuil modules introduced in this subsection correspond to absolutely irreducible mod p representations of $G_{\mathbb{Q}_p}$. We prove it by showing that the restriction of the corresponding representations to the inertia subgroup $I_{\mathbb{Q}_p}$ of $G_{\mathbb{Q}_p}$ is of niveau 3.

Example 2.1. Let $s := (1,2,3)$ be a cycle of length 3 in the symmetric group S_3. For $i = 1,2$ and for a,b,c in $\mathbb{F}^\times$, the Breuil module $\mathcal{M}(s^i,a,b,c)$ is defined as follows: there exist a basis $\underline{e} := (E_1,E_2,E_3)$ for $\mathcal{M}(s^i,a,b,c)$ and a system of generators $\underline{f} := (f_1,f_2,f_3)$ for $\mathrm{Fil}^2\mathcal{M}$ such that

- $\mathcal{M} := \overline{S}(E_1,E_2,E_3)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u^2 & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and
  $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^i \cdot \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$;
- $N : \mathcal{M} \to \mathcal{M}$ is induced by $N(E_1) = N(E_2) = N(E_3) = 0$.

Lemma 2.2. *$\mathcal{M}(s^i,a,b,c)$ is isomorphic to $\mathcal{M}(s^j,\alpha,\beta,\gamma)$ if and only if $i = j$ and $abc = \alpha\beta\gamma$.*

Proof. It is easy to check that if $i \neq j$, then the only morphism between $\mathcal{M}(s^i,a,b,c)$ and $\mathcal{M}(s^j,\alpha,\beta,\gamma)$ is the zero map from the commutativity with ϕ_2. If $i = j$, then the commutativity with ϕ_2 also implies that the morphism is of the form $E_1 \mapsto xE_1, E_2 \mapsto yE_2, E_3 \mapsto zE_3$ for $x,y,z \in \mathbb{F}$. If $i = j = 1$, then, from the commutativity with ϕ_2 again, we have equations $\alpha x = ay, \beta y = bz, \gamma z = cx$, which implies $\alpha\beta\gamma = abc$ if we assume that the morphism is an isomorphism. It is easy to check that the morphism commutes with N since $N(E_1) = N(E_2) = N(E_3) = 0$ and $x,y,z \in \mathbb{F}^\times$. Similarly, one can also get the same result when $i = j = 2$.

Conversely, assume that $abc = \alpha\beta\gamma$. If $i = j = 1$, then the association $E_1 \mapsto E_1, E_2 \mapsto \frac{\alpha}{a}E_2, E_3 \mapsto \frac{\alpha\beta}{ab}E_3$ gives rise to an isomorphism from $\mathcal{M}(s^i,a,b,c)$ to $\mathcal{M}(s^j,\alpha,\beta,\gamma)$, and if $i = j = 2$, then the association $E_1 \mapsto E_1, E_2 \mapsto \frac{b}{\beta}E_2, E_3 \mapsto \frac{bc}{\beta\gamma}E_3$ does so. $\square$

We use Theorem 5.2.2 in [Car] to prove that for the Breuil modules $\mathcal{M}$ in Example 2.1 $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ is absolutely irreducible. To use the theorem, we need a little preparation. Let $n = [\mathbb{F} : \mathbb{F}_p]$ and σ be the absolute arithmetic Frobenius on $\mathbb{F}$, so that the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_p)$ consists of σ^i for $i = 0,1,\ldots,n-1$. The association $k \otimes e \mapsto (k \cdot \sigma^i(e))_i$ gives rise to an isomorphism $\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathbb{F} \underset{\sim}{\longrightarrow} \bigoplus_i \overline{\mathbb{F}}_p$ as i ranges over the integers from 0 to $n-1$. Note that ϕ_r acts on $\overline{\mathbb{F}}_p$ Frobenius-semilinearly and on $\mathbb{F}$ linearly for the Breuil modules over $(\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^p$:

$$\begin{array}{ccccc} (\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^p & \hookleftarrow & \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathbb{F} & \underset{\sim}{\longrightarrow} & \bigoplus_i \overline{\mathbb{F}}_p \\ \circlearrowleft \phi_r & & \circlearrowleft \phi_r & & \circlearrowleft \phi_r \end{array}.$$

We first investigate the action of ϕ_r on $\bigoplus_i \overline{\mathbb{F}}_p$ under the isomorphism above.

Lemma 2.3. *If $(x_i)_i \in \bigoplus_i \overline{\mathbb{F}}_p$, then $\phi_r((x_i)_i) = (x_{i-1}^p)_i$.*

Proof. If $x \otimes y$ is in $\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathbb{F}$, then $\phi_r(x \otimes y) = x^p \otimes y$. Hence, we have $\phi_r((x\sigma^i(y))_i) = (x^p \sigma^i(y))_i = ((x\sigma^{i-1}(y))^p)_i$. The fact that field $\overline{\mathbb{F}}_p$ has characteristic p completes the proof. □

We let $\overline{\mathbb{F}}_p\mathcal{M}(s^i, a, b, c) := \overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathcal{M}(s^i, a, b, c)$ and extend ϕ_2 ϕ-semilinearly and N linearly on $\overline{\mathbb{F}}_p$. Then $\overline{\mathbb{F}}_p\mathcal{M}(s^i, a, b, c)$ is a Breuil module with $(\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^p$-coefficients.

Lemma 2.4. *For all $a, b, c \in \mathbb{F}^\times$, $\overline{\mathbb{F}}_p\mathcal{M}(s^i, a, b, c)$ is isomorphic to $\overline{\mathbb{F}}_p\mathcal{M}(s^i, 1, 1, 1)$ if $[\mathbb{F} : \mathbb{F}_p] = 3m$.*

Proof. By Lemma 2.2, it is enough to show that $\overline{\mathbb{F}}_p\mathcal{M}(s^i, 1, 1, \alpha)$ is isomorphic to $\overline{\mathbb{F}}_p\mathcal{M}(s^i, 1, 1, 1)$ We only prove the case $i = 1$, and the case $i = 2$ is similar. We let f be a morphism from $\overline{\mathbb{F}}_p\mathcal{M}(s, 1, 1, \alpha)$ to $\overline{\mathbb{F}}_p\mathcal{M}(s, 1, 1, 1)$ denoted by

$$E_1 \mapsto (x_i)_i E_1, E_2 \mapsto (y_i)_i E_2, E_3 \mapsto (z_i)_i E_3$$

for $x_i, y_i, z_i \in \overline{\mathbb{F}}_p$. Then, using the action in Lemma 2.3, one can check that f commutes with ϕ_2 if and only if x_i, y_i, z_i satisfy the equations $x_{i-1}^p = y_i$, $y_{i-1}^p = z_i$, and $z_{i-1}^p = \alpha^{p^i} x_i$ for $i \in \mathbb{Z}/n\mathbb{Z}$. But it is easy to check that this system of equations has solutions if and only if x_1, y_1, z_1 satisfy the equations

$$x_1^{p^{3m}} = \alpha^{mp^{3m+1}} x_1,\ y_1^{p^{3m}} = \alpha^{mp^{3m+1}} y_1,\ z_1^{p^{3m}} = \alpha^{mp^{3m+1}} z_1,$$

under the assumption $[\mathbb{F} : \mathbb{F}_p] = 3m$. It is also easy to check that the map f commutes with N since $N(E_1) = N(E_2) = N(E_3) = 0$. □

Lemma 2.4 says that for each $i \in \{1, 2\}$,

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}(s^i, a, b, c))|_{I_{\mathbb{Q}_p}} \simeq \mathrm{T}^*_{\mathrm{st}}(\mathcal{M}(s^i, 1, 1, 1))|_{I_{\mathbb{Q}_p}}$$

for all $a, b, c \in \mathbb{F}^\times$. Hence, it is enough to consider only $\mathcal{M}(s^i, 1, 1, 1)$ to show that $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}(s^i, a, b, c))$ is absolutely irreducible for all $a, b, c \in \mathbb{F}^\times$ and for all $i \in \{1, 2\}$. We let $\mathcal{M}^i := \mathcal{M}(s^i, 1, 1, 1)$ to lighten the notation.

Proposition 2.5. *Let $\bar{\rho} := \mathrm{T}^*_{\mathrm{st}}(\mathcal{M}(s^i, a, b, c)) \otimes_{\mathbb{F}} \overline{\mathbb{F}}_p$. Then*

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \simeq \begin{cases} \omega_3^{2p+1} \oplus \omega_3^{2p^2+p} \oplus \omega_3^{2+p^2} & \text{if } i = 1; \\ \omega_3^{p+2} \oplus \omega_3^{p^2+2p} \oplus \omega_3^{1+2p^2} & \text{if } i = 2, \end{cases}$$

where ω_3 is the fundamental character of level 3. In particular, $\bar{\rho}$ is absolutely irreducible.

Proof. Assume that $3|[\mathbb{F} : \mathbb{F}_p]$ and let $\sigma_0, \sigma_1, \sigma_2$ be the standard idempotents in $\mathbb{F}_{p^3} \otimes \mathbb{F}$ via the isomorphism $\mathbb{F}_{p^3} \otimes \mathbb{F} \simeq \bigoplus_{i=0}^2 \mathbb{F} : a \otimes b \mapsto (\sigma^i(a)b)_i$. Let $\mathbb{F}_{p^3}\mathcal{M}^i := \mathbb{F}_{p^3} \otimes_{\mathbb{F}_p} \mathcal{M}^i$ and extend ϕ_2 ϕ-semilinearly and N linearly on $\mathbb{F}_{p^3}$. Then $\mathbb{F}_{p^3}\mathcal{M}^i$ is naturally a Breuil module defined over $(\mathbb{F}_{p^3} \otimes_{\mathbb{F}_p} \mathbb{F})[u]/u^p$. Assume first $i = 1$.

We let $m := \sigma_0 E_1 + \sigma_1 E_2 + \sigma_2 E_3$. Then it is easy to check that $\overline{S}_3(m)$ is a Breuil submodule of $\mathbb{F}_{p^3}\mathcal{M}$. Since $\phi_2(u^{2-i}\sigma_i E_{i+1}) = \sigma_{i+1}E_{i+2}$ and $N(\sigma_i E_{i+1}) = 0$ for $i \in \mathbb{Z}/3\mathbb{Z}$ (numbered cyclically), it is harmless to assume $\mathbb{F} = \mathbb{F}_p$, i.e., $\overline{S}_3(m)$ is defined over $(\mathbb{F}_{p^3} \otimes_{\mathbb{F}_p} \mathbb{F}_p)[u]/u^p$. We are now ready to apply Theorem 5.2.2 in [Car]: $\mathrm{T}^*_{\mathrm{st}}(\overline{\mathbb{F}}_p \otimes_{\mathbb{F}_p} \overline{S}_3(m)) \simeq \omega_3^{p+2p^2} : I_{\mathbb{Q}_p} \to \mathbb{F}^\times$, and so $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}(s^1, a, b, c))|_{I_{\mathbb{Q}_p}} \simeq \mathrm{T}^*_{\mathrm{st}}(\mathcal{M}^1)|_{I_{\mathbb{Q}_p}}$ has $\omega_3^{p+2p^2}$ as a subrepresentation. Hence, $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}(s^1, a, b, c))|_{I_{\mathbb{Q}_p}} \simeq \omega_3^{2p+1} \oplus \omega_3^{2p^2+p} \oplus \omega_3^{2+p^2}$. For the case $i = 2$, take $m = \sigma_2 E_1 + \sigma_1 E_2 + \sigma_0 E_3$. The rest of the proof for the case $i = 2$ is identical to the case $i = 1$. □

2.2. **Non-simple Breuil modules.** In this subsection, we introduce a few examples of Breuil modules that correspond to reducible representations of $G_{\mathbb{Q}_p}$.

Example 2.6. For a 3×3 invertible matrix A over $\mathbb{F}$, the Breuil module $\mathcal{M}(A)$ is defined as follows: there exist a basis $\underline{e} := (E_1, E_2, E_3)$ for $\mathcal{M}(A)$ and a system of generators $\underline{f} := (f_1, f_2, f_3)$ for $\mathrm{Fil}^2\mathcal{M}$ such that

- $\mathcal{M} := \overline{S}(E_1, E_2, E_3)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix}$ and $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = A$;
- $N : \mathcal{M} \to \mathcal{M}$ is induced by $N(E_1) = N(E_2) = N(E_3) = 0$.

Proposition 2.7. *The corresponding representations to $\mathcal{M}(A)$ are not absolutely irreducible.*

Proof. Assume that $\mathbb{F}$ is big enough so that the characteristic equation of A^T (the transpose of A) has a solution e in $\mathbb{F}$. Note that $e \neq 0$ since A is invertible. We define Breuil modules $\widetilde{\mathcal{M}} := \overline{S}(\widetilde{E})$ of rank 1 as follows:

- $\mathrm{Fil}^2\widetilde{\mathcal{M}}$ is generated by $u\widetilde{E}$;
- $\phi_2 : \mathrm{Fil}^2\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is induced by $u\widetilde{E} \mapsto e\widetilde{E}$;
- $N : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is induced by $N(\widetilde{E}) = 0$.

Let the column vector $(a, b, c)^T$ be an eigenvector associated with the eigenvalue e. Then the association $E_1 \mapsto a\widetilde{E}$, $E_2 \mapsto b\widetilde{E}$, $E_3 \mapsto c\widetilde{E}$ induces a quotient map from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$. Hence, the corresponding representations are reducible. □

The following two examples also correspond to reducible mod p representations of $G_{\mathbb{Q}_p}$. We prove it by constructing a non-trivial morphism between these two modules.

Example 2.8. For a, b, c, d in $\mathbb{F}^\times$, the Breuil module $\mathcal{M}(a, b, c, d)$ is defined as follows: there exist a basis $\underline{e} := (E_1, E_2, E_3)$ for $\mathcal{M}(a, b, c, d)$ and a system of generators $\underline{f} := (f_1, f_2, f_3)$ for $\mathrm{Fil}^2\mathcal{M}$ such that

- $\mathcal{M} = \overline{S}(E_1, E_2, E_3)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d & u \end{pmatrix}$ and $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & c \\ a & 0 & 0 \end{pmatrix}$;
- $N : \mathcal{M} \to \mathcal{M}$ is induced by $N(E_1) = N(E_2) = N(E_3) = 0$.

Example 2.9. For a, b, c, d in $\mathbb{F}^\times$, the Breuil module $\mathcal{M}'(a, b, c, d)$ is defined as follows: there exist a basis $\underline{e} := (E_1, E_2, E_3)$ for $\mathcal{M}'(a, b, c, d)$ and a system of

generators $\underline{f} := (f_1, f_2, f_3)$ for $\mathrm{Fil}^2\mathcal{M}$ such that

- $\mathcal{M}' = \overline{S}(E_1, E_2, E_3)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u & 0 & 0 \\ du & u^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & 0 & c \\ a & 0 & 0 \\ 0 & b & 0 \end{pmatrix}$;
- $N : \mathcal{M}' \to \mathcal{M}'$ is induced by $N(E_1) = N(E_2) = N(E_3) = 0$.

Proposition 2.10. *There is a non-trivial morphism between $\mathcal{M}(a, b, c, d)$ and $\mathcal{M}'(x, y, z, w)$ if $x = -cdw$. Hence, in particular, both of them correspond to reducible representations.*

Proof. We write E_1', E_2', E_3' for the basis for $\mathcal{M}'(x, y, z, w)$. It is routine to check that the association $E_1' \mapsto 0$, $E_2' \mapsto E_2$, $E_3' \mapsto 0$ gives rise to a morphism from $\mathcal{M}'(x, y, z, w)$ to $\mathcal{M}(a, b, c, d)$ if $x = -cdw$. Moreover, this morphism factors through $\widetilde{\mathcal{M}} \to \mathcal{M} : \widetilde{E} \mapsto E_2$ if $e = -cd$, where $\widetilde{\mathcal{M}}$ is the Breuil module of rank 1 defined in the proof of Proposition 2.7. Hence, we have $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) \twoheadrightarrow \mathrm{T}^*_{\mathrm{st}}(\widetilde{\mathcal{M}}) \hookrightarrow \mathrm{T}^*_{\mathrm{st}}(\mathcal{M}')$. □

3. Semi-stable representations with Hodge–Tate weights $(0, 1, 2)$

In the paper [Par], we have classified all the 3-dimensional semi-stable representations of $G_{\mathbb{Q}_p}$ with regular Hodge–Tate weights. The 7 families in (1.1) are the only ones that contain the irreducible semi-stable non-crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, 1, 2)$. In this section, we glue these families together when their Hodge–Tate weights are $(0, 1, 2)$. The following two families $D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$ parameterize all of the families listed in (1.1).

Example 3.1. For $\lambda, \eta \in \mathcal{O}_E$ and $\mathfrak{L}_1, \mathfrak{L}_2 \in E$, we define the admissible filtered (ϕ, N)-modules $D_{[0,\frac{1}{2}]} = D_{[0,\frac{1}{2}]}(\lambda, \eta, \mathfrak{L}_1, \mathfrak{L}_2)$ as follows: there exists a basis $\underline{e} := (e_1, e_2, e_3)$ for $D_{[0,\frac{1}{2}]}$ such that

- $\mathrm{Fil}^i D = \begin{cases} D = E(e_1, e_2, e_3) & \text{if } i \le 0, \\ E(e_1 + \mathfrak{L}_1 e_3, e_2 + \mathfrak{L}_2 e_3) & \text{if } i = 1, \\ E(e_1 + \mathfrak{L}_1 e_3) & \text{if } i = 2, \\ 0 & \text{if } i \ge 3; \end{cases}$
- $\mathrm{Mat}_{\underline{e}}(\phi) = \begin{pmatrix} p\lambda & 0 & 0 \\ 1 & \eta & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ and $\mathrm{Mat}_{\underline{e}}(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$;
- $0 \le v_p(\lambda) \le \frac{1}{2}$ and $2v_p(\lambda) + v_p(\eta) = 2$.

Note that $\eta \ne \lambda$ since $0 \le v_p(\lambda) \le \frac{1}{2} < 1 \le v_p(\eta) \le 2$.

Proposition 3.2. *$D_{[0,\frac{1}{2}]}$ parameterizes $D^{10}_{\mathrm{rk}N=1}$, $D^{12}_{\mathrm{rk}N=1}$, $D^{18}_{\mathrm{rk}N=1}$, and $D^{20}_{\mathrm{rk}N=1}$ for $0 \le v_p(\lambda) \le \frac{1}{2}$ with Hodge–Tate weights $(0, 1, 2)$. Moreover, $D_{[0,\frac{1}{2}]}(\lambda, \eta, \mathfrak{L}_1, \mathfrak{L}_2)$ has a non-trivial proper submodule if and only if either $v_p(\lambda) = 0$ or $v_p(\lambda) = \frac{1}{2}$ and $\mathfrak{L}_2 = 0$.*

Proof. It is immediate that the identity map gives rise to an isomorphism from $D^{10}_{\mathrm{rk}N=1}$ and from $D^{12}_{\mathrm{rk}N=1}$ to $D_{[0,\frac{1}{2}]}$, and so $D^{10}_{\mathrm{rk}N=1}$ and $D^{12}_{\mathrm{rk}N=1}$ are sitting on the $p\lambda = \eta$ part of $D_{[0,\frac{1}{2}]}$. The association $e_1 \mapsto (p\lambda - \eta)e_1 + e_2$, $e_2 \mapsto -e_2$, $e_3 \mapsto (p\lambda - \eta)e_3$ gives rise to an isomorphism from $D^{18}_{\mathrm{rk}N=1}$ and from $D^{20}_{\mathrm{rk}N=1}$ to

$D_{[0,\frac{1}{2}]}$. Then, by replacing $\mathfrak{L}_1$ and $-(p\lambda-\eta)\mathfrak{L}_2$ with $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively, we see that $D^{18}_{\mathrm{rk}N=1}$ and $D^{20}_{\mathrm{rk}N=1}$ cover exactly the $p\lambda \neq \eta$ part of $D_{[0,\frac{1}{2}]}$.

For the second part, we know that only $D^{18}_{\mathrm{rk}N=1}$ and $D^{20}_{\mathrm{rk}N=1}$ contain reducible semi-stable representations by a result of [Par]. Since $D^{18}_{\mathrm{rk}N=1}$ (resp., $D^{20}_{\mathrm{rk}N=1}$ for $0 \leq v_p(\lambda) \leq \frac{1}{2}$) has a submodule if and only if either $v_p(\lambda)=0$ or $v_p(\lambda)=\frac{1}{2}$ (resp., $v_p(\lambda)=0$), the statement is now clear from the association above. □

Example 3.3. For $\lambda, \eta \in \mathcal{O}_E$ and $\mathfrak{L}_1, \mathfrak{L}_2 \in E$, we define the admissible filtered (ϕ, N)-modules $D_{[\frac{1}{2},1]} = D_{[\frac{1}{2},1]}(\lambda, \eta, \mathfrak{L}_1, \mathfrak{L}_2)$ as follows: there exists a basis $\underline{e} := (e_1, e_2, e_3)$ for $D_{[\frac{1}{2},1]}$ such that

- $\mathrm{Fil}^i D = \begin{cases} D = E(e_1, e_2, e_3) & \text{if } i \leq 0, \\ E(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3, e_2) & \text{if } i = 1, \\ E(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3) & \text{if } i = 2, \\ 0 & \text{if } i \geq 3; \end{cases}$
- $\mathrm{Mat}_{\underline{e}}(\phi) = \begin{pmatrix} p\lambda & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 1 & \lambda \end{pmatrix}$ and $\mathrm{Mat}_{\underline{e}}(N) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$;
- $\frac{1}{2} \leq v_p(\lambda) \leq 1$ and $2v_p(\lambda) + v_p(\eta) = 2$.

Note that $p\lambda \neq \eta$ since $0 \leq v_p(\eta) \leq 1 < \frac{3}{2} \leq v_p(p\lambda) \leq 2$.

Proposition 3.4. *$D_{[\frac{1}{2},1]}$ parameterizes $D^4_{\mathrm{rk}N=1}$, $D^6_{\mathrm{rk}N=1}$, $D^{17}_{\mathrm{rk}N=1}$, and $D^{20}_{\mathrm{rk}N=1}$ for $\frac{1}{2} \leq v_p(\lambda) \leq 1$ with Hodge–Tate weights $(0,1,2)$. Moreover, $D_{[\frac{1}{2},1]}(\lambda, \eta, \mathfrak{L}_1, \mathfrak{L}_2)$ has a non-trivial proper submodule if and only if either $v_p(\lambda) = 1$ or $v_p(\lambda) = \frac{1}{2}$ and $\mathfrak{L}_1 = 0$.*

Proof. The identity map gives rise to an isomorphism from $D^4_{\mathrm{rk}N=1}$ to $D_{[\frac{1}{2},1]}$, and so $D^4_{\mathrm{rk}N=1}$ covers the $\eta = \lambda$ and $\mathfrak{L}_1 = 0$ part of $D_{[\frac{1}{2},1]}$. The association $e_1 \mapsto e_1$, $e_2 \mapsto e_2 - \mathfrak{L}_2 e_3$, $e_3 \mapsto e_3$ gives an isomorphism from $D^6_{\mathrm{rk}N=1}$ to $D_{[\frac{1}{2},1]}$, and so we see that $D^6_{\mathrm{rk}N=1}$ covers the $\eta = \lambda$ and $\mathfrak{L}_1 \neq 0$ part of $D_{[\frac{1}{2},1]}$ by replacing $\mathfrak{L}_1$ and $-\mathfrak{L}_1\mathfrak{L}_2$ with $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively. The association $e_1 \mapsto -e_1$, $e_2 \mapsto (\eta-\lambda)e_2 + e_3$, $e_3 \mapsto -e_3$ gives an isomorphism from $D^{17}_{\mathrm{rk}N=1}$ to $D_{[\frac{1}{2},1]}$, so that we see that $D^{17}_{\mathrm{rk}N=1}$ covers the $\eta \neq \lambda$ and $\mathfrak{L}_1 = 0$ part of $D_{[\frac{1}{2},1]}$. The association $e_1 \mapsto -e_1$, $e_2 \mapsto (\eta-\lambda)\mathfrak{L}_2 e_2 + \mathfrak{L}_2 e_3$, $e_3 \mapsto -e_3$ gives an isomorphism from $D^{20}_{\mathrm{rk}N=1}$ to $D_{[\frac{1}{2},1]}$, and so we see that $D^{20}_{\mathrm{rk}N=1}$ covers the $\eta \neq \lambda$ and $\mathfrak{L}_1 \neq 0$ part of $D_{[\frac{1}{2},1]}$ by replacing $(\lambda-\eta)\mathfrak{L}_2$ and $\mathfrak{L}_1 - \mathfrak{L}_2$ with $\mathfrak{L}_1$ and $\mathfrak{L}_2$ respectively.

For the second part, we know that only $D^{17}_{\mathrm{rk}N=1}$ and $D^{20}_{\mathrm{rk}N=1}$ contain reducible semi-stable representations by a result of [Par]. Since $D^{17}_{\mathrm{rk}N=1}$ (resp., $D^{20}_{\mathrm{rk}N=1}$ for $\frac{1}{2} \leq v_p(\lambda) \leq 1$) has a submodule if and only if either $v_p(\lambda) = \frac{1}{2}$ or $v_p(\lambda) = 1$ (resp., $v_p(\lambda) = 1$), the statement is now clear. □

Note that there are no isomorphisms between the modules $D_{[0,\frac{1}{2}]}$ and between the modules $D_{[\frac{1}{2},1]}$ for different values of the parameters $\lambda, \eta, \mathfrak{L}_1, \mathfrak{L}_2$. (See the first part of the proof of the proposition below for the reason.) But there are isomorphisms between $D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$, and this happens between the irreducible parts of $D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$ only when $v_p(\lambda) = \frac{1}{2}$.

Proposition 3.5. *$D_{[0,\frac{1}{2}]}(\lambda,\eta,\mathfrak{L}_1,\mathfrak{L}_2)$ are isomorphic to $D_{[\frac{1}{2},1]}(\lambda',\eta',\mathfrak{L}_1',\mathfrak{L}_2')$ if and only if*

- *$\lambda=\lambda'$ and $\eta=\eta'$ (and so $v_p(\lambda)=\frac{1}{2}$ and $v_p(\eta)=1$);*
- *$(\lambda-\eta)\mathfrak{L}_2=(\eta-p\lambda)\mathfrak{L}_1'$ and $(\eta-p\lambda)(\mathfrak{L}_1-\mathfrak{L}_2')=\mathfrak{L}_2$;*
- *$\mathfrak{L}_2\neq 0\neq\mathfrak{L}_1'$.*

Proof. We start the proof noting that there are no isomorphisms between $D^4_{\mathrm{rk}N=1}$, $D^6_{\mathrm{rk}N=1}$, $D^{10}_{\mathrm{rk}N=1}$, $D^{12}_{\mathrm{rk}N=1}$, $D^{17}_{\mathrm{rk}N=1}$, $D^{18}_{\mathrm{rk}N=1}$, and $D^{20}_{\mathrm{rk}N=1}$ by a result in [Par]. Hence, the isomorphism between $D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$ occurs only between the parts of $D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$ on which $D^{20}_{\mathrm{rk}N=1}$ is sitting, since the $v_p(\lambda)=\frac{1}{2}$ part of $D^{20}_{\mathrm{rk}N=1}$ is embedded both into $D_{[0,\frac{1}{2}]}$ and into $D_{[\frac{1}{2},1]}$. Hence, the isomorphism only occurs when $v_p(\lambda)=\frac{1}{2}=v_p(\lambda')$ and $\mathfrak{L}_2\neq 0\neq\mathfrak{L}_1'$.

During the proof, we use e_i' for the basis of $D_{[\frac{1}{2},1]}$. Let T be an isomorphism from $D_{[0,\frac{1}{2}]}(\lambda,\eta,\mathfrak{L}_1,\mathfrak{L}_2)$ to $D_{[\frac{1}{2},1]}(\lambda',\eta',\mathfrak{L}_1',\mathfrak{L}_2')$. Then T preserves the Jordan form of the Frobenius maps and, in particular, their eigenvalues. Hence, $\lambda=\lambda'$ and $\eta=\eta'$. The commutativity with the monodromy operator N forces that T be of the form $T(e_1)=ae_1'+be_2'+ce_3'$, $T(e_2)=de_2'+ee_3'$, and $T(e_3)=ae_3'$ for some $a,b,c,d,e\in E$. Then the commutativity with the Frobenius maps implies that there exist x and y in E such that $a=x$, $b=(\eta-\lambda)y$, $c=y$, $d=(\eta-p\lambda)(\eta-\lambda)y$, and $e=(\eta-p\lambda)y$. Since T preserves the filtration, we have $(\eta-p\lambda)y+\mathfrak{L}_2x=0$, and $x\mathfrak{L}_1'=(\eta-\lambda)y$, and $y+x\mathfrak{L}_1=x\mathfrak{L}_2'$, which forces $(\lambda-\eta)\mathfrak{L}_2=(\eta-p\lambda)\mathfrak{L}_1'$ and $(\eta-p\lambda)(\mathfrak{L}_1-\mathfrak{L}_2')=\mathfrak{L}_2$.

It is easy to check that the converse holds. The association

$$\begin{cases} e_1\mapsto e_1'+\mathfrak{L}_1'e_2'-(\mathfrak{L}_1-\mathfrak{L}_2')e_3', \\ e_2\mapsto(\lambda-\eta)\mathfrak{L}_2e_2'-\mathfrak{L}_2e_3', \\ e_3\mapsto e_3' \end{cases} \tag{3.1}$$

gives rise to an isomorphism from $D_{[0,\frac{1}{2}]}(\lambda,\eta,\mathfrak{L}_1,\mathfrak{L}_2)$ to $D_{[\frac{1}{2},1]}(\lambda,\eta,\mathfrak{L}_1',\mathfrak{L}_2')$. □

Corollary 3.6. *$D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$ contain all of the 3-dimensional irreducible semi-stable non-crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0,1,2)$.*

Proof. By a result of [Par], the 7 families in (1.1) contain all the irreducible semi-stable non-crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0,1,2)$. By Propositions 3.2 and 3.4, the two families $D_{[0,\frac{1}{2}]}$ and $D_{[\frac{1}{2},1]}$ parameterize all of the 7 families above. □

We end this section by noting that there are also a few families of admissible filtered (ϕ,N)-modules containing only reducible semi-stable non-crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0,1,2)$. See [Par] for details.

4. Sequences

In this section, we introduce certain sequences defined over E and study their various properties and show their convergence. The limits of these sequences will appear as coefficients of our strongly divisible modules.

4.1. **The 1st sequence.** We let $A, B, C, D \in E$ and define two sequences G_m and H_m for $m \geq 0$ recursively as follows: $G_0 = H_0 = 1$ and

$$G_{m+1} = A(AH_m + BG_m)^2;$$
$$H_{m+1} = G_{m+1} + [C(AH_m + BG_m) + DG_m]G_m.$$

To guarantee the convergence of the sequence G_m/H_m and other properties, we need to restrict the regions on which A, B, C, D are defined:

Hyp(1): $v_p(C) - v_p(A(A+B)) > 0$ and $v_p(D) - v_p(A(A+B)^2) > 0$;
Hyp(2): $v_p(BC) - v_p(A(A+B)^2) > 0$ and $v_p(BD) - v_p(A(A+B)^3) > 0$.

By **Hyp**, we always mean all two of the conditions above.

We let $X := \min\{v_p(\frac{C}{A(A+B)}), v_p(\frac{D}{A(A+B)^2})\}$ for brevity. Note that $X > 0$ if we assume **Hyp(1)** and that **Hyp(2)** is equivalent to $v_p(\frac{B}{A+B}) + X > 0$.

Lemma 4.1. *Keep the assumptions* **Hyp**. *Then, for $m \geq 0$,*

(1) $v_p(G_m - H_m) - v_p(G_m) \geq X > 0$;
(2) $v_p(G_m) = v_p(H_m)$ *and* $v_p(AH_m + BG_m) = v_p(A+B) + v_p(H_m)$;
(3) $v_p(G_m H_{m+1} - H_m G_{m+1}) - v_p(H_m H_{m+1}) \geq X + m[v_p(\frac{A}{A+B}) + X]$.

Proof. (1) for m implies $v_p(G_m) = v_p(H_m)$. Since $AH_m + BG_m = (A+B)H_m + B(G_m - H_m)$ and $v_p((A+B)H_m) < X + v_p(B) + v_p(G_m) \leq v_p(B(G_m - H_m))$ by **Hyp**(2) and by part (1), $v_p(AH_m + BG_m) = v_p(A+B) + v_p(H_m)$. Hence, (1) for m implies (2) for m.

We prove (1) by induction. For $m = 0$, it is trivial. Assume that (1) is true for m. Then $v_p(G_m) = v_p(H_m)$ and $v_p(AH_m + BG_m) = v_p((A+B)H_m)$ for m. Hence, we have $v_p(G_{m+1}) = v_p(A) + 2v_p(A+B) + 2v_p(H_m)$ and

$$\begin{aligned} &v_p(G_{m+1} - H_{m+1}) - v_p(G_{m+1}) \\ &\quad = v_p\big([C(AH_m + BG_m) + DG_m]G_m\big) - v_p(G_{m+1}) \\ &\quad \geq \min\big\{v_p\big(C(A+B)\big) + 2v_p(G_m), v_p(D) + 2v_p(G_m)\big\} - v_p(G_{m+1}) \\ &\quad = \min\big\{v_p\big(C/[A(A+B)]\big), v_p\big(D/[A(A+B)^2]\big)\big\} = X > 0. \end{aligned}$$

Hence, (1) holds by induction.

For (3), we induct on m as well. If $m = 0$, then $G_0H_1 - H_0G_1 = H_1 - G_1 = C(A+B) + D$. So it works for $m = 0$.

We claim the following identity: for $m \geq 1$,

$$G_m H_{m+1} - H_m G_{m+1} = -A^2(W_1 + W_2)(G_{m-1}H_m - H_{m-1}G_m),$$

where

$$\begin{cases} W_1 = C(AH_{m-1} + BG_{m-1})(AH_m + BG_m); \\ W_2 = D[(AH_{m-1} + BG_{m-1})G_m + G_{m-1}(AH_m + BG_m)]. \end{cases}$$

Indeed, $G_m H_{m+1} - H_m G_{m+1} = G_m(H_{m+1} - G_{m+1}) + (G_m - H_m)G_{m+1} = [C(AH_m + BG_m) + DG_m]G_m^2 - [C(AH_{m-1} + BG_{m-1}) + DG_{m-1}]G_{m-1}G_{m+1} = CW_1' + DW_2'$, where

$$\begin{cases} W_1' = (AH_m + BG_m)G_m^2 - (AH_{m-1} + BG_{m-1})G_{m-1}G_{m+1}; \\ W_2' = G_m^3 - G_{m-1}^2 G_{m+1}. \end{cases}$$

Then

$$\begin{aligned} W_1' &= (AH_m + BG_m)A(AH_{m-1} + BG_{m-1})^2 G_m \\ &\quad - (AH_{m-1} + BG_{m-1})A(AH_m + BG_m)^2 G_{m-1} \\ &= A^2(AH_{m-1} + BG_{m-1})(AH_m + BG_m)(H_{m-1}G_m - G_{m-1}H_m) \end{aligned}$$

and

$$\begin{aligned} W_2' &= G_m^2 A(AH_{m-1} + BG_{m-1})^2 - G_{m-1}^2 A(AH_m + BG_m)^2 \\ &= A^2[(AH_{m-1} + BG_{m-1})G_m + G_{m-1}(AH_m + BG_m)](H_{m-1}G_m - G_{m-1}H_m). \end{aligned}$$

By letting $W_1(G_{m-1}H_m - H_{m-1}G_m) = -\frac{C}{A^2}W_1'$ and $W_2(G_{m-1}H_m - H_{m-1}G_m) = -\frac{D}{A^2}W_2'$, we complete the proof of the identity.

From the identity, it is now easy to deduce part (3). By part (2),

$$\begin{cases} v_p(W_1) = v_p(C) + 2v_p(A+B) + v_p(H_{m-1}H_m); \\ v_p(W_2) \geq v_p(D) + v_p(A+B) + v_p(H_{m-1}H_m). \end{cases}$$

Thus, we have

$$\begin{aligned} &v_p(G_m H_{m+1} - H_m G_{m+1}) - v_p(H_m H_{m+1}) \\ &\quad \geq 2v_p(A) + \min\{v_p(W_1), v_p(W_2)\} + v_p(G_{m-1}H_m - H_{m-1}G_m) - v_p(H_m H_{m+1}) \\ &\quad \geq \min\left\{v_p\big(C/(A+B)^2\big), v_p\big(D/(A+B)^3\big)\right\} \\ &\qquad + v_p(G_{m-1}H_m - H_{m-1}G_m) - v_p(H_{m-1}H_m) \\ &\quad = \big[v_p\big(A/(A+B)\big) + X\big] + v_p(G_{m-1}H_m - H_{m-1}G_m) - v_p(H_{m-1}H_m). \end{aligned}$$

Hence, (3) holds by induction. □

It is immediate that the conditions **Hyp** imply that $v_p(A/(A+B)) + X > 0$. Hence, part (3) of the lemma says that $v_p(G_m H_{m+1} - H_m G_{m+1}) - v_p(H_m H_{m+1})$ approaches ∞ as m goes to ∞. That is, the sequence G_m/H_m is Cauchy. We let

$$\Delta = \Delta(A, B, C, D) := \lim_{m \mapsto \infty} \frac{G_m}{H_m}.$$

Note that Δ depends on the values of the parameters $A, B, C, D \in E$.

The following lemma will play a crucial role in the following sections.

Lemma 4.2. *Keep the assumptions* **Hyp**.

(1) $v_p(1-\Delta) \geq X > 0$, *in particular,* $\Delta \in 1 + \mathfrak{m}_E$;
(2) $\frac{A+B\Delta}{A+B} \in 1 + \mathfrak{m}_E$;
(3) Δ *satisfies the equation*

$$A(A+B\Delta)^2(1-\Delta) - [C(A+B\Delta) + D\Delta]\Delta^2 = 0.$$

Proof. Case (1) is immediate from case (1) in Lemma 4.1. For case (2), we have

$$\frac{A+B\Delta}{A+B} = \frac{A+B-B(1-\Delta)}{A+B} = 1 - \frac{B(1-\Delta)}{A+B}.$$

By case (1) $v_p\big(\frac{B(1-\Delta)}{A+B}\big) \geq v_p\big(\frac{B}{A+B}\big) + X > 0$. The latter inequality is due to **Hyp**(**2**).

Case (3) is also easy to check. By definition of the sequences, we have

$$\begin{aligned}\frac{G_{m+1}}{H_{m+1}} &= \frac{A(AH_m+BG_m)^2}{A(AH_m+BG_m)^2+[C(AH_m+BG_m)+DG_m]G_m}\\ &= \frac{A(A+B\frac{G_m}{H_m})^2}{A(A+B\frac{G_m}{H_m})^2+[C(A+B\frac{G_m}{H_m})+D\frac{G_m}{H_m}]\frac{G_m}{H_m}}.\end{aligned}$$

By taking limits on both sides, we get

$$\Delta = \frac{A(A+B\Delta)^2}{A(A+B\Delta)^2+[C(A+B\Delta)+D\Delta]\Delta},$$

which immediately gives rise to the equation in part (3). □

4.2. **The 2nd sequence.** We let $A, B, C, D \in E$ and define two sequences G'_m and H'_m for $m \geq 0$ recursively as follows: $G'_0 = H'_0 = 1$ and

$$G'_{m+1} = (AH'_m + BG'_m)^2;$$

$$H'_{m+1} = G'_{m+1} + [C(AH'_m + BG'_m) + DG'_m]H'_m.$$

To guarantee the convergence of the sequence G'_m/H'_m and other properties, we need to restrict the regions on which A, B, C, D are defined:

Hyp′(1): $v_p(C) - v_p(A+B) > 0$ and $v_p(D) - 2v_p(A+B) > 0$;
Hyp′(2): $v_p(BC) - 2v_p(A+B) > 0$ and $v_p(BD) - 3v_p(A+B) > 0$.

By **Hyp′**, we always mean all two of the conditions above.

We let $X' := \min\{v_p(\frac{C}{A+B}), v_p(\frac{D}{(A+B)^2})\}$ for brevity. Note that $X' > 0$ if we assume **Hyp′(1)** and that **Hyp′(2)** is equivalent to $v_p(\frac{B}{A+B}) + X' > 0$.

Lemma 4.3. *Keep the assumptions* **Hyp′**. *Then, for* $m \geq 0$,

(1) $v_p(G'_m - H'_m) - v_p(G'_m) \geq X' > 0$;
(2) $v_p(G'_m) = v_p(H'_m)$ *and* $v_p(AH'_m + BG'_m) = v_p(A+B) + v_p(H'_m)$;
(3) $v_p(G'_m H'_{m+1} - H'_m G'_{m+1}) - v_p(H'_m H'_{m+1}) \geq X' + m\min\{X', v_p(\frac{B}{A+B}) + X'\}$.

Proof. As in Lemma 4.1, (2) for m is immediate from (1) for m.

We prove (1) by induction. For $m = 0$, it is trivial. Assume that (1) is true for m. Then $v_p(G'_m) = v_p(H'_m)$ and $v_p(AH'_m + BG'_m) = v_p((A+B)H'_m)$ for m. Hence, we have $v_p(G'_{m+1}) = 2v_p(A+B) + 2v_p(H'_m)$ and

$$\begin{aligned}&v_p(G'_{m+1} - H'_{m+1}) - v_p(G'_{m+1})\\ &\quad= v_p\big([C(AH'_m + BG'_m) + DG'_m]H'_m\big) - v_p(G'_{m+1})\\ &\quad\geq \min\big\{v_p\big(C(A+B)\big) + 2v_p(G'_m), v_p(D) + 2v_p(G'_m)\big\} - v_p(G'_{m+1})\\ &\quad= \min\big\{v_p\big(C/(A+B)\big), v_p\big(D/(A+B)^2\big)\big\} = X' > 0.\end{aligned}$$

Hence, (1) holds by induction.

For (3), we induct on m as well. If $m = 0$, then $G'_0 H'_1 - H'_0 G'_1 = H'_1 - G'_1 = C(A+B) + D$. So it works for $m = 0$.

We claim the following identity: for $m \geq 1$,

$$G'_m H'_{m+1} - H'_m G'_{m+1} = (W_1 + W_2 + W_3)(G'_{m-1} H'_m - H'_{m-1} G'_m),$$

where

$$\begin{cases} W_1 = BC(AH'_{m-1} + BG'_{m-1})(AH'_m + BG'_m); \\ W_2 = -D(AH'_{m-1} + BG'_{m-1})(AH'_m + BG'_m); \\ W_3 = BD[(AH'_{m-1} + BG'_{m-1})G'_m + G'_{m-1}(AH'_m + BG'_m)]. \end{cases}$$

Indeed, $G'_m H'_{m+1} - H'_m G'_{m+1} = G'_m(H'_{m+1} - G'_{m+1}) + (G'_m - H'_m)G'_{m+1} = [C(AH'_m + BG'_m) + DG'_m]G'_m H'_m - [C(AH'_{m-1} + BG'_{m-1}) + DG'_{m-1}]H'_{m-1}G'_{m+1} = CW'_1 + DW'_2$, where

$$\begin{cases} W'_1 = (AH'_m + BG'_m)G'_m H'_m - (AH'_{m-1} + BG'_{m-1})H'_{m-1}G'_{m+1}; \\ W'_2 = (G'_m)^2 H'_m - G'_{m-1}H'_{m-1}G'_{m+1}. \end{cases}$$

Then, by definition of the sequences,

$$\begin{aligned} W'_1 &= (AH'_m + BG'_m)(AH'_{m-1} + BG'_{m-1})^2 H'_m \\ &\quad - (AH'_{m-1} + BG'_{m-1})(AH'_m + BG'_m)^2 H'_{m-1} \\ &= B(AH'_{m-1} + BG'_{m-1})(AH'_m + BG'_m)(G'_{m-1}H'_m - H'_{m-1}G'_m) \end{aligned}$$

and

$$\begin{aligned} W'_2 &= G'_m H'_m (AH'_{m-1} + BG'_{m-1})^2 - G'_{m-1}H'_{m-1}(AH'_m + BG'_m)^2 \\ &= (AH'_{m-1} + BG'_{m-1})H'_m[(AH'_{m-1} + BG'_{m-1})G'_m - G'_{m-1}(AH'_m + BG'_m)] \\ &\quad + G'_{m-1}(AH'_m + BG'_m)[(AH'_{m-1} + BG'_{m-1})H'_m - H'_{m-1}(AH'_m + BG'_m)] \\ &= A(AH'_{m-1} + BG'_{m-1})H'_m(H'_{m-1}G'_m - G'_{m-1}H'_m) \\ &\quad + B(AH'_m + BG'_m)G'_{m-1}(G'_{m-1}H'_m - H'_{m-1}G'_m) \\ &= -(AH'_{m-1} + BG'_{m-1})(AH'_m + BG'_m)(G'_{m-1}H'_m - H'_{m-1}G'_m) \\ &\quad + B[(AH'_{m-1} + BG'_{m-1})G'_m + G'_{m-1}(AH'_m + BG'_m)](G'_{m-1}H'_m - H'_{m-1}G'_m). \end{aligned}$$

This completes the proof of the identity.

From the identity, it is now easy to deduce part (3). By part (2),

$$\begin{cases} v_p(W_1) = v_p(BC) + 2v_p(A+B) + v_p(H'_{m-1}H'_m); \\ v_p(W_2) = v_p(D) + 2v_p(A+B) + v_p(H'_{m-1}H'_m); \\ v_p(W_3) \geq v_p(BD) + v_p(A+B) + v_p(H'_{m-1}H'_m). \end{cases}$$

Thus, we have

$$\begin{aligned} &v_p(G'_m H'_{m+1} - H'_m G'_{m+1}) - v_p(H'_m H'_{m+1}) \\ &\quad \geq \min\{v_p(W_1), v_p(W_2), v_p(W_3)\} + v_p(G'_{m-1}H'_m - H'_{m-1}G'_m) - v_p(H'_m H'_{m+1}) \\ &\quad \geq \min\left\{v_p\big(BC/(A+B)^2\big), v_p\big(D/(A+B)^2\big), v_p\big(BD/(A+B)^3\big)\right\} \\ &\qquad + v_p(G'_{m-1}H'_m - H'_{m-1}G'_m) - v_p(H'_{m-1}H'_m) \\ &\quad \geq \min\left\{X', v_p\big(B/(A+B)\big) + X'\right\} \\ &\qquad + v_p(G'_{m-1}H'_m - H'_{m-1}G'_m) - v_p(H'_{m-1}H'_m). \end{aligned}$$

Hence, (3) holds by induction. □

Part (3) of the lemma says that $v_p(G'_m H'_{m+1} - H'_m G'_{m+1}) - v_p(H'_m H'_{m+1})$ approaches ∞ as m goes to ∞. That is, the sequence G'_m/H'_m is Cauchy. We let

$$\Delta' = \Delta'(A, B, C, D) := \lim_{m \mapsto \infty} \frac{G'_m}{H'_m}.$$

Note that Δ' depends on the values of the parameters $A, B, C, D \in E$.

The following lemma as well as Lemma 4.2 will play a crucial role in the following sections.

Lemma 4.4. *Keep the assumptions* **Hyp**$'$.

(1) $v_p(1-\Delta') \geq X' > 0$, *in particular,* $\Delta' \in 1+\mathfrak{m}_E$;
(2) $\frac{A+B\Delta'}{A+B} \in 1+\mathfrak{m}_E$;
(3) Δ' *satisfies the equation*

$$(A+B\Delta')^2(1-\Delta') - [C(A+B\Delta') + D\Delta']\Delta' = 0.$$

Proof. It is identical to the proof of Lemma 4.2. □

5. Reduction modulo p of $D_{[0,\frac{1}{2}]}$

In this section, we construct strongly divisible modules for the modules $D_{[0,\frac{1}{2}]}$ in Example 3.1 and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules. We let $\mathfrak{D}_{[0,\frac{1}{2}]} := S \otimes_{\mathbb{Z}_p} D_{[0,\frac{1}{2}]}$. In this section, we write D and $\mathfrak{D}$ for $D_{[0,\frac{1}{2}]}$ and $\mathfrak{D}_{[0,\frac{1}{2}]}$ respectively for brevity.

Recall that we assume $p > 3$ in this paper so that, in particular, $\phi(\gamma) \equiv 0$ modulo $p^4 S$. Since we are concerned only with absolutely irreducible mod p reductions, we also assume $0 < v_p(\lambda) \leq \frac{1}{2}$ (cf. Proposition 3.2).

It is easy to check that

$$\mathrm{Fil}^1\mathfrak{D} = S_E(e_1 + \mathfrak{L}_1 e_3, e_2 + \mathfrak{L}_2 e_3) + \mathrm{Fil}^1 S_E \cdot \mathfrak{D};$$

$$\mathrm{Fil}^2\mathfrak{D} = S_E\big(e_1 + \mathfrak{L}_1 e_3 + \frac{u-p}{p} e_3\big) + \mathrm{Fil}^1 S_E(e_2 + \mathfrak{L}_2 e_3) + \mathrm{Fil}^2 S_E \cdot \mathfrak{D}.$$

(We omit their proofs.) So every element in $\mathrm{Fil}^2\mathfrak{D}$ is of the form

$$C_0(e_1 + \mathfrak{L}_1 e_3 + \frac{u-p}{p} e_3) + C_1(u-p)(e_1 + \mathfrak{L}_1 e_3) \\ + C_2(u-p)(e_2 + \mathfrak{L}_2 e_3) + Ae_1 + Be_2 + Ce_3,$$

where C_0, C_1, C_2 are in E and A, B, C are in $\mathrm{Fil}^2 S_E$. We let

$$\mathfrak{X}_0(C_0, C_1, C_2) := C_0\big(e_1 + \mathfrak{L}_1 e_3 + \frac{u-p}{p} e_3\big) + (u-p)\big(C_1(e_1 + \mathfrak{L}_1 e_3) + C_2(e_2 + \mathfrak{L}_2 e_3)\big).$$

We divide the area in which the parameters of D are defined into 3 pieces as follows: for $\lambda, \eta \in \mathcal{O}_E$ with $0 < v_p(\lambda) \leq \frac{1}{2}$ and $2v_p(\lambda) + v_p(\eta) = 2$ and for $\mathfrak{L}_1, \mathfrak{L}_2 \in E$,

H(0,1): $v_p(\mathfrak{L}_2 + p\lambda) \geq 2 - v_p(\lambda)$ and $v_p(\mathfrak{L}_1 - 1) \geq 1 - v_p(\lambda)$;
H(0,2): $v_p(\mathfrak{L}_2 + p\lambda) \geq 1 + v_p(\mathfrak{L}_1 - 1)$ and $v_p(\mathfrak{L}_1 - 1) < 1 - v_p(\lambda)$;
H(0,3): $v_p(\mathfrak{L}_2 + p\lambda) < 2 - v_p(\lambda)$ and
$$\begin{cases} v_p(\mathfrak{L}_2 + p\lambda) \leq 1 + v_p(\mathfrak{L}_1 - 1) & \text{if } 0 < v_p(\lambda) < \frac{1}{2}, \\ v_p(\mathfrak{L}_2 + p\lambda) < 1 + v_p(\mathfrak{L}_1 - 1) & \text{if } v_p(\lambda) = \frac{1}{2}. \end{cases}$$

Note that these three regions cover the whole $(v_p(\mathfrak{L}_1 - 1), v_p(\mathfrak{L}_2 + p\lambda))$-plane and that if $v_p(\mathfrak{L}_1 - 1) < 1 - v_p(\lambda)$ and $0 < v_p(\lambda) < \frac{1}{2}$, then the conditions $\mathbf{H(0,2)}$ and $\mathbf{H(0,3)}$ intersect on $v_p(\mathfrak{L}_2 + p\lambda) = 1 + v_p(\mathfrak{L}_1 - 1)$. We construct strongly divisible modules for each case in the following three subsections.

5.1. **On the region $\mathbf{H}(\mathbf{0},\mathbf{1})$.** In this subsection, we construct strongly divisible modules in $\mathfrak{D}_{[0,\frac{1}{2}]}$ under the assumption $\mathbf{H}(\mathbf{0},\mathbf{1})$ and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules.

Proposition 5.1. *Keep the assumption* $\mathbf{H}(\mathbf{0},\mathbf{1})$. *Then* $\mathfrak{M}_{[0,\frac{1}{2}]} := S_{\mathcal{O}_E}(E_1,E_2,E_3)$ *is a strongly divisible module in* $\mathfrak{D}_{[0,\frac{1}{2}]}$ *where*

$$\begin{aligned} E_1 &= p\mathrm{e}_1 + \tfrac{1}{\lambda}\mathrm{e}_2 + (\gamma+\mathfrak{L}_1-1)\mathrm{e}_3;\\ E_2 &= \tfrac{1}{p\lambda}(\eta\mathrm{e}_2 + \lambda\mathfrak{L}_2\mathrm{e}_3) - \lambda(\gamma-1)\mathrm{e}_3;\\ E_3 &= \tfrac{p}{\lambda}\mathrm{e}_3. \end{aligned}$$

Proof. During the proof, we write $\mathfrak{M}$ for $\mathfrak{M}_{[0,\frac{1}{2}]}$ for brevity. It is routine to check that

$$\begin{aligned} \phi(E_1) &= p\lambda E_1 + pE_2 - \lambda^2(\gamma+\mathfrak{L}_1-1)E_3\\ &\qquad - \frac{\lambda[\mathfrak{L}_2 - p\lambda(\gamma-1)]}{p}E_3 + \frac{\lambda^2(\phi(\gamma)+\mathfrak{L}_1-1)}{p}E_3;\\ \phi(E_2) &= \eta E_2 - \frac{\lambda\eta[\mathfrak{L}_2 - p\lambda(\gamma-1)]}{p^2}E_3 + \frac{\lambda^2[\mathfrak{L}_2 - p\lambda(\phi(\gamma)-1)]}{p^2}E_3;\\ \phi(E_3) &= \lambda E_3, \end{aligned}$$

and

$$\begin{aligned} N(E_1) &= \lambda E_3 - \lambda[\gamma + (u-p)^{p-1}]E_3;\\ N(E_2) &= \lambda^2[\gamma + (u-p)^{p-1}]E_3;\\ N(E_3) &= 0. \end{aligned}$$

From these computations of $\phi(E_i)$ and $N(E_i)$, it is easy to check that for all $i \in \{1,2,3\}$,

$$\phi(E_i) \equiv 0 \equiv N(E_i) \tag{5.1}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ under the assumption $\mathbf{H}(\mathbf{0},\mathbf{1})$. Hence, $\mathfrak{M}$ is stable under ϕ and N.

Rewriting $\mathfrak{X}_0(C_0,C_1,C_2)$ in terms of E_1,E_2,E_3,

$$\mathfrak{X}_0(C_0,C_1,C_2) \in \mathfrak{Y}_0(C_0,C_1,C_2) + \mathrm{Fil}^p S_E\cdot\mathfrak{D},$$

where

$$\begin{aligned} \mathfrak{Y}_0(C_0,C_1,C_2) := C_0&\left(\frac{1}{p}E_1 - \frac{1}{\eta}E_2 + \frac{\lambda(\mathfrak{L}_2+p\lambda)}{p^2\eta}E_3 - \frac{\lambda(\mathfrak{L}_1-1)}{p^2}E_3 + \frac{\lambda\mathfrak{L}_1}{p}E_3\right)\\ &+ (u-p)\left(\frac{C_1}{p}E_1 - \frac{C_1-p\lambda C_2}{\eta}E_2 + \frac{V}{W}E_3\right). \end{aligned}$$

Here, we let $W = p^2\eta$ and

$$V = \lambda\eta(C_0 + p\mathfrak{L}_1C_1 + p\mathfrak{L}_2C_2) + \lambda(\mathfrak{L}_2+p\lambda)C_1 - \lambda\eta(\mathfrak{L}_1-1)C_1 - p\lambda^2(\mathfrak{L}_2+p\lambda)C_2.$$

One can readily check that $\mathfrak{Y}_0(C_0,C_1,C_2)$ can be rewritten as follows:

$$\mathfrak{Y}_0(C_0,C_1,C_2) = \frac{\lambda C_0}{p^2}\hat{F}_1 + (u-p)\left(\frac{C_1}{p}\hat{F}_2 + \frac{C_1-p\lambda C_2}{\eta}\hat{F}_3\right)$$

where

$$\hat{F}_1 = \frac{p}{\lambda}E_1 - \frac{p^2}{\lambda\eta}E_2 + \frac{\mathfrak{L}_2 + p\lambda}{\eta}E_3 - (\mathfrak{L}_1 - 1)E_3 + p\mathfrak{L}_1 E_3 + (u-p)E_3;$$
$$\hat{F}_2 = E_1 - \frac{\lambda(\mathfrak{L}_1 - 1)}{p}E_3 + \lambda(\mathfrak{L}_1 - 1)E_3 + \frac{\mathfrak{L}_2 + p\lambda}{p}E_3;$$
$$\hat{F}_3 = -E_2 + \frac{\lambda(\mathfrak{L}_2 + p\lambda)}{p^2}E_3 - \frac{\eta\mathfrak{L}_2}{p^2}E_3.$$

It is again easy to check that

$$\hat{F}_1 \equiv uE_3,\ \hat{F}_2 \equiv E_1 - \frac{\lambda(\mathfrak{L}_1 - 1)}{p}E_3,\ \hat{F}_3 \equiv -E_2 + \frac{\lambda(\mathfrak{L}_2 + p\lambda)}{p^2}E_3 \tag{5.2}$$

modulo $\mathfrak{m}_E\mathfrak{M}$. Hence, $\mathfrak{Y}_0(C_0, C_1, C_2) \in \mathrm{Fil}^2\mathfrak{M}$ if and only if

$$v_p(C_0) \geq 2 - v_p(\lambda),\ v_p(C_1) \geq 1,\ v_p(C_1 - p\lambda C_2) \geq v_p(\eta). \tag{5.3}$$

For brevity, we often write $\mathfrak{Y}_0$ for $\mathfrak{Y}_0(C_0, C_1, C_2)$. For any $m \in \mathrm{Fil}^2\mathfrak{D}$, we have $m \in \mathrm{Fil}^2\mathfrak{M}$ if and only if $m \in \mathfrak{Y}_0(C_0, C_1, C_2) + \mathrm{Fil}^2 S\mathfrak{M}$ for some $C_0, C_1, C_2 \in E$ with $\mathfrak{Y}_0(C_0, C_1, C_2) \in \mathfrak{M}$. Therefore, it is enough to check that $\phi(\mathfrak{Y}_0) \in p^2\mathfrak{M}$ whenever $\mathfrak{Y}_0 \in \mathrm{Fil}^2\mathfrak{M}$.

It is also routine to check, by using our computation of $\phi(E_i)$, that

$$\phi(\hat{F}_1) = p^2E_1 - \lambda[p(\gamma - 1) - (u^p - p)]E_3 + \lambda\phi(\gamma)E_3 + \frac{p\lambda^2\phi(\gamma)}{\eta}E_3;$$
$$\phi(\hat{F}_2) = p\lambda E_1 + pE_2 + \frac{\lambda^2\phi(\gamma)}{p}E_3;$$
$$\phi(\hat{F}_3) = -\eta E_2 - \frac{\lambda^2\eta(\gamma - 1)}{p}E_3 + \frac{\lambda^3\phi(\gamma)}{p}E_3,$$

and so we immediately get

$$\frac{1}{p^2}\phi(\mathfrak{Y}_0) \equiv \frac{\lambda C_0}{p^2}E_1 + \frac{C_1}{p}(\gamma - 1)E_2 - \frac{C_1 - p\lambda C_2}{\eta}(\gamma - 1)\left(\frac{\eta}{p}E_2 + \frac{\lambda^2\eta}{p^2}(\gamma - 1)E_3\right) \tag{5.4}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ if the inequalities in (5.3) hold. Hence, $\phi(\mathrm{Fil}^2\mathfrak{M}) \subset p^2\mathfrak{M}$. □

Corollary 5.2. *The mod p reductions of the strongly divisible modules in Proposition 5.1 correspond to the Breuil modules $\mathcal{M}_{[0,\frac{1}{2}]}$ described as follows: there exist a basis $\underline{e} := (\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ for $\mathcal{M}_{[0,\frac{1}{2}]}$ and a system of generators $\underline{f} := (f_1, f_2, f_3)$ for $\mathrm{Fil}^2\mathcal{M}$ such that*

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} 0 & u & 0 \\ 0 & 0 & u \\ u & -\frac{\lambda(\mathfrak{L}_1-1)}{p}u & -\frac{\lambda(\mathfrak{L}_2+p\lambda)}{p^2}u \end{pmatrix}$ *and*

  $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -\frac{\eta}{p} \\ 0 & 0 & \frac{\lambda^2\eta}{p^2} \end{pmatrix};$
- *N is induced by $N(\mathrm{E}_1) = N(\mathrm{E}_2) = N(\mathrm{E}_3) = 0$.*

Proof. We keep the notation as in the proof of Proposition 5.1, and we let $f_1 = \hat{F}_1$ mod $(\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$ and $f_i = (u-p)\hat{F}_i$ mod $(\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$ for $i = 2, 3$. By (5.2), we may let $f_1 = uE_3$, $f_2 = uE_1 - \frac{\overline{\lambda(\mathfrak{L}_1-1)}}{p} uE_3$, $f_3 = uE_2 - \frac{\overline{\lambda(\mathfrak{L}_2+p\lambda)}}{p^2} uE_3$ in $\mathcal{M}_{[0,\frac{1}{2}]} = \mathfrak{M}/(\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$ (cf. (1.2)), which immediately gives rise to $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M})$. It is also immediate to get N and $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2)$ from (5.1) and (5.4) respectively. □

5.2. **On the region $\mathbf{H}(\mathbf{0},\mathbf{2})$.** In this subsection, we construct strongly divisible modules in $\mathfrak{D}_{[0,\frac{1}{2}]}$ under the assumption $\mathbf{H}(\mathbf{0},\mathbf{2})$ and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules.

We specialize the results in Section 4.1 to

$$A = \mathfrak{L}_1 - 1,\ B = -\frac{\eta}{\lambda},\ C = -\frac{[(\mathfrak{L}_2 + p\lambda) - \eta(\mathfrak{L}_1 - 1)]}{\lambda},\ D = -\frac{p\eta}{\lambda}.$$

It is easy to check that these A, B, C, D with the assumption $\mathbf{H}(\mathbf{0},\mathbf{2})$ satisfy the conditions **Hyp** in Section 4.1. We write $\Delta_{5.2}$ for the limit Δ in this case.

Since the results in this subsection heavily rely on Lemma 4.2, we recall the statement specialized to these A, B, C, D with the assumption $\mathbf{H}(\mathbf{0},\mathbf{2})$:

(1) $v_p(1-\Delta_{5.2}) \geq \min\big\{3[v_p(\frac{p}{\lambda}) - v_p(\mathfrak{L}_1 - 1)], v_p\big(\frac{1}{\lambda}[(\mathfrak{L}_2 + p\lambda) - \eta(\mathfrak{L}_1 - 1)]\big) - 2v_p(\mathfrak{L}_1 - 1)\big\} > 0$;
(2) $\frac{\lambda(\mathfrak{L}_1-1)-\eta\Delta_{5.2}}{\lambda(\mathfrak{L}_1-1)-\eta} \in 1 + \mathfrak{m}_E$ and $\frac{\lambda(\mathfrak{L}_1-1)-\eta\Delta_{5.2}}{\lambda(\mathfrak{L}_1-1)} \in 1 + \mathfrak{m}_E$;
(3) $\Delta_{5.2}$ satisfies the equation

$$\begin{aligned}&(\mathfrak{L}_1 - 1)[\lambda(\mathfrak{L}_1 - 1) - \eta\Delta_{5.2}]^2(1 - \Delta_{5.2})\\ &+ \big([(\mathfrak{L}_2 + p\lambda) - \eta(\mathfrak{L}_1 - 1)][\lambda(\mathfrak{L}_1 - 1) - \eta\Delta_{5.2}] + p\lambda\eta\Delta_{5.2}\big)\Delta_{5.2}^2 = 0.\end{aligned}$$

In this subsection, by Lemma 4.2 we always mean this specialized version of the lemma.

Proposition 5.3. *Keep the assumption* $\mathbf{H}(\mathbf{0},\mathbf{2})$. *Then* $\mathfrak{M}_{[0,\frac{1}{2}]} := S_{\mathcal{O}_E}(E_1, E_2, E_3)$ *is a strongly divisible module in* $\mathfrak{D}_{[0,\frac{1}{2}]}$ *where*

$$\begin{aligned}E_1 &= p\mathrm{e}_1 + \frac{1}{\lambda}\mathrm{e}_2 + (\gamma + \mathfrak{L}_1 - 1)\mathrm{e}_3\\ &\quad + \frac{p\Delta_{5.2}(\gamma - 1)}{\lambda[\lambda(\mathfrak{L}_1 - 1) - \eta\Delta_{5.2}]}\left(\frac{\lambda(\mathfrak{L}_1 - 1) - \eta\Delta_{5.2}}{p\lambda(\mathfrak{L}_1 - 1)}(\eta\mathrm{e}_2 + \lambda\mathfrak{L}_2\mathrm{e}_3) - \lambda^2(\gamma - 1)\mathrm{e}_3\right);\\ E_2 &= \frac{\lambda(\mathfrak{L}_1 - 1) - \eta\Delta_{5.2}}{p\lambda(\mathfrak{L}_1 - 1)}(\eta\mathrm{e}_2 + \lambda\mathfrak{L}_2\mathrm{e}_3) - \lambda^2(\gamma - 1)\mathrm{e}_3;\\ E_3 &= [\lambda(\mathfrak{L}_1 - 1) - \eta\Delta_{5.2}]\mathrm{e}_3.\end{aligned}$$

Note that $S_{\mathcal{O}_E}(E_1, E_2, E_3) = S_{\mathcal{O}_E}(E_1, E_2, \lambda(\mathfrak{L}_1 - 1)e_3)$ by part (2) of Lemma 4.2.

Proof. During the proof, we write Δ for $\Delta_{5.2}$ and $\mathfrak{M}$ for $\mathfrak{M}_{[0,\frac{1}{2}]}$ for brevity. It is routine to check that

$$\begin{aligned}
\phi(E_1) &= p\lambda E_1 + \frac{p}{\lambda}E_2 - \frac{p^2\Delta(\gamma-1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_2 \\
&\quad + \frac{\lambda(\phi(\gamma)+\mathfrak{L}_1-1-p\mathfrak{L}_1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3 - \frac{\mathfrak{L}_2}{\lambda(\mathfrak{L}_1-1)}E_3 \\
&\quad - \frac{\big(\mathfrak{L}_2[\lambda(\mathfrak{L}_1-1)-\eta\Delta]-p\lambda^2(\mathfrak{L}_1-1)(\phi(\gamma)-1)\big)\Delta}{(\mathfrak{L}_1-1)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2}E_3 \\
&\quad + \frac{p\Delta\phi(\gamma)}{\lambda[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}\phi(E_2); \\
\phi(E_2) &= \eta E_2 + \frac{(\lambda-\eta)\big(\mathfrak{L}_2[\lambda(\mathfrak{L}_1-1)-\eta\Delta]+p\lambda^2(\mathfrak{L}_1-1)\big)}{p(\mathfrak{L}_1-1)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}E_3 \\
&\quad - \frac{\lambda^2(\lambda\phi(\gamma)-\eta\gamma)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3; \\
\phi(E_3) &= \lambda E_3,
\end{aligned}$$

and

$$\begin{aligned}
N(E_1) &= \frac{p}{\lambda(\mathfrak{L}_1-1)-\eta\Delta} \\
&\quad \times \left[E_3 - [\gamma+(u-p)^{p-1}]\left(\frac{p\Delta}{\lambda}E_2 + E_3 - \frac{p\lambda\Delta(\gamma-1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3\right)\right]; \\
N(E_2) &= \frac{p\lambda^2[\gamma+(u-p)^{p-1}]}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3; \\
N(E_3) &= 0.
\end{aligned}$$

From these computations of $\phi(E_i)$ and $N(E_i)$, it is easy to check that

$$\phi(E_1) \equiv E_3 \text{ and } \phi(E_2) \equiv \phi(E_3) \equiv N(E_1) \equiv N(E_2) \equiv N(E_3) \equiv 0 \tag{5.5}$$

modulo $\mathfrak{m}_E\mathfrak{M}$, under the assumption $\mathbf{H}(\mathbf{0},\mathbf{2})$, using parts (1) and (2) of Lemma 4.2. Hence, $\mathfrak{M}$ is stable under ϕ and N.

Rewriting $\mathfrak{X}_0(C_0,C_1,C_2)$ in terms of E_1, E_2, E_3,

$$\mathfrak{X}_0(C_0,C_1,C_2) \in \mathfrak{Y}_0(C_0,C_1,C_2) + \mathrm{Fil}^p S_E \cdot \mathfrak{D},$$

where

$$\begin{aligned}
&\mathfrak{Y}_0(C_0,C_1,C_2) \\
&\quad := C_0\left(\frac{1}{p}E_1 - \frac{1}{\lambda\eta}E_2 + \frac{(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)}{p\eta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}E_3\right. \\
&\qquad\qquad \left. + \frac{\lambda\Delta}{[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2}E_3 + \frac{\mathfrak{L}_1}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3\right) \\
&\qquad + (u-p)\left(\frac{C_1}{p}E_1 - \frac{[\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1 - p\lambda^2(\mathfrak{L}_1-1)C_2}{\lambda\eta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}E_2 + \frac{V}{W}E_3\right).
\end{aligned}$$

Here, we let $W = p\eta[\lambda(\mathfrak{L}_1 - 1) - \eta\Delta]^2$ and

$$V = \eta[\lambda(\mathfrak{L}_1-1)-\eta\Delta](C_0+p\mathfrak{L}_1C_1+p\mathfrak{L}_2C_2)+(\mathfrak{L}_2+p\lambda)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1+p\lambda\eta\Delta C_1$$
$$-\eta(\mathfrak{L}_1-1)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1 - p\lambda(\mathfrak{L}_2+p\lambda)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_2 - p^2\lambda^2\eta\Delta C_2.$$

One can readily check that

$$\Delta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]\cdot V = \eta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2(\Delta C_0 - p\lambda(\mathfrak{L}_1-1)C_2)$$
$$+[(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)][\lambda(\mathfrak{L}_1-1)-\eta\Delta]\Delta\big([\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1 - p\lambda^2(\mathfrak{L}_1-1)C_2\big)$$
$$+p\lambda\eta\Delta^2\big([\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1 - p\lambda^2(\mathfrak{L}_1-1)C_2\big) + X + Y,$$

where

$$X = p\lambda\eta\big[(\mathfrak{L}_1-1)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2(1-\Delta)$$
$$+\big([(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)][\lambda(\mathfrak{L}_1-1)-\eta\Delta]+p\lambda\eta\Delta\big)\Delta^2\big]C_2$$

and

$$Y = \eta\Delta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2(p\mathfrak{L}_1C_1+p\mathfrak{L}_2C_2).$$

It is clear that $X = 0$ by part (3) of Lemma 4.2.

We often write $\mathfrak{Y}_0$ for $\mathfrak{Y}_0(C_0, C_1, C_2)$ to lighten the notation. Rewriting $\frac{Y}{\Delta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]W}$ and $\frac{C_1}{p}$ as linear combinations of $\frac{C_0}{\lambda\eta}$, $\frac{[\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1-p\lambda^2(\mathfrak{L}_1-1)C_2}{\lambda\eta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}$, and $\frac{\Delta C_0-p\lambda(\mathfrak{L}_1-1)C_2}{p\Delta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}$, one can rewrite $\mathfrak{Y}_0(C_0, C_1, C_2)$ as follows:

$$\mathfrak{Y}_0 = \frac{C_0}{\lambda\eta}\hat{F}_1 + (u-p)\left(\frac{[\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1 - p\lambda^2(\mathfrak{L}_1-1)C_2}{\lambda\eta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}\hat{F}_2 + \frac{\Delta C_0 - p\lambda(\mathfrak{L}_1-1)C_2}{p\Delta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}\hat{F}_3\right)$$

where

$$\hat{F}_2 = \frac{\lambda\eta}{p}E_1 - E_2 + \frac{\lambda[(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)]}{p[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}E_3 + \frac{\lambda^2\eta\Delta}{[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2}E_3 + \frac{\lambda\eta\mathfrak{L}_1}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3;$$
$$\hat{F}_3 = -\lambda\Delta E_1 + E_3 - \frac{p\lambda\Delta\mathfrak{L}_1}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3 - \frac{\Delta\mathfrak{L}_2}{\lambda(\mathfrak{L}_1-1)}E_3;$$
$$\hat{F}_1 = \hat{F}_2 + \frac{\lambda\eta(u-p)}{p[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}(E_3 - \hat{F}_3).$$

It is easy to check that

$$\begin{cases} \hat{F}_3 \equiv E_3; \\ \hat{F}_1 \equiv \hat{F}_2 \equiv -E_2 + \dfrac{(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)}{p(\mathfrak{L}_1-1)}E_3 \end{cases} \tag{5.6}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ under the assumption $\mathbf{H}(\mathbf{0},\mathbf{2})$. Hence, $\mathfrak{Y}_0(C_0, C_1, C_2) \in \mathrm{Fil}^2\mathfrak{M}$ if and only if

$$\begin{cases} v_p(C_0) \geq 2 - v_p(\lambda), \\ v_p\big(\Delta C_0 - p\lambda(\mathfrak{L}_1-1)C_2\big) \geq v_p\big(p\lambda(\mathfrak{L}_1-1)\big), \\ v_p\big(p\lambda^2(\mathfrak{L}_1-1)C_2 - [\lambda(\mathfrak{L}_1-1)-\eta\Delta]C_1\big) \geq 2 + v_p(\mathfrak{L}_1-1). \end{cases} \tag{5.7}$$

For any $m \in \mathrm{Fil}^2\mathfrak{D}$, we have $m \in \mathrm{Fil}^2\mathfrak{M}$ if and only if $m \in \mathfrak{Y}_0(C_0, C_1, C_2) + \mathrm{Fil}^2 S\mathfrak{M}$ for some $C_0, C_1, C_2 \in E$ with $\mathfrak{Y}_0(C_0, C_1, C_2) \in \mathfrak{M}$. Therefore, it is enough to check that $\phi(\mathfrak{Y}_0) \in p^2\mathfrak{M}$ whenever $\mathfrak{Y}_0 \in \mathrm{Fil}^2\mathfrak{M}$.

It is also routine to check, by using our computation of $\phi(E_i)$, that

$$\begin{aligned}\phi(\hat{F}_2) &= \lambda^2\eta E_1 - \frac{p\lambda\eta\Delta(\gamma-1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_2 - \frac{\lambda^2\eta(\gamma-1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3 + \frac{\lambda^2\eta\phi(\gamma)}{p[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}E_3 \\ &\quad + \frac{\lambda^3\eta\Delta\phi(\gamma)}{[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2}E_3 + \frac{\lambda^3\phi(\gamma)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3 + \frac{\eta\Delta\phi(\gamma)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}\phi(E_2); \\ \phi(\hat{F}_3) &= -p\lambda^2\Delta E_1 - p\Delta E_2 + \frac{p^2\lambda\Delta^2(\gamma-1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_2 - \frac{\lambda^2\Delta(\phi(\gamma)+\mathfrak{L}_1-1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3 + \lambda E_3 \\ &\quad + \frac{\lambda\big(\mathfrak{L}_2[\lambda(\mathfrak{L}_1-1)-\eta\Delta] - p\lambda^2(\mathfrak{L}_1-1)(\phi(\gamma)-1)\big)\Delta^2}{(\mathfrak{L}_1-1)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2}E_3 \\ &\quad - \frac{p\Delta^2\phi(\gamma)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}\phi(E_2),\end{aligned}$$

and we have

$$\phi(\hat{F}_1) = \phi(\hat{F}_2) + \frac{\lambda\eta(u^p-p)}{p[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}\left(\lambda E_3 - \phi(\hat{F}_3)\right).$$

We claim that $\phi(\hat{F}_2) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$, $\phi(\hat{F}_3) \equiv -pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$, and $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$. It is easy to check that $\phi(\hat{F}_2) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$. It is also easy to check that $\phi(\hat{F}_3) \equiv -p\Delta E_2 - \frac{\lambda^2\Delta(\mathfrak{L}_1-1)}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}E_3 + \lambda E_3 + \frac{\lambda\left(\mathfrak{L}_2[\lambda(\mathfrak{L}_1-1)-\eta\Delta]+p\lambda^2(\mathfrak{L}_1-1)\right)\Delta^2}{(\mathfrak{L}_1-1)[\lambda(\mathfrak{L}_1-1)-\eta\Delta]^2}E_3$ modulo $p\mathfrak{m}_E\mathfrak{M}$. Then, by the equation in part (3) of Lemma 4.2, $\phi(\hat{F}_3) \equiv -pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$. By the same argument as for $\phi(\hat{F}_2)$, one can compute $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1 - \frac{\lambda\eta\Delta}{\lambda(\mathfrak{L}_1-1)-\eta\Delta}[p(\gamma-1)-(u^p-p)]E_2 - \frac{\lambda^2\eta}{p[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}[p(\gamma-1)-(u^p-p)]E_3$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$. Since $p(\gamma-1) \equiv (u^p-p)$ modulo p^2S, we have $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$.

Therefore, we conclude that

$$\frac{1}{p^2}\phi(\mathfrak{Y}_0) \equiv \frac{\lambda C_0}{p^2}E_1 - \frac{\Delta C_0 - p\lambda(\mathfrak{L}_1-1)C_2}{p\Delta[\lambda(\mathfrak{L}_1-1)-\eta\Delta]}(\gamma-1)E_2 \tag{5.8}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ if the inequalities in (5.7) hold, and so $\phi(\mathrm{Fil}^2\mathfrak{M}) \subset p^2\mathfrak{M}$. □

Corollary 5.4. *The mod p reductions of the strongly divisible modules in Proposition 5.3 correspond to the Breuil modules $\mathcal{M}_{[0,\frac{1}{2}]}$ described as follows: there exist a basis $\underline{e} := (\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ for $\mathcal{M}_{[0,\frac{1}{2}]}$ and a system of generators $\underline{f} := (f_1, f_2, f_3)$ for $\mathrm{Fil}^2\mathcal{M}$ such that*

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)}{p(\mathfrak{L}_1-1)} & u \end{pmatrix}$ *and*

  $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & -\frac{\lambda^2\eta}{p^2} & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$;
- *N is induced by $N(\mathrm{E}_1) = N(\mathrm{E}_2) = N(\mathrm{E}_3) = 0$.*

Proof. We keep the notation as in the proof of Proposition 5.3, and we let $f_1 = (u-p)^2 E_1 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$, $f_2 = \hat{F}_1 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$, and $f_3 = (u-p)\hat{F}_3 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$. By (5.6), we may let $f_1 = u^2 E_1$, $f_2 = E_2 - \frac{(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)}{p(\mathfrak{L}_1-1)} E_3$, and $f_3 = uE_3$ in $\mathcal{M}_{[0,\frac{1}{2}]} = \mathfrak{M}/(\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$ (cf. (1.2)), which immediately gives rise to $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M})$. It is also immediate to get N from (5.5). One can also readily compute $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2)$ by using the results in (5.5) and (5.8). □

5.3. **On the region $\mathbf{H}(\mathbf{0},\mathbf{3})$.** In this subsection, we construct strongly divisible modules in $\mathfrak{D}_{[0,\frac{1}{2}]}$ under the assumption $\mathbf{H}(\mathbf{0},\mathbf{3})$ and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules.

We specialize the results in Section 4.2 to

$$A = \mathfrak{L}_2,\ B = p\lambda,\ C = -\eta(\mathfrak{L}_1 - 1),\ D = -p\eta^2.$$

It is easy to check that these A, B, C, D with the assumption $\mathbf{H}(\mathbf{0},\mathbf{3})$ satisfy the conditions $\mathbf{Hyp}'$ in Section 4.2. We write $\Delta_{5.3}$ for the limit Δ' in this case.

Since the results in this subsection heavily rely on Lemma 4.4, we recall the statement specialized to these A, B, C, D with the assumption $\mathbf{H}(\mathbf{0},\mathbf{3})$:

(1) $v_p(1-\Delta_{5.3}) \geq \min\left\{v_p\big(\eta(\mathfrak{L}_1-1)\big) - v_p(\mathfrak{L}_2+p\lambda), v_p(p\eta^2) - 2v_p(\mathfrak{L}_2+p\lambda)\right\} > 0$;
(2) $\frac{\mathfrak{L}_2+p\lambda\Delta_{5.3}}{\mathfrak{L}_2+p\lambda} \in 1 + \mathfrak{m}_E$;
(3) $\Delta_{5.2}$ satisfies the equation

$$(\mathfrak{L}_2 + p\lambda\Delta_{5.3})^2(1-\Delta_{5.3}) + \eta\big((\mathfrak{L}_1-1)(\mathfrak{L}_2+p\lambda\Delta_{5.3}) + p\eta\Delta_{5.3}\big)\Delta_{5.3} = 0.$$

In this subsection, by Lemma 4.4 we always mean this specialized version of the lemma.

Proposition 5.5. *Keep the assumption* $\mathbf{H}(\mathbf{0},\mathbf{3})$. *Then* $\mathfrak{M}_{[0,\frac{1}{2}]} := S_{\mathcal{O}_E}(E_1, E_2, E_3)$ *is a strongly divisible module in* $\mathfrak{D}_{[0,\frac{1}{2}]}$ *where*

$$\begin{aligned}
E_1 &= p\mathrm{e}_1 + \tfrac{1}{\lambda}\mathrm{e}_2 + (\gamma + \mathfrak{L}_1 - 1)\mathrm{e}_3 + \tfrac{\eta(\gamma-1)}{\lambda^2(\mathfrak{L}_2+p\lambda\Delta_{5.3})}(\eta\mathrm{e}_2 + \lambda\mathfrak{L}_2\mathrm{e}_3);\\
E_2 &= \tfrac{1}{p\lambda}(\eta\mathrm{e}_2 + \lambda\mathfrak{L}_2\mathrm{e}_3) - \lambda\Delta_{5.3}(\gamma-1)\mathrm{e}_3;\\
E_3 &= \tfrac{\lambda(\mathfrak{L}_2+p\lambda\Delta_{5.3})}{p}\mathrm{e}_3.
\end{aligned}$$

Note that $S_{\mathcal{O}_E}(E_1, E_2, E_3) = S_{\mathcal{O}_E}(E_1, E_2, \frac{\lambda(\mathfrak{L}_2+p\lambda)}{p} e_3)$ by part (2) of Lemma 4.4.

Proof. During the proof, we write Δ for $\Delta_{5.3}$ and $\mathfrak{M}$ for $\mathfrak{M}_{[0,\frac{1}{2}]}$ for brevity. It is routine to check that

$$\begin{aligned}
\phi(E_1) = p\lambda E_1 + pE_2 &- \frac{p[\mathfrak{L}_2 - p\lambda\Delta(\gamma-1)]}{\lambda(\mathfrak{L}_2+p\lambda\Delta)}E_3\\
&+ \frac{p[\phi(\gamma) + \mathfrak{L}_1 - 1 - p(\gamma+\mathfrak{L}_1-1)]}{\mathfrak{L}_2+p\lambda\Delta}E_3\\
&- \frac{p^2\eta(\gamma-1)}{\mathfrak{L}_2+p\lambda\Delta}\left(E_2 + \frac{p\Delta(\gamma-1)}{\mathfrak{L}_2+p\lambda\Delta}E_3\right)\\
&+ \frac{p\eta(\phi(\gamma)-1)}{\lambda(\mathfrak{L}_2+p\lambda\Delta)}\left(\phi(E_2) + \frac{p\lambda\Delta(\phi(\gamma)-1)}{\mathfrak{L}_2+p\lambda\Delta}E_3\right);\\
\phi(E_2) = \eta E_2 + E_3 &- \frac{\eta[\mathfrak{L}_2 - p\lambda\Delta(\gamma-1)]}{\lambda(\mathfrak{L}_2+p\lambda\Delta)}E_3 - \frac{p\lambda\Delta\phi(\gamma)}{\mathfrak{L}_2+p\lambda\Delta}E_3;\\
\phi(E_3) = \lambda E_3,&
\end{aligned}$$

and

$$N(E_1) = \frac{p^2}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}\left[E_3 - [\gamma + (u-p)^{p-1}]\left(\eta E_2 + E_3 + \frac{p\eta\Delta(\gamma-1)}{\mathfrak{L}_2 + p\lambda\Delta}E_3\right)\right];$$
$$N(E_2) = \frac{p^2\Delta[\gamma + (u-p)^{p-1}]}{\mathfrak{L}_2 + p\lambda\Delta}E_3;$$
$$N(E_3) = 0.$$

From these computations of $\phi(E_i)$ and $N(E_i)$, it is easy to check that

(5.9)
$$\phi(E_1) \equiv \frac{p(\mathfrak{L}_1 - 1)}{\mathfrak{L}_2 + p\lambda}E_3,\ \phi(E_2) \equiv E_3,\ \text{and } \phi(E_3) \equiv N(E_1) \equiv N(E_2) \equiv N(E_3) \equiv 0$$

modulo $\mathfrak{m}_E\mathfrak{M}$, under the assumption $\mathbf{H}(\mathbf{0},\mathbf{3})$, using parts (1) and (2) of Lemma 4.4. Hence, $\mathfrak{M}$ is stable under ϕ and N.

Rewriting $\mathfrak{X}_0(C_0, C_1, C_2)$ in terms of E_1, E_2, E_3,

$$\mathfrak{X}_0(C_0, C_1, C_2) \in \mathfrak{Y}_0(C_0, C_1, C_2) + \mathrm{Fil}^p S_E \cdot \mathfrak{D},$$

where

$$\begin{aligned}
&\mathfrak{Y}_0(C_0, C_1, C_2)\\
&\quad := C_0\left(\frac{1}{p}E_1 - \frac{1}{\eta}E_2 + \frac{\eta}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}E_2 + \frac{1}{\lambda\eta}E_3\right.\\
&\qquad\qquad \left. - \frac{(\mathfrak{L}_1 - 1) - p\mathfrak{L}_1}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}E_3 - \frac{p\eta\Delta}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)^2}E_3\right)\\
&\qquad + (u-p)\left(\frac{C_1}{p}E_1 - \frac{\lambda(\mathfrak{L}_2 + p\lambda\Delta)(C_1 - p\lambda C_2) - \eta^2 C_1}{\lambda\eta(\mathfrak{L}_2 + p\lambda\Delta)}E_2 + \frac{V}{W}E_3\right).
\end{aligned}$$

Here, we let $W = \lambda\eta(\mathfrak{L}_2 + p\lambda\Delta)^2$ and

$$\begin{aligned}
V = \eta(\mathfrak{L}_2 + p\lambda\Delta)(C_0 + p\mathfrak{L}_1 C_1 + p\mathfrak{L}_2 C_2) + (\mathfrak{L}_2 + p\lambda\Delta)^2 C_1\\
- \eta(\mathfrak{L}_1 - 1)(\mathfrak{L}_2 + p\lambda\Delta)C_1 - p\eta^2\Delta C_1 - p\lambda(\mathfrak{L}_2 + p\lambda\Delta)^2 C_2.
\end{aligned}$$

One can readily check that

$$\Delta \cdot V = (\mathfrak{L}_2 + p\lambda\Delta)[\eta\Delta C_0 + (\mathfrak{L}_2 + p\lambda\Delta)(C_1 - p\lambda\Delta C_2)] + X + Y$$

where

$$X = \big[(\mathfrak{L}_2 + p\lambda\Delta)^2(\Delta - 1) - \eta\big((\mathfrak{L}_1 - 1)(\mathfrak{L}_2 + p\lambda\Delta) + p\eta\Delta\big)\Delta\big]C_1$$

and

$$Y = p\eta\Delta(\mathfrak{L}_2 + p\lambda\Delta)(\mathfrak{L}_1 C_1 + \mathfrak{L}_2 C_2).$$

It is clear that $X = 0$ by part (3) of Lemma 4.4.

We often write $\mathfrak{Y}_0$ for $\mathfrak{Y}_0(C_0, C_1, C_2)$ to lighten the notation. Rewriting C_2 as a linear combination of $\frac{C_0}{\lambda\eta}$, $\frac{C_1}{p}$, and $\frac{\eta\Delta C_0 + (\mathfrak{L}_2 + p\lambda\Delta)(C_1 - p\lambda\Delta C_2)}{\lambda\eta\Delta(\mathfrak{L}_2 + p\lambda\Delta)}$, one can rewrite $\mathfrak{Y}_0(C_0, C_1, C_2)$ as follows:

$$\mathfrak{Y}_0 = \frac{C_0}{\lambda\eta}\hat{F}_1 + (u-p)\left(\frac{C_1}{p}\hat{F}_2 + \frac{\eta\Delta C_0 + (\mathfrak{L}_2 + p\lambda\Delta)(C_1 - p\lambda\Delta C_2)}{\lambda\eta\Delta(\mathfrak{L}_2 + p\lambda\Delta)}\hat{F}_3\right)$$

where

$$\begin{aligned}
\hat{F}_1 =& \frac{\lambda\eta}{p}E_1 - \lambda E_2 + \frac{\eta^2}{\mathfrak{L}_2 + p\lambda\Delta}E_2 + E_3 - \frac{\eta[(\mathfrak{L}_1 - 1) - p\mathfrak{L}_1]}{\mathfrak{L}_2 + p\lambda\Delta}E_3 - \frac{p\eta^2\Delta}{(\mathfrak{L}_2 + p\lambda\Delta)^2}E_3 \\
&+ \frac{\eta(u-p)}{\mathfrak{L}_2 + p\lambda\Delta}\left(\lambda E_2 + \frac{\eta\mathfrak{L}_2}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}E_3\right); \\
\hat{F}_2 =& E_1 + \frac{p\eta}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}E_2 + \frac{p(1-\Delta)}{\eta\Delta}E_2 + \frac{p^2\mathfrak{L}_1}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}E_3 + \frac{p\mathfrak{L}_2}{\lambda^2\Delta(\mathfrak{L}_2 + p\lambda\Delta)}E_3; \\
\hat{F}_3 =& -\lambda E_2 + E_3 - \frac{\eta\mathfrak{L}_2}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}E_3.
\end{aligned}$$

It is easy to check that

$$\begin{cases}
\hat{F}_1 \equiv E_3; \\
\hat{F}_2 \equiv E_1 + \dfrac{p(1-\Delta)}{\eta}E_2 + \dfrac{p}{\lambda^2}E_3 \equiv E_1 - \dfrac{p(\mathfrak{L}_1 - 1)}{\mathfrak{L}_2 + p\lambda}E_2 + \dfrac{p}{\lambda^2}E_3; \\
\hat{F}_3 \equiv E_3
\end{cases} \tag{5.10}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ under the assumption $\mathbf{H}(\mathbf{0},\mathbf{3})$. (Note that the second congruence for $\hat{F}_2$ is due to the equation in part (3) of Lemma 4.4.) Hence, $\mathfrak{Y}_0(C_0, C_1, C_2) \in \mathrm{Fil}^2\mathfrak{M}$ if and only if

$$\begin{cases}
v_p(C_0) \geq v_p(\lambda\eta) = 2 - v_p(\lambda), \\
v_p(C_1) \geq 1, \\
v_p\big(\eta\Delta C_0 + (\mathfrak{L}_2 + p\lambda\Delta)(C_1 - p\lambda\Delta C_2)\big) \geq v_p\big(\lambda\eta(\mathfrak{L}_2 + p\lambda\Delta)\big).
\end{cases} \tag{5.11}$$

For any $m \in \mathrm{Fil}^2\mathfrak{D}$, we have $m \in \mathrm{Fil}^2\mathfrak{M}$ if and only if $m \in \mathfrak{Y}_0(C_0, C_1, C_2) + \mathrm{Fil}^2 S\mathfrak{M}$ for some $C_0, C_1, C_2 \in E$ with $\mathfrak{Y}_0(C_0, C_1, C_2) \in \mathfrak{M}$. Therefore, it is enough to check that $\phi(\mathfrak{Y}_0) \in p^2\mathfrak{M}$ whenever $\mathfrak{Y}_0 \in \mathrm{Fil}^2\mathfrak{M}$.

It is also routine to check, by using our computation of $\phi(E_i)$, that

$$\begin{aligned}
\phi(\hat{F}_1) =& \lambda^2\eta E_1 - \frac{\lambda\eta[p(\gamma-1) - (u^p - p)]}{\mathfrak{L}_2 + p\lambda\Delta}E_3 \\
&- \frac{\lambda\eta^2[p(\gamma-1) - (u^p - p)]}{\mathfrak{L}_2 + p\lambda\Delta}\left(E_2 + \frac{p\Delta(\gamma-1)}{\mathfrak{L}_2 + p\lambda\Delta}E_3\right) + \frac{\lambda\eta\phi(\gamma)}{\mathfrak{L}_2 + p\lambda\Delta}E_3 \\
&- \frac{p\lambda\eta^2\Delta\phi(\gamma)}{(\mathfrak{L}_2 + p\lambda\Delta)^2}E_3 + \frac{p\lambda^2\Delta\phi(\gamma)}{\mathfrak{L}_2 + p\lambda\Delta}E_3 - \frac{p\lambda^2\eta\Delta(u^p - p)\phi(\gamma)}{(\mathfrak{L}_2 + p\lambda\Delta)^2}E_3 \\
&+ \frac{\eta^2\phi(\gamma)}{\mathfrak{L}_2 + p\lambda\Delta}\left(\phi(E_2) + \frac{p\lambda\Delta(\phi(\gamma)-1)}{\mathfrak{L}_2 + p\lambda\Delta}E_3\right); \\
\phi(\hat{F}_2) =& p\lambda E_1 + \frac{p}{\Delta}E_2 - \frac{p^2\eta(\gamma-1)}{\mathfrak{L}_2 + p\lambda\Delta}\left(E_2 + \frac{p\Delta(\gamma-1)}{\mathfrak{L}_2 + p\lambda\Delta}E_3\right) + \frac{p(\phi(\gamma) + \mathfrak{L}_1 - 1)}{\mathfrak{L}_2 + p\lambda\Delta}E_3 \\
&+ \frac{p(1-\Delta)}{\eta\Delta}E_3 + \frac{p\eta\phi(\gamma)}{\lambda(\mathfrak{L}_2 + p\lambda\Delta)}\left(\phi(E_2) + \frac{p\lambda\Delta(\phi(\gamma)-1)}{\mathfrak{L}_2 + p\lambda\Delta}E_3\right) \\
&- \frac{p^2\eta\Delta(\phi(\gamma)-1)}{(\mathfrak{L}_2 + p\lambda\Delta)^2}E_3 - \frac{p^2\lambda(1-\Delta)\phi(\gamma)}{\eta(\mathfrak{L}_2 + p\lambda\Delta)}E_3; \\
\phi(\hat{F}_3) =& -\lambda\eta E_2 - \frac{p\lambda\eta\Delta(\gamma-1)}{\mathfrak{L}_2 + p\lambda\Delta}E_3 + \frac{p\lambda^2\Delta\phi(\gamma)}{\mathfrak{L}_2 + p\lambda\Delta}E_3.
\end{aligned}$$

We claim that $\phi(\hat{F}_3) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$, $\phi(\hat{F}_2) \equiv pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$, and $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$. It is obvious that $\phi(\hat{F}_3) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$, and it is also easy to check that $\phi(\hat{F}_2) \equiv \frac{p}{\Delta}E_2 + \frac{p(\mathfrak{L}_1-1)}{\mathfrak{L}_2+p\lambda\Delta}E_3 + \frac{p(1-\Delta)}{\eta\Delta}E_3 + \frac{p^2\eta\Delta}{(\mathfrak{L}_2+p\lambda\Delta)^2}E_3$ modulo $p\mathfrak{m}_E\mathfrak{M}$. Then, by the equation in part (3) of Lemma 4.4, $\phi(\hat{F}_2) \equiv pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$. Similarly, one can readily check that

$$\begin{aligned}\phi(\hat{F}_1) \equiv \lambda^2\eta E_1 &- \frac{\lambda\eta^2[p(\gamma-1)-(u^p-p)]}{\mathfrak{L}_2+p\lambda\Delta}\left(E_2 + \frac{p\Delta(\gamma-1)}{\mathfrak{L}_2+p\lambda\Delta}E_3\right)\\ &- \frac{\lambda\eta[p(\gamma-1)-(u^p-p)]}{\mathfrak{L}_2+p\lambda\Delta}E_3\end{aligned}$$

modulo $p^2\mathfrak{m}_E\mathfrak{M}$. Since $p(\gamma-1) \equiv (u^p-p)$ modulo p^2S, we have $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$.

Therefore, we conclude that

$$\frac{1}{p^2}\phi(\mathfrak{Y}_0) \equiv \frac{\lambda C_0}{p^2}E_1 + \frac{C_1}{p}(\gamma-1)E_2 \tag{5.12}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ if the inequalities in (5.11) hold, and so $\phi(\mathrm{Fil}^2\mathfrak{M}) \subset p^2\mathfrak{M}$. □

Corollary 5.6. *The mod p reductions of the strongly divisible modules in Proposition 5.5 correspond to the Breuil modules $\mathcal{M}_{[0,\frac{1}{2}]}$ described as follows: there exist a basis $\underline{e} := (\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ for $\mathcal{M}_{[0,\frac{1}{2}]}$ and a system of generators $\underline{f} := (f_1, f_2, f_3)$ for $\mathrm{Fil}^2\mathcal{M}$ such that*

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u & 0 & 0 \\ -\frac{\overline{p(\mathfrak{L}_1-1)}}{\mathfrak{L}_2+p\lambda}u & u^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and*

  $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & 0 & \frac{\overline{\lambda^2\eta}}{p^2} \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$;
- *N is induced by $N(\mathrm{E}_1) = N(\mathrm{E}_2) = N(\mathrm{E}_3) = 0$.*

Proof. We keep the notation as in the proof of Proposition 5.5, and we let $f_1' = (u-p)\hat{F}_2$ mod $(\pi_E, \mathrm{Fil}^pS)\mathfrak{M}$, $f_2 = (u-p)^2E_2$ mod $(\pi_E, \mathrm{Fil}^pS)\mathfrak{M}$, and $f_3 = \hat{F}_1$ mod $(\pi_E, \mathrm{Fil}^pS)\mathfrak{M}$. By (5.10), one can readily check that $f_2 = u^2E_2$, $f_3 = E_3$, and $f_1' = uE_1 - \frac{\overline{p(\mathfrak{L}_1-1)}}{\mathfrak{L}_2+p\lambda}uE_2 + \frac{\overline{p}}{\lambda^2}uE_3$ in $\mathcal{M}_{[0,\frac{1}{2}]} = \mathfrak{M}/(\pi_E, \mathrm{Fil}^pS)\mathfrak{M}$ (cf. (1.2)). Let $f_1 = uE_1 - \frac{\overline{p(\mathfrak{L}_1-1)}}{\mathfrak{L}_2+p\lambda}uE_2$. Then $f_1' = f_1 + \frac{\overline{p}}{\lambda^2}uf_3$. Hence, the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M})$ is immediate from these f_i. It is also immediate to get N from (5.9). $\phi_2(f_1) = \phi(f_1' - \frac{\overline{p}}{\lambda^2}uf_3) = -E_2 + 0 = -E_2$, and so it is now easy to compute $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2)$ by using the results in (5.9) and (5.12). □

6. Reduction modulo p of $D_{[\frac{1}{2},1]}$

In this section, we construct strongly divisible modules for the modules $D_{[\frac{1}{2},1]}$ in Example 3.3 and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules. We let $\mathfrak{D}_{[\frac{1}{2},1]} := S \otimes_{\mathbb{Z}_p} D_{[\frac{1}{2},1]}$. In this section, we write D and $\mathfrak{D}$ for $D_{[\frac{1}{2},1]}$ and $\mathfrak{D}_{[\frac{1}{2},1]}$ respectively for brevity.

Recall that we assume $p > 3$ in this paper so that, in particular, $\phi(\gamma) \equiv 0$ modulo $p^4 S$. Since we are concerned only with absolutely irreducible mod p reductions, we may assume $\frac{1}{2} \leq v_p(\lambda) < 1$ (cf. Proposition 3.4).

It is easy to check that

$$\mathrm{Fil}^1\mathfrak{D} = S_E(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3, e_2) + \mathrm{Fil}^1 S_E \cdot \mathfrak{D};$$

$$\mathrm{Fil}^2\mathfrak{D} = S_E\big(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3 + \frac{u-p}{p} e_3\big) + \mathrm{Fil}^1 S_E(e_2) + \mathrm{Fil}^2 S_E \cdot \mathfrak{D}.$$

(We omit their proofs.) So every element in $\mathrm{Fil}^2\mathfrak{D}$ is of the form

$$C_0(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3 + \frac{u-p}{p} e_3) + C_1(u-p)(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3) + C_2(u-p)e_2 + Ae_1 + Be_2 + Ce_3,$$

where C_0, C_1, C_2 are in E and A, B, C are in $\mathrm{Fil}^2 S_E$. We let

$$\mathfrak{X}_1(C_0, C_1, C_2) := C_0\big(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3 + \frac{u-p}{p} e_3\big) + (u-p)\big(C_1(e_1 + \mathfrak{L}_1 e_2 + \mathfrak{L}_2 e_3) + C_2 e_2\big).$$

We divide the area in which the parameters of D are defined into 3 pieces as follows: for $\lambda, \eta \in \mathcal{O}_E$ with $\frac{1}{2} \leq v_p(\lambda) < 1$ and $2v_p(\lambda) + v_p(\eta) = 2$ and for $\mathfrak{L}_1, \mathfrak{L}_2 \in E$,

H(1,1): $v_p(\mathfrak{L}_1 - \eta) \geq 1$ and $v_p(1 - \mathfrak{L}_2) \geq v_p(\lambda)$;
H(1,2): $v_p(\mathfrak{L}_1 - \eta) \geq 1 - v_p(\lambda) + v_p(1 - \mathfrak{L}_2)$ and $v_p(1 - \mathfrak{L}_2) < v_p(\lambda)$;
H(1,3): $v_p(\mathfrak{L}_1 - \eta) < 1$ and
$$\begin{cases} v_p(\mathfrak{L}_1 - \eta) \leq 1 - v_p(\lambda) + v_p(1 - \mathfrak{L}_2) & \text{if } \frac{1}{2} < v_p(\lambda) < 1, \\ v_p(\mathfrak{L}_1 - \eta) < 1 - v_p(\lambda) + v_p(1 - \mathfrak{L}_2) & \text{if } v_p(\lambda) = \frac{1}{2}. \end{cases}$$

Note that these three regions cover the whole $(v_p(1 - \mathfrak{L}_2), v_p(\mathfrak{L}_1 - \eta))$-plane and that if $v_p(1 - \mathfrak{L}_2) < v_p(\lambda)$ and $\frac{1}{2} < v_p(\lambda) < 1$, then the conditions $\mathbf{H(1,2)}$ and $\mathbf{H(1,3)}$ intersect on $v_p(\mathfrak{L}_1 - \eta) = 1 - v_p(\lambda) + v_p(1 - \mathfrak{L}_2)$. We construct strongly divisible modules for each case in the following three subsections.

6.1. **On the region $\mathbf{H(1,1)}$.** In this subsection, we construct strongly divisible modules in $\mathfrak{D}_{[\frac{1}{2},1]}$ under the assumption $\mathbf{H(1,1)}$ and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules.

Proposition 6.1. *Keep the assumption* $\mathbf{H(1,1)}$. *Then* $\mathfrak{M}_{[\frac{1}{2},1]} := S_{\mathcal{O}_E}(E_1, E_2, E_3)$ *is a strongly divisible module in* $\mathfrak{D}_{[\frac{1}{2},1]}$ *where*

$$\begin{aligned} E_1 &= p\mathrm{e}_1 + \tfrac{\mathfrak{L}_1}{\lambda}(\eta \mathrm{e}_2 + \mathrm{e}_3) + (\gamma + \mathfrak{L}_2 - 1)\mathrm{e}_3 + \tfrac{\eta(\gamma-1)}{\lambda}(\eta \mathrm{e}_2 + \mathrm{e}_3); \\ E_2 &= \tfrac{p}{\lambda}(\eta \mathrm{e}_2 + \mathrm{e}_3); \\ E_3 &= \lambda \mathrm{e}_3. \end{aligned}$$

Proof. During the proof, we write $\mathfrak{M}$ for $\mathfrak{M}_{[\frac{1}{2},1]}$ for brevity. It is routine to check that

$$\phi(E_1) = p\lambda E_1 - \lambda[\mathfrak{L}_1 + \eta(\gamma - 1)]E_2 + \frac{\eta[\mathfrak{L}_1 + \eta(\phi(\gamma) - 1)]}{p} E_2 - p(\gamma + \mathfrak{L}_2 - 1)E_3 + (\phi(\gamma) + \mathfrak{L}_2 - 1)E_3 + \frac{\mathfrak{L}_1 + \eta(\phi(\gamma) - 1)}{\lambda} E_3;$$

$$\phi(E_2) = \eta E_2 + \frac{p}{\lambda} E_3;$$

$$\phi(E_3) = \lambda E_3,$$

and

$$N(E_1) = \frac{p}{\lambda}E_3 - \eta[\gamma + (u-p)^{p-1}]E_2 - \frac{p}{\lambda}[\gamma + (u-p)^{p-1}]E_3;$$
$$N(E_2) = N(E_3) = 0.$$

From these computations of $\phi(E_i)$ and $N(E_i)$, it is easy to see that for all $i \in \{1,2,3\}$,

$$\phi(E_i) \equiv 0 \equiv N(E_i) \tag{6.1}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ under the assumption $\mathbf{H}(\mathbf{1},\mathbf{1})$. Hence, $\mathfrak{M}$ is stable under ϕ and N.

Rewriting $\mathfrak{X}_0(C_0,C_1,C_2)$ in terms of E_1,E_2,E_3,

$$\mathfrak{X}_0(C_0,C_1,C_2) \in \mathfrak{Y}_0(C_0,C_1,C_2) + \mathrm{Fil}^p S_E \cdot \mathfrak{D},$$

where

$$\begin{aligned}\mathfrak{Y}_0&(C_0,C_1,C_2)\\ &:= C_0\left(\frac{1}{p}E_1 - \frac{\mathfrak{L}_1-\eta}{p^2}E_2 + \frac{\lambda\mathfrak{L}_1}{p\eta}E_2 + \frac{1-\mathfrak{L}_2}{p\lambda}E_3 - \frac{(\mathfrak{L}_1-\eta)}{\lambda\eta}E_3 - \frac{(1-\mathfrak{L}_2)}{\lambda}E_3\right)\\ &\quad + (u-p)\left(\frac{C_1}{p}E_1 + \frac{p\lambda\mathfrak{L}_1C_1 + p\lambda C_2 - \eta(\mathfrak{L}_1-\eta)C_1}{p^2\eta}E_2 + \frac{V}{W}E_3\right).\end{aligned}$$

Here, we let $W = p\lambda\eta$ and

$$V = \eta(C_0 + p\mathfrak{L}_2C_1) + \eta(1-\mathfrak{L}_2)C_1 - p(\mathfrak{L}_1C_1 + C_2).$$

One can readily check that $\mathfrak{Y}_0(C_0,C_1,C_2)$ can be rewritten as follows:

$$\mathfrak{Y}_0(C_0,C_1,C_2) = \frac{\eta C_0 - pC_2}{p\lambda\eta}\hat{F}_1 + \frac{\lambda C_2}{p\eta}\hat{F}_2 + \frac{C_1}{p}(u-p)\hat{F}_3$$

where

$$\begin{aligned}\hat{F}_1 &= \lambda E_1 - \frac{\lambda(\mathfrak{L}_1-\eta)}{p}E_2 + \frac{\lambda^2\mathfrak{L}_1}{\eta}E_2 + (1-\mathfrak{L}_2)E_3 - \frac{p(\mathfrak{L}_1-\eta)}{\eta}E_3\\ &\quad - p(1-\mathfrak{L}_2)E_3 + (u-p)E_3;\\ \hat{F}_2 &= \frac{p}{\lambda}E_1 - \frac{\mathfrak{L}_1-\eta}{\lambda}E_2 + \frac{p\mathfrak{L}_1}{\eta}E_2 + \frac{p(1-\mathfrak{L}_2)}{\lambda^2}E_3 - \frac{p^2(\mathfrak{L}_1-\eta)}{\lambda^2\eta}E_3\\ &\quad - \frac{p^2(1-\mathfrak{L}_2)}{\lambda^2}E_3 + (u-p)E_2;\\ \hat{F}_3 &= E_1 - \frac{\mathfrak{L}_1-\eta}{p}E_2 + \frac{\lambda\mathfrak{L}_1}{\eta}E_2 + \frac{1-\mathfrak{L}_2}{\lambda}E_3 - \frac{p(\mathfrak{L}_1-\eta)}{\lambda\eta}E_3 - \frac{p(1-\mathfrak{L}_2)}{\lambda}E_3.\end{aligned}$$

It is again easy to check that

$$\hat{F}_1 \equiv uE_3,\ \hat{F}_2 \equiv uE_2,\ \hat{F}_3 \equiv E_1 - \frac{\mathfrak{L}_1-\eta}{p}E_2 + \frac{1-\mathfrak{L}_2}{\lambda}E_3 \tag{6.2}$$

modulo $\mathfrak{m}_E\mathfrak{M}$. Hence, $\mathfrak{Y}_0(C_0,C_1,C_2) \in \mathrm{Fil}^2\mathfrak{M}$ if and only if

$$v_p(\eta C_0 - pC_2) \geq 1 + v_p(\lambda\eta),\ v_p(C_1) \geq 1,\ v_p(C_2) \geq 1 + v_p(\eta) - v_p(\lambda). \tag{6.3}$$

For brevity, we often write $\mathfrak{Y}_0$ for $\mathfrak{Y}_0(C_0,C_1,C_2)$. For any $m \in \mathrm{Fil}^2\mathfrak{D}$, we have $m \in \mathrm{Fil}^2\mathfrak{M}$ if and only if $m \in \mathfrak{Y}_0(C_0,C_1,C_2) + \mathrm{Fil}^2 S\mathfrak{M}$ for some $C_0,C_1,C_2 \in E$ with $\mathfrak{Y}_0(C_0,C_1,C_2) \in \mathfrak{M}$. Therefore, it is enough to check that $\phi(\mathfrak{Y}_0) \in p^2\mathfrak{M}$ whenever $\mathfrak{Y}_0 \in \mathrm{Fil}^2\mathfrak{M}$.

It is also routine to check, from our computation of $\phi(E_i)$, that

$$\begin{aligned}\phi(\hat{F}_1) &= p\lambda^2 E_1 - \lambda^2\eta(\gamma-1)E_2 + \frac{\lambda\eta^2\phi(\gamma)}{p}E_2 \\ &\quad + (\lambda+\eta)\phi(\gamma)E_3 + \lambda[(u^p-p)-p(\gamma-1)]E_3; \\ \phi(\hat{F}_2) &= p^2E_1 + \frac{\eta^2\phi(\gamma)}{\lambda}E_2 + \frac{p(\lambda+\eta)\phi(\gamma)}{\lambda^2}E_3 + [(u^p-p)-p(\gamma-1)](\eta E_2 + \frac{p}{\lambda}E_3); \\ \phi(\hat{F}_3) &= p\lambda E_1 - \lambda\eta(\gamma-1)E_2 + \frac{\eta^2\phi(\gamma)}{p}E_2 - p(\gamma-1)E_3 + \frac{(\lambda+\eta)\phi(\gamma)}{\lambda}E_3,\end{aligned}$$

and so we immediately get

$$\frac{1}{p^2}\phi(\mathfrak{Y}_0) \equiv \frac{\eta C_0 - pC_2}{p\lambda\eta}\left(\frac{\lambda^2}{p}E_1 - \frac{\lambda^2\eta}{p^2}(\gamma-1)E_2\right) + \frac{\lambda C_2}{p\eta}E_1 - \frac{C_1}{p}(\gamma-1)^2E_3 \tag{6.4}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ if the inequalities in (6.2) hold. Hence, $\phi(\mathrm{Fil}^2\mathfrak{M}) \subset p^2\mathfrak{M}$. □

Corollary 6.2. *The mod p reductions of the strongly divisible modules in Proposition* 6.1 *correspond to the Breuil modules $\mathcal{M}_{[\frac{1}{2},1]}$ described as follows: there exist a basis $\underline{e} := (\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ for $\mathcal{M}_{[\frac{1}{2},1]}$ and a system of generators $\underline{f} := (f_1, f_2, f_3)$ for* $\mathrm{Fil}^2\mathcal{M}$ *such that*

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} 0 & 0 & u \\ 0 & u & -\frac{\overline{\mathfrak{L}_1-\eta}}{p}u \\ u & 0 & \frac{\overline{1-\mathfrak{L}_2}}{\lambda}u \end{pmatrix}$ *and*

  $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} \frac{\overline{\lambda^2}}{p} & 1 & 0 \\ \frac{\overline{\lambda^2\eta}}{p^2} & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$;
- *N is induced by $N(\mathrm{E}_1) = N(\mathrm{E}_2) = N(\mathrm{E}_3) = 0$.*

Proof. We keep the notation as in the proof of Proposition 6.1, and we let $f_i = \hat{F}_i$ mod $(\pi_E, \mathrm{Fil}^pS)\mathfrak{M}$ for $i = 1, 2$ and $f_3 = (u-p)\hat{F}_3$ mod $(\pi_E, \mathrm{Fil}^pS)\mathfrak{M}$. By (6.2), we may let $f_1 = uE_3$, $f_2 = uE_2$, $f_3 = uE_1 - \frac{\overline{\mathfrak{L}_1-\eta}}{p}uE_2 + \frac{\overline{1-\mathfrak{L}_2}}{\lambda}uE_3$ in $\mathcal{M}_{[\frac{1}{2},1]} = \mathfrak{M}/(\pi_E, \mathrm{Fil}^pS)\mathfrak{M}$ (cf. (1.2)), which immediately gives rise to $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M})$. It is also immediate to get N and $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2)$ from (6.1) and (6.4) respectively. □

6.2. **On the region $\mathbf{H}(\mathbf{1},\mathbf{2})$.** In this subsection, we construct strongly divisible modules in $\mathfrak{D}_{[\frac{1}{2},1]}$ under the assumption $\mathbf{H}(\mathbf{1},\mathbf{2})$ and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules.

We specialize the results in Section 4.1 to

$$A = 1-\mathfrak{L}_2,\ B = \frac{p\lambda}{\eta},\ C = \frac{p}{\eta}[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)],\ D = \frac{p^2\lambda}{\eta}.$$

It is easy to check that these A, B, C, D with the assumption $\mathbf{H}(\mathbf{1},\mathbf{2})$ satisfy the conditions **Hyp** in Section 4.1. We write $\Delta_{6.2}$ for the limit Δ in this case.

Since the results in this subsection heavily rely on Lemma 4.2, we recall the statement specialized to these A, B, C, D with the assumption $\mathbf{H}(\mathbf{1},\mathbf{2})$:

(1) $v_p(1-\Delta_{6.2}) \geq \min\big\{3[v_p(\lambda)-v_p(1-\mathfrak{L}_2)], v_p\big(\frac{p}{\eta}[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)]\big) - 2v_p(1-\mathfrak{L}_2)\big\} > 0$;

(2) $\frac{\eta(1-\mathfrak{L}_2)+p\lambda\Delta_{6.2}}{\eta(1-\mathfrak{L}_2)+p\lambda} \in 1+\mathfrak{m}_E$ and $\frac{\eta(1-\mathfrak{L}_2)+p\lambda\Delta_{6.2}}{\eta(1-\mathfrak{L}_2)} \in 1+\mathfrak{m}_E$;

(3) $\Delta_{6.2}$ satisfies the equation

$$\begin{aligned}&(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta_{6.2}]^2(1-\Delta_{6.2})\\ &-p\big([(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)][\eta(1-\mathfrak{L}_2)+p\lambda\Delta_{6.2}]+p\lambda\eta\Delta_{6.2}\big)\Delta_{6.2}^2=0.\end{aligned}$$

In this subsection, by Lemma 4.2 we always mean this specialized version of the lemma.

Proposition 6.3. *Keep the assumption* $\mathbf{H}(\mathbf{1},\mathbf{2})$. *Then* $\mathfrak{M}_{[\frac{1}{2},1]} := S_{\mathcal{O}_E}(E_1,E_2,E_3)$ *is a strongly divisible module in* $\mathfrak{D}_{[\frac{1}{2},1]}$ *where*

$$\begin{aligned}E_1&=p\mathrm{e}_1+\tfrac{\mathfrak{L}_1}{\lambda}(\eta\mathrm{e}_2+\mathrm{e}_3)+(\gamma+\mathfrak{L}_2-1)\mathrm{e}_3+\tfrac{\eta^2(1-\mathfrak{L}_2)(\gamma-1)}{\lambda[\eta(1-\mathfrak{L}_2)+p\lambda\Delta_{6.2}]}(\eta\mathrm{e}_2+\mathrm{e}_3);\\ E_2&=\tfrac{\eta(1-\mathfrak{L}_2)}{p}(\eta\mathrm{e}_2+\mathrm{e}_3)-\lambda\Delta_{6.2}(\gamma-1)\mathrm{e}_3;\\ E_3&=\tfrac{\lambda[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}{p}\mathrm{e}_3.\end{aligned}$$

Note that $S_{\mathcal{O}_E}(E_1,E_2,E_3)=S_{\mathcal{O}_E}\big(E_1,E_2,\frac{p(1-\mathfrak{L}_2)}{\lambda}e_3\big)$ by part (2) of Lemma 4.2.

Proof. During the proof, we write Δ for $\Delta_{6.2}$ and $\mathfrak{M}$ for $\mathfrak{M}_{[\frac{1}{2},1]}$ for brevity. It is routine to check that

$$\begin{aligned}\phi(E_1)&=p\lambda E_1-\frac{p^2\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]+\eta^2(1-\mathfrak{L}_2)(\gamma-1)\big)}{\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}\\ &\quad\times\left(E_2+\frac{p\Delta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right)\\ &\quad+\frac{p\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]+\eta^2(1-\mathfrak{L}_2)(\phi(\gamma)-1)\big)}{\lambda\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}\\ &\quad\times\left(\phi(E_2)+\frac{p\lambda\Delta(\phi(\gamma)-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right)\\ &\quad+\frac{p(\phi(\gamma)+\mathfrak{L}_2-1)-p^2(\gamma+\mathfrak{L}_2-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3;\\ \phi(E_2)&=\eta E_2+E_3+\frac{p\eta\Delta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3-\frac{p\lambda\Delta\phi(\gamma)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3;\\ \phi(E_3)&=\lambda E_3,\end{aligned}$$

and

$$\begin{aligned}N(E_1)&=\frac{p^2}{\lambda[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}\Bigg[E_3-[\gamma+(u-p)^{p-1}]\\ &\qquad\times\left(\eta E_2+E_3+\frac{p\eta\Delta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right)\Bigg];\\ N(E_2)&=\frac{p^2\Delta[\gamma+(u-p)^{p-1}]}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3;\\ N(E_3)&=0.\end{aligned}$$

From these computations of $\phi(E_i)$ and $N(E_i)$, it is easy to check that

$$\begin{aligned}&\phi(E_1)\equiv\frac{p[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)]}{\lambda\eta(1-\mathfrak{L}_2)}E_3,\ \phi(E_2)\equiv E_3,\\ &\phi(E_3)\equiv N(E_1)\equiv N(E_2)\equiv N(E_3)\equiv 0\end{aligned}\tag{6.5}$$

modulo $\mathfrak{m}_E\mathfrak{M}$, under the assumption $\mathbf{H}(\mathbf{1},\mathbf{2})$, using parts (1) and (2) of Lemma 4.2. Hence, $\mathfrak{M}$ is stable under ϕ and N.

Rewriting $\mathfrak{X}_1(C_0,C_1,C_2)$ in terms of E_1,E_2,E_3,

$$\mathfrak{X}_1(C_0,C_1,C_2)\in\mathfrak{Y}_1(C_0,C_1,C_2)+\mathrm{Fil}^p S_E\cdot\mathfrak{D},$$

where

$$\begin{aligned}
&\mathfrak{Y}_1(C_0,C_1,C_2)\\
&\quad:=C_0\Bigg(\frac{1}{p}E_1-\frac{\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]-\eta^2(1-\mathfrak{L}_2)}{\lambda\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}E_2\\
&\qquad\qquad+\frac{p\mathfrak{L}_1}{\eta^2(1-\mathfrak{L}_2)}E_2+\frac{p\Delta\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]-\eta^2(1-\mathfrak{L}_2)\big)}{\lambda\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2}E_3\\
&\qquad\qquad-\frac{p\mathfrak{L}_1}{\lambda\eta^2(1-\mathfrak{L}_2)}E_3+\frac{1-\mathfrak{L}_2+p\mathfrak{L}_2}{\lambda[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}E_3\Bigg)\\
&\qquad+(u-p)\Bigg(\frac{C_1}{p}E_1+\frac{p(\mathfrak{L}_1C_1+C_2)}{\eta^2(1-\mathfrak{L}_2)}E_2\\
&\qquad\qquad\qquad-\frac{\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]-\eta^2(1-\mathfrak{L}_2)\big)C_1}{\lambda\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}E_2+\frac{V}{W}E_3\Bigg).
\end{aligned}$$

Here, we let $W=\lambda\eta^2(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2$ and

$$\begin{aligned}
V&=\eta^2(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta](C_0+p\mathfrak{L}_2C_1)+\eta^2(1-\mathfrak{L}_2)^2[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]C_1\\
&+p\eta\big((\mathfrak{L}_1-\eta)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]+p\lambda\eta\Delta\big)\Delta C_1-p[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2(\mathfrak{L}_1C_1+C_2).
\end{aligned}$$

One can readily check that

$$\begin{aligned}
\Delta\cdot V&=[\eta(1-\mathfrak{L}_2)+p\lambda\Delta](\eta^2(1-\mathfrak{L}_2)\Delta C_0\\
&\qquad+[\eta(1-\mathfrak{L}_2)+p\lambda\Delta][\eta(1-\mathfrak{L}_2)C_1-p\Delta C_2])+X+Y
\end{aligned}$$

where

$$\begin{aligned}
X&=-\eta\big((1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2(1-\Delta)\\
&\qquad-p\big([(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)][\eta(1-\mathfrak{L}_2)+p\lambda\Delta]+p\lambda\eta\Delta\big)\Delta^2\big)C_1
\end{aligned}$$

and

$$Y=p\eta^2\Delta\mathfrak{L}_2(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]C_1-p\Delta\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2C_1.$$

It is clear that $X=0$ by part (3) of Lemma 4.2.

We often write $\mathfrak{Y}_1$ for $\mathfrak{Y}_1(C_0, C_1, C_2)$ to lighten the notation. Rewriting C_2 as a linear combination of $\frac{C_0}{\lambda\eta}$, $\frac{C_1}{p}$, and $\frac{\eta^2(1-\mathfrak{L}_2)\Delta C_0+[\eta(1-\mathfrak{L}_2)+p\lambda\Delta][\eta(1-\mathfrak{L}_2)C_1-p\Delta C_2]}{\lambda\eta^2\Delta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}$, one can rewrite $\mathfrak{Y}_1(C_0, C_1, C_2)$ as follows:

$$\begin{aligned}\mathfrak{Y}_1 &= \frac{C_0}{\lambda\eta}\hat{F}_1 \\ &+(u-p)\left(\frac{C_1}{p}\hat{F}_2 + \frac{\eta^2(1-\mathfrak{L}_2)\Delta C_0 + [\eta(1-\mathfrak{L}_2)+p\lambda\Delta][\eta(1-\mathfrak{L}_2)C_1 - p\Delta C_2]}{\lambda\eta^2\Delta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}\hat{F}_3\right)\end{aligned}$$

where

$$\begin{aligned}\hat{F}_2 &= E_1 - \frac{p\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta] - \eta^2(1-\mathfrak{L}_2)\big)}{\lambda\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}E_2 + \frac{p}{\eta\Delta}E_2 \\ &\quad + \frac{p^2\mathfrak{L}_1}{\lambda\eta^2(1-\mathfrak{L}_2)}(\lambda E_2 - E_3) + \frac{p^2\mathfrak{L}_2}{\lambda[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}E_3; \\ \hat{F}_1 &= \frac{\lambda\eta}{p}\left(\hat{F}_2 - \frac{p}{\eta\Delta}E_2\right) + \frac{p\Delta\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta] - \eta^2(1-\mathfrak{L}_2)\big)}{(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2}E_3 \\ &\quad + \frac{\eta(1-\mathfrak{L}_2)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3 + \frac{\lambda\eta(u-p)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_2; \\ \hat{F}_3 &= -\lambda E_2 + E_3.\end{aligned}$$

It is easy to check that

$$\begin{cases}\hat{F}_1 \equiv E_3; \\ \hat{F}_2 \equiv E_1 - \dfrac{p[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)]}{\lambda\eta(1-\mathfrak{L}_2)}E_2 - \dfrac{p^2(\mathfrak{L}_1-\eta)}{\lambda\eta^2(1-\mathfrak{L}_2)}E_3; \\ \hat{F}_3 \equiv E_3\end{cases} \tag{6.6}$$

modulo $\mathfrak{m}_E\mathfrak{M}$. Hence, $\mathfrak{Y}_1(C_0, C_1, C_2) \in \mathrm{Fil}^2\mathfrak{M}$ if and only if

$$\begin{cases}v_p(C_0) \geq v_p(\lambda\eta) = 2 - v_p(\lambda), \\ v_p(C_1) \geq 1, \\ v_p\big(\eta^2(1-\mathfrak{L}_2)\Delta C_0 + [\eta(1-\mathfrak{L}_2)+p\lambda\Delta][\eta(1-\mathfrak{L}_2)C_1 - p\Delta C_2]\big) \\ \qquad \geq v_p\big(\lambda\eta^3(1-\mathfrak{L}_2)^2\big).\end{cases} \tag{6.7}$$

For any $m \in \mathrm{Fil}^2\mathfrak{D}$, we have $m \in \mathrm{Fil}^2\mathfrak{M}$ if and only if $m \in \mathfrak{Y}_1(C_0, C_1, C_2) + \mathrm{Fil}^2 S\mathfrak{M}$ for some $C_0, C_1, C_2 \in E$ with $\mathfrak{Y}_1(C_0, C_1, C_2) \in \mathfrak{M}$. Therefore, it is enough to check that $\phi(\mathfrak{Y}_1) \in p^2\mathfrak{M}$ whenever $\mathfrak{Y}_1 \in \mathrm{Fil}^2\mathfrak{M}$.

It is also routine to check, by using our computation of $\phi(E_i)$, that

$$\begin{aligned}
\phi(\hat{F}_2) &= p\lambda E_1 - \frac{p^2\eta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}\left(E_2 + \frac{p\Delta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right)\\
&\quad + \frac{p\eta\phi(\gamma)}{\lambda[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}\phi(E_2)\\
&\quad + \frac{p^2\Delta\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]+\eta^2(1-\mathfrak{L}_2)(\phi(\gamma)-1)\big)(\phi(\gamma)-1)}{\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2}E_3\\
&\quad + \frac{p(\phi(\gamma)+\mathfrak{L}_2-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3 - \frac{p^3\lambda\Delta\mathfrak{L}_1\phi(\gamma)}{\eta^2(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}E_3\\
&\quad + \frac{p}{\eta\Delta}\left(\eta E_2 + E_3 - \frac{p\lambda\Delta\phi(\gamma)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right);\\
\phi(\hat{F}_1) &= \lambda^2\eta E_1 - \frac{\lambda\eta^2[p(\gamma-1)-(u^p-p)]}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}\left(E_2 + \frac{p\Delta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right)\\
&\quad + \frac{\lambda\eta\phi(\gamma)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\\
&\quad + \frac{p\lambda\Delta\big(\mathfrak{L}_1[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]-\eta^2(1-\mathfrak{L}_2)\big)\phi(\gamma)}{(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2}E_3\\
&\quad - \frac{p^2\lambda^2\Delta\mathfrak{L}_1\phi(\gamma)}{\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]}E_3\\
&\quad + \frac{\eta^2\phi(\gamma)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}\left(\phi(E_2) + \frac{p\lambda\Delta(\phi(\gamma)-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right)\\
&\quad - \frac{\lambda\eta[p(\gamma-1)-(u^p-p)]}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3 - \frac{p\lambda^2\eta\Delta(u^p-p)\phi(\gamma)}{[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2}E_3;\\
\phi(\hat{F}_3) &= -\lambda\eta E_2 - \frac{p\lambda\eta\Delta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3 + \frac{p\lambda^2\Delta\phi(\gamma)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3.
\end{aligned}$$

We claim that $\phi(\hat{F}_3) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$, $\phi(\hat{F}_2) \equiv pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$, and $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$. It is immediate that $\phi(\hat{F}_3) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$, and it is also easy to check that $\phi(\hat{F}_2) \equiv \frac{p}{\Delta}E_2 - \frac{p^2\Delta\left((\mathfrak{L}_1-\eta)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]+p\lambda\eta\Delta\right)}{\eta(1-\mathfrak{L}_2)[\eta(1-\mathfrak{L}_2)+p\lambda\Delta]^2}E_3 + \frac{p(\mathfrak{L}_2-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3 + \frac{p}{\eta\Delta}E_3$ modulo $p\mathfrak{m}_E\mathfrak{M}$. Then, by part (3) of Lemma 4.2, $\phi(\hat{F}_2) \equiv pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$. Similarly, it is also easy to check that $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1 - \frac{\lambda\eta^2[p(\gamma-1)-(u^p-p)]}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}\left(E_2 + \frac{p\Delta(\gamma-1)}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3\right) - \frac{\lambda\eta[p(\gamma-1)-(u^p-p)]}{\eta(1-\mathfrak{L}_2)+p\lambda\Delta}E_3$. Since $p(\gamma-1) \equiv (u^p-p)$ modulo p^2S, we have $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$.

Therefore, we conclude that

$$\frac{1}{p^2}\phi(\mathfrak{Y}_1) \equiv \frac{\lambda C_0}{p^2}E_1 + (\gamma-1)\frac{C_1}{p}E_2 \tag{6.8}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ if the inequalities in (6.7) hold, and so $\phi(\mathrm{Fil}^2\mathfrak{M}) \subset p^2\mathfrak{M}$. □

Corollary 6.4. *The mod p reductions of the strongly divisible modules in Proposition* 6.3 *correspond to the Breuil modules* $\mathcal{M}_{[\frac{1}{2},1]}$ *described as follows: there exist a basis* $\underline{e} := (\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ *for* $\mathcal{M}_{[\frac{1}{2},1]}$ *and a system of generators* $\underline{f} := (f_1, f_2, f_3)$ *for*

$\mathrm{Fil}^2\mathcal{M}$ *such that*

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u & 0 & 0 \\ -\frac{\overline{p[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)]}}{\lambda\eta(1-\mathfrak{L}_2)}u & u^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ *and*

  $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & 0 & \frac{\overline{\lambda^2\eta}}{p^2} \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$;
- N *is induced by* $N(\mathrm{E}_1) = N(\mathrm{E}_2) = N(\mathrm{E}_3) = 0$.

Proof. We keep the notation as in the proof of Proposition 6.3, and we let $f_1' = (u-p)\hat{F}_2 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$, $f_2 = (u-p)^2 E_2 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$, and $f_3 = \hat{F}_1 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$. By (6.6), one can readily check that $f_1' = uE_1 - \frac{\overline{p[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)]}}{\lambda\eta(1-\mathfrak{L}_2)}uE_2 - \frac{\overline{p^2(\mathfrak{L}_1-\eta)}}{\lambda\eta^2(1-\mathfrak{L}_2)}uE_3$, $f_2 = u^2E_2$, and $f_3 = E_3$ in $\mathcal{M}_{[\frac{1}{2},1]} = \mathfrak{M}/(\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$ (cf. (1.2)). Let $f_1 = uE_1 - \frac{\overline{p[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)]}}{\lambda\eta(1-\mathfrak{L}_2)}uE_2$. Then $f_1' = f_1 - \frac{\overline{p^2(\mathfrak{L}_1-\eta)}}{\lambda\eta^2(1-\mathfrak{L}_2)}uf_3$. The matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M})$ is immediate from these f_i. It is also immediate to get N from (6.5). Note that $\phi_2(uf_3) = 0$, and so $\phi_2(f_1) = \phi_2(f_1' + \frac{\overline{p^2(\mathfrak{L}_1-\eta)}}{\lambda\eta^2(1-\mathfrak{L}_2)}uf_3) = -E_2 + 0 = -E_2$. It is now easy to check $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2)$ by using the results in (6.5) and (6.8). □

6.3. **On the region $\mathbf{H}(1,3)$.** In this subsection, we construct strongly divisible modules in $\mathfrak{D}_{[\frac{1}{2},1]}$ under the assumption $\mathbf{H}(1,3)$ and compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules.

We specialize the results in Section 4.2 to

$$A = \mathfrak{L}_1,\ B = -\eta,\ C = -\lambda(1-\mathfrak{L}_2),\ D = -p\lambda^2.$$

It is easy to check that these A, B, C, D with the assumption $\mathbf{H}(1,3)$ satisfy the conditions $\mathbf{Hyp}'$ in Section 4.2. We write $\Delta_{6.3}$ for the limit Δ' in this case.

Since the results in this subsection heavily rely on Lemma 4.4, we recall the statement specialized to these A, B, C, D with the assumption $\mathbf{H}(1,3)$:

(1) $v_p(1-\Delta_{6.3}) \geq \min\left\{v_p\big(\lambda(1-\mathfrak{L}_2)\big) - v_p(\mathfrak{L}_1-\eta), v_p(p\lambda^2) - 2v_p(\mathfrak{L}_1-\eta)\right\} > 0$;
(2) $\frac{\mathfrak{L}_1-\eta\Delta_{6.3}}{\mathfrak{L}_1-\eta} \in 1+\mathfrak{m}_E$;
(3) $\Delta_{6.3}$ satisfies the equation

$$(\mathfrak{L}_1-\eta\Delta_{6.3})^2(1-\Delta_{6.3}) + \lambda[(1-\mathfrak{L}_2)(\mathfrak{L}_1-\eta\Delta_{6.3}) + p\lambda\Delta_{6.3}]\Delta_{6.3} = 0.$$

In this subsection, by Lemma 4.4 we always mean this specialized version of the lemma.

Proposition 6.5. *Keep the assumption* $\mathbf{H}(1,3)$. *Then* $\mathfrak{M}_{[\frac{1}{2},1]} := S_{\mathcal{O}_E}(E_1, E_2, E_3)$ *is a strongly divisible module in* $\mathfrak{D}_{[\frac{1}{2},1]}$ *where*

$$\begin{aligned} E_1 &= p\mathrm{e}_1 + \frac{\mathfrak{L}_1}{\lambda}(\eta\mathrm{e}_2 + \mathrm{e}_3) + (\gamma + \mathfrak{L}_2 - 1)\mathrm{e}_3 \\ &\quad + \frac{p\Delta_{6.3}(\gamma-1)}{\lambda(\mathfrak{L}_1-\eta\Delta_{6.3})}\left(\frac{\eta(\mathfrak{L}_1-\eta\Delta_{6.3})}{p}(\eta\mathrm{e}_2+\mathrm{e}_3) - \lambda^2(\gamma-1)\mathrm{e}_3\right); \\ E_2 &= \frac{\eta(\mathfrak{L}_1-\eta\Delta_{6.3})}{p}(\eta\mathrm{e}_2+\mathrm{e}_3) - \lambda^2(\gamma-1)\mathrm{e}_3; \\ E_3 &= (\mathfrak{L}_1-\eta\Delta_{6.3})\mathrm{e}_3. \end{aligned}$$

Note that $S_{\mathcal{O}_E}(E_1, E_2, E_3) = S_{\mathcal{O}_E}\big(E_1, \frac{\eta(\mathfrak{L}_1-\eta)}{p}(\eta e_2 + e_3), (\mathfrak{L}_1 - \eta)e_3\big)$ in this case since $v_p(\lambda^2) \geq 1 > v_p(\mathfrak{L}_1 - \eta\Delta_{6.3}) = v_p(\mathfrak{L}_1 - \eta)$ by part (2) of Lemma 4.4.

Proof. During the proof, we write Δ for $\Delta_{6.3}$ and $\mathfrak{M}$ for $\mathfrak{M}_{[\frac{1}{2},1]}$ for brevity. It is routine to check that

$$\begin{aligned}
\phi(E_1) &= p\lambda E_1 + \frac{p}{\lambda}E_2 - \frac{p^2[\mathfrak{L}_1 + \eta\Delta(\gamma-1)]}{\eta(\mathfrak{L}_1 - \eta\Delta)}E_2 + E_3 - \frac{p^2\lambda^2\mathfrak{L}_1(\gamma-1)}{\eta(\mathfrak{L}_1-\eta\Delta)^2}E_3 \\
&\quad + \frac{\lambda(\phi(\gamma) + \mathfrak{L}_2 - 1 - p\mathfrak{L}_2)}{\mathfrak{L}_1 - \eta\Delta}E_3 + \frac{p\lambda^2\Delta(\phi(\gamma)-1)}{(\mathfrak{L}_1-\eta\Delta)^2}E_3 + \frac{p\Delta\phi(\gamma)}{\lambda(\mathfrak{L}_1-\eta\Delta)}\phi(E_2); \\
\phi(E_2) &= \eta E_2 + \frac{\lambda\eta}{p}E_3 + \frac{\lambda^2(\lambda-\eta)}{\mathfrak{L}_1-\eta\Delta}E_3 + \frac{\lambda^2(\eta\gamma - \lambda\phi(\gamma))}{\mathfrak{L}_1-\eta\Delta}E_3; \\
\phi(E_3) &= \lambda E_3,
\end{aligned}$$

and

$$\begin{aligned}
N(E_1) &= \frac{p}{\mathfrak{L}_1-\eta\Delta}\left[E_3 - [\gamma + (u-p)^{p-1}]\left(\frac{p\Delta}{\lambda}E_2 + E_3 - \frac{p\lambda\Delta(\gamma-1)}{\mathfrak{L}_1-\eta\Delta}E_3\right)\right]; \\
N(E_2) &= \frac{p\lambda^2[\gamma + (u-p)^{p-1}]}{\mathfrak{L}_1-\eta\Delta}E_3; \\
N(E_3) &= 0.
\end{aligned}$$

From these computations of $\phi(E_i)$ and $N(E_i)$, it is easy to check that

$$\phi(E_1) \equiv E_3 \text{ and } \phi(E_2) \equiv \phi(E_3) \equiv N(E_1) \equiv N(E_2) \equiv N(E_3) \equiv 0 \tag{6.9}$$

modulo $\mathfrak{m}_E\mathfrak{M}$, under the assumption $\mathbf{H}(\mathbf{1},\mathbf{3})$, using parts (1) and (2) of Lemma 4.4. Hence, $\mathfrak{M}$ is stable under ϕ and N.

We often write $\mathfrak{Y}_1$ for $\mathfrak{Y}_1(C_0, C_1, C_2)$ to lighten the notation. Rewriting $\mathfrak{X}_1(C_0, C_1, C_2)$ in terms of E_1, E_2, E_3,

$$\mathfrak{X}_1(C_0, C_1, C_2) \in \mathfrak{Y}_1(C_0, C_1, C_2) + \mathrm{Fil}^p S_E \cdot \mathfrak{D},$$

where

$$\begin{aligned}
\mathfrak{Y}_1 := C_0\Bigg(&\frac{1}{p}E_1 - \frac{1}{\lambda\eta}E_2 + \frac{p\mathfrak{L}_1}{\eta^2(\mathfrak{L}_1-\eta\Delta)}E_2 + \frac{\lambda\mathfrak{L}_1}{\eta(\mathfrak{L}_1-\eta\Delta)^2}E_3 + \frac{1-\mathfrak{L}_2}{p(\mathfrak{L}_1-\eta\Delta)}E_3 \\
&- \frac{p\lambda^2\mathfrak{L}_1}{\eta^2(\mathfrak{L}_1-\eta\Delta)^2}E_3 - \frac{\mathfrak{L}_1 - \eta + \eta(1-\mathfrak{L}_2)}{\eta(\mathfrak{L}_1-\eta\Delta)}E_3\Bigg) \\
&+ (u-p)\left(\frac{C_1}{p}E_1 + \frac{p\lambda(\mathfrak{L}_1C_1 + C_2) - \eta(\mathfrak{L}_1-\eta\Delta)C_1}{\lambda\eta^2(\mathfrak{L}_1-\eta\Delta)}E_2 + \frac{V}{W}E_3\right).
\end{aligned}$$

Here, we let $W = p\eta^2(\mathfrak{L}_1-\eta\Delta)^2$ and

$$\begin{aligned}
V = \eta^2(\mathfrak{L}_1-\eta\Delta)(C_0 + p\mathfrak{L}_2C_1) + p\lambda\eta\mathfrak{L}_1C_1 + \eta^2(1-\mathfrak{L}_2)(\mathfrak{L}_1-\eta\Delta)C_1 \\
- p^2\lambda^2(\mathfrak{L}_1C_1 + C_2) - p\eta(\mathfrak{L}_1-\eta\Delta)(\mathfrak{L}_1C_1 + C_2).
\end{aligned}$$

One can readily check that

$$\begin{aligned}
\lambda\Delta\cdot V &= \lambda\eta(\mathfrak{L}_1-\eta\Delta)(\eta\Delta C_0 - pC_2) \\
&\quad + [\eta(\mathfrak{L}_1-\eta\Delta)(1-\Delta) - p\lambda^2\Delta][p\lambda C_2 - \eta(\mathfrak{L}_1-\eta\Delta)C_1] + X + Y
\end{aligned}$$

where

$$X = \eta^2\big((\mathfrak{L}_1-\eta\Delta)^2(1-\Delta) + \lambda[(1-\mathfrak{L}_2)(\mathfrak{L}_1-\eta\Delta) + p\lambda\Delta]\Delta\big)C_1$$

and

$$Y = \lambda\Delta[p\eta^2\mathfrak{L}_2(\mathfrak{L}_1 - \eta\Delta) - p^2\lambda^2\mathfrak{L}_1 - p\eta\mathfrak{L}_1(\mathfrak{L}_1 - \eta\Delta)]C_1.$$

It is clear that $X = 0$ by part (3) of Lemma 4.4.

Rewriting C_1 as a linear combination of $\frac{C_0}{\lambda\eta}$, $\frac{\eta(\mathfrak{L}_1-\eta\Delta)C_1-p\lambda C_2}{\lambda\eta^2(\mathfrak{L}_1-\eta\Delta)}$, and $\frac{\eta\Delta C_0-pC_2}{p\eta\Delta(\mathfrak{L}_1-\eta\Delta)}$, one can rewrite $\mathfrak{Y}_1(C_0, C_1, C_2)$ as follows:

$$\mathfrak{Y}_1 = \frac{C_0}{\lambda\eta}\hat{F}_1 + (u-p)\left(\frac{\eta(\mathfrak{L}_1 - \eta\Delta)C_1 - p\lambda C_2}{\lambda\eta^2(\mathfrak{L}_1 - \eta\Delta)}\hat{F}_2 + \frac{\eta\Delta C_0 - pC_2}{p\eta\Delta(\mathfrak{L}_1 - \eta\Delta)}\hat{F}_3\right)$$

where

$$\begin{aligned}
\hat{F}_3 &= E_3 - \lambda\Delta\Bigg(E_1 + \frac{p^2\mathfrak{L}_1}{\eta^2(\mathfrak{L}_1 - \eta\Delta)}E_2 \\
&\qquad + \frac{p\eta^2\mathfrak{L}_2(\mathfrak{L}_1 - \eta\Delta) - p^2\lambda^2\mathfrak{L}_1 - p\eta\mathfrak{L}_1(\mathfrak{L}_1 - \eta\Delta)}{\eta^2(\mathfrak{L}_1 - \eta\Delta)^2}E_3\Bigg); \\
\hat{F}_2 &= -\frac{\eta}{p\Delta}\hat{F}_3 - E_2 + \frac{\eta}{p}E_3 + \frac{\lambda^2}{\mathfrak{L}_1 - \eta\Delta}E_3; \\
\hat{F}_1 &= \frac{\eta}{p\Delta}\left(E_3 - \hat{F}_3\right) - E_2 + \frac{\lambda^2\mathfrak{L}_1}{(\mathfrak{L}_1 - \eta\Delta)^2}E_3 + \frac{\lambda\eta(1-\mathfrak{L}_2)}{p(\mathfrak{L}_1 - \eta\Delta)}E_3 \\
&\quad + \frac{\lambda\eta(u-p)}{p(\mathfrak{L}_1 - \eta\Delta)}\left(E_3 - \hat{F}_3\right).
\end{aligned}$$

It is easy to check that

$$\begin{cases}
\hat{F}_1 \equiv -E_2 + \dfrac{\lambda\eta(1-\mathfrak{L}_2)}{p(\mathfrak{L}_1 - \eta)}E_3; \\
\hat{F}_2 \equiv -E_2 - \dfrac{\eta(1-\Delta)}{p}E_3 \equiv -E_2 + \dfrac{\lambda\eta(1-\mathfrak{L}_2)}{p(\mathfrak{L}_1 - \eta)}E_3; \\
\hat{F}_3 \equiv E_3
\end{cases} \tag{6.10}$$

modulo $\mathfrak{m}_E\mathfrak{M}$. (Note that the second congruence for $\hat{F}_2$ is due to part (3) of Lemma 4.4. In fact, one can readily show that $\hat{F}_1 = \hat{F}_2 + \frac{\lambda\eta}{p(\mathfrak{L}_1-\eta\Delta)}(u-p)\left(E_3 - \hat{F}_3\right)$ by part (3) of Lemma 4.4 again.) Hence, $\mathfrak{Y}_1(C_0, C_1, C_2) \in \mathrm{Fil}^2\mathfrak{M}$ if and only if

$$\begin{cases}
v_p(C_0) \geq v_p(\lambda\eta) = 2 - v_p(\lambda), \\
v_p\big(\eta(\mathfrak{L}_1 - \eta\Delta)C_1 - p\lambda C_2\big) \geq v_p\big(\lambda\eta^2(\mathfrak{L}_1 - \eta\Delta)\big), \\
v_p\big(\eta\Delta C_0 - pC_2\big) \geq v_p\big(p\eta(\mathfrak{L}_1 - \eta\Delta)\big).
\end{cases} \tag{6.11}$$

For any $m \in \mathrm{Fil}^2\mathfrak{D}$, we have $m \in \mathrm{Fil}^2\mathfrak{M}$ if and only if $m \in \mathfrak{Y}_1(C_0, C_1, C_2) + \mathrm{Fil}^2 S\mathfrak{M}$ for some $C_0, C_1, C_2 \in E$ with $\mathfrak{Y}_1(C_0, C_1, C_2) \in \mathfrak{M}$. Therefore, it is enough to check that $\phi(\mathfrak{Y}_1) \in p^2\mathfrak{M}$ whenever $\mathfrak{Y}_1 \in \mathrm{Fil}^2\mathfrak{M}$.

It is also routine to check, by using our computation of $\phi(E_i)$, that

$$\begin{aligned}\phi(\hat{F}_3) &= -p\lambda^2\Delta E_1 - p\Delta E_2 + \frac{p^2\lambda\Delta^2(\gamma-1)}{\mathfrak{L}_1-\eta\Delta}E_2 - \lambda\Delta E_3 - \frac{\lambda^2\Delta(\phi(\gamma)+\mathfrak{L}_2-1)}{\mathfrak{L}_1-\eta\Delta}E_3\\ &\quad - \frac{p\lambda^3\Delta^2(\phi(\gamma)-1)}{(\mathfrak{L}_1-\eta\Delta)^2}E_3 - \frac{p\Delta^2\phi(\gamma)}{\mathfrak{L}_1-\eta\Delta}\phi(E_2) + \lambda E_3 + \frac{p^2\lambda^4\Delta\mathfrak{L}_1\phi(\gamma)}{\eta^2(\mathfrak{L}_1-\eta\Delta)^2}E_3;\\ \phi(\hat{F}_2) &= \lambda^2\eta E_1 - \frac{p\lambda\eta\Delta(\gamma-1)}{\mathfrak{L}_1-\eta\Delta}E_2 + \frac{\lambda^2\eta(\phi(\gamma)+\mathfrak{L}_2-1)}{p(\mathfrak{L}_1-\eta\Delta)}E_3 + \frac{\lambda^3\eta\Delta(\phi(\gamma)-1)}{(\mathfrak{L}_1-\eta\Delta)^2}E_3\\ &\quad + \frac{\eta\Delta\phi(\gamma)}{\mathfrak{L}_1-\eta\Delta}\phi(E_2) - \frac{\lambda\eta(1-\Delta)}{p\Delta}E_3 - \frac{p\lambda^4\mathfrak{L}_1\phi(\gamma)}{\eta(\mathfrak{L}_1-\eta\Delta)^2}E_3\\ &\quad - \frac{\lambda^2\eta(\gamma-1)}{\mathfrak{L}_1-\eta\Delta}E_3 + \frac{\lambda^3\phi(\gamma)}{\mathfrak{L}_1-\eta\Delta}E_3,\end{aligned}$$

and we have

$$\phi(\hat{F}_1) = \phi(\hat{F}_2) + \frac{\lambda\eta(u^p-p)}{p(\mathfrak{L}_1-\eta\Delta)}\left(\lambda E_3 - \phi(\hat{F}_3)\right).$$

We claim that $\phi(\hat{F}_3) \equiv -pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$, $\phi(\hat{F}_2) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$, and $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$. It is easy to check that $\phi(\hat{F}_3) \equiv -p\Delta E_2 + \lambda(1-\Delta)E_3 - \frac{\lambda^2\Delta(\mathfrak{L}_2-1)}{\mathfrak{L}_1-\eta\Delta}E_3 + \frac{p\lambda^3\Delta^2}{(\mathfrak{L}_1-\eta\Delta)^2}E_3$ modulo $p\mathfrak{m}_E\mathfrak{M}$ by using parts (1) and (2) of Lemma 4.4. Now it is immediate that $\phi(\hat{F}_3) \equiv -pE_2$ modulo $p\mathfrak{m}_E\mathfrak{M}$ by part (3) of Lemma 4.4. By the same argument, one can check that $\phi(\hat{F}_2) \equiv 0$ modulo $p\mathfrak{m}_E\mathfrak{M}$. For $\phi(\hat{F}_1)$, one can readily check that $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1 + \frac{\lambda\eta\Delta}{(\mathfrak{L}_1-\eta\Delta)}[(u^p-p)-p(\gamma-1)]E_2 + \frac{\lambda^2\eta}{p(\mathfrak{L}_1-\eta\Delta)}[(u^p-p)-p(\gamma-1)]E_3 + (u^p-p)\Big(\frac{\lambda^2\eta(\Delta-1)}{p(\mathfrak{L}_1-\eta\Delta)}E_3 + \frac{\lambda^3\eta\Delta(\mathfrak{L}_2-1)}{p(\mathfrak{L}_1-\eta\Delta)^2}E_3 - \frac{\lambda^4\eta\Delta^2}{(\mathfrak{L}_1-\eta\Delta)^3}E_3\Big)$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$. By part (3) of Lemma 4.4 again, $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1 + \frac{\lambda\eta\Delta}{(\mathfrak{L}_1-\eta\Delta)}[(u^p-p)-p(\gamma-1)]E_2 + \frac{\lambda^2\eta}{p(\mathfrak{L}_1-\eta\Delta)}[(u^p-p)-p(\gamma-1)]E_3$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$, and then, by using $p(\gamma-1) \equiv (u^p-p)$ modulo p^2S, we have $\phi(\hat{F}_1) \equiv \lambda^2\eta E_1$ modulo $p^2\mathfrak{m}_E\mathfrak{M}$.

Therefore, we conclude that

$$\frac{1}{p^2}\phi(\mathfrak{Y}_1) \equiv \frac{\lambda C_0}{p^2}E_1 - (\gamma-1)\frac{\eta\Delta C_0 - pC_2}{p\eta\Delta(\mathfrak{L}_1-\eta\Delta)}E_2 \tag{6.12}$$

modulo $\mathfrak{m}_E\mathfrak{M}$ if the inequalities in (6.11) hold, and so $\phi(\mathrm{Fil}^2\mathfrak{M}) \subset p^2\mathfrak{M}$. □

Corollary 6.6. *The mod p reductions of the strongly divisible modules in Proposition* 6.5 *correspond to the Breuil modules $\mathcal{M}_{[\frac{1}{2},1]}$ described as follows: there exist a basis $\underline{e} := (\mathrm{E}_1, \mathrm{E}_2, \mathrm{E}_3)$ for $\mathcal{M}_{[\frac{1}{2},1]}$ and a system of generators $\underline{f} := (f_1, f_2, f_3)$ for* $\mathrm{Fil}^2\mathcal{M}$ *such that*

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M}) = \begin{pmatrix} u^2 & 0 & 0\\ 0 & 1 & 0\\ 0 & -\frac{\lambda\eta(1-\mathfrak{L}_2)}{p(\mathfrak{L}_1-\eta)} & u\end{pmatrix}$ *and*

  $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2) = \begin{pmatrix} 0 & -\frac{\lambda^2\eta}{p^2} & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}$;
- *N is induced by $N(\mathrm{E}_1) = N(\mathrm{E}_2) = N(\mathrm{E}_3) = 0$.*

Proof. We keep the notation as in the proof of Proposition 6.5, and we let $f_1 = (u-p)^2 E_1 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$, $f_2 = -\hat{F}_1 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$, and $f_3 = (u-p)\hat{F}_3 \bmod (\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$. By (6.10), we may let $f_1 = u^2 E_1$, $f_2 = E_2 - \frac{\overline{\lambda\eta(1-\mathfrak{L}_2)}}{p(\mathfrak{L}_1-\eta)} E_3$, and $f_3 = uE_3$ in $\mathcal{M}_{[\frac{1}{2},1]} = \mathfrak{M}/(\pi_E, \mathrm{Fil}^p S)\mathfrak{M}$ (cf. (1.2)). The matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^2\mathcal{M})$ is immediate from these f_i. It is also immediate to get N from (6.9). One can also readily compute $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_2)$ by using the results in (6.9) and (6.12). □

7. Main results

In this section, we state and prove the main results in this paper. We determine which of the representations has an absolutely irreducible mod p reduction (cf. Theorems 7.1 and 7.2). We also find out which irreducible representations arise as mod p reductions of semi-stable representations of $G_{\mathbb{Q}_p}$ in Hodge–Tate weights $(0,1,2)$ (cf. Theorem 7.3).

Recall that we denote by $\mathcal{M}_{[0,\frac{1}{2}]}$ (resp., by $\mathcal{M}_{[\frac{1}{2},1]}$) the Breuil modules corresponding to the mod p reductions of strongly divisible modules $\mathfrak{M}_{[0,\frac{1}{2}]}$ (resp,. of strongly divisible modules $\mathfrak{M}_{[\frac{1}{2},1]}$) constructed in Section 5 (resp., in Section 6).

Theorem 7.1. *Let $p > 3$ be a prime number, and assume that $0 < v_p(\lambda) \le \frac{1}{2}$ and $2v_p(\lambda)+v_p(\eta) = 2$. Then the reductions modulo p of the semi-stable representations $\mathrm{V}^*_{\mathrm{st}}(D_{[0,\frac{1}{2}]})$ of $G_{\mathbb{Q}_p}$ are absolutely irreducible if and only if either one of the following holds:*

(1) *$v_p\big((\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)\big) > v_p\big(p(\mathfrak{L}_1-1)\big)$ and $v_p(\mathfrak{L}_1-1) < 1-v_p(\lambda)$, in which case*
$$\mathcal{M}_{[0,\frac{1}{2}]} \simeq \mathcal{M}(s,1,1,-\frac{\overline{\lambda^2\eta}}{p^2});$$

(2) *$v_p(\mathfrak{L}_2+p\lambda) < v_p\big(p(\mathfrak{L}_1-1)\big)$ and $v_p(\mathfrak{L}_2+p\lambda) < v_p(\lambda\eta)$, in which case*
$$\mathcal{M}_{[0,\frac{1}{2}]} \simeq \mathcal{M}(s^2,1,-1,\frac{\overline{\lambda^2\eta}}{p^2}).$$

Proof. One can readily check that the Breuil modules in Corollary 5.2 can be expressed in the form of those in Proposition 2.7 by change of basis so that they are non-simple. If $v_p\big((\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)\big) > v_p\big(p(\mathfrak{L}_1-1)\big)$, then $\frac{\overline{(\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)}}{p(\mathfrak{L}_1-1)} = 0$ in $\mathbb{F}$. Hence, the Breuil modules in Corollary 5.4 are simple if and only if $v_p\big((\mathfrak{L}_2+p\lambda)-\eta(\mathfrak{L}_1-1)\big) > v_p\big(p(\mathfrak{L}_1-1)\big)$ and $v_p(\mathfrak{L}_1-1) < 1-v_p(\lambda)$, by Propositions 2.5 and 2.10. Similarly, if $v_p(\mathfrak{L}_2+p\lambda) < v_p\big(p(\mathfrak{L}_1-1)\big)$, then $\frac{\overline{p(\mathfrak{L}_1-1)}}{\mathfrak{L}_2+p\lambda} = 0$ in $\mathbb{F}$. Hence, the Breuil modules in Corollary 5.6 are simple if and only if $v_p(\mathfrak{L}_2+p\lambda) < v_p\big(p(\mathfrak{L}_1-1)\big)$ and $v_p(\mathfrak{L}_2+p\lambda) < v_p(\lambda\eta)$. □

By Proposition 3.2, the reductions modulo p of the representations corresponding to $D_{[0,\frac{1}{2}]}$ when $v_p(\lambda) = \frac{1}{2}$ and $\mathfrak{L}_2 = 0$ are reducible. This is consistent with the results in Theorem 7.1.

Theorem 7.2. *Let $p > 3$ be a prime number, and assume that $\frac{1}{2} \le v_p(\lambda) < 1$ and $2v_p(\lambda)+v_p(\eta) = 2$. Then the reductions modulo p of the semi-stable representations*

$\mathrm{V}^*_{\mathrm{st}}(D_{[\frac{1}{2},1]})$ *of* $G_{\mathbb{Q}_p}$ *are absolutely irreducible if and only if either one of the following holds:*

(1) $v_p\big(p[(\mathfrak{L}_1-\eta)-\lambda(1-\mathfrak{L}_2)]\big) > v_p\big(\lambda\eta(1-\mathfrak{L}_2)\big)$ *and* $v_p(1-\mathfrak{L}_2) < v_p(\lambda)$, *in which case*
$$\mathcal{M}_{[\frac{1}{2},1]} \simeq \mathcal{M}(s^2, 1, -1, \frac{\overline{\lambda^2\eta}}{p^2});$$
(2) $v_p\big(\lambda(\mathfrak{L}_1-\eta)\big) < v_p\big(p(1-\mathfrak{L}_2)\big)$ *and* $v_p\big(\mathfrak{L}_1-\eta\big) < 1$, *in which case*
$$\mathcal{M}_{[\frac{1}{2},1]} \simeq \mathcal{M}(s, 1, 1, -\frac{\overline{\lambda^2\eta}}{p^2}).$$

Proof. Use the Breuil modules in Section 6. The same argument as in the proof of Theorem 7.1 works. □

By Proposition 3.4, the reductions modulo p of the representations corresponding to $D_{[\frac{1}{2},1]}$ when $v_p(\lambda)=\frac{1}{2}$ and $\mathfrak{L}_1=0$ are reducible. This is consistent with the results in Theorem 7.2.

Theorem 7.3. *Let* $p>3$ *be a prime number. If* $\bar{\rho}: G_{\mathbb{Q}_p} \to \mathrm{GL}_3(\overline{\mathbb{F}}_p)$ *is an irreducible mod* p *reduction of a semi-stable and non-crystalline representation with Hodge–Tate weights* $(0,1,2)$, *then* $\bar{\rho}|_{I_{\mathbb{Q}_p}}$ *is isomorphic to either*
$$\omega_3^{2p+1} \oplus \omega_3^{2p^2+p} \oplus \omega_3^{2+p^2} \quad \text{or} \quad \omega_3^{p+2} \oplus \omega_3^{p^2+2p} \oplus \omega_3^{1+2p^2}$$
where ω_3 *is the fundamental character of level* 3.

Proof. Our computation of Breuil modules says that the Breuil modules in Example 2.1 are exactly the simple Breuil modules that occur as irreducible mod p reductions of the semi-stable non-crystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0,1,2)$ (cf. Theorems 7.1 and 7.2). So it is immediate from Proposition 2.5. □

Remark 7.4. We also check that our results in Theorems 7.1 and 7.2 are consistent with the identification in Proposition 3.5. More precisely, we prove that if $v_p(\lambda)=\frac{1}{2}$, then condition (1) (resp., (2)) in Theorem 7.1 is equivalent to (2) (resp., (1)) in Theorem 7.2 in terms of the identification in Proposition 3.5 (as it should be due to Proposition 3.5).

Indeed, we let $D_{[0,\frac{1}{2}]}=D_{[0,\frac{1}{2}]}(\lambda,\eta,\mathfrak{L}_1,\mathfrak{L}_2)$ and $D_{[\frac{1}{2},1]}=D_{[\frac{1}{2},1]}(\lambda,\eta,\mathfrak{L}_1',\mathfrak{L}_2')$ and assume that $v_p(\lambda)=\frac{1}{2}$. Then, condition (1) in Theorem 7.1 holds if and only if $v_p\big(\mathfrak{L}_2-\eta(\mathfrak{L}_1-1)\big) > v_p\big(p(\mathfrak{L}_1-1)\big)$ and $v_p(\mathfrak{L}_1-1) < v_p(\lambda)$, if and only if $v_p\big(\lambda[\mathfrak{L}_2-(\eta-p\lambda)(\mathfrak{L}_1-1)]\big) > v_p\big(p(\lambda-\eta)(\mathfrak{L}_1-1)\big)$ and $v_p\big((\lambda-\eta)(\mathfrak{L}_1-1)\big) < 1$, if and only if $v_p\big(\lambda(\eta-p\lambda)(1-\mathfrak{L}_2')\big) > v_p\big(p[(\lambda-\eta)(\mathfrak{L}_2'-1)+\mathfrak{L}_1']\big)$ and $v_p\big((\lambda-\eta)(\mathfrak{L}_2'-1)+\mathfrak{L}_1'\big) < 1$ (by the identification in Proposition 3.5), if and only if $v_p\big(\lambda(1-\mathfrak{L}_2')\big) > v_p(\mathfrak{L}_1')$ and $v_p\big(\mathfrak{L}_1'\big) < 1$, if and only if condition (2) in Theorem 7.2 holds.

Similarly, condition (1) in Theorem 7.2 holds if and only if $v_p\big(p[\lambda(1-\mathfrak{L}_2')-\mathfrak{L}_1']\big) > v_p\big(\lambda\eta(1-\mathfrak{L}_2')\big)$ and $v_p(1-\mathfrak{L}_2') < v_p(\lambda)$, if and only if $v_p\big(p[(\lambda-\eta)(1-\mathfrak{L}_2')-\mathfrak{L}_1']\big) > v_p\big(\lambda(\eta-p\lambda)(1-\mathfrak{L}_2')\big)$ and $v_p\big((\eta-p\lambda)(1-\mathfrak{L}_2')\big) < v_p(\lambda\eta)$, if and only if $v_p\big(p(\lambda-\eta)(1-\mathfrak{L}_1)\big) > v_p\big(\lambda[(\eta-p\lambda)(1-\mathfrak{L}_1)+\mathfrak{L}_2]\big)$ and $v_p\big((\eta-p\lambda)(1-\mathfrak{L}_1)+\mathfrak{L}_2\big) < v_p(\lambda\eta)$ (by the identification in Proposition 3.5), if and only if $v_p\big(p(\mathfrak{L}_1-1)\big) > v_p(\mathfrak{L}_2)$ and $v_p(\mathfrak{L}_2) < v_p(\lambda\eta)$, if and only if condition (2) in Theorem 7.1 holds.

Moreover, if $v_p(\lambda) = \frac{1}{2}$, then the images of the strongly divisible modules in Subsection 5.2 (resp., in Subsection 5.3) under the isomorphism (3.1) are homothetic to the strongly divisible modules in Subsection 6.3 (resp., in Subsection 6.2), provided that condition (1) (resp., condition (2)) in Theorem 7.1 and condition (2) (resp., condition (1)) in Theorem 7.2 hold, in terms of the identification in Proposition 3.5, since a p-adic representation whose mod p reduction is absolutely irreducible has a unique Galois stable lattice up to homothety.

Acknowledgements

The author is deeply indebted to David Savitt for kindly suggesting the problem and for his inspiring guidance and encouragement throughout the progress of this work. The author is grateful to Florian Herzig for numerous helpful conversations and encouragement. The author would also like to thank the anonymous referee for a careful reading and many helpful comments and suggestions.

References

[Bre97] Christophe Breuil, *Représentations p-adiques semi-stables et transversalité de Griffiths* (French), Math. Ann. **307** (1997), no. 2, 191–224, DOI 10.1007/s002080050031. MR1428871

[Bre99] Christophe Breuil, *Représentations semi-stables et modules fortement divisibles* (French), Invent. Math. **136** (1999), no. 1, 89–122, DOI 10.1007/s002220050305. MR1681105

[BM] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$ (French, with English and French summaries), Duke Math. J. **115** (2002), no. 2, 205–310, DOI 10.1215/S0012-7094-02-11522-1. With an appendix by Guy Henniart. MR1944572

[Car] Xavier Caruso, *Représentations semi-stables de torsion dans le case* $er < p - 1$ (French, with English summary), J. Reine Angew. Math. **594** (2006), 35–92, DOI 10.1515/CRELLE.2006.035. MR2248152

[CHT] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 1–181, DOI 10.1007/s10240-008-0016-1. With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras. MR2470687

[CF] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables* (French), Invent. Math. **140** (2000), no. 1, 1–43, DOI 10.1007/s002220000042. MR1779803

[EG] Matthew Emerton and Toby Gee, *A geometric perspective on the Breuil-Mézard conjecture*, J. Inst. Math. Jussieu **13** (2014), no. 1, 183–223, DOI 10.1017/S147474801300011X. MR3134019

[EGH] Matthew Emerton, Toby Gee, and Florian Herzig, *Weight cycling and Serre-type conjectures for unitary groups*, Duke Math. J. **162** (2013), no. 9, 1649–1722, DOI 10.1215/00127094-2266365. MR3079258

[GK] Toby Gee and Mark Kisin, *The Breuil-Mézard conjecture for potentially Barsotti-Tate representations*, Forum Math. Pi **2** (2014), e1, 56, DOI 10.1017/fmp.2014.1. MR3292675

[Fon] Jean-Marc Fontaine, *Le corps des périodes p-adiques* (French), Astérisque **223** (1994), 59–111. With an appendix by Pierre Colmez; Périodes p-adiques (Bures-sur-Yvette, 1988). MR1293971

[Kis08] Mark Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. **21** (2008), no. 2, 513–546, DOI 10.1090/S0894-0347-07-00576-0. MR2373358

[Kis09] Mark Kisin, *The Fontaine-Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22** (2009), no. 3, 641–690, DOI 10.1090/S0894-0347-09-00628-6. MR2505297

[Liu] Tong Liu, *On lattices in semi-stable representations: a proof of a conjecture of Breuil*, Compos. Math. **144** (2008), no. 1, 61–88, DOI 10.1112/S0010437X0700317X. MR2388556

[Par] Chol Park, *Regular filtered* (φ, N)*-modules of dimension 3*, J. Number Theory **163** (2016), 406–433, DOI 10.1016/j.jnt.2015.11.017. MR3459578

[Sav] David Savitt, *On a conjecture of Conrad, Diamond, and Taylor*, Duke Math. J. **128** (2005), no. 1, 141–197, DOI 10.1215/S0012-7094-04-12816-7. MR2137952

Korea Institute for Advanced Study, 85 Hoegiro Dondaemun-gu, Seoul 02455, Republic of Korea

E-mail address: cpark@kias.re.kr