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ASYMPTOTIC BEHAVIOR OF DENSITIES OF UNIMODAL CONVOLUTION SEMIGROUPS

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Abstract. We prove the asymptotic formulas for the densities of isotropic unimodal convolution semigroups of probability measures on $\mathbb{R}^d$ under the assumption that its Lévy–Khintchine exponent is regularly varying of index between 0 and 2.

1. Introduction

Studying the asymptotic behavior of the densities of convolution semigroups of probability measures has attracted much attention of experts from mathematics and statistics for many years. The main goal of this paper is to prove the asymptotic formulas for the densities $p(t,x)$ of *isotropic unimodal* convolution semigroups of probability measures on $\mathbb{R}^d$ assuming that the corresponding Lévy–Khintchine exponent $\psi(\xi)$ is regularly varying of index α between 0 and 2.

One of the first results of this kind is due to Pólya (1923, $d = 1$) and to Blumenthal and Getoor (1960, $d > 1$) and provides the asymptotic behavior of the transition density $p_\alpha(t,x)$ of the isotropic α-stable process $\mathbf{X}_\alpha$ in $\mathbb{R}^d$, $\alpha \in (0,2)$. The remarkable scaling property of the process $\mathbf{X}_\alpha$, namely

$$p_\alpha(t,x) = t^{-d/\alpha} p_\alpha\big(1, t^{-1/\alpha}x\big), \qquad t > 0, x \in \mathbb{R}^d,$$

implies that the *strong ratio limit property*

$$\lim_{t|x|^{-\alpha}\to+\infty} \frac{p_\alpha(t,x)}{p_\alpha(t,0)} = 1 \tag{1}$$

holds and that $p_\alpha(t,0) = p_\alpha(1,0)t^{-d/\alpha}$. Concerning the ratio limit property in a general setting we refer to the articles [19], [22] (irreducible and aperiodic Markov chains) and to the articles [26], [27] and [31] (continuous time processes, in particular Lévy processes).

Let us recall that the result of Pólya ($d = 1$) and of Blumenthal and Getoor ($d > 1$) reads as follows:

$$\lim_{|x|\to+\infty} |x|^{d+\alpha} p_\alpha(1,x) = \mathcal{A}_{d,\alpha}, \tag{2}$$

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where

$$\mathcal{A}_{d,\alpha} = \alpha 2^{\alpha-1}\pi^{-d/2-1}\sin\left(\frac{\alpha\,\pi}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+d}{2}\right), \tag{3}$$

equivalently

$$\lim_{t|x|^{-\alpha}\to 0}\frac{p_\alpha(t,x)}{t|x|^{-d-\alpha}} = \mathcal{A}_{d,\alpha}.$$

It follows that[1]

$$p_\alpha(t,x) \asymp \min\left\{p_\alpha(t,0), t|x|^{-d-\alpha}\right\}$$

uniformly in $t>0$ and $x\in\mathbb{R}^d$. Observe that for the Cauchy semigroup ($\alpha=1$) we have the equality

$$p_1(t,x) = \mathcal{A}_{d,1}t\left(t^2+|x|^2\right)^{-\frac{d+\alpha}{2}}.$$

The proofs of statement (2) given by Pólya and by Blumenthal and Getoor were based on Fourier analytic techniques. In [2] Bendikov used a different approach utilizing the Bochner's method of subordination. For completeness we would like to list other related papers [10], [15], [16] and [18] and [24]. We also mention that with the aid of subelliptic estimates developed by Głowacki [11], analogous asymptotic formulas were obtained by Dziubański [9] for strictly stable semigroups of measures in the case of homogeneous Lie groups, in particular $\mathbb{R}^d$.

Let $\mathbf{X}$ be an isotropic Lévy process having transition density $p(t,x)$. Then

$$\int_{\mathbb{R}^d} e^{i\langle\xi,x\rangle}p(t,x)\,\mathrm{d}x = e^{-t\psi(\xi)},$$

where the Lévy–Khintchine exponent ψ is a radial function. We also assume that the Lévy measure has *unimodal* density. As it was recently proved in the paper [7], under weak scaling properties of ψ at infinity

$$p(t,x) \asymp \min\left\{p(t,0), t|x|^{-d}\psi\big(|x|^{-1}\big)\right\} \tag{4}$$

locally in space and time variables. Hence, if the characteristic exponent varies regularly at infinity, one can use (4) to obtain estimates for $p(t,x)$. The results of the present article imply the estimates for large t and $|x|$.

The estimates (4) are also valid for the case when ψ varies regularly at zero. Here, additionally, we need to assume that $e^{-t_0\psi}$ is integrable on $\mathbb{R}^d$ for some $t_0>0$. Indeed, as a consequence of Theorem 4, monotonicity of $p(t,\cdot)$ and Remark 2 (compare the proof of [7, Theorm 21]) one can find $M>0$ such that (4) holds for all $|x|, t>M$.

In this article, for ψ varying regularly of index $\alpha\in(0,2)$ at zero, we prove that

$$\lim_{\substack{|x|\to+\infty\\ t\psi(|x|^{-1})\to 0}}\frac{p(t,x)}{t|x|^{-d}\psi\big(|x|^{-1}\big)} = \mathcal{A}_{d,\alpha}, \tag{5}$$

where the constant $\mathcal{A}_{d,\alpha}$ is given by the formula (3) (see Theorem 4).

The main step in getting the asymptotic formula (5) is to find the asymptotic behavior of tails $\mathbb{P}\big(|X_t|\geq r\big)$. We prove for some positive constant $\mathcal{C}_{d,\alpha}$ that

$$\lim_{\substack{r\to+\infty\\ t\psi(r^{-1})\to 0}}\frac{\mathbb{P}\big(|X_t|\geq r\big)}{t\psi(r^{-1})} = \mathcal{C}_{d,\alpha}.$$

[1] $A\asymp B$ means that $cB\leq A\leq CB$, for some constants $c,C>0$.

This asymptotic expression may be viewed as a uniform in time variant of the Tauberian theorem. In the proof, we interpret the asymptotic behavior of the Laplace transform of $\mathbb{P}\big(|X_t| \geq \sqrt{r}\big)$ as a prescribed distributional limit for a dense class of test functions. Then using equicontinouity of the distributions we conclude the convergence. The above technique was previously used in [3] where subordinated random walks on $\mathbb{Z}^d$ were considered.

Besides the transition density $p(t,x)$ we also investigate the Lévy measure ν of the process $\mathbf{X}$. It is rather usual that ν bears an asymptotic resemblance to $p(t,x)$ since in a vague sense $\nu(x) = \lim_{t\to 0^+} t^{-1}p(t,x)$. Therefore, from (5) we may deduce the following asymptotic behavior of ν (see Theorem 7)

$$\lim_{|x|\to+\infty} \frac{\nu(x)}{|x|^{-d}\psi\big(|x|^{-1}\big)} = \mathcal{A}_{d,\alpha}. \tag{6}$$

The fact that the function ψ appears both in asymptotic formulas for ν and $p(t,x)$ is natural from the point of view of the pseudo-differential calculus and the spectral theory. Moreover, as ψ being connected with the Fourier transform of $p(t,x)$, the asymptotic behavior of ψ at infinity translates into the asymptotic formulas for $p(t,x)$ and ν at zero.

A natural question arises:

> *Assume that the density ν of the Lévy measure of the process $\mathbf{X}$ with the Lévy–Khintchine exponent ψ has the asymptotic behavior of the form* (6) *at infinity. Is it true that ψ has a certain prescribed behavior at zero?*

Surprisingly, the answer is affirmative. It turns out that ψ is necessarily regularly varying (see Theorem 7). This interesting observation is a consequence of an application of the famous Drasin–Shea theorem [8, Theorem 6.2]. In the one-dimensional case this result was partially obtained in the unpublished manuscript by Bendikov [1].

In Subsection 5.2 we investigate the asymptotic behavior of the potential measure provided $d \geq 3$ (see Theorem 8) which allows us to prove that for some positive constant $\tilde{\mathcal{A}}_{d,\alpha}$ we have

$$\lim_{|x|\to+\infty} |x|^d\psi\big(|x|^{-1}\big)G(x) = \tilde{\mathcal{A}}_{d,\alpha},$$

where $G(x)$ is a part of the density of the potential measure absolutely continuous with respect to the Lebesgue measure (see Corollary 3). Such a limit was proved in [23, Theorem 3.1 and Theorem 3.3] for the class of subordinate Brownian motions governed by complete Bernstein functions using the Tauberian theorem for the potential measure of subordinator.

We also prove the corresponding theorems with zero replaced by infinity and vice versa (see Theorem 3, Theorem 5, Theorem 6 and Theorem 9).

A vast class of examples of isotropic unimodal Lévy processes constitute subordinate Brownian motions. For such processes we have $\psi(x) = \phi\big(|x|^2\big)$ where ϕ is the Laplace exponent of the corresponding subordinator. In particular, ϕ is a Bernstein function. For instance the monograph [25] gives many cases and classes of Bernstein functions in its closing list of examples. Moreover, [4, Theorem 2.5] shows that for a given function f such that $f(x)$ and $xf'(x)$ are regularly varying

of index $\beta \in [0,1)$ at zero, one can find a complete Bernstein function ϕ such that

$$\lim_{x\to 0^+} \frac{\phi(x)}{f(x)} = 1.$$

Our results in turn imply the asymptotic formulas for the Lévy measure and the transition density of the corresponding subordinate Brownian motion. Let us observe that the symmetric α-stable processes are subordinate Brownian motions since we may take $\phi(\lambda) = \lambda^\alpha$, $\alpha \in (0,1)$. Thus our theorems may be regarded as significant extensions of the classical result (2).

In the present article we covered the case when the Lévy–Khintchine exponent ψ is α-regular for $\alpha \in (0,2)$. The corresponding results for the case $\alpha = 0$ is the subject of the forthcoming paper [13]. The case $\alpha = 2$ is the ongoing project.

2. Preliminaries

Let $\mathbf{X} = (X_t : t \geq 0)$ be an isotropic Lévy process in $\mathbb{R}^d$, i.e., $\mathbf{X}$ is a cádlág stochastic process with a distribution denoted by $\mathbb{P}$ such that $X_0 = 0$ almost surely, the increments of $\mathbf{X}$ are independent with a radial distribution $p(t,\cdot)$ on $\mathbb{R}^d \setminus \{0\}$. This is equivalent to radiality of the Lévy measure and the Lévy–Khintchine exponent. In particular, the characteristic function of $\mathbf{X}$ has the form

$$\int_{\mathbb{R}^d} e^{i\langle \xi, x\rangle} p(t, \mathrm{d}x) = e^{-t\psi(\xi)}, \tag{7}$$

where

$$\psi(\xi) = \int_{\mathbb{R}^d} \big(1 - \cos\langle \xi, x\rangle\big)\nu(\mathrm{d}x) + \eta|\xi|^2, \tag{8}$$

for some $\eta \geq 0$. We are going to abuse notation by setting $\psi(r)$ for $r > 0$ to be equal to $\psi(\xi)$ for any $\xi \in \mathbb{R}^d$ with $|\xi| = r$. Since the function ψ is not necessarily monotonic, it is convenient to work with ψ^* defined by

$$\psi^*(u) = \sup_{s\in[0,u]} \psi(s)$$

for $u \geq 0$. Let us recall that for $r, u \geq 0$ (see [14, Theorem 2.7])

$$\psi(ru) \leq \psi^*(ru) \leq 2(r^2+1)\psi^*(u). \tag{9}$$

A Borel measure μ is isotropic *unimodal* if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial and radially non-increasing density. A Lévy process $\mathbf{X}$ is isotropic unimodal if $p(t,\cdot)$ is isotropic unimodal for each $t > 0$. In Section 4 we consider a subclass of isotropic processes consisting of isotropic unimodal Lévy processes. They were characterized by Watanabe in [30] as those having the isotropic unimodal Lévy measure. A remarkable property of these processes is (see [7, Proposition 2])

$$\psi^*(u) \leq \pi^2 \psi(u) \tag{10}$$

for all $u \geq 0$.

2.1. **Regular variation.** A function $\ell : [x_0, +\infty) \to (0, +\infty)$, for some $x_0 > 0$, is called *slowly varying at infinity* if for each $\lambda > 0$,

$$\lim_{x\to+\infty} \frac{\ell(\lambda x)}{\ell(x)} = 1.$$

We say that $f : [x_0, +\infty) \to (0, +\infty)$ is *regularly varying of index* $\alpha \in \mathbb{R}$ *at infinity*, if $f(x)x^{-\alpha}$ is slowly varying at infinity. The set of regularly varying functions of index α at infinity is denoted by $\mathcal{R}_\alpha^\infty$. In particular, if $f \in \mathcal{R}_\alpha^\infty$, then

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha, \quad \lambda > 0.$$

A function f is regularly varying of index $\alpha \in \mathbb{R}$ at *zero* if $x \mapsto f\big(x^{-1}\big)^{-1}$ belongs to $\mathcal{R}_\alpha^\infty$. The set of regularly varying functions of index α at zero is denoted by $\mathcal{R}_\alpha^0$. The following property of a slowly varying function at zero appears to be very useful (see [21], see also [5, Theorem 1.5.6]). For every $C > 1$ and $\epsilon > 0$ there is $0 < \delta \le x_0$ such that for all $0 < x, y \le \delta$,

$$\ell(x) \le C\ell(y) \max\{x/y, y/x\}^\epsilon. \tag{11}$$

3. Strong ratio limit theorem

In this section we discuss the strong ratio limit property. Let $\mathbf{X}$ be an isotropic Lévy process in $\mathbb{R}^d$ with the Lévy–Khintchine exponent ψ satisfying

$$e^{-t_0\psi} \in L^1\big(\mathbb{R}^d\big)$$

for some $t_0 > 0$. It is well-known that then $\mathbf{X}$ has a continuous transition density $p(t, x)$ such that for every compact subset $K \subset \mathbb{R}^d$,

$$\lim_{t\to\infty} \frac{p(t, x)}{p(t, 0)} = 1 \tag{12}$$

uniformly with respect to $x \in K$. Our aim is to extend this property to the non-compact case.

Theorem 1. *If $\psi \in \mathcal{R}_\alpha^0$ for some $\alpha \in (0, 2]$ and $e^{-t_0\psi} \in L^1\big(\mathbb{R}^d\big)$, for some $t_0 > 0$, then*

$$\lim_{\substack{t\to+\infty \\ t\psi(|x|^{-1})\to+\infty}} \frac{p(t, x)}{p(t, 0)} = 1.$$

Proof. First, we show that there is $C > 0$ such that for t large enough we have

$$\int_{\mathbb{R}^d} e^{-t\psi(\xi)}|\xi| \, \mathrm{d}\xi \le C \left(\psi^-(1/t)\right)^{d+1}, \tag{13}$$

where $\psi^-(u) = \min\{r \ge 0 : \psi(r) \ge u\}$. We observe that integrability of $e^{-2t_0\psi}$ on $\mathbb{R}^d$ implies that $e^{-2t_0\psi(\xi)} < 1$, $\xi \ne 0$. Moreover, by the Riemann–Lebesgue lemma, $e^{-2t_0\psi(\xi)}$ tends to zero if $|\xi|$ approaches infinity. Hence, for any $\delta > 0$ there is $\gamma > 0$ such that for all $|\xi| \ge \delta$,

$$e^{-2t_0\psi(\xi)} \le 1 - \gamma.$$

Therefore, for $t > 4t_0$,

$$\int_{|\xi|\ge\delta} e^{-t\psi(\xi)}|\xi| \, \mathrm{d}\xi \le (1 - \gamma)^{t/(4t_0)} \int_{|\xi|\ge\delta} e^{-2t_0\psi(\xi)}|\xi| \, \mathrm{d}\xi. \tag{14}$$

By (11), there is $\delta > 0$ such that for all $0 < |x|, |y| \le \delta$,

$$\psi(x) \le 2\psi(y)(|x|/|y|)^\alpha \max\{|x|/|y|, |y|/|x|\}^\epsilon. \tag{15}$$

Since $\psi(\psi^-(1/t)) = 1/t$ and for $t > t_0$ sufficiently large $\psi^-(1/t) < \delta$, we get

$$2t\psi(\xi) \ge \big(|\xi|\psi^-(1/t)^{-1}\big)^\alpha \min\Big\{|\xi|\psi^-(1/t)^{-1}, |\xi|^{-1}\psi^-(1/t)\Big\}^\epsilon.$$

Hence, by the change of variables $u = \xi\psi^-(1/t)^{-1}$ we obtain[2]

$$\int_{|\xi|\le\delta} e^{-t\psi(\xi)}|\xi|\,\mathrm{d}\xi \lesssim \big(\psi^-(1/t)\big)^{d+1}. \tag{16}$$

By putting (16) together with (14) we arrive at (13).

Now, by the Fourier inversion formula we get, for $x \in \mathbb{R}^d$,

$$|D_x p(t,x)| \lesssim \int_{\mathbb{R}^d} e^{-t\psi(\xi)}|\langle \xi, x\rangle|\,\mathrm{d}\xi.$$

Hence, by the mean value theorem and the estimate (13)

$$|p(t,0) - p(t,x)| \le \sup_{0\le\theta\le 1} |D_x p(t,\theta x)| \lesssim |x|\big(\psi^-(1/t)\big)^{d+1}.$$

Because

$$p(t,0) \gtrsim \int_{|\xi|\le\psi^-(1/t)} e^{-t\psi(\xi)}\,\mathrm{d}\xi \gtrsim \big(\psi^-(1/t)\big)^d, \tag{17}$$

we obtain

$$|p(t,0) - p(t,x)| \le Cp(t,0)|x|\psi^-(1/t). \tag{18}$$

Therefore, it is enough to show that

$$\lim_{t\psi(|x|^{-1})\to+\infty} |x|\psi^-(1/t) = 0. \tag{19}$$

By (9), for $\lambda \ge 1$ and $u \ge 0$ we have

$$\psi^*(\lambda u) \le 4\lambda^2\psi^*(u),$$

thus

$$\psi^-(\lambda u) \ge \frac{1}{2}\sqrt{\lambda}\psi^-(u).$$

By taking $\lambda = t\psi\big(|x|^{-1}\big)$ and $u = 1/t$ we obtain

$$|x|\psi^-(1/t) \le \frac{1}{2}\big(t\psi\big(|x|^{-1}\big)\big)^{-1/2}, \tag{20}$$

and the proof is finished. □

Remark 1. Instead of regular variation at zero one may assume that there are $\alpha \in (0,2]$ and $c > 0$ such that for all $\lambda \ge 1$, $x \in \mathbb{R}^d$,

$$\psi(\lambda x) \ge c\,\lambda^\alpha\psi(x). \tag{21}$$

Then for each $t > 0$, $e^{-t\psi}$ is integrable on $\mathbb{R}^d$, (see [7, Lemma 7]). Hence, the measure $p(t,\mathrm{d}x)$ has a smooth and integrable density $p(t,x)$ (see [17, Theorem 1]). In particular, we have (13), which again implies (18). Using (19) we conclude that

$$\lim_{t\psi(|x|^{-1})\to\infty} \frac{p(t,x)}{p(t,0)} = 1.$$

[2]We write $A \lesssim B$ if there is an absolute constant $C > 0$ such that $A \le CB$.

For the case when the Lévy–Khintchine exponent ψ varies regularly at infinity we have the following.

Proposition 1. *If $\psi \in \mathcal{R}_\alpha^\infty$ for some $\alpha \in (0,2]$, then*

$$\lim_{\substack{|x|\to 0\\ t\psi(|x|^{-1})\to+\infty}} \frac{p(t,x)}{p(t,0)} = 1.$$

Proof. By (11) we have that for any $\underline{x} > 0$ there is $c > 0$, if $x \geq \underline{x}$ and $\lambda \geq 1$, then

$$\psi(\lambda x) \geq c\,\lambda^{\alpha/2}\psi(x).$$

Whence, by [7, Lemma 16], for any $T > 0$ there is $C_T > 0$ such that for all $t \in (0,T)$,

$$\int_{\mathbb{R}^d} e^{-t\psi(\xi)}|\xi|\,\mathrm{d}\xi \leq C_T\left(\psi^-(1/t)\right)^{d+1}.$$

Again, by the mean value theorem and (17) we get

$$|p(t,0) - p(t,x)| \lesssim |x|p(t,0)\psi^-(1/t), \tag{22}$$

for $t \in (0,T)$.

Let $\epsilon > 0$. By (12), there is $T > 0$ such that for all $t \geq T$ and $|x| \leq \epsilon$,

$$|p(t,x) - p(t,0)| < \epsilon p(t,0).$$

Therefore, by (20) and (22), it is enough to take $t\psi\big(|x|^{-1}\big) \geq C_T^2\epsilon^{-2}$. □

Remark 2. Writing the Fourier inversion formula in polar coordinates we obtain

$$p(t,0) = \frac{2^{1-d}}{\pi^{d/2}\Gamma(d/2)}\int_0^\infty e^{-t\psi(r)}r^{d-1}\,\mathrm{d}r.$$

Hence, after the change of variables we get

$$p(t,0) = \frac{2^{1-d}}{d\pi^{d/2}\Gamma(d/2)}\int_0^\infty e^{-tr}\,\mathrm{d}\left\{\left(\psi^-(r)\right)^d\right\}.$$

Next, applying Karamata's theory [5, Theorem 1.7.1 and 1.7.1'] we conclude

$$\psi \in \mathcal{R}_\alpha^0 \Longrightarrow \lim_{t\to+\infty}\frac{p(t,0)}{\left(\psi^-(1/t)\right)^d} = \frac{2^{1-d}\Gamma(1+d/\alpha)}{d\pi^{d/2}\Gamma(d/2)}$$

and

$$\psi \in \mathcal{R}_\alpha^\infty \Longrightarrow \lim_{t\to 0^+}\frac{p(t,0)}{\left(\psi^-(1/t)\right)^d} = \frac{2^{1-d}\Gamma(1+d/\alpha)}{d\pi^{d/2}\Gamma(d/2)}.$$

Remark 3. Let us mention that in the case when $t\psi\big(|x|^{-1}\big)$ tends neither to zero nor to infinity, the strong ratio limit property may fail. To illustrate this it is enough to consider the Cauchy semigroup with the transition density $p_1(t,x)$ (cf. (2)), for which we have

$$\frac{p_1(t,x)}{p_1(t,0)} = \left(1 + \frac{|x|^2}{t^2}\right)^{-\frac{d+1}{2}}. \tag{23}$$

Now, if $t|x|^{-1}$ goes to η, $\eta \in (0,+\infty)$, then the limit of the ratio (23) depends on η.

4. Asymptotic behavior of tails

Suppose that $\mathbf{X} = (X_t : t \geq 0)$ is an isotropic Lévy process in $\mathbb{R}^d$ with the Lévy–Khintchine exponent ψ. In this section we prove a Tauberian-like theorem for tails of $\mathbf{X}$. For $t > 0$, we set

$$F_t(r) = \mathbb{P}\big(|X_t| \geq \sqrt{r}\big), \quad r \geq 0.$$

For a function $f : [0, +\infty) \to \mathbb{C}$ its Laplace transform $\mathcal{L}f$ is defined by

$$\mathcal{L}f(\lambda) = \int_0^\infty e^{-\lambda r} f(r)\,\mathrm{d}r.$$

Lemma 1. *If $\psi \in \mathcal{R}_\alpha^0$ for some $\alpha \in [0, 2]$, then*

$$\lim_{\substack{\lambda \to 0^+ \\ t\psi(\sqrt{\lambda}) \to 0}} \frac{\lambda \mathcal{L}F_t(\lambda)}{t\psi(\sqrt{\lambda})} = 2^\alpha \frac{\Gamma\big((d+\alpha)/2\big)}{\Gamma(d/2)}.$$

Proof. Let us observe that

$$\lambda \mathcal{L}F_t(\lambda) = \int_{\mathbb{R}^d} \big(1 - e^{-\lambda|x|^2}\big) p(t, \mathrm{d}x).$$

Since

$$1 - e^{-|x|^2} = (4\pi)^{-d/2} \int_{\mathbb{R}^d} \big(1 - \cos\langle x, \xi\rangle\big) e^{-\frac{|\xi|^2}{4}}\,\mathrm{d}\xi, \tag{24}$$

by the Fubini–Tonelli theorem and (7) we get

$$\begin{aligned}\lambda \mathcal{L}F_t(\lambda) &= (4\pi)^{-d/2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \big(1 - \cos\langle x, \xi\sqrt{\lambda}\rangle\big) p(t, \mathrm{d}x) e^{-\frac{|\xi|^2}{4}}\,\mathrm{d}\xi \\ &= (4\pi)^{-d/2} \int_{\mathbb{R}^d} \big(1 - e^{-t\psi(\xi\sqrt{\lambda})}\big) e^{-\frac{|\xi|^2}{4}}\,\mathrm{d}\xi.\end{aligned}$$

Therefore, using polar coordinates we obtain

$$\lambda \mathcal{L}F_t(\lambda) = \frac{2^{1-d}}{\Gamma(d/2)} \int_0^\infty \big(1 - e^{-t\psi(r\sqrt{\lambda})}\big) e^{-\frac{r^2}{4}} r^{d-1}\,\mathrm{d}r. \tag{25}$$

We claim that for every $\epsilon > 0$ there are $\delta > 0$ and $C = C(\delta) > 0$ such that for all $r > 0$ and $0 < u \leq \delta$,

$$\psi(ru) \leq C\psi(u)\big(r^2 + r^{-\epsilon}\big). \tag{26}$$

For $\alpha > 0$, we recall that (see [5, Theorem 1.5.3])

$$\lim_{u \to 0^+} \frac{\psi^*(u)}{\psi(u)} = 1.$$

Hence, by (9), there is $C > 0$,

$$\psi(ru) \leq \psi^*(ru) \leq 2(r^2 + 1)\psi^*(u) \leq C(r^2 + 1)\psi(u).$$

For $\alpha = 0$, by (11), there is $\delta > 0$ such that if $ru, u \leq \delta$, then

$$\psi(ru) \leq 2\psi(u) \max\{r^\epsilon, r^{-\epsilon}\}.$$

Otherwise, $ru \geq \delta$ and (9) implies that

$$\psi(ru) \leq \psi^*(1)\big(2 + \delta^{-2}\big) r^2 u^2.$$

Again, by (11), for every $\epsilon > 0$ there is $C > 0$ such that

$$\psi(u) \geq Cu^\epsilon,$$

we obtain

$$\psi(ru) \lesssim r^2\psi(u)u^{2-\epsilon} \lesssim r^2\psi(u),$$

proving the claim (26).

Next, by (26), there is $C > 0$ such that for all $r \geq 0$ and $\lambda \leq \delta$,

$$\frac{1 - e^{-t\psi(r\sqrt{\lambda})}}{t\psi(\sqrt{\lambda})} \leq \frac{\psi(r\sqrt{\lambda})}{\psi(\sqrt{\lambda})} \leq C(r^2 + r^{-\epsilon}).$$

Hence, by the dominated convergence theorem

$$\lim_{\substack{\lambda\to 0^+\\ t\psi(\sqrt{\lambda})\to 0}} \frac{1}{t\psi(\sqrt{\lambda})} \int_0^\infty \big(1 - e^{-t\psi(r\sqrt{\lambda})}\big)e^{-r^2/4}r^{d-1}\,\mathrm{d}r = 2^{d+\alpha-1}\Gamma\big((d+\alpha)/2\big),$$

because for each $r > 0$,

$$\lim_{\substack{\lambda\to 0^+\\ t\psi(\sqrt{\lambda})\to 0}} \frac{1 - e^{-t\psi(r\sqrt{\lambda})}}{t\psi(r\sqrt{\lambda})} \cdot \frac{\psi(r\sqrt{\lambda})}{\psi(\sqrt{\lambda})} = r^\alpha. \qquad \square$$

The following theorem provides the asymptotic behavior of F_t at infinity. Here, we have to exclude $\alpha = 2$ which is natural since for the Brownian motion $\psi(x) = |x|^2$ and the tail decays exponentially.

Theorem 2. *If $\psi \in \mathcal{R}_\alpha^0$ for some $\alpha \in [0, 2)$, then*

$$\lim_{\substack{r\to 0^+\\ t\psi(r)\to 0}} \frac{\mathbb{P}\big(|X_t| \geq r^{-1}\big)}{t\psi(r)} = \mathcal{C}_{d,\alpha}, \tag{27}$$

where

$$\mathcal{C}_{d,\alpha} = 2^\alpha \frac{\Gamma\big((d+\alpha)/2\big)}{\Gamma(d/2)\Gamma(1-\alpha/2)}.$$

Proof. Let us define

$$\mathcal{F}_t(x) = \int_0^x F_t(r)\,\mathrm{d}r, \quad x \geq 0.$$

Then

$$\mathcal{F}_t(x) \leq e\int_0^x e^{-r/x}F_t(r)\,\mathrm{d}r \leq e\mathcal{L}F_t(x^{-1}).$$

Hence, by (25)

$$\mathcal{F}_t(x) \lesssim xt\int_0^\infty \psi\big(sx^{-1/2}\big)e^{-\frac{s^2}{4}}s^{d-1}\,\mathrm{d}s. \tag{28}$$

Let $\delta > 0$. For $(t, r) \in (0, +\infty) \times (0, \delta)$ we define a tempered distribution $\Lambda_{t,r} \in \mathcal{S}'\big([0, +\infty)\big)$ by setting

$$\Lambda_{t,r}(f) = \frac{r}{t\psi\big(r^{1/2}\big)}\int_0^\infty f(x)\mathcal{F}_t\big(r^{-1}x\big)\,\mathrm{d}x, \quad f \in \mathcal{S}\big([0,\infty)\big).$$

Recall that the space $\mathcal{S}\big([0, +\infty)\big)$ consists of Schwartz functions on $\mathbb{R}$ restricted to $[0, +\infty)$, and $\mathcal{S}'\big([0, +\infty)\big)$ consists of tempered distributions supported by $[0, +\infty)$; see [28] for details.

We claim that the family $\big(\Lambda_{t,r} : t > 0, r \in (0,\delta)\big)$ is equicontinuous. Indeed, by (28) and (26), for each $\epsilon > 0$ there is $\delta > 0$ such that

$$\begin{aligned}\Lambda_{t,r}(f) &\lesssim \int_0^\infty |f(x)| x \int_0^\infty \frac{\psi\big(sr^{1/2}x^{-1/2}\big)}{\psi\big(r^{1/2}\big)} e^{-\frac{s^2}{4}} s^{d-1}\,\mathrm{d}s\,\mathrm{d}x \\ &\lesssim \int_0^\infty |f(x)| \int_0^\infty \big(s^2 + s^{-\epsilon}x^{1-\epsilon/2}\big) e^{-\frac{s^2}{4}} s^{d-1}\,\mathrm{d}s\,\mathrm{d}x,\end{aligned}$$

for all $t > 0$ and $r \in (0,\delta)$. Hence,

$$\Lambda_{t,r}(f) \lesssim \sup_{x\geq 0} \big|(1+x)^3 f(x)\big|.$$

Next, for any $\tau > 0$ we set $f_\tau(x) = e^{-\tau x}$. Then

$$\Lambda_{t,r}(f_\tau) = \frac{1}{t\tau^2\psi\big(r^{1/2}\big)} \tau r \mathcal{L}F_t(\tau r) = \frac{\psi\big(r^{1/2}\tau^{1/2}\big)}{\tau^2\psi\big(r^{1/2}\big)} \cdot \frac{\tau r \mathcal{L}F_t(\tau r)}{t\psi\big(\tau^{1/2}r^{1/2}\big)}.$$

In particular, by Lemma 1 we obtain

$$\lim_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \Lambda_{t,r}(f_\tau) = C'_{d,\alpha}\tau^{\alpha/2-2} = \frac{C'_{d,\alpha}}{\Gamma(2-\alpha/2)} \int_0^\infty e^{-\tau x} x^{1-\alpha/2}\,\mathrm{d}x,$$

where

$$C'_{d,\alpha} = 2^\alpha \frac{\Gamma\big((d+\alpha)/2\big)}{\Gamma(d/2)}.$$

Since $\mathcal{B}$, the linear span of the set $\big\{f_\tau : \tau > 0\big\}$, is dense in $\mathcal{S}\big([0,+\infty)\big)$ and the family $\big(\Lambda_{t,r} : t > 0, r \in (0,\delta)\big)$ is equicontinuous on $\mathcal{S}\big([0,+\infty)\big)$, we conclude that for any $f \in \mathcal{S}\big([0,+\infty)\big)$,

$$\lim_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \Lambda_{t,r}(f) = \frac{C'_{d,\alpha}}{\Gamma(2-\alpha/2)} \int_0^\infty f(x) x^{1-\alpha/2}\,\mathrm{d}x.$$

For completeness of argument we provide a sketch of the proof that $\mathcal{B}$ is dense in $\mathcal{S}\big([0,+\infty)\big)$, for details we refer to [28] and [29]. By the Hahn–Banach theorem it is enough to show that if $\Lambda \in \mathcal{S}'\big([0,+\infty)\big)$ and $\Lambda(\phi) = 0$ for all $\phi \in \mathcal{B}$, then Λ is the zero functional. Assume that Λ vanishes on $\mathcal{B}$ and let $\mathcal{L}\Lambda(z) = \Lambda(e^{-z\cdot})$, $\operatorname{Re} z > 0$, be the Laplace transform of Λ. Since $\Lambda = 0$ on $\mathcal{B}$ we get $\mathcal{L}\Lambda(\lambda) = 0$ for $\lambda > 0$. But the Laplace transform is analytic in the half-plane $\operatorname{Re} z > 0$, whence $\mathcal{L}\Lambda(z) = 0$, for $\operatorname{Re} z > 0$. Using the connection between Laplace and Fourier transforms,

$$\widehat{\Lambda}(\xi) = \lim_{\lambda\to 0^+} \mathcal{L}\Lambda(\lambda + i\xi), \quad \text{in } \mathcal{S}'\big([0,+\infty)\big),$$

we obtain that $\widehat{\Lambda} = 0$. Hence, Λ is the zero functional as desired.

Now, we claim that

$$\lim_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \frac{r\mathcal{F}_t\big(r^{-1}\big)}{t\psi(\sqrt{r})} = \frac{C'_{d,\alpha}}{\Gamma(2-\alpha/2)}. \tag{29}$$

For a given $\epsilon > 0$ choose $\phi_+ \in \mathcal{S}\big([0,+\infty)\big)$ such that

$$\phi_+(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } 1+\varepsilon \leq x. \end{cases}$$

We have

$$\frac{r\mathcal{F}_t\big(r^{-1}\big)}{t\psi(\sqrt{r})} = \frac{r}{t\psi(\sqrt{r})}\int_0^{1/r} \mathrm{d}\mathcal{F}_t(s) \le \frac{r}{t\psi(\sqrt{r})}\int_0^{1/r} \phi_+(s/r)\,\mathrm{d}\mathcal{F}_t(s)$$
$$\le \frac{r}{t\psi(\sqrt{r})}\int_0^{\infty} \phi_+(s/r)\,\mathrm{d}\mathcal{F}_t(s),$$

thus

$$\frac{r\mathcal{F}_t\big(r^{-1}\big)}{t\psi(\sqrt{r})} \le -\Lambda_{t,r}(\phi'_+).$$

Hence,

$$\limsup_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \frac{r\mathcal{F}_t\big(r^{-1}\big)}{t\psi(\sqrt{r})} \le \frac{-C'_{d,\alpha}}{\Gamma(2-\alpha/2)}\int_0^{\infty} s^{1-\alpha/2}\phi'_+(s)\,\mathrm{d}s$$
$$= \frac{C'_{d,\alpha}}{\Gamma(1-\alpha/2)}\int_0^{\infty} s^{-\alpha/2}\phi_+(s)\,\mathrm{d}s$$
$$\le \frac{C'_{d,\alpha}}{\Gamma(2-\alpha/2)}(1+\varepsilon)^{1-\alpha/2}.$$

Similarly, taking $\phi_- \in \mathcal{S}\big([0,+\infty)\big)$ such that

$$\phi_-(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1-\varepsilon,\\ 0 & \text{for } 1 \le x,\end{cases}$$

we may show that

$$\liminf_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \frac{r\mathcal{F}_t\big(r^{-1}\big)}{t\psi(\sqrt{r})} \ge \frac{C'_{d,\alpha}}{\Gamma(2-\alpha/2)}(1-\varepsilon)^{1-\alpha/2}.$$

This proves (29).

To show (27) we adapt the proof of the monotone density theorem (see, e.g., [5, Theorem 1.7.2]). Let $\epsilon > 0$. The function $F_t(s)$ is non-increasing, therefore

$$\mathcal{F}_t\big(r^{-1}\big) - \mathcal{F}_t\big((1-\epsilon)r^{-1}\big) = \int_{(1-\epsilon)/r}^{1/r} F_t(s)\,\mathrm{d}s \ge \epsilon r^{-1}F_t\big(r^{-1}\big) \tag{30}$$

and

$$\mathcal{F}_t\big((1+\epsilon)r^{-1}\big) - \mathcal{F}_t\big(r^{-1}\big) = \int_{1/r}^{(1+\epsilon)/r} F_t(s)\,\mathrm{d}s \le \epsilon r^{-1}F_t\big(r^{-1}\big). \tag{31}$$

By (29),

$$\lim_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \frac{r\mathcal{F}_t\big((1-\epsilon)r^{-1}\big)}{t\psi(\sqrt{r})} = \frac{C'_{d,\alpha}(1-\epsilon)^{1-\alpha/2}}{\Gamma(2-\alpha/2)}.$$

Hence, (30) implies

$$\limsup_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \frac{F_t\big(r^{-1}\big)}{t\psi(\sqrt{r})} \le \frac{C'_{d,\alpha}}{\Gamma(2-\alpha/2)}\cdot\frac{1-(1-\epsilon)^{1-\alpha/2}}{\epsilon}.$$

Similarly, by (31) we get

$$\liminf_{\substack{r\to 0^+\\ t\psi(\sqrt{r})\to 0}} \frac{F_t\big(r^{-1}\big)}{t\psi(\sqrt{r})} \geq \frac{C'_{d,\alpha}}{\Gamma(2-\alpha/2)} \cdot \frac{(1+\epsilon)^{1-\alpha/2}-1}{\epsilon}.$$

Finally, by taking ϵ tending to zero we obtain (27). □

By the same line of reasoning as in the proofs of Lemma 1 and Theorem 2 we may show the corresponding results if the Lévy–Khintchine exponent ψ varies regularly at infinity.

Lemma 2. *If $\psi \in \mathcal{R}_\alpha^\infty$ for $\alpha \in [0,2]$, then*

$$\lim_{\substack{\lambda\to+\infty\\ t\psi(\sqrt{\lambda})\to 0}} \frac{\lambda\mathcal{L}F_t(\lambda)}{t\psi(\sqrt{\lambda})} = 2^\alpha \frac{\Gamma\big((d+\alpha)/2\big)}{\Gamma(d/2)}.$$

Theorem 3. *If $\psi \in \mathcal{R}_\alpha^\infty$ for some $\alpha \in [0,2)$, then*

$$\lim_{\substack{r\to+\infty\\ t\psi(r)\to 0}} \frac{\mathbb{P}\big(|X_t| \geq r^{-1}\big)}{t\psi(r)} = \mathcal{C}_{d,\alpha}.$$

5. Asymptotic behavior of densities

Suppose that $\mathbf{X} = (X_t : t \geq 0)$ is an isotropic *unimodal* Lévy process in $\mathbb{R}^d$, i.e., a process having a rotationally invariant and radially non-increasing density function $p(t,\,\cdot\,)$ on $\mathbb{R}^d \setminus \{0\}$.

5.1. Asymptotics of transition density.

In the following theorem we give the asymptotic behavior of the transition density $p(t,x)$.

Theorem 4. *If $\psi \in \mathcal{R}_\alpha^0$, for some $\alpha \in (0,2)$, then*

$$\lim_{\substack{|x|\to+\infty\\ t\psi(|x|^{-1})\to 0}} \frac{p(t,x)}{|x|^{-d} t\psi\big(|x|^{-1}\big)} = \mathcal{A}_{d,\alpha},$$

where

$$\mathcal{A}_{d,\alpha} = \alpha 2^{\alpha-1}\pi^{-d/2-1}\sin\Big(\frac{\alpha\pi}{2}\Big)\Gamma\Big(\frac{\alpha}{2}\Big)\Gamma\Big(\frac{d+\alpha}{2}\Big).$$

Proof. In the proof we adapt the argument from [5, Theorem 1.7.2]. For any $0 < a < b$, we have

$$F_t(a/r) - F_t(b/r) = c_d \int_{\sqrt{a/r}}^{\sqrt{b/r}} u^{d-1} p(t,u)\,\mathrm{d}u,$$

where

$$c_d = 2\frac{\pi^{d/2}}{\Gamma(d/2)}.$$

Since the function $u \mapsto p(t,u)$ is non-increasing, we get

$$\frac{F_t(a/r) - F_t(b/r)}{t\psi(\sqrt{r})} \geq \frac{c_d}{d} \cdot \frac{p(t,\sqrt{b/r})}{t\psi(\sqrt{r})} \cdot \frac{b^{d/2}-a^{d/2}}{r^{d/2}} \tag{32}$$

and

$$\frac{F_t(a/r) - F_t(b/r)}{t\psi(\sqrt{r})} \le \frac{c_d}{d} \cdot \frac{p(t, \sqrt{a/r})}{t\psi(\sqrt{r})} \cdot \frac{b^{d/2} - a^{d/2}}{r^{d/2}}. \tag{33}$$

By Theorem 2, we have

$$\lim_{\substack{r\to 0^+ \\ t\psi(\sqrt{r})\to 0}} \frac{F_t(a/r) - F_t(b/r)}{t\psi(\sqrt{r})} = \mathcal{C}_{d,\alpha}\big(a^{-\alpha/2} - b^{-\alpha/2}\big).$$

Hence, (32) gives

$$\limsup_{\substack{r\to 0^+ \\ t\psi(\sqrt{r})\to 0}} \frac{p(t, \sqrt{b/r})}{r^{d/2} t\psi(\sqrt{r})} \le d \cdot \frac{\mathcal{C}_{d,\alpha}}{c_d} \cdot \frac{a^{-\alpha/2} - b^{-\alpha/2}}{b^{d/2} - a^{d/2}}.$$

Taking $b = 1$, $a = 1 - \epsilon$ and letting ϵ tend to zero, we obtain

$$\lim_{\substack{r\to 0^+ \\ t\psi(\sqrt{r})\to 0}} \frac{p\big(t, r^{-1/2}\big)}{r^{d/2} t\psi(\sqrt{r})} \le \alpha \cdot \frac{\mathcal{C}_{d,\alpha}}{c_d}.$$

Similarly, using (33), one can show that

$$\liminf_{\substack{r\to 0^+ \\ t\psi(\sqrt{r})\to 0}} \frac{p\big(t, r^{-1/2}\big)}{r^{d/2} t\psi(\sqrt{r})} \ge \alpha \cdot \frac{\mathcal{C}_{d,\alpha}}{c_d}.$$

Finally, by the Euler's reflection formula we conclude the proof. □

Using Lemma 2 and Theorem 3 we obtain the following asymptotic formula.

Theorem 5. *If $\psi \in \mathcal{R}_\alpha^\infty$ for some $\alpha \in (0, 2)$, then*

$$\lim_{\substack{x\to 0 \\ t\psi(|x|^{-1})\to 0}} \frac{p(t, x)}{|x|^{-d} t\psi\big(|x|^{-1}\big)} = \mathcal{A}_{d,\alpha}.$$

Next, let us denote by $\nu(x)$ the density function of the Lévy measure ν associated to the process $\mathbf{X}$. The following theorem gives the asymptotics of $\nu(x)$ as well as the equivalence of asymptotics of $p(t, x)$ and $\nu(x)$ with regular variation of the Lévy–Khintchine exponent.

Theorem 6. *Let $\mathbf{X} = (X_t : t \ge 0)$ be an isotropic unimodal Lévy process on $\mathbb{R}^d$ with the characteristic exponent ψ and the Lévy density ν. Then the following are equivalent:*

(i) *$\psi \in \mathcal{R}_\alpha^\infty$ for some $\alpha \in (0, 2)$;*
(ii) *there is $c > 0$,*

$$\lim_{\substack{x\to 0 \\ t\psi(|x|^{-1})\to 0}} \frac{p(t, x)}{|x|^{-d} t\psi\big(|x|^{-1}\big)} = c;$$

(iii) *there is $c > 0$,*

$$\lim_{x\to 0} \frac{\nu(x)}{|x|^{-d} \psi\big(|x|^{-1}\big)} = c.$$

Proof. We observe that Theorem 5 yields the implication (i) $\Rightarrow$ (ii). Also, the implication (ii) $\Rightarrow$ (iii) follows because

$$\lim_{t\to 0^+} t^{-1}p(t,x) = \nu(x) \tag{34}$$

vaguely on $\mathbb{R}^d \setminus \{0\}$. Indeed, let $\epsilon > 0$, then there exists $\delta > 0$ such that for $|x| \le \delta$ and $t\psi(|x|^{-1}) < \delta$,

$$c - \epsilon \le \frac{t^{-1}p(t,x)}{|x|^{-d}\psi\big(|x|^{-1}\big)} \le c + \epsilon.$$

Hence, by taking t approaching zero we get

$$c - \epsilon \le \frac{\nu(x)}{|x|^{-d}\psi\big(|x|^{-1}\big)} \le c + \epsilon$$

for $0 < |x| \le \delta$.

To prove that (iii) implies (i), we use the Drasin–Shea Theorem (see [8, Theorem 6.2], see also [5, Theorem 5.2.1]). Let $u_0 = \big(d^{-1/2}, \dots, d^{-1/2}\big) \in \mathbb{R}^d$ and $r > 0$. Notice that (iii) forces the Gaussian part to vanish. Using the polar coordinates we may write

$$\begin{aligned}\psi(r) = \psi(ru_0) &= \int_0^\infty \int_{\mathbb{S}^{d-1}} \big(1 - \cos(\rho r\langle u_0, u\rangle)\big)\sigma(\mathrm{d}u)\rho^{d-1}\nu(\rho)\,\mathrm{d}\rho \\ &= \int_0^\infty k(\rho^{-1}r)\rho^{-d}\nu\big(\rho^{-1}\big)\rho^{-1}\,\mathrm{d}\rho,\end{aligned}$$

where σ denotes the spherical measure on the unite sphere $\mathbb{S}^{d-1}$ in $\mathbb{R}^d$, and

$$k(r) = \int_{\mathbb{S}^{d-1}} \big(1 - \cos(r\langle u_0, u\rangle)\big)\sigma(\mathrm{d}u). \tag{35}$$

Let us recall the definition of the *Mellin convolution* (see, e.g., [5, Section 4.1]), defined for two functions $f, g : [0,\infty) \to \mathbb{C}$ by the formula

$$\mathcal{M}(f,g)(x) = \int_0^\infty f\big(t^{-1}x\big)g(t)t^{-1}\,\mathrm{d}t.$$

Then, by setting $f(r) = r^{-d}\nu\big(r^{-1}\big)$, we may write

$$\psi(r) = \mathcal{M}(k,f)(r).$$

Since

$$0 \le k(r) \le (1 \wedge r^2)\sigma\big(\mathbb{S}^{d-1}\big), \tag{36}$$

the Mellin transform $\check{k}$ where

$$\check{k}(z) = \int_0^\infty t^{-z-1}k(t)\,\mathrm{d}t,$$

is absolutely convergent on the strip $\{z \in \mathbb{C} : 0 < \operatorname{Re} z < 2\}$. Moreover, $x \mapsto \nu(x)$ is non-increasing and integrable on $\big\{x \in \mathbb{R}^d : |x| \ge 1\big\}$, thus

$$\lim_{r\to 0^+} f(r) = 0.$$

By (iii), there is $\delta > 0$ such that for all $|x| \le \delta$,

$$(c-\epsilon)|x|^{-d}\psi\big(|x|^{-1}\big) \le \nu(x) \le (c+\epsilon)|x|^{-d}\psi\big(|x|^{-1}\big).$$

Hence, [7, Theorem 26] implies that ψ satisfies weak upper and lower scaling, i.e., there are $C > 0$, and $\overline{\beta}, \underline{\beta} \in (0,2)$, and $r_0 \geq 0$ such that for all $r \geq r_0$,

$$C^{-1} r^{\underline{\beta}} \leq \psi(r) \leq C r^{\overline{\beta}}.$$

Therefore, if $r \geq \max\{\delta^{-1}, r_0\}$ we obtain

$$C^{-1}(c-\epsilon) r^{\underline{\beta}} \leq r^{-d} \nu\big(r^{-1}\big) \leq C(c+\epsilon) r^{\overline{\beta}}.$$

Thus,

$$\rho = \limsup_{r \to +\infty} \frac{\log f(r)}{\log r} \in (0,2),$$

and f has bounded decrease (see [5, Section 2.1]). Moreover,

$$\lim_{r \to +\infty} \frac{\mathcal{M}(k,f)(r)}{f(r)} = \lim_{x \to 0} \frac{\psi\big(|x|^{-1}\big)}{|x|^{-d} \nu(x)} = c^{-1}.$$

Therefore, we may apply the Drasin–Shea theorem to conclude that $f \in \mathcal{R}_\rho^\infty$, which translates to $\psi \in \mathcal{R}_\rho^\infty$. □

Corollary 1. *If $\psi \in \mathcal{R}_\alpha^\infty$, then*

$$\lim_{\substack{x \to 0 \\ t\psi(|x|^{-1}) \to 0}} \frac{p(t,x)}{t\nu(x)} = 1.$$

To prove an analogue of the above theorem in the case when $\psi \in \mathcal{R}_\alpha^0$ we shall need the following lemma.

Lemma 3. *Suppose that there are $c > 0$ and $M > 1$ such that*

$$\nu(x) \geq c|x|^{-d} \psi\big(|x|^{-1}\big)$$

whenever $|x| \geq M$. Then there are $C > 1$, and $\underline{\beta}, \overline{\beta} \in (0,2)$ such that for all $0 < \lambda, r \leq 1$,

$$C^{-1} \lambda^{\underline{\beta}} \psi(r) \leq \psi(\lambda r) \leq C \lambda^{\overline{\beta}} \psi(r). \tag{37}$$

Proof. We adapt the proof of [7, Theorem 26] to the current setting. First, let us notice that

$$\lim_{r \to 0^+} \frac{r^2}{\psi(r)} = 0. \tag{38}$$

Indeed, by [12, Corollary 1], for $0 < r < M^{-1}$,

$$\psi^*(r) \geq \frac{r^2}{24d} \int_{r|x| \leq 1} |x|^2 \nu(x)\, \mathrm{d}x \gtrsim r^2 \int_M^{r^{-1}} u \psi\big(u^{-1}\big)\, \mathrm{d}u.$$

Moreover, by (9) and (10), for $u \geq 1$,

$$\pi^2 \psi(u^{-1}) \geq \psi^*(u^{-1}) \geq \frac{1}{2(1+u^2)} \psi^*(1).$$

Thus,

$$\psi(r) \gtrsim r^2 \int_M^{r^{-1}} \frac{u}{1+u^2}\, \mathrm{d}u,$$

which implies (38).

In view of (38), we may assume that $\mathbf{X}$ is pure-jump, i.e., $\eta = 0$ in (8). Let us consider a function $\phi : [0, +\infty) \to \mathbb{R}$ defined by

$$\phi(\lambda) = \int_0^\infty \frac{\lambda}{\lambda + s} \nu\big(s^{-1/2}\big) s^{-1-d/2} \, \mathrm{d}s.$$

Then

$$\phi(\lambda) = \int_0^\infty \big(1 - e^{-\lambda r}\big) \mu(r) \, \mathrm{d}r,$$

where

$$\mu(r) = \int_0^\infty e^{-rs} \nu\big(s^{-1/2}\big) s^{-d/2} \, \mathrm{d}s.$$

By [25, the proof of Theorem 6.2], the function ϕ is a complete Bernstein function. According to [7, estimates (28)–(30)], there is $C > 0$ such that

$$C^{-1}\phi(\lambda) \le \psi(\sqrt{\lambda}) \le C\phi(\lambda), \quad \text{for all } \lambda > 0, \tag{39}$$

and

$$\mu(r) \le C r^{-2} \phi'\big(r^{-1}\big), \quad \text{for all } r > 0,$$

and

$$\nu(x) \le C|x|^{-d+2} \mu\big(|x|^2\big), \quad \text{for all } x \neq 0.$$

Therefore, for $|x| > M$ we get for some $c > 0$,

$$cC^{-1}\phi\big(|x|^{-2}\big) \le c\psi\big(|x|^{-1}\big) \le |x|^d \nu(x) \le C^2 |x|^{-2} \phi'\big(|x|^{-2}\big). \tag{40}$$

Hence, there is $\overline{\beta} > 0$ such that for all $\lambda \in \big(0, M^{-2}\big)$,

$$\overline{\beta}\phi(\lambda) \le \lambda \phi'(\lambda).$$

In particular, the function $\lambda \mapsto \lambda^{-\overline{\beta}}\phi(\lambda)$ is non-decreasing on $\big(0, M^{-2}\big)$, thus, for all $u \in (0, 1)$ and $\lambda \in \big(0, M^{-2}\big)$

$$(u\lambda)^{-\overline{\beta}} \phi(u\lambda) \le \lambda^{-\overline{\beta}} \phi(\lambda),$$

which implies the upper bound of (37) due to (39) and continuity of ψ. From (38) we may deduce that $\overline{\beta} < 2$.

Next, we show the lower scaling at zero. As ϕ is a complete Bernstein function, we have that $\phi_1 : [0, +\infty) \to \mathbb{R}$, where

$$\phi_1(\lambda) = \frac{\lambda}{\phi(\lambda)},$$

is a special Bernstein function. Since $\mathbf{X}$ is a pure-jump Lévy process,

$$\lim_{|x| \to +\infty} \frac{\psi(x)}{|x|^2} = 0,$$

Thus, by (39) we conclude that

$$\lim_{\lambda \to +\infty} \phi_1(\lambda) = +\infty.$$

Moreover, $\phi(0) = 0$. Hence, the potential measure of the subordinator with the Laplace exponent ϕ_1 (see [25, (10.9) and Theorem 10.3]) is absolutely continuous with the density function

$$f(s) = \int_s^\infty \mu(u) \, \mathrm{d}u.$$

We observe that, by (40), for $s > M^2$,

$$\mu(s) \geq cC^{-2}\phi\big(s^{-1}\big),$$

thus,

$$\begin{aligned} f(s) &\geq cC^{-2}\int_s^\infty \phi\big(u^{-1}\big)u^{-1}\,\mathrm{d}u \\ &\geq cC^{-2}\int_s^\infty \phi'\big(u^{-1}\big)u^{-2}\,\mathrm{d}u \\ &= cC^{-2}\phi\big(s^{-1}\big). \end{aligned} \tag{41}$$

Since $\mathcal{L}f = 1/\phi_1$, by [7, Lemma 5], there is $D > 0$ such that for $s > 0$,

$$f(s) \leq D\frac{\phi_1'\big(s^{-1}\big)}{s^2\phi_1^2\big(s^{-1}\big)}.$$

Hence, by (41), there is $\underline{\beta} > 0$ such that for $\lambda \in \big(0, M^{-2}\big)$,

$$\underline{\beta}\phi_1(\lambda) \leq \lambda\phi_1'(\lambda). \tag{42}$$

Therefore, for all $u \in (0,1)$ and $\lambda \in \big(0, M^{-2}\big)$ we get

$$\frac{(u\lambda)^{1-\underline{\beta}}}{\phi(u\lambda)} = (u\lambda)^{-\underline{\beta}}\phi_1(u\lambda) \leq \lambda^{-\underline{\beta}}\phi_1(\lambda) = \frac{\lambda^{1-\underline{\beta}}}{\phi(\lambda)},$$

which implies the left inequality in (37). Let us observe that since ϕ_1 is concave, we have

$$\lambda\phi_1'(\lambda) \leq \phi_1(\lambda),$$

thus, (42) forces $\underline{\beta} < 1$. □

Theorem 7. *Let $\mathbf{X} = (X_t : t \geq 0)$ be an isotropic unimodal Lévy process on $\mathbb{R}^d$ with the characteristic exponent ψ and the Lévy density ν. Then the following are equivalent:*

(i) *$\psi \in \mathcal{R}_\alpha^0$, for some $\alpha \in (0,2)$;*

(ii) *there is $c > 0$*

$$\lim_{\substack{|x|\to+\infty \\ t\psi(|x|^{-1})\to 0}} \frac{p(t,x)}{|x|^{-d}t\psi\big(|x|^{-1}\big)} = c;$$

(iii) *there is $c > 0$*

$$\lim_{|x|\to+\infty} \frac{\nu(x)}{|x|^{-d}\psi\big(|x|^{-1}\big)} = c.$$

Proof. The proof is similar to Theorem 6. First, in view of Theorem 4 and (34), it is enough to show that (iii) implies (i).

Here, again we use the Drasin–Shea theorem. In light of Lemma 3 we may assume that the process $\mathbf{X}$ is pure-jump. Let $u_0 = \big(d^{-1/2}, \ldots, d^{-1/2}\big) \in \mathbb{R}^d$. For $r > 0$ we can write

$$\psi\big(r^{-1}\big) = \mathcal{M}(K,F)(r),$$

where $F(r) = r^d\nu(r)$ and

$$K(r) = \int_{\mathbb{S}^{d-1}} \big(1 - \cos\big(r^{-1}\langle u_0, u\rangle\big)\big)\sigma(\mathrm{d}u).$$

Notice that the function F does not vanish at zero. Therefore, instead of F we consider

$$\tilde{F}(r) = \begin{cases} F(r) & \text{if } r \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since

$$0 \leq \mathcal{M}(K,F)(r) - \mathcal{M}(K,\tilde{F})(r) \leq Cr^{-2} \int_0^1 t^{1+d}\nu(t)\,\mathrm{d}t,$$

by (38), we have

$$\lim_{r\to+\infty} \frac{\mathcal{M}(K,F)(r)}{\mathcal{M}(K,\tilde{F})(r)} = 1.$$

By (iii), for $\epsilon > 0$ there is $M_0 > 1$ such that for all $|x| \geq M_0$,

$$(c-\epsilon)|x|^{-d}\psi\big(|x|^{-1}\big) \leq \nu(x) \leq (c+\epsilon)|x|^{-d}\psi\big(|x|^{-1}\big).$$

Hence, by Lemma 3, there are $C > 1$, and $\underline{\beta}, \overline{\beta} \in (0,2)$, such that for all $|x| \geq M_0$,

$$C^{-1}(c-\epsilon)|x|^{-\underline{\beta}}\psi\big(M_0^{-1}\big) \leq |x|^d\nu(x) \leq C(c+\epsilon)|x|^{-\overline{\beta}}\psi\big(M_0^{-1}\big).$$

In particular,

$$\rho = \limsup_{r\to+\infty} \frac{\log \tilde{F}(r)}{\log r} \in (-2,0),$$

and $\tilde{F}$ has bounded decrease. Again, by (iii), we get

$$\lim_{r\to+\infty} \frac{\mathcal{M}(K,\tilde{F})(r)}{\tilde{F}(r)} = \lim_{r\to+\infty} \frac{\mathcal{M}(K,F)(r)}{F(r)} = \lim_{|x|\to+\infty} \frac{|x|^{-d}\psi\big(|x|^{-1}\big)}{\nu(x)} = c^{-1}.$$

Now, we may apply the Drasin–Shea theorem to obtain $\tilde{F} \in \mathcal{R}_\rho^\infty$. Therefore, $F \in \mathcal{R}_\rho^\infty$, which implies $\psi \in \mathcal{R}_\rho^0$. □

Corollary 2. *If $\psi \in \mathcal{R}_\alpha^0$, then*

$$\lim_{\substack{|x|\to+\infty \\ t\psi(|x|^{-1})\to 0}} \frac{p(t,x)}{t\nu(x)} = 1.$$

5.2. **Asymptotics of Green function.** In this subsection we assume $d \geq 3$. Therefore, the isotropic unimodal Lévy process $\mathbf{X} = (X_t : t \geq 0)$ is transient and the associated potential measure G is well-defined,

$$G(x,A) = \int_0^\infty \mathbb{P}_x(X_t \in A)\,\mathrm{d}t,$$

where $\mathbb{P}_x$ is the standard measure $\mathbb{P}(\,\cdot\,|X_0 = x)$ and $A \subset \mathbb{R}^d$ is a Borel set. We set $G(A) = G(0,A)$. We also use the same notation G for the density of the part of the potential measure absolutely continuous with respect to the Lebesgue measure. Thus, we have $G(x,y) = G(0,y-x)$. We set $G(x) = G(0,x)$.

Theorem 8. *Assume that $\psi \in \mathcal{R}_\alpha^0$, for some $\alpha \in [0,2]$. Then*

$$\lim_{r\to+\infty} \psi\big(r^{-1}\big)G\big(\{x : |x| \leq r\}\big) = \tilde{\mathcal{C}}_{d,\alpha},$$

where

$$\tilde{\mathcal{C}}_{d,\alpha} = 2^{-\alpha}\frac{\Gamma\big((d-\alpha)/2\big)}{\Gamma(d/2)\Gamma(1+\alpha/2)}.$$

Proof. Let us define

$$f(r) = G\big(\{x : |x| \le \sqrt{r}\}\big) = \int_0^\infty \int_{|x|\le\sqrt{r}} p(t, \mathrm{d}x)\,\mathrm{d}t.$$

Hence, by the Fubini–Tonelli theorem we have

$$\mathcal{L}f(\lambda) = \lambda^{-1} \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda|x|^2} p(t, \mathrm{d}x)\,\mathrm{d}t.$$

By (24), the second application of the Fubini–Tonelli theorem gives

$$\begin{aligned}\mathcal{L}f(\lambda) &= (4\pi)^{-d/2}\lambda^{-1} \int_0^\infty \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \cos\langle x, \xi\sqrt{\lambda}\rangle\, p(t, \mathrm{d}x) e^{-\frac{|\xi|^2}{4}}\,\mathrm{d}\xi\,\mathrm{d}t \\ &= (4\pi)^{-d/2}\lambda^{-1} \int_0^\infty \int_{\mathbb{R}^d} e^{-t\psi(\xi\sqrt{\lambda})} e^{-\frac{|\xi|^2}{4}}\,\mathrm{d}\xi\,\mathrm{d}t,\end{aligned}$$

where in the last equality we have used (7). Finally, after integration with respect to t with the help of polar coordinates we may write

$$\begin{aligned}\mathcal{L}f(\lambda) &= (4\pi)^{-d/2}\lambda^{-1} \int_{\mathbb{R}^d} e^{-\frac{|\xi|^2}{4}} \frac{\mathrm{d}\xi}{\psi(\xi\sqrt{\lambda})} \\ &= \frac{2^{1-d}}{\Gamma(d/2)}\lambda^{-1} \int_0^\infty e^{-\frac{r^2}{4}} r^{d-1} \frac{\mathrm{d}r}{\psi(r\sqrt{\lambda})}.\end{aligned}$$

Since the process $\mathbf{X}$ is unimodal, ψ and ψ^* are comparable by (10). Therefore, by (9)

$$\psi(\sqrt{\lambda}) \le \psi^*(\sqrt{\lambda}) \le 2(r^{-2}+1)\psi^*(r\sqrt{\lambda}) \le 2\pi^2(r^{-2}+1)\psi(r\sqrt{\lambda}).$$

Because $d \ge 3$, by the dominated convergence we obtain

$$\lim_{\lambda\to 0^+} \frac{\lambda\mathcal{L}f(\lambda)}{\psi(\sqrt{\lambda})} = 2^{-\alpha}\frac{\Gamma\big((d-\alpha)/2\big)}{\Gamma(d/2)}.$$

Hence, by the Tauberian theorem (see, e.g., [5, Theorem 1.7.1]) we conclude the proof. □

Corollary 3. *Assume that $\psi \in \mathcal{R}_\alpha^0$, for some $\alpha \in (0, 2]$. Then*

$$\lim_{|x|\to+\infty} |x|^d \psi\big(|x|^{-1}\big) G(x) = \tilde{\mathcal{A}}_{d,\alpha},$$

where

$$\tilde{\mathcal{A}}_{d,\alpha} = 2^{-\alpha}\pi^{-d/2}\frac{\Gamma\big((d-\alpha)/2\big)}{\Gamma(\alpha/2)}.$$

Proof. For the proof we use Theorem 8 and the line of reasoning from the proof of Theorem 4. □

Theorem 9. *Assume that $\psi \in \mathcal{R}_\alpha^\infty$, for some $\alpha \in [0, 2]$. Then*

$$\lim_{r\to 0^+} \psi\big(r^{-1}\big) G\big(\{x : |x| \le r\}\big) = \tilde{\mathcal{C}}_{d,\alpha}.$$

Corollary 4. *Assume that $\psi \in \mathcal{R}_\alpha^\infty$, for some $\alpha \in (0, 2]$. Then*

$$\lim_{x\to 0} |x|^d \psi\big(|x|^{-1}\big) G(x) = \tilde{\mathcal{A}}_{d,\alpha}.$$

5.3. **Examples.** We begin with a result which is helpful in verifying whether the Lévy–Khintchine exponent ψ is regularly varying.

Proposition 2. *Let* $\mathbf{X}$ *be a pure-jump isotropic unimodal Lévy processes with the Lévy density* $\nu(r) = r^{-d}g(r^{-1})$*, for some* $g : (0,\infty) \to [0,\infty)$*.*

(i) *If* $g \in \mathcal{R}_\alpha^0$ *for some* $\alpha \in (0,2)$*, then* $\psi \in \mathcal{R}_\alpha^0$ *and*

$$\lim_{r\to 0^+} \frac{g(r)}{\psi(r)} = \mathcal{A}_{d,\alpha}.$$

(ii) *If* $g \in \mathcal{R}_\alpha^\infty$ *for some* $\alpha \in (0,2)$*, then* $\psi \in \mathcal{R}_\alpha^\infty$ *and*

$$\lim_{r\to +\infty} \frac{g(r)}{\psi(r)} = \mathcal{A}_{d,\alpha}.$$

Proof. For $r > 0$, we have (see (35))

$$\psi(r) = \int_0^\infty k(r\rho)g(\rho^{-1})\frac{\mathrm{d}\rho}{\rho} = \int_0^\infty k(\rho)g(r\rho^{-1})\frac{\mathrm{d}\rho}{\rho}.$$

Assume that $g \in \mathcal{R}_\alpha^0$. Let $0 < \varepsilon < 2-\alpha$, then, by (11), there is $0 < \delta \le 1$ such that for $r, r\rho^{-1} \le \delta$,

$$g(r\rho^{-1}) \le 2g(r)\rho^{-\alpha-\varepsilon}.$$

Moreover, by (36) we get

$$\int_0^{r\delta^{-1}} k(\rho)g(r\rho^{-1})\frac{\mathrm{d}\rho}{\rho} = \int_0^{\delta^{-1}} k(r\rho)g(\rho^{-1})\frac{\mathrm{d}\rho}{\rho} \lesssim r^2.$$

Hence, the first claim of the proposition is a consequence of the dominated convergence theorem and the fact that

$$\mathcal{A}_{d,\alpha} = \int_0^\infty k(\rho)\frac{\mathrm{d}\rho}{\rho^{1+\alpha}}.$$

In a similar way one can prove the second claim. □

Example 1. Let $\mathbf{X}$ be the relativistic stable process, i.e., $\psi(x) = (|x|^2+1)^{\alpha/2} - 1$, $\alpha \in (0,2)$. We have

$$\lim_{\substack{x\to 0\\ t|x|^{-\alpha}\to 0}} \frac{p(t,x)}{t|x|^{-d-\alpha}} = \mathcal{A}_{d,\alpha}. \tag{43}$$

Example 2. Let $\nu(r) = r^{-d}g(r)$, $r \in (0,\infty)$ and $\alpha, \alpha_1 \in (0,2)$. The same limit as (43) exists and equals 1 for

(i) truncated stable process: $g(r) = r^{-\alpha}\mathbb{1}_{(0,1)}(r)$;
(ii) tempered stable process: $g(r) = r^{-\alpha}e^{-r}$;
(iii) isotropic Lamperti stable process: $g(r) = re^{\delta r}(e^r-1)^{-\alpha-1}$, $\delta < \alpha+1$;
(iv) layered stable process: $g(r) = r^{-\alpha}\mathbb{1}_{(0,1)(r)} + r^{-\alpha_1}\mathbb{1}_{[1,\infty)}(r)$.

Since distributions of processes (i)–(iii) have a finite second moment, the dominated convergence theorem yields

$$\lim_{r\to 0^+} r^{-2}\psi(r) = 2^{-1}\int_0^\infty \rho g(\rho)\,\mathrm{d}\rho = c_1,$$

which implies, for $d \ge 3$,

$$\lim_{|x|\to+\infty} |x|^{2-d}G(x) = c_1\tilde{\mathcal{A}}_{d,2}.$$

Example 3. Let

$$\psi(\xi) = |\xi|^\alpha \log^\beta(1 + |\xi|^\gamma),$$

where $\gamma, \alpha, \alpha + 2\beta \in (0, 2)$. We note that ψ is the Lévy–Khintchine exponent of a subordinate Brownian motion; see [25, Theorem 12.14, Proposition 7.10, Proposition 7.1, Corollary 7.9, Section 13 and examples 1 and 26 from Section 15.2]. For this process we have

$$\lim_{\substack{x\to 0\\ t|x|^{-\alpha}\log^\beta|x|\to 0}} \frac{p(t,x)}{t|x|^{-d-\alpha}\log^\beta|x|} = \gamma^\beta \mathcal{A}_{d,\alpha}$$

and

$$\lim_{\substack{|x|\to+\infty\\ t|x|^{-\alpha-\gamma\beta}\to 0}} \frac{p(t,x)}{t|x|^{-d-\alpha-\gamma\beta}} = \mathcal{A}_{d,\alpha+\gamma\beta}.$$

Example 4. Let $\alpha \in (0, 2)$ and $\beta \in \mathbb{R}$ and $\mathbf{X}$ be isotropic unimodal with the Lévy density $\nu(r) = r^{-d-\alpha}\log^\beta(1 + e^\beta + r)$. Then

$$\lim_{\substack{|x|\to\infty\\ t|x|^{-\alpha}\log^\beta|x|\to 0}} \frac{p(t,x)}{t|x|^{-d-\alpha}\log^\beta|x|} = 1.$$

Example 5. Let $\mathbf{X}$ be the gamma variance process, i.e., $\psi(\xi) = \log(1+|\xi|^2)$. Then

$$\lim_{\substack{r\to 0^+\\ t|\log r|\to 0}} \frac{\mathbb{P}(|X_t| > r)}{t|\log r|} = 2.$$

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